EQUIVARIANT KODAIRA EMBEDDING FOR CR MANIFOLDS WITH CIRCLE ACTION

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ABSTRACT. We consider a compact CR manifold with a transversal CR locally free circle action endowed with an S^1-equivariant positive CR line bundle. We prove that a certain weighted Fourier-Szegő kernel of the CR sections in the high tensor powers admits a full asymptotic expansion. As a consequence, we establish an equivariant Kodaira embedding theorem.

CONTENTS

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The goal of this paper is to study the Szegő kernel and the equivariant embedding of CR manifolds with circle action. The embedding of CR manifolds in general is a subject with long tradition. One paradigm is the embedding theorem of compact strictly pseudoconvex CR manifolds. A famous theorem of Louis Boutet de Monvel [7] asserts that such manifolds can be embedded by CR maps into the complex Euclidean space, provided the dimension of the manifold is greater than or equal to five.

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In dimension three there are non-embeddable compact strictly pseudoconvex CR manifolds (see e. g. Burns [10], where the boundary of the non-fillable example of strictly pseudoconcave manifold by Grauert [16], Andreotti-Siu [2] and Rossi [35] is shown to be non-embeddable). However, if the manifold admits a circle action, then it is embeddable, by a theorem of Lempert [27]. In the study of CR functions, which would eventually provide an embedding, it is natural to look to the orthogonal projector on the space of square integrable CR functions, called Szegő projector. The Schwartz kernel of this projector is called Szegő kernel. In this spirit, a proof based on the Szegő kernel of Lempert's embedding theorem was given in [25, Theorem 1.13]. Using the Szegő kernel of the Fourier components it was recently shown in [17, Theorem 1.2] that there exists an *equivariant* embedding of strictly pseudoconvex CR manifolds with circle action.

Leaving the territory of strictly pseudoconvex CR manifolds, the natural idea arrises to embed CR manifolds into the projective space by means of CR sections of a CR line bundle of positive curvature [15, 17, 20, 21, 22, 23, 25, 26, 31, 33]. This is the analogue of the *Kodaira embedding theorem* from complex geometry. In the case of CR manifolds we have to use an analytic method, while Kodaira's original proof relied on cohomology vanishing theorems. Analytic proofs of the Kodaira embedding theorem for Kähler and symplectic manifolds, based on the Bergman kernel asymptotics, were given in [9, 28]. In this paper we will use Szegő kernel analogues on CR manifolds of the Bergman kernel asymptotics on Kähler or symplectic manifolds [11, 24, 28, 29, 36, 38, 40]. A motivating example is the quadric

$$\left\{[z]\in\mathbb{CP}^{N-1};\,|z_1|^2+\ldots+|z_q|^2-|z_{q+1}|^2-\ldots-|z_N|^2=0\right\}$$

which is a CR manifold possessing a positive line bundle and a circle action. In [9, 36, 40] the Szegő kernel on a strictly pseudoconvex CR manifold with trivial line bundle [8] (see also [25]) was used to study the Bergman kernel on a Kähler manifold, while here we study the Szegő kernel for tensor powers of a CR line bundle.

We are thus led to the problem of *equivariant Kodaira embedding* of CR manifolds with circle action, which will be the subject of this paper. We will prove that a certain weighted Fourier-Szegő kernel admits a full asymptotic expansion and by using these asymptotics, we will show that if X admits a transversal CR locally free S^1-action and there is a S^1-equivariant positive CR line bundle L over X, then X can be CR embedded into projective space *without any assumption* of the Levi form. In particular, when X is Levi-flat, we improve to C^∞ the regularity in the Kodaira embedding theorem of Ohsawa and Sibony (see Corollary 1.4).

Let us now formulate our main results. We refer to Section 2 for some standard notations and terminology used here. Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n-1$, $n\geqslant 2$, endowed with a locally free S^1-action $S^1\times X\to X$, $(e^{i\theta},x)\mapsto e^{i\theta}x$ and we let T be the infinitesimal generator of the S^1-action.

We assume that this S^1-action is transversal CR, that is, T preserves the CR structure $T^{1,0}X$, and T and $T^{1,0}X\oplus\overline{T^{1,0}X}$ generate the complex tangent bundle to X. In our paper we will make systematic use of appropriate coordinates introduced by Baouendi-Rothschild-Treves [4, Theorem II.1, Proposition I.2]. Namely, if X admits a transversal CR locally free S^1-action, then for each point $p\in X$ there exist a coordinate neighborhood U with coordinates $(x_1,\ldots,x_{2n-1})$, centered at $p=0$, there exist $\eta>0$, $\varepsilon_0>0$ and $\phi\in C^\infty(D,\mathbb{R})$ independent of θ, where $D:=\{(z,\theta)\in U:|z|<\eta,|\theta|<\varepsilon_0\}\subset U$, such that, by setting

$$z_j=x_{2j-1}+ix_{2j},\quad j=1,\ldots,n-1,\qquad x_{2n-1}=\theta,$$

we have

$$T = \frac{\partial}{\partial \theta} \text{ on } D, \tag{1.1}$$

the vector fields

$$Z_j = \frac{\partial}{\partial z_j} + i\frac{\partial \phi}{\partial z_j}(z, \overline{z})\frac{\partial}{\partial \theta}, \;\; j = 1, \ldots, n-1, \tag{1.2}$$

form a frame of $T^{1,0}X$ over D. From this it follows that the vector fields $\overline{Z}_1, \ldots, \overline{Z}_{n-1}$ form a frame of $T^{0,1}X$ over D and they annihilate the functions

$$z_1, \ldots, z_{n-1}, \zeta := \theta + i\phi,$$

which are thus CR functions. The map

$$D \longrightarrow \mathbb{C}^n, \;\; p \longmapsto (z_1(p), \ldots, z_{n-1}(p), \zeta(p))$$

is therefore a CR embedding, so X is locally embeddable. Actually, by [4, Theorem II.1], any abstract CR structure invariant under a transversal Lie group action is locally embeddable. We call $(x_1, \ldots, x_{2n-1})$ canonical coordinates, D canonical coordinate patch and $(D, (z, \theta), \phi)$ a BRT trivialization.

In this paper, we work with locally trivializable CR vector bundles, see Definition 2.5. Such bundles admit local CR frames and transitions matrices with CR entries. We consider in the following S^1-equivariant CR bundles, that is, CR bundles endowed with a CR S^1-action lifting the S^1-action on X (see Definition 2.6). Such bundles admit local S^1-equivariant CR frames, called rigid CR frames, so there is a family of trivializations which cover X such that the entries of the transition matrices are CR functions annihilated by T, see Proposition 2.7. The operator T extends to an operator on $C^\infty(X, L)$, see (2.21). We consider further an S^1-equivariant Hermitian metric on L. Then for every rigid frame $\{f_1, \ldots, f_r\}$, the inner product of f_j and f_ℓ are annihilated by T for any j, ℓ.

Let L be an S^1-equivariant CR line bundle over X and let L^k be the k-th power of L, which is also an S^1-equivariant CR line bundle. Let

$$\overline{\partial}_b : \Omega^{0,q}(X, L^k) \to \Omega^{0,q+1}(X, L^k),$$

be the tangential Cauchy-Riemann operator with values in L^k. The action of S^1 commutes with $\overline{\partial}_b$. We will therefore obtain information about $\overline{\partial}_b$ by decomposing the spaces of sections under the group action. For every $m \in \mathbb{Z}$, we consider the Fourier component of the space of smooth sections $C^\infty(X, L^k)$ consisting of equivariant CR sections of L^k of frequency m, that is,

$$C^\infty_m(X, L^k) := \left\{u \in C^\infty(X, L^k);\, Tu = imu\right\}, \tag{1.3}$$

and the corresponding Fourier component of the space of CR sections,

$$\mathcal{H}^0_{b,m}(X, L^k) := \left\{u \in C^\infty_m(X, L^k);\, \overline{\partial}_b u = 0\right\}. \tag{1.4}$$

Since X is a compact manifold we have for every $m \in \mathbb{Z}$ (see [22, Theorem 1.23] and also Theorem 3.7) that

$$\dim \mathcal{H}^0_{b,m}(X, L^k) < \infty. \tag{1.5}$$

For $\lambda > 0$, put

$$\mathcal{H}^0_{b,\leq\lambda}(X, L^k) := \bigoplus_{|m|\leq\lambda} \mathcal{H}^0_{b,m}(X, L^k). \tag{1.6}$$

We assume further that L is endowed with an S^1-equivariant Hermitian metric h. The curvature of (L,h) at a point $x\in X$ is denoted by R^L_x (cf. Definition 2.11), and (L,h) is called positive if R^L_x is positive definite at any point $x\in X$. The Hermitian metric on L^k induced by h is denoted by h^k. Working with a positive line bundle L we will embed the manifold X by using weighted projections on the Fourier components (cf. (1.9)) of sections of $\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$ for $\delta>0$ sufficiently small.

The bundle $\mathbb{C}TX$ is S^1-equivariant and we can take an S^1-equivariant Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ such that

$$T^{1,0}X\perp T^{0,1}X,\;\; T\perp(T^{1,0}X\oplus T^{0,1}X),\;\; \langle\,T\,|\,T\,\rangle=1$$

and $\langle\,u\,|v\,\rangle$ is real if u,v are real tangent vectors (see Theorem 2.14). We denote by dv_X the volume form induced by $\langle\,\cdot\,|\,\cdot\,\rangle$.

Let $\omega_0\in C^\infty(X,T^*X)$ be the real 1-form of unit length annihilating $T^{1,0}X\oplus T^{0,1}X$ and satisfying $\omega_0(T)=-1$. The Levi form $\mathcal{L}_x$ at a point $x\in X$ is the Hermitian quadratic form on $T^{1,0}_xX$ given by $\mathcal{L}_x(U,\overline{V})=-\frac{1}{2i}\langle\,d\omega_0(x)\,,\,U\wedge\overline{V}\,\rangle$, $U,V\in T^{1,0}_xX$.

Let $(\,\cdot\,|\,\cdot\,)_k=(\,\cdot\,|\,\cdot\,)$ be the L^2 inner product on $C^\infty(X,L^k)$ induced by h^k and dv_X. Let $L^2(X,L^k)$ be the completion of $C^\infty(X,L^k)$ with respect to $(\,\cdot\,|\,\cdot\,)$. We extend $(\,\cdot\,|\,\cdot\,)$ to $L^2(X,L^k)$.

For every $m\in\mathbb{Z}$, let $L^2_m(X,L^k)\subset L^2(X,L^k)$ be the completion of $C^\infty_m(X,L^k)$ with respect to $(\,\cdot\,|\,\cdot\,)$. Let

$$Q^{(0)}_{m,k}:L^2(X,L^k)\to L^2_m(X,L^k) \tag{1.7}$$

be the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)$. We have the Fourier decomposition

$$L^2(X,L^k)=\bigoplus_{m\in\mathbb{Z}}L^2_m(X,L^k).$$

We first construct a bounded operator on $L^2(X,L^k)$ by putting a weight on the components of the Fourier decomposition with the help of a cut-off function. Fix $\delta>0$ and a function

$$\tau_\delta\in C^\infty_0((-\delta,\delta)),\;\; 0\leq\tau_\delta\leq 1,\;\; \tau_\delta=1 \text{ on } \left[-\frac{\delta}{2},\frac{\delta}{2}\right]. \tag{1.8}$$

We define the weighted projector on the Fourier components by

$$\begin{aligned}&F_{k,\delta}:L^2(X,L^k)\to L^2(X,L^k),\\ &\qquad u\mapsto\sum_{m\in\mathbb{Z}}\tau_\delta\left(\frac{m}{k}\right)Q^{(0)}_{m,k}(u).\end{aligned} \tag{1.9}$$

For every $\lambda>0$, we consider the partial Szegő projector

$$\Pi_{k,\leq\lambda}:L^2(X,L^k)\to\mathcal{H}^0_{b,\leq\lambda}(X,L^k) \tag{1.10}$$

which is the orthogonal projection on the space of equivariant CR functions of degree less than λ. Finally, we consider the weighted Fourier-Szegő operator

$$P_{k,\delta}:=F_{k,\delta}\circ\Pi_{k,\leq k\delta}\circ F_{k,\delta}:L^2(X,L^k)\to\mathcal{H}^0_{b,\leq k\delta}(X,L^k). \tag{1.11}$$

The Schwartz kernel of $P_{k,\delta}$ with respect to dv_X is the smooth function $P_{k,\delta}(x,y)\in L^k_x\otimes(L^k_y)^*$ satisfying (cf. Section 2.2, [28, B.2])

$$(P_{k,\delta}u)(x)=\int_X P_{k,\delta}(x,y)u(y)\,dv_X(y)\,,\;\; u\in L^2(X,L^k). \tag{1.12}$$

Let $f_j = f_j^{(k)}$, $j = 1, \ldots, d_k$, be an orthonormal basis of $\mathcal{H}^0_{b,\leq k\delta}(X, L^k)$. Then

$$\begin{aligned} P_{k,\delta}(x,y) &= \sum_{j=1}^{d_k} (F_{k,\delta} f_j)(x) \otimes \big((F_{k,\delta} f_j)(y)\big)^*, \\ P_{k,\delta}(x,x) &= \sum_{j=1}^{d_k} \big|(F_{k,\delta} f_j)(x)\big|^2_{h^k}, \end{aligned} \tag{1.13}$$

(see Lemma 4.1) and these representations are independent of the chosen orthonormal basis. It should be noticed that the partial Szegő kernel $\sum_{j=1}^{d_k} |f_j(x)|^2_{h^k}$ does not admit an asymptotic expansion in general, hence the necessity of using the weighted projector $F_{k,\delta}$. This is discussed in Section 3.5.

In order to describe the Fourier-Szegő kernel $P_{k,\delta}(x,y)$ we will localize $P_{k,\delta}$ with respect to a local rigid CR frame s of L on an open set $D \subset X$. We define the weight of the metric h on L with respect to s to be the function $\Phi \in C^\infty(D)$ satisfying $|s|^2_h = e^{-2\Phi}$. We have an isometry

$$U_{k,s} : L^2(D) \to L^2(D, L^k), \;\; u \longmapsto u e^{k\Phi} s^k, \tag{1.14}$$

with inverse $U_{k,s}^{-1} : L^2(D, L^k) \to L^2(D)$, $\alpha \mapsto e^{-k\Phi} s^{-k} \alpha$. The localization of $P_{k,\delta}$ with respect to the trivializing rigid CR section s is given by

$$P_{k,\delta,s} : L^2_{\rm comp}(D) \to L^2(D), \;\; P_{k,\delta,s} = U_{k,s}^{-1} P_{k,\delta} U_{k,s}, \tag{1.15}$$

where $L^2_{\rm comp}(D)$ is the subspace of elements of $L^2(D)$ with compact support in D. Let $P_{k,\delta,s}(x,y) \in C^\infty(D \times D)$ be the Schwartz kernel of $P_{k,\delta,s}$ with respect to dv_X, defined as in (1.12). The first main result of this work describes the structure of the localized Fourier-Szegő kernel $P_{k,\delta,s}(x,y)$.

Theorem 1.1. *Let X be a compact CR manifold with a CR transversal locally free S^1-action and let (L,h) be an S^1-equivariant positive CR line bundle on X. Consider a point $p \in X$ and a canonical coordinate neighborhood $(D, x = (x_1, \ldots, x_{2n-1}))$ centered at $p = 0$. Let s be a local rigid CR frame of L on D and set $|s|^2_h = e^{-2\Phi}$. Fix $\delta > 0$ small enough and $D_0 \Subset D$. Then*

$$P_{k,\delta,s}(x,y) = \int_{\mathbb{R}} e^{ik\varphi(x,y,t)} g(x,y,t,k) dt + O(k^{-\infty}) \;\text{ on } D_0 \times D_0, \tag{1.16}$$

where $\varphi \in C^\infty(D \times D \times (-\delta,\delta))$ is a phase function such that for some constant $c > 0$ we have

$$\begin{aligned} & d_x\varphi(x,y,t)|_{x=y} = -2\mathrm{Im}\,\overline{\partial}_b\Phi(x) + t\omega_0(x), \;\; d_y\varphi(x,y,t)|_{x=y} = 2\mathrm{Im}\,\overline{\partial}_b\Phi(x) - t\omega_0(x), \\ & \mathrm{Im}\,\varphi(x,y,t) \geq c|z-w|^2, \;\; (x,y,t) \in D \times D \times (-\delta,\delta), x = (z, x_{2n-1}), y = (w, y_{2n-1}), \\ & \mathrm{Im}\,\varphi(x,y,t) + \left|\tfrac{\partial\varphi}{\partial t}(x,y,t)\right|^2 \geq c\,|x-y|^2, \; (x,y,t) \in D \times D \times (-\delta,\delta), \\ & \varphi(x,y,t) = 0 \text{ and } \tfrac{\partial\varphi}{\partial t}(x,y,t) = 0 \text{ if and only if } x = y, \end{aligned} \tag{1.17}$$

and $g(x,y,t,k) \in S^n_{\rm loc}(1; D \times D \times (-\delta,\delta)) \cap C^\infty_0(D \times D \times (-\delta,\delta))$ is a symbol with expansion

$$g(x,y,t,k) \sim \sum_{j=0}^{\infty} g_j(x,y,t) k^{n-j} \text{ in } S^n_{\rm loc}(1; D \times D \times (-\delta,\delta)), \tag{1.18}$$

and for $x \in D_0$ and $|t| < \delta$ we have

$$g_0(x,x,t) = (2\pi)^{-n} \left|\det\big(R^L_x - 2t\mathcal{L}_x\big)\right| |\tau_\delta(t)|^2 . \tag{1.19}$$

We refer the reader to Section 2.2 for the notations in semi-classical analysis used in Theorem 1.1. The determinant of a Hermitian quadratic form $\mathcal{R}_x$ on $T^{1,0}_xX$ is defined by $\det \mathcal{R}_x = \lambda_1 \ldots \lambda_{n-1}$ where $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $\mathcal{R}_x$ with respect to $\langle \cdot \,|\, \cdot \rangle$.

It should be noticed that the integral in the classical Boutet de Monvel-Sjöstrand's description [8] of the Szegő kernels for strictly pseudoconvex domains is $\mathbb{R}_+$, whereas the integral in our expansion (1.16) is $(-\delta, \delta)$. The difference is that in [8] one works with the Szegő projector on the infinite dimensional space of CR functions, while here we use the weighted Fourier-Szegő projector (1.11) on a finite dimensional space of sections of L^k, where the cut-off function τ_δ at frequency level plays an essential role. Moreover, $P_{k,\delta,s}(x, y)$ are semi-classical kernels (with a k in the phase) compared to the kernel in [8] where there is no semi-classical parameter. We refer to Section 3.4 for a comparison to Szegő/Bergman kernels in other geometric situations.

From Theorem 1.1, we deduce the asymptotics of the kernel $P_{k,\delta}(x, y)$ on the diagonal. Note that $P_{k,\delta}(x, x) = P_{k,\delta,s}(x, x)$.

Corollary 1.2. *In the conditions of Theorem 1.1 we have as $k \to \infty$,*

$$P_{k,\delta}(x, x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j(x) \text{ in } S^n_{\mathrm{loc}}(1; X) \tag{1.20}$$

where $b_j(x) \in C^\infty(X)$, $j = 0, 1, 2, \ldots$, and

$$b_0(x) = (2\pi)^{-n} \int_{\mathbb{R}} \left|\det\left(R^L_x - 2t\mathcal{L}_x\right)\right| |\tau_\delta(t)|^2 \, dt, \tag{1.21}$$

with $\tau_\delta(t) \in C^\infty_0(\mathbb{R})$ introduced in (1.8).

The leading term b_0 of the asymptotics (1.20) reflects thus the interplay between the curvature of L and the Levi form of the base manifold X, a phenomenon first noticed in [23]. This is a new feature compared to the asymptotics of the Bergman kernel of a positive line bundle over a complex manifold where only the curvature of L appears [11, 24, 28, 29, 38, 40]. Note however that for a Levi-flat manifold X we have

$$b_0(x) = (2\pi)^{-n} \int_{\mathbb{R}} \left|\det\left(R^L_x\right)\right| |\tau_\delta(t)|^2 \, dt, \tag{1.22}$$

which is an integrated version of the leading term for complex manifolds, reflecting the fact that Levi-flat manifolds are foliated by complex manifolds.

Back to the general case, observe that for δ small enough and $|t| \leq \delta$ we have for all $x \in X$ $|\det(R^L_x - 2t\mathcal{L}_x)| > 0$, due to the positivity of L and compactness of X. Hence $b_0(x) > 0$ on X and we deduce from (1.20) that for k large enough (see also Lemma 4.2),

$$d_k = \int_X \sum_{j=1}^{d_k} |f^k_j(x)|^2 \, dv_X(x) \geq \int_X P_{k,\delta}(x, x) \, dv_X(x) \gtrsim k^n.$$

Since we also have $d_k \lesssim k^n$ by [22, Theorem 1.4], we see that the spaces $\mathcal{H}_{k,\leq k\delta}(X, L^k)$ have maximal growth, $d_k \sim k^n$.

We define now the Kodaira map. Consider an open set $D \subset X$ with

$$\bigcup_{-\pi \leq \theta \leq \pi} e^{i\theta} D \subset D\,, \tag{1.23}$$

and let $s : D \to L$ be a local rigid CR trivializing section on D, see Proposition 2.7. For any $u \in C^\infty(X, L^k)$ we write $u(x) = s^k(x) \otimes \widetilde{u}(x)$ on D, with $\widetilde{u} \in C^\infty(D)$. Let $\{f_j\}_{j=1}^{d_k}$ be an orthonormal basis of $\mathcal{H}^0_{b,\leq k\delta}(X, L^k)$ with respect to $(\,\cdot\,|\,\cdot\,)$ such that $f_j \in \mathcal{H}^0_{b,m_j}(X, L^k)$ and set $g_j = F_{k,\delta} f_j$, $1 \leq j \leq d_k$. The Kodaira map is defined on D by

$$\begin{aligned} &\Phi_{k,\delta} : D \longrightarrow \mathbb{CP}^{d_k-1}, \\ &\quad x \longmapsto \big[F_{k,\delta} f_1, \ldots, F_{k,\delta} f_{d_k}\big] := \big[\widetilde{g}_1(x), \ldots, \widetilde{g}_{d_k}(x)\big], \ \text{ for } x \in D. \end{aligned} \tag{1.24}$$

By the proof of [22, Lemma 1.22] there exist an open cover of X with sets D satisfying (1.23). Thus we have a well-defined global map

$$\Phi_{k,\delta} : X \longrightarrow \mathbb{CP}^{d_k-1}, \quad x \longmapsto \big[F_{k,\delta} f_1, \ldots, F_{k,\delta} f_{d_k}\big]. \tag{1.25}$$

Since $g_j \in \mathcal{H}^0_{b,m_j}(X, L^k)$ we have $T\,\widetilde{g}_j = i m_j \widetilde{g}_j$ hence

$$g_j(e^{i\theta}x) = s^k(e^{i\theta}x) \otimes \widetilde{g}_j(e^{i\theta}x) = s^k(e^{i\theta}x) \otimes e^{i m_j \theta}\widetilde{g}_j(x).$$

Thus

$$\begin{aligned} \Phi_{k,\delta}(e^{i\theta}x) &= [\widetilde{g}_1(e^{i\theta}x), \cdots, \widetilde{g}_{d_k}(e^{i\theta}x)] = [e^{i m_1\theta}\widetilde{g}_1(x), \cdots, e^{i m_{d_k}\theta}\widetilde{g}_{d_k}(x)] \\ &= \big[e^{i m_1\theta}\Phi^1_{k,\delta}(x), \ldots, e^{i m_{d_k}\theta}\Phi^{d_k}_{k,\delta}(x)\big] \end{aligned} \tag{1.26}$$

We are thus led to consider *weighted diagonal* S^1-actions on $\mathbb{CP}^N$, that is, actions for which there exists $(m_1, \ldots, m_N, m_{N+1}) \in \mathbb{N}_0^{N+1}$ such that for all $\theta \in [0, 2\pi)$,

$$e^{i\theta}[z_1, \ldots, z_{N+1}] = \big[e^{i m_1\theta} z_1, \ldots, e^{i m_{N+1}\theta} z_{N+1}\big], \ \ [z_1, \ldots, z_{N+1}] \in \mathbb{CP}^N. \tag{1.27}$$

Theorem 1.3. *Let $(X, T^{1,0}X)$ be a compact CR manifold with a transversal CR locally free S^1-action. Assume there is an S^1-equivariant positive CR line bundle (L, h) over X. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there exists $k(\delta)$ so that for $k > k(\delta)$ and any orthonormal basis $\{f_j\}_{j=1}^{d_k}$ of $\mathcal{H}^0_{b,\leq k\delta}(X, L^k)$ with respect to $(\,\cdot\,|\,\cdot\,)$ such that $f_j \in \mathcal{H}^0_{b,m_j}(X, L^k)$, the map $\Phi_{k,\delta}$ introduced in* (1.25) *is a smooth CR embedding which is S^1-equivariant with respect to the weighted diagonal action defined by $(m_1, \ldots, m_{d_k}) \in \mathbb{N}_0^{d_k}$ as in* (1.27)*, that is,*

$$\Phi_{k,\delta}(e^{i\theta}x) = e^{i\theta}\Phi_{k,\delta}(x), \ \ x \in X, \ \ \theta \in [0, 2\pi).$$

In particular, the image $\Phi_{k,\delta}(X) \subset \mathbb{CP}^{d_k-1}$ is a CR submanifold with an induced weighted diagonal locally free S^1-action.

In [21, Theorem 1.11], it was proved that if X admits a transversal CR locally free S^1-action and there is an S^1-equivariant positive CR line bundle L over X, then X can be CR embedded into projective space under the assumption that condition $Y(0)$ holds on X. The condition $Y(0)$ is needed in [21] to insure that the spaces $H^0(X, L^k)$ are finite dimensional. The Kodaira map defined by these spaces is proved to be a CR embedding for k large enough. The embeddings are thus not S^1-equivariant. In Theorem 1.3 we remove the Levi curvature assumption $Y(0)$ used in [21] and also obtain an equivariant embedding. We achieve this by working with Fourier components of the spaces $H^0(X, L^k)$ (cf. (1.5), (1.6)), which are finite dimensional even if $H^0(X, L^k)$ are not. The ample spaces of CR sections are the direct sums $\mathcal{H}^0_{b,\leq k\delta}(X, L^k)$ of Fourier components $\mathcal{H}^0_{b,m}(X, L^k)$ for $|m| \leq k\delta$, which tend to fill $H^0(X, L^k)$ for $k \to \infty$. We consider the weighted projector on the Fourier components (1.9), the associated weighted Fourier-Szegő operator (1.11) and the resulting equivariant Kodaira map (1.25). The main technical ingredient is then Theorem 1.1, which provides a precise description of the kernel of the weighted Fourier-Szegő operator.

An interesting case when $Y(0)$ doesn't hold but Theorem 1.3 applies is the case of Levi-flat CR manifolds.

Corollary 1.4. *Let X be a compact Levi-flat CR manifold. Assume that X admits a transversal CR locally free S^1-action and an S^1-equivariant positive CR line bundle. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there exists $k(\delta)$ so that for $k > k(\delta)$ the map $\Phi_{k,\delta}$ introduced in* (1.25) *is a C^∞ CR embedding of X in $\mathbb{CP}^{d_k-1}$ which is S^1-equivariant with respect to weighted diagonal actions.*

Ohsawa and Sibony [32, 33] constructed for every $\kappa \in \mathbb{N}$ a CR projective embedding of class C^κ of a Levi-flat CR manifold by using $\overline{\partial}$-estimates. The first and third authors [26] gave a Szegő kernel proof of Ohsawa and Sibony's result. A natural question is whether we can improve the regularity to $\kappa = \infty$. Adachi [1] showed that the answer is no, in general. The analytic difficulty of this problem comes from the fact that the Kohn Laplacian is not hypoelliptic on Levi flat manifolds. Corollary 1.4 shows that one can find C^∞ CR embeddings of Levi flat manifolds in the equivariant setting.

When X is strongly pseudoconvex, it is known [34, Theorem 1.11] that there is a S^1-equivariant positive CR line bundle over X. We deduce from Theorem 1.3:

Corollary 1.5. *Let $(X, T^{1,0}X)$ be a compact strongly pseudoconvex CR manifold with a transversal CR locally free S^1-action. Then there exists smooth CR embeddings $\Phi_{k,\delta}$ of X in $\mathbb{CP}^{d_k-1}$ which are S^1-equivariant with respect to weighted diagonal actions (cf. Theorem 1.3).*

We illustrate Corollary 1.3 in Example 4.9.

This paper is organized as follows. In Section 2 we recall the necessary notions and results from semiclassical analysis and theory of CR manifolds with circle action. In Section 3 we prove the asymptotics of the Fourier-Szegő kernel (Theorem 1.1 and Corollary 1.2). Section 4 deals with the Kodaira embedding theorem.

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2. Preliminaries

2.1. **Some standard notations.** We use the following notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R};\, x \geq 0\}$. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$ we set $|\alpha| = \alpha_1 + \ldots + \alpha_m$. For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ we write

$$x^\alpha = x_1^{\alpha_1} \ldots x_m^{\alpha_m}, \quad \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_m}^{\alpha_m} = \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$
$$D_{x_j} = \frac{1}{i}\partial_{x_j}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_m}^{\alpha_m}, \quad D_x = \frac{1}{i}\partial_x .$$

Let $z = (z_1, \ldots, z_m)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, m$, be coordinates of $\mathbb{C}^m$, where $x = (x_1, \ldots, x_{2m}) \in \mathbb{R}^{2m}$ are coordinates in $\mathbb{R}^{2m}$. Throughout the paper we also use the notation $w = (w_1, \ldots, w_m) \in \mathbb{C}^m$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, m$, where $y = (y_1, \ldots, y_{2m}) \in \mathbb{R}^{2m}$. We write

$$z^\alpha = z_1^{\alpha_1}\ldots z_m^{\alpha_m}\,,\quad \overline{z}^\alpha = \overline{z}_1^{\alpha_1}\ldots \overline{z}_m^{\alpha_m}\,,$$
$$\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\Big)\,,\quad \partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\Big),$$
$$\partial_z^\alpha = \partial_{z_1}^{\alpha_1}\ldots\partial_{z_m}^{\alpha_m} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}\,,\quad \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1}\ldots\partial_{\overline{z}_m}^{\alpha_m} = \frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha}\,.$$

Let X be a C^∞ orientable paracompact manifold. We let TX and T^*X denote the tangent bundle of X and the cotangent bundle of X respectively. The complexified tangent bundle of X and the complexified cotangent bundle of X will be denoted by $\mathbb{C}TX$ and $\mathbb{C}T^*X$ respectively. We write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TX and T^*X. We extend $\langle\,\cdot\,,\cdot\,\rangle$ $\mathbb{C}$-bilinearly to $\mathbb{C}TX\times\mathbb{C}T^*X$.

Let E be a C^∞ vector bundle over X. The fiber of E at $x\in X$ will be denoted by E_x. Let F be another vector bundle over X. We write $F\boxtimes E^*$ to denote the vector bundle over $X\times X$ with fiber over $(x,y)\in X\times X$ consisting of the linear maps from E_x to F_y.

Let $Y\subset X$ be an open set. The spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y,E)$ and $\mathscr{D}'(Y,E)$ respectively. Let $\mathscr{E}'(Y,E)$ be the subspace of $\mathscr{D}'(Y,E)$ whose elements have compact support in Y. For $m\in\mathbb{R}$, we let $H^m(Y,E)$ denote the Sobolev space of order m of sections of E over Y. Put

$$H^m_{\rm loc}(Y,E) = \big\{u\in\mathscr{D}'(Y,E);\ \varphi u\in H^m(Y,E),\ \text{for all } \varphi\in C^\infty_0(Y)\big\}\,,$$
$$H^m_{\rm comp}(Y,E) = H^m_{\rm loc}(Y,E)\cap\mathscr{E}'(Y,E)\,.$$

2.2. **Definitions and notations from semi-classical analysis.** We recall the Schwartz kernel theorem [19, Theorems 5.2.1, 5.2.6], [37, p. 296], [28, B.2]. Let E and F be smooth vector bundles over X. Let Y be an open set of X. Let $A(\cdot,\cdot)\in\mathscr{D}'(Y\times Y,F\boxtimes E^*)$. For any fixed $u\in C^\infty_0(Y,E)$, the linear map $C^\infty_0(Y,F^*)\ni v\mapsto (A(\cdot,\cdot),v\otimes u)\in\mathbb{C}$ defines a distribution $Au\in\mathscr{D}'(Y,F)$. The operator $A: C^\infty_0(Y,E)\to\mathscr{D}'(Y,F)$, $u\mapsto Au$, is linear and continuous.

The Schwartz kernel theorem asserts that, conversely, for any continuous linear operator $A: C^\infty_0(Y,E)\to\mathscr{D}'(Y,F)$ there exists a unique distribution $A(\cdot,\cdot)\in\mathscr{D}'(Y\times Y,F\boxtimes E^*)$ such that $(Au,v) = (A(\cdot,\cdot),v\otimes u)$ for any $u\in C^\infty_0(Y,E)$, $v\in C^\infty_0(Y,F^*)$. The distribution $A(\cdot,\cdot)$ is called the Schwartz distribution kernel of A. We say that A is properly supported if the canonical projections on the two factors restricted to $\operatorname{Supp} A(\cdot,\cdot)\subset Y\times Y$ are proper.

The following two statements are equivalent:

(a) A can be extended to a continuous operator $A:\mathscr{E}'(Y,E)\to C^\infty(Y,F)$,
(b) $A(\cdot,\cdot)\in C^\infty(Y\times Y,F\boxtimes E^*)$.

If A satisfies (a) or (b), we say that A is a *smoothing operator*. Furthermore, A is smoothing if and only if for all $N\geq 0$ and $s\in\mathbb{R}$, $A: H^s_{\rm comp}(Y,E)\to H^{s+N}_{\rm loc}(Y,F)$ is continuous.

Let A be a smoothing operator. Then for any volume form $d\mu$, the Schwartz kernel of A is represented by a smooth kernel $K\in C^\infty(Y\times Y,F\boxtimes E^*)$, called the Schwartz kernel of A with respect to $d\mu$, such that

$$(Au)(x) = \int_M K(x,y)u(y)\,d\mu(y)\,,\quad \text{for any } u\in C^\infty_0(Y,E)\,. \tag{2.1}$$

Then A can be extended as a linear continuous operator $A:\mathscr{E}'(Y,E)\to C^\infty(Y,F)$ by setting $(Au)(x) = \big(u(\cdot),K(x,\cdot)\big)$, $x\in Y$, for any $u\in\mathscr{E}'(Y,E)$.

Let W_1, W_2 be open sets in $\mathbb{R}^N$ and let E and F be complex Hermitian vector bundles over W_1 and W_2 respectively. Let $s, s' \in \mathbb{R}$ and $n_0 \in \mathbb{R}$. For a k-dependent continuous function $F_k : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F)$ we write

$$F_k = O(k^{n_0}) : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F),$$

if for any $\chi_0 \in C^\infty(W_2), \chi_1 \in C^\infty_0(W_1)$, there is a positive constant $c > 0$ independent of k, such that

$$\|(\chi_0 F_k \chi_1)u\|_{s'} \leq ck^{n_0}\|u\|_s, \quad \text{for all } u \in H^s_{\rm loc}(W_1, E), \tag{2.2}$$

where $\|\cdot\|_s$ denotes the usual Sobolev norm of order s. We write

$$F_k = O(k^{-\infty}) : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F),$$

if $F_k = O(k^{-N}) : H^s_{\rm comp}(W_1, E) \to H^{s'}_{\rm loc}(W_2, F)$, for every $N > 0$.

A k-dependent continuous operator $A_k : C^\infty_0(W_1, E) \to \mathscr{D}'(W_2, F)$ is called k-negligible on $W_2 \times W_1$ if for k large enough A_k is smoothing and for any $K \Subset W_2 \times W_1$, any multi-indices α, β and any $N \in \mathbb{N}$ there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_y A_k(x, y)\right| \leq C_{K,\alpha,\beta,N} k^{-N} \ \text{ on } K. \tag{2.3}$$

We write in this case

$$A_k(x, y) = O(k^{-\infty}) \ \text{ on } W_2 \times W_1,$$

or

$$A_k = O(k^{-\infty}) \ \text{ on } W_2 \times W_1.$$

If $A_k, B_k : C^\infty_0(W_1, E) \to \mathscr{D}'(W_2, F)$ are k-dependent continuous operators, we write $A_k = B_k + O(k^{-\infty})$ if $A_k - B_k = O(k^{-\infty})$ on $W_2 \times W_1$.

Let $A_k : L^2(X, L^k) \to L^2(X, L^k)$ be a continuous operator. Let s, s_1 be local rigid CR frames of L on open sets $D_0 \Subset M$, $D_1 \Subset M$ respectively, $|s|^2_h = e^{-2\Phi}$, $|s_1|^2_h = e^{-2\Phi_1}$. The localization of A_k (with respect to the trivializing rigid CR sections s and s_1) is given by

$$A_{k,s,s_1} : L^2(D_1) \cap \mathscr{E}'(D_1) \to L^2(D), \ u \longmapsto e^{-k\Phi} s^{-k} A_k(s^k_1 e^{k\Phi_1} u) = U^{-1}_{k,s} A_k U_{k,s_1}, \tag{2.4}$$

and let $A_{k,s,s_1}(x, y) \in \mathscr{D}'(D \times D_1)$ be the distribution kernel of A_{k,s,s_1}. Let $\sigma, \sigma', n_0 \in \mathbb{R}$. We write

$$A_k = O(k^{n_0}) : H^\sigma(X, L^k) \to H^{\sigma'}(X, L^k),$$

if for all local rigid CR frames s, s_1 on D and D_1 respectively, we have

$$A_{k,s,s_1} = O(k^{n_0}) : H^\sigma_{\rm comp}(D_1) \to H^{\sigma'}_{\rm loc}(D).$$

We write

$$A_k = O(k^{-\infty}) : H^\sigma(X, L^k) \to H^{\sigma'}(X, L^k),$$

if for all local rigid CR frames s, s_1 on D and D_1 respectively, we have

$$A_{k,s,s_1} = O(k^{-\infty}) : H^\sigma_{\rm comp}(D_1) \to H^{\sigma'}_{\rm loc}(D).$$

We write

$$A_k = O(k^{-\infty})$$

if for all local rigid CR frames s, s_1 on D and D_1 respectively, we have

$$A_{k,s,s_1}(x, y) = O(k^{-\infty}) \ \text{ on } D \times D_1.$$

When $s = s_1$, $D = D_1$, we write $A_{k,s} := A_{k,s,s}$, $A_{k,s}(x, y) := A_{k,s,s}(x, y)$.

We recall the definition of the semi-classical symbol spaces [14, Chapter 8]:

Definition 2.1. Let W be an open set in $\mathbb{R}^N$. Let

$$S(1;W) := \Big\{a \in C^\infty(W) \,|\, \text{for all} \alpha \in \mathbb{N}_0^N : \sup_{x\in W} |\partial^\alpha a(x)| < \infty\Big\},$$

$$S^0_{\rm loc}(1;W) := \Big\{(a(\cdot,k))_{k\in\mathbb{R}} \,|\, \text{for all } \alpha \in \mathbb{N}_0^N, \text{for all } \chi \in C_0^\infty(W) : \sup_{k\in\mathbb{R},k\geq 1} \sup_{x\in W} |\partial^\alpha(\chi a(x,k))| < \infty\Big\}.$$

For $m \in \mathbb{R}$ let

$$S^m_{\rm loc}(1) := S^m_{\rm loc}(1;W) = \Big\{(a(\cdot,k))_{k\in\mathbb{R}} \,|\, (k^{-m}a(\cdot,k)) \in S^0_{\rm loc}(1;W)\Big\}.$$

Hence $a(\cdot,k) \in S^m_{\rm loc}(1;W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in C_0^\infty(W)$, there exists $C_\alpha > 0$ independent of k, such that $|\partial^\alpha(\chi a(\cdot,k))| \leq C_\alpha k^m$ on W.

Consider a sequence $a_j \in S^{m_j}_{\rm loc}(1)$, $j \in \mathbb{N}_0$, where $m_j \searrow -\infty$, and let $a \in S^{m_0}_{\rm loc}(1)$. We say that

$$a(\cdot,k) \sim \sum_{j=0}^\infty a_j(\cdot,k) \text{ in } S^{m_0}_{\rm loc}(1),$$

if for every $\ell \in \mathbb{N}_0$ we have $a - \sum_{j=0}^{\ell} a_j \in S^{m_{\ell+1}}_{\rm loc}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\rm loc}(1) = S^{-\infty}_{\rm loc}(1;W) := \cap_m S^m_{\rm loc}(1)$.

We say that $a(\cdot,k) \in S^m_{\rm loc}(1)$ is a classical symbol on W of order m if

$$a(\cdot,k) \sim \sum_{j=0}^\infty k^{m-j}a_j \text{ in } S^m_{\rm loc}(1), \quad a_j(x) \in S^0_{\rm loc}(1),\ j = 0,1\ldots. \tag{2.5}$$

The set of all classical symbols on W of order m is denoted by $S^m_{\rm loc\,,cl}(1) = S^m_{\rm loc\,,cl}(1;W)$.

Definition 2.2. Let W be an open set in $\mathbb{R}^N$. A semi-classical pseudodifferential operator on W of order m with classical symbol is a k-dependent continuous operator $A_k : C_0^\infty(W) \to C^\infty(W)$ such that the distribution kernel $A_k(x,y)$ is given by the oscillatory integral

$$\begin{aligned} A_k(x,y) =& \frac{k^N}{(2\pi)^N}\int e^{ik\langle x-y,\eta\rangle}a(x,y,\eta,k)d\eta + O(k^{-\infty}),\\ & a(x,y,\eta,k) \in S^m_{\rm loc\,,cl}(1;W\times W\times\mathbb{R}^N).\end{aligned} \tag{2.6}$$

We shall identify A_k with $A_k(x,y)$. It is clear that A_k has a unique continuous extension $A_k : \mathscr{E}'(W) \to \mathscr{D}'(W)$. It is well-known that (see [14, Chapter 7]) that there is a symbol

$$\alpha(x,\eta,k) \in S^m_{\rm loc\,,cl}(1;W\times\mathbb{R}^N) = S^m_{\rm loc\,,cl}(1;T^*W) \tag{2.7}$$

unique up to an element in $S^{-\infty}_{\rm loc}(1)$ such that

$$A_k(x,y) = \frac{k^N}{(2\pi)^N}\int e^{ik\langle x-y,\eta\rangle}\alpha(x,\eta,k)d\eta + O(k^{-\infty}). \tag{2.8}$$

2.3. **CR manifolds with circle action.** Let X be a real manifold and let $\mathbb{C}TX$ denote its complexified tangent bundle. Let F be a complex subbundle of $\mathbb{C}TX$. We say that F is totally complex if $F \cap \overline{F} = 0$, where the bar denotes complex-conjugation in $\mathbb{C}TX$ and 0 is the zero section of $\mathbb{C}TX$. We say that F is involutive if $[C^\infty(X,F), C^\infty(X,F)] \subset C^\infty(X,F)$, that is, if the space of smooth sections of F is closed under Lie brackets.

A CR manifold of hypersurface type is a pair $(X, T^{1,0}X)$, where X is a a smooth real manifold of dimension $2n-1$, $n \geq 2$, and $T^{1,0}X$ is a subbundle of rank $n-1$ of the complexified tangent bundle $\mathbb{C}TX$, which is totally complex and involutive. The bundle $T^{1,0}X$ is called CR structure and we set $T^{0,1}X=\overline{T^{1,0}X}$. Throughout this paper we work only with CR manifolds of hypersurface type, which we call simply CR manifolds.

We assume that X admits a S^1-action $S^1\times X\to X$, $(e^{i\theta},x)\mapsto e^{i\theta}x$. The global real vector field $T\in C^\infty(X,TX)$ induced by the S^1-action is given by

$$(Tu)(x)=\frac{\partial}{\partial\theta}\left(u(e^{i\theta}x)\right)|_{\theta=0},\ \ u\in C^\infty(X). \tag{2.9}$$

Definition 2.3. We say that the S^1-action $e^{i\theta}$ is CR if $[T,C^\infty(X,T^{1,0}X)]\subset C^\infty(X,T^{1,0}X)$ and the S^1-action is transversal if for each $x\in X$, $\mathbb{C}T(x)\oplus T^{1,0}_x(X)\oplus T^{0,1}_xX=\mathbb{C}T_xX$. Moreover, we say that the S^1-action is locally free if $T\neq 0$ everywhere.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$ respectively. Define the vector bundle of $(0,q)$ forms by $T^{*0,q}X=\Lambda^q(T^{*0,1}X)$. Let $D\subset X$ be an open subset. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega^{0,q}_0(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D. Similarly, if E is a vector bundle over D, then we let $\Omega^{0,q}(D,E)$ denote the space of smooth sections of $T^{*0,q}X\otimes E$ over D and let $\Omega^{0,q}_0(D,E)$ be the subspace of $\Omega^{0,q}(D,E)$ whose elements have compact support in D.

Fix $\theta_0\in]-\pi,\pi[$ close to 0. Let

$$de^{i\theta_0}:\mathbb{C}T_xX\to\mathbb{C}T_{e^{i\theta_0}x}X$$

denote the differential map of $e^{i\theta_0}:X\to X$. Since the S^1-action is CR, we can check that

$$\begin{aligned}&de^{i\theta_0}:T^{1,0}_xX\to T^{1,0}_{e^{i\theta_0}x}X,\\ &de^{i\theta_0}:T^{0,1}_xX\to T^{0,1}_{e^{i\theta_0}x}X,\\ &de^{i\theta_0}(T(x))=T(e^{i\theta_0}x).\end{aligned} \tag{2.10}$$

Let $(e^{i\theta_0})^*:\Lambda^r(\mathbb{C}T^*X)\to\Lambda^r(\mathbb{C}T^*X)$ be the pull-back map of $e^{i\theta_0}$, $r=0,1,\ldots,2n-1$. From (2.10), it is easy to see that for every $q=0,1,\ldots,n$,

$$(e^{i\theta_0})^*:T^{*0,q}_{e^{i\theta_0}x}X\to T^{*0,q}_xX. \tag{2.11}$$

Let $u\in\Omega^{0,q}(X)$. Define (see also (2.38))

$$Tu:=\frac{\partial}{\partial\theta}\big((e^{i\theta})^*u\big)|_{\theta=0}\in\Omega^{0,q}(X). \tag{2.12}$$

For every $\theta\in[0,2\pi)$ and every $u\in C^\infty(X,\Lambda^r(\mathbb{C}T^*X))$, we write $u(e^{i\theta}x):=(e^{i\theta})^*u(x)$. It is clear that for every $u\in C^\infty(X,\Lambda^r(\mathbb{C}T^*X))$ the Fourier expansion of u reads

$$u(x)=\sum_{m\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{i\theta}x)e^{-im\theta}d\theta. \tag{2.13}$$

Let $\overline\partial_b:\Omega^{0,q}(X)\to\Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. Since the S^1-action is CR, it is straightforward to see that (see also (2.39))

$$T\overline\partial_b=\overline\partial_bT\ \text{ on }\Omega^{0,q}(X).$$

Definition 2.4. Let $D\subset U$ be an open set. We say that a function $u\in C^\infty(D)$ is rigid if $Tu=0$. We say that a function $u\in C^\infty(X)$ is Cauchy-Riemann (CR for short) if $\overline\partial_bu=0$. We say that $u\in C^\infty(X)$ is rigid CR if $\overline\partial_bu=0$ and $Tu=0$.

In this paper we use the following notion of CR vector bundles.

Definition 2.5. Let X be a CR manifold of hypersurface type. A smooth complex vector bundle (F, π, X) of rank r over X is called CR vector bundle if F has the structure of a smooth abstract CR manifold of hypersurface type, the map $\pi : F \to X$ is a CR map, and for each point of X there exist an open neighborhood U and a smooth trivialization of $F|_U$ which is a CR diffeomorphism (that is, the map and its inverse are CR). We define a smooth CR section of F over an open subset D of X as a smooth section $s : D \to F$ which is a CR map. A CR frame of F over an open subset U of X is a smooth frame $\{f^1, f^2, \ldots, f^r\}$ of $F|_U$ where each f^k is a CR section.

If F is a CR vector bundle then each point has a neighborhood U with a CR frame of F over U. Let $(U_j)_j$ be an open cover of X with CR frames $\{f_j^1, f_j^2, \ldots, f_j^r\}$ of F over U_j and let $\{g_{jk}\}_{j,k}$ be the cocycle of transition matrices between these frames. Then the entries of the matrices $g_{jk} : U_j \cap U_k \to Gl(r, \mathbb{C})$ are CR functions. Note that CR manifolds with transversal S^1-action are locally embeddable and there exist locally plenty of CR functions.

In CR geometry there exist a more general notion of CR vector bundle [18, 39] which does not require the CR local triviality (the definitions in [18] and [39] are equivalent). There are indeed examples of CR vector bundles in the sense of [18, 39] which are not locally CR trivializable (see e. g. [18, p. 279]). The goal of our paper is to prove a Kodaira embedding theorem, so to work with very ample line bundles, whose global CR sections give an embedding in the projective space. Such bundles are locally CR trivializable, so we restrict here to the notion introduced in Definition 2.5.

Definition 2.6. Let X be a CR manifold endowed with an S^1-action and let (F, π, X) be a CR vector bundle of rank r over X. We say the S^1-action on X can be lifted to F that is there exists an S^1 action on F still denoted by $e^{i\theta}$ such that

$$\pi(e^{i\theta} \circ v(x)) = e^{i\theta} \circ x, \;\; v(x) \in F_x, \; x \in X.$$

A lifting is called a CR bundle lifting in F if for each $e^{i\theta}$, the map $e^{i\theta} : F \to F$ is a CR bundle map. Such a bundle is called S^1-equivariant CR vector bundle.

Proposition 2.7. *Let (F, π, X) be a S^1-equivariant vector bundle. Then there exists in the neighborhood of each point a rigid CR local frame of F. In particular, there exists an open cover $(U_j)_j$ of X and trivializing frames $\{f_j^1, f_j^2, \ldots, f_j^r\}$ on each U_j such that the corresponding transition matrices are rigid CR.*

Proof. In order to ease notation we denote S^1 by G. Since X is a CR manifold of hypersurface type with transversal S^1-action, for any $x \in X$, by the slice theorem, we have a diffeomorphism of a $G = S^1$-neighbourhood of x in X, $U \to G \times_{G_x} N$ here $N = T_xX/T(Gx)$ and $G_x = \{g \in G : gx = x\}$ and $g \in G_x$ acts on N as $dg : N \to N$.

Now $T^{1,0}X$ induces a subbundle $T^{1,0}N$ of $T(e \times N) \otimes \mathbb{C}$ by projection as $TX = T(Gx) \oplus T(e \times N)$. Since $\dim T(Gx) = 1$, for a frame $\{w_j\}$ of $T^{1,0}N$ the associated frame of $T^{1,0}X$ is

$$w_j^H = w_j + a_j(g, y)u \text{ for } [g, y] \in G \times_{G_x} B(0, 1) \tag{2.14}$$

and $B(0, \epsilon) \subset N$ the ball of center 0 with radius ϵ, w_j does not depend on g and $u \in T(Gx) \otimes \mathbb{C}$ the vector field generated by $0 \neq K \in \mathrm{Lie}(S^1)$. Thus $[w_j, u] = 0$. By Definition,

$$[w_j + a_j(g, y)u, w_k + a_k(g, y)u] = [w_j, w_k] + (w_j a_k(g, y))u - (w_k a_j(g, y))u \in T^{1,0}X, \tag{2.15}$$

thus $[w_j, w_k] \in T^{1,0}N$. This means that $T^{1,0}N$ defines a complex structure of N.

Let F be an S^1-equivariant CR vector bundle on X. Then for

$$\widetilde{F} = \sigma^* F \text{ with } \sigma : G \times N \to G \times_{G_x} N \text{ the natural projection,} \tag{2.16}$$

$\widetilde{F}$ is a S^1-equivariant CR vector bundle on $G \times B(0,1)$ induced by the CR structure on $G \times_{G_x} B(0,1)$ as above.

Now the G-equivariant sections of $\widetilde{F}$ induce a vector bundle $\widetilde{F}_G$ on N and

$$C^\infty(N, \widetilde{F}_G) = C^\infty(G \times N, \widetilde{F})^G, \tag{2.17}$$

where $C^\infty(G \times N, \widetilde{F})^G$ denote the space of G-invariant sections of $\widetilde{F}$ on $G \times N$.

For $w_j \in T^{1,0}N$, $s \in C^\infty(N, \widetilde{F}_G) = C^\infty(G \times N, \widetilde{F})^G$, we define

$$\overline{\partial}^{\widetilde{F}_G}_{\widetilde{w}_j} s := \overline{\partial}^{\widetilde{F}}_{b,\widetilde{w}^H_j} s. \tag{2.18}$$

Then

$$(\overline{\partial}^{\widetilde{F}_G})^2 = 0, \tag{2.19}$$

and it defines a holomorphic structure on $\widetilde{F}_G$ over $B(0,1) \subset N$. Now for a holomorphic frame $\{f_j\}$ of $\widetilde{F}_G$ over $B(0,1) \subset N$, we see by (2.18) that the corresponding lift $f_j \in C^\infty(G \times N, \widetilde{F})^G$ fulfills the relations

$$\overline{\partial}^{\widetilde{F}}_b f_j = 0, \quad L_{K^X} f_j = 0 \text{ for } 0 \neq K \in \mathrm{Lie}(S^1). \tag{2.20}$$

Here K^X denotes the vector field on X generated by $K \in \mathrm{Lie}(S^1)$. Now for these choice of frames, the transition functions are CR and they are annihilated by L_{K^X}. □

If F is an S^1-equivariant vector bundle, we can define the operator T on $\Omega^{0,q}(X,F)$. Indeed, every $u \in \Omega^{0,q}(X,F)$ can be written on U_j as $u = \sum u_\ell \otimes f^\ell_j$ with $u_\ell \in \Omega^{0,q}(U)$ and we set

$$Tu = \sum T u_\ell \otimes f^\ell_j. \tag{2.21}$$

Then Tu is well-defined as element of $\Omega^{0,q}(X,F)$, since the entries of the transition matrices between different frames $\{f^1_j, f^2_j, \ldots, f^r_j\}$ are annihilated by T.

Example 2.8. Let X be a compact CR manifold with a locally free transversal CR S^1 action. We study here the bundle $T^{1,0}X$ by using the canonical BRT coordinates [4, Theorem II.1, Proposition I.2]. Let $(D, (z,\theta), \phi)$ be a BRT trivialization defined in (1.2). Then on D,

$$\begin{aligned} T &= \frac{\partial}{\partial \theta}, \\ Z_j &= \frac{\partial}{\partial z_j} + i \frac{\partial \phi}{\partial z_j}(z, \overline{z}) \frac{\partial}{\partial \theta}, \ \ j = 1, \ldots, n-1, \end{aligned} \tag{2.22}$$

where $\{Z_j : j = 1, \ldots, n-1\}$ is a frame of $T^{1,0}X$ over D. We always assume that $\phi(0,0) = 0$. Let $(\tilde{D}, (w,\eta), \tilde{\phi})$ be another BRT trivialization. Then on $\tilde{D}$,

$$\begin{aligned} T &= \frac{\partial}{\partial \eta}, \\ \tilde{Z}_j &= \frac{\partial}{\partial w_j} + i \frac{\partial \tilde{\phi}}{\partial w_j}(w, \overline{w}) \frac{\partial}{\partial \eta}, \ \ j = 1, \ldots, n-1, \end{aligned} \tag{2.23}$$

where $\{\tilde{Z}_j : j = 1, \ldots, n-1\}$ is a frame of $T^{1,0}X$ over $\tilde{D}$. We have on $D \cap \tilde{D}$,

$$\tilde{Z}_j = \sum_{k=1}^{n-1} c_{j,k} Z_k \tag{2.24}$$

where $c_{j,k} \in C^\infty(D \cap \tilde{D})$ are smooth and the matrix $(c_{j,k})$ is invertible. Since

$$[Z_j, T] = 0, \quad [\tilde{Z}_j, T] = 0, \qquad j \in \{1, \ldots, n-1\}, \tag{2.25}$$

we conclude from (2.24) that

$$Tc_{j,k} = 0, \quad j, k \in \{1, \ldots, n-1\}. \tag{2.26}$$

On the other hand, using (2.24) we obtain

$$[\tilde{Z}_j, \overline{\tilde{Z}}_k] = \sum_{\ell,m} c_{j,\ell}(Z_\ell\, \overline{c}_{k,m})\overline{Z}_m - \sum_{\ell,m} \overline{c}_{k,\ell}(\overline{Z}_\ell\, c_{j,m})Z_m + \sum_{\ell,m} c_{j,\ell}\overline{c}_{k,m}[Z_\ell, \overline{Z}_m]. \tag{2.27}$$

Note that

$$[Z_j, \overline{Z}_k] \in \mathbb{C}T, \quad [\tilde{Z}_j, \overline{\tilde{Z}}_k] \in \mathbb{C}T, \qquad j, k \in \{1, \ldots, n-1\}, \tag{2.28}$$

so we conclude from (2.27) that

$$\sum_{\ell} \overline{c}_{k,\ell}\big(\overline{Z}_\ell\, c_{j,m}\big) = 0, \tag{2.29}$$

for all $k, \ell, m \in \{1, \ldots, n-1\}$. Since the matrix $(\overline{c}_{k,\ell})$ is invertible we deduce from (2.29) that

$$\overline{Z}_\ell\, c_{j,m} = 0, \quad \ell, j, m \in \{1, \ldots, n-1\}, \tag{2.30}$$

that is, $c_{j,m}$ are CR functions. Therefore, (2.26) and (2.30) show that $c_{j,k}$ are rigid CR functions on $D \cap \tilde{D}$ for all $j, k \in \{1, \ldots, n-1\}$. Thus, arranging an atlas of BRT trivializations, we see that $\{Z_j\}_{j=1}^{n-1}$ are rigid CR frames as in Proposition 2.7.

Although we don't use later, let us note the following relations. Write $z = z(w, \overline{w}, \eta)$, $\theta = \theta(w, \overline{w}, \eta)$ in the coordinates $(w, \overline{w}, \eta)$. Then $\partial z/\partial\eta = Tz = \partial z/\partial\theta = 0$. Since z_k are CR functions we obtain by (2.23),

$$\frac{\partial z_k}{\partial \overline{w}_j} = \overline{\tilde{Z}}_j z_k = 0, \quad j, k \in \{1, \cdots, n-1\},$$

hence $z = z(w) = H(w)$, where H is a biholomorphic map. Since $\partial\theta/\partial\eta = 1$, we see that $\theta = \eta + G(w, \overline{w})$, where $G(w, \overline{w})$ is a real valued smooth function. Thus, the coordinate transformation from (w, η) to (z, θ) is given by

$$z = H(w), \;\; \theta = \eta + G(w, \overline{w}). \tag{2.31}$$

We write $c_{j,k} = c_{j,k}(w, \overline{w})$ in the coordinates $(w, \overline{w}, \eta)$ (recall that $c_{j,k}$ is independent of η due to (2.26)). From (2.24) and (2.31) it follows that

$$c_{j,k} = \frac{\partial H_k}{\partial w_j}(w), \quad \tilde{\phi}(w, \overline{w}) = \phi\big(H(w), \overline{H(w)}\,\big). \tag{2.32}$$

In particular, the functions $c_{j,k}$ are holomorphic in w, which follows also from (2.30). We have thus a complete description of the change between two canonical coordinate systems.

Example 2.9. Let X be a compact CR manifold with a locally free transversal CR S^1-action. Then $T^{1,0}X$ and the determinant bundle $\det T^{1,0}X$ are S^1-equivariant CR bundles.

Example 2.10. Let M be a compact complex manifold of $\dim M = n$ and let $(L,h) \to M$ be a Hermitian line bundle. We denote by e the local holomorphic trivializing section of L defined on a local holomorphic coordinate chart (U,z), $|e|_h^2 = e^{-2\phi(z)}$. Let (L^*,h^*) be the dual line bundle of (L,h) and let e^* be the dual frame of e. Let (z,t) be the local coordinates on L^*. Then the boundary of the Grauert-tube with respect to (L,h) is given by $X = \{v \in L^* : |v|^2_{h^{-1}} = 1\}$. X is a compact CR manifold with a natural CR structure $T^{1,0}X := T^{1,0}L^* \cap \mathbb{C}TX$. A natural transversal CR S^1 action on X is given by $e^{i\theta} \circ (z,t) = (z, e^{i\theta}t)$ and locally

$$\begin{aligned} &T^{1,0}X = \mathrm{Span}_{\mathbb{C}}\Big\{\frac{\partial}{\partial z_j} + i\frac{\partial \phi}{\partial z_j}(z,\overline{z})\frac{\partial}{\partial\theta}, j = 1,\ldots,n\Big\},\\ &T = \frac{\partial}{\partial\theta}\,. \end{aligned} \tag{2.33}$$

It is easy to check that the natural S^1-action given on X is transversal and CR. Let E be a holomorphic vector bundle over M. Then the restriction of the pull back $\pi^*E|_X$ on X is an S^1-equivariant CR vector bundle over X.

From now on, let L be an S^1-equivariant CR line bundle over X. We fix an open covering $(U_j)_j$ and a family $(s_j)_j$ of rigid CR frames s_j on U_j. Let L^k be the k-th tensor power of L. Then $(s_j^{\otimes k})_j$ are rigid CR frames for L^k.

The tangential Cauchy-Riemann operator $\overline{\partial}_b : \Omega^{0,q}(X,L^k) \to \Omega^{0,q+1}(X,L^k)$ is well-defined. Since L^k is S^1-equivariant, we can also define Tu for every $u \in \Omega^{0,q}(X,L^k)$ and we have

$$T\overline{\partial}_b = \overline{\partial}_b T \ \text{ on } \Omega^{0,q}(X,L^k). \tag{2.34}$$

For every $m \in \mathbb{Z}$, let

$$\Omega^{0,q}_m(X,L^k) := \left\{u \in \Omega^{0,q}(X,L^k);\, Tu = imu\right\}. \tag{2.35}$$

For $q = 0$, we write $C^\infty_m(X,L^k) := \Omega^{0,0}_m(X,L^k)$.

Let h be an S^1-equivariant Hermitian metric on L. The local weight of h with respect to a local rigid CR frame s of L over an open subset $D \subset X$ is the function $\Phi \in C^\infty(D,\mathbb{R})$ for which

$$|s(x)|^2_{h^L} = e^{-2\Phi(x)}, x \in D. \tag{2.36}$$

We denote by Φ_j the weight of h with respect to s_j.

Definition 2.11. Let L be an S^1-equivariant CR line bundle and let h be an S^1-equivariant Hermitian metric on L. The curvature of (L,h) is the Hermitian quadratic form $R^L = R^{(L,h)}$ on $T^{1,0}X$ defined by

$$R^L_p(U,\overline{V}) = \left\langle d(\overline{\partial}_b\Phi_j - \partial_b\Phi_j)(p), U \wedge \overline{V}\,\right\rangle,\ \ U,V \in T^{1,0}_pX,\ \ p \in U_j. \tag{2.37}$$

Due to [23, Proposition 4.2], R^L is a well-defined global Hermitian form, since the transition functions between different frames s_j are annihilated by T.

Definition 2.12. We say that (L,h) is positive if the associated curvature R^L_x is positive definite at every $x \in X$.

Example 2.13. Let $(E,h) \xrightarrow{\pi} M$ be a Hermitian line bundle over a projective manifold M and let $X = \{v \in E : h(v) = 1\}$ be the circle bundle over M. Let (L,h^L) be a positive line bundle over M. Then the restriction of the pull back $(\pi^*L|_X, \pi^*h^L)$ on X is a positive CR line bundle over X with curvature $\pi^*R^{(L,h^L)}|_{T^{1,0}X}$. Thus all Grauert tubes over projective manifolds admit S^1-equivariant positive CR line bundles.

For the following result we refer to [13, Theorem 2.10].

Theorem 2.14. *On every S^1-equivariant vector bundle F over X there exists an S^1-equivariant Hermitian metric $\langle \cdot | \cdot \rangle_F$.*

Since $T^{1,0}X$ is S^1-equivariant, Theorem 2.14 shows that there is an S^1-equivariant Hermitian metic on $T^{1,0}X$. From now on, we take a S^1-equivariant Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T \,|\, T \rangle = 1$. The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ induces by duality a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0,q)$ forms $T^{*0,q}X, q = 0, 1 \cdots, n-1$. We shall also denote all these induced metrics by $\langle \cdot | \cdot \rangle$. For every $v \in T^{*0,q}X$, we write $|v|^2 := \langle v \,|\, v \rangle$.

The Hermitian metrics on $T^{*0,q}X$ and L induce Hermitian metrics on $T^{*0,q}X \otimes L^k$, $q = 0, 1, \ldots, n$. We shall also denote these induced metrics by $\langle \cdot | \cdot \rangle_{h^k}$. For $f \in \Omega^{0,q}(X, L^k)$, we denote the pointwise norm $|f(x)|^2_{h^k} := \langle f(x) \,|\, f(x) \rangle_{h^k}$. Let $dv_X = dv_X(x)$ the volume form on X induced by the fixed Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$. Then we get natural global L^2 inner products $(\cdot | \cdot)$ on $\Omega^{0,q}(X, L^k)$ and $\Omega^{0,q}(X)$ respectively. We denote by $L^2(X, T^{*0,q}X \otimes L^k)$ and $L^2(X, T^{*0,q}X)$ the completions of $\Omega^{0,q}(X, L^k)$ and $\Omega^{0,q}(X)$ with respect to $(\cdot | \cdot)$.

Similarly, for each $m \in \mathbb{Z}$, we denote by $L^2_m(X, T^{*0,q}X \otimes L^k)$ and $L^2_m(X, T^{*0,q}X)$ the completions of $\Omega^{0,q}_m(X, L^k)$ and $\Omega^{0,q}_m(X)$ with respect to $(\cdot | \cdot)$. We extend $(\cdot | \cdot)$ and $(\cdot | \cdot)$ to $L^2(X, T^{*0,q}X \otimes L^k)$ and $L^2(X, T^{*0,q}X)$ in the standard way. For $f \in \Omega^{0,q}(X, L^k)$ or $f \in \Omega^{0,q}(X)$, we denote $\|f\|^2 := (\, f \,|\, f \,)$.

2.4. **Expression of T and $\overline{\partial}_b$ in BRT trivializations.** In a BRT trivialization $(D, (z, \theta), \phi)$, we have a useful formula for the operator T on $\Omega^{0,q}(X)$ defined by (2.12). It is clear that

$$\{d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}, 1 \leq j_1 < \ldots < j_q \leq n-1\}$$

is a rigid frame of $T^{*0,q}X$ on D so for $u \in \Omega^{0,q}(X)$ we write

$$u = \sum_{j_1 < \ldots < j_q} u_{j_1 \ldots j_q} d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q} \text{ on } D.$$

Then we can check that

$$Tu = \sum_{j_1 < \ldots < j_q} (Tu_{j_1 \ldots j_q}) d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q} \text{ on } D. \tag{2.38}$$

Note that in terms of the BRT trivialization $(D, (z, \theta), \phi)$, we have

$$\overline{\partial}_b = \sum_{j=1}^{n-1} d\overline{z}_j \wedge \Big(\frac{\partial}{\partial \overline{z}_j} - i \frac{\partial \phi}{\partial \overline{z}_j}(z, \overline{z}) \frac{\partial}{\partial \theta} \Big). \tag{2.39}$$

3. SZEGŐ KERNEL ASYMPTOTICS

In this section, we will prove Theorem 1.1. We first introduce some notations. Let

$$\overline{\partial}^*_b : \Omega^{0,q+1}(X, L^k) \to \Omega^{0,q}(X, L^k)$$

be the formal adjoint of $\overline{\partial}_b$ with respect to $(\cdot | \cdot)$. Since $\langle \cdot | \cdot \rangle$ and h are S^1-equivariant, we can check that

$$\begin{aligned} &T\overline{\partial}^*_b = \overline{\partial}^*_b T \text{ on } \Omega^{0,q}(X, L^k),\ q = 1, 2, \ldots, n-1, \\ &\overline{\partial}^*_b : \Omega^{0,q+1}_m(X, L^k) \to \Omega^{0,q}_m(X, L^k), \text{ for all } m \in \mathbb{Z}. \end{aligned} \tag{3.1}$$

Put

$$\Box^{(q)}_{b,k} := \overline{\partial}_b\overline{\partial}^*_b + \overline{\partial}^*_b\overline{\partial}_b : \Omega^{0,q}(X, L^k) \to \Omega^{0,q}(X, L^k). \tag{3.2}$$

From (2.34) and (3.1), we have

$$\begin{aligned} &T\Box^{(q)}_{b,k} = \Box^{(q)}_{b,k}T \ \text{ on } \Omega^{0,q}(X, L^k),\ q = 0, 1, \dots, n-1,\\ &\Box^{(q)}_{b,k} : \Omega^{0,q}_m(X, L^k) \to \Omega^{0,q}_m(X, L^k), \quad \text{for all } m \in \mathbb{Z}. \end{aligned} \tag{3.3}$$

Let $\Pi_k : L^2(X) \to \operatorname{Ker}\Box^{(0)}_{b,k}$ be the orthogonal projection (the Szegő projector).

Definition 3.1. Let $A_k : L^2(X, L^k) \to L^2(X, L^k)$ be a continuous operator. Let $D \Subset X$. We say that $\Box^{(0)}_{b,k}$ has $O(k^{-n_0})$ small spectral gap on D with respect to A_k if for every $D' \Subset D$, there exist $C_{D'} > 0$, $n_0, p \in \mathbb{N}$, $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ and $u \in C^\infty_0(D', L^k)$, we have

$$\|A_k(I - \Pi_k)u\| \leq C_{D'}\, k^{n_0}\sqrt{(\,(\Box^{(0)}_{b,k})^p u\,|\,u\,)}\,.$$

Fix $\lambda > 0$ and let $\Pi_{k,\leq\lambda}$ be as in (1.10).

Definition 3.2. Let $A_k : L^2(X, L^k) \to L^2(X, L^k)$ be a continuous operator. We say that $\Pi_{k,\leq\lambda}$ is k-negligible away the diagonal with respect to A_k on $D \Subset X$ if for any $\chi, \chi_1 \in C^\infty_0(D)$ with $\chi_1 = 1$ on some neighborhood of $\operatorname{Supp}\chi$, we have

$$\big(\chi A_k(1-\chi_1)\big)\Pi_{k,\leq\lambda}\big(\chi A_k(1-\chi_1)\big)^* = O(k^{-\infty}) \quad \text{on } D,$$

where $\big(\chi A_k(1-\chi_1)\big)^* : L^2(X, L^k) \to L^2(X, L^k)$ is the Hilbert space adjoint of $\chi A_k(1-\chi_1)$ with respect to $(\,\cdot\,|\,\cdot\,)$.

Fix $\delta > 0$ and let $F_{k,\delta}$ be as in (1.9).

Theorem 3.3 ([21, Theorem 1.13]). *With the notations and assumptions used above, let s be a local rigid CR frame of L on a canonical coordinate patch $D \Subset X$ with canonical coordinates $x = (z, \theta) = (x_1, \dots, x_{2n-1})$, $|s|^2_h = e^{-2\Phi}$. Let $\delta > 0$ be a small constant so that $R^L_x - 2t\mathcal{L}_x$ is positive definite, for every $x \in X$ and $|t| \leq \delta$. Let $F_{k,\delta}$ be as in* (1.9) *and let $F_{k,\delta,s}$ be the localized operator of $F_{k,\delta}$ given by* (2.4). *Assume that:*

(I) $\Box^{(0)}_{b,k}$ *has $O(k^{-n_0})$ small spectral gap on D with respect to $F_{k,\delta}$.*
(II) $\Pi_{k,\leq\delta k}$ *is k-negligible away the diagonal with respect to $F_{k,\delta}$ on D.*
(III) $F_{k,\delta,s} - B_k = O(k^{-\infty}) : H^r_{\mathrm{comp}}(D) \to H^r_{\mathrm{loc}}(D)$*, for all $r \in \mathbb{N}_0$, where*

$$B_k = \frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik\langle x-y,\eta\rangle}\alpha(x,\eta,k)d\eta + O(k^{-\infty})$$

is a classical semi-classical pseudodifferential operator on D of order 0 with

$$\begin{aligned}&\alpha(x,\eta,k) \sim \textstyle\sum_{j=0}^{\infty}\alpha_j(x,\eta)k^{-j} \text{ in } S^0_{\mathrm{loc}}(1; T^*D),\\ &\alpha_j(x,\eta) \in C^\infty(T^*D), \quad j = 0, 1, \dots,\end{aligned}$$

and for every $(x,\eta) \in T^*D$, $\alpha(x,\eta,k) = 0$ if $\big|\langle\,\eta\,|\,\omega_0(x)\,\rangle\big| > \delta$. Fix $D_0 \Subset D$. Then

$$P_{k,\delta,s}(x,y) = \int e^{ik\varphi(x,y,t)}g(x,y,t,k)dt + O(k^{-\infty}) \ \text{ on } D_0 \times D_0, \tag{3.4}$$

where $\varphi(x,y,t)\in C^\infty(D\times D\times(-\delta,\delta))$ *is as in* (1.17) *and*

$$g(x,y,t,k)\in S^n_{\rm loc}(1;D\times D\times(-\delta,\delta))\cap C^\infty_0(D\times D\times(-\delta,\delta)),$$

$$g(x,y,t,k)\sim\sum_{j=0}^\infty g_j(x,y,t)k^{n-j} \text{ in } S^n_{\rm loc}(1;D\times D\times(-\delta,\delta))$$

is as in (1.18)*, where* $P_{k,\delta,s}$ *is given by* (1.15)*.*

In view of Theorem 3.3, we see that to prove Theorem 1.1, we only need to prove that (I), (II) and (III) in Theorem 3.3 hold if $\delta>0$ is small enough. Until further notice, we fix $\delta>0$ small enough so that $R^L_x-2s\mathcal{L}_x$ is positive definite for every $x\in X$ and $|s|\leq\delta$. We first prove (I) in Theorem 3.3 holds.

3.1. Small spectral gap of the Kohn Laplacian.

For $m\in\mathbb{Z}$, let

$$Q^{(q)}_{m,k}:L^2(X,T^{*0,q}X\otimes L^k)\to L^2_m(X,T^{*0,q}X\otimes L^k) \tag{3.5}$$

be the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)$. Let $\tau_\delta\in C^\infty_0((-\delta,\delta))$ be as in (1.8). Similar to (1.9), let $F^{(q)}_{k,\delta}$ be the continuous operator given by

$$F^{(q)}_{k,\delta}:L^2(X,T^{*0,q}X\otimes L^k)\to L^2(X,T^{*0,q}X\otimes L^k),\quad u\longmapsto\sum_{m\in\mathbb{Z}}\tau_\delta\Big(\frac{m}{k}\Big)Q^{(q)}_{m,k}u\,. \tag{3.6}$$

Note that $F_{k,\delta}=F^{(0)}_{k,\delta}$. It is not difficult to see that for every $m\in\mathbb{Z}$, we have

$$\begin{aligned}&\left\|TQ^{(q)}_{m,k}u\right\|=|m|\left\|Q^{(q)}_{m,k}u\right\|,\quad\text{for all } u\in L^2(X,T^{*0,q}X\otimes L^k),\\&\left\|TF^{(q)}_{k,\delta}u\right\|\leq k\delta\left\|F^{(q)}_{k,\delta}u\right\|,\quad\text{for all } u\in L^2(X,T^{*0,q}X\otimes L^k),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&Q^{(q)}_{m,k}:\Omega^{0,q}(X,L^k)\to\Omega^{0,q}_m(X,L^k),\\&F^{(q)}_{k,\delta}:\Omega^{0,q}(X,L^k)\to\bigoplus_{m\in\mathbb{Z}\cap[-k\delta,k\delta]}\Omega^{0,q}_m(X,L^k).\end{aligned} \tag{3.8}$$

Since the Hermitian metrics $\langle\,\cdot\,|\,\cdot\,\rangle$ and h^k are all rigid, it follows as in [20, Section 5]:

$$\begin{aligned}&\Box^{(q)}_{b,k}Q^{(q)}_{m,k}=Q^{(q)}_{m,k}\Box^{(q)}_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\ \text{for all } m\in\mathbb{Z},\\&\Box^{(q)}_{b,k}F^{(q)}_{k,\delta}=F^{(q)}_{k,\delta}\Box^{(q)}_{b,k}\text{ on }\Omega^{0,q}(X,L^k),\\&\overline\partial_bQ^{(q)}_{m,k}=Q^{(q+1)}_{m,k}\overline\partial_b\text{ on }\Omega^{0,q}(X,L^k),\ \text{for all } m\in\mathbb{Z},\ q=0,1,\ldots,n-2,\\&\overline\partial_bF^{(q)}_{k,\delta}=F^{(q+1)}_{k,\delta}\overline\partial_b\text{ on }\Omega^{0,q}(X,L^k),\ q=0,1,\ldots,n-2,\\&\overline\partial^*_bQ^{(q)}_{m,k}=Q^{(q-1)}_{m,k}\overline\partial^*_b\text{ on }\Omega^{0,q}(X,L^k),\ \text{for all } m\in\mathbb{Z},\ q=1,\ldots,n-1,\\&\overline\partial^*_bF^{(q)}_{k,\delta}=F^{(q-1)}_{k,\delta}\overline\partial^*_b\text{ on }\Omega^{0,q}(X,L^k),\ q=1,\ldots,n-1.\end{aligned} \tag{3.9}$$

By elementary Fourier analysis, it is straightforward to see that for every $u\in\Omega^{0,q}(X,L^k)$,

$$\begin{aligned}&\lim_{N\to\infty}\sum_{m=-N}^NQ^{(q)}_{m,k}u\to u\ \text{ in } C^\infty\text{ topology},\\&\sum_{m=-N}^N\left\|Q^{(q)}_{m,k}u\right\|^2\leq\|u\|^2,\quad\text{for all } N\in\mathbb{N}_0.\end{aligned} \tag{3.10}$$

Thus, for every $u \in L^2(X, T^{*0,q}X \otimes L^k)$,

$$\begin{aligned} &\lim_{N\to\infty} \sum_{m=-N}^{N} Q^{(q)}_{m,k} u \to u \text{ in } L^2(X, T^{*0,q}X \otimes L^k), \\ &\sum_{m=-N}^{N} \left\| Q^{(q)}_{m,k} u \right\|^2 \leq \|u\|^2, \quad \text{for all } N \in \mathbb{N}_0. \end{aligned} \tag{3.11}$$

We will use the following result.

Theorem 3.4 ([21, Theorem 9.4]). *With the assumptions and notations above, let $q \geq 1$. If $\delta > 0$ is small enough, then for every $u \in \Omega^{0,q}(X, L^k)$, we have*

$$\big\|\Box^{(q)}_{b,k} F^{(q)}_{k,\delta} u\big\|^2 \geq c_1 k^2 \big\|F^{(q)}_{k,\delta} u\big\|^2, \tag{3.12}$$

where $c_1 > 0$ is a constant independent of k and u.

Now, we assume that $\delta > 0$ is small enough so that (3.12) holds. For $q = 0, 1, \ldots, n-1$, put

$$\begin{aligned} &\Omega^{0,q}_{\leq k\delta}(X, L^k) := \bigoplus_{\substack{m\in\mathbb{Z} \\ |m|\leq k\delta}} \Omega^{0,q}_m(X, L^k), \\ &L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k) := \bigoplus_{\substack{m\in\mathbb{Z} \\ |m|\leq k\delta}} L^2_m(X, T^{*0,q}X \otimes L^k). \end{aligned} \tag{3.13}$$

We write $C^\infty_{\leq k\delta}(X, L^k) := \Omega^{0,0}_{\leq k\delta}(X, L^k)$, $L^2_{\leq k\delta}(X, L^k) := L^2_{\leq k\delta}(X, T^{*0,0}X \otimes L^k)$. It is clear that

$$\Box^{(q)}_{b,k} : \Omega^{0,q}_{\leq k\delta}(X, T^{*0,q}X \otimes L^k) \to \Omega^{0,q}_{\leq k\delta}(X, T^{*0,q}X \otimes L^k).$$

We will denote by $\Box^{(q)}_{b,\leq k\delta}$ the restriction of $\Box^{(q)}_{b,k}$ to the space $\Omega^{0,q}_{\leq k\delta}(X, L^k)$. We extend $\Box^{(q)}_{b,\leq k\delta}$ to $L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k)$ by

$$\Box^{(q)}_{b,\leq k\delta} : \mathrm{Dom}\, \Box^{(q)}_{b,\leq k\delta} \subset L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k) \to L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k), \tag{3.14}$$

with $\mathrm{Dom}\, \Box^{(q)}_{b,\leq k\delta} := \{u \in L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k);\ \Box^{(q)}_{b,\leq k\delta} u \in L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k)\}$, where for any $u \in L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k)$, $\Box^{(q)}_{b,\leq k\delta} u$ is defined in the sense of distributions.

In general, the Kohn Laplacian may not be subelliptic. If the CR manifold admits a transversal CR S^1-action, then the Kohn Laplacian is in fact transversal elliptic in the sense of Atiyah [3].

Lemma 3.5. *We have* $\mathrm{Dom}\, \Box^{(q)}_{b,\leq k\delta} = L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k) \bigcap H^2(X, T^{*0,q}X \otimes L^k)$.

Proof. It is clearly that $L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k) \bigcap H^2(X, T^{*0,q}X \otimes L^k) \subset \mathrm{Dom}\, \Box^{(q)}_{b,\leq k\delta}$. We only need to prove that $\mathrm{Dom}\, \Box^{(q)}_{b,\leq k\delta} \subset L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k) \bigcap H^2(X, T^{*0,q}X \otimes L^k)$. Let $u \in \mathrm{Dom}\, \Box^{(q)}_{b,\leq k\delta}$. Put $v = \Box^{(q)}_{b,\leq k\delta} u \in L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k)$. We have $(\Box^{(q)}_{b,\leq k\delta} - T^2)u = v - T^2 u \in L^2(X, T^{*0,q}X \otimes L^k)$ since $\|T^2 u\| \leq k^2\delta^2 \|u\|$. Since $(\Box^{(q)}_{b,\leq k\delta} - T^2)$ is elliptic, we have $u \in H^2(X, T^{*0,q}X \otimes L^k)$. The lemma follows. ∎

Theorem 3.6. *The operator $\Box^{(q)}_{b,\leq k\delta}$ defined in* (3.14) *is self-adjoint.*

Proof. Let $(\Box^{(q)}_{b,\leq k\delta})^* : \mathrm{Dom}\, (\Box^{(q)}_{b,\leq k\delta})^* \subset L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k) \to L^2_{\leq k\delta}(X, T^{*0,q}X \otimes L^k)$ be the Hilbert space adjoint of $\Box^{(q)}_{b,\leq k\delta}$. Let $v \in \mathrm{Dom}\, (\Box^{(q)}_{b,\leq k\delta})^*$. Then, by definition of the Hilbert space adjoint of $\Box^{(q)}_{b,\leq k\delta}$, it is easy to see that $\Box^{(q)}_{b,\leq k\delta}v\in L^2_{\leq k\delta}(X,T^{*0,q}X\otimes L^k)$ and hence $v\in\operatorname{Dom}\Box^{(q)}_{b,\leq k\delta}$ and $\Box^{(q)}_{b,\leq k\delta}v=(\Box^{(q)}_{b,\leq k\delta})^*v$.

From Lemma 3.5, we can check that

$$(\Box^{(q)}_{b,\leq k\delta}g\,|\,f\,)=(\,g\,|\,\Box^{(q)}_{b,\leq k\delta}f\,),\quad\text{for all } g,f\in\operatorname{Dom}\Box^{(q)}_{b,\leq k\delta}. \tag{3.15}$$

From (3.15), we deduce that $\operatorname{Dom}\Box^{(q)}_{b,\leq k\delta}\subset\operatorname{Dom}(\Box^{(q)}_{b,\leq k\delta})^*$ and $\Box^{(q)}_{b,\leq k\delta}u=(\Box^{(q)}_{b,\leq k\delta})^*u$, for all $u\in\operatorname{Dom}\Box^{(q)}_{b,\leq k\delta}$. The theorem follows. □

Let $\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}\subset[0,\infty[$ denote the spectrum of $\Box^{(q)}_{b,\leq k\delta}$. For any $\lambda>0$, put

$$\Pi^{(q)}_{k,\leq k\delta,\leq\lambda}:=E^{(q)}_{\leq k\delta}([0,\lambda]),\quad \Pi^{(q)}_{k,\leq k\delta,>\lambda}:=E^{(q)}_{\leq k\delta}(]\lambda,\infty[),$$

where $E^{(q)}_{\leq k\delta}$ denotes the spectral measure for $\Box^{(q)}_{b,\leq k\delta}$. We set

$$\Pi_{k,\leq k\delta,\leq\lambda}:=\Pi^{(0)}_{k,\leq k\delta,\leq\lambda},\quad \Pi_{k,\leq k\delta,>\lambda}:=\Pi^{(0)}_{k,\leq k\delta,>\lambda}\,.$$

Theorem 3.7. $\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}$ *is a discrete subset of* $[0,\infty[$ *, for any* $\nu\in\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}$*,* ν *is an eigenvalue of* $\Box^{(q)}_{b,\leq k\delta}$ *and the eigenspace*

$$\mathcal{E}^q_{\leq k\delta,\nu}(X,L^k):=\big\{u\in\operatorname{Dom}\Box^{(q)}_{b,\leq k\delta};\ \Box^{(q)}_{b,\leq k\delta}u=\nu u\big\}$$

is finite dimensional with $\mathcal{E}^q_{\leq k\delta,\nu}(X,L^k)\subset\Omega^{0,q}_{\leq k\delta}(X,L^k)$*.*

Proof. Fix $\lambda>0$. We claim that $\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}\bigcap[0,\lambda]$ is discrete. If not, we can find an orthonormal system $\{f_j\in\operatorname{Range}E^{(q)}_{\leq k\delta}([0,\lambda]);j\in\mathbb{N}\}$, i. e. $(\,f_j\,|\,f_\ell\,)=\delta_{j,\ell}$ for all $j,\ell\in\mathbb{N}$. Note that

$$\big\|\Box^{(q)}_{b,\leq k\delta}f_j\big\|\leq\lambda\big\|f_j\big\|,\quad j=1,2,\ldots. \tag{3.16}$$

From (3.16), we have

$$\big\|(\Box^{(q)}_{b,\leq k\delta}-T^2)f_j\big\|\leq(\lambda+k^2\delta^2)\big\|f_j\big\|,\quad j=1,2,\ldots. \tag{3.17}$$

Since $\Box^{(q)}_{b,\leq k\delta}-T^2$ is a second order elliptic operator, there is $C_\delta>0$ independent of j such that

$$\|f_j\|_2\leq C_\delta,\quad j=1,2,\ldots, \tag{3.18}$$

where $\|\cdot\|_2$ denotes the usual Sobolev norm of order 2. From (3.18), we can apply Rellich's theorem and find subsequence $\{f_{j_s}\}^\infty_{s=1}$, such that $f_{j_s}\to f$ in $L^2_{\leq k\delta}(X,T^{*0,q}X\otimes L^k)$. This is a contradiction to the fact that $\{f_j;j\in\mathbb{N}\}$ is orthonormal. Thus, $\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}\bigcap[0,\lambda]$ is discrete and therefore $\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}$ is a discrete subset of $[0,\infty[$.

Let $r\in\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}$. Since $\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}$ is discrete, $\Box^{(q)}_{b,\leq k\delta}-r$ has L^2 closed range. If $\Box^{(q)}_{b,\leq k\delta}-r$ is injective, then $\operatorname{Range}(\Box^{(q)}_{b,\leq k\delta}-r)=L^2_{\leq k\delta}(X,T^{*0,q}X\otimes L^k)$ and

$$(\Box^{(q)}_{b,\leq k\delta}-r)^{-1}:L^2_{\leq k\delta}(X,T^{*0,q}X\otimes L^k)\to L^2_{\leq k\delta}(X,T^{*0,q}X\otimes L^k)$$

is continuous. We get a contradiction. Hence r is an eigenvalue of $\Box^{(q)}_{b,\leq k\delta}$.

For any $\nu\in\operatorname{Spec}\Box^{(q)}_{b,\leq k\delta}$, put

$$\mathcal{E}^q_{\leq k\delta,\nu}(X,L^k):=\big\{u\in\operatorname{Dom}\Box^{(q)}_{b,\leq k\delta};\ \Box^{(q)}_{b,\leq k\delta}u=\nu u\big\}.$$

We can repeat the argument before and conclude that $\mathcal{E}^q_{\leq k\delta,\nu}(X,E)$ is finite dimensional. Let $u\in\mathcal{E}^q_{\leq k\delta,\nu}(X,L^k)$. Then, $\Box^{(q)}_{b,\leq k\delta}u=\nu u$. For $m\in\mathbb{Z}$, put $u_m:=Q^{(q)}_{m,k}u\in L^2_m(X,T^{*0,q}X\otimes L^k)$. We have $u=\sum_{m\in\mathbb{Z},|m|\leq k\delta}u_m$. We can check that

$$\Box^{(q)}_{b,k}u_m=\nu u_m,\quad\text{for all } m\in\mathbb{Z}.$$

Hence

$$(\Box^{(q)}_{b,k}-T^2)u_m=(\nu+m^2)u_m,\quad\text{for all } m\in\mathbb{Z}. \tag{3.19}$$

From (3.19), we can apply some standard argument in partial differential operator and deduce that $u_m\in\Omega^{0,q}_m(X,L^k)$. Thus, $u\in\Omega^{0,q}_{\leq k\delta}(X,L^k)$ and hence $\mathcal{E}^q_{\leq k\delta,\nu}(X,L^k)\subset\Omega^{0,q}_{\leq k\delta}(X,L^k)$. The theorem follows. $\Box$

For every $\mu\in\operatorname{Spec}\Box^{(0)}_{b,\leq k\delta}$ let

$$\Pi_{k,\leq k\delta,\mu}:L^2(X,L^k)\to\mathcal{E}^0_{\leq k\delta,\mu}(X,L^k)$$

be the orthogonal projection. For $\mu=0$, it is clearly that $\Pi_{k,\leq k\delta,0}=\Pi_{k,\leq k\delta}$, where $\Pi_{k,\leq k\delta}$ is given by (1.10). We have

Theorem 3.8. *With the assumptions and notations above, if $\epsilon_0>0$ is small enough, then for every $u\in C^\infty(X,L^k)$, we have*

$$F_{k,\delta}\Pi_{k,\leq k\delta,\mu}u=0,\quad\text{for all } \mu\in\operatorname{Spec}\Box^{(0)}_{b,\leq k\delta},\ 0<\mu\leq k\epsilon_0, \tag{3.20}$$

and

$$\big\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})u\big\|\leq\frac{1}{k\epsilon_0}\big\|\Box^{(0)}_{b,k}u\big\|. \tag{3.21}$$

Proof. Let $\epsilon_0>0$ be a small constant. For $u\in L^2_{\leq k\delta}(X,L^k)$, we have

$$(I-\Pi_{k,\leq k\delta})u=\sum_{\substack{\mu\in\operatorname{Spec}\Box^{(0)}_{b,\leq k\delta}\\ 0<\mu\leq k\epsilon_0}}\Pi_{k,\leq k\delta,\mu}u+\Pi_{k,\leq k\delta,>k\epsilon_0}u. \tag{3.22}$$

We claim that for every $\mu\in\operatorname{Spec}\Box^{(0)}_{b,\leq k\delta}, 0<\mu\leq k\epsilon_0$ and every $u\in C^\infty(X,L^k)$,

$$F_{k,\delta}\Pi_{k,\leq k\delta,\mu}u=0 \tag{3.23}$$

if $\epsilon_0>0$ is small enough. Fix $\mu\in\operatorname{Spec}\Box^{(0)}_{b,\leq k\delta}\bigcap(0,k\delta]$ and $u\in C^\infty(X,L^k)$. From (3.9) and (3.12), we have

$$\Big\|\Box^{(1)}_{b,k}F^{(1)}_{k,\delta}\overline{\partial}_b\Pi_{k,\leq k\delta,\mu}u\Big\|^2\geq c_1k^2\Big\|F^{(1)}_{k,\delta}\overline{\partial}_b\Pi_{k,\leq k\delta,\mu}u\Big\|^2, \tag{3.24}$$

where $c_1>0$ is a constant independent of k and u. It is easy to see that

$$\Box^{(1)}_{b,k}F^{(1)}_{k,\delta}\overline{\partial}_b\Pi_{k,\leq k\delta,\mu}u=\mu F^{(1)}_{k,\delta}\overline{\partial}_b\Pi_{k,\leq k\delta,\mu}u.$$

Thus,

$$\Big\|\Box^{(1)}_{b,k}F^{(1)}_{k,\delta}\overline{\partial}_b\Pi_{k,\leq k\delta,\mu}u\Big\|^2\leq k^2\epsilon_0^2\Big\|F^{(1)}_{k,\delta}\overline{\partial}_b\Pi_{k,\leq k\delta,\mu}u\Big\|^2. \tag{3.25}$$

From (3.24) and (3.25), we conclude that if $\epsilon_0>0$ is small enough then

$$F^{(1)}_{k,\delta}\overline{\partial}_b\Pi_{k,\leq k\delta,\mu}u=\overline{\partial}_bF_{k,\delta}\Pi_{k,\leq k\delta,\mu}u=0.$$

Hence,

$$F_{k,\delta}\Pi_{k,\leq k\delta,\mu}u=\frac{1}{\mu}\Box^{(0)}_{b,k}F_{k,\delta}\Pi_{k,\leq k\delta,\mu}u=0. \tag{3.26}$$

From (3.26), the claim (3.23) follows. We get (3.20).

Let $Q^{(0)}_{\leq k\delta}:L^2(X,L^k)\to L^2_{\leq k\delta}(X,L^k)$ be the orthogonal projection. From (3.22) and (3.23), if $\epsilon_0>0$ is small enough, then

$$\begin{aligned}
&\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})u)\|\\
&=\left\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})(Q^{(0)}_{\leq k\delta}u)\right\|\\
&=\left\|F_{k,\delta}\Pi_{k,\leq k\delta,>k\epsilon_0}(Q^{(0)}_{\leq k\delta}u)\right\|\\
&\leq\left\|\Pi_{k,\leq k\delta,>k\epsilon_0}(Q^{(0)}_{\leq k\delta}u)\right\|\\
&\leq\frac{1}{k\epsilon_0}\left\|\Box^{(0)}_{b,k}\Pi^{(0)}_{k,\leq k\delta,>k\epsilon_0}(Q^{(0)}_{\leq k\delta}u)\right\|\\
&=\frac{1}{k\epsilon_0}\left\|\Pi^{(0)}_{k,\leq k\delta,>k\epsilon_0}\Box^{(0)}_{b,k}(Q^{(0)}_{\leq k\delta}u)\right\|\leq\frac{1}{k\epsilon_0}\left\|\Box^{(0)}_{b,k}u\right\|,
\end{aligned} \tag{3.27}$$

for every $u\in C^\infty(X,L^k)$. From (3.27), (3.21) follows. □

Theorem 3.9. *$\Box^{(0)}_{b,k}$ has a $O(k^{-n_0})$ small spectral gap on X with respect to $F_{k,\delta}$.*

Proof. Let $u\in C^\infty(X,L^k)$. It is easy to see that

$$F_{k,\delta}(I-\Pi_k)u=F_{k,\delta}(I-\Pi_{k,\leq k\delta})u. \tag{3.28}$$

From (3.28) and (3.21), the theorem follows. □

3.2. **The weighted projector $F_{k,\delta}$ on a canonical coordinate patch.** Let $D\subset X$ be a canonical coordinate patch and let $x=(x_1,\ldots,x_{2n-1})$ be canonical coordinates on D. We identify D with $W\times]-\varepsilon,\varepsilon[\subset\mathbb{R}^{2n-1}$, where W is some open set in $\mathbb{R}^{2n-2}$ and $\varepsilon>0$. Until further notice, we work with canonical coordinates $x=(x_1,\ldots,x_{2n-1})$. Let $\eta=(\eta_1,\ldots,\eta_{2n-1})$ be the dual coordinates of x. Let s be a local rigid CR frame of L on D, $|s|^2_h=e^{-2\Phi}$. Let $F_{k,\delta,s}$ be the localized operator of $F_{k,\delta}$ constructed by (2.4). Put

$$B_k=\frac{k^{2n-1}}{(2\pi)^{2n-1}}\int e^{ik\langle x-y,\eta\rangle}\tau_\delta(\eta_{2n-1})d\eta. \tag{3.29}$$

Lemma 3.10. *We have*

$$F_{k,\delta,s}-B_k=O(k^{-\infty}):H^r_{\rm comp}(D)\to H^r_{\rm loc}(D),\quad\text{for all } r\in\mathbb{N}_0.$$

Proof. We also write $y=(y_1,\ldots,y_{2n-1})$ to denote the canonical coordinates x. We claim that on D,

$$F_{k,\delta,s}u(y)=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\tau_\delta\Big(\frac{m}{k}\Big)e^{imy_{2n-1}}\int_{-\pi}^{\pi}e^{-imt}u(e^{it}y')dt,\quad\text{for all } u\in C^\infty_0(D), \tag{3.30}$$

where $y'=(y_1,\ldots,y_{2n-2},0)$ and for convenience we just write $y'=(y_1,\ldots,y_{2n-2})$ if there is no confusion. Note that if $|t|\leq\varepsilon$, then $e^{it}y'=(y',t)$. Let $u\in C^\infty_0(D)$. By (2.13) and the definition of $F_{k,\delta,s}$ we have

$$
\begin{aligned}
u(y) &= \sum_{m\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imt}u(e^{it}y)dt \ \text{ on } D,\\
(F_{k,\delta,s}u)(y) &= \sum_{m\in\mathbb{Z}} \tau\Big(\frac{m}{k}\Big)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imt}u(e^{it}y)dt \ \text{ on } D.
\end{aligned}
\tag{3.31}
$$

Fix $m\in\mathbb{Z}$. Since $T=\frac{\partial}{\partial y_{2n-1}}$ on D, we have

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imt}u(e^{it}y)dt = e^{imy_{2n-1}}u_m(y') \ \text{ on } D,
\tag{3.32}
$$

where $u_m(y')\in C^\infty(D)$ is independent of y_{2n-1}. Taking $y_{2n-1}=0$ in (3.32), we get

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imt}u(e^{it}y')dt = u_m(y').
\tag{3.33}
$$

From (3.32) and (3.33), we have

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imt}u(e^{it}y)dt = e^{imy_{2n-1}}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-imt}u(e^{it}y')dt \ \text{ on } D.
\tag{3.34}
$$

From (3.34) and (3.31), we get the claim and also the formula

$$
u(y) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{imy_{2n-1}}\int_{-\pi}^{\pi} e^{-imt}u(e^{it}y')dt \ \text{ on } D.
\tag{3.35}
$$

Fix $D'\Subset D$ and let $\chi(y_{2n-1})\in C_0^\infty(]-\varepsilon,\varepsilon[)$ such that $\chi(y_{2n-1})=1$ for every $(y',y_{2n-1})\in D'$. Let $R_k: C_0^\infty(D')\to C^\infty(D')$ be the continuous operator given by

$$
\begin{aligned}
&(2\pi)^2R_ku=\\
&\sum_{m\in\mathbb{Z}}\int_{|t|\leq\pi} e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}\rangle+im(y_{2n-1}-t)}\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)(1-\chi(y_{2n-1}))u(e^{it}x')dtd\eta_{2n-1}dy_{2n-1}.
\end{aligned}
\tag{3.36}
$$

By using integration by parts with respect to η_{2n-1}, it is easy to see that the integral (3.36) is well-defined. Moreover, we can integrate by parts with respect to η_{2n-1} and y_{2n-1} several times and conclude that

$$
R_k = O(k^{-\infty}): H^r_{\rm comp}(D')\to H^r_{\rm loc}(D'), \ \text{ for all } r\in\mathbb{N}_0.
\tag{3.37}
$$

Now, we claim that

$$
B_k+R_k = F_{k,\delta,s} \ \text{ on } C_0^\infty(D').
\tag{3.38}
$$

Let $u \in C_0^\infty(D')$. From (3.29) and the Fourier inversion formula, we have

$$\begin{aligned}
B_k u(x) &= \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int e^{ik\langle x-y,\eta\rangle} \tau_\delta(\eta_{2n-1}) u(y) dy d\eta \\
&= \frac{k^{2n-1}}{(2\pi)^{2n-1}} \int \Big(\int e^{ik\langle x'-y',\eta'\rangle} u(y', y_{2n-1}) dy' d\eta' \Big) e^{ik\langle x_{2n-1}-y_{2n-1},\eta\rangle} \tau_\delta(\eta_{2n-1}) dy_{2n-1} d\eta \\
&= \frac{k}{2\pi} \int e^{ik\langle x_{2n-1}-y_{2n-1},\eta\rangle} u(x', y_{2n-1}) \tau_\delta(\eta_{2n-1}) dy_{2n-1} d\eta_{2n-1} \\
&= \frac{1}{2\pi} \int e^{i\langle x_{2n-1}-y_{2n-1},\eta\rangle} u(x', y_{2n-1}) \tau_\delta(\frac{\eta_{2n-1}}{k}) dy_{2n-1} d\eta_{2n-1} \\
&= \frac{1}{2\pi} \int e^{i\langle x_{2n-1}-y_{2n-1},\eta\rangle} u(x', y_{2n-1}) \chi(y_{2n-1}) \tau_\delta(\frac{\eta_{2n-1}}{k}) dy_{2n-1} d\eta_{2n-1},
\end{aligned} \tag{3.39}$$

where $\eta' = (\eta_1, \ldots, \eta_{2n-2})$, $d\eta' = d\eta_1 \cdots d\eta_{2n-2}$, $dy' = dy_1 \cdots dy_{2n-2}$. From (3.35) and (3.39), we get

(3.40)
$$B_k u(x) = \frac{1}{(2\pi)^2} \sum_{m\in\mathbb{Z}} \int_{|t|\leq\pi} e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}\rangle} \tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big) \chi(y_{2n-1}) e^{im(y_{2n-1}-t)} u(e^{it}x') dt d\eta_{2n-1} dy_{2n-1}.$$

From (3.40) and (3.36), we have

$$\begin{aligned}
&(B_k + R_k)u(x) \\
&= \frac{1}{(2\pi)^2} \sum_{m\in\mathbb{Z}} \int_{|t|\leq\pi} e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}\rangle} \tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big) e^{imy_{2n-1}} e^{-imt} u(e^{it}x') dt d\eta_{2n-1} dy_{2n-1}.
\end{aligned} \tag{3.41}$$

Note that the following formula holds for every $m \in \mathbb{Z}$,

$$\int e^{imy_{2n-1}} e^{-iy_{2n-1}\eta_{2n-1}} dy_{2n-1} = 2\pi \delta_m(\eta_{2n-1}), \tag{3.42}$$

where the integral is defined as an oscillatory integral and δ_m is the Dirac measure at m. Using (3.30), (3.42) and the Fourier inversion formula, (3.41) becomes

$$(B_k + R_k)u(x) = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \tau_\delta\Big(\frac{m}{k}\Big) e^{ix_{2n-1}m} \int_{|t|\leq\pi} e^{-imt} u(e^{it}x') dt = F_{k,\delta,s} u(x). \tag{3.43}$$

From (3.43), the claim (3.38) follows. From (3.38) and (3.37), the lemma follows. □

From Lemma 3.10, we see that the condition (III) in Theorem 3.3 holds.

Lemma 3.11. *Let $D \subset X$ be a canonical coordinate patch of X. Then, $\Pi_{k,\leq k\delta}$ is k-negligible away the diagonal with respect to $F_{k,\delta}$ on D.*

Proof. Let $\chi, \chi_1 \in C_0^\infty(D)$, $\chi_1 = 1$ on some neighbourhood of $\operatorname{Supp} \chi$. Let $u \in \mathcal{H}^0_{b,\leq k\delta}(X, L^k)$ with $\|u\| = 1$. By [22, Theorem 2.4]) there exists $C > 0$ independent of k and u such that

$$|u(x)|^2_{h^k} \leq Ck^n, \quad \text{for all } x \in X. \tag{3.44}$$

Let $x=(x_1,\ldots,x_{2n-1})=(x',x_{2n-1})$ be canonical coordinates on D. Put $v=(1-\chi_1)u$. It is straightforward to see that on D,

$$\begin{split}&(2\pi)^2\chi F_{k,\delta}(1-\chi_1)u(x)\\ &=\sum_{\substack{m\in\mathbb{Z}\\ |m|\leq 2k\delta}}\int_{|t|\leq\pi}e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}\rangle}\chi(x)\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)e^{im(y_{2n-1}-t)}v(e^{it}x')dt\,d\eta_{2n-1}\,dy_{2n-1}.\end{split}\tag{3.45}$$

Let $\varepsilon>0$ be a small constant so that for every $(x_1,\ldots,x_{2n-1})\in\operatorname{Supp}\chi$, we have

$$(x_1,\ldots,x_{2n-2},y_{2n-1})\in\{x\in D;\ \chi_1(x)=1\}\,,\ \text{ for all } |y_{2n-1}-x_{2n-1}|<\varepsilon.\tag{3.46}$$

Let $\psi\in C^\infty_0((-1,1))$, $\psi=1$ on $\big[-\frac{1}{2},\frac{1}{2}\big]$. Put

$$\begin{split}I_0(x)=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z},|m|\leq 2k\delta}\int_{|t|\leq\pi}e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}\rangle}\Big(1-\psi\Big(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon}\Big)\Big)\chi(x)\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)e^{imy_{2n-1}}\\ \times e^{-imt}v(e^{it}x')dtd\eta_{2n-1}dy_{2n-1},\end{split}\tag{3.47}$$

$$\begin{split}I_1(x)=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{|t|\leq\pi}e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}>}\psi\Big(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon}\Big)\chi(x)\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)e^{imy_{2n-1}}\\ \times e^{-imt}v(e^{it}x')dtd\eta_{2n-1}dy_{2n-1},\end{split}\tag{3.48}$$

and

$$\begin{split}I_2(x)=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z},|m|>2k\delta}\int_{|t|\leq\pi}e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}\rangle}\psi\Big(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon}\Big)\chi(x)\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)e^{imy_{2n-1}}\\ \times e^{-imt}v(e^{it}x')dtd\eta_{2n-1}dy_{2n-1}.\end{split}\tag{3.49}$$

It is clear that on D,

$$\chi F_{k,\delta}(1-\chi_1)u(x)=I_0(x)+I_1(x)-I_2(x).\tag{3.50}$$

By using integration by parts with respect to η_{2n-1} several times and (3.44), we conclude that for every $N>0$ and $m\in\mathbb{N}$, there is $C_{N,m}>0$ independent of u and k such that

$$\|I_0(x)\|_{C^m(D)}\leq C_{N,m}k^{-N}.\tag{3.51}$$

Similarly, by using integration by parts with respect to y_{2n-1} several times and (3.44), we conclude that for every $N>0$ and $m\in\mathbb{N}$, there is $\widetilde{C}_{N,m}>0$ independent of u and k such that

$$\|I_2(x)\|_{C^m(D)}\leq \widetilde{C}_{N,m}k^{-N}.\tag{3.52}$$

Again, from (3.35), we can check that

$$I_1(x)=\frac{1}{2\pi}\int e^{i\langle x_{2n-1}-y_{2n-1},\eta_{2n-1}\rangle}\psi\Big(\frac{x_{2n-1}-y_{2n-1}}{\varepsilon}\Big)\chi(x)\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)v(x',y_{2n-1})d\eta_{2n-1}dy_{2n-1}.\tag{3.53}$$

From (3.46) and (3.53), we deduce that

$$I_1(x)=0 \text{ on } D.\tag{3.54}$$

From (3.50), (3.51), (3.52) and (3.54), we conclude that for every $N > 0$ and $m \in \mathbb{N}$, there is a $\hat{C}_{N,m} > 0$ independent of u and k such that

$$\|\chi F_{k,\delta}(1-\chi_1)u(x)\|_{C^m(D)} \leq \hat{C}_{N,m} k^{-N}. \tag{3.55}$$

From (3.44) and (3.55), it is not difficult to see that

$$\sum_{j=1}^{d_k} |\chi F_{k,\delta}(1-\chi_1)f_j(x)|^2_{h^k} = O(k^{-\infty}) \text{ on } D, \tag{3.56}$$

where $\{f_1, \ldots, f_{d_k}\}$ is an orthonormal basis for $\mathcal{H}^0_{b,\leq k\delta}(X, L^k)$. From (3.56), the lemma follows. □

From Theorem 3.9, Lemma 3.10 and Lemma 3.11, we see that the conditions (I) , (II) and (III) in Theorem 3.3 holds. The proof of Theorem 1.1 is completed.

Proof of Corollary 1.2. We use the same notations as in Theorem 1.1. On the diagonal $x = y$, by (1.17) we have $\varphi(x, x, t) = 0$. From (1.16) we have

$$P_{k,\delta}(x) = P_{k,\delta,s}(x, x) = \int_{\mathbb{R}} g(x, x, t, k)dt + O(k^{-\infty}) \text{ on } D_0. \tag{3.57}$$

Recall that $g(x, x, t, k) = 0$ if $|t| > \delta$ and hence $\int_{\mathbb{R}} g(x, x, t, k)dt = \int_{-\delta}^{\delta} g(x, x, t, k)dt$. Combining with (1.18), there exist $b_j(x) \in C^\infty(D_0)$, $j \in \mathbb{N}_0$, such that

$$P_{k,\delta}(x) = P_{k,\delta,s}(x, x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j(x) \text{ in } S^n_{\rm loc}(1; D_0). \tag{3.58}$$

Let D_1 be another canonical coordinate neighborhood and s_1 be another local rigid CR frame of L on D_1. Then from Theorem 1.1 and the above argument, on D_1 we have

$$P_{k,\delta}(x) = P_{k,\delta,s_1}(x, x) \sim \sum_{j=0}^{\infty} k^{n-j} \hat{b}_j(x) \text{ in } S^n_{\rm loc}(1; D_1), \tag{3.59}$$

where $\hat{b}_j(x) \in C^\infty(D_1)$, $j \in \mathbb{N}_0$. Since on $D \cap D_1$, we have $P_{k,\delta,s}(x, x) = P_{k,\delta,s_1}(x, x) = P_{k,\delta}(x)$, (3.58) and (3.59) yield $b_j(x) = \hat{b}_j(x)$ on $D \cap D_1$, for all $j \in \mathbb{N}_0$. Hence, $b_j(x) \in C^\infty(X)$, for all $j \in \mathbb{N}_0$, and we get the conclusion of Corollary 1.2. □

3.3. **Properties of the phase function.** In this section, we collect some properties of the phase φ in Theorem 1.1. We will use the same notations as in Theorem 1.1.

In view of (1.17), we see that $\operatorname{Im}\varphi(x, y, t) \geq 0$. Moreover, we can estimate $\operatorname{Im}\varphi(x, y, t)$ in some local coordinates.

Theorem 3.12. *With the assumptions and notations used in Theorem 1.1, fix $p \in D$. We take canonical coordinates $x = (x_1, \ldots, x_{2n-1})$ defined in a small neighbourhood of p so that $x(p) = 0$, $\omega_0(p) = -dx_{2n-1}$ and $T^{1,0}_p X \oplus T^{0,1}_p X = \left\{\sum_{j=1}^{2n-2} a_j \frac{\partial}{\partial x_j};\ a_j \in \mathbb{C}, j = 1, \ldots, 2n-2\right\}$. If D is small enough, then there is $c > 0$ such that for $(x, y, t) \in D \times D \times (-\delta, \delta)$,*

$$\begin{aligned} &\operatorname{Im}\varphi(x, y, t) \geq c\,|x' - y'|^2, \quad \text{for all } (x, y, t) \in D \times D \times (-\delta, \delta), \\ &\operatorname{Im}\varphi(x, y, t) + \left|\frac{\partial \varphi}{\partial t}(x, y, t)\right|^2 \geq c\big(|x_{2n-1} - y_{2n-1}|^2 + |x' - y'|^2\big), \end{aligned} \tag{3.60}$$

where $x' = (x_1, \ldots, x_{2n-2})$, $y' = (y_1, \ldots, y_{2n-2})$, $|x' - y'|^2 = \sum_{j=1}^{2n-2} |x_j - y_j|^2$.

For the proof of Theorem 3.12, we refer the reader to the proof of Theorem 4.24 in [21].

In Section 4.4 of [21], the first author determined the tangential Hessian of $\varphi(x, y, t)$. We denote as usual $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, canonical local coordinates of X. We also use $y = (y_1, \ldots, y_{2n-1})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n-1$.

Theorem 3.13. *With the assumptions and notations used in Theorem 1.1, fix $(p, p, t_0) \in D \times D \times (-\delta, \delta)$, and let $\overline{Z}_{1,t_0}, \ldots, \overline{Z}_{n-1,t_0}$ be an orthonormal rigid frame of $T^{1,0}_x X$ varying smoothly with x in a neighbourhood of p, for which the Hermitian quadratic form $R^L_x - 2t_0\mathcal{L}_x$ is diagonal at p, that is,*

$$R^L_p\big(\overline{Z}_{j,t_0}(p), Z_{k,t_0}(p)\big) - 2t_0\mathcal{L}_p\big(\overline{Z}_{j,t_0}(p), Z_{k,t_0}(p)\big) = \lambda_j(t_0)\delta_{j,k}, \quad j, k = 1, \ldots, n-1,$$

where $\lambda_j(t_0) > 0$, $j = 1, \ldots, n-1$. Let s be a local rigid CR frame of L defined in some small neighbourhood of p such that

$$\begin{aligned}
& x(p) = 0, \ \ \omega_0(p) = -dx_{2n-1}, \ \ T = \frac{\partial}{\partial x_{2n-1}} = \frac{\partial}{\partial \theta}, \\
& \Big\langle \frac{\partial}{\partial x_j}(p) \,|\, \frac{\partial}{\partial x_k}(p) \Big\rangle = 2\delta_{j,k}, \quad j, k = 1, \ldots, 2n-2, \\
& \overline{Z}_{j,t_0}(p) = \frac{\partial}{\partial z_j} + i\sum_{k=1}^{n-1} \tau_{j,k}\overline{z}_k \frac{\partial}{\partial x_{2n-1}} + O(|z|^2), \quad j = 1, \ldots, n-1, \\
& \Phi(x) = \frac{1}{2}\sum_{l,k=1}^{n-1} \mu_{k,l} z_k \overline{z}_l + \sum_{l,k=1}^{n-1} \big(a_{l,k} z_l z_k + \overline{a}_{l,k}\overline{z}_l\overline{z}_k\big) + O(|z|^3),
\end{aligned} \tag{3.61}$$

where $\tau_{j,k}, \mu_{j,k}, a_{j,k} \in \mathbb{C}$, $\mu_{j,k} = \overline{\mu}_{k,j}$, $j, k = 1, \ldots, n-1$. Then there exists a neighbourhood of (p, p) such that

$$\begin{aligned}
\varphi(x, y, t_0) = {} & t_0(-x_{2n-1} + y_{2n-1}) - \frac{i}{2}\sum_{j,l=1}^{n-1} (a_{l,j} + a_{j,l})(z_j z_l - w_j w_l) \\
& + \frac{i}{2}\sum_{j,l=1}^{n-1} (\overline{a}_{l,j} + \overline{a}_{j,l})(\overline{z}_j\overline{z}_l - \overline{w}_j\overline{w}_l) + \frac{it_0}{2}\sum_{j,l=1}^{n-1} (\overline{\tau}_{l,j} - \tau_{j,l})(z_j\overline{z}_l - w_j\overline{w}_l) \\
& - \frac{i}{2}\sum_{j=1}^{n-1} \lambda_j(t_0)(z_j\overline{w}_j - \overline{z}_j w_j) + \frac{i}{2}\sum_{j=1}^{n-1} \lambda_j(t_0)\,|z_j - w_j|^2 \\
& + (-x_{2n-1} + y_{2n-1}) f(x, y, t_0) + O(|(x, y)|^3),
\end{aligned} \tag{3.62}$$

where f is smooth in a neighborhood of (p, p, t_0) and $f(0, 0, t_0) = 0$

3.4. **Comparison with other Szegő kernel expansions.** Let us recall that the Szegő kernel $\Pi(x, y)$ of the boundary X of a relatively compact strictly pseudoconvex domain G is a Fourier integral operator with complex phase, by a result of Boutet de Monvel-Sjöstrand [8]. Here $\Pi : L^2(X) \to \mathcal{H}^0_b(X)$ is the orthogonal projection on the space of CR functions on X (Szegő projector). In particular, $\Pi(x, y)$ is smooth outside the diagonal $x = y$ of $X \times X$ and there is a precise description of the singularity on the diagonal $x = y$, where $\Pi(x, y)$ has a certain asymptotic expansion. More precisely, let $G = \{\rho < 0\} \Subset G'$ be a strictly pseudoconvex domain in a $(n+1)$-dimensional complex manifold G', where $\rho \in \mathscr{C}^\infty(G')$ is a defining function of G. Then by taking an almost-analytic extension $\varphi = \varphi(x, y) : G' \times G' \to \mathbb{C}$ of ρ, see [8, (1.1)-(1.3)], we have

$$\Pi(x, y) = \int_0^\infty e^{i\varphi(x,y)t} s(x, y, t)dt + R(x, y), \tag{3.63}$$

where $s(x, y, t) \in S^n(X \times X \times \mathbb{R}_+)$ and $R(x, y)$ is a smooth function. Assume now that X is the strictly pseudoconvex CR manifold given by the boundary of the unit disc bundle of E^*, where E is a positive line bundle over a compact complex manifold M. Then X admits a natural S^1 action and we define as in (1.3), (1.4) the spaces of equivariant CR functions $\mathcal{H}^0_{b,m}(X) := \{u \in C^\infty(X); Tu = imu,\ \overline{\partial}_b u = 0\}$ for the trivial line bundle L over X. Then $\mathcal{H}^0_{b,m}(X)$ is isomorphic to the space of holomorphic sections $H^0(M, E^m)$ of E^m over M. By an appropriate choice of metric data these spaces are also isometric for the corresponding L^2 inner products, see [40]. We have an orthogonal decomposition $\mathcal{H}^0_b(X) = \oplus_{m\in\mathbb{N}_0}\mathcal{H}^0_{b,m}(X)$ and we decompose accordingly the Szegő pojector $\Pi = \sum_{m\in\mathbb{N}_0} \Pi_m$, where $\Pi_m : L^2(X) \to \mathcal{H}^0_{b,m}(X)$ is the orthogonal projection. We can thus see the analogies and differences between the expressions (3.63) of the Szegő kernel $\Pi(x, y)$ and (1.16) of the Fourier-Szegő kernel $P_{k,\delta,s}(x, y)$. In (3.63) we integrate over $\mathbb{R}_+$ and this corresponds to a sum over all $m \in \mathbb{N}_0$, while in (1.16) we have an integral over $(-\delta, \delta)$ which corresponds to a sum only over $m \in \mathbb{Z}$, $|m| \le k\delta$. Of course, in (3.63) there is no semi-classical parameter k as in (1.16), since we work with the trivial line bundle.

Using (3.63) and the stationary phase formula it is shown in [11, 40] that $\Pi_m(x, y)$ have an asymptotic expansion for $m \to \infty$. Moreover, they corrrespond to the Bergman kernels $B_m(z, w)$ of $H^0(M, E^m)$, which have accordingly the form $B_{m,s}(z, w) = e^{im\varphi(z,w)}b(z, w, m)$, where $b(z, w, m) \sim \sum_{j=0}^\infty m^{n-1-j} b_j(z, w)$ in $S^{n-1}_{\rm loc}(1; D \times D)$, see also [25, 28, 29]. Here $D \subset M$ is an open set over which we have a trivializing section s of E and $B_{m,s}(z, w)$ is the corresponding localization of $B_m(z, w)$. In this respect, the form (1.16) bears resemblance to the expansion of the Bergman kernel on a complex manifold, but we have to integrate since our space $\mathcal{H}^0_{b,\le k\delta}(X, L^k)$ consists of all the components $\mathcal{H}^0_{b,m}(X, L^k)$ with $|m| \le k\delta$.

There is also an expansion of the Szegő kernel for a positive line bundle L over an arbitrary Levi-flat CR manifold [26, Theorem 1.2]. The Szegő kernel $\Pi_k(x, y)$ of the projector on the CR sections of L^k is close in the semiclassical limit to an approximate Szegő projector $\mathcal{S}_k$, which has an asymptotic expansion in Sobolev spaces, given locally by the operator S_k with kernel

$$S_k(x, y) = \int_{\mathbb{R}} e^{ik\psi(x,y,u)} s(x, y, u, k)du. \tag{3.64}$$

The integral $\int_{\mathbb{R}} du$ in (3.64) arises due to the transversal direction to the leaves of the Levi foliation. This result implies the Kodaira embedding theorem for Levi-flat CR manifolds [26]. In the presence of a S^1 action we can refine this result and work with the Fourier-Szegő projector $P_{k,\delta}$ with the asymptotics (1.16) and (1.20) with leading term (1.22).

Finally, we refer to [30] for the relation between heat kernels and Szegő kernels for non-necessarily positive line bundles.

3.5. **The necessity of the weighted Fourier-Szegő operator** $P_{k,\delta}$. We now give a simple example to show that the partial Szegő kernel of $H^0_{b,\le k\delta}(X, L^k)$, doesn't have an asymptotic expansion, hence the need for the weighted projector $F_{k,\delta}$ and weighted Fourier-Szegő operator $P_{k,\delta}$.

Let (L,h) be a positive holomorphic line bundle over a compact complex manifold M of dimension $n-1$. Then $X:=M\times S^1$ is a Levi-flat CR manifold of dimension $2n-1$ with transversal CR S^1 action $e^{i\theta}$ and the pull-back of (L,h) by the projection $M\times S^1\to X$ is a positive CR line bundle over X, denoted again (L,h). For $k>0$, let $g_1^{(k)},\ldots,g_{r_k}^{(k)}$ be an orthonormal basis of the space $H^0(M,L^k)$ of global holomorphic sections with values in L^k. By the asymptotic expansion of the Bergman kernel of L^k [11, 38, 40] (see also [28, 29]) we have in any C^ℓ topology on M,

$$\sum_{j=1}^{r_k}\left|g_j^{(k)}(x)\right|^2_{h^k}\sim k^{n-1}b_0(x)+k^{n-2}b_1(x)+\ldots,\quad k\to\infty. \tag{3.65}$$

For each $m\in\mathbb{Z}$, $\left\{f_{j,m}^{(k)}(x,\theta):=\frac{1}{\sqrt{2\pi}}g_j^{(k)}(x)e^{im\theta}\right\}_{j=1}^{r_k}$ is an orthonormal basis of $H^0_{b,m}(X,L^k)$. Hence, $\left\{f_{j,m}^{(k)}(x,\theta);\, m\in\mathbb{Z},|m|\leq k\delta\right\}$ is an orthonormal basis of the space $H^0_{b,\leq k\delta}(X,L^k)$, whose cardinal is denoted by d_k. Thus, the Szegő kernel of $H^0_{b,\leq k\delta}(X,L^k)$ is given by

$$\sum_{\substack{1\leq j\leq d_k\\|m|\leq k\delta}}\left|f_{j,m}^{(k)}(x)\right|^2_{h^k}=\frac{1}{\pi}[k\delta]\sum_{j=1}^{r_k}\left|g_j^{(k)}(x)\right|^2_{h^k}, \tag{3.66}$$

where $[k\delta]$ is the Gauss' symbol denoting the integral part of $k\delta$. The difficulty comes from the fact that the function $\delta\mapsto[k\delta]$ does not admit an asymptotic expansion in k. To get asymptotic expansion, we consider

$$\sum_{\substack{1\leq j\leq d_k\\|m|\leq k\delta}}\left|(F_{\delta,k}f_{j,m}^{(k)})(x)\right|^2_{h^k}=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\left|\tau_\delta\Big(\frac{m}{k}\Big)\right|^2\sum_{j=1}^{r_k}\left|g_j^{(k)}(x)\right|^2_{h^k}. \tag{3.67}$$

We need the following.

Theorem 3.14. *Let $g\in C_0^\infty(\mathbb{R})$. Then there exists a sequence $(b_j)_{j\in\mathbb{N}_0}$ of complex numbers such that for every $N\in\mathbb{N}$ there exists $C_N>0$ with*

$$\left|\frac{1}{k}\sum_{m\in\mathbb{Z}}g\Big(\frac{m}{k}\Big)-\Big(b_0+\frac{b_1}{k}+\cdots+\frac{b_N}{k^N}\Big)\right|\leq C_Nk^{-(N+1)}\ \textit{for every } k\in\mathbb{N}. \tag{3.68}$$

Proof. Let $b_0=\int_{\mathbb{R}}g(x)dx$. By Taylor expansion, we have

$$\left|\frac{1}{k}\sum_{m\in\mathbb{Z}}g\Big(\frac{m}{k}\Big)-\int g(x)dx\right|=\left|\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\Big(g\Big(\frac{m}{k}\Big)-g(x)\Big)dx\right|=\left|\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\int_x^{\frac{m}{k}}g'(t)dtdx\right|$$
$$\leq\frac{1}{k}\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\left|g'(t)\right|dt=\frac{1}{k}\int\left|g'(t)\right|dt.$$

We have proved that (3.68) holds for $N=1$. Assume that (3.68) holds for $N\leq N_0$, $N_0\in\mathbb{N}$. We are going to prove that (3.68) holds for $N=N_0+1$. By Taylor expansion, we have

(3.69)

$$\begin{split}&\frac{1}{k}\sum_{m\in\mathbb{Z}}g\Big(\frac{m}{k}\Big)-\int g(x)dx=\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\Big(g\Big(\frac{m}{k}\Big)-g(x)\Big)dx\\&=\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\sum_{j=1}^{N_0+2}\frac{1}{j!}\Big(\frac{m}{k}-x\Big)^jg^{(j)}(x)dx+\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\int_x^{\frac{m}{k}}\frac{1}{(N_0+2)!}\Big(\frac{m}{k}-x\Big)^{N_0+2}g^{(N_0+3)}(t)dtdx.\end{split}$$

We have

$$\begin{split}&\left|\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\int_x^{\frac{m}{k}}\frac{1}{(N_0+2)!}\Big(\frac{m}{k}-x\Big)^{N_0+2}g^{(N_0+3)}(t)dtdx\right|\\&\leq\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\int_x^{\frac{m}{k}}\frac{1}{(N_0+2)!}\Big(\frac{m}{k}-x\Big)^{N_0+2}\left|g^{(N_0+3)}(t)\right|dtdx\\&\leq\frac{k^{-(N_0+3)}}{(N_0+3)!}\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\left|g^{(N_0+3)}(t)\right|dt=\frac{k^{-(N_0+3)}}{(N_0+3)!}\int\left|g^{(N_0+3)}(t)\right|dt,\end{split}\tag{3.70}$$

for every $k\in\mathbb{N}$. Let $h\in C_0^\infty(\mathbb{R})$. We claim that for every $j=1,2,\ldots,N_0+2$, we have

$$\left|\sum_{m\in\mathbb{Z}}\int\Big(\frac{m}{k}-x\Big)^jh(x)dx-\Big(b_{j,1}k^{-1}+b_{j,2}k^{-2}+\cdots+b_{j,N_0+1}k^{-(N_0+1)}\Big)\right|\leq C_{N_0}k^{-(N_0+2)},\tag{3.71}$$

for every $k\in\mathbb{N}$, where $C_{N_0}>0$, $b_{j,s}\in\mathbb{C}$, $j=1,\ldots,N_0+2$, $s=1,\ldots,N_0+1$, are constants independent of k. We have

$$\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\Big(\frac{m}{k}-x\Big)^{N_0+2}h(x)dx\leq k^{-(N_0+2)}\int|h(x)|\,dx.$$

Hence (3.71) holds for $j=N_0+2$ with $b_{N_0+2,1}=\cdots=b_{N_0+2,N_0+1}=0$. Assume that (3.71) holds for $j\geq s_0$, $s_0\in\mathbb{N}$, $2\leq s_0\leq N_0+2$. Let $j=s_0-1$. By using integration by parts, we have

$$\begin{split}&\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\Big(\frac{m}{k}-x\Big)^{s_0-1}h(x)dx\\&=\sum_{m\in\mathbb{Z}}\frac{-1}{s_0}\Big(\frac{m}{k}-x\Big)^{s_0}h(x)|_{\frac{m-1}{k}}^{\frac{m}{k}}+\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\frac{1}{s_0}\Big(\frac{m}{k}-x\Big)^{s_0}h'(x)dx\\&=\frac{1}{s_0}\Big(\frac{1}{k}\Big)^{s_0}\sum_{m\in\mathbb{Z}}h\Big(\frac{m-1}{k}\Big)+\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\frac{1}{s_0}\Big(\frac{m}{k}-x\Big)^{s_0}h'(x)dx.\end{split}\tag{3.72}$$

By the induction assumption, (3.68) holds for $N\leq N_0$, we have

$$\begin{split}&\left|\frac{1}{s_0}\Big(\frac{1}{k}\Big)^{s_0}\sum_{m\in\mathbb{Z}}h\Big(\frac{m-1}{k}\Big)-k^{-s_0}\Big(d_{s_0,1}k^{-1}+d_{s_0,2}k^{-2}+\cdots+d_{s_0,N_0}k^{-N_0}\Big)\right|\\&\leq\hat{C}_{N_0}k^{-(N_0+1)-s_0}\leq\hat{C}_{N_0}k^{-(N_0+2)},\end{split}\tag{3.73}$$

for every $k\in\mathbb{N}$, where $\hat{C}_{N_0}>0$, $d_{s_0,t}\in\mathbb{C}$, $t=1,\ldots,N_0$, are constants independent of k. By the induction assumption, (3.71) holds for $j\geq s_0$, we have

(3.74)
$$\left|\sum_{m\in\mathbb{Z}}\int_{\frac{m-1}{k}}^{\frac{m}{k}}\frac{1}{s_0}\Big(\frac{m}{k}-x\Big)^{s_0}h'(x)dx-\Big(e_{s_0,1}k^{-1}+e_{s_0,2}k^{-2}+\cdots+e_{s_0,N_0+1}k^{-(N_0+1)}\Big)\right|\leq\widetilde{C}_{N_0}k^{-(N_0+2)},$$

for every $k\in\mathbb{N}$, where $\widetilde{C}_{N_0}>0$, $e_{s_0,t}\in\mathbb{C}$, $t=1,\ldots,N_0+1$, are constants independent of k. From (3.72), (3.73) and (3.74), the claim (3.71) holds for $j=s_0-1$. The claim (3.71) follows thus by induction. From (3.71), (3.70) and (3.69), we see that (3.68) holds for $N=N_0+1$ which finishes the induction proof of (3.68). □

Applying (3.68) for $g=\tau_\delta^2$ we obtain the asymptotic expansion

$$\sum_{m\in\mathbb{Z}}\left|\tau_\delta\Big(\frac{m}{k}\Big)\right|^2\sim k\int_{\mathbb{R}}|\tau_\delta(t)|^2\,dt+a_0+a_{-1}k^{-1}+\ldots,\ \ k\to\infty. \tag{3.75}$$

Combining (3.65) with (3.75) we obtain an asymptotic expansion of (3.67) in k. This shows the necessity of introducing the weighted operators $F_{\delta,k}$ and $P_{\delta,k}$.

4. Equivariant Kodaira embedding

In this section we will prove Theorem 1.3. Let

$$f_1\in\mathcal{H}^0_{b,\leq k\delta}(X,L^k),\ldots,f_{d_k}\in\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$$

be an orthonormal basis of $\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$ with respect to $(\,\cdot\,|\,\cdot\,)$. On D, we write

$$F_{k,\delta}f_j=s^k\otimes\widetilde{g}_j,\ \ \widetilde{g}_j\in C^\infty(D),\ \ j=1,2,\ldots,d_k.$$

Lemma 4.1. *We have*

$$\begin{aligned}P_{k,\delta,s}(x,y)&=\sum_{j=1}^{d_k}e^{-k\Phi(x)}\widetilde{g}_j(x)\overline{\widetilde{g}_j(y)}e^{-k\Phi(y)},\\ P_{k,\delta,s}(x,x)&=\sum_{j=1}^{d_k}|\widetilde{g}_j(x)|^2e^{-2k\Phi(x)}=\sum_{j=1}^{d_k}\left|(F_{k,\delta}f_j)(x)\right|^2_{h^k}.\end{aligned} \tag{4.1}$$

In particular, (1.13) *holds.*

Lemma 4.2. *Let $\delta>0$ be a small constant. Then there exist $C_0>0$ and $k_0\in\mathbb{N}$ such that for all $k\geq k_0$ and all $x\in X$ we have*

$$\sum_{j=1}^{d_k}\left|F_{k,\delta}f_j(x)\right|^2_{h^k}\geq C_0k^n. \tag{4.2}$$

Moreover, there is $c_0>0$ and $k_0\in\mathbb{N}$ such that for $k\geq k_0$ and every $x\in X$, there exists $j_0\in\{1,2,\ldots,d_k\}$ with

$$\left|F_{k,\delta}f_{j_0}(x)\right|^2_{h^k}\geq c_0. \tag{4.3}$$

Proof. Theorem 1.1 immediately implies the first assertion. We only need to prove the second. By [22, Theorem 1.4] we know that there exists $C_1>0$ such that

$$\dim\mathcal{H}^0_{b,\leq k\delta}(X,L^k)=d_k\leq C_1k^n, \tag{4.4}$$

where $C_1 > 0$ is a constant independent of k. From (4.4) and (4.2), we have for every $x \in X$,

$$\begin{aligned} &C_1 k^n \sup \left\{ |F_{k,\delta} f_j(x)|^2_{h^k} ; \, j = 1, 2, \ldots, d_k \right\} \\ &\geq d_k \sup \left\{ |F_{k,\delta} f_j(x)|^2_{h^k} ; \, j = 1, 2, \ldots, d_k \right\} \\ &\geq \sum_{j=1}^{d_k} |F_{k,\delta} f_j(x)|^2_{h^k} \geq C_0 k^n, \end{aligned}$$

which yields (4.3). □

The (modified) Kodaira map $\Phi_{k,\delta} : X \to \mathbb{CP}^{d_k - 1}$ introduced in (1.25) is explicitly defined as follows. For $x_0 \in X$, let s be a local rigid CR frame of L on an open neighbourhood $D \subset X$ of x_0, $|s(x)|^2_h = e^{-2\Phi}$. On D, put $F_{k,\delta} f_j(x) = s^k \widetilde{f}_j(x)$, $\widetilde{f}_j(x) \in C^\infty(D)$, $j = 1, \ldots, d_k$. Then,

$$\Phi_{k,\delta}(x_0) = \left[\widetilde{f}_1(x_0), \ldots, \widetilde{f}_{d_k}(x_0) \right] \in \mathbb{CP}^{d_k - 1}. \tag{4.5}$$

In view of (4.3), we see that $\Phi_{k,\delta}$ is well-defined as a smooth CR map from X to $\mathbb{CP}^{d_k-1}$. We wish to prove that $\Phi_{k,\delta}$ is an embedding for k large enough. Since X is compact, a smooth map is an embedding if and only if it is an injective immersion.

Theorem 4.3. *The map $\Phi_{k,\delta}$ is an immersion for k large enough.*

To prove Theorem 4.3, we need some preparations. Fix $p \in X$ and let s be a local rigid CR frame of L on a canonical coordinate patch D, $p \in D$, $|s|^2_h = e^{-2\Phi}$, with canonical local coordinates $x = (x_1, \ldots, x_{2n-1}) = (z, \theta)$. We take canonical coordinates x and s so that (3.61) hold. We identify D with an open set in $\mathbb{R}^{2n-1}$. For $r > 0$, put

$$D_r := \left\{ x \in \mathbb{R}^{2n-1}; \, |x_j| < r, j = 1, 2, \ldots, 2n-1 \right\}.$$

For $x = (x_1, \ldots, x_{2n-1})$, we consider the rescaling map

$$F_k^* x := \left(\frac{x_1}{\sqrt{k}}, \ldots, \frac{x_{2n-2}}{\sqrt{k}}, \frac{x_{2n-1}}{k} \right).$$

Note that we rescale with a factor $\frac{1}{\sqrt{k}}$ in the direction of the complex variables z and by a factor $\frac{1}{k}$ in the real direction θ. Such anisotropic rescaling was used already in [23, Section 2.2]. The rescaling by $\frac{1}{\sqrt{k}}$ in the direction of the complex variables is very natural and was used in [6, 28].

From (3.62), we can check that

$$k i \varphi(0, F_k^* y, t) = i y_{2n-1} t + i \varphi_0(w, t) + r_k(y, t) \quad \text{on } D_{\log k}, \tag{4.6}$$

where $r_k(y, t) \in C^\infty(D_{\log k} \times (-\delta, \delta))$ and $\varphi_0(w, t) \in C^\infty(\mathbb{R}^{2n-2} \times (-\delta, \delta))$ independent of k. Moreover, for every $\alpha \in \mathbb{N}^{2n-1}$, we have

$$\lim_{k \to \infty} \sup_{(y,t) \in D_{\log k} \times (-\delta, \delta)} \left| \partial^\alpha_y r_k(y, t) \right| = 0, \tag{4.7}$$

and there exists $C > 0$ such that

$$\begin{aligned} &\varphi_0(0, t) = 0, \quad \text{for all } t \in (-\delta, \delta), \\ &\varphi_0(w, t) = \varphi_0(-w, t), \quad \text{for all } (w, t) \in \mathbb{R}^{2n-2} \times (-\delta, \delta), \\ &\int e^{-\operatorname{Im} \varphi_0(w,t)} dy_1 \ldots dy_{2n-2} \leq C < \infty, \quad \text{for all } t \in (-\delta, \delta). \end{aligned} \tag{4.8}$$

Take $\chi \in C_0^\infty((-1,1))$ with $0 \le \chi \le 1$, $\chi(x) = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\chi(t) = \chi(-t)$, for every $t \in \mathbb{R}$. For $\epsilon > 0$, put $\chi_\epsilon(x) := \epsilon^{-1}\chi(\epsilon^{-1}x)$. Let $g_0(x,y,t) \in C^\infty(D \times D \times (-\delta,\delta))$ be as (1.18). We can check that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int e^{i\varphi_0(w,t)+iy_{2n-1}t}g_0(0,0,t)\chi_\epsilon(y_1)\ldots\chi_\epsilon(y_{2n-2})dydt\\
&=\int e^{iy_{2n-1}t}g_0(0,0,t)dtdy_{2n-1}\int_{\mathbb{R}^{2n-2}}\chi(y_1)\ldots\chi(y_{2n-2})dy_1\ldots dy_{2n-2}\\
&=(2\pi)g_0(0,0,0)\int_{\mathbb{R}^{2n-2}}\chi(y_1)\ldots\chi(y_{2n-2})dy_1\ldots dy_{2n-2}\neq 0,
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int e^{i\varphi_0(w,t)+iy_{2n-1}t}g_0(0,0,t)\epsilon^{-2}\left|y_j\right|^2\chi_\epsilon(y_1)\ldots\chi_\epsilon(y_{2n-1})dydt\\
&=\int e^{iy_{2n-1}t}g_0(0,0,t)dtdy_{2n-1}\int_{\mathbb{R}^{2n-2}}\left|y_j\right|^2\chi(y_1)\ldots\chi(y_{2n-2})dy_1\ldots dy_{2n-2}\\
&=(2\pi)g_0(0,0,0)\int_{\mathbb{R}^{2n-2}}\left|y_j\right|^2\chi(y_1)\ldots\chi(y_{2n-2})dy_1\ldots dy_{2n-2}\neq 0,\quad j=1,2,\ldots,2n-2,
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int e^{i\varphi_0(w,t)+iy_{2n-1}t}(-it)g_0(0,0,t)\chi_\epsilon(y_1)\ldots\chi_\epsilon(y_{2n-1})y_{2n-1}dydt\\
&=\int e^{iy_{2n-1}t}(-ity_{2n-1})g_0(0,0,t)dtdy_{2n-1}\int_{\mathbb{R}^{2n-2}}\chi(y_1)\ldots\chi(y_{2n-2})dy_1\ldots dy_{2n-2}\\
&=(2\pi)g_0(0,0,0)\int_{\mathbb{R}^{2n-2}}\chi(y_1)\ldots\chi(y_{2n-2})dy_1\ldots dy_{2n-2}\neq 0.
\end{aligned}\tag{4.11}$$

From (4.9), (4.10) and (4.11), we deduce that there exists $\epsilon_0 > 0$ small enough such that

$$\begin{aligned}
&\int e^{i\varphi_0(w,t)+iy_{2n-1}t}g_0(0,0,t)\chi_{\epsilon_0}(y_1)\ldots\chi_{\epsilon_0}(y_{2n-2})dydt =: V_0\neq 0,\\
&\int e^{i\varphi_0(w,t)+iy_{2n-1}t}g_0(0,0,t)\left|y_j\right|^2\chi_{\epsilon_0}(y_1)\ldots\chi_{\epsilon_0}(y_{2n-2})dydt =: V_j\neq 0,\quad j=1,2,\ldots,2n-2,\\
&\int e^{i\varphi_0(w,t)+iy_{2n-1}t}(-ity_{2n-1})g_0(0,0,t)\chi_{\epsilon_0}(y_1)\ldots\chi_{\epsilon_0}(y_{2n-2})dydt =: V\neq 0.
\end{aligned}\tag{4.12}$$

Assume that $p \in D_0 \Subset D$. We need

Lemma 4.4. *With the notations above, there is a $K_0 > 0$ independent of the point p such that for all $k \ge K_0$, we can find*

$$g_k^j \in \mathcal{H}^0_{b,\le k\delta}(X, L^k),\quad j=1,\ldots,n,$$

such that if we put $F_{k,\delta}g^j_k = s^k\widetilde{g}^j_k$ *on* D, $j = 1, \ldots, n$, *then*

$$\begin{aligned}
&\sum_{j=1}^{n}\left|(e^{-k\Phi}\widetilde{g}^j_k)(p)\right| = 0,\\
&\left|\frac{1}{\sqrt{k}}\partial_{\overline{z}_t}\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right| \leq \epsilon_k,\ \ j = 1, 2, \ldots, n,\ \ t = 1, 2, \ldots, n-1,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_t}\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right| \leq \epsilon_k,\ \ j = 1, 2, \ldots, n,\ \ t = 1, 2, \ldots, n-1,\ \ j \neq t,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_j}\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right| \geq C_0,\ \ j = 1, \ldots, n-1,\\
&\left|\frac{1}{k}\partial_{x_{2n-1}}\big(e^{-k\Phi}\widetilde{g}^n_k\big)(0)\right| \geq C_0,
\end{aligned} \tag{4.13}$$

where $C_0 > 0$ *is a constant independent of* k *and the point* p, ϵ_k *is a sequence independent of* p *with* $\lim_{k\to\infty}\epsilon_k = 0$.

Proof. Let $\chi \in C^\infty_0(\mathbb{R})$ and $\epsilon_0 > 0$ be as in (4.12). Put

$$u_k := \Pi_{k,\leq k\delta}F_{k,\delta}\left(e^{k\Phi(w)}\chi_{\epsilon_0}(\sqrt{k}y_1)\ldots\chi_{\epsilon_0}(\sqrt{k}y_{2n-2})\chi\Big(\frac{k}{\log k}y_{2n-1}\Big)\frac{s^k(w)}{m(y)}\right), \tag{4.14}$$

where $w = (w_1, \ldots, w_{n-1})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n-1$, and $m(y)dy = dv_X(y)$ on D. We put $F_{k,\delta}u_k = s^k\widetilde{u}_k$ on D. In view of Theorem 1.1, we see that on D_0,

$$e^{-k\Phi(z)}\widetilde{u}_k(x) = \int e^{ik\varphi(x,y,t)}g(x,y,t,k)\chi_{\epsilon_0}(\sqrt{k}y_1)\ldots\chi_{\epsilon_0}(\sqrt{k}y_{2n-2})\chi(\frac{k}{\log k}y_{2n-1})dydt + O(k^{-\infty}). \tag{4.15}$$

From (4.6), (4.7), (4.15) and (4.12), we can check that

$$\begin{aligned}
&\lim_{k\to\infty}\int e^{ik\varphi(0,y,t)}g(0,y,t,k)\chi_{\epsilon_0}(\sqrt{k}y_1)\ldots\chi_{\epsilon_0}(\sqrt{k}y_{2n-2})\chi(\frac{k}{\log k}y_{2n-1})dydt\\
&= \lim_{k\to\infty}\int e^{ik\varphi(0,F^*_ky,t)}k^{-n}g(0,F^*_ky,t,k)\chi_{\epsilon_0}(y_1)\ldots\chi_{\epsilon_0}(y_{2n-2})\chi(\frac{y_{2n-1}}{\log k})dtdy\\
&= \int e^{i\varphi_0(w,t)+iy_{2n-1}t}g_0(0,0,t)\chi_{\epsilon_0}(y_1)\ldots\chi_{\epsilon_0}(y_{2n-2})dydt = V_0 \neq 0.
\end{aligned} \tag{4.16}$$

From (4.16) and since that X is compact, it is easy to see that there is a $k_0 > 0$ independent of the point p such that for all $k \geq k_0$, we have

$$\frac{1}{A_0} \leq \left|e^{-k\Phi(0)}\widetilde{u}_k(0)\right| \leq A_0, \tag{4.17}$$

where $A_0 > 1$ is a constant independent of k and the point p. From now on, we assume that $k \geq k_0$. From (3.62) and (4.15), we can check that

$$\lim_{k\to\infty}\frac{1}{k}\partial_{x_{2n-1}}(e^{-k\Phi}\widetilde{u}_k)(0) = \int e^{i\varphi_0(w,t)+ity_{2n-1}}(-it)g_0(0,0,t)\chi_{\epsilon_0}(y_1)\ldots\chi_{\epsilon_0}(y_{2n-2})dydt = 0, \tag{4.18}$$

and for $j = 1, \dots, n-1$,

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{\sqrt{k}}\partial_{z_j}(e^{-k\Phi}\widetilde{u}_k)(0)\\&=\int e^{i\varphi_0(w,t)+ity_{2n-1}}(-i)\lambda_j(t)(y_{2j-1}-iy_{2j})g_0(0,0,t)\chi_{\epsilon_0}(y_1)\dots\chi_{\epsilon_0}(y_{2n-2})dydt=0,\end{aligned}\tag{4.19}$$

$$\lim_{k\to\infty}\frac{1}{\sqrt{k}}\partial_{\overline{z}_j}(e^{-k\Phi}\widetilde{u}_k)(0)=0.\tag{4.20}$$

From (4.18), (4.19) and (4.20), it is easy to see that

$$\begin{aligned}&\left|\frac{1}{k}\partial_{x_{2n-1}}(e^{-k\Phi}\widetilde{u}_k)(0)\right|\leq\delta_k,\\&\left|\frac{1}{\sqrt{k}}\partial_{z_j}(e^{-k\Phi}\widetilde{u}_k)(0)\right|\leq\delta_k,\ \ j=1,\dots,n-1,\\&\left|\frac{1}{\sqrt{k}}\partial_{\overline{z}_j}(e^{-k\Phi}\widetilde{u}_k)(0)\right|\leq\delta_k,\ \ j=1,\dots,n-1,\end{aligned}\tag{4.21}$$

where δ_k is a sequence independent of the point p with $\lim_{k\to\infty}\delta_k=0$.

Put

$$v_k^n:=\Pi_{k,\leq k\delta}F_{k,\delta}\left(e^{k\Phi(w)}ky_{2n-1}\chi_{\epsilon_0}(\sqrt{k}y_1)\dots\chi_{\epsilon_0}(\sqrt{k}y_{2n-2})\chi\Big(\frac{k}{\log k}y_{2n-1}\Big)\frac{1}{m(y)}\right).\tag{4.22}$$

We put $F_{k,\delta}v_k^n=s^k\widetilde{v}_k^n$ on D. In view of Theorem 1.1, we see that on D_0 we have

$$\begin{aligned}&e^{-k\Phi(z)}\widetilde{v}_k^n(x)\\&=\int e^{ik\varphi(x,y,t)}g(x,y,t,k)ky_{2n-1}\chi_{\epsilon_0}(\sqrt{k}y_1)\dots\chi_{\epsilon_0}(\sqrt{k}y_{2n-2})\chi\Big(\frac{k}{\log k}y_{2n-1}\Big)dydt+O(k^{-\infty}).\end{aligned}\tag{4.23}$$

From (4.23), it is easy to see that there is $E_0>0$ independent of k and the point p such that

$$\left|e^{-k\Phi(0)}\widetilde{v}_k^n(0)\right|\leq E_0.\tag{4.24}$$

From (3.62) and (4.12), we can check that

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{k}\partial_{x_{2n-1}}(e^{-k\Phi}\widetilde{v}_k^n)(0)\\&=\int e^{i\varphi_0(w,t)+ity_{2n-1}}(-ity_{2n-1})g_0(0,0,t)\chi_{\epsilon_0}(y_1)\dots\chi_{\epsilon_0}(y_{2n-2})dydt=V\neq0,\end{aligned}\tag{4.25}$$

and for $j=1,\dots,n-1$,

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{\sqrt{k}}\partial_{z_j}(e^{-k\Phi}\widetilde{v}_k^n)(0)\\&=\int e^{i\varphi_0(w,t)+ity_{2n-1}}(-i)\lambda_j(t)(y_{2j-1}-iy_{2j})y_{2n-1}g_0(0,0,t)\chi_{\epsilon_0}(y_1)\dots\chi_{\epsilon_0}(y_{2n-2})dydt=0,\end{aligned}\tag{4.26}$$

$$\lim_{k\to\infty}\frac{1}{\sqrt{k}}\partial_{\overline{z}_j}(e^{-k\Phi}\widetilde{v}_k^n)(0)=0.\tag{4.27}$$

From (4.25), (4.26) and (4.27), it is easy to see that there is a $k_1 > k_0$ independent of the point p such that for all $k \geq k_1$, we have

$$\begin{aligned}
&\left|\frac{1}{k}\partial_{x_{2n-1}}(e^{-k\Phi}\widetilde{v}_k^n)(0)\right| \geq B_0,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_j}(e^{-k\Phi}\widetilde{v}_k^n)(0)\right| \leq \hat{\delta}_k, \ \ j=1,\ldots,n-1,\\
&\left|\frac{1}{\sqrt{k}}\partial_{\overline{z}_j}(e^{-k\Phi}\widetilde{v}_k^n)(0)\right| \leq \hat{\delta}_k, \ \ j=1,\ldots,n-1,
\end{aligned} \tag{4.28}$$

where $B_0 > 0$ is a constant independent of k and the point p and $\hat{\delta}_k$ is a sequence independent of the point p with $\lim_{k\to\infty}\hat{\delta}_k = 0$. From now on, we assume that $k \geq k_1$. Put

$$g_k^n := v_k^n - \frac{(e^{-k\Phi}\widetilde{v}_k^n)(0)}{(e^{-k\Phi}\widetilde{u}_k)(0)}u_k \in \mathcal{H}^0_{b,\leq k\delta}(X, L^k).$$

Put $F_{k,\delta}g_k^n = s^k\widetilde{g}_k^n$ on D. From (4.17), (4.21), (4.24) and (4.28), we see that there is a constant $k_1 > 0$ independent of k and the point p such that

$$\begin{aligned}
&\left|\big(e^{-k\Phi}\widetilde{g}_k^n\big)(0)\right| = 0,\\
&\left|\frac{1}{\sqrt{k}}\partial_{\overline{z}_t}\big(e^{-k\Phi}\widetilde{g}_k^n\big)(0)\right| \leq \epsilon_k, \ \ t=1,\ldots,n-1,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_t}\big(e^{-k\Phi}\widetilde{g}_k^n\big)(0)\right| \leq \epsilon_k, \ \ t=1,\ldots,n-1,\\
&\left|\frac{1}{k}\partial_{x_{2n-1}}\big(e^{-k\Phi}\widetilde{g}_k^n\big)(0)\right| \geq C_0,
\end{aligned} \tag{4.29}$$

where $C_0 > 0$ is a constant independent of k and the point p, ϵ_k is a sequence independent of p with $\lim_{k\to\infty}\epsilon_k = 0$.

Fix $j \in \{1, 2, \ldots, n-1\}$. Put

$$v_k^j := \Pi_{k,\leq k\delta}F_{k,\delta}\left(e^{k\Phi(w)}\sqrt{k}(y_{2j-1}+iy_{2j})\chi_{\epsilon_0}(\sqrt{k}y_1)\ldots\chi_{\epsilon_0}(\sqrt{k}y_{2n-2})\chi\Big(\frac{k}{\log k}y_{2n-1}\Big)\frac{1}{m(y)}\right).$$

We put $F_{k,\delta}v_k^j = s^k\widetilde{v}_k^j$ on D. In view of Theorem 1.1, we see that

$$\begin{aligned}
e^{-k\Phi(z)}\widetilde{v}_k^j(x) = \int e^{ik\varphi(x,y,t)}&g(x,y,t,k)\sqrt{k}(y_{2j-1}+iy_{2j})\\
&\times \chi_{\epsilon_0}(\sqrt{k}y_1)\ldots\chi_{\epsilon_0}(\sqrt{k}y_{2n-2})\chi\Big(\frac{k}{\log k}y_{2n-1}\Big)dydt + O(k^{-\infty}) \ \text{ on } D_0.
\end{aligned} \tag{4.30}$$

From (4.30), it is easy to see that there is a constant $E_1 > 0$ independent of k and the point p such that

$$\left|e^{-k\Phi(0)}\widetilde{v}_k^j(0)\right| \leq E_1. \tag{4.31}$$

Moreover, from (3.62), (4.12) and (4.30), we can repeat the proof of (4.28) with minor changes and deduce that there is a $\hat{k}_0 > 0$ independent of the point p such that for all $k \geq \hat{k}_0$, we have

$$\begin{aligned}
&\left|\frac{1}{k}\partial_{x_{2n-1}}(e^{-k\Phi}\widetilde{v}^j_k)(0)\right|\leq\widetilde{\delta}_k,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_j}(e^{-k\Phi}\widetilde{v}^j_k)(0)\right|\geq B_1,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_t}(e^{-k\Phi}\widetilde{v}^j_k)(0)\right|\leq\widetilde{\delta}_k,\ \ j,t=1,\ldots,n-1,\ \ j\neq t,\\
&\left|\frac{1}{\sqrt{k}}\partial_{\overline{z}_t}(e^{-k\Phi}\widetilde{v}^j_k)(0)\right|\leq\widetilde{\delta}_k,\ \ j,t=1,\ldots,n-1,
\end{aligned}\tag{4.32}$$

where $B_1>0$ is a constant independent of k and the point p and $\widetilde{\delta}_k$ is a sequence independent of the point p with $\lim_{k\to\infty}\widetilde{\delta}_k=0$. Put

$$g^j_k:=v^j_k-\frac{(e^{-k\Phi}\widetilde{v}^j_k)(0)}{(e^{-k\Phi}\widetilde{u}_k)(0)}u_k\in\mathcal{H}^0_{b,\leq k\delta}(X,L^k).\tag{4.33}$$

Put $F_{k,\delta}g^j_k=s^k\widetilde{g}^j_k$ on D. From (4.17), (4.21), (4.31) and (4.32), we see that there is a constant $\hat{k}_1>0$ independent of k and the point p such that

$$\begin{aligned}
&\left|\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right|=0,\\
&\left|\frac{1}{\sqrt{k}}\partial_{\overline{z}_t}\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right|\leq\epsilon_k,\ \ t=1,\ldots,n-1,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_t}\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right|\leq\epsilon_k,\ \ t=1,\ldots,n-1,\ \ t\neq j,\\
&\left|\frac{1}{\sqrt{k}}\partial_{z_j}\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right|\geq C_0,\\
&\left|\frac{1}{k}\partial_{x_{2n-1}}\big(e^{-k\Phi}\widetilde{g}^j_k\big)(0)\right|\leq\epsilon_k,
\end{aligned}\tag{4.34}$$

where $C_0>0$ is a constant independent of k and the point p, ϵ_k is a sequence independent of p with $\lim_{k\to\infty}\epsilon_k=0$.

From (4.29) and (4.34), the lemma follows. □

Proof of Theorem 4.3. We are going to prove that if k is large enough then the map

$$d\Phi_{k,\delta}(x):T_xX\to T_{\Phi_{k,\delta}(x)}\mathbb{CP}^{d_k-1},$$

is injective. Fix $p\in X$ and let s be a local rigid CR frame of L on a canonical coordinate patch D, $p\in D$, $|s|^2_h=e^{-2\Phi}$, with canonical local coordinates $x=(x_1,\ldots,x_{2n-1})=(z,\theta)$. We take local coordinate x and s so that (3.61) hold. From Lemma 4.2, we may assume that

$$\left|\big(e^{-k\Phi}\widetilde{f}_1\big)(p)\right|^2\geq c_0,\tag{4.35}$$

where $F_{k,\delta}f_j=s^k\widetilde{f}_j$ on D, $j=1,\ldots,d_k$ and $c_0>0$ is a constant independent of k and the point p. Let $g^1_k,\ldots,g^n_k\in\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$ be as in Lemma 4.4. From (4.13), it is not difficult to see that there is a $\hat{K}_0>0$ independent of the point p such that $f_1,g^1_k,\ldots,g^n_k$ are linearly independent over $\mathbb{C}$. Put

$$
\begin{aligned}
&p_k^j = \frac{e^{-k\Phi}\widetilde{g}_k^j}{e^{-k\Phi}\widetilde{f}_1}, \quad j = 1, \ldots, n,\\
&p_k^j = \alpha_k^{2j-1} + i\alpha_k^{2j}, \quad \alpha_k^{2j-1} = \operatorname{Re} p_k^j, \quad \alpha_k^{2j} = \operatorname{Im} p_k^j, \quad j = 1, \ldots, n-1,
\end{aligned} \tag{4.36}
$$

where $F_{k,\delta} g_k^j = s^k \widetilde{g}_k^j$ on D, $j = 1, \ldots, n$. From (4.13) and (4.35), it is not difficult to see that there is a $\widetilde{K}_0 > 0$ independent of the point p such that for all $k \geq \widetilde{K}_0$, we have

$$
\left|\partial_{z_t} p_k^t(p)\right| \geq c_1\sqrt{k}, \quad t = 1, \ldots, n-1, \quad \left|\partial_{x_{2n-1}} p_k^n(p)\right| \geq c_1 k, \tag{4.37}
$$

and

$$
\begin{aligned}
&\sup\left\{\left|\partial_{x_{2n-1}} p_k^t(p)\right|, \left|\partial_{z_s} p_k^t(p)\right|, \left|\partial_{z_s} p_k^n(p)\right|;\ s, t = 1, \ldots, n-1, s \neq t\right\}\\
&\quad + \sup\left\{\left|p_k^t(p)\right|, \left|\partial_{\overline{z}_s} p_k^t(p)\right|;\ s = 1, \ldots, n-1, t = 1, \ldots, n\right\} \leq \varepsilon_k,
\end{aligned} \tag{4.38}
$$

where ε_k is a sequence independent of the point p with $\lim_{k\to\infty} \varepsilon_k = 0$. From (4.37), (4.38) and some elementary linear algebra argument, we conclude that there is a $K_1 > 0$ independent of the point p such that for every $k \geq K_1$, the linear map $A_k : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n}$ represented by the matrix

$$
\begin{bmatrix}
\partial_{x_1}\big(e^{-k\Phi}\alpha_k^1\big)(p) & \partial_{x_2}\big(e^{-k\Phi}\alpha_k^1\big)(p) & \ldots & \partial_{x_{2n-1}}\big(e^{-k\Phi}\alpha_k^1\big)(p)\\
\partial_{x_1}\big(e^{-k\Phi}\alpha_k^2\big)(p) & \partial_{x_2}\big(e^{-k\Phi}\alpha_k^2\big)(p) & \ldots & \partial_{x_{2n-1}}\big(e^{-k\Phi}\alpha_k^2\big)(p)\\
\vdots & \vdots & \vdots & \vdots\\
\partial_{x_1}\big(e^{-k\Phi}\alpha_k^{2n}\big)(p) & \partial_{x_2}\big(e^{-k\Phi}\alpha_k^{2n}\big)(p) & \ldots & \partial_{x_{2n-1}}\big(e^{-k\Phi}\alpha_k^{2n}\big)(p)
\end{bmatrix},
$$

is injective. Hence the differential of the map

$$
X \ni x \longmapsto \left(\frac{\widetilde{g}_k^1}{\widetilde{f}_1}(x), \ldots, \frac{\widetilde{g}_k^n}{\widetilde{f}_1}(x)\right) \in \mathbb{C}^n
$$

at p is injective if $k \geq K_1$. From this and some elementary linear algebra arguments, we conclude that the differential of the map

$$
X \ni x \longmapsto \left(\frac{\widetilde{f}_2}{\widetilde{f}_1}(x), \ldots, \frac{\widetilde{f}_{d_k}}{\widetilde{f}_1}(x)\right) \in \mathbb{C}^{d_k}
$$

at p is injective if $k \geq K_1$. Theorem 4.3 follows. □

In the rest of this section, we will prove that for k large enough, the map $\Phi_{k,\delta} : X \to \mathbb{CP}^{d_k-1}$ is injective. We need some preparations. Let $(D, (z,\theta), \phi)$ be a BRT trivialization. We write $x = (z,\theta) = (x_1, \ldots, x_{2n-1})$, $x' = (x_1, \ldots, x_{2n-2}, 0)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$. We need

Lemma 4.5. *With the notations above, for every $u_k \in C^\infty(X, L^k)$, we have*

$$
\begin{aligned}
&(F_{k,\delta}u_k)(x)\\
&= \frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int e^{i(x_{2n-1}-y_{2n-1})\eta_{2n-1}}\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)e^{imy_{2n-1}}e^{-im\theta}u_k(e^{i\theta}x')d\theta d\eta_{2n-1}dy_{2n-1} \ \text{ on } D.
\end{aligned} \tag{4.39}
$$

Proof. Put $\tau_{\delta,k}(\eta_{2n-1}) := \tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)$. By the Fourier inversion formula, we have

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int e^{i(x_{2n-1}-y_{2n-1})\eta_{2n-1}}\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)e^{imy_{2n-1}}e^{-im\theta}u_k(e^{i\theta}x')d\theta d\eta_{2n-1}dy_{2n-1}\\
&=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int \hat{\tau}_{\delta,k}(y_{2n-1}-x_{2n-1})e^{imy_{2n-1}}e^{-im\theta}u_k(e^{i\theta}x')d\theta dy_{2n-1}\\
&=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int \hat{\tau}_{\delta,k}(y_{2n-1})e^{imy_{2n-1}+imx_{2n-1}}e^{-im\theta}u_k(e^{i\theta}x')d\theta dy_{2n-1}\\
&=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int \tau_{\delta,k}(m)e^{imx_{2n-1}}e^{-im\theta}u(e^{i\theta}x')d\theta\\
&=\frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int \tau_\delta\Big(\frac{m}{k}\Big)e^{-im\theta}u(e^{i\theta}x)d\theta\\
&=F_{k,\delta}u_k,
\end{aligned}\tag{4.40}$$

where $\hat{\tau}_{\delta,k}$ denotes the Fourier transform of $\tau_{\delta,k}$. From (4.40), the lemma follows. □

Lemma 4.6. *With the notations above, let $u_k\in C^\infty(X,L^k)$. Assume that there are constants $C>0$ and $M>0$ independent of k such that $|u_k(x)|^2_{h^k}\le Ck^M$, for every $x\in X$. If $\operatorname{Supp}u_k\bigcap D=\emptyset$ for every k, then $F_{k,\delta}u_k=O(k^{-\infty})$ on D.*

Proof. Assume that $D=U\times(-\epsilon_0,\epsilon_0)$, where U is an open set in $\mathbb{C}^{n-1}$ and $\epsilon_0>0$. Fix $D'\Subset D$ and let $\chi(y_{2n-1})\in C^\infty_0((-\epsilon_0,\epsilon_0))$ such that $\chi(y_{2n-1})=1$ for every $(y',y_{2n-1})\in D'$. Let

$$\begin{aligned}
R_ku_k(x)=&\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{|\theta|\le\pi}e^{i(x_{2n-1}-y_{2n-1})\eta_{2n-1}}\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)\\
&\times(1-\chi(y_{2n-1}))e^{imy_{2n-1}}e^{-im\theta}u_k(e^{i\theta}x')d\theta d\eta_{2n-1}dy_{2n-1},
\end{aligned}\tag{4.41}$$

where $x\in D'$. Since $\chi(y_{2n-1})=1$ for every $(y',y_{2n-1})\in D'$, we can integrate by parts with respect to η_{2n-1} several times and deduce that

$$R_ku_k(x)=O(k^{-\infty})\ \text{ on } D'.\tag{4.42}$$

From (4.39) and (4.41), we have

$$\begin{aligned}
&(F_{k,\delta}u_k-R_ku_k)(x)\\
&=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}}\int_{|\theta|\le\pi}e^{i(x_{2n-1}-y_{2n-1})\eta_{2n-1}}\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)\chi(y_{2n-1})e^{imy_{2n-1}}e^{-im\theta}u_k(e^{i\theta}x')d\theta d\eta_{2n-1}dy_{2n-1}\\
&=\frac{1}{(2\pi)}\int e^{i(x_{2n-1}-y_{2n-1})\eta_{2n-1}}\tau_\delta\Big(\frac{\eta_{2n-1}}{k}\Big)\chi(y_{2n-1})u_k(x_1,\ldots,x_{2n-2},y_{2n-1})d\eta_{2n-1}dy_{2n-1}=0
\end{aligned}\tag{4.43}$$

since $\operatorname{Supp}u_k\bigcap D=\emptyset$. From (4.42) and (4.43), the lemma follows. □

We need the following CR peak sections lemma.

Lemma 4.7. *Let $p\neq q$ be two points in X and let $\{x_k\}$, $\{y_k\}$ be two sequences in X with $x_k\to p$, $y_k\to q$. Then there exist $v_k\in\mathcal{H}^0_{b,\le k\delta}(X,L^k)$ such that $u_k=F_{k,\delta}v_k$ satisfies for k large enough,*

$$|u_k(x_k)|^2_{h^k}\ge 1,\ \ |u_k(y_k)|^2_{h^k}\le\frac{1}{2}\,.\tag{4.44}$$

Proof. Let $(D,(z,\theta),\phi)$ be a BRT trivialization with $p\in D$ and $q\notin D$. We may assume that $p=(0,0)$. As before, let

$$f_1\in\mathcal{H}^0_{b,\leq k\delta}(X,L^k),\ldots,f_{d_k}\in\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$$

be an orthonormal basis for $\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$ with respect to $(\,\cdot\,|\,\cdot\,)$. Let s be a local rigid CR frame of L on an open neighbourhood $D\subset X$ of p, $|s(x)|^2_h=e^{-2\Phi}$. Let $\chi\in C^\infty_0(D)$, $\chi=1$ on D_0, where $D_0\subset D$ is an open set of p. On D, put $F_{k,\delta}f_j(x)=s^k\widetilde{f}_j(x)$, $\widetilde{f}_j(x)\in C^\infty(D)$, $j=1,\ldots,d_k$. Assume that $\{x_k\}\Subset D_0$. Let

$$\widetilde{v}_k(x)=s^k(x)\otimes\sum_{j=1}^{d_k}\chi(x)(F_{k,\delta}f_j)(x)\overline{\widetilde{f}_j(x_k)}e^{-k\Phi(x_k)}\in C^\infty_0(D,L^k)\subset C^\infty(X,L^k). \tag{4.45}$$

In view of Theorem 1.1, we see that

$$\widetilde{v}_k(x)=s^k(x)\otimes\chi(x)\int e^{ik\varphi(x,x_k,t)+k\Phi(x)}g(x,x_k,t,k)dt+O(k^{-\infty})\ \text{ on } D. \tag{4.46}$$

Since $\int e^{ik\varphi(x,x_k,t)}g(x,x_k,t,k)dt=O(k^{-\infty})$ on $D\setminus D_0$ and

$$\Box^{(0)}_{b,k}(s^k(x)\otimes\textstyle\int e^{ik\varphi(x,x_k,t)+k\Phi(x)}g(x,x_k,t,k)dt)=O(k^{-\infty})\text{ on } D,$$

we conclude that

$$\Box^{(0)}_{b,k}\widetilde{v}_k=O(k^{-\infty})\ \text{ on } D. \tag{4.47}$$

Let $v_k=\Pi_{k,\leq k\delta}\widetilde{v}_k\in\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$ and let $u_k=F_{k,\delta}v_k=F_{k,\delta}\Pi_{k,\leq k\delta}\widetilde{v}_k$. From (3.21) and (4.47), we can check that

$$\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|=O(k^{-\infty}). \tag{4.48}$$

Form Kohn's estimates or the arguments in the proof [12, Theorem 8.3.5], we see that for every $s\in\mathbb{N}$, there is a constant $C_{s,k}>0$ such that

$$\|u\|_{s+1,k}\leq C_{s,k}(\|\Box^{(0)}_{b,k}u\|_{s,k}+\|Tu\|_{s,k}+\|u\|_{0,k}),\ \text{ for all } u\in C^\infty(X,L^k), \tag{4.49}$$

where $\|\cdot\|_{s,k}$ denotes the standard Sobolev norm of order s on the Sobolev space $H^s(X,L^k)$. There is condition $Y(q)$ in the assumption of [12, Theorem 8.3.5] in order that $\|Tu\|_{s,k}$ can be controlled by $\|\Box^{(q)}_{b,k}u\|_{s,k}$ and $\|u\|_{0,k}$. Moreover, the constant $C_{s,k}$ can be bounded by the $C^{P_s}(X)$-norm of the volume form on X, the Hermitian metric of L and the coefficients of $\Box^{(0)}_{b,k}$ and T on X, where $P_s\in\mathbb{N}$ only depends on s. Hence, there is a constant $C_s>0$ independent of k and $N_s\in\mathbb{N}$ such that $C_{s,k}\leq C_sk^{N_s}$, for all $k\in\mathbb{N}$. From this observation and (4.49), we deduce that

$$\begin{aligned}&\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s+1,k}\\&\leq C_sk^{N_s}\Big(\|\Box^{(0)}_{b,k}F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s,k}+\|TF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s,k}+\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{0,k}\Big).\end{aligned} \tag{4.50}$$

We claim that for every $s\in\mathbb{N}$, there are $\widetilde{N}_s\in\mathbb{N}$, $\widetilde{C}_s>0$ independent of k such that

$$\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s,k}\leq\widetilde{C}_sk^{\widetilde{N}_s}\Big(\sum_{j=0}^{s}\Big\|(\Box^{(0)}_{b,k})^jF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\Big\|\Big). \tag{4.51}$$

Taking $s=0$ in (4.50) and using the estimate

$$\|TF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|\leq k\delta\,\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|\,,$$

we get the claim (4.51) for $s = 1$. Assume that (4.51) holds for all $s \in \mathbb{N}$ with $s \leq s_0$, for some $s_0 \in \mathbb{N}$. Hence,

$$\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s_0,k} \leq \widetilde{C}_{s_0}k^{\widetilde{N}_{s_0}}\Big(\sum_{j=0}^{s_0}\Big\|(\Box^{(0)}_{b,k})^jF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\Big\|\Big). \tag{4.52}$$

Taking $s = s_0$ in (4.50), we get

$$\begin{aligned}&\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s_0+1,k}\\ &\leq C_{s_0}k^{N_{s_0}}\Big(\|\Box^{(0)}_{b,k}F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s_0,k}+\|TF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s_0,k}+\|F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{0,k}\Big).\end{aligned} \tag{4.53}$$

By substituting $\Box^{(0)}_{b,k}F_{k,\delta}(I-\Pi_{k,\leq k\delta})\tilde{v}_k$ to (4.52) and note that

$$F_{k,\delta}(I-\Pi_{k,\leq k\delta})(\Box^{(0)}_{b,k}F_{k,\delta}(I-\Pi_{k,\leq k\delta})\tilde{v}_k) = \Box^{(0)}_{b,k}F_{k,\delta}(I-\Pi_{k,\leq k\delta})\tilde{v}_k,$$

we get

$$\Big\|\Box^{(0)}_{b,k}F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\Big\|_{s_0,k} \leq \widetilde{C}_{s_0}k^{\widetilde{N}_{s_0}}\Big(\sum_{j=1}^{s_0+1}\Big\|(\Box^{(0)}_{b,k})^jF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\Big\|\Big). \tag{4.54}$$

By substituting $TF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k = F_{k,\delta}(I-\Pi_{k,\leq k\delta})T\widetilde{v}_k$ to (4.52), we get

$$\|TF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\|_{s_0,k} \leq \widetilde{C}_{s_0}k^{\widetilde{N}_{s_0}}\Big(\sum_{j=0}^{s_0}\Big\|(\Box^{(0)}_{b,k})^jF_{k,\delta}(I-\Pi_{k,\leq k\delta})T\widetilde{v}_k\Big\|\Big). \tag{4.55}$$

From (4.53), (4.54), (4.55) and note that

$$\Big\|(\Box^{(0)}_{b,k})^jF_{k,\delta}(I-\Pi_{k,\leq k\delta})T\widetilde{v}_k\Big\| \leq k\delta\Big\|(\Box^{(0)}_{b,k})^jF_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k\Big\|, \ \text{ for every } j \in \mathbb{N}_0.$$

we get the claim (4.51) for $s = s_0 + 1$. By induction, the claim (4.51) follows.

From (4.47), (4.48) and (4.51), we deduce that

$$F_{k,\delta}(I-\Pi_{k,\leq k\delta})\widetilde{v}_k = O(k^{-\infty}) \tag{4.56}$$

and thus

$$u_k = F_{k,\delta}\widetilde{v}_k + O(k^{-\infty}). \tag{4.57}$$

Let $\widetilde{\chi} \in C^\infty_0(D)$, $\widetilde{\chi}(x_k) = 1$ for each k and $\chi = 1$ on $\operatorname{Supp}\widetilde{\chi}$. We can repeat the proof of (3.56) with minor change and deduce that

$$\sum_{j=1}^{d_k}|\widetilde{\chi}F_{k,\delta}(1-\chi)F_{k,\delta}f_j(x)|^2_{h^k} = O(k^{-\infty}) \text{ on } D \tag{4.58}$$

and hence

$$\begin{aligned}|F_{k,\delta}\widetilde{v}_k(x_k)|^2_{h^k} &= \Big|\sum_{j=1}^{d_k}(F_{k,\delta}f_j)(x_k)\overline{\widetilde{f}_j(x_k)}e^{-k\Phi(x_k)}\Big|^2_{h^k}+O(k^{-\infty})\\ &= \int e^{ik\varphi(x_k,x_k,t)}g(x_k,x_k,t,k)dt+O(k^{-\infty})\\ &\geq Ck^n,\end{aligned} \tag{4.59}$$

where $C > 0$ is a constant independent of k. Note that $\mathrm{Supp}\,\widetilde{v}_k \subset D$ and $q \notin D$. From this observation and Lemma 4.6, we deduce that

$$|F_{k,\delta}\widetilde{v}_k(y_k)|^2_{h^k} = O(k^{-\infty}). \tag{4.60}$$

From (4.57), (4.59) and (4.60), the lemma follows. □

Theorem 4.8. *The map $\Phi_{k,\delta}$ is injective for k large enough.*

Proof. We assume that the claim of the theorem is not true. We can find $x_{k_j}, y_{k_j} \in X$, $x_{k_j} \neq y_{k_j}$, $0 < k_1 < k_2 < \dots$, $\lim_{j\to\infty} k_j = \infty$, such that $\Phi_{k_j,\delta}(x_{k_j}) = \Phi_{k_j,\delta}(y_{k_j})$, for each j. We may suppose that there are $x_k, y_k \in X$, $x_k \neq y_k$, such that $\Phi_{k,\delta}(x_k) = \Phi_{k,\delta}(y_k)$, for each k. We may assume that $x_k \to p \in X$, $y_k \to q \in X$, as $k \to \infty$. If $p \neq q$. From Lemma 4.7, we can find $u_k = F_{k,\delta}f_k$, $v_k = F_{k,\delta}g_k$, $f_k, g_k \in \mathcal{H}^0_{b,\leq k\delta}(X, L^k)$ such that for k large, we have

$$|u_k(x_k)|^2_{h^k} \geq 1, \ \ |u_k(y_k)|^2_{h^k} \leq \frac{1}{2}, \tag{4.61}$$

and

$$|v_k(y_k)|^2_{h^k} \geq 1, \ \ |v_k(x_k)|^2_{h^k} \leq \frac{1}{2}. \tag{4.62}$$

Now, $\Phi_{k,\delta}(x_k) = \Phi_{k,\delta}(y_k)$ implies that

$$|u_k(x_k)|^2_{h^k} = r_k\,|u_k(y_k)|^2_{h^k}\,, \ \ |v_k(x_k)|^2_{h^k} = r_k\,|v_k(y_k)|^2_{h^k}\,,$$

where $r_k \in \mathbb{R}_+$, for each k. We deduce from (4.61) that $r_k \geq 2$, for k large. But (4.62) implies that $r_k \leq \frac{1}{2}$, for k large. We get a contradiction. Thus, we must have $p = q$.

Let $\{f_j\}_{j=1}^{d_k}$ be an orthonormal basis of $\mathcal{H}^0_{b,\leq k\delta}(X, L^k)$. Let s be a local rigid CR frame of L on a BRT trivialization $(D, (z, \theta), \phi)$, $p \in D$, $|s|^2_h = e^{-2\Phi}$, $F_{k,\delta}f_j = s^k \otimes \widetilde{f}_j, j = 1, \dots, d_k$. Since both x_k and y_k converge to p, one can assume that $x_k, y_k \in D$, for every k. Since $\Phi_{k,\delta}(x_k) = \Phi_{k,\delta}(y_k)$, there is a $\lambda_k \in \mathbb{C}$, such that $e^{-k\Phi(x_k)}\widetilde{f}_j(x_k) = \lambda_k e^{-k\Phi(y_k)}$, for each k, and we may assume that $|\lambda_k| \geq 1$, for each k, and hence

$$e^{-k\Phi(x_k)}\widetilde{f}_j(x_k) = \lambda_k e^{-k\Phi(y_k)}\widetilde{f}_j(y_k), \lambda_k \in \mathbb{C}, |\lambda_k| \geq 1. \tag{4.63}$$

In fact, if for some x_k, y_k, $|\lambda_k| < 1$ one can replace x_k by y_k and y_k by x_k. This implies that

$$P_{k,\delta,s}(x_k, y_k) = \lambda_k P_{k,\delta,s}(y_k, y_k), \lambda_k \in \mathbb{C}, |\lambda_k| \geq 1. \tag{4.64}$$

We will show that (4.64) is impossible. Write $x_k = (z^k, x^k_{2n-1}) = (x^k_1, \dots, x^k_{2n-1})$, $y_k = (w^k, y^k_{2n-1}) = (y^k_1, \dots, y^k_{2n-1})$ and

$$z^k = (z^k_1, \dots, z^k_{n-1}), \ \ w^k = (w^k_1, \dots, w^k_{n-1}).$$

Let

$$\limsup_{k\to\infty} k|z^k - w^k|^2 = M \in [0, \infty].$$

By definition, there is a subsequence $(k_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{j\to\infty} k_j|z^{k_j} - w^{k_j}|^2 = \limsup_{k\to\infty} k|z^k - w^k|^2 = M.$$

Without loss of generality we can assume that

$$\lim_{k\to\infty} k|z^k - w^k|^2 = M, M \in [0, \infty].$$

Case I: $M \in (0, \infty]$. First we assume that $M = \infty$. From (3.4) we have

$$\limsup_{k\to\infty} k^{-n}|P_{k,\delta,s}(x_k, y_k)| \leq \limsup_{k\to\infty} \int e^{-k\mathrm{Im}\varphi(x_k,y_k,t)} |g_0(x_k, y_k, t)| \, dt. \tag{4.65}$$

Combining with (4.65) and the fact $\mathrm{Im}\varphi(x_y, y_k, s) \geq c|z^k - w^k|^2$ in (1.17) we have

$$\limsup_{k\to\infty} k^{-n}|P_{k,\delta,s}(x_k, y_k)| = 0.$$

This is a contradiction with $\lim_{k\to\infty} k^{-n}P_{k,\delta,s}(y_k, y_k) = \int g_0(p, p, t)dt \neq 0$ and the assumption (4.64). Thus we have $M < \infty$. From (3.4) we have

$$\lim_{k\to\infty} k^{-n}|P_{k,\delta,s}(x_k, y_k)| \leq e^{-cM} \int g_0(p, p, t)dt \tag{4.66}$$

for some positive constant c. On the other hand $\lim_{k\to\infty} k^{-n}|P_{k,\delta,s}(y_k, y_k)| = \int g_0(p, p, t)dt$. This is a contradiction with (4.64). Thus we have $M = 0$, that is

$$\lim_{k\to\infty} k|z^k - w^k|^2 = 0. \tag{4.67}$$

Set

$$\widehat{\widehat{\alpha_k}} = \sqrt{-1}\sum_{j=1}^{n-1}\left[\frac{\partial\phi(z^k)}{\partial\overline{z}_j}(\overline{z}_j^k - \overline{w}_j^k) - \frac{\partial\phi(z^k)}{\partial z_j}(z_j^k - w_j^k)\right] \in \mathbb{R}. \tag{4.68}$$

Recall that

$$\omega_0(x) = -dx_{2n-1} + i\sum_{j=1}^{n-1}\Big(\frac{\partial\phi}{\partial\overline{z}_j}(z)d\overline{z}_j - \frac{\partial\phi}{\partial z_j}(z)dz_j\Big), \quad x = (z, \theta).$$

Let

$$\limsup_{k\to\infty} k\left|y_{2n-1}^k - x_{2n-1}^k + \widehat{\widehat{\alpha_k}}\right| = N \in [0, \infty].$$

There is a subsequence $(k_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{j\to\infty} k_j\left|y_{2n-1}^{k_j} - x_{2n-1}^{k_j} + \widehat{\widehat{\alpha_{k_j}}}\right| = \limsup_{k\to\infty} k\left|y_{2n-1}^k - x_{2n-1}^k + \widehat{\widehat{\alpha_k}}\right|.$$

Without loss of generality we assume that

$$\lim_{k\to\infty} k\left|y_{2n-1}^k - x_{2n-1}^k + \widehat{\widehat{\alpha_k}}\right| = N \in [0, \infty]. \tag{4.69}$$

Case II:

$$\lim_{k\to\infty} k|z^k - w^k|^2 = 0; \lim_{k\to\infty} k\left|y_{2n-1}^k - x_{2n-1}^k + \widehat{\widehat{\alpha_k}}\right| = N \in (0, \infty]. \tag{4.70}$$

First we assume $N = \infty$. From (3.4) we have

$$k^{-n}P_{k,\delta,s}(x_k, y_k) = k^{-n}\int e^{ik\varphi(x_k,y_k,t)}g(x_k, y_k, t, k)dt + r_k, \tag{4.71}$$

where $|r_k| = O(k^{-\infty})$. By the second property in (1.17) we have

$$\begin{aligned}\varphi(x, y, t) =& t(y_{2n-1} - x_{2n-1}) + ti\sum_{j=1}^{n-1}\left[\frac{\partial\phi}{\partial\overline{z}_j}(z)(\overline{z}_j - \overline{w}_j) - \frac{\partial\phi}{\partial z_j}(z)(z_j - w_j)\right] \\ &+ i\sum_{j=1}^{n-1}\left[\frac{\partial\Phi}{\partial\overline{z}_j}(z)(\overline{z}_j - \overline{w}_j) - \frac{\partial\Phi}{\partial z_j}(z)(z_j - w_j)\right] + O(|x - y|^2).\end{aligned} \tag{4.72}$$

Note that

$$k|x_k-y_k|^2 \lesssim k|z^k-w^k|^2+k|y^k_{2n-1}-x^k_{2n-1}+\widehat{\widehat{\alpha_k}}|^2 \lesssim k|z^k-w^k|^2+k|y^k_{2n-1}-x^k_{2n-1}+\widehat{\widehat{\alpha_k}}|\epsilon_k, \tag{4.73}$$

where $\epsilon_k \to 0$. From (4.72), (4.73) and the assumption $N=\infty$ we have

$$\lim_{k\to\infty} k\frac{\partial\varphi(x_k,y_k,t)}{\partial t}=\infty. \tag{4.74}$$

Substituting (4.72) to (4.71), by (4.74) and integrating by parts with respect to t we have

$$\limsup_{k\to\infty} k^{-n}|P_{k,\delta,s}(x_k,y_k)|=0. \tag{4.75}$$

This is a contradiction with (4.64) since $\lim_{k\to\infty} k^{-n}P_{k,\delta,s}(y_k,y_k)=\int g_0(p,p,t)dt\neq 0$. Second, we assume $N<\infty$. Since $\lim\limits_{k\to\infty} k|z^k-w^k|^2=0$ and $\lim\limits_{k\to\infty} k|y^k_{2n-1}-x^k_{2n-1}+\widehat{\widehat{\alpha_k}}|=N<\infty$, by (4.73) we have that $\lim\limits_{k\to\infty} k|x_k-y_k|^2=0$. Substituting (4.72) to (4.71) we have that

(4.76)

$$\begin{aligned}\limsup_{k\to\infty} k^{-n}|P_{k,\delta,s}(x_k,y_k)| &\leq \limsup_{k\to\infty} k^{-n}\left|\int e^{ik\left[t(y_{2n-1}-x_{2n-1}+\widehat{\widehat{\alpha_k}})+O(|x_k-y_k|^2)\right]}g(x_k,y_k,t,k)dt\right|\\ &\leq \left|\int e^{iNt}g_0(p,p,t)dt\right|.\end{aligned}$$

Since $\left|\int e^{iNt}g_0(p,p,t)dt\right|<\int g_0(p,p,t)dt=\lim_{k\to\infty}k^{-n}P_{k,\delta,s}(y_k,y_k)$, combining this with (4.64) and (4.76) we get a contradiction. Thus we have $N=0$.

Case III:

$$\lim_{k\to\infty} k|z^k-w^k|^2=0;\ \lim_{k\to\infty} k|y^k_{2n-1}-x^k_{2n-1}+\widehat{\widehat{\alpha_k}}|=0. \tag{4.77}$$

Define

$$A_k(u)=\left|P_{k,\delta,s}(ux_k+(1-u)y_k,y_k)\right|^2,\ B_k(u)=P_{k,\delta,s}(ux_k+(1-u)y_k,ux_k+(1-u)y_k)\cdot P_{k,\delta,s}(y_k,y_k).$$

Set $H_k(u)=\frac{A_k(u)}{B_k(u)}$. By Schwartz inequality, we have $0\leq H_k(u)\leq 1$. Since $H_k(0)=H_k(1)=1$, then there exists a $u_k\in(0,1)$ such that $H_k^{''}(u_k)\geq 0$. By direct calculation,

$$H_k^{''}(u_k)=\frac{A_k^{''}(u_k)}{B_k(u_k)}-2\frac{A_k^{'}(u_k)B_k^{'}(u_k)}{B_k^2(u_k)}-\frac{A_k(u_k)B_k^{''}(u_k)}{B_k^2(u_k)}+2\frac{A_k(u_k)B_k^{'2}(u_k)}{B_k^3(u_k)}. \tag{4.78}$$

Write $\alpha_k(u)=P_{k,\delta,s}(ux_k+(1-u)y_k,y_k)$. Then $A_k(u)=|\alpha_k(u)|^2$, $A_k'(u)=\alpha_k'(u)\overline{\alpha_k(u)}+\alpha_k(u)\overline{\alpha_k'(u)}$ and

$$A_k''(u_k)=\alpha_k''(u_k)\overline{\alpha_k(u_k)}+2|\alpha_k'(u_k)|^2+\alpha_k(u_k)\overline{\alpha_k''(u_k)}. \tag{4.79}$$

By Theorem 3.3, we have

$$\alpha_k(u)=\int e^{ik\varphi(ux_k+(1-u)y_k,y_k,t)}g(ux_k+(1-u)y_k,y_k,t,k)dt+\gamma_k(u), \tag{4.80}$$

where $\gamma_k(u)=O(k^{-\infty})$. Write $\beta_k(u)=\int e^{ik\varphi(ux_k+(1-u)y_k,y_k,t)}g(ux_k+(1-u)y_k,y_k,t,k)dt$ and then

(4.81)

$$\begin{aligned}A_k''(u_k)&=2|\beta_k'(u_k)|^2+\beta_k''(u_k)\overline{\beta_k(u_k)}+\overline{\beta''(u_k)}\beta_k(u_k)+2\beta_k'(u_k)\overline{\gamma_k'(u_k)}+2\gamma_k'(u_k)\overline{\beta_k'(u_k)}\\ &+\beta_k''(u_k)\overline{\gamma_k(u_k)}+\gamma_k(u_k)\overline{\beta_k''(u_k)}+\overline{\beta_k(u_k)}\gamma_k''(u_k)+\beta_k(u_k)\overline{\gamma_k''(u_k)}\\ &+\gamma_k''(u_k)\overline{\gamma_k(u_k)}+\gamma_k(u_k)\overline{\gamma''(u_k)}+2|\gamma_k'(u_k)|^2.\end{aligned}$$

Set

$$\widehat{\alpha}_k(u) = i\sum_{j=1}^{n-1}\left[\frac{\partial\phi}{\partial\overline{z}_j}\Big|_{w^k+u(z^k-w^k)}\cdot(\overline{z}^k_j-\overline{w}^k_j)-\frac{\partial\phi}{\partial z_j}\Big|_{w^k+u(z^k-w^k)}\cdot(z^k_j-w^k_j)\right]. \tag{4.82}$$

By the mean value theorem,

$$|k(y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k))-k(y^k_{2n-1}-x^k_{2n-1}+\widehat{\widehat{\alpha_k}})| = k|\widehat{\widehat{\alpha_k}}-\widehat{\alpha}_k(u_k)| \lesssim k|z^k-w^k|^2. \tag{4.83}$$

Then (4.77) and (4.83) implies that

$$\lim_{k\to\infty} k|y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k)| = 0. \tag{4.84}$$

By direct calculation we have that

$$\begin{aligned}
&2|\beta'_k(u_k)|^2+\beta''_k(u_k)\overline{\beta_k(u_k)}+\overline{\beta''_k(u_k)}\beta_k(u_k)\\
&=2k^{2n+2}\left[\left|\int tg_0(p,p,t)dt\right|^2-\int g_0(p,p,t)t^2dt\cdot\int g_0(p,p,t)dt\right](y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k))^2\\
&\quad-2k^{2n+1}\int\left[\sum_{j,l=1}^{2n-2}\frac{\partial^2\mathrm{Im}\varphi(p,p,t)}{\partial x_j\partial x_l}(x^k_j-y^k_j)(x^k_l-y^k_l)\right]g_0(p,p,t)dt\\
&\quad+o(k^{2n})O(k|z^k-w^k|^2+k^2|y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k)|^2).
\end{aligned} \tag{4.85}$$

By (3.62) there exists $c>0$ such that for δ sufficiently small the following holds,

$$\int\left[\sum_{j,l=1}^{2n-2}\frac{\partial^2\mathrm{Im}\varphi(p,p,t)}{\partial x_j\partial x_l}(x^k_j-y^k_j)(x^k_l-y^k_l)\right]g_0(p,p,t)dt \ge c|z^k-w^k|^2. \tag{4.86}$$

By Hölder's inequality, $\left|\int tg_0(p,p,t)dt\right|^2 < \int t^2g_0(p,p,t)dt\cdot\int g_0(p,p,t)dt$, so by combining (4.85) and (4.86) there exists $c_1>0$ such that

$$\begin{aligned}
&\limsup_{k\to\infty} k^{-2n}\left[k|z^k-w^k|^2+k^2(y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k))^2\right]^{-1}\times\\
&\qquad\left[2|\beta'_k(u_k)|^2+\beta''_k(u_k)\overline{\beta_k(u_k)}+\overline{\beta''_k(u_k)}\beta_k(u_k)\right] < -c_1 < 0.
\end{aligned} \tag{4.87}$$

By direct calculation we have that

$$\limsup_{k\to\infty} k^{-2n}\left[k|z^k-w^k|^2+k^2(y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k))^2\right]^{-1}C_k = 0, \tag{4.88}$$

where

$$\begin{aligned}
C_k =&2\beta'_k(u_k)\overline{\gamma'_k(u_k)}+2\gamma'_k(u_k)\overline{\beta'_k(u_k)}+\beta''_k(u_k)\overline{\gamma_k(u_k)}+\gamma_k(u_k)\overline{\beta''_k(u_k)}+\\
&\overline{\beta_k(u_k)}\gamma''_k(u_k)+\beta_k(u_k)\overline{\gamma''_k(u_k)}+\gamma''_k(u_k)\overline{\gamma_k(u_k)}+\gamma_k(u_k)\overline{\gamma''(u_k)}+2|\gamma'_k(u_k)|^2.
\end{aligned}$$

Combining (4.87), (4.88) and (4.81) there exists $c_2>0$ such that

$$\limsup_{k\to\infty}\left[k|z^k-w^k|^2+k^2(y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k))^2\right]^{-1}\frac{A''_k(u_k)}{B_k(u_k)} < -c_2 < 0. \tag{4.89}$$

It is straightforward to see that

$$\begin{aligned}&\limsup_{k\to\infty}\left[k|z^k-w^k|^2+k^2(y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k))^2\right]^{-1}\times\\&\left\{2\frac{|A_k'(u_k)|\cdot|B_k'(u_k)|}{B_k^2(u_k)}+\frac{|A_k(u_k)|\cdot|B_k''(u_k)|}{B_k^2(u_k)}+2\frac{|A_k(u_k)|\cdot|B_k'^2(u_k)|}{B_k^3(u_k)}\right\}=0.\end{aligned}\tag{4.90}$$

From (4.89) and (4.90) we have

$$\limsup_{k\to\infty}\left[k|z^k-w^k|^2+k^2(y^k_{2n-1}-x^k_{2n-1}+\widehat{\alpha}_k(u_k))^2\right]^{-1}H_k''(u_k)<0.\tag{4.91}$$

This is a contradiction with $H_k''(u_k)\geq 0$. □

Proof of Theorem 1.3. Since X is compact Theorems 4.3 and 4.8 implies the modified Kodaira map $\Phi_{k,\delta}$ defined in (4.5) is an embedding. For different $m_1,m_2\in\mathbb{Z}$, $\mathcal{H}^0_{b,m_1}(X,L^k)\perp\mathcal{H}^0_{b,m_2}(X,L^k)$, thus we can choose an orthonormal basis $\{f_j\}_{j=1}^{d_k}$ of $\mathcal{H}^0_{b,\leq k\delta}(X,L^k)$ such that $f_j\in\mathcal{H}^0_{b,m_j}(X,L^k)$ with $m_j\in\mathbb{Z}$ and $|m_j|\leq k\delta$ for each $1\leq j\leq d_k$. Then $F_{k,\delta}f_j\in\mathcal{H}^0_{b,m_j}(X)$ for each j. For any $p\in X$, from the argument in the proof of [22, Lemma 1.20] we can find a local trivialization W which is an S^1 invariant neighborhood of p and local trivializing rigid CR section s of L on W. Then $F_{k,\delta}f_j=s^k\otimes\widetilde{f}_j$ on W with $\widetilde{f}_j\in C^\infty(W),1\leq j\leq d_k$. Since $F_{k,\delta}f_j\in\mathcal{H}^0_{b,m_j}(X,L^k)$, we have $T\widetilde{f}_j=im_j\widetilde{f}_j$. Then for any $\theta\in[0,2\pi)$ we have $\widetilde{f}_j(e^{i\theta}p)=e^{im_j\theta}\widetilde{f}_j(p)$. Thus,

$$\Phi_{k,\delta}(e^{i\theta}p)=\left[\widetilde{f}_1(e^{i\theta}p),\ldots,\widetilde{f}_{d_k}(e^{i\theta}p)\right]=\left[e^{im_1\theta}\widetilde{f}_1(p),\ldots,e^{im_{d_k}\theta}\widetilde{f}_{d_k}(p)\right]=e^{i\theta}\Phi_{k,\delta}(p),$$

so we get the conclusion of Theorem 1.3. □

Proof of Corollaries 1.4 and 1.5. They are immediate consequences of Theorem 1.3. □

We close with an application of Corollary 1.5.

Example 4.9. Let $(X,T^{1,0}X)$ be a compact CR manifold of dimension 3 with a transversal CR locally free S^1-action. Assume that X admits a S^1-equivariant positive CR line bundle L. For example, if X is strongly pseudoconvex, there is an S^1-equivariant positive CR line bundle over X. Take $Z\in C^\infty(X,T^{1,0}X)$ such that Z_x is a basis for $T^{1,0}_xX$, for every $x\in X$. Let h be a distribution on X with $Th=0$ and Zh smooth (note that it is possible that there is a non-smooth function h such that Zh is smooth). Hence, $Zh\in C^\infty(X)$. Consider $\hat{T}^{1,0}X:=\operatorname{span}\langle Z+(Zh)T\rangle$. Then, $(X,\hat{T}^{1,0}X)$ is a compact CR manifold of dimension 3 with a transversal CR locally free S^1-action. Moreover, L is still an S^1-equivariant positive CR line bundle over $(X,\hat{T}^{1,0}X)$. To see this, let s be a rigid CR frame with respect to $T^{1,0}X$ and $|s|^2=e^{-2\phi}$. Then s is still a rigid CR frame with respect to $\hat{T}^{1,0}X$. Let $\hat{\overline{\partial}}_b$ be the tangential Cauchy-Riemann operator with respect to $\hat{T}^{1,0}X$ and $\hat{\partial}_b$ its conjugate. Then the curvature of L is given by $\hat{R}^L=2\hat{\partial}_b\hat{\overline{\partial}}_b\phi$. Since $\hat{\overline{\partial}}_b\phi=(\overline{Z}+\overline{Z}hT)\phi=\overline{Z}\phi d\overline{z}$ we have $\hat{\partial}_b\hat{\overline{\partial}}_b\phi=Z\overline{Z}\phi\,dz\wedge d\overline{z}=\partial_b\overline{\partial}_b\phi>0$.

From Theorem 1.3, we deduce that there exists smooth CR embeddings $\Phi_{k,\delta}$ of $(X,\hat{T}^{1,0}X)$ in $\mathbb{CP}^{d_k-1}$ which are S^1-equivariant with respect to weighted diagonal actions.

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