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# Bergman kernels and equidistribution for sequences of line bundles on Kähler manifolds

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ABSTRACT

Given a sequence of positive Hermitian holomorphic line bundles $(L_p, h_p)$ on a Kähler manifold $X$, we establish the asymptotic expansion of the Bergman kernel of the space of global holomorphic sections of $L_p$, under a natural convergence assumption on the sequence of curvatures $c_1(L_p, h_p)$. We then apply this to study the asymptotic distribution of common

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zeros of random sequences of $m$-tuples of sections of $L_p$ as $p \to +\infty$.



**Contents**



## 0. Introduction

The familiar setting of geometric quantization is a compact Kähler manifold $(X, \omega)$ with Kähler form $\omega$ endowed with a Hermitian holomorphic line bundle $(L, h^L)$, called prequantum line bundle, satisfying the prequantization condition

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L = c_1(L, h^L), \tag{0.1}$$

where $R^L$ denotes the curvature of the Chern connection on $(L, h^L)$ and $c_1(L, h^L)$ denotes the Chern curvature form of $(L, h^L)$. The existence of the prequantum line bundle $(L, h^L)$ allows to consider the Hilbert space of holomorphic sections $H^0(X, L)$ and construct a correspondence between smooth objects on $X$ (classical observables) and operators on $H^0(X, L)$ (quantum observables) [2,20], stated in terms of the semi-classical limit in which Planck's constant tends to zero. Changing Planck's constant is equivalent to rescaling the Kähler form, and this is achieved by taking tensor powers $L^p = L^{\otimes p}$ of the line bundle, since the curvature of $L^p$ is $pR^L$. In this case the Planck constant corresponds to $\hbar = 1/p$. A pivotal role in this correspondence is played by the orthogonal projection on $H^0(X, L^p)$, see [28,30]. Its integral kernel, called Bergman kernel, admits a full asymptotic expansion as $p \to +\infty$ to any order (cf. [5,12,28,29,36,37]).

Condition (0.1) is an integrality condition: a prequantum bundle exists if and only if the de Rham cohomology class $[\omega]$ is integral, $[\omega] \in H^2(X, \mathbb{Z})$. What can one do in general if $\omega$ is a not necessarily integral Kähler form? We can then associate to $\omega$ a more general sequence of positive line bundles $(L_p, h_p)$ such that their curvatures only approximate multiples of $\omega$. Such a sequence can be thought as a "prequantization" of the nonintegral Kähler form $\omega$.

In this paper we establish the asymptotic expansion of the Bergman kernel of the holomorphic space $H^0(X, L_p)$ on Kähler manifold $X$ under a natural approximation assumption of $\omega$ by the curvatures of the positive line bundles $L_p$.

There are several motivations and possible applications of our result. The first one is the Tian-Yau-Donaldson program [21,36], that studies the connection between the existence of Kähler metrics of constant scalar curvature and $K$-stability. If the metric is polarized by a line bundle $L$ (0.1), the expansion of the Bergman kernel of $H^0(X, L^p)$ plays a central role through its second coefficient that equals the scalar curvature of the given metric (cf. (0.4)).

A further motivation is geometric quantization with the goal of constructing a strict deformation quantization and obtaining the correspondence principle between classical and quantum observables. Since on such a deformation the commutator of operators corresponds *asymptotically* to $i\hbar$ times the Poisson bracket in the classical limit [28,30,33], the construction using a general sequence of approximating line bundle is quite natural. This is linked to Fedosov's "asymptotic operator representations" [22]. We will pursue this direction elsewhere.

Another motivation comes from questions arising around the "transcendental" holomorphic Morse inequalities [14,26]. The goal here is to extend to cohomology classes of type $(1, 1)$, which are in general not algebraic or even analytic, certain asymptotic cohomology estimates known for tensor powers of line bundles. The estimates of the asymptotic cohomology functions should involve Monge-Ampère integrals of the given cohomology class.

Let $(X, \vartheta, J)$ be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$ with Kähler form $\vartheta$ and complex structure $J$. Let $(L_p, h_p)$, $p \geqslant 1$, be a sequence of holomorphic Hermitian line bundles on $X$ with smooth Hermitian metrics $h_p$. Let $\nabla^{L_p}$ be the Chern connection on $(L_p, h_p)$ with curvature $R^{L_p} = (\nabla^{L_p})^2$. Denote by $c_1(L_p, h_p)$ the Chern curvature form of $(L_p, h_p)$. Let $g^{TX}(\cdot, \cdot) = \vartheta(\cdot, J\cdot)$ be the Riemannian metric on $TX$ induced by $\vartheta$ and $J$. The Riemannian volume form $dv_X$ has the form $dv_X = \vartheta^n/n!$. We endow the space $\mathscr{C}^\infty(X, L_p)$ of smooth sections of $L_p$ with the inner product

$$\langle s_1, s_2 \rangle := \int_X \langle s_1(x), s_2(x) \rangle \, \frac{\vartheta^n}{n!}\,, \quad s_1, s_2 \in \mathscr{C}^\infty(X, L_p), \tag{0.2}$$

and we set $\|s\|^2_{L^2} = \langle s, s \rangle$. We denote by $\mathcal{L}^2(X, L_p)$ the completion of $\mathscr{C}^\infty(X, L_p)$ with respect to this norm. Let $H^0(X, L_p)$ be the space of holomorphic sections of $L_p$ and let $P_p : \mathcal{L}^2(X, L_p) \to H^0(X, L_p)$ be the orthogonal projection. The integral kernel $P_p(x, x')$ $(x, x' \in X)$ of $P_p$ with respect to $dv_X(x')$ is smooth and is called the Bergman kernel. The restriction of the Bergman kernel to the diagonal of $X$ is the Bergman kernel function of $H^0(X, L_p)$, which we still denote by $P_p$, i.e., $P_p(x) = P_p(x, x)$.

The main result of this paper is as follows.

**Theorem 0.1.** *Let $(X, \vartheta)$ be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$. Let $(L_p, h_p)$, $p \geqslant 1$, be a sequence of holomorphic Hermitian line bundles on $X$ with smooth Hermitian metrics $h_p$. Let $\omega$ be a Kähler form on $X$ such that*

$$A_p^{-1} c_1(L_p, h_p) = \omega + O(A_p^{-a}), \text{ as } p \to +\infty, \text{ in the } \mathscr{C}^\infty\text{-topology}, \tag{0.3}$$

*where $a > 0$, $A_p > 0$ and $\lim_{p\to+\infty} A_p = +\infty$. Then as $p \to +\infty$, in the $\mathscr{C}^\infty$- topology,*

$$P_p(x) = A_p^n \boldsymbol{b}_0(x) + A_p^{n-1} \boldsymbol{b}_1(x) + \ldots + A_p^{n+\lfloor -a \rfloor+1} \boldsymbol{b}_{-\lfloor -a \rfloor -1}(x) + O(A_p^{n-a}), \tag{0.4}$$

*where $\boldsymbol{b}_0(x) = \omega^n/\vartheta^n$ and $\boldsymbol{b}_1 = \frac{1}{8\pi}(\omega^n/\vartheta^n) r^X_\omega$, where $r^X_\omega$ is the scalar curvature of $\omega$, and $\lfloor -a \rfloor$ the integer part of $-a$.*

Note that the following general result was obtained in [6, Theorem 1.2]. Let $(X, \vartheta)$ be a compact Kähler manifold of dimension $n$ and $(L_p, h_p)$, $p \geqslant 1$, be a sequence of holomorphic Hermitian line bundles on $X$ with singular Hermitian metrics $h_p$ that satisfy $c_1(L_p, h_p) \geqslant a_p \, \vartheta$, where $a_p > 0$ and $\lim_{p\to+\infty} a_p = +\infty$. If $A_p = \int_X c_1(L_p, h_p) \wedge \vartheta^{n-1}$ denotes the mass of the current $c_1(L_p, h_p)$, then $\frac{1}{A_p} \log P_p \to 0$ in $\mathcal{L}^1(X, \vartheta^n)$ as $p \to +\infty$. Theorem 0.1 refines this result under the stronger assumptions that the metrics are smooth and (0.3) holds.

Our assumption (0.3) means that for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$\left| A_p^{-1} c_1(L_p, h_p) - \omega \right|_{\mathscr{C}^k} \leqslant C_k A_p^{-a}, \tag{0.5}$$

where the $\mathscr{C}^k$-norm is induced by the Levi-Civita connection $\nabla^{TX}$. We will give several natural examples of sequences $(L_p, h_p)$ as above. The most straightforward is $(L_p, h_p) = (L^{\otimes p}, h^{\otimes p})$ for some fixed prequantum line bundle $(L, h)$. Then it follows from (0.1) that (0.3) holds for $A_p = p$ and all $a > 0$. In this case we recover from (0.4) the known result on asymptotic expansion of Bergman kernel of $H^0(X, L^p)$ (cf. [5,12,28,36,37]). Other examples include $(L_p, h_p) = (L^{\otimes p}, h_p)$ where $h_p$ is not necessarily the product $h^p$, e. g. $h_p = h^p e^{-\varphi_p}$, with suitable weights $\varphi_p$, or tensor powers of several bundles, see Theorem 0.3.

Our approximation assumption (0.3) (or (0.5)) is natural in the following sense. Given a Kähler form $\omega$ one can first approximate its cohomology class $[\omega] \in H^2(X, \mathbb{R})$ by integral classes in $H^2(X, \mathbb{Z})$ by using diophantine approximation (Kronecker's lemma) and then one constructs smooth forms representing these approximating classes. By [14], [26, Théorème 1.3, p. 57] condition (0.5) holds true for any $k$, $A_p = p$ and $a = 1+1/\beta_2(X)$, where $\beta_2(X)$ denotes the second Betti number of $X$, but in general with a non-necessarily holomorphic Hermitian line bundle $(L_p, h_p)$. In this paper we show that if there is a good diophantine approximation with holomorphic line bundles we obtain corresponding good asymptotics of the Bergman kernel.

The proof of Theorem 0.1 follows the general lines from the case of tensor powers $L^p$ with important technical changes. The hypothesis (0.3) says roughly that the curvatures $c_1(L_p, h_p)$ grow to infinity at rate $A_p$ with respect to $\omega$. This implies via the Lichnerowicz formula that the corresponding Dirac operators $D_p$ have a spectral gap of size $a_p \to \infty$, and this allows to show, via finite propagation speed for the wave operator, that the asymptotics have local nature and can be obtained by a scaling techniques from geometric data brought to $\mathbb{C}^n$. In the case of tensor powers $L^p$ the arguments are facilitated by the fact that once a local trivialization of $L$ is chosen, it induces trivializations for all $L^p$ and straightforward formulas for the connection forms $\Gamma^{L^p} = p\Gamma^L$ and the curvatures $c_1(L^p, h^p) = pc_1(L, h)$. The asymptotics are computed from the asymptotics of a local model, given by freezing the curvature of $L$ at a given point.

In the general case of a family $L_p$ we only have a condition on the growth of $c_1(L_p, h_p)$. Moreover, it is not a priori clear what would be the local model. The first technical challenge is to choose in a coherent manner trivializations and connection forms $\Gamma^{L_p}$ for each $L_p$ with control of their Sobolev norms. In this way we reduce the study of the asymptotics to a local problem, and we can construct a local potential of the metric $\omega$, which provides a "local prequantum bundle" (a trivial bundle with non-trivial metric) for $\omega$. We then consider the local restrictions of $(L_p, h_p)$ as real powers of the local prequantum bundle and prove that their Bergman kernels converge to the Bergman kernel of the model operator given by its curvature, that is by $\omega$. Thus, the convergence of the curvatures $c_1(L_p, h_p)$ in (0.3) forces the convergence of the Bergman kernels $P_p$ and we can deduce their asymptotics (0.4).

If $0 < a < 1$, then Theorem 0.1 reduces to the following result.

**Corollary 0.2.** *Let $(X, \vartheta)$ be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$. Let $(L_p, h_p)$, $p \geqslant 1$, be a sequence of holomorphic Hermitian line bundles on $X$ with smooth Hermitian metrics $h_p$. Assume that there exists a Kähler form $\omega$ on $X$ such that*

$$A_p^{-1} c_1(L_p, h_p) = \omega + O(A_p^{-a}), \text{ as } p \to +\infty, \text{ in the } \mathscr{C}^\infty\text{- topology,} \tag{0.6}$$

*where $0 < a < 1$, $A_p > 0$ and $\lim_{p\to+\infty} A_p = +\infty$. Then*

$$P_p(x) = A_p^n \boldsymbol{b}_0(x) + O(A_p^{n-a}), \text{ as } p \to +\infty, \text{ in the } \mathscr{C}^\infty\text{- topology,} \tag{0.7}$$

*where $\boldsymbol{b}_0(x) = \omega^n/\vartheta^n$.*

Note that a similar result was obtained in [6, Theorem 1.3] under different approximation assumptions.

An interesting situation when the previous results apply is when $L_p$ equals a product of tensor powers of several holomorphic line bundles, $L_p = F_1^{m_{1,p}} \otimes \ldots \otimes F_k^{m_{k,p}}$, where $\{m_{j,p}\}_p$, $1 \leqslant j \leqslant k$, are sequences in $\mathbb{N}$ such that $m_{j,p} = r_j\, p + O(p^{1-a})$ as $p \to +\infty$, where $a \geqslant 2$ and $r_j > 0$ are given. This means that $(m_{1,p}, \ldots, m_{k,p}) \in \mathbb{N}^k$ approximate the semiclassical ray $\mathbb{R}_{>0}\cdot(r_1,\dots,r_k)\in\mathbb{R}^k_{>0}$ with a remainder $O(p^{1-a})$, as $p\to+\infty$ (cf. also [6, Corollary 5.11]).

**Theorem 0.3.** *Let $(X,\vartheta)$ be a compact Kähler manifold of $\dim_{\mathbb{C}} X=n$. Let $(F_j,h^{F_j})$ be smooth holomorphic Hermitian line bundles on $X$ with $c_1(F_j,h^{F_j})\geqslant 0$ for $1\leqslant j\leqslant k$ and one of them is strictly positive, say, $c_1(F_1,h^{F_1})\geqslant\varepsilon\vartheta$ for some $\varepsilon>0$. Let $r_j>0$, $1\leqslant j\leqslant k$, be positive real numbers and set $\omega=\sum_{j=1}^k r_jc_1(F_j,h^{F_j})$. Assume that there exist sequences $\{m_{j,p}\}_p$, $1\leqslant j\leqslant k$, in $\mathbb{N}$ and $a\geqslant 2$, $C>0$ such that*

$$\Big|\frac{m_{j,p}}{p}-r_j\Big|\leqslant\frac{C}{p^a},\quad 1\leqslant j\leqslant k,\ \textit{for } p>1. \tag{0.8}$$

*Let $P_p$ be the Bergman kernel function of $H^0(X,F_1^{m_{1,p}}\otimes\ldots\otimes F_k^{m_{k,p}})$. Then*

$$P_p(x)=p^n\boldsymbol{b}_0(x)+p^{n-1}\boldsymbol{b}_1(x)+\ldots+p^{n-k}\boldsymbol{b}_k(x)+O(p^{n-a}), \tag{0.9}$$
$$\textit{as } p\to+\infty,\ \textit{in the } \mathscr{C}^\infty\textit{- topology,}$$

*where $k=-\lfloor -a\rfloor-1$, $\boldsymbol{b}_0(x)=\omega^n/\vartheta^n$ and $\boldsymbol{b}_1=\frac{1}{8\pi}(\omega^n/\vartheta^n)r^X_\omega$, where $r^X_\omega$ is the scalar curvature of $\omega$.*

We apply Theorem 0.1 to study the asymptotic distribution of common zeros of random sequences of $m$-tuples of sections of $L_p$ as $p\to+\infty$, see [6–10,18,19,34,35] for previous results and references. Let $(X,\omega)$ be a compact Kähler manifold of dimension $n$ and let $(L_p,h_p)$, $p\geqslant 1$, be a sequence of Hermitian holomorphic line bundles on $X$. To study the equidistribution problem in a more general frame, we assume that the metrics $h_p$ are of class $\mathscr{C}^2$ and verify condition (0.5) for $k=0$. Namely, there exist a Kähler form $\omega$ on $X$ and $a>0$, $C_0>0$, such that for every $p\geqslant 1$ we have

$$\big|A_p^{-1}c_1(L_p,h_p)-\omega\big|_{\mathscr{C}^0}\leqslant C_0A_p^{-a}\,,\ \text{ where } A_p>0 \text{ and } \lim_{p\to+\infty}A_p=+\infty. \tag{0.10}$$

As before we endow the space of global holomorphic sections $H^0(X,L_p)$ with the inner product (0.2) and we set $\|s\|_p^2=\langle s,s\rangle_p$, $d_p=\dim H^0(X,L_p)$. Let $P_p$ be the Bergman kernel function of $H^0(X,L_p)$. We assume that there exist $M_0>1$ and $p_0\in\mathbb{N}$ such that

$$\frac{A_p^n}{M_0}\leqslant P_p(x)\leqslant M_0A_p^n, \tag{0.11}$$

holds for every $x\in X$ and $p>p_0$. Note that, under the stronger hypothesis (0.3), condition (0.11) follows easily from Theorem 0.1 (see (0.4)).

Given $m\in\{1,\dots,n\}$ and $p\geqslant 1$ we consider the multi-projective space

$$\mathbb{X}_{p,m}:=\big(\mathbb{P}H^0(X,L_p)\big)^m, \tag{0.12}$$

equipped with the probability measure $\sigma_{p,m}$ which is the $m$-fold product of the Fubini-Study volume on $\mathbb{P}H^0(X, L_p) \simeq \mathbb{P}^{d_p-1}$. If $s \in H^0(X, L_p)$ we denote by $[s = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{s = 0\} \subset X$, and we let

$$[\mathbf{s}_p = 0] := [s_{p1} = 0] \wedge \ldots \wedge [s_{pm} = 0]\,, \text{ for } \mathbf{s}_p = (s_{p1}, \ldots, s_{pm}) \in \mathbb{X}_{p,m},$$

whenever this current is well-defined (see Section 3). We also consider the probability space

$$(\mathbb{X}_{\infty,m}, \sigma_{\infty,m}) := \prod_{p=1}^{\infty} (\mathbb{X}_{p,m}, \sigma_{p,m})\,.$$

In the above setting, we have the following theorem:

**Theorem 0.4.** *Let $(X, \vartheta)$ be a compact Kähler manifold of dimension $n$ and let $(L_p, h_p)$, $p \geqslant 1$, be a sequence of Hermitian holomorphic line bundles on $X$ with metrics $h_p$ of class $\mathscr{C}^2$. Assume that conditions (0.10) and (0.11) hold. Then there exist $C > 0$ and $p_1 \in \mathbb{N}$ such that for every $\beta > 0$, $m \in \{1, \ldots, n\}$ and $p > p_1$ there exists a subset $E^{\beta}_{p,m} \subset \mathbb{X}_{p,m}$ with the following properties:*

*(i) $\sigma_{p,m}(E^{\beta}_{p,m}) \leqslant C A_p^{-\beta}$;*

*(ii) if $\mathbf{s}_p \in \mathbb{X}_{p,m} \setminus E^{\beta}_{p,m}$ then, for any $(n-m, n-m)$ form $\phi$ of class $\mathscr{C}^2$ on $X$,*

$$\Big|\Big\langle \frac{1}{A_p^m}\,[\mathbf{s}_p = 0] - \omega^m, \phi\Big\rangle\Big| \leqslant C\left((\beta+1)\,\frac{\log A_p}{A_p} + A_p^{-a}\right)\|\phi\|_{\mathscr{C}^2}\,. \tag{0.13}$$

*Moreover, if $\sum_{p=1}^{\infty} A_p^{-\beta} < +\infty$ then estimate (0.13) holds for $\sigma_{\infty,m}$-a.e. sequence $\{\mathbf{s}_p\}_{p\geqslant 1} \in \mathbb{X}_{\infty,m}$ provided that $p$ is large enough.*

The question of characterizing the positive closed currents on $X$ which can be approximated by currents of integration along analytic subsets of $X$, and its local version as well, are important problems in pluripotential theory and have many applications. Results in this direction are obtained in [8,11]. Theorem 0.4 shows in particular that the smooth positive closed form $\omega^m$ can be approximated by currents of integration along analytic subsets of $X$ of dimension $n - m$, for each $m \in \{1, \ldots, n\}$.

The paper is organized as follows. In Section 1 we show that the asymptotic expansion of the Bergman kernel can be localized. In Section 2 we establish the asymptotic expansion of the Bergman kernel near the diagonal and then prove Theorem 0.1. The proof of Theorem 0.4 is given in Section 3, using the technique of meromorphic transforms of Dinh and Sibony [19], as in the papers [10,11].

## 1. Localization of the problem

In this Section we show that the problem has local nature by using the spectral gap for the Dirac operator (as consequence of the Lichnerowicz formula) and the finite propagation speed for wave operators.

### 1.1. Lichnerowicz formula

The complex structure $J$ induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles. Denote by $\Omega^{0,j}(X, L_p)$ the space of smooth $(0,j)$-forms over $X$ with values in $L_p$ and set $\Omega^{0,\bullet}(X, L_p) = \oplus_{j=0}^{n}\Omega^{0,j}(X, L_p)$. We still denote by $\langle\cdot,\cdot\rangle$ the fibrewise metric on $\Lambda(T^{*(0,1)}X) \otimes L_p$ induced by $g^{TX}$ and $h_p$.

The $L^2$-scalar product on $\Omega^{0,\bullet}(X, L_p)$ is given by

$$\langle s_1, s_2\rangle := \int_X \langle s_1(x), s_2(x)\rangle \,\frac{\vartheta^n}{n!}\,, \quad s_1, s_2 \in \Omega^{0,\bullet}(X, L_p), \tag{1.1}$$

and we set $\|s\|^2_{L^2} = \langle s, s\rangle$. Let $\overline{\partial}^{L_p,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L_p}$ with respect to the scalar product (1.1). The Dolbeault-Dirac operator is given by

$$D_p := \sqrt{2}\Big(\overline{\partial}^{L_p} + \overline{\partial}^{L_p,*}\Big) : \Omega^{0,\bullet}(X, L_p) \to \Omega^{0,\bullet}(X, L_p). \tag{1.2}$$

The Kodaira Laplacian

$$\square^{L_p} := \overline{\partial}^{L_p}\overline{\partial}^{L_p,*} + \overline{\partial}^{L_p,*}\overline{\partial}^{L_p} : \Omega^{0,\bullet}(X, L_p) \to \Omega^{0,\bullet}(X, L_p)\,, \tag{1.3}$$

preserves the $\mathbb{Z}$-grading on $\Omega^{0,\bullet}(X, L_p)$. It is an essentially self-adjoint operator on the space $\mathcal{L}^2_{0,\bullet}(X, L_p)$, the $L^2$-completion of $\Omega^{0,\bullet}(X, L_p)$. We have

$$D_p^2 = 2\,\square^{L_p}. \tag{1.4}$$

For any $v \in TX$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $v^*_{1,0} \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then $c(v) = \sqrt{2}(v^*_{1,0} \wedge -i_{v_{0,1}})$ defines the Clifford action of $v$ on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and $i$ denote the exterior and interior product respectively.

Let $\nabla^{TX}$ denote the Levi-Civita connection on $(TX, g^{TX})$, then its induced connection on $T^{(1,0)}X$ is the Chern connection $\nabla^{T^{(1,0)}X}$ on $(T^{(1,0)}X, h^{T^{(1,0)}X})$, where $h^{T^{(1,0)}X}$ is the Hermitian metric on $T^{(1,0)}X$ induced by $g^{TX}$. The Chern connection $\nabla^{T^{(1,0)}X}$ on $T^{(1,0)}X$ induces naturally a connection $\nabla^{\Lambda(T^{*(0,1)}X)}$ on $\Lambda(T^{*(0,1)}X)$. Then by [28, p. 31] we have for an orthonormal frame $\{e_j\}_{j=1}^{2n}$ of $(X, g^{TX})$,

$$D_p = \sum_{j=1}^{2n} c(e_j)\nabla_{e_j}^{\Lambda(T^{*(0,1)}X)\otimes L_p}, \tag{1.5}$$

with

$$\nabla^{\Lambda(T^{*(0,1)}X)\otimes L_p} = \nabla^{\Lambda(T^{*(0,1)}X)} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{L_p}. \tag{1.6}$$

Denote by $\Delta^{\Lambda(T^{*(0,1)}X)\otimes L_p}$ the Bochner Laplacian on $\Lambda(T^{*(0,1)}X) \otimes L_p$. Then

$$\Delta^{\Lambda(T^{*(0,1)}X)\otimes L_p} = -\sum_{j=1}^{2n} \Big[\nabla_{e_j}^{\Lambda(T^{*(0,1)}X)\otimes L_p}\nabla_{e_j}^{\Lambda(T^{*(0,1)}X)\otimes L_p} - \nabla_{\nabla^{TX}_{e_j}e_j}^{\Lambda(T^{*(0,1)}X)\otimes L_p}\Big]. \tag{1.7}$$

Let $K_X = \det(T^{*(1,0)}X)$ be the canonical line bundle on $X$. The Chern connection $\nabla^{T^{(1,0)}}X$ on $T^{(1,0)}X$ induces the Chern connection $\nabla^{K_X^*}$ on $K_X^* = \det(T^{(1,0)}X)$. Denote by $R^{K_X^*}$ the curvature of $\nabla^{K^*}$ and by $r^X$ the scalar curvature of $(X, g^{TX})$. The Lichnerowicz formula (cf. [28, (1.4.29)]) reads

$$D_p^2 = \Delta^{\Lambda(T^{*(0,1)}X)\otimes L_p} + \frac{r^X}{4} + \frac{1}{2}\Big(R^{L_p} + \frac{1}{2}R^{K_X^*}\Big)(e_i,e_j)c(e_i)c(e_j). \tag{1.8}$$

We have used Einstein's summation convention of summing over repeated indices from 1 to $2n$ (or $n$) without sum symbol. This is also used throughout the paper.

### *1.2. Spectral gap of the Dirac operator*

As in the case of powers $L^p$ of a single line bundle we have a spectral gap for the square $D_p^2$ of the Dirac operator acting on $L_p$. The result and the proof are analogous to [27, Theorem 1.1], [28, Theorem 1.5.5]. For a Hermitian holomorphic line bundle $(L,h)$ on $X$ set

$$a_L := \inf\left\{\frac{R_x^L(u,\overline{u})}{|u|^2_{g^{TX}}} : x \in X, u \in T_x^{(1,0)}X \setminus \{0\}\right\}. \tag{1.9}$$

Note that $a_L(x) = \inf\left\{R_x^L(u,\overline{u})/|u|^2_{g^{TX}} : u \in T_x^{(1,0)}X \setminus \{0\}\right\}$ is the smallest eigenvalue of the curvature form $R_x^L$ with respect to $g_x^{TX}$ for $x \in X$ and $a_L = \inf_{x\in X} a_L(x)$. We also set

$$C_X := \inf\left\{\frac{R_x^{K_X^*}(u,\overline{u})}{|u|^2_{g^{TX}}} : x \in X, u \in T_x^{(1,0)}X \setminus \{0\}\right\}. \tag{1.10}$$

Here the curvature on the anticanonical line bundle $K_X^*$ is induced by the Kähler form $\vartheta$. Since $X$ is compact, the quantities $a_L$ and $C_X$ are finite.

We denote by $\mathrm{Spec}(A)$ the spectrum of a self-adjoint operator $A$ on a Hilbert space.

**Theorem 1.1.** *Let $(X, \vartheta)$ be a compact Kähler manifold. Then for every Hermitian holomorphic line bundle $(L, h)$ on $X$, the Dirac operator $D = D_L$ on $L$ satisfies the estimate*

$$\left\|Ds\right\|_{L^2}^2 \geqslant 2(a_L + C_X)\left\|s\right\|_{L^2}^2, \quad s \in \Omega^{>0}(X, L) := \bigoplus_{j=1}^{n} \Omega^{0,j}(X, L). \tag{1.11}$$

*Moreover,* $\operatorname{Spec}(D^2) \subset \{0\} \cup [2(a_L + C_X), +\infty)$.

**Proof.** We will use the Bochner-Kodaira-Nakano formula [28, (1.4.63)]. The Chern connection on $K_X^* \otimes L$ is given by

$$\nabla^{K_X^* \otimes L} = \nabla^{K_X^*} \otimes \operatorname{Id} + \operatorname{Id} \otimes \nabla^L, \tag{1.12}$$

and its curvature is

$$R^{K_X^* \otimes L} = R^{K_X^*} \otimes \operatorname{Id} + \operatorname{Id} \otimes R^L. \tag{1.13}$$

Let $\{w_j\}_{j=1}^n$ be an orthonormal frame of $T^{(1,0)}X$. Then [28, (1.4.63)] reads

$$\square^L s = \Delta^{0,\bullet} s + R^{L \otimes K_X^*}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, \quad \text{for } s \in \Omega^{0,\bullet}(X, L), \tag{1.14}$$

where $\Delta^{0,\bullet}$ is a holomorphic Kodaira type Laplacian. More precisely, $\Delta^{0,\bullet}$ is the conjugate by a fiberwise isometry of the $(1,0)$-Laplacian $\overline{\square}^{L \otimes K_X^*}$ (cf. [28, Remark 1.4.16]). Thus $\langle \Delta^{0,\bullet} s, s \rangle \geqslant 0$ so (1.14) yields

$$\left\|\overline{\partial}^L s\right\|_{L^2}^2 + \left\|\overline{\partial}^{L,*} s\right\|_{L^2}^2 \geqslant \left\langle R^{K_X^* \otimes L}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s \right\rangle, \quad s \in \Omega^{0,\bullet}(X, L). \tag{1.15}$$

By (1.9) we have

$$\left\langle R^L(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s \right\rangle \geqslant a_L \left\|s\right\|_{L^2}^2, \quad s \in \Omega^{>0}(X, L). \tag{1.16}$$

By (1.10) we have

$$\left\langle R^{K_X^*}(w_j, \overline{w}_k)\overline{w}^k \wedge i_{\overline{w}_j} s, s \right\rangle \geqslant C_X \left\|s\right\|_{L^2}^2, \quad s \in \Omega^{>0}(X, L). \tag{1.17}$$

Then (1.11) follows immediately from (1.4) and (1.15)–(1.16). Since $X$ is compact, $D^2$ has a discrete spectrum consisting of eigenvalues of finite multiplicity. Let $s \in \mathscr{C}^\infty(X, L)$ be an eigensection of $D^2$ with $D^2 s = \lambda s$ and $\lambda \neq 0$, then $Ds \neq 0$ and

$$D^2(Ds) = \lambda Ds. \tag{1.18}$$

Now $Ds \in \Omega^{0,1}(X, L)$, so by (1.11) we have $\lambda \geqslant 2(a_L + C_X)$. $\square$

For a sequence $(L_p, h_p)$, $p \geqslant 1$, let us denote

$$a_p := a_{L_p}, \tag{1.19}$$

with $a_{L_p}$ in (1.9). We have thus

$$\big\|D_p s\big\|^2_{L^2} \geqslant 2(a_p + C_X)\big\|s\big\|^2_{L^2}, \;\; s \in \Omega^{>0}(X, L_p) = \bigoplus_{j=1}^{n} \Omega^{0,j}(X, L_p). \tag{1.20}$$

Note that under hypothesis (0.3) we have $\lim_{p\to+\infty} a_p = +\infty$. As a consequence of Theorem 1.1 we obtain a Kodaira-Serre vanishing theorem for the sequence $L_p$.

**Corollary 1.2.** *Let $(X, \vartheta)$ be a compact Kähler manifold and let $(L_p, h_p)$, $p \geqslant 1$, be a sequence of holomorphic Hermitian line bundles on $X$ such that $\lim_{p\to+\infty} a_p = +\infty$. Then for $p$ large enough the Dolbeault cohomology groups of $L_p$ satisfy*

$$H^{0,j}(X, L_p) = 0, \quad \text{for } j \neq 0. \tag{1.21}$$

*Hence the kernel of $D_p^2$ is concentrated in degree 0 for $p$ large enough, i.e.,*

$$\operatorname{Ker}(D_p^2) = H^0(X, L_p), \;\; p \gg 1. \tag{1.22}$$

**Proof.** By Hodge theory we know that

$$\operatorname{Ker} D_p^2\big|_{\Omega^{0,j}(X,L_p)} \simeq H^{0,j}(X, L_p), \tag{1.23}$$

where $H^{0,\bullet}(X, L_p)$ denotes the Dolbeault cohomology groups. Thus (1.21) follows from (1.20). Moreover, (1.23) and (1.21) yield (1.22). $\square$

We also note the following generalization of the Kodaira embedding theorem. The proof follows the pattern of the classical proof given in [23, p. 189].

**Corollary 1.3.** *Let $(X, \vartheta)$ be a compact Kähler manifold and let $(L_p, h_p)$, $p \geqslant 1$, be a sequence of holomorphic Hermitian line bundles on $X$ such that $\lim_{p\to+\infty} a_p = +\infty$. Then for $p$ large enough the Kodaira map*

$$\Phi_p : X \to \mathbb{P}(H^0(X, L_p)^*), \quad x \longmapsto \{s \in H^0(X, L_p) : s(x) = 0\},$$

*is well defined and a holomorphic embedding.*

We also need a generalization for non-compact manifolds of Theorem 1.1. Let $(X, \vartheta)$ be a Hermitian manifold and let $(L, h)$ be a Hermitian holomorphic line bundle on $X$.

If $(X,\vartheta)$ is complete, then the square $D^2$ of the Dirac operator on $L$ is essentially self-adjoint by the Andreotti-Vesentini lemma [28, Lemma 3.3.1] and we denote by $\mathrm{Dom}(D^2)$ the domain of its self-adjoint extension.

**Theorem 1.4.** *Let $(X,\vartheta)$ be a complete Kähler manifold with Ricci curvature $R^{K_X^*}$ bounded from below, that is, the infimum $C_X$ in (1.10) is finite. Then for every Hermitian holomorphic line bundles $(L,h)$ on $X$ with $a_L>-\infty$ we have*

$$\big\|Ds\big\|^2_{L^2}\geqslant 2(a_L+C_X)\big\|s\big\|^2_{L^2},\ \ s\in\mathrm{Dom}(D^2)\cap\bigoplus_{j=1}^n\mathcal{L}^2_{0,j}(X,L). \tag{1.24}$$

*Moreover,* $\mathrm{Spec}(D^2)\subset\{0\}\cup[2(a_L+C_X),+\infty)$.

**Proof.** The proof follows from the proofs of Theorem 1.5 and [28, Theorem 6.1.1]. □

*1.3. Localization of the problem*

Let $a^X$ be the injectivity radius of $(X,g^{TX})$, and $\varepsilon_0\in(0,a^X/4)$. We denote by $B^X(x,\varepsilon_0)$ and $B^{T_xX}(0,\varepsilon_0)$ the open ball in $X$ and $T_xX$ with the center $x$ and radius $\varepsilon_0$, respectively. Then we identify $B^{T_xX}(0,\varepsilon_0)$ with $B^X(x,\varepsilon_0)$ by the exponential map $Z\mapsto\exp^X_x(Z)$ for $Z\in T_xX$. Let $f:\mathbb{R}\to[0,1]$ be a smooth even function such that $f(v)=1$ for $|v|\leqslant\varepsilon_0/2$ and $f(v)=0$ for $|v|\geqslant\varepsilon_0$. Set

$$F(a)=\Big(\int_{-\infty}^{\infty}f(v)dv\Big)^{-1}\int_{-\infty}^{\infty}e^{iva}f(v)dv. \tag{1.25}$$

Then $F(a)$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0)=1$.

**Proposition 1.5.** *For any $l,m\in\mathbb{N}$, $\varepsilon_0\in(0,a^X/4)$, there exists $C_{l,m,\varepsilon_0}>0$ such that for $p\geqslant 1$ and $x,x'\in X$,*

$$\big|F(D_p)(x,x')-P_p(x,x')\big|_{\mathscr{C}^m(X\times X)}\leqslant C_{l,m,\varepsilon_0}A_p^{-l}, \tag{1.26}$$

$$\big|P_p(x,x')\big|_{\mathscr{C}^m(X\times X)}\leqslant C_{l,m,\varepsilon_0}A_p^{-l},\ \ \textit{if } d(x,x')\geqslant\varepsilon_0. \tag{1.27}$$

*Here the $\mathscr{C}^m$ norm is induced by $\nabla^{L_p}$ and $\nabla^{TX}$.*

**Proof.** We adapt here the proof of [12, Proposition 4.1], [28, Proposition 4.1.5]. For $a\in\mathbb{R}$, set

$$\phi_p(a)=\mathbb{1}_{[\sqrt{a_p},+\infty)}(|a|)F(a). \tag{1.28}$$

For $a_p>-2C_X$ we have by Theorem 1.1 and (1.19),

$$F(D_p) - P_p = \phi_p(D_p), \tag{1.29}$$

here we use the fact that the eigenspaces of the elliptic operator $D_p$ furnish complete orthonormal systems for $\mathcal{L}^2_{0,\bullet}(X, L_p)$, i.e., there is a Hilbert space decomposition of $\mathcal{L}^2_{0,\bullet}(X, L_p)$ into the sum of the eigenspaces of $D_p$. By (1.25), for any $m \in \mathbb{N}$ there exists $C_m > 0$ such that

$$\sup_{a\in\mathbb{R}} |a|^m |F(a)| \leqslant C_m. \tag{1.30}$$

Since $X$ is compact there exist $\{x_i\}_{i=1}^r$ such that $\{\mathcal{U}_i := B^X(x_i, \varepsilon_0)\}_{i=1}^r$ is a covering of $X$. We identify $B^{T_{x_i}X}(0, \varepsilon_0)$ with $B^X(x_i, \varepsilon_0)$ by the exponential map as above. For $Z \in B^{T_{x_i}X}(0, \varepsilon_0)$ we identify

$$(L_p)_Z \cong (L_p)_{x_i}\,, \quad \Lambda(T^{*(0,1)}X)_Z \cong \Lambda(T^{*(0,1)}_{x_i}X),$$

by parallel transport along the curve $[0, 1] \ni u \mapsto uZ$ with respect to the connection $\nabla^{L_p}$ and $\nabla^{\Lambda(T^{*(0,1)}X)}$, respectively.

Let $\{e_j\}_{j=1}^{2n}$ be an orthonormal basis of $T_{x_i}X$. Let $\tilde{e}_j(Z)$ be the parallel transport of $e_j$ with respect to $\nabla^{TX}$ along the above curve. Let $\Gamma^{L_p}$, $\Gamma^{\Lambda(T^{*(0,1)}X)}$ be the corresponding connection forms of $\nabla^{L_p}$ and $\nabla^{\Lambda(T^{*(0,1)}X)}$ with respect to any fixed frame for $L_p$ and $\Lambda(T^{*(0,1)}X)$ which is parallel along the above curve under the trivialization on $\mathcal{U}_i$.

Denote by $\nabla_U$ the ordinary differentiation operator on $T_{x_i}X$ in the direction $U$. By (1.5),

$$D_p = \sum_{j=1}^{2n} c(\tilde{e}_j)\big(\nabla_{\tilde{e}_j} + \Gamma^{L_p}(\tilde{e}_j) + \Gamma^{\Lambda(T^{*(0,1)}X)}(\tilde{e}_j)\big). \tag{1.31}$$

Let $\{\rho_i\}$ be a partition of unity subordinate to $\{\mathcal{U}_i\}$. For $\ell \in \mathbb{N}$, let $\mathbf{H}^\ell_p$ be the set of sections of $L_p$ over $X$ which lie in the $\ell$-th Sobolev space. We define a Sobolev norm on the $\ell$-th Sobolev space $\mathbf{H}^\ell_p$ by

$$\|s\|_{\mathbf{H}^\ell_p} = \sum_{j=1}^{r} \sum_{k=0}^{\ell} \sum_{i_1,\ldots,i_k=1}^{2n} \big\|\nabla_{e_{i_1}} \ldots \nabla_{e_{i_k}}(\rho_j s)\big\|_{L^2}. \tag{1.32}$$

Denote by $\mathcal{R} = \sum_j Z_j e_j$ the radial vector field. By [28, (1.2.32)], $L_{\mathcal{R}}\Gamma^{L_p} = i_{\mathcal{R}}R^{L_p}$. Set

$$\Gamma^{L_p} = \sum_{j=1}^{2n} a_j(Z)dZ_j, \quad a_j \in \mathscr{C}^\infty(\mathcal{U}_i). \tag{1.33}$$

Then

$$(L_{\mathcal{R}}\Gamma^{L_p})_Z = \sum_{j,k=1}^{2n} \Big(Z_k \frac{\partial a_j}{\partial Z_k}(Z)\Big) dZ_j + \sum_{j=1}^{2n} a_j(Z) dZ_j. \tag{1.34}$$

Evaluating at the point $tZ$ yields

$$\frac{\partial}{\partial t}\big(ta_j(tZ)\big)dZ_j = (L_{\mathcal{R}}\Gamma^{L_p})_{tZ} = (i_{\mathcal{R}}R^{L_p})_{tZ}. \tag{1.35}$$

From (1.35) we obtain immediately $\Gamma_0^{L_p} = 0$ and (cf. also [16, (2-16)])

$$\Gamma_Z^{L_p} = \int_0^1 (L_{\mathcal{R}}\Gamma^{L_p})_{tZ}\, dt = \int_0^1 (i_{\mathcal{R}}R^{L_p})_{tZ}\, dt, \tag{1.36}$$

which allows us to estimate the term $\Gamma^{L_p}$ in (1.31). From (0.5), (1.31), (1.32) and (1.36),

$$\|s\|_{\mathbf{H}^1_p} \leqslant C\big(\|D_p s\|_{L^2} + A_p\|s\|_{L^2}\big). \tag{1.37}$$

The quantity $A_p$ appears in (1.37) through formula (1.31) for $D_p$, which involves the connection form $\Gamma^{L_p}$ given by (1.36), where we can apply the estimate (0.5) of the curvature.

Let $Q$ be a differential operator of order $m$ with scalar principal symbol and with compact support in $\mathcal{U}_i$, then

$$\big[D_p, Q\big] = \sum_{j=1}^{2n} \big[c(\tilde{e}_j)\Gamma^{L_p}(\tilde{e}_j), Q\big] + \sum_{j=1}^{2n} \big[c(\tilde{e}_j)(\nabla_{\tilde{e}_j} + \Gamma^{\Lambda(T^{*(0,1)}X)}(\tilde{e}_j)), Q\big], \tag{1.38}$$

where the sums are differential operators of orders $m-1$ and $m$, respectively. By (1.37) and (1.38),

$$\begin{aligned}\big\|Qs\big\|_{\mathbf{H}^1_p} &\leqslant C\big(\|D_pQs\|_{L^2} + A_p\|Qs\|_{L^2}\big) \\ &\leqslant C\big(\|QD_ps\|_{L^2} + A_p\|s\|_{\mathbf{H}^m_p}\big).\end{aligned} \tag{1.39}$$

Due to (1.39), for every $m \in \mathbb{N}$ there exists $C'_m > 0$ such that for $p \geqslant 1$,

$$\big\|s\big\|_{\mathbf{H}^{m+1}_p} \leqslant C'_m\big(\|D_ps\|_{\mathbf{H}^m_p} + A_p\|s\|_{\mathbf{H}^m_p}\big). \tag{1.40}$$

This means that

$$\|s\|_{\mathbf{H}^{m+1}_p} \leqslant C'_m A_p^{m+1} \sum_{j=0}^{m+1} A_p^{-j}\|D_p^j s\|_{L^2}\,. \tag{1.41}$$

Using the Sobolev estimate (1.41) we can now repeat the proof of [12, Proposition 4.1], [28, Proposition 4.1.5] and conclude the proof of (1.26).

By (1.25) we have

$$F(D_p) = \Big( \int_{-\infty}^{\infty} f(v)dv \Big)^{-1} \int_{-\varepsilon_0}^{\varepsilon_0} \cos(vD_p) f(v)dv. \tag{1.42}$$

By [28, Theorem D.2.1] (finite propagation speed of solutions of hyperbolic equations), $F(D_p)(x,x') = 0$ if $d(x,x') \geqslant \varepsilon$. This yields (1.27). $\square$

Proposition 1.5 shows that for $\varepsilon \in (0, a^X/4)$, $P_p(x,x')$ is negligible of order $O(A_p^{-\infty})$ outside a neighborhood $\{(x,x') \in X \times X : d(x,x') < \varepsilon\}$ of the diagonal of $X \times X$. In this neighborhood $P_p(x,x')$ is approximated up to $O(A_p^{-\infty})$ by $F(D_p)(x,x')$. In the next section (Proposition 2.3) we show that given $x_0 \in X$ and $\varepsilon \in (0, a^X/4)$, the kernel $F(D_p)(x,x')$ for $x, x' \in B^X(x_0,\varepsilon)$ is close up to a term of order $O(A_p^{-\infty})$ to the Bergman kernel $P_p^0(x,x')$ constructed from the local data given by restrictions of $(L_p, h_p)$ and $\vartheta$ to $B^X(x_0,\varepsilon)$. Thus the asymptotics of $P_p(x,x')$ for $x, x' \in B^X(x_0,\varepsilon)$ depend only on the local geometric data in $B^X(x_0,\varepsilon)$, up to a negligible term $O(A_p^{-\infty})$.

## 2. Asymptotic expansion of Bergman kernel

In this Section we establish the asymptotic expansion of the Bergman kernel and then prove Theorem 0.1.

### *2.1. Uniform trivialization*

To get uniform estimates of the Bergman kernel in terms of $p$, we adapt the approach of [16, §2] by using holomorphic coordinates instead of normal coordinates as it was done in [12, §4.2] and [28, §4.1.3]. The advantage of using holomorphic coordinates is that we can recover the holomorphic structure of a Hermitian holomorphic line bundle under a unitary trivialization by solving the $\overline{\partial}$-equation for the local weight function as in Lemma 2.1, and then we can transfer the problem of having to deal with different holomorphic line bundles by working directly with the corresponding weight functions.

Let $\psi : \mathcal{U} \to V \subset \mathbb{C}^n$, $\mathcal{U} \subset X$, be a holomorphic local chart on $X$ such that $V$ is convex and $0 \in V$ (we identify $\mathcal{U}$ with $V$ and by abuse of notation, we sometimes write $x$ instead of $\psi(x)$). Let $B(x,\varepsilon)$ be the standard ball in $\mathbb{C}^n$ with center $x$ and radius $\varepsilon$. Then for $x \in \frac{1}{2}V := \{y \in \mathbb{C}^n, 2y \in V\}$, we will use the holomorphic coordinates induced by $\psi$ and let $0 < \varepsilon_0 < 1$ be such that $B(0, 8\varepsilon_0) \subset V$.

Recall that the Riemannian volume form $dv_X$ is given by $\vartheta^n/n!$. The $L^2$-norm on $B(0,\varepsilon) \subset V$ is given by

$$\|s\|^2_{L^2,\varepsilon} = \int_{B(0,\varepsilon)} |s(y)|^2 dv_X(y). \tag{2.1}$$

Let $S_p$ be a unitary section of $(L_p, h_p)$ which is parallel with respect to $\nabla^{L_p}$ along the curve $[0,1] \ni u \mapsto uZ$ for $|Z| \leqslant 8\varepsilon_0$.

**Lemma 2.1.** *For each $p \geqslant 1$ there exists a holomorphic frame $\sigma_p := e^{-f_p} S_p$ of $L_p$ over $B(0, 8\varepsilon_0)$ such that for any $k \in \mathbb{N}$ there exists $C_k > 0$ independent of $p$, so that*

$$\|f_p\|_{\mathscr{C}^k(B(0,6\varepsilon_0))} \leqslant C_k \|R^{L_p}\|_{k+n+1}, \tag{2.2}$$

*where $\|\cdot\|_{k+n+1}$ denotes the Sobolev norm.*

**Proof.** Denote by $\Gamma^{L_p}$ the connection form of $\nabla^{L_p}$ with respect to the frame $S_p$ of $L_p$ and by $(\Gamma^{L_p})^{0,1}$ the $(0,1)$-part of $\Gamma^{L_p}$. As $\overline{\partial}(\Gamma^{L_p})^{0,1} = 0$, by [15, Chapter VIII, Theorem 6.1 and (6.4)], there exists $f_p \in \mathscr{C}^\infty(B(0, 8\varepsilon_0))$ orthogonal to $\operatorname{Ker}(\overline{\partial})$ in the $L^2$-space satisfying

$$\overline{\partial} f_p = (\Gamma^{L_p})^{0,1}, \tag{2.3}$$

and

$$\|f_p\|_{L^2, 8\varepsilon_0} \leqslant c_1 \|\Gamma^{L_p}\|_{L^2, 8\varepsilon_0}, \tag{2.4}$$

where $c_1$ is a constant independent of $p$. Using elliptic estimates we obtain

$$\|f_p\|_{k+1,\epsilon} \leqslant c_{2,k,\epsilon,\epsilon'} \big(\|\overline{\partial} f_p\|_{k,\epsilon'} + \|f_p\|_{L^2,\epsilon'}\big), \tag{2.5}$$

where $\|\cdot\|_{k,\varepsilon}$ denotes the Sobolev norm on the Sobolev space $\mathbf{H}^k(B(0,\varepsilon))$ and $c_{2,k,\epsilon,\epsilon'}$ is a constant dependent on $k$, $0 < \epsilon < \epsilon' < 1$ and independent of $p$. Denote by $\varphi_p$ be the real part of $f_p$. From (2.3) we know that $\sigma_p := e^{-f_p} S_p$ forms a holomorphic frame of $L_p$ on $B(0, 8\varepsilon_0)$ with

$$|\sigma_p|^2_{h_p}(Z) = e^{-2\varphi_p(Z)}. \tag{2.6}$$

The estimate (2.2) follows from (1.36), (2.3)-(2.5), and the Sobolev embedding theorem. □

**Remark 2.2.** Note that on a Stein manifold $M$ we have $H^1(M, \mathscr{O}^*) \cong H^2(M, \mathbb{Z})$ due to Cartan's theorem B (see e.g. [25, p. 201]), thus any holomorphic line bundle $L$ over a Stein contractible manifold, for example a coordinate ball, is holomorphically trivial (this is due to Oka [31]). Lemma 2.1 gives a proof with estimates of this result over a coordinate ball.

For $x \in B(0, 2\varepsilon_0)$ consider the holomorphic family of holomorphic local coordinates

$$\psi_x : \psi^{-1}(B(x, 4\varepsilon_0)) \to B(0, 4\varepsilon_0), \quad \psi_x(y) := \psi(y) - x\,. \tag{2.7}$$

Consider the holomorphic family of holomorphic trivializations of $L_p$ associated with the coordinate $\psi_x$ and the frame $\sigma_p$. These trivializations are given by

$$\Psi_{p,x} : L_p|_{\psi^{-1}(B(x,4\varepsilon_0))} \to B(0,4\varepsilon_0) \times \mathbb{C} \tag{2.8}$$

with $\Psi_{p,x}(y, v_p) := (\psi_x(y), v_p/\sigma_p(y))$ for $v_p$ a vector in the fiber of $L_p$ over the point $y$.

Consider a point $x_0 \in B(0, 2\varepsilon_0)$. Denote by $f_{p,x_0} = f_p \circ \psi_{x_0}^{-1}$, $\varphi_{p,x_0} = \varphi_p \circ \psi_{x_0}^{-1}$ the function $f_p$, $\varphi_p$ in (2.6) in local coordinate $\psi_{x_0}$. Denote by $\varphi^{[1]}_{p,x_0}$ and $\varphi^{[2]}_{p,x_0}$ the first and second order Taylor expansion of $\varphi_{p,x_0}$, i.e.,

$$\begin{aligned} \varphi^{[1]}_{p,x_0}(Z) &:= \sum_{j=1}^{n} \Big( \frac{\partial \varphi_p}{\partial z_j}(x_0) z_j + \frac{\partial \varphi_p}{\partial \overline{z}_j}(x_0) \overline{z}_j \Big), \\ \varphi^{[2]}_{p,x_0}(Z) &:= \operatorname{Re} \sum_{j,k=1}^{n} \Big( \frac{1}{2} \frac{\partial^2 \varphi_p}{\partial z_j \partial z_k}(x_0) z_j z_k + \frac{\partial^2 \varphi_p}{\partial z_j \partial \overline{z}_k}(x_0) z_j \overline{z}_k \Big), \end{aligned} \tag{2.9}$$

where we write $z = (z_1, \ldots, z_n)$ the complex coordinate of $Z$.

Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\rho(t) = 1 \quad \text{if} \quad |t| < 2; \;\; \rho(t) = 0 \quad \text{if} \quad |t| > 4. \tag{2.10}$$

We denote in the sequel $X_0 = \mathbb{R}^{2n} \simeq T_{x_0} X$ and equip $X_0$ with the metric $g^{TX_0}(Z) := g^{TX}\big(\psi_{x_0}^{-1}(\rho(|Z|/\varepsilon_0) Z)\big)$. Now let $0 < \varepsilon < \varepsilon_0$ be determined and define

$$\phi_{p,\varepsilon}(Z) := \rho(|Z|/\varepsilon) \varphi_{p,x_0}(Z) + \big(1 - \rho(|Z|/\varepsilon)\big)\big(\varphi_p(x_0) + \varphi^{[1]}_{p,x_0}(Z) + \varphi^{[2]}_{p,x_0}(Z)\big). \tag{2.11}$$

Let $h^{L_{p,0}}_{\varepsilon}$ be the metric on $L_{p,0} = X_0 \times \mathbb{C}$ defined by

$$|1|^2_{h^{L_{p,0}}_{\varepsilon}}(Z) := e^{-2\phi_{p,\varepsilon}(Z)}. \tag{2.12}$$

Let $\nabla^{L_{p,0}}_{\varepsilon}$ be the Chern connection of $(L_{p,0}, h^{L_{p,0}}_{\varepsilon})$ and let $R^{L_{p,0}}_{\varepsilon}$ be the curvature of $\nabla^{L_{p,0}}_{\varepsilon}$. By (0.5), (2.2) and the Taylor expansion of the function $f_{p,x_0}$, there exists $C > 0$ independent of $p$ such that for $x_0 \in B(0, 2\varepsilon_0)$, $|Z| \leqslant 4\varepsilon$, $0 \leqslant j \leqslant 2$, we have

$$\Big| f_{p,x_0}(Z) - \big(f_p(x_0) + f^{[1]}_{p,x_0}(Z) + f^{[2]}_{p,x_0}(Z)\big) \Big|_{\mathscr{C}^j} \leqslant C A_p |Z|^{3-j}. \tag{2.13}$$

From (0.5) and (1.19), we may assume that $a_p/A_p \geqslant \mu_0$ holds for all $p \in \mathbb{N}^*$, here $\mu_0$ is a constant depending only on $\omega$. By (2.11)–(2.13), there exists $0 < \varepsilon < \varepsilon_0$ small enough such that the following estimate holds for every $x_0 \in B(0, 2\varepsilon_0)$, we have $B^X(x_0, \varepsilon) \subset U$ and

$$\inf \Big\{ \sqrt{-1} R^{L_{p,0}}_{\varepsilon,Z}(u, Ju)/|u|^2_{g^{TX_0}} : u \in T_Z X_0 \setminus \{0\} \text{ and } Z \in X_0 \Big\} \geqslant \frac{4}{5} a_p. \tag{2.14}$$

In the sequel we fix $\varepsilon > 0$ small such that (2.14) holds. Let

$$D_p^{X_0} = \sqrt{2}\Big(\overline{\partial}^{L_{p,0}} + (\overline{\partial}^{L_{p,0}})^*\Big) \tag{2.15}$$

be the Dolbeault-Dirac operator on $X_0$ associated to the above data, where $(\overline{\partial}^{L_{p,0}})^*$ is the adjoint of $\overline{\partial}^{L_{p,0}}$ with respect to the metrics $g^{TX_0}$ and $h_\varepsilon^{L_{p,0}}$. Note that over the ball $B(x_0, 2\varepsilon)$, $D_p$ is just the restriction of $D_p^{X_0}$.

Let $\nabla^{T^{(1,0)}X_0}$ be the holomorphic Hermitian connection on $(T^{(1,0)}X_0, h^{T^{(1,0)}X_0})$ with curvature $R^{T^{(1,0)}X_0}$. It induces naturally a connection $\nabla^{T^{(0,1)}X_0}$ on $T^{(0,1)}X_0$. Set $\widetilde{\nabla}^{TX_0} = \nabla^{T^{(1,0)}X_0} \oplus \nabla^{T^{(0,1)}X_0}$. Then $\widetilde{\nabla}^{TX_0}$ is a connection on $TX_0 \otimes_{\mathbb{R}} \mathbb{C}$.

Let $T_0$ be the torsion of the connection $\widetilde{\nabla}^{TX_0}$ and $T_{0,as}$ be the anti-symmetrization of the tensor $V, W, Y \to \langle T_0(V,W), Y\rangle$. Let $\nabla^{\mathrm{Cl}_0}$ be the Clifford connection on $\Lambda(T^{*(0,1)}X_0)$ (cf. [28, (1.3.5)]). Define the operator ${}^c(\cdot)$ on $\Lambda(T^*X_0) \otimes_{\mathbb{R}} \mathbb{C}$ by ${}^c(e^{i_1} \wedge \ldots \wedge e^{i_j}) = c(e_{i_1}) \ldots c(e_{i_j})$ for $1 \leqslant i_1 < \ldots < i_j \leqslant 2n$. Set

$$\nabla_U^{A_0} = \nabla_U^{\mathrm{Cl}_0} - \frac{1}{4}\, {}^c(i_U T_{0,as}). \tag{2.16}$$

Then as explained in [28, (1.4.27)-(1.4.28)], $\nabla^{A_0}$ preserves the $\mathbb{Z}$-grading on $\Lambda(T^{*(0,1)}X_0)$. Let $\nabla^{A_0 \otimes L_{p,0}}$ be the connection on $\Lambda(T^{*(0,1)}X_0) \otimes L_{p,0}$ induced by $\nabla^{A_0}$ and $\nabla_\varepsilon^{L_{p,0}}$ as in (1.6). Denote by $\Delta^{A_0 \otimes L_{p,0}}$ the Bochner Laplacian on $\Lambda(T^{*(0,1)}X_0) \otimes L_{p,0}$ associated with $\nabla^{A_0 \otimes L_{p,0}}$. By [3, (2.29)], [28, (1.2.51), (1.4.29)], we have

$$\begin{aligned}(D_p^{X_0})^2 = \Delta^{A_0 \otimes L_{p,0}} + \frac{r^{X_0}}{4} + {}^c\Big(R^{L_{p,0}} + \frac{1}{2}\mathrm{Tr}[R^{T^{(1,0)}X_0}]\Big) \\ - \frac{1}{4}\, {}^c(dT_{0,as}) - \frac{1}{8}\big|T_{0,as}\big|^2,\end{aligned} \tag{2.17}$$

where the norm $|A|$ for $A \in \Lambda^3(T^*X_0)$ is given by $|A|^2 = \sum_{i<j<k} |A(e_i, e_j, e_k)|^2$. By Theorem 1.4 we get from (2.14) the existence of $C > 0$ such that for any $p \in \mathbb{N}^*$,

$$\mathrm{Spec}\big((D_p^{X_0})^2\big) \subset \{0\} \cup [a_p - C, +\infty). \tag{2.18}$$

Note that $(D_p^{X_0})^2$ preserves the $\mathbb{Z}$-grading on $\Omega^{0,\bullet}(X_0, L_p)$.

Let $S_{p,x_0}$ be a unit section of $(L_{p,0}, h_{p,0})$ which is parallel with respect to $\nabla_\varepsilon^{L_{p,0}}$ along the curve $[0,1] \ni u \to uZ$ for any $Z \in X_0$. As in [16, p. 78] we choose

$$S_{p,x_0}(Z) = \exp\left(-\varphi_p(x_0) + 2\int_0^1 (i_Z \partial\phi_{p,\varepsilon})_{tZ}\, dt\right) 1. \tag{2.19}$$

The unit frame $S_{p,x_0}$ provides an isometry $L_{p,0} \simeq \mathbb{C}$. Let $P_p^0$ be the orthogonal projection

$$P_p^0 : L^2(X_0, L_{p,0}) \simeq L^2(X_0, \mathbb{C}) \to \operatorname{Ker} D_p^{X_0}\,, \tag{2.20}$$

and let $P_p^0(x,x')$ be the smooth kernel of $P_p^0$ with respect to the volume form $dv_{X_0}(x')$.

**Proposition 2.3.** *For any $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ such that for $x,x'\in B^{T_{x_0}X}(0,\varepsilon)$,*

$$\left|P_p^0(x,x')-P_p(x,x')\right|_{\mathscr{C}^m}\leqslant C_{l,m}A_p^{-l}. \tag{2.21}$$

**Proof.** Using (1.25) and (2.18), we know that $P_p^0-F(D_p)$ verifies also (1.26) for $x,x'\in B^{T_{x_0}X}(0,\varepsilon)$, thus we get (2.21). □

This shows that the asymptotics of $P_p(x,x')$ depend only on the local geometric data given by the restrictions of $(L_p,h_p)$ and $\vartheta$ to a neighborhood of a fixed point up to a negligible term $O(A_p^{-\infty})$.

*2.2. A family of holomorphic line bundles on $X_0$*

Up to now we have shown that $P_p(x,x')$ has the same asymptotics as the local Bergman kernel $P_p^0(x,x')$ associated to the local model $D_p^{X_0}$ of $D_p$. The next task is to find the asymptotics of $P_p^0(x,x')$ in terms of a local model defined by the Kähler form $\omega$. In the case of powers $L^p$ of a line bundle $L$, we used in [12,16,28,29] the scaling technique of Bismut-Lebeau [4] with scaling parameter $1/\sqrt{p}$. This allows to use continuous parameter $t$ in the neighborhood of 0, take derivatives with respect to $t$ as in [28, (4.1.56), (4.1.58)] and set $t=1/\sqrt{p}$ at the end. In the general case of an arbitrary sequence $(L_p,h_p)$ we will construct a local smooth family $(L_\tau)_\tau$ of Hermitian holomorphic line bundles that interpolates between $(L_p,h_p)$ and the Hermitian holomorphic line bundle $L_0$ with curvature $\omega$ (cf. (2.43)). This allows to perform analysis with respect to $\tau$. The associated Bergman kernels will then interpolate between the Bergman kernel $P_p^0(x,x')$ and the limit Bergman kernel.

The idea is that the proof of Lemma 2.1 and (0.3) allow us to approximate the line bundle $L_p|_{B^X(x,4\varepsilon_0)}$ by powers of a locally defined holomorphic line bundle with curvature $\omega$. Note that this is not possible globally on $X$ since $\omega$ is not an integral class. Although $\omega$ is not the curvature of a line bundle we define on $X_0$ a "connection form" $\Gamma$ inspired by (1.36). We set

$$\Gamma=-2\pi\sqrt{-1}\int_0^1(i_{\mathcal{R}}\omega)_{tZ}dt. \tag{2.22}$$

Since we work with complex coordinates, the $(0,1)$-part of $\Gamma$ is

$$\Gamma^{0,1}=-2\pi\sqrt{-1}\int_0^1(i_z\omega)_{tZ}dt, \tag{2.23}$$

with $z = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$. Set

$$\alpha_p = \frac{1}{A_p} c_1(L_p, h_p) - \omega. \tag{2.24}$$

**Lemma 2.4.** *There exists $f \in \mathscr{C}^\infty(B(0, 8\varepsilon_0))$ such that for any $k \in \mathbb{N}$ there exists $C_k > 0$ so that*

$$\overline{\partial} f = \Gamma^{0,1}, \quad \|f\|_{\mathscr{C}^k(B(0,6\varepsilon_0))} \leqslant C_k \|\omega\|_{k+n+1}, \tag{2.25}$$

*and for every $p \geqslant 1$ we have*

$$\left\| \frac{1}{A_p} f_p - f \right\|_{\mathscr{C}^k(B(0,6\varepsilon_0))} \leqslant C_k \|\alpha_p\|_{k+n+1}. \tag{2.26}$$

**Proof.** By the argument (1.33)-(1.36), we get

$$\frac{\sqrt{-1}}{2\pi} d\Gamma_Z = \int_0^1 t(d i_{\mathcal{R}} \omega)_{tZ} dt = \int_0^1 t(L_{\mathcal{R}} \omega)_{tZ} dt = \omega_Z, \tag{2.27}$$

and

$$i_{\mathcal{R}} \Gamma = 0, \ \Gamma|_{Z=0} = 0. \tag{2.28}$$

As $\omega$ is a $(1,1)$-form, from (2.23) and (2.27) we get

$$\overline{\partial} \Gamma^{0,1} = 0. \tag{2.29}$$

Again by [15, Chapter VIII, Theorem 6.1 and (6.4)], there exists $f \in \mathscr{C}^\infty(B(0, 8\varepsilon_0))$ orthogonal to $\mathrm{Ker}(\overline{\partial})$ in $L^2$-space satisfying

$$\overline{\partial} f = \Gamma^{0,1} \tag{2.30}$$

and

$$\|f\|_{L^2, 8\varepsilon_0} \leqslant c_1 \|\Gamma\|_{L^2, 8\varepsilon_0}. \tag{2.31}$$

Applying the above procedure for $\frac{1}{A_p}(\Gamma^{L_p})^{0,1} - \Gamma^{0,1}$, we obtain also

$$\left\| \frac{1}{A_p} f_p - f \right\|_{L^2, 8\varepsilon_0} \leqslant c_1 \left\| \frac{1}{A_p} (\Gamma^{L_p})^{0,1} - \Gamma^{0,1} \right\|_{L^2, 8\varepsilon_0}, \tag{2.32}$$

where $c_1$ is a constant independent of $p$. Using elliptic estimates as in (2.5), we get Lemma 2.4 from (2.30), (2.31) and (2.32). $\square$

Denote by $\varphi$ the real part of $f$ and identify $f, \varphi$ as functions on $U$ via $\psi$. From (2.25) we know $\sigma = e^{-f}$ forms a holomorphic frame of $L_0 = \mathbb{C}|_{B(0,8\varepsilon_0)}$ which is trivial as smooth line bundle with metric $h$,

$$|\sigma|_h^2(Z) = e^{-2\varphi(Z)} \tag{2.33}$$

and its curvature of Chern connection on $(\mathbb{C}|_{B(0,8\varepsilon_0)}, h)$ is

$$R^{L_0} = -2\pi\sqrt{-1}\omega. \tag{2.34}$$

For $0 \leqslant \tau \leqslant 1$, set

$$f_{\tau,p} = f + \tau\Big(\frac{1}{A_p} f_p - f\Big) A_p^a, \quad \phi_{\tau,p} = \mathrm{Re} f_{\tau,p}. \tag{2.35}$$

Now we get a smooth family of holomorphic line bundles $L_{\tau,p}$ on $B(0, 8\varepsilon_0)$ given by

$$B(0, 8\varepsilon_0) \times \mathbb{C} \to L_{\tau,p}, \quad (y, v) \to (y, ve^{-f_{\tau,p}}). \tag{2.36}$$

Thus we have the isomorphism of holomorphic line bundles on $B(0, 8\varepsilon_0)$ via (2.8) and (2.36):

$$L^{A_p}_{A_p^{-a},p} = L_p \ \text{ and } \ L_{0,p} = L_0. \tag{2.37}$$

Thus for $x \in B(0, 2\varepsilon_0)$,

$$\begin{gathered} B(0, 4\varepsilon_0) \times \mathbb{C} \to B(x, 4\varepsilon_0) \times \mathbb{C} = L_{\tau,p}|_{B(x,4\varepsilon_0)} \\ (y, v) \mapsto (y + x, ve^{-f_{\tau,p}(x+y)}) \end{gathered} \tag{2.38}$$

is an isomorphism of holomorphic line bundles.

For $x_0 \in B(0, 2\varepsilon_0)$ endowed with the local coordinates $\psi_{x_0}$ from (2.7), denote by

$$\varphi_{x_0} = \varphi \circ \psi_{x_0}^{-1}, \quad f_{x_0} = f \circ \psi_{x_0}^{-1}, \tag{2.39}$$

and we can define $\varphi_{x_0}^{[1]}, \varphi_{x_0}^{[2]}, \phi_\varepsilon, f_{x_0}^{[1]}, f_{x_0}^{[2]}$ as in (2.9) and (2.11) by replacing $\varphi_p, f_p$ by $\varphi, f$. Then we have for $x_0 \in B(0, 2\varepsilon_0)$, $|Z| \leqslant 4\varepsilon$, $0 \leqslant j \leqslant 2$,

$$\Big| f_{x_0}(Z) - \big(f(x_0) + f_{x_0}^{[1]}(Z) + f_{x_0}^{[2]}(Z)\big)\Big|_{\mathscr{C}^j} \leqslant C|Z|^{3-j}. \tag{2.40}$$

Let $\phi_{\tau,p,\varepsilon}$ be defined as in (2.11) by replacing $f_p$ by $f_{\tau,p}$. Let $h^{L_\tau}_{\varepsilon,p}$ be the metric on $L_\tau = X_0 \times \mathbb{C}$ defined by

$$|1|^2_{h^{L_\tau}_{\varepsilon,p}}(Z) = e^{-2\phi_{\tau,p,\varepsilon}(Z)}. \tag{2.41}$$

Let $\nabla^{L_\tau}_{\tau,p}$ be the Chern connection on $(L_\tau, h^{L_\tau}_{\varepsilon,p})$ and $R^{L_\tau}_{\tau,p}$ be the curvature of $\nabla^{L_\tau}_{\tau,p}$, then by (0.5), (2.26), (2.35) and (2.40), we can take $\varepsilon$ small enough and $\tau_0 > 0$ such that (2.14) holds and for any $p$ large enough, $x \in B(0, 2\varepsilon_0)$ and $\tau \in [0, \tau_0]$,

$$\inf\left\{\sqrt{-1}R^{L_\tau}_{\tau,p,Z}(u, Ju)/|u|^2_{g^{TX_0}} : u \in T_Z X_0 \setminus \{0\} \text{ and } Z \in X_0\right\} \geqslant \frac{4}{5}\mu_0. \tag{2.42}$$

We can summarize the above discussion as follows. We work on the trivial holomorphic line bundle $L_\tau = X_0 \times \mathbb{C}$ with the canonical section 1, and the metric $h^{L_\tau}_{\varepsilon,p}$ defined by (2.41). Then the curvature $R^{L_\tau}_{\tau,p}$ of the Chern connection $\nabla^{L_\tau}_{\tau,p}$ on $(L_\tau, h^{L_\tau}_{\varepsilon,p})$ verifies (2.42), and for $x_0 \in B(0, 2\varepsilon_0)$, we have the isometry of holomorphic line bundles on $B(0, 4\varepsilon_0)$

$$(L_\tau, h^{L_\tau}_{\varepsilon,p})^{\otimes A_p}\big|_{\tau=A_p^{-a}} \simeq (L_p, h_p), \quad \text{and} \quad (L_\tau, h^{L_\tau}_{\varepsilon,p})\big|_{\tau=0} \simeq (L_0, h). \tag{2.43}$$

Since $L_\tau$ is trivial we can consider real powers of $L_\tau$. This means that the global weight of the Hermitian metric is multiplied by the corresponding real number. Let $\nabla^{L_\tau^N}_{\tau,p}$ be the Chern connection on $(L_\tau, h^{L_\tau}_{\varepsilon,p})^{\otimes N}$ for $N \in \mathbb{R}_{>0}$. Let $D^{X_0}_{p,\tau,N}$ be the Dolbeault-Dirac operator defined in (2.15) associated with the holomorphic Hermitian line bundle $(L_\tau, h^{L_\tau}_{\varepsilon,p})^{\otimes N}$.

Now for $(L_\tau, h^{L_\tau}_{\varepsilon,p})$, we use as frame a unit section $S_{\tau,p,x_0}$ of $(L_\tau, h^{L_\tau}_{\varepsilon,p})$ which is parallel with respect to $\nabla^{L_\tau}_{\tau,p}$ along the curve $[0,1] \ni u \to uZ$, in particular, we can take as in (2.19):

$$S_{\tau,p,x_0}(Z) = \exp\left(-\phi_{\tau,p}(x_0) + 2\int_0^1 (i_Z \partial\phi_{\tau,p,\varepsilon})_{tZ}\, dt\right) 1. \tag{2.44}$$

The unit frame $S_{\tau,p,x_0}$ provides an isometry $L_\tau \simeq \mathbb{C}$, where the trivial line bundle $\mathbb{C}$ is endowed with the canonical metric. Thus under this identification we consider $(D^{X_0}_{p,\tau,N})^2$ acting on $\mathscr{C}^\infty(X_0, \mathbb{C})$ and we have

$$D^{X_0}_p = D^{X_0}_{p,A_p^{-a},A_p}. \tag{2.45}$$

Let $P_{0,p,\tau,N}$ be the orthogonal projection

$$P_{0,p,\tau,N} : L^2(X_0, L^N_\tau) \simeq L^2(X_0, \mathbb{C}) \to \operatorname{Ker}(D^{X_0}_{p,\tau,N})^2 \tag{2.46}$$

and $P_{0,p,\tau,N}(x, x')$ be the smooth kernel of $P_{0,p,\tau,N}$ with respect to $dv_{X_0}(x')$. Let $dv_{TX}$ be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = \kappa(Z) dv_{TX}(Z), \tag{2.47}$$

with $\kappa(0) = 1$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbb{C})$, $Z \in \mathbb{R}^{2n}$ and $t = \frac{1}{\sqrt{N}}$, set

$$
\begin{aligned}
(\delta_t s)(Z) =& s(Z/t),\\
\nabla_{\tau,t,\bullet} =& \delta_t^{-1} t \kappa^{1/2} \nabla^{L^N_\tau}_{\tau,p} \kappa^{-1/2} \delta_t, \\
\mathscr{L}^t_{\tau,2} =& \delta_t^{-1} t^2 \kappa^{1/2} (D^{X_0}_{p,\tau,N})^2 \kappa^{-1/2} \delta_t.
\end{aligned}
\tag{2.48}
$$

Denote by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|_0$ the inner product and the $L^2$-norm on $\mathscr{C}^\infty(X_0,\mathbb{C})$ induced by $dv_{TX}(Z)$. For $s \in \mathscr{C}^\infty_0(X_0,\mathbb{C})$, set

$$
\begin{aligned}
\|s\|^2_{t,0} &:= \|s\|^2_0 = \int_{\mathbb{R}^{2n}} |s(Z)|^2 dv_{TX}(Z), \\
\|s\|^2_{\tau,t,m} &:= \sum_{l=0}^m \sum_{j_1,\dots,j_l=1}^{2n} \|\nabla_{\tau,t,e_{j_1}} \dots \nabla_{\tau,t,e_{j_l}} s\|^2_{t,0} \,.
\end{aligned}
\tag{2.49}
$$

Let $\mathcal{P}_{0,p,\tau,t}(Z,Z') = \mathcal{P}_{0,p,\tau,t,x_0}(Z,Z')$ be the smooth kernel of the spectral projection

$$
\mathcal{P}_{0,p,\tau,t} : (L^2(X_0,\mathbb{C}), \|\cdot,\cdot\|_0) \to \mathrm{Ker}(\mathscr{L}^t_{\tau,2})
\tag{2.50}
$$

with respect to $dv_{TX}(Z')$.

*2.3. Asymptotics of the scaled operators and Bergman kernel*

Let $\{w_j\}_{j=1}^n$ be an orthonormal basis of $T^{(1,0)}_{x_0}X$. Then

$$
e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \;\text{ and }\; e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j), j = 1,\dots,n,
\tag{2.51}
$$

form an orthonormal basis of $T_{x_0}X$. Set

$$
\begin{aligned}
\nabla_{0,\bullet} =& \nabla_\bullet + \frac{1}{2}\gamma_{x_0}(Z,\cdot), \text{ with } \gamma = -2\pi\sqrt{-1}\omega, \\
\mathscr{L}^0_2 =& -\sum_{j=1}^{2n} (\nabla_{0,e_j})^2 - \gamma_{x_0}(w_j,\overline{w}_j).
\end{aligned}
\tag{2.52}
$$

Then by [28, Theorem 4.1.7], we have:

**Theorem 2.5.** *The operator $\mathscr{L}^t_{0,2}$ has the following expansion as $t \to 0$,*

$$
\mathscr{L}^t_{0,2} = \mathscr{L}^0_2 + \sum_{r=1}^k t^r \mathcal{O}_r + \mathscr{O}(t^{k+1}),
\tag{2.53}
$$

*where $\mathcal{O}_r$ are second order differential operators on $\mathscr{C}^\infty(\mathbb{R}^{2n},\mathbb{C})$.*

Now we discuss the eigenvalues and eigenfunctions of $\mathscr{L}_2^0$ in detail. We choose $\{w_j\}_{j=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)}X$ such that

$$\gamma_{x_0}(w_j, \overline{w}_j) = a_j, \quad a_j > 0. \tag{2.54}$$

Let $\{w^j\}_{j=1}^n$ be its dual basis. Then $\{e_j\}_{j=1}^{2n}$ given by (2.51) forms an orthonormal basis of $T_{x_0}X$. We use the coordinates on $\mathbb{R}^{2n} \simeq T_{x_0}X$ induced by $e_j$ as

$$\mathbb{R}^{2n} \ni (Z_1, \ldots, Z_{2n}) \longmapsto \sum_{j=1}^{2n} Z_j e_j \in T_{x_0}X. \tag{2.55}$$

In what follows we also introduce the complex coordinates $z = (z_1, \ldots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Thus $Z = z + \overline{z}$, and $w_j = \sqrt{2}\frac{\partial}{\partial z_j}$, $\overline{w}_j = \sqrt{2}\frac{\partial}{\partial \overline{z}_j}$. We will also identify $z$ to $\sum_j z_j \frac{\partial}{\partial z_j}$ and $\overline{z}$ to $\sum_j \overline{z}_j \frac{\partial}{\partial \overline{z}_j}$ when we consider $z$ and $\overline{z}$ as vector fields. Remark that

$$\Big|\frac{\partial}{\partial z_j}\Big|^2 = \Big|\frac{\partial}{\partial \overline{z}_j}\Big|^2 = \frac{1}{2}, \quad \text{so that } \ |z|^2 = |\overline{z}|^2 = \frac{1}{2}|Z|^2. \tag{2.56}$$

It is very useful to rewrite $\mathscr{L}_2^0$ by using the creation and annihilation operators. Set

$$b_j = -2\nabla_{0,\frac{\partial}{\partial z_j}}, \quad b_j^+ = 2\nabla_{0,\frac{\partial}{\partial \overline{z}_j}}, \quad b = (b_1, \ldots, b_n). \tag{2.57}$$

Then by (2.52) and (2.54), we have

$$b_j = -2\frac{\partial}{\partial z_j} + \frac{1}{2}a_j\overline{z}_j, \quad b_j^+ = 2\frac{\partial}{\partial \overline{z}_j} + \frac{1}{2}a_j z_j. \tag{2.58}$$

Then

$$\mathscr{L}_2^0 = \sum_{j=1}^n b_j b_j^+. \tag{2.59}$$

Let $\mathscr{P} : \big(\mathcal{L}^2(\mathbb{R}^{2n}), \|\cdot\|_{\mathcal{L}^2}\big) \to \operatorname{Ker}(\mathscr{L}_2^0)$ be the orthogonal projection. Denote by $\mathscr{P}(x, y)$ the Schwartz kernel of $\mathscr{P}$. By [28, (4.1.84)],

$$\mathscr{P}(Z, Z') = \prod_{j=1}^n \frac{a_j}{2\pi}\exp\Big[-\frac{1}{4}\sum_{j=1}^n a_j\big(|z_j|^2 + |z_j'|^2 - 2z_j\overline{z}_j'\big)\Big]. \tag{2.60}$$

In particular,

$$\mathscr{P}(0,0) = \prod_{j=1}^n \frac{a_j}{2\pi} = \frac{\omega^n}{\vartheta^n}\cdot \tag{2.61}$$

### 2.4. Proof of Theorem 0.1

By the arguments of [28, p. 194] we know that the constant in [28, Theorem 4.1.16] is uniformly bounded if with respect to a fixed metric $g_0^{TX}$, the $\mathscr{C}^{2n+m+m'+r+4}$-norms on $B(0,6\varepsilon_0)$ of the metric $h^L$ are bounded (here we use the power $N$ as parameter). Moreover, the $\mathscr{C}^{m'}$-norms in [28, (4.1.58)] can also include the parameters if the $\mathscr{C}^{m'}$-norms with respect to the parameter and $x_0 \in B(0,2\varepsilon_0)$ of the derivatives of the data $h^L$ with order $\leqslant 2n+m+r+4$ are bounded. Here we have a family of metrics $h^{L_\tau}_{\varepsilon,p}$ which certainly verify these conditions for $\tau \in [0,\tau_0]$ and $p$ large enough. Thus applying [28, Theorem 4.1.16] with parameter $\tau \in [0,\tau_0]$ and $r=0$, we obtain the following statement.

**Theorem 2.6.** *For any $m, m' \in \mathbb{N}$, $q>0$, there exists $C>0$ such that for $t \in [0,t_0]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant q$, $\tau \in [0,\tau_0]$,*

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}} \frac{\partial}{\partial \tau} \mathcal{P}_{0,p,\tau,t} \right|_{\mathscr{C}^{m'}(B(0,2\varepsilon_0))} \leqslant C. \tag{2.62}$$

By (2.62) we get

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha Z'^{\alpha'}} (\mathcal{P}_{0,p,0,t} - \mathcal{P}_{0,p,A_p^{-a},t})(Z,Z') \right|_{\mathscr{C}^{m'}(B(0,2\varepsilon_0))} \leqslant C A_p^{-a}. \tag{2.63}$$

Taking $Z = Z' = 0$ in (2.63) yields

$$\left| \mathcal{P}_{0,p,0,t}(x_0,x_0) - \mathcal{P}_{0,p,A_p^{-a},t}(x_0,x_0) \right|_{\mathscr{C}^{m'}(B(0,2\varepsilon_0))} \leqslant C A_p^{-a}. \tag{2.64}$$

By [28, (4.1.96)] we have

$$P_{0,p,\tau,N}(0,0) = N^n \mathcal{P}_{0,p,\tau,t}(0,0). \tag{2.65}$$

Moreover, $\mathcal{P}_{0,p,0,t}(x_0,x_0)$ does not depend on $p$ and by the argument in [28, §4.1.4-4.1.5] and Theorem 2.5, we see that for any $k \in \mathbb{N}$ there exists $C>0$ such that for $t \in [0,t_0]$, we have

$$\left| \mathcal{P}_{0,p,0,t}(x_0,x_0) - \sum_{r=0}^{k} t^{2r} \boldsymbol{b}_r(x_0) \right|_{\mathscr{C}^{m'}(B(0,2\varepsilon_0))} \leqslant C t^{2k+2}, \tag{2.66}$$

with $\boldsymbol{b}_r(x_0)$ is computed exactly as in [28, Theorem 4.1.21] associated with $-2\pi\sqrt{-1}\omega$. In particular, $\boldsymbol{b}_0(x) = \mathscr{P}(0,0)$ is given by (2.61). By (2.45) we have also

$$P_p^0(x_0,x_0) = P_{0,p,A_p^{-a},A_p}(x_0,x_0). \tag{2.67}$$

Proposition 2.3, relations (2.64), (2.65), (2.66) for $t = \frac{1}{\sqrt{A_p}}$ and (2.67), yield the conclusion of Theorem 0.1.

## 3. Equidistribution of zeros of random sections

In this section we prove Theorem 0.4. We assume throughout this section the setting of Theorem 0.4 and let $m \in \{1, \ldots, n\}$. We will denote by $\omega_{\text{FS}}$ the Fubini-Study form on a projective space $\mathbb{P}^d$, normalized so that $\omega_{\text{FS}}^d$ is a probability measure.

Let us start by introducing notation and recalling some facts needed for the proof. If $\{S_j^p\}_{j=1}^{d_p}$ is an orthonormal basis of $H^0(X, L_p)$ then the Bergman kernel function $P_p$ of $H^0(X, L_p)$ is given by

$$P_p(x) = \sum_{j=1}^{d_p} |S_j^p(x)|_{h_p}^2 , \;\; x \in X. \tag{3.1}$$

Let $U$ be a contractible Stein open set in $X$ and write $S_j^p = f_j^p e_p$, where $e_p$ is a local holomorphic frame of $L_p$ and $f_j^p$ is a holomorphic function on $U$. The *Fubini-Study current* $\gamma_p$ of $H^0(X, L_p)$ is defined by

$$\gamma_p \,|_U = \frac{1}{2}\, dd^c \log \sum_{j=1}^{d_p} |f_j^p|^2 , \tag{3.2}$$

where $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. These are positive closed currents of bidegree $(1,1)$, smooth away from the base locus $\operatorname{Bs} H^0(X, L_p)$ of $H^0(X, L_p)$. We have

$$\gamma_p = c_1(L_p, h_p) + \frac{1}{2}\, dd^c \log P_p \,. \tag{3.3}$$

Let $\Phi_p : X \dashrightarrow \mathbb{P}^{d_p - 1}$ be the Kodaira map defined by the basis $\{S_j^p\}_{j=1}^{d_p}$, so

$$\Phi_p(x) = [f_1^p(x) : \ldots : f_{d_p}^p(x)] \;\text{ for } x \in U. \tag{3.4}$$

Then $\gamma_p = \Phi_p^*(\omega_{\text{FS}})$.

If $s \in H^0(X, L_p)$ we denote by $[s = 0]$ the current of integration (with multiplicities) along the analytic hypersurface $\{s = 0\}$. One has the Lelong-Poincaré formula (see [28, Theorem 2.3.3])

$$[s = 0] = c_1(L_p, h_p) + dd^c \log |s|_{h_p} \,. \tag{3.5}$$

Recall that $\mathbb{X}_{p,m} = \big(\mathbb{P} H^0(X, L_p)\big)^m$, $d_p = \dim H^0(X, L_p)$. Set

$$d_{p,m} := \dim \mathbb{X}_{p,m} = m(d_p - 1). \tag{3.6}$$

Let $\pi_k : \mathbb{X}_{p,m} \to \mathbb{P} H^0(X, L_p)$ be the canonical projection onto the $k$-th factor. We endow $\mathbb{X}_{p,m}$ with the Kähler form

$$\omega_{p,m} := c_{p,m}\big(\pi_1^*\omega_{\rm FS} + \ldots + \pi_m^*\omega_{\rm FS}\big),$$

where the constant $c_{p,m}$ is chosen so that $\omega_{p,m}^{d_{p,m}} = \sigma_{p,m}$ is a probability measure on $\mathbb{X}_{p,m}$. It follows that

$$c_{p,m} = \left(\frac{\big((d_p-1)!\big)^m}{d_{p,m}!}\right)^{1/d_{p,m}}. \tag{3.7}$$

**Lemma 3.1.** *In the hypotheses of Theorem 0.4, the following hold for $p > p_0$:*

*(i) $\gamma_p$ are smooth $(1,1)$ forms on $X$.*

*(ii) For $\sigma_{p,m}$-a.e. $\mathbf{s}_p = (s_{p1},\ldots,s_{pm}) \in \mathbb{X}_{p,m}$ we have that the analytic set $\{s_{pi_1} = 0\} \cap \ldots \cap \{s_{pi_k} = 0\}$ has pure dimension $n-k$ for each $1 \leqslant k \leqslant m$ and $1 \leqslant i_1 < \ldots < i_k \leqslant m$. In particular the current $[\mathbf{s}_p = 0] := [s_{p1} = 0] \wedge \ldots \wedge [s_{pm} = 0]$ is well defined and is equal to the current of integration with multiplicities over the common zero set $\{\mathbf{s}_p = 0\} := \{s_{p1} = 0\} \cap \ldots \cap \{s_{pm} = 0\}$.*

**Proof.** By (0.11) we have $P_p(x) > 0$ for all $x \in X$ and $p > p_0$, hence $\operatorname{Bs} H^0(X, L_p) = \emptyset$ and $(i)$ follows from (3.3). Since $\operatorname{Bs} H^0(X, L_p) = \emptyset$ for $p > p_0$, [9, Proposition 4.1] implies that, for $\sigma_{p,m}$-a.e. $\mathbf{s}_p = (s_{p1},\ldots,s_{pm}) \in \mathbb{X}_{p,m}$, the analytic hypersurfaces $\{s_{p1} = 0\},\ldots,\{s_{pm} = 0\}$ are in general position, i.e. $\{s_{pi_1} = 0\} \cap \ldots \cap \{s_{pi_k} = 0\}$ has dimension at most $n-k$ for each $1 \leqslant k \leqslant m$ and $1 \leqslant i_1 < \ldots < i_k \leqslant m$. Hence

$$R := [s_{pi_1} = 0] \wedge \ldots \wedge [s_{pi_k} = 0] \tag{3.8}$$

is a well defined positive closed current of bidegree $(k,k)$ by [13, Corollary 2.11], supported in the set $\{s_{pi_1} = 0\} \cap \ldots \cap \{s_{pi_k} = 0\}$. Moreover, by the Lelong-Poincaré formula (3.5),

$$\int_X R \wedge \vartheta^{n-k} = \int_X c_1(L_p, h_p)^k \wedge \vartheta^{n-k} > 0\,.$$

So $\{s_{pi_1} = 0\} \cap \ldots \cap \{s_{pi_k} = 0\} \neq \emptyset$, hence it has pure dimension $n-k$. The last assertion of $(ii)$ now follows from [13, Corollary 2.11, Proposition 2.12]. $\square$

The proof of Theorem 0.4 uses results of Dinh and Sibony [19, Section 3.1] on meromorphic transforms. As in [19, Example 3.6 (c)], [10, Section 4.2], [17] we consider the meromorphic transform $\Phi_{p,1}$ from $X$ to $\mathbb{P}H^0(X, L_p)$ defined by its graph $\Gamma_{p,1} = \big\{(x,s) \in X \times \mathbb{P}H^0(X, L_p) : s(x) = 0\big\}$. This is related to the Kodaira map $\Phi_p$ from (3.4). Its $m$-fold product $\Phi_{p,m}$ (see [19, Section 3.3]) is the meromorphic transform from $X$ to $\mathbb{X}_{p,m}$ with graph

$$\Gamma_{p,m} = \big\{(x, s_{p1},\ldots,s_{pm}) \in X \times \mathbb{X}_{p,m} :\ s_{p1}(x) = \ldots = s_{pm}(x) = 0\big\}.$$

Using Lemma 3.1 (*ii*) and arguing as in [10, Section 4.2], it follows that $\Phi_{p,m}$ is a meromorphic transform of codimension $n-m$, with fibers

$$\Phi_{p,m}^{-1}(\mathbf{s}_p)=\{x\in X:\ s_{p1}(x)=\ldots=s_{pm}(x)=0\}\,,\ \text{ where }\ \mathbf{s}_p=(s_{p1},\ldots,s_{pm})\in\mathbb{X}_{p,m}\,.$$

Moreover, for $\mathbf{s}_p\in\mathbb{X}_{p,m}$ generic, the current

$$\Phi_{p,m}^*(\delta_{\mathbf{s}_p})=[\mathbf{s}_p=0]=[s_{p1}=0]\wedge\ldots\wedge[s_{pm}=0]=\Phi_{p,1}^*(\delta_{s_{p1}})\wedge\ldots\wedge\Phi_{p,1}^*(\delta_{s_{pm}})$$

is a well defined positive closed current of bidegree $(m,m)$ on $X$. Here $\delta_x$ denotes the Dirac mass at a point $x$, and $F^*(T)$ denotes the pull-back of a current $T$ by a meromorphic transform $F$ as defined in [19, Section 3.1]. Following the proof of [9, Theorem 1.2] (see also [10, Lemma 4.5]), we can show that

$$\Phi_{p,m}^*(\sigma_{p,m})=\gamma_p^m\,,\ \text{ for all } p>p_0.$$

We consider the intermediate degrees of $\Phi_{p,m}$ of order $d_{p,m}$, resp. $d_{p,m}-1$ [19, Section 3.1]:

$$\delta_{p,m}^1:=\int_X\Phi_{p,m}^*(\omega_{p,m}^{d_{p,m}})\wedge\vartheta^{n-m}\,,\ \ \delta_{p,m}^2:=\int_X\Phi_{p,m}^*(\omega_{p,m}^{d_{p,m}-1})\wedge\vartheta^{n-m+1}. \tag{3.9}$$

We have

$$\delta_{p,m}^1=\int_X c_1(L_p,h_p)^m\wedge\vartheta^{n-m}\,,\ \ \delta_{p,m}^2=\frac{1}{c_{p,m}}\int_X c_1(L_p,h_p)^{m-1}\wedge\vartheta^{n-m+1}. \tag{3.10}$$

Indeed, using (3.3), we infer by above that

$$\delta_{p,m}^1=\int_X\gamma_p^m\wedge\vartheta^{n-m}=\int_X c_1(L_p,h_p)^m\wedge\vartheta^{n-m},$$

and the formula for $\delta_{p,m}^2$ follows as in the proof of [10, Lemma 4.4]. We will need the following estimates:

**Lemma 3.2.** *(i) For every $p\geqslant 1$ and $m\in\{1,\ldots,n\}$, we have $\frac{1}{2em}<c_{p,m}<\frac{2e}{m}$.*

*(ii) There exist constants $M_1>1$ and $p_1>p_0$ such that, for every $p>p_1$, we have*

$$M_1^{-1}A_p^n\leqslant d_p\leqslant M_1A_p^n\,, \tag{3.11}$$

$$M_1^{-1}A_p^m\leqslant\delta_{p,m}^1\leqslant M_1A_p^m\,,\ \ M_1^{-1}A_p\leqslant\frac{\delta_{p,m}^1}{\delta_{p,m}^2}\leqslant M_1A_p\,,\ \ \forall\, m\in\{1,\ldots,n\}\,. \tag{3.12}$$

**Proof.** $(i)$ We have that [32, p. 200]

$$e^{\frac{7}{8}} < \frac{k!}{\left(\frac{k}{e}\right)^k \sqrt{k}} \leqslant e\,, \quad \text{for every } k \geqslant 1.$$

Since $k^{\frac{1}{2k}} < 2$ this implies that $\frac{k}{e} < \big(k!\big)^{\frac{1}{k}} < 2k$. Hence by (3.7) and (3.6),

$$\frac{1}{2em} < c_{p,m} = \frac{\big((d_p-1)!\big)^{1/(d_p-1)}}{\big(d_{p,m}!\big)^{1/d_{p,m}}} < \frac{2e}{m}\,.$$

$(ii)$ We infer from (0.10) that there exists $p_1 \in \mathbb{N}$ such that

$$\frac{\omega}{2} \leqslant \frac{1}{A_p} c_1(L_p, h_p) \leqslant 2\omega\,, \quad \text{for all } p > p_1. \tag{3.13}$$

By (3.10) we obtain

$$2^{-m} A_p^m \int\limits_X \omega^m \wedge \vartheta^{n-m} \leqslant \delta^1_{p,m} \leqslant 2^m A_p^m \int\limits_X \omega^m \wedge \vartheta^{n-m},$$

which readily implies the first estimate from (3.12). Using this and part $(i)$, we obtain the estimate on $\delta^1_{p,m}/\delta^2_{p,m}$ from (3.12), by increasing the constant $M_1$. Finally, using (0.11), we get

$$\frac{A_p^n}{M_0} \int\limits_X \frac{\vartheta^n}{n!} \leqslant d_p = \int\limits_X P_p \frac{\vartheta^n}{n!} \leqslant M_0 A_p^n \int\limits_X \frac{\vartheta^n}{n!}\,, \quad \text{for all } p > p_0. \quad \square$$

Our next result deals with the part of the proof of Theorem 0.4 which uses the Dinh-Sibony meromorphic transform technique and equidistribution theorem [19, Theorem 4.1, Lemma 4.2 (d)]. For $p > p_0$, $m \in \{1, \dots, n\}$ and $\varepsilon > 0$, let

$$E_{p,m}(\varepsilon) := \bigcup_{\|\phi\|_{\mathscr{C}^2} \leqslant 1} \big\{ \mathbf{s}_p \in \mathbb{X}_{p,m} : \big|\big\langle [\mathbf{s}_p = 0] - \gamma_p^m, \phi \big\rangle\big| \geqslant A_p^m \varepsilon \big\}, \tag{3.14}$$

where $\phi$ is a $(n-m, n-m)$ form of class $\mathscr{C}^2$ on $X$. We also assume that the set of $\mathbf{s}_p \in \mathbb{X}_{p,m}$ for which the current $[\mathbf{s}_p = 0]$ is not well defined is contained in $E_{p,m}(\varepsilon)$. Note that, by Lemma 3.1, this is a set of measure 0 since $p > p_0$.

**Proposition 3.3.** *In the hypotheses of Theorem 0.4, there exist constants $\nu, \alpha, \zeta > 0$ and $p_1 > p_0$, such that for every $p > p_1$, $m \in \{1, \dots, n\}$ and $\varepsilon > 0$ we have*

$$\sigma_{p,m}(E_{p,m}(\varepsilon)) \leqslant \nu\, A_p^{\zeta}\, e^{-\alpha A_p \varepsilon}.$$

**Proof.** Fix $m \in \{1, \dots, n\}$. We apply [19, Lemma 4.2 (d)] to the sequence of meromorphic transforms $\Phi_{p,m} : (X, \vartheta) \dashrightarrow (\mathbb{X}_{p,m}, \omega_{p,m})$ of codimension $n - m$ and the probability measures $\sigma_{p,m} = \omega_{p,m}^{d_{p,m}}$ on $\mathbb{X}_{p,m}$. Let

$$E'_{p,m}(\varepsilon) := \bigcup_{\|\phi\|_{\mathscr{C}^2} \leqslant 1} \big\{ \mathbf{s}_p \in \mathbb{X}_{p,m} : \big| \big\langle [\mathbf{s}_p = 0] - \gamma_p^m, \phi \big\rangle \big| \geqslant \delta^1_{p,m} \varepsilon \big\},$$

where $p > p_0$ and $\delta^1_{p,m}$ is the degree of $\Phi_{p,m}$ defined in (3.9). By [19, Lemma 4.2 (d)] it follows that

$$\sigma_{p,m}(E'_{p,m}(\varepsilon)) \leqslant \Delta_p(\eta_{\varepsilon,p}), \ \text{ where } \eta_{\varepsilon,p} := \varepsilon \, \frac{\delta^1_{p,m}}{\delta^2_{p,m}} - 3R_p.$$

Here

$$R_p := R(\mathbb{X}_{p,m}, \omega_{p,m}, \sigma_{p,m}), \ \ \Delta_p(t) := \Delta(\mathbb{X}_{p,m}, \omega_{p,m}, \sigma_{p,m}, t), \ \text{ where } t > 0,$$

are quantities defined in [19, Sections 2.1, 2.2] and are related to the Alexander-Dinh-Sibony capacity [1,19,24]. We recall their definition in the present situation. Let $\mathcal{S}_{p,m}$ denote the class of quasiplurisubharmonic functions $\varphi$ on $\mathbb{X}_{p,m}$ such that $dd^c\varphi \geqslant -m\omega_{p,m}$ and $\int_{\mathbb{X}_{p,m}} \varphi \, d\sigma_{p,m} = 0$. Then

$$R_p = \sup \Big\{ \max_{\mathbb{X}_{p,m}} \varphi : \ \varphi \in \mathcal{S}_{p,m} \Big\}, \ \Delta_p(t) = \sup \{ \sigma_{p,m}(\varphi < -t) : \ \varphi \in \mathcal{S}_{p,m} \}.$$

By the appendix of [19] (see also [10, Lemma 4.6]) we infer that

$$R_p \leqslant \nu' m \big(1 + \log d_{p,m}\big), \ \ \Delta_p(t) \leqslant \nu' \big(d_{p,m}\big)^{\zeta'} e^{-\alpha' t}, \ t > 0,$$

where $\nu', \zeta', \alpha' > 0$ are constants depending only on $m$. Let $M_1, p_1$ be as in Lemma 3.2. Then by (3.12) we have for $p > p_1$,

$$\eta_{\varepsilon,p} \geqslant \frac{\varepsilon A_p}{M_1} - 3R_p \geqslant \frac{\varepsilon A_p}{M_1} - 3\nu' m \big(1 + \log d_{p,m}\big).$$

Hence

$$\sigma_{p,m}(E'_{p,m}(\varepsilon)) \leqslant \Delta_p(\eta_{\varepsilon,p}) \leqslant \nu'' \big(d_{p,m}\big)^{\zeta''} e^{-\alpha'' A_p \varepsilon},$$

where $\nu'', \zeta'' > 0$ are constants depending only on $m$ and $\alpha'' = \alpha'/M_1$. Using again (3.12) we have $\delta^1_{p,m} \leqslant M_1 A_p^m$, so $E_{p,m}(\varepsilon) \subset E'_{p,m}(\varepsilon/M_1)$. Therefore

$$\sigma_{p,m}(E_{p,m}(\varepsilon)) \leqslant \sigma_{p,m}(E'_{p,m}(\varepsilon/M_1)) \leqslant \nu'' \big(d_{p,m}\big)^{\zeta''} e^{-\alpha'' A_p \varepsilon / M_1}.$$

Since by (3.11), $d_{p,m} < m d_p \leqslant m M_1 A_p^n$ for $p > p_1$, the conclusion follows. $\square$

**Proposition 3.4.** *In the hypotheses of Theorem 0.4, there exist $C > 0$ and $p_1 \in \mathbb{N}$ such that for every $\beta > 0$, $m \in \{1, \dots, n\}$ and $p > p_1$ there exists a subset $E^\beta_{p,m} \subset \mathbb{X}_{p,m}$ with the following properties:*

*(i)* $\sigma_{p,m}(E^\beta_{p,m}) \leqslant C A_p^{-\beta}$;

*(ii) if* $\mathbf{s}_p \in \mathbb{X}_{p,m} \setminus E^\beta_{p,m}$ *then, for any* $(n-m, n-m)$ *form* $\phi$ *of class* $\mathscr{C}^2$ *on* $X$,

$$\Big|\frac{1}{A_p^m}\Big\langle [\mathbf{s}_p = 0] - \gamma_p^m, \phi \Big\rangle\Big| \leqslant C\,(\beta+1)\,\frac{\log A_p}{A_p}\,\|\phi\|_{\mathscr{C}^2}\,.$$

*Moreover, if* $\sum_{p=1}^\infty A_p^{-\beta} < +\infty$ *then the last estimate holds for* $\sigma_{\infty,m}$*-a.e. sequence* $\{\mathbf{s}_p\}_{p\geqslant 1} \in \mathbb{X}_{\infty,m}$ *provided that* $p$ *is large enough.*

**Proof.** For every $\beta > 0$, $m \in \{1, \dots, n\}$ and $p > p_1$, let

$$\varepsilon_p = \frac{(\beta+\zeta)\log A_p}{\alpha A_p}\,,\;\; E^\beta_{p,m} := E_{p,m}(\varepsilon_p),$$

where $p_1, \alpha, \zeta$ are as in Proposition 3.3 and the set $E_{p,m}(\varepsilon)$ is defined in (3.14). By Proposition 3.3, we have that

$$\sigma_{p,m}(E^\beta_{p,m}) \leqslant \nu\, A_p^\zeta\, e^{-\alpha A_p \varepsilon_p} = \nu A_p^{-\beta}.$$

If $\mathbf{s}_p = (s_{p1}, \dots, s_{pm}) \in \mathbb{X}_{p,m} \setminus E^\beta_{p,m}$ then, by the definition of $E^\beta_{p,m}$, the current $[\mathbf{s}_p = 0] = [s_{p1} = 0] \wedge \ldots \wedge [s_{pm} = 0]$ is well defined and

$$\Big|\frac{1}{A_p^m}\Big\langle [\mathbf{s}_p = 0] - \gamma_p^m, \phi \Big\rangle\Big| \leqslant \varepsilon_p\,\|\phi\|_{\mathscr{C}^2}\,,$$

for any $(n-m, n-m)$ form $\phi$ of class $\mathscr{C}^2$. So assertions $(i)$ and $(ii)$ hold with the constant $C := \max\big\{\nu, \frac{1}{\alpha}, \frac{\zeta}{\alpha}\big\}$. The last assertion follows from these using the Borel-Cantelli lemma (see e.g. the proof of [10, Theorem 4.2]). $\square$

**Proposition 3.5.** *In the hypotheses of Theorem 0.4, there exist $C > 0$ and $p_1 \in \mathbb{N}$ such that for every $m \in \{1, \dots, n\}$, $p > p_1$ and every $(n-m, n-m)$ form $\phi$ of class $\mathscr{C}^2$ on $X$, we have*

$$\Big|\Big\langle \frac{\gamma_p^m}{A_p^m} - \omega^m, \phi \Big\rangle\Big| \leqslant C\left(\frac{\log A_p}{A_p} + A_p^{-a}\right)\|\phi\|_{\mathscr{C}^2}\,.$$

**Proof.** There exists $c > 0$ such that for every real $(n-m, n-m)$ form $\phi$ of class $\mathscr{C}^2$, $m \in \{1, \dots, n\}$, and every real $(1,1)$ form $\theta$ on $X$ one has

$$-c\|\phi\|_{\mathscr{C}^2}\,\vartheta^{n-m+1} \leqslant dd^c\phi \leqslant c\|\phi\|_{\mathscr{C}^2}\,\vartheta^{n-m+1}, \tag{3.15}$$

$$-c\|\phi\|_{\mathscr{C}^0}\|\theta\|_{\mathscr{C}^0}\,\vartheta^{n-m+1}\leqslant\phi\wedge\theta\leqslant c\|\phi\|_{\mathscr{C}^0}\|\theta\|_{\mathscr{C}^0}\,\vartheta^{n-m+1}. \tag{3.16}$$

For $p>p_0$ let

$$\mathcal{R}_p:=\frac{\gamma_p^m}{A_p^m}-\omega^m\,,\ \ \rho_p:=\sum_{j=0}^{m-1}\frac{\gamma_p^j}{A_p^j}\wedge\omega^{m-1-j}\,,\ \ \alpha_p:=\frac{c_1(L_p,h_p)}{A_p}-\omega\,.$$

By (0.10), respectively by (3.3), we have that

$$\|\alpha_p\|_{\mathscr{C}^0}\leqslant\frac{C_0}{A_p^a}\,,\qquad \frac{\gamma_p}{A_p}-\omega=\alpha_p+\frac{1}{2A_p}\,dd^c\log P_p\,.$$

Hence if $\phi$ is a real $(n-m,n-m)$ form of class $\mathscr{C}^2$ we obtain that

$$\langle\mathcal{R}_p,\phi\rangle=\Big\langle\Big(\frac{\gamma_p}{A_p}-\omega\Big)\wedge\rho_p,\phi\Big\rangle=\int_X\rho_p\wedge\alpha_p\wedge\phi+\int_X\frac{\log P_p}{2A_p}\,\rho_p\wedge dd^c\phi\,. \tag{3.17}$$

Using (3.16) we infer that

$$-\frac{c\,C_0}{A_p^a}\,\|\phi\|_{\mathscr{C}^0}\,\vartheta^{n-m+1}\leqslant\alpha_p\wedge\phi\leqslant\frac{cC_0}{A_p^a}\,\|\phi\|_{\mathscr{C}^0}\,\vartheta^{n-m+1},$$

hence

$$\Big|\int_X\rho_p\wedge\alpha_p\wedge\phi\Big|\leqslant\frac{cC_0}{A_p^a}\,\|\phi\|_{\mathscr{C}^0}\int_X\rho_p\wedge\vartheta^{n-m+1}. \tag{3.18}$$

By (3.15), the total variation of the signed measure $\rho_p\wedge dd^c\phi$ verifies

$$|\rho_p\wedge dd^c\phi|\leqslant c\|\phi\|_{\mathscr{C}^2}\,\rho_p\wedge\vartheta^{n-m+1}.$$

Therefore

$$\Big|\int_X\frac{\log P_p}{2A_p}\,\rho_p\wedge dd^c\phi\Big|\leqslant c\|\phi\|_{\mathscr{C}^2}\int_X\frac{|\log P_p|}{2A_p}\,\rho_p\wedge\vartheta^{n-m+1}.$$

We choose $p_1>p_0$ such that (3.13) holds for $p>p_1$ and $A_p>M_0$ for $p>p_1$, where $M_0$ is the constant from (0.11). By (0.11) it follows that $A_p^{n-1}\leqslant P_p\leqslant A_p^{n+1}$, so $|\log P_p|\leqslant(n+1)\log A_p$, hold on $X$ for $p>p_1$. We infer that

$$\Big|\int_X\frac{\log P_p}{2A_p}\,\rho_p\wedge dd^c\phi\Big|\leqslant\frac{nc\|\phi\|_{\mathscr{C}^2}\log A_p}{A_p}\int_X\rho_p\wedge\vartheta^{n-m+1}\ \text{ for } p>p_1. \tag{3.19}$$

Using (3.3) and (3.13) we have, for $p>p_1$ and $0\leqslant j\leqslant m-1$, that

$$\int_X \frac{\gamma_p^j}{A_p^j} \wedge \omega^{m-1-j} \wedge \vartheta^{n-m+1} = \int_X \frac{c_1(L_p, h_p)^j}{A_p^j} \wedge \omega^{m-1-j} \wedge \vartheta^{n-m+1} \leqslant 2^j \int_X \omega^{m-1} \wedge \vartheta^{n-m+1}.$$

Hence

$$\int_X \rho_p \wedge \vartheta^{n-m+1} = \sum_{j=0}^{m-1} \int_X \frac{\gamma_p^j}{A_p^j} \wedge \omega^{m-1-j} \wedge \vartheta^{n-m+1} < 2^m \int_X \omega^{m-1} \wedge \vartheta^{n-m+1}. \tag{3.20}$$

By (3.17), (3.18), (3.19) and (3.20) we conclude that if $p > p_1$ then

$$|\langle \mathcal{R}_p, \phi \rangle| \leqslant 2^m \Big( \frac{c\, C_0}{A_p^a} \, \|\phi\|_{\mathscr{C}^0} + \frac{nc \log A_p}{A_p} \, \|\phi\|_{\mathscr{C}^2} \Big) \int_X \omega^{m-1} \wedge \vartheta^{n-m+1},$$

for every $m \in \{1, \dots, n\}$ and every real $(n-m, n-m)$ form $\phi$ of class $\mathscr{C}^2$. This implies the proposition. □

**Proof of Theorem 0.4.** Theorem 0.4 follows at once from Propositions 3.4 and 3.5. □

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## References

[1] H. Alexander, Projective Capacity, in: Recent Developments in Several Complex Variables, Proc. Conf., Princeton Univ., Princeton, N. J., 1979, in: Ann. of Math. Stud., vol. 100, Princeton Univ. Press, Princeton, N. J., 1981, pp. 3–27.
[2] F.A. Berezin, General concept of quantization, Commun. Math. Phys. 40 (1975) 153–174.
[3] J.-M. Bismut, A local index theorem for non-Kähler manifolds, Math. Ann. 284 (1989) 681–699.
[4] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, Publ. Math. Inst. Hautes Études Sci. 74 (1991), ii+298 pp., 1992.
[5] D. Catlin, The Bergman kernel and a theorem of Tian, in: Analysis and Geometry in Several Complex Variables, Katata, 1997, in: Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1–23.
[6] D. Coman, X. Ma, G. Marinescu, Equidistribution for sequences of line bundles on normal Kähler spaces, Geom. Topol. 21 (2) (2017) 923–962.
[7] D. Coman, G. Marinescu, Convergence of Fubini-study currents for orbifold line bundles, Int. J. Math. 24 (7) (2013) 1350051.
[8] D. Coman, G. Marinescu, On the approximation of positive closed currents on compact Kähler manifolds, Math. Rep. (Bucur.) 15(65) (4) (2013) 373–386.
[9] D. Coman, G. Marinescu, Equidistribution results for singular metrics on line bundles, Ann. Sci. Éc. Norm. Supér. (4) 48 (3) (2015) 497–536.
[10] D. Coman, G. Marinescu, V.-A. Nguyên, Hölder singular metrics on big line bundles and equidistribution, Int. Math. Res. Not. (16) (2016) 5048–5075.

[11] D. Coman, G. Marinescu, V.-A. Nguyên, Approximation and equidistribution results for pseudo-effective line bundles, J. Math. Pures Appl. (9) 115 (2018) 218–236.
[12] X. Dai, K. Liu, X. Ma, On the asymptotic expansion of Bergman kernel, J. Differ. Geom. 72 (2006) 1–41.
[13] J.-P. Demailly, Monge-Ampère operators, Lelong numbers and intersection theory, in: Complex Analysis and Geometry, Plenum, New York, 1993, pp. 115–193.
[14] J.-P. Demailly, Holomorphic Morse inequalities and asymptotic cohomology groups: a tribute to Bernhard Riemann, Milan J. Math. 78 (1) (2010) 265–277.
[15] J.-P. Demailly, Complex Analytic and Differential Geometry, Open Content Book, 2012.
[16] T.-C. Dinh, X. Ma, V.-A. Nguyen, On the asymptotic behavior of Bergman kernels for positive line bundles, Pac. J. Math. 29 (2017) 71–89.
[17] T.-C. Dinh, X. Ma, G. Marinescu, Equidistribution and convergence speed for zeros of holomorphic sections of singular Hermitian line bundles, J. Funct. Anal. 271 (11) (2016) 3082–3110.
[18] T.-C. Dinh, G. Marinescu, V. Schmidt, Equidistribution of zeros of holomorphic sections in the non-compact setting, J. Stat. Phys. 148 (2012) 113–136.
[19] T.-C. Dinh, N. Sibony, Distribution des valeurs de transformations méromorphes et applications, Comment. Math. Helv. 81 (1) (2006) 221–258.
[20] S.K. Donaldson, Planck's constant in complex and almost-complex geometry, in: XIIIth International Congress on Mathematical Physics, London, 2000, Int. Press, Boston, MA, 2001, pp. 63–72.
[21] S.K. Donaldson, Scalar curvature and projective embeddings. I, J. Differ. Geom. 59 (3) (2001) 479–522.
[22] B. Fedosov, Deformation Quantization and Index Theory, Mathematical Topics, vol. 9, Akademie Verlag, Berlin, 1996, 325 pp.
[23] Ph. Griffiths, J. Harris, Principles of Algebraic Geometry, Reprint of the 1978 Original, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, xiv+813 pp.
[24] V. Guedj, A. Zeriahi, Intrinsic capacities on compact Kähler manifolds, J. Geom. Anal. 15 (2005) 607–639.
[25] L. Hörmander, An Introduction to Complex Analysis in Several Variables, third edition, North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990, xii+254 pp.
[26] L. Laeng, Estimations spectrales asymptotiques en géométrie hermitienne, Thèse de Doctorat de l'Université Joseph-Fourier - Grenoble I, http://tel.archives-ouvertes.fr/tel-00002098/en/, 2002.
[27] X. Ma, G. Marinescu, The Spin$^c$ Dirac operator on high tensor powers of a line bundle, Math. Z. 240 (2002) 651–664.
[28] X. Ma, G. Marinescu, Holomorphic Morse Inequalities and Bergman Kernels, Progress in Mathematics, vol. 254, Birkhäuser Boston, Inc., Boston, MA, 2007.
[29] X. Ma, G. Marinescu, Generalized Bergman kernels on symplectic manifolds, Adv. Math. 217 (2008) 1756–1815.
[30] X. Ma, G. Marinescu, Toeplitz operators on symplectic manifolds, J. Geom. Anal. 18 (2) (2008) 565–611.
[31] K. Oka, Sur les fonctions des plusieurs variables. III: deuxième problème de cousin, J. Sci. Hiroshima Univ. 9 (1939) 7–19.
[32] W. Rudin, Principles of Mathematical Analysis, third edition, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976.
[33] M. Schlichenmaier, Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization, in: Conférence Moshé Flato 1999, vol. II (Dijon), in: Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 289–306.
[34] B. Shiffman, S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles, Commun. Math. Phys. 200 (1999) 661–683.
[35] B. Shiffman, S. Zelditch, Number variance of random zeros on complex manifolds, Geom. Funct. Anal. 18 (2008) 1422–1475.
[36] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, J. Differ. Geom. 32 (1990) 99–130.
[37] S. Zelditch, Szegö kernels and a theorem of Tian, Int. Math. Res. Not. 6 (1998) 317–331.