EQUDISTRIBUTION AND CONVERGENCE SPEED FOR ZEROS OF HOLOMORPHIC SECTIONS OF SINGULAR HERMITIAN LINE BUNDLES

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ABSTRACT. We establish the equidistribution of zeros of random holomorphic sections of powers of a semipositive singular Hermitian line bundle, with an estimate of the convergence speed.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to study the convergence speed of the zero-divisors of random sequences of holomorphic sections in high tensor powers of a holomorphic line bundle endowed with a singular Hermitian metric.

Distribution of zeros of random polynomials is a classical subject, starting with the papers of Bloch-Pólya, Littlewood-Offord, Hammersley, Kac and Erdös-Turán, see e.g., [3, 4, 5, 25] for a review and complete references. After the work of Nonnenmacher-Voros [19, 20], general methods were developed by Shiffman-Zelditch [26] and Dinh-Sibony [16] to describe the asymptotic distribution of zeros of random holomorphic sections of a positive line bundle over a projective manifold endowed with a smooth positively curved metric. The paper [16] gives moreover a good estimate of the convergence speed and applies to general measures (e. g., equidistribution of complex zeros of homogeneous polynomials with real coefficients). These methods were extended to the non-compact setting in [14]. Some important technical tools for higher dimension used in the previous works were introduced by Fornæss-Sibony [17].

In [6] it was shown that the equidistribution results from [16, 26] extend to the case of a singular Hermitian holomorphic line bundle with strictly positive curvature current.

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We will start with an abstract statement. For an arbitrary complex vector space V we denote by $\mathbb{P}(V)$ the projective space of 1-dimensional subspaces of V. For $v \in V$ we denote by $[v]$ its class in $\mathbb{P}(V)$. Fix now a vector space V of complex dimension $d+1$. Recall that there is a canonical identification of $\mathbb{P}(V^*)$ with the Grassmannian $G_d(V)$ of hyperplanes in $\mathbb{P}(V)$, given by $\mathbb{P}(V^*) \ni [\xi] \longmapsto H_\xi := \mathbb{P}(\ker \xi) \in G_d(V)$, for $\xi \in V^* \setminus \{0\}$. If V is endowed with a Hermitian metric, then we denote by $\omega_{\rm FS}$ the induced Fubini-Study form on projective spaces $\mathbb{P}(V)$ normalized so that $\sigma_{\rm FS} := \omega_{\rm FS}^d$ is a probability measure. We also use the same notations for $\mathbb{P}(V^*)$.

Fix an integer $1 \leqslant k \leqslant n$. We consider on $\mathbb{P}(V^*)^k$ the Haar measure $\sigma_{\rm MP}$ associated with the natural action of the unitary group on the factors of $\mathbb{P}(V^*)^k$ (cf. (3.10)). If $\xi = (\xi_1, \dots, \xi_k)$ is a point in $\mathbb{P}(V^*)^k$, denote by H_ξ the intersection of the hyperplanes H_{ξ_i} in $\mathbb{P}(V)$. The following extension of Theorem 3.2 stated below can be obtained using the ideas in [14] and [16].

Theorem 1.1. *Let (X, ω_X) be a compact Kähler manifold of dimension n and let V be a Hermitian complex vector space of dimension $d+1$. Let $\Phi : X \dashrightarrow \mathbb{P}(V)$ be a meromorphic map. Then, there exist $c > 0$ depending only on (X, ω_X) and $m > 0$ depending only on k such that for any $\gamma > 0$ there is a subset E_γ of $\mathbb{P}(V^*)^k$ with the following properties*

(a) $\sigma_{\rm MP}(E_\gamma) \leqslant c\, d^m e^{-\gamma/c}$.

(b) *For ξ outside E_γ, the current $\Phi^*[H_\xi]$ is well-defined and*

$$\|\Phi^*[H_\xi] - \Phi^*(\omega_{\rm FS}^k)\|_{-2} \leqslant \gamma m_{k-1}, \tag{1.1}$$

where m_k denotes the mass of the current $\Phi^*(\omega_{\rm FS}^k)$ (cf. (3.1), (3.2) for the definitions of the semi-norm $\| \quad \|_{-2}$ and mass on currents).

Consider now a holomorphic line bundle $L \to X$ on a compact Kähler manifold (X, ω_X) endowed with a singular Hermitian metric h^L. Let K_X be the canonical line bundle on X. Let (F, h^F) be an auxiliary Hermitian holomorphic line bundle endowed with a smooth metric h^F. These metrics and the volume form ω_X^n induce an L^2 scalar product (2.7) on the space of sections of $L^p \otimes F$ and we denote by $H^0_{(2)}(X, L^p \otimes F)$ the space of holomorphic L^2 sections (cf. (2.8)). These spaces are finite dimensional Hilbert spaces endowed with the scalar product (2.7).

This induces Fubini-Study metrics $\omega_{\rm FS}$ and probability measures $\sigma_{\rm FS}$ on the spaces $\mathbb{P}(H^0_{(2)}(X, L^p \otimes F))$ and also multi-projective metrics $\omega_{\rm MP}$ and natural probability measures $\sigma_p := \sigma_{p,\rm MP}$ on $\mathbb{P}(H^0_{(2)}(X, L^p \otimes F))^k$ (see (3.10)). Consider the probability space

$$(\Omega_k(L, F), \sigma_\infty) := \prod_{p=1}^{\infty} \big(\mathbb{P}(H^0_{(2)}(X, L^p \otimes F))^k, \sigma_p\big). \tag{1.2}$$

Although we don't indicate explicitly, these spaces depend on h^L, h^F. If F is trivial we just write $(\Omega_k(L), \sigma_\infty)$.

We have the following equidistribution result with speed estimate for the zeros of random L^2 holomorphic sections of big line bundles endowed with semipositively curved metrics. For a holomorphic section s of a line bundle we denote by $\mathrm{Div}(s)$ the associated divisor and by $[\mathrm{Div}(s)]$ the current of integration on $\mathrm{Div}(s)$. We refer to Definition 3.1 for the notion of convergence speed of currents.

Theorem 1.2. *Let (X,ω_X) be a compact Kähler manifold of dimension n, and let L be a holomorphic line bundle endowed with a singular metric h^L such that $c_1(L,h^L)\geq 0$ on X.*

(i) Assume that L is big and let $\widetilde{h}^L$ be a singular Hermitian metric on L with $c_1(L,\widetilde{h}^L)\geq \varepsilon\omega_X$ for some $\varepsilon>0$. Assume that $h^L\leqslant A\,\widetilde{h}^L$ for some constant $A>0$. Then for σ_∞-almost every sequence $([s_p])\in(\Omega_1(L),\sigma_\infty)$, $(\frac{1}{p}[\operatorname{Div}(s_p)])$ converges to $c_1(L,h^L)$ on X as $p\to\infty$ with speed $O\big(\frac{1}{p}\log p\big)$.

(ii) Let $U\subset X$ be a relatively compact open set such that $c_1(L,h^L)\geq\varepsilon\omega_X$ on a neighborhood of $\overline{U}$ for some $\varepsilon>0$. Then for σ_∞-almost every sequence $([s_p])\in(\Omega_1(L,K_X),\sigma_\infty)$, $(\frac{1}{p}[\operatorname{Div}(s_p)])$ converges to $c_1(L,h^L)$ on U as $p\to\infty$ with speed $O\big(\frac{1}{p}\log p\big)$.

The assumption $h^L\leqslant A\,\widetilde{h}^L$ in (i) means that h^L is less singular than the positively curved metric $\widetilde{h}^L$. Note that the assumptions in (i) and (ii) are necessary. Without them there could be very few sections in $H^0(X,L^p)$ or $H^0(X,L^p\otimes K_X)$, respectively, that is, their dimension could be bounded independently of p.

We consider next continuous Hermitian metrics on ample line bundles. Let L be an ample line bundle over a compact Kähler manifold X of dimension n. Let h^L_0 be a smooth Hermitian metric on L such that $\alpha=c_1(L,h^L_0)$ is a Kähler form. Let h^L be a continuous Hermitian metric on L which is associated with a continuous function φ by $h^L=h^L_0e^{-2\varphi}$. We call φ a global weight of h. We do not assume that the curvature current $c_1(L,h^L)$ is positive (it is not of order 0 in general).

Define the equilibrium weight $\varphi_{\rm eq}$ associated with the continuous weight φ as the upper envelope of all α-psh functions (cf. (2.1)) smaller than φ on X,

$$\varphi_{\rm eq}:X\to[-\infty,\infty),\ \ \varphi_{\rm eq}(x):=\sup{}^*\Big\{\psi(x):\psi\in PSH(X,\alpha),\,\psi\leq\varphi\ \text{ on } X\Big\}\tag{1.3}$$

where the star denotes upper semi-continuous regularization. (The upper semi-continuous regularization of a function ψ is $\psi^*(x)=\limsup_{y\to x}\psi(y)$.) The equilibrium first Chern form is defined by

$$\omega_{\rm eq}:=\alpha+dd^c\varphi_{\rm eq}\,.\tag{1.4}$$

The equilibrium metric on L is given by $h^L_{\rm eq}=h^L_0e^{-2\varphi_{\rm eq}}$; it satisfies $c_1(L,h^L_{\rm eq})=\omega_{\rm eq}$. The wedge-products $\omega^k_{\rm eq}$, $1\leq k\leq n$, are well-defined on the set where $\varphi_{\rm eq}$ is locally bounded [1]. The equilibrium measure is given by $\mu_{\rm eq}=\omega^n_{\rm eq}$. When X is the projective line $\mathbb{P}^1$ and L is the hyperplane line bundle $\mathcal{O}(1)$, the measure $\mu_{\rm eq}$ is a minimizer of the weighted logarithmic energy [24].

The following result generalizes a result by Berman [2] where smooth weights φ were considered. It shows that the equilibrium weight of a global Hölder weight can be uniformly approximated by global Fubini-Study weights, with speed estimate.

Theorem 1.3. *Let (X,ω_X) be a compact Kähler manifold, (L,h^L_0) be an ample line bundle endowed with a smooth metric h^L_0 such that $c_1(L,h^L_0)$ is a Kähler form. Let $h^L=h^L_0e^{-2\varphi}$ be a singular metric on L, such that φ is Hölder continuous on X. Then the equilibrium weight $\varphi_{\rm eq}$ is continuous on X. Moreover, the global Fubini-Study weights φ_p given by* (4.4) *converge to $\varphi_{\rm eq}$ with estimate*

$$\|\varphi_p-\varphi_{\rm eq}\|_\infty=O\left(\frac{1}{p}\log p\right)\,,\ \ p\to\infty\,,\tag{1.5}$$

where $\|\cdot\|_\infty$ *denotes the supremum norm on* X*. In particular, for any* $1 \leq k \leq n$ *we have* $\frac{1}{p^k}\omega_p^k \to \omega_{\rm eq}^k$ *on* X *as* $p \to \infty$ *with speed* $O\big(\frac{1}{p}\log p\big)$*.*

Corollary 1.4. *Let* (X, ω_X)*,* (L, h^L) *and* U *be as in Theorem 1.3. Let* $1 \leq k \leq n$*. Then for* σ_∞*-almost every sequence* $(S_p) \in (\Omega_k(L), \sigma_\infty)$*,* $S_p = ([s_p^{(1)}], \ldots, [s_p^{(k)}])$*, the sequence of currents of integration on the common zeros* $\frac{1}{p^k}\big[s_p^{(1)} = \ldots = s_p^{(k)} = 0\big]$ *converges to* $\omega_{\rm eq}^k$ *on* X *as* $p \to \infty$ *with speed* $O\big(\frac{1}{p}\log p\big)$*.*

The paper is organized like follows. In Section 2 we recall the notions of Bergman kernel and Fubini-Study currents in the context of singular Hermitian metrics. In Section 3 we describe a general setting for the equidistribution of zeros, which also delivers precise information about the convergence speed. In Section 4 we apply these results to semipositive Hermitian metrics and prove Theorem 1.2. Finally, in Section 5 we consider the case of arbitrary singular metrics and prove Theorem 1.3 and Corollary 1.4.

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2. Preliminaries

Let X be a complex manifold. We assume that the reader is acquainted with the notion of plurisubharmonic (henceforth abbreviated psh) function $\varphi : X \to [-\infty, \infty)$, see [13, Ch. I (5.1)]. Recall that psh functions are locally integrable ([13, Ch. I (4.17), (5.3)]). A function $\varphi : X \to [-\infty, \infty)$ is called quasi-psh if it is locally given as the sum of a psh and a smooth function.

We also assume that the reader is familiar to the notion of positive current (in the sense of Lelong, i. e., non-negative, see [13, Ch. III (1.13)], [18, B.2.11]). For a positive current β we write $\beta \geq 0$. If α is a closed real current of bidegree $(1,1)$ on X we define the space of α-psh functions as

$$PSH(X, \alpha) := \big\{\varphi : X \to [-\infty, \infty) : \varphi \text{ quasi-psh}, \, dd^c\psi + \alpha \geq 0\big\}. \tag{2.1}$$

Here $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$, hence $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

Let (X, ω_X) be a compact Kähler manifold of dimension n and consider a holomorphic line bundle $L \to X$. Let $U \subset X$ be an open set for which there exists a local holomorphic frame $e_L : U \longrightarrow L$.

Let h^L be a smooth Hermitian metric on L. Recall that the first Chern form $c_1(L, h^L)$ of h is defined by

$$c_1(L, h^L)\mid_U = -dd^c \log |e_L|_{h^L} = \frac{i}{2\pi} R^L, \tag{2.2}$$

where R^L is the curvature of the holomorphic Hermitian connection ∇^L on (L, h^L).

If h^L is a singular Hermitian metric on L then we set

$$|e_L|^2_{h^L} = e^{-2\varphi}, \tag{2.3}$$

where the function $\varphi \in L^1_{loc}(U)$ is called the local weight of the metric h with respect to the frame e_L (see [10], also [18, p. 97]). The curvature of h^L,

$$c_1(L, h^L)\mid_U = dd^c\varphi\,, \tag{2.4}$$

is a well-defined closed (1,1) current on X. The cohomology class of $c_1(L,h^L)$ in $H^{1,1}(X,\mathbb{R})$ does not depend on the choice of h^L. This is the Chern class of L and we denote it by $c_1(L)$.

We say that the metric h^L is semipositively curved if $c_1(L,h^L)$ is a positive current. Equivalently, the local weights φ given by (2.3) are (equal almost everywhere) to psh functions. Recall that a line bundle L is said to be pseudoeffective if it admits a (singular) semipositively curved metric h^L (see [10]).

Let L be a holomorphic line bundle and h^L_0 be a smooth metric on L. Set $\alpha = c_1(L,h^L_0)$. Let us denote by $\mathrm{Met}^+(L)$ the set of semipositively curved metrics on L. There exists a bijection

$$PSH(X,\alpha) \longrightarrow \mathrm{Met}^+(L)\,,\;\; \varphi \longmapsto h^L_\varphi = h^L_0 e^{-2\varphi}, \tag{2.5}$$

and $c_1(L,h^L_\varphi) = \alpha + dd^c\varphi$.

Let (F,h^F) be an auxiliary Hermitian holomorphic line bundle endowed with a smooth metric h^F. We denote by

$$h_p = (h^L)^{\otimes p} \otimes h^F, \tag{2.6}$$

the metric induced by h^L, h^F on $L^p \otimes F$. Consider the space $L^2(X, L^p\otimes F)$ of L^2 sections of $L^p\otimes F$ relative to the metric h_p and the volume form ω_X^n on X, endowed with the inner product

$$(s,s')_p = \int_X \langle s,s'\rangle_{h_p}\,\omega_X^n\,,\quad \text{where}\;\; s,s'\in L^2(X,L^p\otimes F). \tag{2.7}$$

We let $\|s\|_p^2 = (s,s)_p$. Let us denote by

$$H^0_{(2)}(X,L^p\otimes F) := \Big\{ s\in L^2(X,L^p\otimes F) : s \text{ holomorphic}\Big\} \tag{2.8}$$

the space of L^2-holomorphic sections of $L^p\otimes F$. In the same way, let $L^2_{q,r}(X,L^p\otimes F)$ be the space of L^2-integrable (q,r)-forms with values in $L^p\otimes F$ relative to h_p and ω_X. We will add 'loc' for spaces of locally L^2-integrable forms when X is not compact.

For a section $s\in H^0_{(2)}(X,L^p\otimes F)$ we denote by $\mathrm{Div}(s)$ the divisor defined by s (cf. [18, (2.1.4)]) and by $[\mathrm{Div}(s)]$ the current of integration on $\mathrm{Div}(s)$ (cf. [13, Ch. III (2.5)], [18, (B.2.16)]). Note that for two non-zero elements $s,s'\in H^0_{(2)}(X,L^p\otimes F)$ which are in the same equivalence class in $\mathbb{P}(H^0_{(2)}(X,L^p\otimes F))$ we have $\mathrm{Div}(s)=\mathrm{Div}(s')$, so Div is well-defined on $\mathbb{P}(H^0_{(2)}(X,L^p\otimes F))$.

Assume now that (L,h^L) is a holomorphic line bundle endowed with semipositively curved singular metric. Denote by $\Sigma\subset X$ the set of points where h^L is not bounded. This set has zero Lebesgue mass. Let $\{s_j^p\}_{j=1}^{d_p}$ be an orthonormal basis of $H^0_{(2)}(X,L^p\otimes F)$. Let B_p be the Bergman kernel function defined by

$$B_p(x) = \sum_{j=1}^{d_p} |s_j^p(x)|^2_{h_p}\,,\quad x\in X\setminus\Sigma\,, \tag{2.9}$$

where h_p is given by (2.6). Let $s_j^p = f_j^p e_L^{\otimes p}\otimes e_F$, where $f_j^p\in\mathcal{O}(U)$ and e_L, e_F are holomorphic frames of L, F on U. Let φ' be the local weight of h^F with respect to e_F, defined as in (2.3). Then on $U \setminus \Sigma$ the following holds

$$\log B_p = \log\Big(\sum_{j=1}^{d_p} |f_j^p|^2\Big) - 2p\,\varphi - 2\,\varphi'\,. \tag{2.10}$$

The right-hand side of (2.10) is a difference of psh (hence locally integrable) functions on U, so defines an element in $L^1_{loc}(U,\omega_X^n)$. Therefore, $\log B_p$ defines an element in $L^1(X,\omega_X^n)$.

The Kodaira map is the meromorphic map given by

$$\begin{aligned}&\Phi_p : X \dashrightarrow \mathbb{P}\big(H^0_{(2)}(X, L^p\otimes F)^*\big)\,,\\ &\Phi_p(x) = \big\{s\in H^0_{(2)}(X,L^p\otimes F) : s(x)=0\big\}\,,\ \ x\in X\setminus Bs_p\,,\end{aligned} \tag{2.11}$$

where a point in $\mathbb{P}\big(H^0_{(2)}(X,L^p\otimes F)^*\big)$ is identified with a hyperplane through the origin in $H^0_{(2)}(X,L^p\otimes F)$ and $Bs_p=\{x\in X : s(x)=0 \text{ for all } s\in H^0_{(2)}(X,L^p\otimes F)\}$ is the base locus of $H^0_{(2)}(X,L^p\otimes F)$. We define the *Fubini-Study currents* by

$$\omega_p = \Phi_p^*(\omega_{\rm FS}), \tag{2.12}$$

where $\omega_{\rm FS}$ denotes the Fubini-Study $(1,1)$-form on $\mathbb{P}\big(H^0_{(2)}(X,L^p\otimes F)^*\big)$. They are positive closed $(1,1)$-currents obtained by pulling back the Fubini-Study form $\omega_{\rm FS}$. The current ω_p is in fact given by an L^1-form on X, which is smooth outside the set of indeterminacy of Φ_p, see Lemma 2.1 below. We have

$$\omega_p\,|_U = \frac{1}{2}\,dd^c\log\Big(\sum_{j=1}^{d_p}|f_j^p|^2\Big)\,, \tag{2.13}$$

hence by (2.10)

$$\frac{1}{2}\,dd^c\log B_p = \omega_p - p\,c_1(L,h^L) - c_1(F,h^F)\,. \tag{2.14}$$

We used above the following basic property that we will give a proof for the reader's convenience.

Lemma 2.1. *Let $\Phi : Y \dashrightarrow Z$ be a meromorphic map between two compact complex manifolds Y, Z of dimensions ℓ and m respectively. Let α be a smooth (q,r)-form on Z with $0\leq q,r\leq \min(\ell,m)$. Then the (q,r)-current $\Phi^*(\alpha)$ on Y is well-defined and given by a (q,r)-form with L^1 coefficients which is smooth outside the indeterminacy set of Φ.*

Proof. Recall that for a meromorphic map $\Phi : Y \dashrightarrow Z$ ([18, Definition 2.1.19], [22]) there is an analytic subset I of Y such that Φ is holomorphic on $Y\setminus I$ and the closure of the graph of Φ over $Y\setminus I$ is an irreducible analytic subset of dimension ℓ of $Y\times Z$, called the graph of Φ. The smallest set I with this property is called the indeterminacy set of Φ. Since Y is a manifold, I is of codimension at least two [22, p. 333]. Denote by Γ the graph of Φ. It defines, by integration on its regular part $\operatorname{reg}(\Gamma)$, a positive closed current $[\Gamma]$ of bi-dimension (ℓ,ℓ) in $Y\times Z$ [13, p. 140].

Denote by π_Y,π_Z the natural projections from $Y\times Z$ to Y and Z respectively. The pull-back $\Phi^*(\alpha)$ is defined by

$$\Phi^*(\alpha) := (\pi_Y)_*(\pi_Z^*(\alpha)\wedge[\Gamma]). \tag{2.15}$$

This is the formal definition for any current α. It makes sense when the wedge-product in the last expression is well-defined because here the operator $(\pi_Y)_*$ is well-defined on all currents. In our setting, since $\pi_Z^*(\alpha)$ is smooth, the current $\Phi^*(\alpha)$ is well-defined. More precisely, if β is a smooth form of bidegree $(\ell-q,\ell-r)$ on Y then

$$\langle \Phi^*(\alpha),\beta\rangle = \int_{\mathrm{reg}(\Gamma)} \pi_Z^*(\alpha)\wedge\pi_Y^*(\beta). \tag{2.16}$$

Note that the 2ℓ-dimensional volume of Γ is finite [13, p. 140].

Formula (2.16) shows that the current $\Phi^*(\alpha)$ extends continuously to the space of test forms β with continuous coefficients. So $\Phi^*(\alpha)$ is a current of order 0. If V is a proper analytic subset of Y, then $\Gamma\cap\pi_Y^{-1}(V)$ is a proper analytic subset of Γ, so $\Gamma\cap\pi_Y^{-1}(y)$ has zero 2ℓ-dimensional volume. Therefore, the last formula implies that $\Phi^*(\alpha)$ has no mass on V, in particular, this current has no mass on the indeterminacy set I.

If β has compact support in $Y\setminus I$, since π_Y defines a bi-holomorphic map from $\Gamma\setminus\pi_Y^{-1}(I)$ to $Y\setminus I$, the last integral is equal to the integral on $Y\setminus I$ of the form $(\pi_Y)_*(\pi_Z)^*(\alpha)\wedge\beta$. The last expression is equal to $(\Phi|_{Y\setminus I})^*(\alpha)\wedge\beta$, where $(\Phi|_{Y\setminus I})^*(\alpha)$ is the pull-back of the smooth form α by the holomorphic map $\Phi|_{Y\setminus I}$. We conclude that the current $\Phi^*(\alpha)$ is equal on $Y\setminus I$ to the smooth form $(\Phi|_{Y\setminus I})^*(\alpha)$. Finally, since $\Phi^*(\alpha)$ is of order 0 and has no mass on I, the form $(\Phi|_{Y\setminus I})^*(\alpha)$ has L^1 coefficients and is equal, in the sense of currents on Y, to $\Phi^*(\alpha)$. This completes the proof of the lemma. □

Note that the lemma can be extended to meromorphic maps between open manifolds provided that π_Y is proper on $\pi_Z^{-1}(\mathrm{supp}(\alpha))\cap\Gamma$. Moreover, by definition, if α is closed and/or positive then $\Phi^*(\alpha)$ is also closed and/or positive.

3. Abstract setting for equidistribution

We will only consider the case of compact Kähler manifolds but it is certainly easy to extend the results to the case of manifolds of Fujiki class and even open manifolds satisfying some properties of concavity.

Let (X,ω_X) be a compact Kähler manifold of dimension n. Recall that we can introduce several semi-norms on the set of currents of order 0 on X. If U is an open subset of X, α is a strictly positive number and T is a current of order 0 on X, define

$$\|T\|_{U,-\alpha} := \sup\big|\langle T,u\rangle\big| \tag{3.1}$$

where the supremum is taken over smooth test forms u with support in U and such that their $\mathscr{C}^\alpha$-norm satisfies $\|u\|_{\mathscr{C}^\alpha}\leq 1$.

For simplicity, we will drop the letter U when $U=X$. In this case, $\|\cdot\|_{-\alpha}$ is a norm and the associated topology coincides with the weak topology on any set of currents with mass bounded by a fixed constant. We will only consider the case $\alpha=2$ and we will be interested in estimates on $\|\cdot\|_{U,-2}$. The other cases can be obtained as a consequence, e.g., if $\alpha<2$, we can use the theory of interpolation between Banach spaces [16], [28].

Definition 3.1. Let (c_p) be a sequence of positive numbers converging to 0. Let $\{T_p : p\in\mathbb{N}\}$ and T be currents on X with mass bounded by a fixed constant. We say that the sequence (T_p) converges on U to T with speed (c_p) if $\|T_p-T\|_{U,-2}\leqslant c_p$ for p large enough. We also say that the sequence converges with speed $O(c_p)$ if it converges with speed (Cc_p) for some $C\geq 0$.

Recall that a current of order 0 is an element in the dual of the space of continuous forms. The mass of such currents is the norm dual to the $\mathscr{C}^0$ norm on forms. However, for a positive (q,q)-current T on (X,ω_X), it is more convenient to use the following notion of mass

$$\|T\| = \left\langle T, \omega_X^{n-q} \right\rangle \tag{3.2}$$

which is equivalent to the above mass-norm. The advantage is that when T is positive closed, its mass only depends on its cohomology class in $H^{q,q}(X,\mathbb{R})$.

The following result was obtained in [14, Theorem 4], where we assumed that the map Φ has generically maximal rank n, but the proof there is valid without this condition.

Theorem 3.2. *Let (X,ω_X) be a compact Kähler manifold of dimension n and let V be a Hermitian complex vector space of dimension $d{+}1$. Consider a meromorphic map $\Phi : X \to \mathbb{P}(V)$. Then there exists $c>0$ depending only on (X,ω_X) such that for any $\gamma>0$ there is a subset E_γ of $\mathbb{P}(V^*)$ with the following properties:*

(a) $\sigma_{_{\rm FS}}(E_\gamma) \leq c\, d^2\, e^{-\gamma/c}$.

(b) *For ξ outside E_γ, the current $\Phi^*[H_\xi]$ is well-defined and*

$$\left\|\Phi^*[H_\xi] - \Phi^*(\omega_{_{\rm FS}})\right\|_{-2} \leqslant \gamma\,. \tag{3.3}$$

Consider now holomorphic Hermitian line bundles (L,h^L), (F,h^F) such that h^L is a singular Hermitian metric. We have $H^0_{(2)}(X,L^p\otimes F) \subset H^0(X,L^p\otimes F)$, thus $d_p := \dim H^0_{(2)}(X,L^p\otimes F) < \infty$. We assume that $d_p \geq 1$. Note that there exists $C>0$ such that $d_p \leqslant Cp^n$ for all $p\in\mathbb{N}$, where $C>0$ is a constant depending only on (X,ω_X), $c_1(L)$, $c_1(F)$. This follows from the holomorphic Morse inequalities [18, Theorem 1.7.1] or the Siegel Lemma [18, Lemma 2.2.6].

We have the following consequence of the above result (compare also [14, Theorem 2]).

Corollary 3.3. *Let (X,ω_X) be a compact Kähler manifold of dimension n and let (L,h^L) be a singular Hermitian holomorphic line bundle on X. Let (F,h^F) be a holomorphic line bundle with smooth Hermitian metric. Then there is $c=c(X,L,F)>0$ depending only on (X,ω_X) and $c_1(L)$, $c_1(F)$, with the following property. For any sequence of positive numbers λ_p, there are subsets $E_p\subset\mathbb{P}(H^0_{(2)}(X,L^p\otimes F))$ such that for p large enough*

$$\sigma_p(E_p) \leqslant c\,p^{2n} e^{-\lambda_p/c}\,, \tag{3.4}$$

$$\left\|[\operatorname{Div}(s)] - \omega_p\right\|_{-2} \leqslant \lambda_p\,, \ \textit{for any } [s]\in\mathbb{P}(H^0_{(2)}(X,L^p\otimes F))\setminus E_p\,. \tag{3.5}$$

Let (λ_p) be a sequence of positive numbers such that

$$\liminf_{p\to\infty} \frac{\lambda_p}{\log p} > (2n+1)c\,. \tag{3.6}$$

Then for σ_∞-almost every sequence $([s_p])\in\Omega_1(L,F)$, the estimate (3.5) holds for $s=s_p$ and p large enough.

Proof. We apply Theorem 3.2 for $V = H^0(X,L^p\otimes F)^*$ and for $\Phi=\Phi_p$, where Φ_p is the Kodaira map (2.11). The first assertion is a direct consequence of Theorem 3.2. We prove now the second assertion. The hypothesis (3.6) on $\lambda_p/\log p$ and (3.4) guarantee that

$$\sum_{p=1}^{\infty} \sigma_p(E_p) \leqslant c' \sum_{p=1}^{\infty} \frac{1}{p^{\delta}} < \infty$$

for some $c' > 0$ and $\delta > 1$. Hence the set

$$E = \big\{([s_p]) \in \Omega_1(L,F) : [s_p] \in E_p \text{ for an infinite number of indices } p\big\} \tag{3.7}$$

satisfies $\sigma_\infty(E) = 0$. Indeed, for every $N \geq 0$, it is contained in the set

$$\big\{([s_p]) \in \Omega_1(L,F) : [s_p] \in E_p \text{ for at least one index } p \geq N\big\}$$

which is of σ_∞-measure at most equal to

$$\sum_{p=N}^{\infty} \sigma_p(E_p) \leq c' \sum_{p=N}^{\infty} \frac{1}{p^{\delta}} = O(N^{1-\delta}). \tag{3.8}$$

Therefore, the second assertion of the corollary. □

We easily deduce from Corollary 3.3 the following.

Corollary 3.4. *Let (X, ω_X) be a compact Kähler manifold of dimension n and let (L, h^L) be a singular Hermitian holomorphic line bundle on X. Let (F, h^F) be a holomorphic line bundle with a smooth Hermitian metric. Let $c = c(X, L, F)$ be the constant given by Corollary 3.3 and let (λ_p) be a sequence of positive numbers satisfying*

$$\liminf_{p\to\infty} \frac{\lambda_p}{\log p} > (2n+1)c\,, \quad \lim_{p\to\infty} \frac{\lambda_p}{p} = 0\,. \tag{3.9}$$

Let $U \subset X$ be an open set. Assume that $(\frac{1}{p}\omega_p)$ converges to a current Θ in U with speed (c_p). Then for σ_∞-almost every sequence $([s_p]) \in \Omega_1(L,F)$, $(\frac{1}{p}[\mathrm{Div}(s_p)])$ converges to Θ on U with speed $\big(c_p + \frac{\lambda_p}{p}\big)$ as $p \to \infty$.

We consider now products of projective spaces. Let $\pi_i : \mathbb{P}(V^*)^k \to \mathbb{P}(V^*)$, $i = 1, \ldots, k$, be the canonical projections from the multi-projective space $\mathbb{P}(V^*)^k \simeq (\mathbb{P}^d)^k$ onto its factors. As usual we denote by ω_{FS} the Fubini-Study form on $\mathbb{P}(V^*)$. Consider the Kähler form and volume form on $\mathbb{P}(V^*)^k$,

$$\omega_{\mathrm{MP}} := c_{d,k} \sum_{i=1}^{k} \pi_i^*(\omega_{\mathrm{FS}})\,, \quad \sigma_{\mathrm{MP}} := \omega_{\mathrm{MP}}^{kd}\,, \tag{3.10}$$

where $c_{d,k}$ is the positive constant so that the volume form σ_{MP} defines a probability measure. The constant $c_{d,k}$ is given by the formula

$$(c_{d,k})^{-dk} = \binom{dk}{d}\binom{dk-d}{d}\cdots\binom{2d}{d} = \frac{(dk)!}{(d!)^k}. \tag{3.11}$$

Using Stirling's formula $n! \simeq \sqrt{2\pi n}\, n^n e^{-n}$, one can show that $c_{d,k}$ is smaller than 1 and larger than a strictly positive constant depending only on k. The measure σ_{MP} is the Haar measure associated with the natural action of the unitary group on the factors of $\mathbb{P}(V^*)^k$.

We give now the proof of Theorem 1.1. Recall that a *quasi-psh* function is locally the difference between a psh function and a smooth function. A quasi-psh function u on $\mathbb{P}(V^*)^k$ is ω_{MP}*-psh* if it satisfies $dd^c u \geq -\omega_{\mathrm{MP}}$, i.e., $dd^c u + \omega_{\mathrm{MP}}$ is a positive current. We need the following result from [16, Proposition A.9].

Lemma 3.5. *There are $c > 0$, $\alpha > 0$ and $m > 0$ depending only on k such that if u is an $\omega_{\rm MP}$-psh function on $\mathbb{P}(V^*)^k$ with $\int u d\sigma_{\rm MP} = 0$, then*

$$u \leq c(1+\log d) \quad and \quad \sigma_{\rm MP}\{u < -t\} \leq c\, d^m e^{-\alpha t} \quad for \;\; t \geq 0. \tag{3.12}$$

Lemma 3.6. *Let Σ be a closed subset of $\mathbb{P}(V^*)^k$ and let u be an L^1 function which is continuous on $\mathbb{P}(V^*)^k \setminus \Sigma$. Let γ be a positive constant. Suppose there is a positive closed $(1,1)$-current S of mass 1 on $\mathbb{P}(V^*)^k$ such that $-S \leqslant dd^c u \leqslant S$ and $\int u d\sigma_{\rm MP} = 0$. Then, there are $c > 0$, $\alpha > 0$, $m > 0$ depending only on k and a Borel set $E' \subset \mathbb{P}(V^*)^k$ depending only on S and γ such that*

$$\sigma_{\rm MP}(E') \leqslant c\, d^m e^{-\alpha\gamma} \quad and \quad |u(a)| \leqslant \gamma \quad for \quad a \notin \Sigma \cup E'. \tag{3.13}$$

Proof. By Künneth's formula, the cohomology group $H^{1,1}(\mathbb{P}(V^*)^k, \mathbb{R})$ is generated by the classes of $\pi_i^*(\omega_{\rm FS})$ with $i = 1, \ldots, k$. Therefore, there are $\lambda_i \geq 0$ such that the class $\{S\}$ of S is equal to $\sum \lambda_i \{\pi_i^*(\omega_{\rm FS})\}$. The mass of S can be computed cohomologically. If we identify the top bi-degree cohomology group $H^{kd,kd}(\mathbb{P}(V^*)^k, \mathbb{R})$ with $\mathbb{R}$ in the canonical way, this mass is equal to the cup product $\{S\} \smile \{\omega_{\rm MP}\}^{kd-1}$ and then a direct computation gives

$$\sum_{i=1}^k \lambda_i (c_{d,k})^{kd-1} \binom{dk-1}{d-1} \binom{dk-d}{d} \cdots \binom{2d}{d} = \sum_{i=1}^k \lambda_i (c_{d,k})^{-1} k^{-1}. \tag{3.14}$$

We used here that $\{\omega_{\rm FS}\}^d = 1$ in $H^{d,d}(\mathbb{P}(V^*), \mathbb{R}) \simeq \mathbb{R}$. Since $c_{d,k} \leq 1$ and S is of mass 1, we deduce that $\lambda_i \leq k$.

By the dd^c-lemma [18, Lemma 1.5.1], there is a unique quasi-psh function v such that

$$dd^c v = S - \sum_{i=1}^k \lambda_i \pi_i^*(\omega_{\rm FS}) \quad \text{and} \quad \int v d\sigma_{\rm MP} = 0. \tag{3.15}$$

We have $dd^c v + \lambda\omega_{\rm MP} \geq S$ for some constant $\lambda > 0$ depending only on k. Define $w := \lambda^{-1}(u+v)$. We have $dd^c w \geq -\omega_{\rm MP}$. Since u is continuous outside Σ, the latter property implies that w is equal outside Σ to a quasi-psh function. We still denote this quasi-psh function by w. Applying Lemma 3.5 to w instead of u, we obtain that

$$u = \lambda w - v \leqslant c\lambda(1+\log d) - v. \tag{3.16}$$

Let E' denote the set $\{v < -\gamma + c\lambda(1+\log d)\}$ which does not depend on u. Clearly, $u \leqslant \gamma$ outside $\Sigma \cup E'$. The same property applied to $-u$ implies that $|u| \leqslant \gamma$ outside $\Sigma \cup E'$. It remains to bound the size of E'. Lemma 3.5 applied to $\lambda^{-1} v$ yields

$$\sigma_{\rm MP}(E') \leq c\, d^m \exp\big(-\alpha\lambda^{-1}\gamma + c\alpha(1+\log d)\big). \tag{3.17}$$

This is the desired inequality for (other) suitable constants c, α and m. □

End of the proof of Theorem 1.1. Let $\Phi : X \dashrightarrow \mathbb{P}(V)$ be a meromorphic map and let $\Gamma \subset X \times \mathbb{P}(V)$ its graph. We define

$$\widetilde{X} = \big\{(x,\xi) \in X \times \mathbb{P}(V^*)^k : \exists v \in \mathbb{P}(V) \text{ such that } (x,v) \in \Gamma,\, v \in H_\xi\big\}. \tag{3.18}$$

Recall that for $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{P}(V^*)^k$ we denote $H_\xi = H_{\xi_1} \cap \ldots \cap H_{\xi_k}$ the intersection of the hyperplanes H_{ξ_i} in $\mathbb{P}(V)$. The set $\widetilde{X}$ is a compact analytic subset in $X \times \mathbb{P}(V^*)^k$, of dimension $n + (d-1)k$. Let Π_1 and Π_2 denote the natural projections from $\widetilde{X}$ onto X and $\mathbb{P}(V^*)^k$ respectively.

Lemma 3.7. *Let $\Sigma \subset \mathbb{P}(V^*)^k$ be the set of points ξ such that $\widetilde{X} \cap \Pi_2^{-1}(\xi) \neq \emptyset$ and one of the following properties holds:*

(a) $\dim H_\xi > d-k$;

(b) $\dim \widetilde{X} \cap \Pi_2^{-1}(\xi) > n-k$;

(c) $\dim H_\xi = d-k$, $\dim \widetilde{X} \cap \Pi_2^{-1}(\xi) = n-k$ *but the last intersection is not transversal at a generic point.*

Then Σ is contained in a proper analytic subset of $\mathbb{P}(V^*)^k$.

Proof. If Π_2 is not surjective, the lemma is clear because Σ is contained in $\Pi_2(\widetilde{X})$ which is a proper analytic subset of $\mathbb{P}(V^*)^k$. Assume that Π_2 is surjective. So $\widetilde{X} \cap \Pi_2^{-1}(\xi) \neq \emptyset$ for every ξ. Observe that the set Σ_1 of ξ satisfying (a) is a proper analytic subset of $\mathbb{P}(V^*)^k$. Thus, we only consider parameters ξ outside Σ_1.

Let $\tau : \widehat{X} \to \widetilde{X}$ be a singularity resolution for $\widetilde{X}$ and define $\widehat{\Pi}_2 := \Pi_2 \circ \tau$. The last map is a holomorphic surjective map between compact complex manifolds. So by Bertini-Sard type theorem, there is a proper analytic subset Σ_2 of $\mathbb{P}(V^*)^k$ such that $\widehat{\Pi}_2$ is a submersion outside $\widehat{\Pi}_2^{-1}(\Sigma_2)$. Indeed, Σ_2 is the set of critical values of $\widehat{\Pi}_2$ which is analytic. Sard's theorem implies that it is a proper analytic subset of $\mathbb{P}(V^*)^k$. It follows that for $\xi \notin \Sigma_1 \cup \Sigma_2$ the fiber $\widehat{\Pi}_2^{-1}(\xi)$ has dimension $n-k$, i.e., the minimal dimension for the fibers of $\widehat{\Pi}_2$. So $\Pi_2^{-1}(\xi)$, which is the image of $\widehat{\Pi}_2^{-1}(\xi)$ by τ, is also of minimal dimension $n-k$. Therefore, such parameters ξ do not satisfy (b).

Let E denote the exceptional analytic subset in $\widehat{X}$, i.e., the pull-back of the singularities of $\widetilde{X}$ by τ. Since $\dim E < \dim \widehat{X}$, arguing as above, we obtain a proper analytic subset Σ_3 of $\mathbb{P}(V^*)^k$ such that for ξ outside Σ_3, the dimension of $E \cap \widehat{\Pi}_2^{-1}(\xi)$ is at most equal to $n-k-1$. Since τ is locally bi-holomorphic outside E, for $\xi \notin \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, the intersection $\widetilde{X} \cap \Pi_2^{-1}(\xi)$ is transverse outside the image by τ of $E \cap \widehat{\Pi}_2^{-1}(\xi)$, which is of dimension at most $n-k-1$. Such parameters ξ do not satisfy (c). The lemma follows. □

From now on, we only consider $\xi \in \mathbb{P}(V^*)^k \setminus \Sigma$, where Σ is defined in Lemma 3.7. The current $[(\Pi_2)^*(\xi)]$ is then well-defined and we have

$$\Phi^*[H_\xi] = (\Pi_1)_*\big([(\Pi_2)^*(\xi)]\big). \tag{3.19}$$

There exists $C>0$ such that for any test smooth real $(n-k,n-k)$-form φ on X with $\|\varphi\|_{\mathscr{C}^2} \leq C$ we have

$$-\,\omega_X^{n-k+1} \leq dd^c\varphi \leq \omega_X^{n-k+1}. \tag{3.20}$$

Take such a φ and define $v := (\Pi_2)_*(\Pi_1)^*(\varphi)$. This is a function on $\mathbb{P}(V^*)^k$ whose value at $\xi \in \mathbb{P}(V^*)^k \setminus \Sigma$ is the integration of $(\Pi_1)^*(\varphi)$ on the fiber $\Pi_2^{-1}(\xi)$. So, we have

$$v(\xi) = \langle \Phi^*[H_\xi], \varphi \rangle\,. \tag{3.21}$$

Hence, v is continuous on $\mathbb{P}(V^*)^k \setminus \Sigma$. Since the form $\omega_{\rm FS}$ on $\mathbb{P}(V)$ is the average of $[H_{\xi_i}]$ with respect to the measure $\sigma_{\rm FS}$ on $\xi_i \in \mathbb{P}(V^*)$, the average of $[H_\xi]$ with respect to the measure $\sigma_{\rm MP}$ on $\xi \in \mathbb{P}(V^*)^k$ is equal to $\omega_{\rm FS}^k$. Thus, the mean value of v is

$$M_v := \int v\, d\sigma_{\rm MP} = \langle \Phi^*(\omega_{\rm FS}^k), \varphi \rangle\,. \tag{3.22}$$

So we need to prove that $|v - M_v| \leq \gamma m_{k-1}$ outside a set E_γ of $\sigma_{\rm MP}$-measure less than $c\, d^m e^{-\gamma/c}$ which does not depend on φ. This implies Theorem 1.1.

Define

$$T := (\Pi_2)_*(\Pi_1)^*(\omega_X^{n-k+1}). \tag{3.23}$$

This is a positive closed $(1,1)$-current on $\mathbb{P}(V^*)^k$ and we have, thanks to the above property (3.20) of $dd^c\varphi$, that

$$-T \leq dd^c v \leq T. \tag{3.24}$$

Let ϑ be the mass of T. We can apply Lemma 3.6 to the function $u := \vartheta^{-1}(v - M_v)$, $S := \vartheta^{-1}T$ and to $\vartheta^{-1}\gamma m_{k-1}$ instead of γ. We can take $E_\gamma = \Sigma \cup E'$ which does not depend on φ. Since Σ is of measure 0, in order to get from Lemma 3.6 the desired estimate on $\sigma_{\text{MP}}(E_\gamma) = \sigma_{\text{MP}}(E')$, it is enough to show that ϑ is bounded above by m_{k-1} times a constant which only depends on k.

We have

$$\|T\| = \big\langle (\Pi_2)_*(\Pi_1)^*(\omega_X^{n-k+1}), \omega_{\text{MP}}^{kd-1} \big\rangle = \big\langle \omega_X^{n-k+1}, (\Pi_1)_*(\Pi_2)^*(\omega_{\text{MP}}^{kd-1}) \big\rangle. \tag{3.25}$$

Let $\widetilde{\mathbb{P}(V)}$ denote the set of points $(x,\xi) \in \mathbb{P}(V) \times \mathbb{P}(V^*)^k$ such that $x \in H_{\xi_i}$ for every i. Denote by Π_1' and Π_2' the natural projections from $\widetilde{\mathbb{P}(V)}$ onto $\mathbb{P}(V)$ and $\mathbb{P}(V^*)^k$. By construction, we have

$$(\Pi_1)_*(\Pi_2)^*(\omega_{\text{MP}}^{kd-1}) = \Phi^*\big((\Pi_1')_*(\Pi_2')^*(\omega_{\text{MP}}^{kd-1})\big). \tag{3.26}$$

By definition of m_{k-1}, it is enough to check that $(\Pi_1')_*(\Pi_2')^*(\omega_{\text{MP}}^{kd-1})$ is bounded by ω_{FS}^{k-1} times a constant depending only on k.

We obtain with a direct computation

$$\omega_{\text{MP}}^{kd-1} = c_{d,k}^{kd-1} \binom{dk-1}{d-1} \binom{dk-d}{d} \cdots \binom{2d}{d} \sum_{i=1}^{k} \Theta_i = (c_{d,k})^{-1} k^{-1} \sum_{i=1}^{k} \Theta_i, \tag{3.27}$$

where

$$\Theta_i := \pi_1^*(\omega_{\text{FS}}^d) \wedge \ldots \wedge \pi_{i-1}^*(\omega_{\text{FS}}^d) \wedge \pi_i^*(\omega_{\text{FS}}^{d-1}) \wedge \pi_{i+1}^*(\omega_{\text{FS}}^d) \wedge \ldots \wedge \pi_k^*(\omega_{\text{FS}}^d). \tag{3.28}$$

We will show that $(\Pi_1')_*(\Pi_2')^*(\Theta_i) = \omega_{\text{FS}}^{k-1}$ and this implies the theorem.

For simplicity, assume that $i = 1$. Since $(\Pi_1')_*(\Pi_2')^*(\Theta_1)$ is invariant under the action of the unitary group, it is equal to a constant times ω_{FS}^{k-1}. So we only have to check that the constant is 1 or equivalently the mass of $(\Pi_1')_*(\Pi_2')^*(\Theta_1)$ is 1. Recall that the mass of a positive closed current depends only on its cohomology class. Therefore, in the definition of Θ_1, we can replace ω_{FS}^{d-1} with the current of integration on a generic projective line ℓ and each ω_{FS}^d with the Dirac mass of a generic point, say ξ_j, for $j = 2, \ldots, k$. The current Θ_1 is in the same cohomology class as the current of integration on

$$\ell \times \{\xi_2\} \times \cdots \times \{\xi_k\}$$

that we denote by Θ_1'.

It is not difficult to see that $(\Pi_1')_*(\Pi_2')^*(\Theta_1')$ is the current of integration on the projective subspace $H_{\xi_2} \cap H_{\xi_3} \cap \ldots \cap H_{\xi_k}$. So it is clear that its mass is equal to 1. This completes the proof of the Theorem 1.1. □

The following property of the constants m_k is useful.

Lemma 3.8. *There is $c > 0$ depending only on (X, ω_X) such that $m_k \leq c\, m_1^k$ for $1 \leq k \leq n$.*

Proof. Observe that by Lemma 2.1, the currents $\Phi^*(\omega^k_{\rm FS})$ are given by L^1 forms and they are smooth on some Zariski open set U of X where Φ is holomorphic. Moreover, we have $\Phi^*(\omega^k_{\rm FS}) = \Phi^*(\omega_{\rm FS})^k$ on U. Since $\Phi^*(\omega^k_{\rm FS})$ is given by an L^1 form, it has no mass outside U.

By [15, Lemma 2.2], there is $C > 0$ depending only on (X, ω_X) such that if T and S are positive closed currents on X which are smooth in an open set U then the mass $\|T \wedge S\|_U$ of $T \wedge S$ on U is bounded by $C\|T\|\|S\|$. By Skoda's extension theorem [27, Théorème 1], positive closed currents of finite mass can be extended by 0 through analytic sets. So if U is a Zariski open set, the form $T \wedge S$ extends by 0 to a positive closed current on X with mass bounded by $C\|T\|\|S\|$. This allows us to apply inductively the mass estimate for $T \wedge S$ to the case of product of several positive closed currents.

Observe that $\Phi^*(\omega^k_{\rm FS}) = \Phi^*(\omega_{\rm FS}) \wedge \Phi^*(\omega^{k-1}_{\rm FS})$ on U, so, by induction on k, we deduce from the above discussion that $m_k \leq C^{k-1} m_1^k$. The lemma follows. □

In the case where $V = H^0_{(2)}(X, L^p \otimes F)^*$, we have $m_1 = O(p)$ and therefore $m_k = O(p^k)$. This together with Theorem 1.1 imply the following corollary. Consider the Kodaira map $\Phi_p : X \dashrightarrow \mathbb{P}\big(H^0_{(2)}(X, L^p \otimes F)^*\big)$ defined in (2.11). The pull-back $\Phi_p^*(\omega^k_{\rm FS})$ of the current $\omega^k_{\rm FS}$ is given by an L^1 form equal to ω_p^k on a dense Zariski open set (here ω_p is the Fubini-Study current (2.12)).

Corollary 3.9. *There are $c = c(X, L, F) > 0$ and $m = m(X, L, F) > 0$ depending only on (X, ω_X) and $c_1(L)$, $c_1(F)$, with the following property. For any sequence λ_p, there are subsets E_p of $\mathbb{P}(H^0_{(2)}(X, L^p \otimes F))^k$ such that for p large enough*

(a) $\sigma_p(E_p) \leqslant c\, p^m e^{-\lambda_p/c}$.

(b) *For $S_p = ([s_p^{(1)}], \ldots, [s_p^{(k)}])$ in $\mathbb{P}(H^0_{(2)}(X, L^p \otimes F))^k \setminus E_p$, we have*

$$\Big\| \frac{1}{p^k} \big[s_p^{(1)} = \ldots = s_p^{(k)} = 0 \big] - \frac{1}{p^k} \Phi_p^*(\omega^k_{\rm FS}) \Big\|_{-2} \leqslant \frac{\lambda_p}{p} \cdot \tag{3.29}$$

In particular, when

$$\liminf_{p \to \infty} \frac{\lambda_p}{\log p} > (m+1)c\,, \tag{3.30}$$

for σ_∞-almost every sequence $(S_p) \in \Omega_k(L, F)$, the above estimate holds for p large enough. If $\frac{1}{p^k}\Phi_p^*(\omega^k_{\rm FS})$ converge to a current Θ_k in some open set U with speed (c_p) as $p \to \infty$, then $\frac{1}{p^k}\big[s_p^{(1)} = \ldots = s_p^{(k)} = 0\big]$ converge to Θ_k on U with speed $\big(c_p + \lambda_p/p\big)$ as $p \to \infty$.

Note that the constants in the corollary can be chosen independently of k because $1 \leq k \leq n = \dim X$. The corollary can be applied in the situation of Corollaries 1.4 and 4.4. In that cases, we have $\Theta_k = c_1(L, h^L)^k$ on U.

4. Semi-positive curved metrics on big line bundles

Let (X, ω_X) be a compact Kähler manifold of dimension n. Let (L, h^L) be a holomorphic line bundle endowed with a singular metric h^L. Fix a smooth Hermitian metric h_0^L on L and let $\alpha = c_1(L, h_0^L)$ denote its first Chern form. We can write

$$h^L = e^{-2\varphi} h_0^L\,, \text{ i.e., } |s|^2_{h^L} = |s|^2_{h_0^L} e^{-2\varphi} \text{ for any section } s \text{ of } L\,, \tag{4.1}$$

where φ is an L^1 function on X with values in $\mathbb{R} \cup \{\pm\infty\}$. We assume that the curvature of h^L is semipositive, that is, $c_1(L, h^L) = dd^c\varphi + \alpha$ is a positive current. So the function φ

is α-psh, i.e., φ is quasi-psh and satisfies $dd^c\varphi \geq -\alpha$. Define

$$\omega := c_1(L, h^L) = dd^c\varphi + \alpha. \tag{4.2}$$

We also assume that the line bundle L is big. So, there is a metric

$$\widetilde{h}^L = e^{-2\varphi'} h_0^L \tag{4.3}$$

such that $\omega' := dd^c\varphi' + \alpha \geq \varepsilon\omega_X$ for some $\varepsilon > 0$ (cf. [18, Theorem 2.3.30]). Let B_p be the Bergman function in (2.9) associated with $(L^p, (h^L)^{\otimes p})$. The function

$$\varphi_p := \varphi + \frac{1}{2p}\log B_p \tag{4.4}$$

is quasi-psh and by (2.14) satisfies

$$\frac{1}{p}\,\omega_p = dd^c\varphi_p + \alpha, \tag{4.5}$$

where ω_p are the Fubini-Study currents (2.12). We call the functions φ_p *global Fubini-Study weights*.

We we will use the L^2-estimates of Andreotti-Vesentini-Hörmander for $\overline{\partial}$ in the following form (cf. [9, Théorème 5.1]).

Theorem 4.1 (L^2-estimates for $\overline{\partial}$). *(i) Let (X, ω_X) be a Kähler manifold of dimension n which admits a complete Kähler metric. Let (L, h^L) be a singular Hermitian holomorphic line bundle and let $\lambda : X \to [0, +\infty)$ be a continuous function such that $c_1(L, h^L) \geq \lambda\omega_X$. Then for any form $g \in L^2_{n,1}(X, L, loc)$ satisfying*

$$\overline{\partial} g = 0\,, \quad \int_X \lambda^{-1}|g|^2\,\omega_X^n < +\infty \tag{4.6}$$

there exists $u \in L^2_{n,0}(X, L)$ with $\overline{\partial} u = g$ and

$$\int_X |u|^2\,\omega_X^n \leq \int_X \lambda^{-1}|g|^2\,\omega_X^n\,. \tag{4.7}$$

(ii) Let (X, ω_X) be a complete Kähler manifold of dimension n and let (L, h^L) be a singular Hermitian line bundle. Assume that there exists $C > 0$ such that

$$c_1(L, h^L) + c_1(K_X^*, h^{K_X^*}) \geq C\omega_X$$

where $h^{K_X^*}$ is the metric induced by ω_X on the anti-canonical bundle K_X^*. Then for any form $g \in L^2_{0,1}(X, L)$ satisfying $\overline{\partial} g = 0$ there exists $u \in L^2_{0,0}(X, L)$ with

$$\overline{\partial} u = g\,, \quad \int_X |u|^2\,\omega_X^n \leqslant \frac{1}{C}\int_X |g|^2\,\omega_X^n\,. \tag{4.8}$$

We will also need the following.

Lemma 4.2. *Let ψ be a negative psh function on a neighborhood of the unit ball B in $\mathbb{C}^n$. Define*

$$\psi'(z) := \sup_{B(z,\rho^4)} \psi, \tag{4.9}$$

where $B(z,\rho^4)$ denotes the ball of center z and radius ρ^4. Then there is $c>0$ depending on ψ such that for ρ small enough

$$\Big|\int_B \psi' dZ\Big| \geq \Big|\int_B \psi dZ\Big| - c\rho, \tag{4.10}$$

where dZ denotes the Lebesgue measure on $\mathbb{C}^n$.

Proof. In the last integral, we can replace B by $B(0,1-2\rho^2)$ because by Cauchy-Schwarz inequality, the associated error is $O(\rho)$; we use here that psh functions are locally L^2-integrable. So, we have to prove that

$$\Big|\int_B \psi' dZ\Big| \geq \Big|\int_{B(0,1-2\rho^2)} \psi dZ\Big| - c\rho. \tag{4.11}$$

It is enough to check for some (other) constant c and for ρ small enough that

$$\Big|\int_B \psi' dZ\Big| \geq (1-c\rho)\Big|\int_{B(0,1-2\rho^2)} \psi dZ\Big|. \tag{4.12}$$

We claim that

$$\rho^{-8n}\int_{B(z,\rho^4)} \psi' dZ \leqslant (1-c\rho)\rho^{-4n}\int_{B(z,\rho^2)} \psi dZ. \tag{4.13}$$

The inequality can be rewritten as

$$\rho^{-8n}\int_{B(0,\rho^4)} \psi'(z+t)dZ(t) \leqslant (1-c\rho)\rho^{-4n}\int_{B(0,\rho^2)} \psi(z+t)dZ(t). \tag{4.14}$$

Recall that ψ and ψ' are negative. Therefore, taking integrals in z of both sides of the last inequality over $B(0,1-\rho^2)$ and using Fubini's theorem for the variables z and t, we obtain the desired inequality (4.12). It remains to prove the claim.

Fix x in $B(z,\rho^4)$. It is enough to check that

$$\psi'(x) \leqslant (1-c\rho)n!\pi^{-n}\rho^{-4n}\int_{B(z,\rho^2)} \psi dZ. \tag{4.15}$$

Note that the last expression is $1-c\rho$ times the average of ψ on $B(z,\rho^2)$.

By definition, there is $y\in B(z,2\rho^4)$ such that $\psi(y)=\psi'(x)$. So, there is a holomorphic automorphism τ of $B(z,\rho^2)$ such that $\tau(y)=z$ and $\|\tau-\mathrm{id}\|_{\mathscr{C}^1}=O(\rho)$ (cf. [23, p. 25-28]). Applying the sub-mean inequality to the psh function $\widetilde{\psi}:=\psi\circ\tau^{-1}$ at z we have

$$\psi'(x)=\widetilde{\psi}(z) \leqslant n!\,\pi^{-n}\rho^{-4n}\int_{B(z,\rho^2)} \widetilde{\psi} dZ = n!\,\pi^{-n}\rho^{-4n}\int_{B(z,\rho^2)} \psi\tau^*(dZ). \tag{4.16}$$

Observe that since $\|\tau-\mathrm{id}\|_{\mathscr{C}^1}=O(\rho)$,

$$\tau^*(dZ) \geq (1-c\rho)dZ \tag{4.17}$$

for some $c>0$. The lemma follows. □

The following result gives us a situation where Corollary 3.3 applies. It refines [6, Theorem 5.1], where it is shown that $\frac{1}{p}\log B_p \to 0$ in $L^1(X,\omega_X^n)$ for the Bergman kernel B_p on powers L^p of a big line bundle L over a compact Kähler manifold (X,ω_X).

Theorem 4.3. *Let (X,ω_X) be a compact Kähler manifold of dimension n. Let L be a big holomorphic line bundle and let h^L, $\widetilde{h}^L$ be singular Hermitian metrics on L such that $c_1(L,h^L)\geq 0$ and $c_1(L,\widetilde{h}^L)\geq\varepsilon\omega_X$ for some $\varepsilon>0$. Assume there is $A>0$ such that $h^L\leq A\,\widetilde{h}^L$. Then*

$$\big\|\log B_p\big\|_{L^1(X)}=O(\log p)\,,\quad p\to\infty\,. \tag{4.18}$$

Hence $\dfrac{1}{p}\omega_p\to c_1(L,h^L)$ as $p\to\infty$ with speed $O(\frac{1}{p}\log p)$.

Proof. Since we work only on L, we set in this proof for simplicity $h=h^L$, $\widetilde{h}=\widetilde{h}^L$. Let $x\in X$ and $U_0\subset X$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_L of L. Let ψ be the psh weight of h on U_0 relative to e_L, $|e_L|^2_h=e^{-2\psi}$. Likewise, let ψ' be the psh weight of $\widetilde{h}$ on U_0 relative to e_L, $|e_L|^2_{\widetilde{h}}=e^{-2\psi'}$. Multiplying the section e_L with a constant allows us to assume that $\psi\leq 0$. Fix $r_0>0$ so that the ball $V:=B(x,2r_0)$ of center x and radius $2r_0$ is relatively compact in U_0 and let $U:=B(x,r_0)$. By [6, Theorem 5.1] and its proof (following [11]) there exists $C_1>0$ so that

$$\log B_p(z)\leq\log(C_1r^{-2n})+2p\Big(\sup_{B(z,r)}\psi-\psi(z)\Big) \tag{4.19}$$

holds for all $p\geq 1$, $0<r<r_0$ and $z\in U$ with $\psi(z)>-\infty$.

Choose $r=1/p^4$. By applying Lemma 4.2 to ψ we obtain from (4.19) that the integral on U of the positive part of the right hand side of (4.19) is smaller than $C_2\log p+C_2$ for some $C_2>0$. Hence, in order to prove (4.18) it remains to bound the negative part of $\log B_p$.

Multiplying $\widetilde{h}$ with a constant allows us to assume that $A=1$. So we have $h\leq\widetilde{h}$ and $\psi'\leq\psi$. Consider an integer p_0 (to be chosen momentarily). Write $L^p=L^{p-p_0}\otimes L^{p_0}$ and consider on L^p, $p>p_0$, the metric

$$H_p:=h^{\otimes(p-p_0)}\otimes\widetilde{h}^{\otimes p_0}\,,\qquad h_p:=h^{\otimes p}. \tag{4.20}$$

Then

$$c_1(L^p,H_p)=(p-p_0)c_1(L,h)+p_0c_1(L,\widetilde{h})\geq p_0\varepsilon\omega_X\,. \tag{4.21}$$

The weight of the metric H_p with respect to the frame $e_L^{\otimes p}$ is $\Psi_p:=(p-p_0)\psi+p_0\psi'$ and we have $|e_L^{\otimes p}|^2_{H_p}=e^{-2\Psi_p}$.

Following [12, Section 9], we proceed as in [6, Theorem 5.1] to show that there exist $C_1>0$ and $p_0\in\mathbb{N}$ such that for all $p>p_0$ and all $z\in U$ with $\Psi_p(z)>-\infty$ there is a section $s_{z,p}\in H^0_{(2)}(X,L^p)$ with $s_{z,p}(z)\neq 0$ and

$$\int_X|s_{z,p}|^2_{H_p}\,\omega_X^n\leq C_1|s_{z,p}(z)|^2_{H_p}\,. \tag{4.22}$$

Let us prove the existence of $s_{z,p}$ as above. By the Ohsawa-Takegoshi extension theorem [21] there exists $C'>0$ (depending only on x) such that for any $z\in U$ and any $p\in\mathbb{N}$ one can find a holomorphic function $v_{z,p}$ on V with $v_{z,p}(z)\neq 0$ and

$$\int_V|v_{z,p}|^2e^{-2\Psi_p}\omega_X^n\leq C'|v_{z,p}(z)|^2e^{-2\Psi_p(z)}\,. \tag{4.23}$$

The function $v_{z,p}$ can be identified to a local section of L^p satisfying an estimate similar to (4.22).

We shall now solve the $\overline{\partial}$-equation with L^2-estimates in order to modify $v_{z,p}$ and get a global section $s_{z,p}$ of L^p over X. Let $\theta \in \mathscr{C}^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \theta \leq 1$, $\theta(t) = 1$ for $|t| \leq \frac{1}{2}$, $\theta(t) = 0$ for $|t| \geq 1$. Define the quasi-psh function φ_z on X by

$$\varphi_z(y) = \begin{cases} n\theta\big(\frac{|y-z|}{r_0}\big)\log\frac{|y-z|}{r_0}\,, & \text{for } y \in U_0\,, \\ 0, & \text{for } y \in X \setminus B(z, r_0)\,. \end{cases} \tag{4.24}$$

We apply Theorem 4.1 (ii) for (X, ω_X) and $(L^p, H_p\, e^{-\varphi_z})$. Note that there exists $C_3 > 0$ such that $dd^c\varphi_z \geq -C_3\omega_X$ for all $z \in U$. We have

$$c_1(L^p, H_p\, e^{-\varphi_z}) = (p - p_0)\, c_1(L, h^L) + p_0\, c_1(L, \widetilde{h}^L) + dd^c\varphi_z \geq (p_0\varepsilon - C_3)\omega_X\,. \tag{4.25}$$

Since p_0 is large enough, we have $(p_0\varepsilon - C_3)\omega_X + c_1(K_X^*, h^{K_X^*}) \geq C_3\,\omega_X$. Thus,

$$c_1(L^p, H_p\, e^{-\varphi_z}) + c_1(K_X^*, h^{K_X^*}) \geq C_3\,\omega_X\,, \quad \text{for any } p \geq p_0\,. \tag{4.26}$$

Consider the form

$$g \in L^2_{0,1}(X, L^p),\ \ g = \overline{\partial}\big(v_{z,p}\,\theta\big(\tfrac{|y-z|}{r_0}\big)e_L^{\otimes p}\big), \tag{4.27}$$

which vanishes outside V and also on $B(z, r_0/2)$. By (4.23), (4.27) and $\Psi_p(z) > -\infty$, we get

$$\begin{aligned} \int_X |g|^2_{H_p}\, e^{-2\varphi_z}\omega_X^n &= \int_{V\setminus B(z,r_0/2)} |v_{z,p}|^2 |\overline{\partial}\theta(\tfrac{|y-z|}{r_0})|^2 e^{-2\Psi_p} e^{-2\varphi_z}\omega_X^n \\ &\leqslant C'' \int_V |v_{z,p}|^2 e^{-2\Psi_p}\omega_X^n \leqslant C''C'\, |v_{z,p}(z)|^2 e^{-2\Psi_p(z)} < \infty, \end{aligned} \tag{4.28}$$

where $C'' > 0$ is a constant that depends only on x. By Theorem 4.1 (ii), (4.26) and (4.28), for each $p \geq p_0$ there exists $u \in L^2_{0,0}(X, L^p)$ such that $\overline{\partial}u = g$ and

$$\int_X |u|^2_{H_p}\, e^{-2\varphi_z}\,\omega_X^n \leqslant \frac{1}{C_3}\int_X |g|^2_{H_p}\, e^{-2\varphi_z}\omega_X^n\,. \tag{4.29}$$

Since g is smooth, u is also smooth. Near z, $e^{-2\varphi_z(y)} = r_0^{2n}|y-z|^{-2n}$ is not integrable, thus $u(z) = 0$. Define

$$s_{z,p} := v_{z,p}\,\theta\big(\tfrac{|y-z|}{r_0}\big)e_L^{\otimes p} - u. \tag{4.30}$$

Then

$$\overline{\partial}s_{z,p} = 0, \quad s_{z,p}(z) = v_{z,p}(z)e_L^{\otimes p}(z) \neq 0, \quad s_{z,p} \in H^0_{(2)}(X, L^p). \tag{4.31}$$

Since $\varphi_z \leqslant 0$ on X, by (4.23), (4.28), (4.29) and (4.30), we get

$$\begin{aligned} \int_X |s_{z,p}|^2_{H_p}\,\omega_X^n &\leqslant 2\left(\int_V |v_{z,p}|^2 e^{-2\Psi_p}\omega_X^n + \int_X |u|^2_{H_p} e^{-2\varphi_z}\,\omega_X^n\right) \\ &\leqslant 2C'\left(1 + \frac{C''}{C_3}\right)|v_{z,p}(z)|^2 e^{-2\Psi_p(z)} = C_1 |s_{z,p}(z)|^2_{H_p}\,, \end{aligned}$$

with a constant $C_1 > 0$ that depends only on x. This concludes the proof of (4.22).

By dividing both sides of (4.22) by a constant, we obtain the existence of sections $s_{z,p} \in H^0(X, L^p)$, $p > p_0$, such that

$$\int_X |s_{z,p}|^2_{H_p}\,\omega_X^n = 1\,, \quad |s_{z,p}(z)|^2_{H_p} \geq \frac{1}{C_1}. \tag{4.32}$$

Since $\widetilde{h} \geq h$, the first property of (4.32) and (4.20) imply

$$\int_X |s_{z,p}|^2_{h_p}\,\omega_X^n \leqslant 1\,. \tag{4.33}$$

Then (4.1), (4.3), (4.20) and the second property of (4.32) yield

$$|s_{z,p}(z)|^2_{h_p} \geq C_1^{-1} e^{2p_0(\psi'(z)-\psi(z))} = C_1^{-1} e^{2p_0(\varphi'(z)-\varphi(z))}. \tag{4.34}$$

Recall now (see e. g., [6, Lemma 3.1]) that

$$\begin{aligned} B_p(x) &= \max\{|s(x)|^2_{h_p} :\ s \in H^0_{(2)}(X, L^p),\ \|s\|_p = 1\} \\ &= \max\{|s(x)|^2_{h_p} :\ s \in H^0_{(2)}(X, L^p),\ \|s\|_p \leq 1\}\,. \end{aligned} \tag{4.35}$$

It follows from (4.33)-(4.35) that there exists $C_5 > 0$ such that

$$\log B_p(z) \geq \log |s_{z,p}(z)|^2_{h_p} \geq 2p_0\big(\varphi'(z) - \varphi(z)\big) - C_5 =: \eta(z)\,, \tag{4.36}$$

where $\eta \in L^1(X, \omega_X^n)$, $\eta \leq 0$. Hence $\log B_p \geq \eta$ a.e. on X. The result follows. □

Corollary 4.4. *Let (X, ω_X) be a compact Kähler manifold of dimension n. Let L be a big holomorphic line bundle and let h^L, $\widetilde{h}^L$ be as in Theorem 4.3. Let U be an open subset of X.*
(i) Assume that the global weight φ' of $\widetilde{h}^L$ given by (4.3) *is bounded on a neighborhood of $\overline{U}$. Then*

$$\|\varphi_p - \varphi\|_{L^1(U)} = O\left(\frac{1}{p}\log p\right)\,,\ \ p \to \infty, \tag{4.37}$$

and for every $1 \leq k \leq n$ we have

$$\frac{1}{p^k}\omega_p^k \to c_1(L, h^L)^k\,,\ \ p \to \infty,\ \ \text{on } U. \tag{4.38}$$

(ii) Assume moreover, φ is Hölder continuous on a neighborhood of $\overline{U}$. Then

$$\|\varphi_p - \varphi\|_{U,\infty} = O\left(\frac{1}{p}\log p\right)\,,\ \ p \to \infty\,, \tag{4.39}$$

and (4.38) *holds with speed $O\big(\frac{1}{p}\log p\big)$.*

Hence for σ_∞-almost every sequence $(S_p) \in (\Omega_k(L), \sigma_\infty)$, $S_p = ([s_p^{(1)}], \dots, [s_p^{(k)}])$,

$$\frac{1}{p^k}\big[s_p^{(1)} = \ldots = s_p^{(k)} = 0\big] \to c_1(L, h^L)^k\,,\ \ p \to \infty,\ \ \text{on } U \text{ with speed } O\left(\frac{1}{p}\log p\right). \tag{4.40}$$

Proof. Since φ' is bounded on a neighborhood of $\overline{U}$, φ is also bounded in that neighborhood. We see in the above proof that (4.37) holds and $\varphi_p + c/p \geq \varphi$ for some $c > 0$. On the set where φ and φ_p are locally bounded the wedge-products ω^k and ω_p^k are well-defined for any $1 \leq k \leq n$ by (4.2), (4.5) and [1]. Thus (4.38) holds.

Assume moreover that φ is Hölder continuous on a neighborhood of $\overline{U}$. Observe that the function η in (4.36) is bounded on U, thus, taking $r = 1/p^\ell$ with ℓ large enough in (4.19), yields (4.39). Finally, (4.40) follows from Corollary 3.9. □

Note that under the assumptions of Corollary 4.4 (i) we do not obtain an estimate of the convergence speed in (4.38). To get this, the assumption of Hölder continuity in item (ii) is necessary.

We can state a result similar to Theorem 4.3 in the case of adjoint line bundles $L^p \otimes K_X$. We do not suppose that the base manifold is compact, so the space of L^2 holomorphic sections could be infinite dimensional. However, the definitions (2.9) and (2.13) of the Bergman kernel function and Fubini-Study currents carry over without change. Theorem 4.5 refines [8, Theorem 3.1], where it is shown that $\frac{1}{p}\log B_p \to 0$ in $L^1(U, \omega_X^n)$.

Theorem 4.5. *Let (X, ω_X) be a Kähler manifold of dimension n which admits a (possibly different) complete Kähler metric. Let L be a holomorphic line bundle and let h^L be a singular Hermitian metric on L such that $c_1(L, h^L) \geq 0$. Let $U \subset X$ be a relatively compact open set such that $c_1(L, h^L) \geq \varepsilon\omega_X$ on a neighborhood of $\overline{U}$ for some $\varepsilon > 0$. Let B_p and ω_p be the Bergman kernel function and Fubini-Study current associated with $H^0_{(2)}(X, L^p \otimes K_X)$. Then*

$$\big\| \log B_p \big\|_{L^1(U)} = O(\log p)\,, \quad p \to \infty\,. \tag{4.41}$$

Hence $\dfrac{1}{p}\omega_p \to c_1(L, h^L)$ on U as $p \to \infty$ with speed $O\big(\frac{1}{p}\log p\big)$.

Proof. The proof is similar to the proof of Theorem 4.3, with some simplifications due to the fact that we don't need an auxiliary metric $\widetilde{h}^L$. The Kähler metric ω_X induces a metric on the canonical line bundle K_X that we denote by h^{K_X}. We denote by h_p the metric induced by h^L and h^{K_X} on $L^p \otimes K_X$. Let U' be a neighborhood of $\overline{U}$ on which the hypothesis $c_1(L, h^L) \geq \varepsilon\omega_X$ holds. We let $x \in U$ and $U_0 \subset U'$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_L of L and e' of K_X. Let ψ be a psh weight of h^L. Fix $r_0 > 0$ so that the ball $V := B(x, 2r_0) \Subset U_0$ and let $W := B(x, r_0)$.

Following the arguments of [6, Theorem 5.1] (or, more precisely, [7, Theorem 4.2], where forms with values in $L^p \otimes K_X$ are considered) we show that there exist $C = C(W) > 0$ and $p_0 = p_0(W) \in \mathbb{N}$ so that

$$-\log C \leq \log B_p(z) \leq \log(Cr^{-2n}) + 2p\Big(\max_{B(z,r)} \psi - \psi(z)\Big) \tag{4.42}$$

holds for all $p > p_0$, $0 < r < r_0$ and $z \in W$ with $\psi(z) > -\infty$.

The right-hand side estimate follows as in [6, Theorem 5.1]; it holds for all p and does not require the hypothesis that X is compact.

We prove next the lower estimate from (4.42). We proceed like in the proof of [7, Theorem 4.2] to show that there exist $C_2 = C_2(W) > 0$, $p_0 = p_0(W) \in \mathbb{N}$ such that for all $p > p_0$ and all $z \in W$ with $\psi(z) > -\infty$ there exists $s_{z,p} \in H^0_{(2)}(X, L^p \otimes K_X)$ with $s_{z,p}(z) \neq 0$ and

$$\|s_{z,p}\|_p^2 \leq C_2 |s_{z,p}(z)|^2_{h_p}\,, \tag{4.43}$$

where $\|s\|_p$ is the L^2 norm defined in (2.7). This is done exactly as in [7, Theorem 4.2]; the main point is again the Ohsawa-Takegoshi extension theorem and the solution of the $\overline{\partial}$-equation by the L^2 method from Theorem 4.1 (i). Observe that (4.35) and (4.43) yield the desired lower estimate

$$\log B_p(z) = \max_{\|s\|_p=1} \log|s(z)|^2_{h_p} \geq -\log C_2\,, \quad \text{for } p > p_0,\ z \in W \text{ and } \psi(z) > -\infty. \tag{4.44}$$

Since U is relatively compact we can choose C_2 and p_0 such that $\log B_p \geq -\log C_2$ holds a.e. on U for all $p > p_0$. As in the proof of Theorem 4.3, we use the estimate from above in (4.42) and Lemma 4.2 to show the existence of $C_1 = C_1(U') > 0$ such that for all $p \in \mathbb{N}^*$,

$$\int_U (\log B_p)\,\omega_X^n \le C_1 \log p + C_1.$$

This completes the proof of Theorem 4.5. □

Proof of Theorem 1.2. Combining Theorem 4.3 and Corollary 3.4 applied to the case where (F, h^F) is the trivial line bundle and $\lambda_p = (2n+2)c\log p$, we obtain item (i). Theorem 4.5 and Corollary 3.4 for $(F, h^F) = (K_X, h^{K_X})$ and the same λ_p as above yield item (ii). □

5. Approximation of Hölder continuous weights

In this section we prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. The continuity can be deduced directly from the estimate (1.5).

We prove now (1.5). Write as above $L^p = L^{p-p_0} \otimes L^{p_0}$ with the metric $H_{e,p} := (h^L_{\rm eq})^{\otimes(p-p_0)} \otimes (h^L_0)^{\otimes p_0}$. As in Section 4 (see (4.32)), given a point $x_0 \in X$, there exists a neighborhood $U(x_0)$ and $C > 0$ such that for any $z \in U(x_0)$, one can find a holomorphic section $s_{z,p} \in H^0(X, L^p)$ satisfying

$$\int_X |s_{z,p}|^2_{H_{e,p}} \omega_X^n \le C\,, \quad |s_{z,p}(z)|_{H_{e,p}} = 1. \tag{5.1}$$

Since $\varphi_{\rm eq}$ and φ are bounded and $\varphi_{\rm eq} \le \varphi$, we deduce from (5.1) that there exists $C > 0$ such that

$$\int_X |s_{z,p}|^2_{h_p} \omega_X^n \le C\,, \quad |s_{z,p}(z)|_{h_p} \ge C^{-1} e^{p(\varphi_{\rm eq}-\varphi)}. \tag{5.2}$$

It follows from (4.35) and (5.2) that there exists $c > 0$ such that we have

$$\frac{1}{2p}\log B_p \ge \varphi_{\rm eq} - \varphi - \frac{c}{p} \quad \text{on } X. \tag{5.3}$$

Since $\varphi_p = \varphi + \frac{1}{2p}\log B_p$, we obtain that

$$\varphi_p - \varphi_{\rm eq} \ge -\frac{c}{p} \quad \text{on } X. \tag{5.4}$$

The estimate from above for $\varphi_p - \varphi_{\rm eq}$ is obtained using the submean inequality. Since φ_p is α-psh, by (1.3), it is enough to show that $\varphi_p \le \varphi + \frac{c\log p}{p}$ on X which is equivalent to $B_p \le p^{2c}$ for some $c > 0$. Fix a point a in X. Consider an arbitrary holomorphic section $s \in H^0(X, L^p)$ such that

$$\int_X |s|^2_{h_p} \omega_X^n = 1. \tag{5.5}$$

By (4.35), we only have to check that

$$|s(a)|^2_{h_p} \le p^{2c}. \tag{5.6}$$

Fix local holomorphic coordinates z around a with $|z| \le 1$ and a holomorphic frame of L such that s is represented by a holomorphic function f and the metric h^L is represented by $e^{-\psi}$ with ψ is Hölder continuous and $\psi(0) = 0$. So, we have for some $C, \alpha > 0$

$$|\psi(z)| \le C|z|^\alpha. \tag{5.7}$$

Since s has unit L^2-norm, the integral

$$\int_{|z|\leq p^{-1/\alpha}} |f(z)|^2 e^{-2Cp|z|^\alpha} dZ$$

is bounded by a constant independent of p. It follows that the integral of $|f|^2$ on the ball $B(0, p^{-1/\alpha})$ is bounded, because the function $e^{2Cp|z|^\alpha}$ is bounded there. Therefore, by the submean inequality, we get

$$|s(a)|^2_{h_p} = |f(0)|^2 \leq C' p^{2n/\alpha}. \tag{5.8}$$

This completes the proof. □

Proof of Corollary 1.4. Theorem 1.3 together with Corollary 3.9 applied to

$$\lambda_p = (m+2)c \log p$$

imply immediately the result. □

Example 5.1. Let us discuss here the important example of the line bundle $L = \mathcal{O}(1)$ over $X = \mathbb{P}^n$. The global holomorphic sections of $L^p =: \mathcal{O}(p)$ are given by homogeneous polynomials of degree p on $\mathbb{C}^{n+1}$:

$$H^0(\mathbb{P}^n, \mathcal{O}(p)) \cong \big\{ f \in \mathbb{C}[w_0, \dots, w_n] : f \text{ homogeneous}, \deg f = p \big\} =: R_p. \tag{5.9}$$

There exists a smooth metric $h_{\text{FS}} = h^{\mathcal{O}(1)}_{\text{FS}}$ on $\mathcal{O}(1)$ such that the Fubini-Study Kähler form on $\mathbb{P}^n$ is defined as the first Chern form associated to $(\mathcal{O}(1), h_{\text{FS}})$,

$$\omega_{\text{FS}} = \frac{i}{2\pi} R^{\mathcal{O}(1)}. \tag{5.10}$$

Let $\text{Met}^+(\mathcal{O}(1))$ be the set of all semipositively curved singular metrics on $\mathcal{O}(1)$. By (2.5) we know that there exists a bijection

$$PSH(\mathbb{P}^n, \omega_{\text{FS}}) \longrightarrow \text{Met}^+(\mathcal{O}(1)), \ \ \varphi \longmapsto h_\varphi = h_{\text{FS}} e^{-2\varphi}, \tag{5.11}$$

and $c_1(\mathcal{O}(1), h_\varphi) = \omega_{\text{FS}} + dd^c\varphi$. Moreover, $PSH(\mathbb{P}^n, \omega_{\text{FS}})$ is in one-to-one correspondence to the Lelong class $\mathcal{L}(\mathbb{C}^n)$ of entire psh functions with logarithmic growth,

$$\mathcal{L}(\mathbb{C}^n) = \Big\{ \psi \in PSH(\mathbb{C}^n) : \text{ there is } C_\psi \in \mathbb{R} \text{ such that } \psi(z) \leq \tfrac{1}{2}\log(1+|z|^2) + C_\psi \text{ for } z \in \mathbb{C}^n \Big\},$$

and the map $\mathcal{L}(\mathbb{C}^n) \to PSH(\mathbb{P}^n, \omega_{\text{FS}})$ is given by $\psi \mapsto \varphi$ where

$$\varphi = \begin{cases} \psi(w) - \frac{1}{2}\log(1+|w|^2), & w \in \mathbb{C}^n, \\ \limsup\limits_{z \to w, z \in \mathbb{C}^n} \varphi(z), & w \in \mathbb{P}^n \setminus \mathbb{C}^n. \end{cases}$$

Here we use the usual embeding of $\mathbb{C}^n$ in $\mathbb{P}^n$. Let $h \in \text{Met}^+(\mathcal{O}(1))$ and let $\varphi \in PSH(\mathbb{P}^n, \omega_{\text{FS}})$ such that $h = h_{\text{FS}} e^{-2\varphi}$. Then

$$H^0_{(2)}(\mathbb{P}^n, \mathcal{O}(p)) = \Big\{ f \in H^0(\mathbb{P}^n, \mathcal{O}(p)) : \int_{\mathbb{P}^n} |f|^2_{h^p_{\text{FS}}} e^{-2p\varphi} \omega^n_{\text{FS}} < \infty \Big\} =: R_p(\varphi). \tag{5.12}$$

We denote as usual by ω_p the Fubini-Study current associated with $H^0_{(2)}(\mathbb{P}^n, \mathcal{O}(p))$ by (2.12) and let φ_p be the Fubini-Study global weights (4.4). Note that if φ is bounded, $R_p(\varphi) = R_p$ (as sets but in general not as Hilbert spaces).

We have the following immediate consequence of Theorem 1.2 and Corollary 4.4.

Corollary 5.2. *(i) Let $\varphi \in PSH(\mathbb{P}^n, \omega_{\rm FS})$. Assume there exists $\widetilde{\varphi} \in PSH(\mathbb{P}^n, \omega_{\rm FS})$ such that*

$$\varphi \geq \widetilde{\varphi} \text{ and } (1-\varepsilon)\omega_{\rm FS} + dd^c\widetilde{\varphi} \geq 0\,, \text{ for some } \varepsilon > 0.$$

Then for σ_∞-almost every sequence $[s_p] \in \mathbb{P}\big(R_p(\varphi)\big)$ of homogeneous polynomials, $\big(\frac{1}{p}[\mathrm{Div}(s_p)]\big)$ converges to $\omega_{\rm FS} + dd^c\varphi$ on $\mathbb{P}^n$ as $p \to \infty$ with speed $O\big(\frac{1}{p}\log p\big)$.
(ii) Let U be an open subset of $\mathbb{P}^n$. Assume that $\widetilde{\varphi}$ is bounded on a neighborhood of $\overline{U}$. Then

$$\|\varphi_p - \varphi\|_{L^1(U)} = O\left(\frac{1}{p}\log p\right)\,,\ p \to \infty, \tag{5.13}$$

and for every $1 \leq k \leq n$ we have (not necessarily with speed estimate),

$$\frac{1}{p^k}\omega_p^k \to (\omega_{\rm FS} + dd^c\varphi)^k\,,\ p \to \infty,\ \text{on } U. \tag{5.14}$$

(iii) Assume moreover that φ is Hölder continuous on a neighborhood of $\overline{U}$. Then

$$\|\varphi_p - \varphi\|_{U,\infty} = O\left(\frac{1}{p}\log p\right)\,,\ p \to \infty\,, \tag{5.15}$$

and (5.14) *holds with speed $O\big(\frac{1}{p}\log p\big)$.*

Hence for σ_∞-almost every sequence $([s_p^{(1)}], \ldots, [s_p^{(k)}]) \in \mathbb{P}\big(R_p(\varphi)\big)^k$ of k-tuples of homogeneous polynomials we have as $p \to \infty$,

$$\frac{1}{p^k}\big[s_p^{(1)} = \ldots = s_p^{(k)} = 0\big] \to (\omega_{\rm FS} + dd^c\varphi)^k\,,\ \text{on } U \text{ with speed } O\left(\frac{1}{p}\log p\right). \tag{5.16}$$

Theorem 1.3 and Corollary 1.4 imply the following.

Corollary 5.3. *Let φ be a Hölder continuous function on $\mathbb{P}^n$. Then:*

(i) The equilibrium weight $\varphi_{\rm eq}$ is continuous on $\mathbb{P}^n$ and the global Fubini-Study weights φ_p given by (4.4) *converge to $\varphi_{\rm eq}$ uniformly with speed $O\big(\frac{1}{p}\log p\big)$.*

(ii) For any $1 \leq k \leq n$ we have $\frac{1}{p^k}\omega_p^k \to \omega_{\rm eq}^k$ on X as $p \to \infty$ with speed $O\big(\frac{1}{p}\log p\big)$.

(iii) Let $1 \leq k \leq n$. For σ_∞-almost every sequence $([s_p^{(1)}], \ldots, [s_p^{(k)}]) \in \mathbb{P}\big(R_p(\varphi)\big)^k$,

$$\frac{1}{p^k}\big[s_p^{(1)} = \ldots = s_p^{(k)} = 0\big] \to \omega_{\rm eq}^k\,,\ \text{on } \mathbb{P}^n \text{ with speed } O\left(\frac{1}{p}\log p\right). \tag{5.17}$$

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