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Moser-Trudinger inequalities and complex Monge-Ampère equation

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Abstract. We present a version of the Moser-Trudinger inequality in the setting of complex geometry. As a very particular case, our result already gives a new Moser-Trudinger inequality for functions in the Sobolev space $W^{1,2}$ of a domain in $\mathbb{R}^2$. We also deduce a new necessary condition for the existence of a Hölder continuous solution of the complex Monge-Ampère equation with right-hand side a given measure on a compact Kähler manifold.

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1. Introduction

Moser-Trudinger inequalities are important in Functional Analysis and Partial Differential Equations. There exist various versions of the Moser-Trudinger inequalities, see [2,4,21,22,25] and the references therein, to cite just a few. We just recall here a well-known version of this inequality in the real two-dimensional setting.

Let Ω be a domain in $\mathbb{C} \approx \mathbb{R}^2$. Let $W^{1,2}(\Omega)$ be the Sobolev space of square integrable functions on Ω whose partial derivatives of order one are also square integrable on Ω. We will denote by Leb the Lebesgue measure in the Euclidean spaces.

Theorem 1.1 ([21]). *Let K be a compact subset of Ω. There exist strictly positive constants α and c such that*

$$\int_K e^{\alpha|u|^2} d\,\mathrm{Leb} \leq c \tag{1.1}$$

for every $u \in W^{1,2}(\Omega)$ of $W^{1,2}$-norm at most 1.

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In higher real dimension n, in the present literature, a similar inequality holds if we consider the Sobolev space $W^{1,n}$ instead of $W^{1,2}$ and the term $|u|^2$ in (1.1) is replaced by $|u|^{\frac{n}{n-1}}$.

Our aim in this paper is to give a version of the classical Moser-Trudinger inequality in the setting of complex geometry. Our result is already new in the case of complex dimension one as shown by Corollary 1.4 below and the comment following it. Let us first set some notation.

Let n be a positive integer and Ω be a domain in $\mathbb{C}^n \approx \mathbb{R}^{2n}$. Let $W^{1,2}(\Omega)$ be the set of square integrable functions on Ω whose partial derivatives of order one are also square integrable on Ω. Let $W^{1,2}_*(\Omega)$ be the set of $u \in W^{1,2}(\Omega)$ such that there exists a closed positive $(1,1)$-current $T = T_u$ of bounded mass on Ω with

$$i\partial u \wedge \overline{\partial} u \leq T. \tag{1.2}$$

This functional space was introduced in [10] in the context of complex dynamics and studied in more detail in [26], see [7, 29] for recent applications to complex dynamics. For $u \in W^{1,2}_*(\Omega)$, put

$$\|u\|_*^2 := \|u\|_{L^2}^2 + \inf\big\{\|T\| : T \text{ satisfies (1.2)}\big\}.$$

Note that by the compactness of the space of closed positive currents, the last infimum is actually a minimum. The last formula defines a norm on $W^{1,2}_*(\Omega)$ that becomes a Banach space with respect to this norm ([26, Proposition 1]). In dimension one, we have $W^{1,2}_*(\Omega) = W^{1,2}(\Omega)$.

We set $d^c = \frac{1}{2\pi}(\overline{\partial} - \partial)$, so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$. In the complex one-dimensional case, dd^c is simply the Laplace operator. We will need some notions from the pluripotential theory. We refer to [1,3,5,18,19,24] for an introduction to this topic. Recall that a *plurisubharmonic* (or p.s.h. for short) function v on Ω satisfies that $dd^c v$ is a closed positive $(1,1)$-current on Ω. A subset A of Ω is said to be *pluripolar* if there exists a p.s.h. function $v \not\equiv -\infty$ on Ω such that $A \subset \{v = -\infty\}$. By a classical result of Josefson [16, 19], a locally pluripolar set is pluripolar. Hence, we don't need to precise the ambient domain when we talk about a pluripolar set, see also [28].

Let $v_1, \ldots, v_n$ be bounded p.s.h. functions on Ω. It is classical in the pluripotential theory that the intersection $dd^c v_1 \wedge \cdots \wedge dd^c v_n$ is well-defined and is a positive measure on Ω having no mass on pluripolar subsets of Ω. In general, that measure can be *singular* with respect to the Lebesgue measure on Ω, see for example [14, 27]. Here is our main result.

Theorem 1.2. *Let Ω be a domain in $\mathbb{C}^n$ and K be a compact subset of Ω. Let $v_1, \ldots, v_n$ be p.s.h. functions which are Hölder continuous of exponent $\beta \in (0,1)$ on Ω. Let $u \in W^{1,2}_*(\Omega)$. Assume that $\|v_j\|_{\mathscr{C}^\beta} \leq 1$ for $1 \leq j \leq n$ and $\|u\|_* \leq 1$. Then there exist strictly positive constants α and c depending on Ω, K, β but independent of $u, v_1, \ldots, v_n$ such that*

$$\int_K e^{\alpha|u|^2} dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq c. \tag{1.3}$$

In particular, u belongs to L^p_{loc} *with respect to the measure* $dd^c v_1 \wedge \cdots \wedge dd^c v_n$ *for every* $p \in [1, \infty)$.

Let us stress two features of Theorem 1.2. Firstly, unlike other known higher dimensional versions of Moser-Trudinger inequalities, we get the term $|u|^2$ as in the complex one-dimensional case. Secondly, (1.3) holds for much more general measures than the Lebesgue measure. Let us make some more comments about (1.3). Firstly, since the elements in $W^{1,2}_*(\Omega)$ are *a priori* only measurable functions with respect to Lebesgue measures, it is not obvious that the integral in the left-hand side of (1.3) makes sense. To simplify the situation, one can consider for the moment u continuous in (1.3) with values in $\mathbb{R} \cup \{\pm\infty\}$ and the last inequality tells us that the integral is bounded uniformly in u. Actually, every $u \in W^{1,2}_*(\Omega)$ can be represented, in a canonical way, by Borel functions defined on Ω except possibly a pluripolar subset of Ω, and such two representatives are equal outside a pluripolar set. So the integral in (1.3) is the integral of one of such representatives of u and it is independent of the choice of such a representative, see Theorem 2.10 below and [26]. Note that the Hölder continuity of v_j in Theorem 1.2 is necessary; see Example 3.7 for a counter-example if v_j's are merely continuous.

We now present some consequences of Theorem 1.2. Let (X, ω) be a compact Kähler manifold of dimension n. Let φ be a bounded ω-p.s.h. function on X. Recall that φ is called ω-p.s.h. if φ is locally the sum of a p.s.h. and a smooth function and $dd^c\varphi + \omega \geq 0$ in the sense of currents. The measure $\mu := (dd^c\varphi + \omega)^n$ is called *a Monge-Ampère measure*. These measures are central objects of study in complex geometry and pluripotential theory. If φ is Hölder continuous, μ is called a *Monge-Ampère measure with Hölder potentials*. The following is a direct consequence of our main result.

Corollary 1.3. *Let X be a compact Kähler manifold. Let* μ *be a Monge-Ampère measure with Hölder potentials on X. Then there exist strictly positive constants* α *and c such that*

$$\int_X e^{\alpha |u|^2} d\mu \leq c$$

for every $u \in W^{1,2}_*(X)$ *with* $\|u\|_* \leq 1$.

Here we define $W^{1,2}_*(X)$ in a way similar to that of $W^{1,2}_*(\Omega)$. The last result gives us a necessary condition to test whether a given measure is a Monge-Ampère measure with Hölder potentials. We refer to [8,17,20,30] and the references therein for related results. The readers can also consult [6, Theorems 2.1 and 4.6] for a Moser-Trudinger type inequality for quasi-psh functions of finite energy.

We give now another application of Theorem 1.2. Let Y be a smooth *generic Cauchy-Riemann* (real) submanifold of Ω, *i.e.*, given any point $a \in Y$, the tangent space of Y at a is not contained in any complex hyperplane of the tangent space of Ω at the point. The simplest example is $Y := \mathbb{R}^n \cap \Omega$, where $\mathbb{R}^n \approx \mathbb{R}^n + i\,0 \hookrightarrow \mathbb{C}^n := \mathbb{R}^n + i\mathbb{R}^n$. Let K be a compact subset of Y. Since $\mathbb{C}^n \subset \mathbb{P}^n$ (the complex projective space of dimension n), using [27], we see that the restriction of a Lebesgue measure of Y to K is a Monge-Ampère measure with Hölder potentials on Ω (here by a Lebesgue measure, we mean the volume with respect to a smooth Riemannian metric on Y). Hence, Theorem 1.2 immediately gives us the following result.

Corollary 1.4. *Let Ω be a domain in $\mathbb{C}^n$ and Y be a smooth generic Cauchy-Riemann submanifold of Ω. Let K be a compact subset of Y and let* Leb *denote a fixed Lebesgue measure on Y. Then there exist strictly positive constants α and c such that*

$$\int_K e^{\alpha|u|^2}\, d\,\mathrm{Leb} \leq c$$

for every $u \in W^{1,2}_*(\Omega)$ with $\|u\|_* \leq 1$.

Observe that the last result is already new even if we apply it to the simplest situation where $\Omega = \mathbb{D}$ is the unit disc in $\mathbb{C}$ and $K \Subset \mathbb{R} \cap \mathbb{D}$. We also obtain from the last result applied to $Y = \Omega$ the following corollary.

Corollary 1.5. *Let $(u_k)_k$ be a bounded sequence in $W^{1,2}_*(\Omega)$ converging to a function $u \in W^{1,2}_*(\Omega)$ in the sense of currents. Then u_k converges to u in L^p_{loc} for every $1 \leq p < +\infty$.*

In the next section, we present some facts about the space $W^{1,2}_*$. In the last section, we prove the main result and Corollary 1.5. Our proof of Theorem 1.2 consists of two steps. Firstly, we use the Hölder continuity of $v_1, \ldots, v_n$ and arguments similar to those in [8, 9] to reduce the question to the case where $v_1, \ldots, v_n$ are smooth. In the second step, we use slicing of currents to make a reduction to a lower dimension case and then we apply Theorem 1.1. The case with v_j smooth is enough to obtain Corollary 1.5.

2. Properties of functions in the complex Sobolev space

Let Ω be a domain in $\mathbb{C}^n$. We present properties of the space $W^{1,2}_*(\Omega)$. To simplify the notation, we write $W^{1,2}_*$ instead of $W^{1,2}_*(\Omega)$ if no confusion arises. We will use some basic results in pluripotential theory and refer the reader to [1,3,5,19,24] for details.

Standard regularization. The following approximation of functions and currents will be used several times in this paper. Let χ be a smooth non-negative cut-off radial function with support in a ball $\mathbb{B}(0, R)$ of center 0 and radius R in $\mathbb{C}^n$ such that $\int_{\mathbb{C}^n} \chi\, d\,\mathrm{Leb} = 1$. For every real number $\varepsilon > 0$, put $\chi_\varepsilon(x) := \varepsilon^{-2n}\chi(\varepsilon^{-1}x)$. We have $\int_{\mathbb{C}^n} \chi_\varepsilon\, d\,\mathrm{Leb} = 1$ and the support of χ_ε is contained in the ball of center 0 and radius $R\varepsilon$. For every function $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the convolution

$$u_\varepsilon(x) := u * \chi_\varepsilon(x) = \int_{\mathbb{C}^n} u(x-y)\chi_\varepsilon(y)\, d\,\mathrm{Leb}(y).$$

This is a smooth function defined on $\Omega_\varepsilon := \{x : \text{dist}(x, \partial\Omega) > R\varepsilon\}$. We call it *a standard regularization* of u. Since χ is non-negative radial, if u is p.s.h., u_ε is smooth, p.s.h. and decreases to u when ε decreases to 0, thanks to the submean inequality.

Let T be a closed positive $(1,1)$-current on Ω. Write $T = dd^c\varphi$ locally and define $T_\varepsilon := dd^c\varphi_\varepsilon$, where φ_ε is the standard regularization of φ. Observe that T_ε is independent of the choice of φ (because if φ' is another local potential of T, then $dd^c(\varphi' - \varphi) = 0$, hence we obtain $dd^c\varphi'_\epsilon = dd^c\varphi_\epsilon$). Therefore, we obtain a closed positive $(1,1)$-current T_ε defined on Ω_ε that we also call *a standard regularization* of T.

Wedge-product of currents and continuity. Let R be a closed positive current on Ω. Recall that if v is a bounded p.s.h. function, then $dd^cv \wedge R := dd^c(vR)$ is a closed positive current. Hence, for bounded p.s.h. functions $v_1, \dots, v_l$, we can define inductively $dd^cv_1 \wedge \dots \wedge dd^cv_l \wedge R$. It is well-known that both $v_1dd^cv_2 \wedge \dots \wedge dd^cv_l \wedge R$ and $dd^cv_1 \wedge \dots \wedge dd^cv_l \wedge R$ depend continuously on $v_1, \dots, v_l$ by taking sequences of p.s.h. functions decreasing to $v_1, \dots, v_l$. So, we can apply this property for the standard regularization of v_j described above.

Let v_1 and v_2 be bounded p.s.h. functions on Ω. If $A > 0$ is a large enough constant, we have $v_j + A \geq 0$ and hence $(v_j + A)^2$ and $(v_1 + v_2 + 2A)^2$ are p.s.h. functions. It follows that $(v_1 - v_2)^2$ is the difference of two bounded p.s.h. functions because we can write

$$(v_1 - v_2)^2 = \left[2(v_1 + A)^2 + 2(v_2 + A)^2\right] - (v_1 + v_2 + 2A)^2.$$

This, together with the identity $dd^cv^2 = 2(dv \wedge d^cv + v\,dd^cv)$, allow us to define

$$\begin{aligned} d(v_1 - v_2) \wedge d^c(v_1 - v_2) \wedge R := \frac{1}{2}dd^c(v_1 - v_2)^2 \\ \wedge R - (v_1 - v_2)dd^c(v_1 - v_2) \wedge R. \end{aligned} \tag{2.1}$$

Capacity and convergence in capacity. Let K be a Borel subset of Ω. Recall that the *capacity* of K in Ω is the quantity

$$\text{cap}(K, \Omega) := \sup\left\{\int_K (dd^cv)^n : 0 \leq v \leq 1 \quad \text{p.s.h. on } \Omega\right\}.$$

This notion was introduced in [1]. Every pluripolar set in Ω is of zero capacity in Ω. Recall that Ω is called *hyperconvex* if there exists a continuous p.s.h. function $\rho : \Omega \to (-\infty, 0)$ such that $\{\rho < c\}$ is relatively compact in Ω for every constant $c < 0$. Examples of such domains are balls in $\mathbb{C}^n$. If Ω is hyperconvex, then a subset A of Ω is pluripolar if and only if $\text{cap}^*(A, \Omega) = 0$, where

$$\text{cap}^*(A, \Omega) := \inf\left\{\text{cap}(U, \Omega) : A \subset U \subset \Omega,\ U \text{ open}\right\},$$

see [1, 19].

Let u_k be a Borel function defined everywhere on Ω except on a pluripolar subset of Ω for $k \in \mathbb{N}$. We say that $(u_k)_{k\in\mathbb{N}}$ is a *Cauchy sequence with respect to capacity* if for every constant $\delta > 0$, every open subset U in Ω and every compact set K in U, we have

$$\lim_{N\to\infty} \sup_{\{k,l\geq N\}} \operatorname{cap}\big(K \cap \{|u_k - u_l| \geq \delta\}, U\big) = 0. \tag{2.2}$$

Similarly, given a Borel function u defined on Ω except maybe on a pluripolar set, we say that u_k *converges to u in capacity as $k \to \infty$* or u is *a capacity limit of* $(u_k)_k$ if for every open set $U \subset \Omega$, every compact set $K \Subset U$, and every constant $\delta > 0$, we have

$$\operatorname{cap}\big(K \cap \{|u_k - u| \geq \delta\}, U\big) \to 0 \tag{2.3}$$

as $k \to \infty$, see [19] for details. One can check that capacity limits of a given sequence only differ on pluripolar sets. Notice also that if u_k converges to some function u in capacity as $k \to \infty$, then $(u_k)_k$ is a Cauchy sequence with respect to capacity. The above two notions are local because

$$\operatorname{cap}(K \cup K', \Omega) \leq \operatorname{cap}(K, \Omega) + \operatorname{cap}(K', \Omega)$$

for $K \subset \Omega$, $K' \subset \Omega$ and $\operatorname{cap}(K, \Omega) \leq \operatorname{cap}(K, \Omega')$ when $K \subset \Omega' \subset \Omega$.

Two other notions of convergence. Let $u_k \in W^{1,2}_*$ for $k \in \mathbb{N}$ and $u \in W^{1,2}_*$. We say that $u_k \to u$ in the *weak* topology of $W^{1,2}_*$ if $u_k \to u$ in the sense of distributions and $\|u_k\|_*$ is uniformly bounded. Since $W^{1,2}_*$ is continuously embedded in $W^{1,2}$, by Rellich's theorem, we have $u_k \to u$ in $L^{\frac{2n}{n-1}}_{\text{loc}}$ (or L^p_{loc} for every $1 \leq p < \infty$ when $n = 1$). In particular, we have $u_k \to u$ in L^2_{loc}. Note that Corollary 1.5 in Introduction gives a much stronger property. Assume that $u_k \to u$ weakly in $W^{1,2}_*$ as above. Assume also that $i\partial u_k \wedge \overline{\partial} u_k \leq T_k$ for some closed positive $(1,1)$-current T_k converging to a current T. Then we have $i\partial u \wedge \overline{\partial} u \leq T$, see [26, page 251].

We say that $u_k \to u$ *nicely* if $u_k \to u$ weakly in $W^{1,2}_*$ and for every $x \in \Omega$, there exist an open neighbourhood U_x of x and a p.s.h. function φ_k on U_x such that

$$i\partial u_k \wedge \overline{\partial} u_k \leq dd^c \varphi_k$$

for every k and φ_k decreases to some p.s.h. function on U_x.

For $K \subset \Omega$ and R a current on Ω, we denote by $\|R\|_K$ the mass of R on K. We will need the following important estimates, see [3, 19].

Lemma 2.1. *Let $K \Subset \Omega$ be a compact set. Let $v_1, \ldots, v_m$ be bounded p.s.h. functions on Ω and φ another p.s.h. function on Ω. Let R be a closed positive (p,p)-current on Ω with $0 \leq p \leq n-1$. Then, there exists a positive constant c depending only on K and Ω such that*

$$\|dd^c v_1 \wedge \cdots \wedge dd^c v_m \wedge R\|_K \leq c\|v_1\|_{L^\infty(\Omega)} \cdots \|v_m\|_{L^\infty(\Omega)} \|R\|_\Omega,$$

for $1 \leq m \leq n-p$ *and*

$$\begin{aligned}&\|d(v_1-v_2)\wedge d^c(v_1-v_2)\wedge R\|_K\\ \leq& c\big(\|v_1\|_{L^\infty(\Omega)}+\|v_2\|_{L^\infty(\Omega)}\big)\|v_1-v_2\|_{L^\infty(\Omega)}\|R\|_\Omega,\end{aligned}$$

and for every constant $N>0$,

$$\operatorname{cap}\big(\{\varphi\leq -N\}\cap K,\Omega\big)\leq cN^{-1}\|\varphi\|_{L^1(\Omega)}.$$

Proof. For the first and third estimates, we refer to [19, page 8 and Proposition 1.10], see also [3]. The proof of the first estimate is based on an induction on m and the use of integration by parts. We use the same techniques together with (2.1) in order to get the second estimate. We give here the details as we will need them later.

Since the problem is local, we can assume that Ω is the unit ball. By subtracting from v_1, v_2 a same constant, we can assume that

$$v_j\leq -\frac{1}{2}\max\big(\|v_1\|_{L^\infty(\Omega)},\|v_2\|_{L^\infty(\Omega)}\big).$$

Let $A>0$ be equal to a large enough constant (depending on K) times $\|v_1\|_{L^\infty(\Omega)}+\|v_2\|_{L^\infty(\Omega)}$. Then we replace v_j by $v_j':=\max(v_j, A(\|x\|^2-1))$, where x denotes the standard coordinates in $\mathbb{C}^n$. We can choose A large enough so that $v_j'=v_j$ on K and $v_j'=A(\|x\|^2-1)$ outside a fixed ball L such that $K\Subset L\Subset\Omega$. Since $|v_1'-v_2'|\leq|v_1-v_2|$, this step doesn't affect our problem. In this way, we can assume for simplicity that $v_1=v_2=A(\|x\|^2-1)$ outside L and hence $v_1-v_2=0$ there. Let ω denote the standard Kähler form on $\mathbb{C}^n$. We have using (2.1)

$$\begin{aligned}\|d(v_1-v_2)\wedge d^c(v_1-v_2)\wedge R\|_K\leq&\int_\Omega d(v_1-v_2)\wedge d^c(v_1-v_2)\wedge R\wedge\omega^{n-p-1}\\ =\frac{1}{2}\int_\Omega dd^c(v_1-v_2)^2\wedge R\wedge\omega^{n-p-1}&-\int_\Omega(v_1-v_2)dd^c(v_1-v_2)\wedge R\wedge\omega^{n-p-1}.\end{aligned}$$

As $v_1-v_2=0$ on $\Omega\setminus L$, the first integral in the last line vanishes by integration by parts. The second one is bounded by

$$\|v_1-v_2\|_{L^\infty(\Omega)}\big[\|dd^cv_1\wedge R\|_L+\|dd^cv_2\wedge R\|_L\big].$$

We obtain the second estimate in the lemma by applying the first one to L instead of K. □

We have the following elementary property of a Cauchy sequence with respect to capacity.

Lemma 2.2. *Let $(u_k)_k$ be a sequence of continuous functions on Ω. Assume that $(u_k)_k$ is a Cauchy sequence with respect to capacity. Then there exists a Borel function u_∞ defined everywhere on Ω except on a pluripolar set such that:*

(i) *u_k converges to u_∞ in capacity; if u_k converges to another function u'_∞ in capacity, then $u'_\infty = u_\infty$ outside a pluripolar set;*
(ii) *There exists a sequence $(j_k)_k \subset \mathbb{N}$ converging to ∞ such that $u_{j_k} \to u_\infty$ pointwise except on a pluripolar set as $k \to \infty$;*
(iii) *If $(j'_k)_k \subset \mathbb{N}$ is another sequence converging to ∞ such that $u_{j'_k}$ converges pointwise to some function u'_∞ outside a pluripolar set, then $u'_\infty = u_\infty$ outside a pluripolar set.*

Proof. Assume (ii) for the moment. We explain how to get (i) and (iii). The second assertion in (i) is clear. Let $(u_{j_k})_k$ be a subsequence of $(u_k)_k$ such that $u_{j_k} \to u_\infty$ pointwise except on a pluripolar set. Let U be an open subset of Ω and K a compact subset of U. Let $\varepsilon > 0$ be a constant. By (2.2), there exists $N \in \mathbb{N}$ big enough so that

$$\operatorname{cap}\big(K \cap \{|u_{l_1} - u_{l_2}| \geq \delta/2\}, U\big) \leq \varepsilon$$

for every $l_1, l_2 \geq N$. Applying the last inequality to $l_2 = j_k$ and letting $k \to \infty$ give

$$\operatorname{cap}\big(K \cap \{|u_{l_1} - u_\infty| \geq \delta\}, U\big) \leq \varepsilon$$

for every $l_1 \geq N$. This implies that $u_{l_1} \to u_\infty$ in capacity as $l_1 \to \infty$. Hence, (i) follows. Let $(j'_k)_k$ and u'_∞ be as in (iii). Then by the above arguments, we get $u_k \to u'_\infty$ in capacity. Hence, $u'_\infty = u_\infty$ outside a pluripolar set.

It remains to prove (ii). Let $(\Omega_s)_{s\in\mathbb{N}}$ be a countable covering of Ω by open balls. As observed above, Ω_s is hyperconvex for every s. Let $(\Omega'_s)_{s\in\mathbb{N}}$ be another covering of Ω by open balls such that $\Omega'_s \Subset \Omega_s$ for every s. Fix $s \in \mathbb{N}$. Let $\delta > 0$ be a constant. By (2.2), there exists a sequence $(j^s_k) \to \infty$ such that for every k, we have

$$\operatorname{cap}\left(E^s_k \cap \Omega'_s, \Omega_s\right) \leq \operatorname{cap}\left(E^s_k \cap \overline{\Omega}'_s, \Omega_s\right) \leq \delta/2^k,$$

where $E^s_k := \left\{|u_{j^s_k} - u_{j^s_{k+1}}| > 2^{-k}\right\}$ which is an open set. Hence, for $E^s_\delta := \cup_{k=1}^\infty E^s_k$, the sequence $(u_{j^s_k})_k$ converges uniformly on $\Omega'_s \backslash E^s_\delta$.

Observe that $\operatorname{cap}(E^s_\delta \cap \Omega'_s, \Omega_s) \leq \delta$. For $\delta = 1/m$, by a diagonal argument, we can assume that $u_{j^s_k}$ converges uniformly on $\Omega'_s \backslash E^s_{1/m}$ for every m. Hence, $u_{j^s_k}$ converges pointwise on $\Omega'_s \backslash (\cap_m E^s_{1/m})$. Since

$$\operatorname{cap}\left(E^s_{1/m} \cap \Omega'_s, \Omega_s\right) \leq 1/m$$

for every m, we obtain that $\operatorname{cap}^*(\cap_m E^s_{1/m} \cap \Omega'_s, \Omega_s) = 0$. Hence, $\cap_m E^s_{1/m} \cap \Omega'_s$ is pluripolar. This implies that $u_{j^s_k}$ converges pointwise on Ω'_s except on a pluripolar set.

Applying the above arguments to $s = 1, 2, \ldots$ and using a diagonal argument again, we obtain a sequence $(j_k)_k \subset \mathbb{N}$ converging to ∞ such that u_{j_k} converges pointwise on Ω'_s except on a pluripolar set for every s. Thus, (ii) follows. This finishes the proof. □

The following result provides a good regularization for functions in $W_*^{1,2}$, see [26].

Lemma 2.3. *Let u be a function in $W_*^{1,2}(\Omega)$ and let u_ε be the standard regularization of u as above. Then we have $\|u_\varepsilon\|_{*,\Omega_\varepsilon} \leq \|u\|_{*,\Omega}$, $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq \|u\|_{L^\infty(\Omega)}$, and if $u \geq 0$, we have $u_\varepsilon \geq 0$ for every ε. Moreover, for every sequence $(\varepsilon_k)_k$ decreasing to 0, we have $u_{\varepsilon_k} \to u$ nicely in $W_*^{1,2}(\Omega')$ for every open set $\Omega' \Subset \Omega$.*

Proof. It is clear from the definition of u_ε that $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq \|u\|_{L^\infty(\Omega)}$ and $u_\varepsilon \geq 0$ if $u \geq 0$. It is also clear that $\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \|u\|_{L^2(\Omega)}$ and $u_\varepsilon \to u$ in $L^2(\Omega')$ for every $\Omega' \Subset \Omega$. Let T be a closed positive $(1,1)$-current on Ω with $\|T\|$ minimal such that (1.2) holds. Denote by T_ε the standard regularization of T. By [26, Lemma 5]) (see also the proof of Lemma 3.3 below), we have $i\partial u_\varepsilon \wedge \overline{\partial} u_\varepsilon \leq T_\varepsilon$ on Ω_ε. So we deduce that $u_\varepsilon \in W_*^{1,2}(\Omega_\varepsilon)$ and its $*$-norm bounded by $\|u\|_{*,\Omega}$. We conclude that $u_{\varepsilon_k} \to u$ nicely in $W_*^{1,2}(\Omega')$ as we have seen that if we write $T = dd^c\varphi$, then φ_ε decreases to φ when ε decreases to 0. This finishes the proof. □

Note that Lipschitz functions belong to $W_*^{1,2}$. The following result shows that $W_*^{1,2}$ is closed under basic operations on functions and allows us to produce functions in this space, see [10].

Lemma 2.4. *Let $\tau : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $u \in W_*^{1,2}$. Define* $u^\pm := \max\{\pm u, 0\}$.

(i) *We have $\tau(u) \in W_*^{1,2}$ and $\|\tau(u)\|_* \leq c(|\tau(0)| + \|u\|_*)$ for some constant $c > 0$ independent of u. In particular, we have $u^+, u^-, |u| \in W_*^{1,2}$ and $\max\{u_1, u_2\} \in W_*^{1,2}$ if $u_1, u_2 \in W_*^{1,2}$;*

(ii) *If $u_k \to u$ weakly in $W_*^{1,2}$, then $\tau(u_k) \to \tau(u)$ weakly in $W_*^{1,2}$. If $u_k \to u$ nicely in $W_*^{1,2}$, then $\tau(u_k) \to \tau(u)$ nicely in $W_*^{1,2}$;*

(iii) *Assume that Ω is bounded. Let v be a p.s.h. function on an open neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ such that $0 \leq v \leq 1$. Then v belongs to $W_*^{1,2}(\Omega)$ and $\|v\|_*$ is bounded by a constant depending only on Ω and $\widetilde{\Omega}$.*

Proof. As in [10, Proposition 4.1 and Lemma 4.2], we easily obtain (i) and (ii) using that $|\tau(t)| \leq |\tau(0)| + A|t|$ and $i\partial\tau(u) \wedge \overline{\partial}\tau(u) \leq A^2 i\partial u \wedge \overline{\partial} u$ if τ is A-Lipschitz. We also used here that the maps $t \mapsto t^+, t^-, |t|$ are 1-Lipschitz and $\max\{u_1, u_2\} = (u_1 - u_2)^+ + u_2$. The assertion (iii) is a direct consequence of (i) by using $i\partial v \wedge \overline{\partial} v \leq i\partial\overline{\partial} v^2$ and by observing that v^2 is a p.s.h function. □

We will need the following estimates, see also Proposition 2.13 below. We note that results related to the inequality (2.4) below were proved in [23, 26].

Lemma 2.5. *Let $u \in W^{1,2}_* \cap \mathscr{C}^0(\Omega)$ with $\|u\|_* \leq 1$ and let $v_1, \dots, v_n$ be p.s.h. functions on Ω with values in $[0,1]$. Let K be a compact subset of Ω. Then there is a constant $c > 0$ depending only on K and Ω such that*

$$\int_K u^2 dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq c, \tag{2.4}$$

and if $i\partial u \wedge \overline{\partial} u \leq dd^c \varphi$ for some p.s.h. function φ on Ω such that $0 \leq \varphi \leq 1$, then

$$\int_K u^2 dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq c \left(\int_\Omega u^2 (dd^c \varphi + \omega)^n \right)^{1/2^n},$$

where ω is the standard Kähler form on $\mathbb{C}^n$.

Proof. By regularization, we can assume that u is smooth. The point here is that the constant c is independent of u. Let T be a closed positive $(1,1)$-current so that $\|T\| \leq 1$ and

$$i\partial u \wedge \overline{\partial} u \leq T.$$

In order to get (2.4) it is enough to prove by induction on $0 \leq l \leq n$ that

$$\int_K u^2 dd^c v_1 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l} \leq c$$

for some constant c depending only on K and Ω. The case where $l = 0$ is clear, see the beginning of this section. Assume this property for $l-1$ instead of l and for every K. We now prove it for l.

Since the problem is local, we can assume that Ω is the unit ball in $\mathbb{C}^n$. Fix a compact set K in Ω. As in the proof of Lemma 2.1, we can assume that all v_j are smooth outside a compact set L with $K \Subset L \Subset \Omega$ and $\|v_j\|_{\mathscr{C}^2(\Omega\setminus L)} \leq 1$. Fix a smooth function $0 \leq \tau \leq 1$ with $\tau = 1$ on a neighbourhood of L such that τ is supported by a compact set K' in Ω. Define

$$I_{\max} := \sup_{v'_j} \int_\Omega \tau u^2 dd^c v'_1 \wedge \cdots \wedge dd^c v'_l \wedge \omega^{n-l},$$

where the supremum is taken over all p.s.h. functions v'_j with $0 \leq v'_j \leq 1$ and $\|v'_j\|_{\mathscr{C}^2(\Omega\setminus L)} \leq 1$. Define

$$\begin{aligned} I &:= \int_\Omega \tau u^2 dd^c v_1 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l} \\ \text{and} \quad R &:= dd^c v_2 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l}. \end{aligned} \tag{2.5}$$

Since u is smooth, we can perform integration by parts to obtain

$$I = -\int_\Omega u^2 d\tau \wedge d^c v_1 \wedge R - 2\int_\Omega \tau u du \wedge d^c v_1 \wedge R.$$

Denote by I_1, I_2 the first and second integral in the right-hand side of the last equality.

Observe that $d\tau \wedge d^c v_1 \wedge R$ is a bounded form because $d\tau$ vanishes outside $K' \setminus L$ and $\|v_j\|_{\mathscr{C}^2(K'\setminus L)} \leq 1$. Therefore, $|I_1|$ is bounded by a constant because u has bounded $L^2(\Omega)$-norm. For I_2, since $i\partial u \wedge \overline{\partial} u \leq T$, we obtain

$$du \wedge d^c u \wedge R = \pi^{-1} i\partial u \wedge \overline{\partial} u \wedge R \leq R \wedge T.$$

This, together with the Cauchy-Schwarz inequality and Lemma 2.1, give

$$\begin{aligned}|I_2| &\leq \Big(\int_\Omega \tau du \wedge d^c u \wedge R\Big)^{1/2} \Big(\int_\Omega \tau u^2 dv_1 \wedge d^c v_1 \wedge R\Big)^{1/2} \\ &\leq \Big(\int_\Omega \tau R \wedge T\Big)^{1/2} \Big(\int_\Omega \tau u^2 dd^c v_1^2 \wedge R\Big)^{1/2} \\ &\leq \Big(\int_\Omega \tau u^2 dd^c v_1^2 \wedge R\Big)^{1/2} \lesssim (I_{\max})^{1/2}\end{aligned}$$

(for the last inequality, we used that v_1^2 is p.s.h., $0 \leq v_1^2 \leq 1$ and $\|v_1^2\|_{\mathscr{C}^2(\Omega\setminus L)}$ is bounded). It follows that $|I| = |I_1 + 2I_2| \lesssim 1 + (I_{\max})^{1/2}$ for every v_j as above. Therefore, we deduce from the definition of $I_{\max}$ that $I_{\max} \lesssim 1 + (I_{\max})^{1/2}$. Thus, $I_{\max}$ is bounded by a constant and the inequality (2.4) is proved.

We now prove the second inequality in the lemma. Using (2.4), we can reduce slightly Ω and assume that

$$Q := \int_\Omega u^2 (dd^c\varphi + \omega)^n$$

is bounded by a constant. It follows that $Q \lesssim Q^{1/2^l}$ for $0 \leq l \leq n$. We can now follow the main lines of the proof of (2.4) but we replace each ω by $dd^c\varphi + \omega$. Denote by I', R', I_1', I_2' the quantities defined as I, R, I_1, I_2 but we replace ω by $dd^c\varphi + \omega$. With the same arguments, we get $|I_1'| \lesssim Q \lesssim Q^{1/2^l}$. The estimate of $|I_2'|$ is slightly different. By Cauchy-Schwarz inequality, the inequality $idu \wedge d^c u \leq dd^c\varphi$, Lemma 2.1 and the induction hypothesis, we get

$$\begin{aligned}|I_2'| &\leq \Big(\int_\Omega \tau u^2 du \wedge d^c u \wedge R'\Big)^{1/2} \Big(\int_\Omega \tau dv_1 \wedge d^c v_1 \wedge R'\Big)^{1/2} \\ &\lesssim \Big(\int_\Omega \tau u^2 dd^c\varphi \wedge R'\Big)^{1/2} \lesssim \Big(Q^{1/2^{l-1}}\Big)^{1/2},\end{aligned}$$

where we bound $\int_\Omega \tau dv_1 \wedge d^c v_1 \wedge R'$ by using Lemma 2.1 and the fact that $0 \leq v_j, \phi \leq 1$. We conclude that $|I'| = |I'_1 + 2I'_2| \lesssim Q^{1/2^l}$ which ends the proof of the lemma. □

Lemma 2.6. *Let $(u_k)_k \subset W^{1,2}_* \cap \mathscr{C}^0(\Omega)$ be a sequence converging to 0 weakly in $W^{1,2}_*$. For each $j = 1, \dots, n$, let $(v_{j,k})_k$ be a sequence of bounded p.s.h. functions on Ω decreasing to some bounded p.s.h. function v_j when k tends to infinity. Then we have for every compact set K in Ω*

$$\lim_{k\to\infty} \int_K |u_k| dd^c v_{1,k} \wedge \cdots \wedge dd^c v_{n,k} = 0.$$

In particular, if we assume moreover that $(u_k)_k$ is uniformly bounded, then

$$\lim_{k\to\infty} \int_K u_k^2 dd^c v_{1,k} \wedge \cdots \wedge dd^c v_{n,k} = 0.$$

Proof. By Lemma 2.1, the mass of $dd^c v_{1,k} \wedge \cdots \wedge dd^c v_{n,k}$ on K is bounded by a constant. Therefore, the second assertion of the lemma is a direct consequence of the first one. We prove now the first assertion.

By replacing u_k with $|u_k|$, we can assume that $u_k \geq 0$, see Lemma 2.4. Then, using the standard regularization, we can assume that u_k is smooth. Since the problem is local, as in the proof of Lemma 2.1, we can assume that $\|v_{j,k}\|_{\mathscr{C}^2(\Omega\setminus L)} \leq 1$ for some compact set L such that $K \Subset L \Subset \Omega$. We can also assume that

$$i\partial u_k \wedge \overline{\partial} u_k \leq T_k$$

for some closed positive (1, 1)-current T_k such that $\|T_k\| \leq 1$. Choose a smooth non-negative function τ with compact support in Ω which is equal to 1 in a neighbourhood of L. It is enough to prove by induction on l that

$$\lim_{k\to\infty} \int_\Omega \tau u_k dd^c v_{1,k} \wedge \cdots \wedge dd^c v_{l,k} \wedge \omega^{n-l} = 0.$$

The case where $l = 0$ is clear because $u_k \to 0$ in L^2_{loc} by hypothesis, see the beginning of this section. Assume that the desired property holds for $l - 1$ instead of l. We need to prove it for l.

Define

$$R_k := dd^c v_{2,k} \wedge \cdots \wedge dd^c v_{l,k} \wedge \omega^{n-l} \quad \text{and} \quad R := dd^c v_2 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l}.$$

Let $v_{1,k,\varepsilon}$ and $v_{1,\varepsilon}$ be the standard regularizations of $v_{1,k}$ and v_1 for $\varepsilon > 0$ a small constant (since τ has compact support, we can reduce Ω slightly in order to avoid problems near the boundary of Ω). Define $v'_{1,k,\varepsilon} := v_{1,k} - v_{1,k,\varepsilon}$. Observe that when k tends to infinity, $v_{1,k,\varepsilon}$ decreases to $v_{1,\varepsilon}$ and hence $v'_{1,k,\varepsilon}$ tends to $v'_{1,\varepsilon} := v_1 - v_{1,\varepsilon}$ pointwise. Write

$$\int_\Omega \tau u_k dd^c v_{1,k} \wedge R_k = \int_\Omega \tau u_k dd^c v_{1,k,\varepsilon} \wedge R_k + \int_\Omega \tau u_k dd^c v'_{1,k,\varepsilon} \wedge R_k.$$

As k tends to infinity, the first term in the last sum converges to 0 by the induction hypothesis. Denote the second term by $I_k(\varepsilon)$. It remains to check that

$$\lim_{\varepsilon\to 0} \limsup_{k\to\infty} I_k(\varepsilon) = 0.$$

By integration by parts, we get

$$I_k(\varepsilon) = -\int_\Omega u_k d\tau \wedge d^c v'_{1,k,\varepsilon} \wedge R_k - \int_\Omega \tau du_k \wedge d^c v'_{1,k,\varepsilon} \wedge R_k.$$

The first integral in the last line tends to 0 when k tends to infinity because $u_k \to 0$ in $L^2_{\mathrm{loc}}(\Omega)$, $d\tau$ vanishes outside L and $d\tau \wedge d^c v'_{1,k,\varepsilon} \wedge R_k$ is bounded uniformly on $\Omega \setminus L$. The second integral, denoted by $J_k(\varepsilon)$, satisfies the following estimates, thanks to the Cauchy-Schwarz inequality

$$\begin{aligned}|J_k(\varepsilon)|^2 &\le \Big(\int_\Omega \tau du_k \wedge d^c u_k \wedge R_k\Big)\Big(\int_\Omega \tau dv'_{1,k,\varepsilon} \wedge d^c v'_{1,k,\varepsilon} \wedge R_k\Big)\\ &\lesssim \Big(\int_\Omega \tau T_k \wedge R_k\Big)\Big(\int_\Omega \tau dv'_{1,k,\varepsilon} \wedge d^c v'_{1,k,\varepsilon} \wedge R_k\Big).\end{aligned}$$

The first factor in the last line is uniformly bounded, thanks to Lemma 2.1 and the fact that $\|T_k\| \le 1$. By integration by parts, the second factor is equal to

$$-\int_\Omega v'_{1,k,\varepsilon} d\tau \wedge d^c v'_{1,k,\varepsilon} \wedge R_k - \int_\Omega \tau v'_{1,k,\varepsilon} \wedge dd^c v'_{1,k,\varepsilon} \wedge R_k.$$

Taking $k \to \infty$ and then $\varepsilon \to 0$, we see that the first term tends to 0 because $d\tau$ vanishes outside L and $v'_{1,k,\varepsilon} d\tau \wedge d^c v'_{1,k,\varepsilon} \wedge R_k$ is smooth and tends uniformly to 0 on $\Omega \setminus L$ thanks to properties of the convolution operator. By continuity of wedge-product described at the beginning of this section, when k tends to infinity, the second term tends to

$$\int_\Omega \tau v'_{1,\varepsilon} \wedge dd^c v'_{1,\varepsilon} \wedge R.$$

Finally, when ε decreases to 0, since $v_{1,\varepsilon}$ decreases to v_1, the last expression tends to 0, again, by using the continuity of the wedge-product. This ends the proof of the lemma. □

We have the following result which will be extended later in Proposition 2.13 to every $u \in W^{1,2}_*$ which is not necessary continuous.

Lemma 2.7. *Let τ be a smooth function with compact support in Ω. Then there is a constant $c > 0$ such that for every $u \in W^{1,2}_* \cap \mathscr{C}^0(\Omega)$ with $\|u\|_* \leq 1$ and every p.s.h. functions $v_1, \dots, v_{n-1}, w_1, w_2$ on Ω with values in $[0, 1]$, we have*

$$\left| \int_\Omega \tau u dd^c(w_1 - w_2) \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{n-1} \right| \leq c\|w_1 - w_2\|^{1/2}_{L^\infty(\Omega)}.$$

Proof. Let T be as in the proof of Lemma 2.5. As in that lemma, by regularisation, we can assume u smooth. Define $w := w_1 - w_2$ and $R := dd^c v_1 \wedge \cdots \wedge dd^c v_{n-1}$. Let $U \Subset \Omega$ be an open set containing the support of τ. By integration by parts, we get

$$\int_\Omega \tau u dd^c w \wedge R = -\int_\Omega u d\tau \wedge d^c w \wedge R - \int_\Omega \tau du \wedge d^c w \wedge R.$$

Denote by I_1, I_2 the first and second term in the right-hand side of the last equality. By Cauchy-Schwarz inequality, (2.1) and Lemma 2.1, one has

$$\begin{aligned} |I_2| &\lesssim \|du \wedge d^c u \wedge R\|^{1/2}_U \|dw \wedge d^c w \wedge R\|^{1/2}_U \\ &\lesssim \|R \wedge T\|^{1/2}_U \|w\|^{1/2}_{L^\infty(\Omega)} \lesssim \|w\|^{1/2}_{L^\infty(\Omega)}. \end{aligned}$$

Similarly,

$$|I_1| \leq \left\|u^2 d\tau \wedge d^c\tau \wedge R\right\|^{1/2}_U \|dw \wedge d^c w \wedge R\|^{1/2}_U \lesssim \|w\|^{1/2}_{L^\infty(\Omega)}$$

by Lemmas 2.1 and 2.5. The result follows. □

In general, the potentials of a current T satisfying (1.2) are not locally bounded. We will introduce below an operator which produces, from a bounded function $u \in W^{1,2}_*$, new bounded functions in $W^{1,2}_*$ such that their associated (1, 1)-currents have bounded potentials.

Lemma 2.8. *Let $u \in W^{1,2}_*$ be such that $|u| \leq 1$ and let T be as in (1.2). Assume moreover that $T = dd^c\varphi$ for some negative p.s.h. function φ on Ω. Define $\varphi_N := \max\{\varphi, -N\} + N$ for a constant $N \geq 1$. Then the function $w := \varphi_N u$ belongs to $W^{1,2}_*(\Omega')$ for every open set $\Omega' \Subset \Omega$ and it satisfies*

$$i\partial w \wedge \overline{\partial} w \leq 2\pi N^2 dd^c\left(\varphi_N^2 + \varphi_{N+1}\right).$$

Proof. Observe that $0 \leq \varphi_N \leq N$ and both φ_N and φ_N^2 are p.s.h. So the estimate in the lemma implies that w belongs to $W^{1,2}_*(\Omega')$ and we only need to prove this estimate.

Particular Case. Consider first the case where φ is continuous. Since $\partial w = u\partial\varphi_N + \varphi_N\partial u$, by Cauchy-Schwarz inequality, we have

$$i\partial w \wedge \overline{\partial} w \leq 2u^2 i\partial\varphi_N \wedge \overline{\partial}\varphi_N + 2\varphi_N^2 i\partial u \wedge \overline{\partial} u \leq 2\pi N^2 \left(dd^c\varphi_N^2 + dd^c\varphi\right).$$

This implies the desired estimate because $\varphi = \varphi_{N+1}$ on the open set $\{\varphi > -N-1\}$ which contains the closed set $\{\varphi \geq -N\}$ and w is supported by the last one.

General Case. Denote by u_ε and φ_ε the standard regularizations of u and φ. We reduce slightly the domain Ω in order to avoid problems near the boundary. We have seen in the proof of Lemma 2.3 that $i\partial u_\varepsilon \wedge \overline{\partial} u_\varepsilon \leq dd^c\varphi_\varepsilon$. Define as above the functions $\varphi_{\varepsilon,N}$ and w_ε associated to φ_ε and u_ε. We obtain from the last case that

$$i\partial w_\varepsilon \wedge \overline{\partial} w_\varepsilon \leq 2\pi N^2 dd^c \left(\varphi_{\varepsilon,N}^2 + \varphi_{\varepsilon,N+1}\right).$$

When ε decreases to 0, it is easy to see that w_ε converges almost everywhere to w and the right-hand side of the last inequality converges to $2\pi N^2 dd^c(\varphi_N^2 + \varphi_{N+1})$ because φ_ε decreases to φ. The desired inequality in the lemma follows, see also the beginning of this section for the weak convergence in $W_*^{1,2}$. □

The following lemma gives a link between the nice convergence and the convergence in capacity. A related result was given in [26].

Lemma 2.9. *Let $(u_k)_k \in W_*^{1,2} \cap \mathscr{C}^0(\Omega)$ be a sequence converging nicely to 0. Then u_k converges to 0 in capacity as $k \to \infty$.*

Proof. Since the problem is local, we can assume that there are closed positive $(1,1)$-currents T_k and negative p.s.h. functions φ_k such that

$$i\partial u_k \wedge \overline{\partial} u_k \leq T_k = dd^c\varphi_k \quad \text{and} \quad \|T_k\| \leq 1. \tag{2.6}$$

Since $u_k \to 0$ nicely, we can also assume that φ_k decreases to some negative p.s.h. function φ on Ω. Let K be a compact subset of Ω and $\delta > 0$. We need to show that

$$\lim_{k\to\infty} \operatorname{cap}(\{|u_k| \geq \delta\} \cap K, \Omega) = 0.$$

Consider a large positive constant N. By definition of capacity and Lemma 2.5, we have

$$\operatorname{cap}(\{|u_k| \geq N\} \cap K, \Omega) \lesssim N^{-2} \sup\left\{\int_K u_k^2 (dd^c v)^n,\ 0 \leq v \leq 1 \text{ p.s.h.}\right\} \lesssim N^{-2}.$$

On the other hand, by Lemma 2.4, $\min(|u_k|, N)$ converges to 0 nicely. Therefore, replacing u_k with $\min(|u_k|, N)$ allows us to assume that the sequence $(u_k)_k$ is uniformly bounded.

Define $\varphi_{k,N} := \max(\varphi_k, -N) + N$ and $u'_k := \varphi_{k,N} u_k$. By Lemma 2.8, we have $i\partial u'_k \wedge \overline{\partial} u'_k \leq dd^c\varphi'_k$ with $\varphi'_k := 2\pi N^2(\varphi_{k,N}^2 + \varphi_{k,N+1})$. Observe that φ'_k

decreases to the bounded p.s.h. function $\varphi' := 2\pi N^2(\varphi_N^2 + \varphi_{N+1})$ as φ_k decreases to the p.s.h. function φ. Thus, u'_k converges to 0 nicely. Since $\varphi_k \geq \varphi$, we have $\varphi_{k,N} \geq 1$ on $\{\varphi \geq -N+1\}$. It follows that $|u'_k| \geq |u_k|$ on $\{\varphi \geq -N+1\}$. On the other hand, using Lemma 2.1, we obtain

$$\operatorname{cap}\big(\{\varphi < -N+1\} \cap K, \Omega\big) \lesssim N^{-1}.$$

It follows that

$$\operatorname{cap}\big(\{|u_k| \geq \delta\} \cap K, \Omega\big) \lesssim \operatorname{cap}\big(\{|u'_k| \geq \delta\} \cap K, \Omega\big) + N^{-1}.$$

Therefore, since the last estimate holds for every N, we only need to check that

$$\lim_{k\to\infty} \operatorname{cap}(\{|u'_k| \geq \delta\} \cap K, \Omega) = 0.$$

By definition of capacity and Lemma 2.5, we have

$$\begin{aligned}\operatorname{cap}\big(\{|u'_k| \geq \delta\} \cap K, \Omega\big) &\leq \delta^{-2} \sup \left\{ \int_K u_k'^2 (dd^c v)^n,\ 0 \leq v \leq 1 \text{ p.s.h.} \right\} \\ &\lesssim \left(\int_\Omega u_k'^2 (dd^c \varphi'_k + \omega)^n \right)^{1/2^n}.\end{aligned}$$

By Lemma 2.6, the last integral tends to 0 as k tends to infinity. The lemma follows. □

Here is the main result of this section which generalizes results by Vigny in [26], see also Corollary 2.11 below.

Theorem 2.10. *Let $u \in W_*^{1,2}$. Then there exists a Borel function $\tilde{u}$ defined everywhere on Ω except on a pluripolar set such that $\tilde{u} = u$ almost everywhere and the following properties hold:*

(i) *For every open set $U \subset \Omega$ and every sequence $(u_k)_k \subset W_*^{1,2}(U) \cap \mathscr{C}^0(U)$ such that $u_k \to u$ nicely in $W_*^{1,2}(U)$, we have $u_k \to \tilde{u}$ in capacity as $k \to \infty$. In particular, there exists a subsequence $(u_{j_k})_k$ of $(u_k)_k$ such that u_{j_k} converges pointwise to $\tilde{u}$ everywhere on U except on a pluripolar set;*
(ii) *For every constant $\varepsilon > 0$, there exists an open subset U of Ω with* $\operatorname{cap}(U, \Omega) \leq$ *ε such that $\tilde{u}$ is continuous on $\Omega \backslash U$;*
(iii) *If $\tilde{u}'$ is another Borel function satisfying (i), then $\tilde{u}' = \tilde{u}$ on Ω except on a pluripolar set.*

We see that $\tilde{u}$ is unique modulo pluripolar sets. In analogy with the case of p.s.h. functions, we consider the above property (ii) as a quasi-continuity property of functions in $W_*^{1,2}$.

Proof. By Lemma 2.2, (iii) is a direct consequence of (i). Moreover, (ii) follows from (i) by using exactly the arguments to prove the quasi-continuity of p.s.h. functions, see [19, Th. 1.13]. We now prove (i) and start with the construction of $\tilde{u}$.

Let u_ε be the standard regularization of u. Choose a sequence of positive numbers $(\varepsilon_k)_k$ decreasing to 0. We first prove the following claim.

Claim. $(u_{\varepsilon_k})_k$ is a Cauchy sequence in $W^{1,2}_*(\Omega')$ with respect to capacity for every open set $\Omega' \Subset \Omega$.

Assume by contradiction that the claim is not true. By replacing $(\varepsilon_k)_k$ with a subsequence, we have the following property for some compact set K and some open set U with $K \Subset U \Subset \Omega$:

$$\mathrm{cap}\big(\{|u_{\varepsilon_{2m}} - u_{\varepsilon_{2m+1}}| > \delta\} \cap K, U\big) \geq \kappa$$

for every $m = 1, 2, \ldots$, where δ and κ are some positive numbers.

By Lemma 2.3, we have $u_{\varepsilon_k} \to u$ nicely. So we can write locally $i\partial u_{\varepsilon_k} \wedge \overline{\partial} u_{\varepsilon_k} \leq dd^c\psi_k$ for some p.s.h. function ψ_k which decreases to a p.s.h. function when k tends to infinity. Observe that by Cauchy-Schwarz inequality

$$\begin{aligned}
&i\partial\big(u_{\varepsilon_{2m}} - u_{\varepsilon_{2m+1}}\big) \wedge \overline{\partial}\big(u_{\varepsilon_{2m}} - u_{\varepsilon_{2m+1}}\big) \\
&\leq 2i\partial u_{\varepsilon_{2m}} \wedge \overline{\partial} u_{\varepsilon_{2m}} + 2i\partial u_{\varepsilon_{2m+1}} \wedge \overline{\partial} u_{\varepsilon_{2m+1}} \\
&\leq dd^c\big(2\psi_{2m} + 2\psi_{2m+1}\big).
\end{aligned}$$

Therefore, $u_{\varepsilon_{2m}} - u_{\varepsilon_{2m+1}}$ tends to 0 nicely and hence in capacity, according to Lemma 2.9. This contradicts the above estimate on capacity for $u_{\varepsilon_{2m}} - u_{\varepsilon_{2m+1}}$ and completes the proof of the claim.

We apply Lemma 2.2 to the sequence $(u_{\varepsilon_k})_k$ and obtain a function $\tilde{u}$ equal almost everywhere to u such that $u_{\varepsilon_k} \to u$ in capacity. Now, the function $\tilde{u}$ is constructed. It remains to prove the first part of the assertion (i). Since $u_k \to u$ nicely, we can write locally $i\partial u_k \wedge \overline{\partial} u_k \leq dd^c\varphi_k$ for some p.s.h. function φ_k decreasing to a p.s.h. function when k goes to infinity. Using that both $i\partial u_{\varepsilon_k} \wedge \overline{\partial} u_{\varepsilon_k}$ and $i\partial u_k \wedge \overline{\partial} u_k$ are bounded by $dd^c(\psi_k + \varphi_k)$, we see that the sequence

$$u_{\varepsilon_1}, u_1, u_{\varepsilon_2}, u_2, \ldots$$

converges to u nicely. As above, we can show that this is a Cauchy sequence with respect to capacity. Therefore, Lemma 2.2 implies that $u_k \to \tilde{u}$ in capacity. □

From now on, by a *good representative* of u, we always mean a function $\tilde{u}$ as in Theorem 2.10. It coincides with the representative constructed by Vigny in [26]. Theorem 2.10(i) shows that the good representatives do not depend on the coordinates on Ω. Therefore, this notion is well defined for functions on manifolds. If no confusion arises, when refer to functions in $W^{1,2}_*$, *we often use implicitly their good representatives*. The requirement that u_k is continuous in Theorem 2.10(i) is actually superfluous as shown in the following result.

Corollary 2.11. *Let $u \in W^{1,2}_*$ and $u_k \in W^{1,2}_*$ for $k \in \mathbb{N}$. Assume that $u_k \to u$ nicely as $k \to \infty$. Then $\tilde{u}_k \to \tilde{u}$ in capacity as $k \to \infty$, where $\tilde{u}, \tilde{u}_k$ are good representatives of u and u_k respectively (we often say that $u_k \to u$ in capacity for simplicity).*

Proof. Note that the problem is local and we can always reduce the domain Ω in order to avoid problems near the boundary. By hypothesis, we can write $i\partial u_k \wedge \overline{\partial} u_k \leq dd^c\varphi_k$ for some p.s.h. function φ_k decreasing to a p.s.h. function when k tends to infinity.

We can apply Lemma 2.3 for u_k instead of u. Then we apply Theorem 2.10(i) for the obtained sequence of functions. We deduce the existence of $u'_k \in W^{1,2}_* \cap \mathscr{C}^\infty(\Omega)$ such that

$$\|u'_k - u_k\|_{L^2} \leq 1/k, \quad \operatorname{cap}\Big(\big\{|u'_k - \tilde{u}_k| \geq 1/k\big\} \cap K, \Omega\Big) \leq 1/k, \quad \|u'_k\|_* \leq c$$

for some constant c independent of k and $i\partial u'_k \wedge \overline{\partial} u'_k \leq dd^c\varphi'_k$, where φ'_k is a p.s.h. function. We can obtain φ'_k from φ_k using the standard regularization, see the proof of Lemma 2.3. Since the sequence $(\varphi_k)_k$ decreases to a p.s.h. function, we can choose u'_k (inductively on $k = 1, 2, \ldots$) so that $(\varphi'_k)_k$ also decreases to some p.s.h. function. It follows that $u'_k \to u$ nicely in $W^{1,2}_*$. This allows us to apply Theorem 2.10(i) again to infer that $u'_k \to \tilde{u}$ in capacity. Finally, the above capacity estimate (involving $u'_k - \tilde{u}_k$) implies the result. □

We also need the following observation in order to work directly with good representatives.

Lemma 2.12. *Let $\tau : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Let $u \in W^{1,2}_*$ and $\tilde{u}$ a good representative of u. Then, $\tau(\tilde{u})$ is a good representative of $\tau(u) \in W^{1,2}_*$. In particular, the functions $\tilde{u}^+, \tilde{u}^-, |\tilde{u}|$ are good representatives of $u^+, u^-, |u|$, and if $u_1, u_2 \in W^{1,2}_*$, then $\max\{\tilde{u}_1, \tilde{u}_2\}$ is a good representative of $\max\{u_1, u_2\}$, where $\tilde{u}_j$ is a good representative of u_j for $j = 1, 2$.*

Proof. Let $u_k := u_{\varepsilon_k}$ be as in Lemma 2.3. By Lemma 2.4(ii), we have $\tau(u_k) \to \tau(u)$ nicely. Thus, the result is a direct consequence of the second assertion of Theorem 2.10(i). □

Let $v_1, \ldots, v_n$ be bounded p.s.h. functions on Ω and define $\mu := dd^c v_1 \wedge \cdots \wedge dd^c v_n$. Note that μ is a positive measure having no mass on pluripolar sets because the capacity of every pluripolar set is zero. Theorem 2.10 allows us to integrate any non-negative $u \in W^{1,2}_*$ against μ by putting $\langle \mu, u \rangle := \langle \mu, \tilde{u} \rangle$. The definition is independent of the choice of a good representative $\tilde{u}$ of u. More generally, we can defined in the same way $\langle \mu, \phi(u) \rangle$ for any positive Borel function ϕ defined everywhere on $\mathbb{R}$.

For every set $A \subset \Omega$ and every signed measure ν, denote by $\|\nu\|_A$ the mass of ν on A. The following properties will be useful in practice.

Proposition 2.13. *The estimate* (2.4) *and Lemma* 2.7 *hold for all functions* $u \in W_*^{1,2}$ *with* $\|u\|_* \leq 1$ *which are not necessarily continuous. Moreover, if* $u_k \to u$ *in* $W_*^{1,2}$ *nicely, then*

$$\lim_{k\to\infty} \|(u_k - u)\mu\|_K = 0$$

for every compact set $K \subset \Omega$.

Proof. We first prove (2.4) for every $u \in W_*^{1,2}$ with $\|u\|_* \leq 1$. By Lemmas 2.4 and 2.12, without loss of generality, we can suppose that u is a bounded function. The point here is that the constants involving in our estimates do not depend on u. By Lemma 2.3, we can find a sequence of smooth $u_k \in W_*^{1,2}$ (shrinking Ω if necessary) so that

$$\|u_k\|_{L^\infty} \leq \|u\|_{L^\infty}, \quad \|u_k\|_* \leq \|u\|_*$$

and $u_k \to u$ nicely. By Theorem 2.10 and extracting a subsequence if necessary, we can assume that $u_k \to u$ pointwise except on a pluripolar set. This together with Lebesgue's dominated convergence theorem gives

$$\int_K |u|^2 d\mu = \lim_{k\to\infty} \int_K |u_k|^2 d\mu.$$

The last integral is bounded uniformly by a constant times $\|u_k\|_*^2$ according to Lemma 2.5. Hence, (2.4) holds for every u.

Observe that Lemma 2.7 for general u can be obtained using the above functions u_k and the last assertion in the proposition. Therefore, it remains to prove this assertion. Since $\mu \leq (dd^c(v_1 + \cdots + v_n))^n$, we can replace all v_j by $v_1 + \cdots + v_n$ and assume that μ is a Monge-Ampère measure with bounded potential. Using Lemmas 2.4 and 2.12, we can assume that u_k and u are non-negative. Let N be a big constant. Define

$$u_{k,N} := \min\{u_k, N\} \quad \text{and} \quad u_N := \min\{u, N\}.$$

We have $0 \leq u_{k,N} \leq N$, $u_{k,N} = u_k$ on $\{u_k \geq N\}$ and similar properties for u in place of u_k. Observe that $u_{k,N} \to u_N$ nicely as $k \to \infty$, see Lemma 2.4(ii). Hence, by Corollary 2.11, we have $u_{k,N} \to u_N$ in capacity.

Using the first assertion in the proposition, we have

$$\|(u_k - u_{k,N})\mu\|_K \leq \int_{K\cap\{u_k\geq N\}} u_k d\mu \leq N^{-1} \int_K |u_k|^2 d\mu \lesssim 1/N$$

and a similar estimate for $(u - u_N)\mu$. Together with the equality

$$u_k - u = (u_k - u_{k,N}) + (u_{k,N} - u_N) + (u_N - u),$$

we infer

$$\|(u_k - u)\mu\|_K \leq \|(u_{k,N} - u_N)\mu\|_K + O(1/N).$$

Denote by I_k the left-hand side of the last inequality. Let $\delta > 0$ be a small constant. Using that μ is a Monge-Ampère measure with bounded potential and the inequalities $0 \leq u_{k,N}, u_N \leq N$, we deduce from the last estimate that

$$\begin{aligned} I_k &\leq \left(\int_{K \cap \{|u_{k,N} - u_N| \geq \delta\}} |u_{k,N} - u_N| d\mu \right) \\ &\quad + \left(\int_{K \cap \{|u_{k,N} - u_N| < \delta\}} |u_{k,N} - u_N| d\mu \right) + c/N \\ &\leq c \Big[N \operatorname{cap}\big(K \cap \{|u_{k,N} - u_N| \geq \delta\}, \Omega \big) + \delta + 1/N \Big] \end{aligned}$$

for some constant c independent of k, N, δ. Letting k tend to infinity, since $u_{k,N} \to u_N$ in capacity, we obtain

$$\limsup_{k \to \infty} I_k \leq c[\delta + 1/N]$$

for every $\delta, N > 0$. Thus, $\lim_{k \to \infty} I_k = 0$ and the proof is complete. □

3. Proof of the main results

We will consider Corollary 1.5 at the end of this section. The proof of Theorem 1.2 consists of two main steps. In Step 1, we show how to reduce the question to the case where $v_1, \dots, v_n$ are smooth. In Step 2, we prove the desired result in the latter case. Here is the precise formulation for Step 1.

Proposition 3.1. *If Theorem 1.2 holds for $v_j(x) = \|x\|^2$ for every $1 \leq j \leq n$, then it holds (possibly with different constants α and C) for every Hölder continuous p.s.h. function $v_1, \dots, v_n$ with Hölder exponent $\beta \in (0, 1]$ on Ω such that $\|v_j\|_{\mathscr{C}^\beta} \leq 1$ for $1 \leq j \leq n$.*

For every Hölder continuous function v on Ω, recall that the standard Hölder norm $\|v\|_{\mathscr{C}^\beta}$ is given by

$$\|v\|_{\mathscr{C}^\beta} := \sup_{x, y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\beta}.$$

Proof. Without loss of generality, we can assume $u \geq 0$, see Lemma 2.4. Let $K \Subset \Omega$ be a compact set. By hypothesis, there exist strictly positive constants α and c such that for every $u \in W^{1,2}_*(\Omega)$ with $\|u\|_* \leq 1$, we have

$$\int_K e^{\alpha u^2} d \operatorname{Leb} \leq c. \tag{3.1}$$

Let ω be the standard Kähler form on $\mathbb{C}^n$. Let l be an integer in $[0, n]$. Put $u_N := \min\{u, N\}$ which is in $W^{1,2}_*$ with bounded $*$-norm.

Claim. There exist positive constants α and c such that for every constant $N > 0$, we have

$$\int_K (u - u_N) dd^c v_1 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l} \leq c e^{-\alpha N^2},$$

uniformly in p.s.h. functions $v_1, \ldots, v_l$ on Ω and $u \in W^{1,2}_*(\Omega)$ such that $\|v_j\|_{\mathscr{C}^\beta} \leq 1$ for every $1 \leq j \leq l$ and $\|u\|_* \leq 1$.

Note that since $u - u_N \geq 1$ on $\{u \geq N + 1\}$, the claim with $l = n$ implies the following inequality

$$\int_{\{u \geq N+1\} \cap K} dd^c v_1 \wedge \cdots \wedge dd^c v_n \leq c e^{-\alpha N^2}.$$

From this estimate, we easily deduce the desired assertion (1.3) (we change the constants α and c if necessary).

It remains to prove the claim and this will be done by induction on l. When $l = 0$, the claim is a direct consequence of (3.1) (again, we change the constants c and α if necessary). Assume that the claim holds for $l - 1$ instead of l. We need to prove it for l. Choose a non-negative smooth function τ supported by a compact set $K' \Subset \Omega$ such that $\tau = 1$ on K. Since $u - u_N \geq 0$, we only need to bound the integral

$$I := \int_\Omega \tau (u - u_N) dd^c v_1 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l}.$$

Let $v_{1,\varepsilon}$ be the standard regularization of v_1 for $0 < \varepsilon < 1$, see the beginning of Section 2. As $v_{1,\varepsilon}$ is obtained from v_1 by convolution and $\|v_1\|_{\mathscr{C}^\beta} \leq 1$, we have $\|v_{1,\varepsilon} - v_1\|_{L^\infty} \lesssim \varepsilon^\beta$ and $\|v_{1,\varepsilon}\|_{\mathscr{C}^2} \lesssim \varepsilon^{-2}$. By induction hypothesis applied to K' instead of K, one gets

$$I_1 := \int_\Omega \tau (u - u_N) dd^c v_{1,\varepsilon} \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l} \lesssim \varepsilon^{-2} e^{-\alpha N^2}$$

for some constant $\alpha > 0$. Define

$$I_2 := \int_\Omega \tau (u - u_N) dd^c (v_1 - v_{1,\varepsilon}) \wedge dd^c v_2 \wedge \cdots \wedge dd^c v_l \wedge \omega^{n-l}.$$

By Lemma 2.7 and Proposition 2.13, we have

$$I_2 \lesssim \|v_1 - v_{1,\varepsilon}\|_{L^\infty}^{1/2} \lesssim \varepsilon^{\beta/2}.$$

Since $I = I_1 + I_2$, we deduce that

$$I \lesssim \varepsilon^{-2} e^{-\alpha N^2} + \varepsilon^{\beta/2}.$$

Letting $\varepsilon := e^{-2(4+\beta)^{-1}\alpha N^2}$ gives $I \lesssim e^{-\beta(4+\beta)^{-1}\alpha N^2}$. We obtain the desired claim by changing α to $\beta(4 + \beta)^{-1}\alpha$. This ends the proof of the proposition. $\square$

It remains to prove Theorem 1.2 for $v_j = \|x\|^2$. In this case, $\mu := dd^c v_1 \wedge \cdots \wedge dd^c v_n$ is the standard volume form on Ω. The idea is to use suitable slicing in order to reduce the problem to the case of dimension 1. We first recall some facts about the slicing theory of closed positive currents. We refer to [11, 13] for details. Our setting is simpler because we only work with (1, 1)-currents.

Slicing theory. Let U and V be bounded open subsets of $\mathbb{C}^{m_1}$ and $\mathbb{C}^{m_2}$ respectively. Let $\pi_U : U \times V \to U$ and $\pi_V : U \times V \to V$ be the natural projections. Observe that if R is a form with L^1_{loc} coefficients (which is not necessarily closed or positive), we can always define the restriction R_z of R to the fiber $\pi_V^{-1}(z)$ for almost every $z \in V$ (with respect to the Lebesgue measure on V).

Consider now a closed positive (1, 1)-current R on $U \times V$. Write $R = dd^c w$ locally, where w is a p.s.h. function. For $z \in V$, we define the slice R_z of R on $\pi_V^{-1}(z)$ to be $dd^c\big(w(\cdot, z)\big)$ which is a closed positive (1, 1)-current on $\pi_V^{-1}(z)$. Let A be the set of z so that $w(\cdot, z) \equiv -\infty$. Observe that A is pluripolar and for $z \notin A$, the slice R_z is well-defined. One can see that the definition of the slice R_z is independent of the choice of a local potential w of R.

Let χ_{m_2} be a non-negative smooth radial function with compact support on $\mathbb{C}^{m_2}$ such that $\int_{\mathbb{C}^{m_2}} \chi d\,\text{Leb} = 1$ and for every constant $\varepsilon > 0$, we put $\chi_{m_2,\varepsilon}(z) := \varepsilon^{-2m_2}\chi_{m_2}(\varepsilon^{-1}z)$. The following result is straightforward. We just notice that (iii) is a direct consequence of (ii).

Lemma 3.2.

(i) *Let Φ be a smooth form of suitable bi-degree with compact support in $U \times V$. Let $\Theta(z)$ be a smooth volume form on V. Then we have*

$$\langle R, \Phi \wedge \Theta(z)\rangle = \int_{z \in Z} \langle R_z, \Phi\rangle \Theta(z);$$

(ii) *Then for $z_0 \notin A$, we have*

$$\lim_{\varepsilon \to 0} R \wedge \pi_V^*\big(\chi_\varepsilon(z - z_0)\,\text{Leb}(z)\big) = R_{z_0},$$

where we identified R_{z_0} with a current on $U \times V$ (when R is a (1, 1)-form with L^1_{loc} coefficients which is not necessarily closed or positive, then the same conclusion holds for almost every $z \in V$);

(iii) *Let R' be another closed positive (1, 1)-current on $U \times V$ or a real (1, 1)-form with L^1_{loc} coefficients. Assume that $R' \le R$ on $U \times V$. Then for almost every $z \in Z$, we have $R'_z \le R_z$.*

We continue the proof of Theorem 1.2 for $v_j = \|x\|^2$. We will need the following lemma.

Lemma 3.3. *Let $\eta(x, z)$ be a (1, 0)-form with L^2 coefficients on $U \times V$. Let T be a closed positive (1, 1)-current of mass at most 1 on $U \times V$ such that $i\,\eta \wedge \overline{\eta} \le T$ on*

$U \times V$. Define $\tilde{\eta}(x) := \int_{z\in V} \eta(x,z) d$ Leb(z). Then there exists a positive constant C independent of η and T such that

$$i\,\tilde{\eta} \wedge \overline{\tilde{\eta}} \le C\,(\pi_U)_*\big(T \wedge \mathrm{Leb}(z)\big),$$

where Leb(z) *denotes both the Lebesgue measure and the standard volume form on V. Moreover, $(\pi_U)_*\big(T \wedge \mathrm{Leb}(z)\big)$ is a closed positive $(1,1)$-current on U whose mass is bounded by a constant independent of η and T.*

Proof. Let Φ be a weakly positive smooth form of right bi-degree with compact support on U. Using the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\langle i\,\tilde{\eta} \wedge \overline{\tilde{\eta}}, \Phi\rangle &= \int_{(z,z')\in V^2} \langle i\,\eta_z \wedge \overline{\eta}_{z'}, \Phi\rangle d\,\mathrm{Leb}(z,z') \\
&\le \frac{1}{2}\int_{(z,z')\in V^2} \big(\langle i\,\eta_z \wedge \overline{\eta}_z, \Phi\rangle + \langle i\,\eta_{z'} \wedge \overline{\eta}_{z'}, \Phi\rangle\big) d\,\mathrm{Leb}(z,z') \\
&\lesssim \int_V \langle i\,\eta_z \wedge \overline{\eta}_z, \Phi\rangle d\,\mathrm{Leb}(z) \\
&= \int_{U\times V} i\,\eta \wedge \overline{\eta} \wedge \mathrm{Leb}(z) \wedge \pi_V^*(\Phi) \le \big\langle (\pi_U)_*\big(T \wedge \mathrm{Leb}(z)\big), \Phi\big\rangle.
\end{aligned} \tag{3.2}$$

This implies the first assertion in the lemma.

If $\omega(x)$ denotes the standard Kähler form on U, then the mass of $(\pi_U)_*\big(T \wedge \mathrm{Leb}(z)\big)$ is equal to the mass of the measure

$$T \wedge \mathrm{Leb}(z) \wedge \omega(x)^{m_1-1}.$$

Clearly, this mass is bounded by a constant because the mass of T is at most equal to 1 by assumption. It remains to show that $(\pi_U)_*\big(T \wedge \mathrm{Leb}(z)\big)$ is closed. Let $0 \le \chi_k(z) \le 1$ be a sequence of smooth functions with compact support in V which increases to 1. We have

$$(\pi_U)_*\big(T \wedge \mathrm{Leb}(z)\big) = \lim_{k\to\infty} (\pi_U)_*\big(T \wedge \chi_k(z)\,\mathrm{Leb}(z)\big).$$

Observe that $\chi_k(z)\,\mathrm{Leb}(z)$ is closed because it is of maximal degree in z. Therefore, the current $T \wedge \chi_k(z)\,\mathrm{Leb}(z)$ is also closed. Since π_U is proper on the support of $T \wedge \chi_k(z)\,\mathrm{Leb}(z)$, we deduce that $(\pi_U)_*\big(T \wedge \chi_k(z)\,\mathrm{Leb}(z)\big)$ is closed and the last identity implies the result. □

Lemma 3.4. *Let U, V be open subsets in $\mathbb{C}^{m_1}, \mathbb{C}^{m_2}$ respectively. Let u be a locally integrable function in $U \times V$ such that $\partial u \in L^2_{\mathrm{loc}}(U \times V)$. Let T be a closed positive $(1,1)$-current on $U \times V$ such that $i\,\partial u \wedge \overline{\partial u} \le T$. Then, for almost every $z \in V$, we have that $\partial(u|_{U\times\{z\}}) \in L^2_{\mathrm{loc}}(U)$ and*

$$i\,\partial(u|_{U\times\{z\}}) \wedge \overline{\partial}(u|_{U\times\{z\}}) \le T|_{U\times\{z\}}. \tag{3.3}$$

Proof. By Fubini's theorem, for almost everywhere z, the forms $\partial(u|_{U\times\{z\}})$ and $\partial(u|_{U\times\{z\}}) \wedge \overline{\partial}(u|_{U\times\{z\}})$ are equal to the slice of ∂u, $\partial u \wedge \overline{\partial} u$ along $U \times \{z\}$, respectively. This combined with Lemma 3.2 gives the desired assertion. □

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. We will need the following basic observation which can be deduced using the Riesz representation of subharmonic functions (see [15, Theorem 3.3.6]).

Lemma 3.5. *Let φ be a negative subharmonic function on $\mathbb{D}$ and $\varphi(0) \geq -1$. Let K be a compact subset of $\mathbb{D}$. Then there exists a constant C independent of φ such that $\|i\partial\overline{\partial}\varphi\|_K \leq C$.*

End of the proof of Theorem 1.2. We prove (1.3) by induction. For $n = 1$, this is Theorem 1.1. We assume that (1.3) holds for every dimension at most $n-1$. We need to prove that (1.3) holds for dimension n. Let $u \in W^{1,2}_*$ with $\|u\|_* = 1$. Let T be a closed positive current of bi-degree $(1,1)$ such that

$$i\partial u \wedge \overline{\partial} u \leq T \tag{3.4}$$

and $\|T\|_\Omega \leq 1$. Let $K \Subset \Omega$. Since our problem is local, by solving the dd^c-equation, without loss of generality, we can assume that $T = dd^c\varphi$ for some p.s.h. function φ on Ω and $\|\varphi\|_{L^1(\Omega)} \leq C$, where C is a constant independent of T (see [29, Lemma 2.1] for example). Thus, for every constant $M > 0$ and $F_M := \{|\varphi| \leq M\}$, we get

$$\mathrm{Leb}(\Omega\backslash F_M) = \int_{\{|\varphi|>M\}} \omega^n \leq M^{-1}\int_{\{|\varphi|>M\}} |\varphi|\omega^n \lesssim M^{-1}. \tag{3.5}$$

By decomposing K into the union of a finite number of small compact sets, it suffices to consider the case where the diameter of K is as small as we want. Hence, by using a change of coordinates, we can assume that the closure of the unit ball $\mathbb{B}$ is contained in Ω and K is contained in $\{3/4 \leq \|x\| \leq 4/5\}$. By (3.5), we see that if M is big enough, there is a point $a \in \mathbb{B}$ with $\|a\| \leq 1/100$ so that $\varphi(a) > -M$. Using a linear change of coordinates again, we can assume furthermore that $a = 0$. Thus, $\varphi(0) > -M$ for some fixed constant M big enough and independent of u, T.

The set of complex lines passing through the origin is parameterized by the complex projective space $\mathbb{P}^{n-1}$. For $y \in \mathbb{P}^{n-1}$, denote by L_y the complex line given by y and $\mathbb{D}_y := \mathbb{B} \cap L_y$ which is the unit disc of L_y. Put $A_y := \{x \in \mathbb{B} : \|x\| > 1/10\} \cap \mathbb{D}_y$. We can identify A_y with an annulus $A := \{z \in \mathbb{D} : |z| > 1/10\}$ in $\mathbb{D}$ (here we consider a linear isometry from $\mathbb{D}_y$ to the unit disk $\mathbb{D}$). Let $u_y := u|_{L_y}$ for every $y \in \mathbb{P}^{n-1}$ and

$$\underline{u}(y) := \int_{z\in A_y} u_y(z)d\,\mathrm{Leb}(z), \quad \tilde{u}_y := u_y - \underline{u}(y).$$

The last functions are well-defined for almost every y. Denote by $T|_{\mathbb{D}_y}$ the slice of T along $\mathbb{D}_y$. Recall that $T|_{\mathbb{D}_y} = dd^c(\varphi|_{\mathbb{D}_y})$. Observe that since $\varphi|_{\mathbb{D}_y}(0) =$

$\varphi(0) > -M$, the mass of $T|_{\mathbb{D}_y}$ on $\mathbb{D}_y$ is bounded by a constant independent of y, T (note here that $\overline{\mathbb{B}} \subset \Omega$). On the other hand, by Lemma 3.4, for almost every y, we have that $u_y \in W^{1,2}_*(\mathbb{D}_y)$ and using the definition of $\tilde{u}_y$,

$$\|\tilde{u}_y\|_*^2 \lesssim \|\tilde{u}_y\|^2_{L^2} + \|T|_{\mathbb{D}_y}\| \lesssim \|T|_{\mathbb{D}_y}\|$$

by Poincaré's inequality ([12, page 275]). It follows that

$$\|\tilde{u}_y\|_* \lesssim \|T|_{\mathbb{D}_y}\| \lesssim 1. \tag{3.6}$$

Let $\alpha > 0$ be a fixed small constant. Let $A' := \{z \in \mathbb{D} : 2/3 \leq |z| \leq 5/6\}$ which is compact in A. Let A'_y be the image of A' under the natural identification $\mathbb{D}_y \approx \mathbb{D}$. By construction, we have $K \subset \cup_{y \in \mathbb{P}^{n-1}} A'_y$. Note that $|u_y|^2 \leq 2(|\tilde{u}_y|^2 + |\underline{u}(y)|^2)$. By this and Fubini's theorem, we get

$$\begin{aligned}
\int_K e^{\alpha|u|^2} d\,\text{Leb} &\lesssim \int_{y \in \mathbb{P}^{n-1}} \left(\int_{A'_y} e^{\alpha|u_y|^2} d\,\text{Leb} \right) d\,\text{Leb}(y) \\
&\lesssim \int_{y \in \mathbb{P}^{n-1}} \left(\int_{A'_y} e^{2\alpha(|\tilde{u}_y|^2 + |\underline{u}(y)|^2)} d\,\text{Leb} \right) d\,\text{Leb}(y) \\
&\lesssim \int_{y \in \mathbb{P}^{n-1}} \left(\int_{A'_y} e^{4\alpha|\tilde{u}_y|^2} d\,\text{Leb} \right) d\,\text{Leb}(y) \\
&\quad + \int_{y \in \mathbb{P}^{n-1}} e^{4\alpha|\underline{u}(y)|^2} d\,\text{Leb}(y)
\end{aligned}$$

by Hölder's inequality. Denote by I_1, I_2 the first and second terms respectively in the right-hand side of the last inequality. By (3.6) and induction hypothesis, the number I_1 is uniformly bounded in u for some constant $\alpha > 0$ independent of u (note that $A'_y \approx A' \Subset A \approx A_y$). It remains to check that I_2 is also uniformly bounded.

Cover $\mathbb{P}^{n-1}$ by a finite number of small local charts U such that $U \times A$ is identified with a chart in $\mathbb{B}$ and $\{y\} \times A$ is identified with A_y for every $y \in U$. Thus, we get

$$\|\underline{u}\|_{L^2(U)} \lesssim \|u\|_{L^2(\mathbb{B})} \lesssim 1.$$

By (3.4) we can apply Lemma 3.3 to $\eta(y,z) := u(y,z)$ on $U \times A$. Hence one obtains

$$i\partial\underline{u} \wedge \overline{\partial}\underline{u} \lesssim \pi_*\big(T \wedge (i dz \wedge d\bar{z})\big),$$

where π denotes the natural projection from $U \times A$ to the first component. Moreover, the right-hand side of the last inequality is a closed positive $(1,1)$-current on U with bounded mass. We deduce that $\underline{u} \in W^{1,2}_*(U)$ with $*$-norm uniformly bounded. By induction hypothesis applied to $\underline{u}$, the integral I_2 is bounded uniformly for some constant $\alpha > 0$ (we can slightly reduce U in order to apply the induction hypothesis). This finishes the proof. □

We have the following result.

Proposition 3.6. *Let K be a compact subset of an open set Ω in $\mathbb{C}^n$. Let $(u_k)_k \subset W_*^{1,2}(\Omega)$ be a sequence converging weakly to a function $u \in W_*^{1,2}(\Omega)$. Assume that $\|u_k\|_* \leq 1$. Then there is a positive constant α depending only on K and Ω such that*

$$\lim_{k\to\infty} \left\| e^{\alpha(u_k-u)^2} - 1 \right\|_{L^1(K)} = 0.$$

Proof. Observe that $\|u\|_* \leq 1$ and hence $\|u_k - u\|_* \leq 2$. Define $v_k := u_k - u$, and for $N \in \mathbb{R}_{>0}$, define $K_{k,N} := \{|v_k| \geq N\} \cap K$. Note that $\|v_k\|_*$ is bounded uniformly in k. By this and Theorem 1.2, there are positive constants α and c such that (we change the constant α in order to get the factor 2)

$$\int_K e^{2\alpha|v_k|^2} d\,\mathrm{Leb} \leq c.$$

Using that $|v_k| \geq N$ on $K_{k,N}$, we deduce that

$$\mathrm{Leb}(K_{k,N}) \leq e^{-2\alpha N^2} \int_{K_{k,N}} e^{2\alpha|v_k|^2} d\,\mathrm{Leb} \leq c e^{-2\alpha N^2}$$

and by Cauchy-Schwarz inequality

$$\int_{K_{k,N}} e^{\alpha|v_k|^2} d\,\mathrm{Leb} \leq \mathrm{Leb}(K_{k,N})^{1/2} \left(\int_{K_{k,N}} e^{2\alpha|v_k|^2} d\,\mathrm{Leb} \right)^{1/2} \leq c e^{-\alpha N^2}. \quad (3.7)$$

On another hand, on $K \setminus K_{k,N}$ with N fixed, we have $|v_k| \leq N$ and hence $e^{\alpha v_k^2} - 1 \lesssim v_k^2$. As mentioned at the beginning of Section 2, Rellich's theorem implies that $\|v_k\|_{L^2(K)} \to 0$. We deduce that

$$\lim_{k\to\infty} \left\| e^{\alpha v_k^2} - 1 \right\|_{L^1(K\setminus K_{k,N})} = 0.$$

This, together with (3.7), imply that

$$\limsup_{k\to\infty} \left\| e^{\alpha v_k^2} - 1 \right\|_{L^1(K)} \leq c e^{-\alpha N^2}.$$

Since this estimate holds for every N, the proposition follows. □

Proof of Corollary 1.5. By Hölder's inequality, we can assume that $p \geq 2$. Then the corollary is a direct consequence of Proposition 3.6 because $|t|^p \lesssim e^{\alpha t^2} - 1$. □

Example 3.7. We consider the 1-dimensional case. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. Let $u := (1 - \log|z|)^{1/3}$ for $z \in \mathbb{D}$. One can check directly that $u \in W_*^{1,2}(\mathbb{D})$. Consider $f(x) := x^{-1}(-\log|x|)^{-3}$ for $x \in (0, 1/2]$, and the positive measure $\mu := \mathbf{1}_{[0,1/2]} f(x)dx$. Observe that $\log|z|$ is in $L^{3/2}(\mu)$ but not in $L^3(\mu)$. We have $\mu = dd^c v$ for $v(w) := \int_{\mathbb{D}} \log|z - w| d\mu(w)$. Using the concrete form of f, one can prove that v is continuous on $\mathbb{D}$, and $e^{\alpha u^2}$ is not locally integrable with respect to μ for any $\alpha > 0$.

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