

THE FIRST COEFFICIENTS OF THE ASYMPTOTIC
EXPANSION OF THE BERGMAN KERNEL
OF THE Spin^c DIRAC OPERATOR

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We establish the existence of the asymptotic expansion of the Bergman kernel associated to the spin^c Dirac operators acting on high tensor powers of line bundles with non-degenerate mixed curvature (negative and positive eigenvalues) by extending [15]. We compute the second coefficient \mathbf{b}_1 in the asymptotic expansion using the method of [24].

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0. Introduction

In [15, Theorem 4.18'], Dai, Liu and Ma established the full off-diagonal asymptotic expansion of the Bergman kernel of the spin^c Dirac operator on high tensor powers of a line bundle on a compact symplectic manifold. This paper is a continuation of their work.

For some applications, it is necessary to compute the first two coefficients \mathbf{b}_0 , \mathbf{b}_1 of the asymptotic expansion. The approach of [15] is to relate the heat kernel and the Bergman kernel expansions. The computation of the coefficients of the Bergman kernel expansion is done by using the corresponding coefficients of the heat kernel. Thus, \mathbf{b}_0 is calculated in [15, Theorem 1.1] and \mathbf{b}_1 in [15, Theorem 1.3], the latter only in the Kähler case. These results brought a new proof of [14, 22, 27, 28].

Considering the symplectic case, the coefficient of t^1 in the Taylor expansion of the rescaled operator does not vanish, thus it is complicate to compute \mathbf{b}_1 in this way. However, we developed in [24, Sec. 1.5] a method of formal power series to

compute the coefficients for the renormalized Bochner–Laplacian. The main result of this paper is Theorem 2.1, where we compute the coefficient \mathbf{b}_1 in the asymptotic expansion of the Bergman kernel associated to the spin^c Dirac operators by applying the method in [24, Secs. 1.5, 2.3]. Comparing with [24, Sec. 2.3], the contribution from the coefficient of t^1 in the Taylor expansion of the rescaled operator L_2^t in (2.7) is quite complicate here. But after taking the trace in (2.2), we refine the Hermitian scalar curvature of [17]. For more details on our approach we also refer the readers to our recent book [25].

Another feature of the paper is that we do not suppose that the almost complex structure polarizes the curvature of the line bundle, that is we allow bundles with mixed curvature (negative and positive eigenvalues). In Sec. 1, we explain that the arguments in [15] still work well under this condition (cf. (1.6)). Then we compute the coefficient \mathbf{b}_1 in Sec. 2.

1. Bergman Kernel of the Spin^c Dirac Operator

This section is organized as follows. In Sec. 1.1, we recall the Lichnerowicz formula for the spin^c Dirac operator D_p . As a consequence we exhibit the spectral gap for D_p^2 without assumption that the almost complex structure J polarizes the symplectic form ω . This is done in Sec. 1.2. We explain in Sec. 1.3 the full off-diagonal asymptotic expansion for the Bergman kernel. Then we show how to handle the operator $\bar{\partial} + \bar{\partial}^*$, which is the content of Sec. 1.4, we explain also its relation to the tangential Cauchy–Riemann complex in Sec. 1.5.

1.1. The spin^c Dirac operator

Let (X, ω) be a compact connected symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$, where $R^L = (\nabla^L)^2$ is the curvature of (L, ∇^L) . Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E .

Let J be an almost complex structure on TX and g^{TX} be a Riemannian metric on X compatible with J , i.e. $g^{TX}(\cdot, \cdot) = g^{TX}(J\cdot, J\cdot)$. We designate by ∇^{TX} the Levi–Civita connection on (TX, g^{TX}) and by R^{TX} and r^X its curvature and scalar curvature, respectively.

The almost complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Then $P^{(1,0)} = \frac{1}{2}(1 - \sqrt{-1}J)$ is the projection from $TX \otimes_{\mathbb{R}} \mathbb{C}$ onto $T^{(1,0)}X$. For any $v \in TX \otimes_{\mathbb{R}} \mathbb{C}$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\bar{v}_{1,0}^* \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then $c(v) = \sqrt{2}(\bar{v}_{1,0}^* \wedge -i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where \wedge and i denote the exterior and interior product respectively.

Let $\nabla^{T^{(1,0)}X} = P^{(1,0)}\nabla^{TX}P^{(1,0)}$ be the connection on $T^{(1,0)}X$ induced by ∇^{TX} , with curvature $R^{T^{(1,0)}X}$. Let ∇^{Cliff} be the Clifford connection on $\Lambda(T^{*(0,1)}X)$

induced canonically by ∇^{TX} (cf. [23, Sec. 2])^a and let R^{Cliff} be its curvature. Let $\nabla^X J$ be the covariant derivative of J induced by ∇^{TX} . Let $\{e_i\}_{1 \leq i \leq 2n}$ be an orthonormal basis of TX . From the definitions, we get (cf. also [24, (0.12)])

$$\begin{aligned}
 R^{T^{(1,0)}X} &= P^{1,0} \left[R^{TX} - \frac{1}{4}(\nabla^X J) \wedge (\nabla^X J) \right] P^{1,0}, \\
 R^{\text{Cliff}} &= \frac{1}{4} \langle R^{TX} e_l, e_m \rangle c(e_l) c(e_m) + \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}].
 \end{aligned}
 \tag{1.1}$$

Let ∇^{E_p} be the connection on $E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$ induced by ∇^{Cliff} , ∇^L and ∇^E . The spin^c Dirac operator $D_p := \sum_i c(e_i) \nabla_{e_i}^{E_p}$ and acts on $\Omega^{0,\bullet}(X, L^p \otimes E) = \bigoplus_{q=0}^n \Omega^{0,q}(X, L^p \otimes E)$, the direct sum of spaces of $(0, q)$ -forms with values in $L^p \otimes E$. We content ourselves to describe the operator D_p^2 in the sequel.

Let $\langle \cdot, \cdot \rangle_{E_p}$ be the metric on E_p induced by g^{TX} , h^L and h^E . Let dv_X be the Riemannian volume form of (TX, g^{TX}) . The L^2 -scalar product on $\Omega^{0,\bullet}(X, L^p \otimes E)$, the space of smooth sections of E_p , is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_{E_p} dv_X(x).
 \tag{1.2}$$

We denote the corresponding norm with $\| \cdot \|_{L^2}$.

Let $(\nabla^{E_p})^*$ be the formal adjoint of ∇^{E_p} with respect to (1.2). Set

$$\mathbf{c}(R) = \frac{1}{2} \left(R^E + \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}] \right) (e_l, e_m) c(e_l) c(e_m).
 \tag{1.3}$$

Then the Lichnerowicz formula [4, Theorem 3.52] (cf. [23, Theorem 2.2]) for D_p^2 is

$$D_p^2 = (\nabla^{E_p})^* \nabla^{E_p} + \frac{1}{4} r^X + \frac{1}{2} p R^L(e_l, e_m) c(e_l) c(e_m) + \mathbf{c}(R).
 \tag{1.4}$$

The Bergman kernel $P_p(x, x')$, $(x, x' \in X)$, is the smooth kernel with respect to $dv_X(x')$ of the orthogonal projection P_p from $\Omega^{0,\bullet}(X, L^p \otimes E)$ on $\text{Ker } D_p$. Then $P_p(x, x)$ is an element of $\text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$.

1.2. Spectral gap of the spin^c Dirac operator

We choose the almost complex structure J such that ω is J -invariant, i.e. $\omega(\cdot, \cdot) = \omega(J\cdot, J\cdot)$. But we do not suppose that $\omega(\cdot, J\cdot)$ is positive in Secs. 1.2–1.4. This is the difference comparing with the assumption in [15, 23, 24].

Let $\mathbf{J} : TX \rightarrow TX$ be the skew-adjoint linear map which satisfies the relation

$$\omega(u, v) = g^{TX}(\mathbf{J}u, v)
 \tag{1.5}$$

for $u, v \in TX$. Then J commutes with \mathbf{J} . Thus $\mathbf{J} \in \text{End}(T^{(1,0)}X)$, and for any $x \in X$, we can diagonalize \mathbf{J}_x , i.e. find an orthonormal basis $\{w_j\}_{j=1}^n$ of $T^{(1,0)}X$

^aIn [23, (2.3)], one missed a term $+\frac{1}{2} \text{Tr}_{T^{(1,0)}X} \Gamma$ in the right-hand side of the first line, and the second line should be read as “ $= d + \sum_{lm} \{ \langle \Gamma w_l, \bar{w}_m \rangle \bar{w}^m \wedge i \bar{w}_l + \dots$ ”.

such that $\mathbf{J}_x w_j = \frac{\sqrt{-1}}{2\pi} a_j(x) w_j$ with $a_j(x) \in \mathbb{R}$. As ω is non-degenerate, the number of negative eigenvalue of $\mathbf{J}_x \in \text{End}(T_x^{(1,0)} X)$ does not depend on x , and we denote it by q (in [15, 23, 24], we suppose that $q = 0$). From now on, we assume that

$$\mathbf{J}_x w_j = \frac{\sqrt{-1}}{2\pi} a_j w_j, \quad a_j(x) < 0 \text{ for } j \leq q \text{ and } a_j(x) > 0 \text{ for } j > q. \quad (1.6)$$

Then the vectors $\{w_j\}_{j=1}^q$ span a sub-bundle W of $T^{(1,0)} X$. Set

$$\begin{aligned} \omega_d(x) &= - \sum_{j=1}^n a_j \bar{w}^j \wedge i \bar{w}_j + \sum_{j=1}^q a_j = \sum_{j=1}^q a_j i \bar{w}_j \wedge \bar{w}^j - \sum_{j=q+1}^n a_j \bar{w}^j \wedge i \bar{w}_j, \\ \tau(x) &= \pi \text{Tr} |_{TX} (-\mathbf{J}^2)^{1/2} = \sum_{j=1}^n |a_j| = - \sum_{j=1}^q a_j + \sum_{j=q+1}^n a_j, \\ \mu_0 &= \inf_{x \in X, j} |a_j(x)|. \end{aligned} \quad (1.7)$$

Then we have

$$\frac{1}{2} R^L(e_l, e_m) c(e_l) c(e_m) = -2\omega_d - \tau. \quad (1.8)$$

The following result extends [23, Theorem 2.5] to the current situation. We denote by $\Omega^{\neq q}(X, L^p \otimes E) = \bigoplus_{k \neq q} \Omega^{0,k}(X, L^p \otimes E)$.

Theorem 1.1. *There exists $C > 0$ such that for any $p \in \mathbb{N}$*

$$\|D_p s\|_{L^2}^2 \geq (2p\mu_0 - C) \|s\|_{L^2}^2, \quad \text{for } s \in \Omega^{\neq q}(X, L^p \otimes E). \quad (1.9)$$

Proof. By (1.4) and (1.8), for $s \in \Omega^{0,\bullet}(X, L^p \otimes E)$,

$$\begin{aligned} \|D_p s\|_{L^2}^2 &= \{ \|\nabla^{\Lambda^{0,\bullet} \otimes L^p \otimes E} s\|_{L^2}^2 - p \langle \tau(x) s, s \rangle \} - 2p \langle \omega_d s, s \rangle \\ &\quad + \left\langle \left(\frac{1}{4} r^X + \mathbf{c}(R) \right) s, s \right\rangle. \end{aligned} \quad (1.10)$$

We consider now $s \in \mathcal{C}^\infty(X, L^p \otimes E')$, where $E' = E \otimes \Lambda(T^{*(0,1)} X)$. We apply [23, Corollary 2.4] which is a direct consequence of the Lichnerowicz formula (cf. also [20, Theorem 1], [10, Theorem 2.1], [13, Theorem 4.4]), to the almost complex structure $J' \in \text{End}(TX)$ defined by

$$J' w_j = -\sqrt{-1} w_j \text{ for } j \leq q \text{ and } J' w_j = \sqrt{-1} w_j \text{ for } j > q,$$

then we get there exists $C > 0$ such that for any $p > 0$, $s \in \mathcal{C}^\infty(X, L^p \otimes E')$, we have

$$\|\nabla^{L^p \otimes E'} s\|_{L^2}^2 - p \langle \tau(x) s, s \rangle \geq -C \|s\|_{L^2}^2. \quad (1.11)$$

If $s \in \Omega^{\neq q}(X, L^p \otimes E)$, the second term of (1.10), $-2p \langle \omega_d s, s \rangle$ is bounded below by $2p\mu_0 \|s\|_{L^2}^2$, while the third term of (1.10) is $\mathcal{O}(\|s\|_{L^2}^2)$. The proof of (1.9) is completed. \square

Set

$$\begin{aligned} D_p^+ &= D_p|_{\Omega^{0,\text{even}}}, \quad D_p^- = D_p|_{\Omega^{0,\text{odd}}}, \\ o_q &= - \text{ if } q \text{ is even; } \quad o_q = + \text{ if } q \text{ is odd.} \end{aligned} \quad (1.12)$$

By using the trick of the proof of McKean–Singer formula and Theorem 1.1, the same proof as in [23, Sec. 3] gives the following extension of [23, Theorem 1.1]. For any operator A , we denote by $\text{Spec}(A)$ the spectrum of A .

Theorem 1.2. *There exists $C > 0$ such that for $p \in \mathbb{N}$,*

$$\text{Spec}(D_p^2) \subset \{0\} \cup [2p\mu_0 - C, +\infty[. \tag{1.13}$$

For p large enough, we have

$$\text{Ker } D_p^{2q} = \{0\}. \tag{1.14}$$

1.3. Off-diagonal asymptotic expansion of Bergman kernel

The existence of the spectral gap expressed in Theorem 1.2 allows us to obtain immediately as in [15, Proposition 4.1] the far off-diagonal behavior of the Bergman kernel. Namely, for any $l, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $C_{l,m,\varepsilon} > 0$ such that for $p \geq 1$, $x, x' \in X$, $d(x, x') > \varepsilon$,

$$|P_p(x, x')|_{\mathcal{C}^m(X \times X)} \leq C_{l,m,\varepsilon} p^{-l}. \tag{1.15}$$

We denote by $I_{\det(\overline{W}^*) \otimes E}$ the orthogonal projection from $\mathbf{E} := \Lambda(T^{*(0,1)}X) \otimes E$ onto $\det(\overline{W}^*) \otimes E$. Let $\pi : TX \times_X TX \rightarrow X$ be the natural projection from the fiberwise product of TX on X . Let $\nabla^{\text{End}(\mathbf{E})}$ be the connection on $\text{End}(\Lambda(T^{*(0,1)}X) \otimes E)$ induced by ∇^{Cliff} and ∇^E .

We use now normal coordinates centered at $x_0 \in Y$. We identify L_Z , E_Z and $(E_p)_Z$ for $Z \in B^{T_{x_0}X}(0, \varepsilon)$ to L_{x_0}, E_{x_0} and $(E_p)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and ∇^{E_p} along the curve $\gamma_Z : [0, 1] \ni u \rightarrow \exp_{x_0}^X(uZ)$. Under this identification and (1.15), we will view $P_p(x, x')$ as a smooth section $P_{p,x_0}(Z, Z')$, ($Z, Z' \in B^{T_{x_0}X}(0, \varepsilon)$), of $\pi^*(\text{End}(\Lambda(T^{*(0,1)}X) \otimes E))$ on $TX \times_X TX$. And $\nabla^{\text{End}(\mathbf{E})}$ induces naturally a \mathcal{C}^m -norm for the parameter $x_0 \in X$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_X(Z) = \kappa(Z)dv_{TX}(Z), \tag{1.16}$$

with $\kappa(0) = 1$. We denote by $\det_{\mathbb{C}}$ for the determinant function on the complex bundle $T^{(1,0)}X$, and $|\mathbf{J}_{x_0}| = (-\mathbf{J}_{x_0}^2)^{1/2}$. Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U . On $T_{x_0}X \simeq \mathbb{R}^{2n}$, set

$$L_{2,\mathbb{C}}^0 = - \sum_j \left(\nabla_{e_j} + \frac{1}{2} R_{x_0}^L(Z, e_j) \right)^2 - \tau_{x_0}. \tag{1.17}$$

Let $P(Z, Z')$ be the Bergman kernel of $L_{2,\mathbb{C}}^0$, i.e. the smooth kernel of the orthogonal projection from $L^2(\mathbb{R}^{2n}, \mathbb{C})$ onto $\text{Ker } L_{2,\mathbb{C}}^0$. Then for $Z, Z' \in T_{x_0}X$,

$$P(Z, Z') = \det_{\mathbb{C}}(|\mathbf{J}_{x_0}|) \exp\left(-\frac{\pi}{2} \langle |\mathbf{J}_{x_0}|(Z - Z'), (Z - Z') \rangle - \pi \sqrt{-1} \langle \mathbf{J}_{x_0} Z, Z' \rangle\right). \tag{1.18}$$

The main result of this part is the following extension of [15, Theorem 4.18'].

Theorem 1.3. *There exist polynomials $J_r(Z, Z') \in \text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ ($x_0 \in X$), in Z, Z' with the same parity as r and with $\deg J_r \leq 3r$, whose coefficients are polynomials in $R^{TX}, R^{T(1,0)X}, R^E$ (and R^L) and their derivatives of order $\leq 2r - 1$ (respectively, $\leq 2r$) and reciprocals of linear combinations of eigenvalues of \mathbf{J} at x_0 , such that by setting*

$$P_{x_0}^{(r)}(Z, Z') = J_r(Z, Z')P(Z, Z'), \quad J_0(Z, Z') = \det_{\mathbb{C}}(|\mathbf{J}_{x_0}|)I_{\det(\overline{W}^*) \otimes E}, \quad (1.19)$$

the following statement holds: there exists $C'' > 0$ such that for any $k, m, m' \in \mathbb{N}$, there exist $N \in \mathbb{N}$ and $C > 0$ with

$$\begin{aligned} & \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_p(Z, Z') - \sum_{r=0}^k P^{(r)}(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-1}(Z') p^{-r/2} \right) \right|_{\mathcal{E}^{m'}(X)} \\ & \leq C p^{-(k+1-m)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-\sqrt{C''\mu_0} \sqrt{p}|Z - Z'|) + \mathcal{O}(p^{-\infty}). \end{aligned} \quad (1.20)$$

for any $\alpha, \alpha' \in \mathbb{N}^n$, with $|\alpha| + |\alpha'| \leq m$, any $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leq \varepsilon$ and any $x_0 \in X, p \geq 1$.

Here $\mathcal{E}^{m'}(X)$ is the $\mathcal{E}^{m'}$ -norm for the parameter $x_0 \in X$. We say that a term $T = \mathcal{O}(p^{-\infty})$ if for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that the \mathcal{E}^{l_1} -norm of T is dominated by $C_{l,l_1} p^{-l}$.

Proof. By using Theorems 1.1 and 1.2, we know the arguments in [15, Secs. 4.1–4.3] go through without any change. By (1.4), (1.8), as in [15, (4.105)], the corresponding limit operator here is still

$$L_2^0 = L_{2,\mathbb{C}}^0 - 2\omega_{d,x_0}. \quad (1.21)$$

Let $e^{-uL_{2,c}^0}(Z, Z'), e^{-uL_2^0}(Z, Z')$ be the smooth kernels of $e^{-uL_{2,c}^0}, e^{-uL_2^0}$ with respect to $dv_{TX}(Z')$. Now from (1.21) (cf. [8, (6.37), (6.38)]), we need to replace [15, (4.106)] by the following equations

$$\begin{aligned} e^{-uL_{2,c}^0}(Z, Z') &= \det_{\mathbb{C}} \left(\frac{|\mathbf{J}_{x_0}|}{1 - e^{-4\pi u|\mathbf{J}_{x_0}|}} \right) \exp \left(-\frac{1}{2} \left\langle \frac{\pi|\mathbf{J}_{x_0}|}{\tanh(2\pi u|\mathbf{J}_{x_0}|)} Z, Z \right\rangle \right. \\ &\quad - \frac{1}{2} \left\langle \frac{\pi|\mathbf{J}_{x_0}|}{\tanh(2\pi u|\mathbf{J}_{x_0}|)} Z', Z' \right\rangle \\ &\quad \left. + \left\langle \frac{\pi|\mathbf{J}_{x_0}|}{\sinh(2\pi u|\mathbf{J}_{x_0}|)} e^{-2\pi\sqrt{-1}u\mathbf{J}_{x_0}} Z, Z' \right\rangle \right), \end{aligned} \quad (1.22)$$

$$e^{-uL_2^0}(Z, Z') = e^{-uL_{2,c}^0}(Z, Z') e^{2u\omega_{d,x_0}}.$$

Observe that for $\omega_{d,x_0} \in \text{End}(\Lambda(T^{*(0,1)}X)_{x_0})$, by (1.7),

$$\begin{aligned} \text{Ker } \omega_{d,x_0} &= \det(\overline{W}^*)_{x_0}, \\ \omega_{d,x_0} &\leq -\mu_0 \quad \text{on } (\det(\overline{W}^*)_{x_0})^\perp = \text{the orthogonal complement of } \det(\overline{W}^*). \end{aligned} \quad (1.23)$$

Thus from [15, Secs. 4.4, 4.5], we get Theorem 1.3. □

If we take $Z, Z' = 0$ in Theorem 1.3, we infer the following result by the same argument in [15, Sec. 4.5].

Theorem 1.4. *There exist smooth coefficients $\mathbf{b}_r(x) \in \text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$ which are polynomials in $R^{TX}, R^{T^{(1,0)}X}, R^E$ (and R^L) and their derivatives of order $\leq 2r - 1$ (respectively, $\leq 2r$) and reciprocals of linear combinations of eigenvalues of \mathbf{J} at x , such that*

$$\mathbf{b}_0 = \det_{\mathbb{C}}(|\mathbf{J}|)I_{\det(\overline{W}^*) \otimes E} \tag{1.24}$$

and for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ with

$$\left| P_p(x, x) - \sum_{r=0}^k \mathbf{b}_r(x) p^{n-r} \right|_{\mathcal{C}^l} \leq C_{k,l} p^{n-k-1}, \tag{1.25}$$

for any $x \in X$ and $p \in \mathbb{N}$. Moreover, the expansion is uniform in that for any $k, l \in \mathbb{N}$, there is an integer s such that if all data $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E)$ run over a set which are bounded in \mathcal{C}^s and with g^{TX} bounded below, there exists the constant $C_{k,l}$ independent of g^{TX} , and the \mathcal{C}^l -norm in (1.25) includes also the derivatives on the parameters.

1.4. Holomorphic case revisited

In this section, we suppose that (X, J) is a complex manifold with complex structure J and E, L are holomorphic vector bundles on X . We assume that ∇^E, ∇^L are the holomorphic Hermitian (i.e. Chern) connections on $(E, h^E), (L, h^L)$ and moreover, $\omega := \frac{\sqrt{-1}}{2\pi} R^L$ defines a symplectic form on X . Therefore the signature of the curvature R^L (i.e. number of negative and positive eigenvalues) with respect to any Riemannian metric compatible with J will be the same. Let g^{TX} be any Riemannian metric on TX compatible with J . Since g^{TX} is not necessarily Kähler, $\mathbf{J} \neq J$ in (1.5) in general. Recall the number q is defined by (1.6), i.e. is the number of negative eigenvalues of R^L . Set

$$\Theta(X, Y) = g^{TX}(JX, Y). \tag{1.26}$$

Then the 2-form Θ need not to be closed (the convention here is different to [7, (2.1)] by a factor -1).

Let $\bar{\partial}^{L^p \otimes E, *}$ be the formal adjoint of the Dolbeault operator $\bar{\partial}^{L^p \otimes E}$ on the Dolbeault complex $\Omega^{0, \bullet}(X, L^p \otimes E)$ with the scalar product induced by g^{TX}, h^L, h^E as in (1.2). Set

$$D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *}). \tag{1.27}$$

We denote by $\square^{L^p \otimes E} = \bar{\partial}^{L^p \otimes E} \bar{\partial}^{L^p \otimes E, *} + \bar{\partial}^{L^p \otimes E, *} \bar{\partial}^{L^p \otimes E}$ the Kodaira–Laplacian. Then $D_p^2 = 2\square^{L^p \otimes E}$ it is twice the Kodaira–Laplacian and preserves the \mathbb{Z} -grading of $\Omega^{0, \bullet}(X, L^p \otimes E)$. By Hodge theory, we know that for any $k, p \in \mathbb{N}$,

$$\text{Ker } D_p|_{\Omega^{0, k}} = \text{Ker } D_p^2|_{\Omega^{0, k}} \simeq H^{0, k}(X, L^p \otimes E), \tag{1.28}$$

where $H^{0,\bullet}(X, L^p \otimes E)$ is the Dolbeault cohomology. Here D_p is not a spin^c Dirac operator on $\Omega^{0,\bullet}(X, L^p \otimes E)$, and D_p^2 is not a renormalized Bochner–Laplacian as in [24], so we cannot apply directly Theorems 1.1 and 1.2. Now we explain how to recover the conclusions of these theorems in the case of D_p . The first step is to exhibit the spectral gap.

Theorem 1.5. *The statements of Theorems 1.1 and 1.2 still hold for the operator D_p defined by (1.27). In particular, for p large enough,*

$$H^{0,k}(X, L^p \otimes E) = 0 \quad \text{for } k \neq q. \tag{1.29}$$

Proof. As we will use [7, Theorem 2.3] to study the Bergman kernel in the sequel, we prove Theorem 1.5 by explaining [7, Theorem 2.3].

Let S^{-B} denote the 1-form with values in the antisymmetric elements of $\text{End}(TX)$ which satisfies

$$\langle S^{-B}(U)V, W \rangle = \frac{\sqrt{-1}}{2}((\partial - \bar{\partial})\Theta)(U, V, W), \quad \text{for } U, V, W \in TX. \tag{1.30}$$

The Bismut connection ∇^{-B} on TX is defined by

$$\nabla^{-B} = \nabla^{TX} + S^{-B}. \tag{1.31}$$

Then by [7, Proposition 2.5], ∇^{-B} preserves the metric g^{TX} and the complex structure of TX . Let ∇^{\det} be the holomorphic Hermitian connection on $K_X^* := \det(T^{(1,0)}X)$ whose curvature is denoted R^{\det} . Then these two connections induce naturally an unique connection on $\Lambda(T^{*(0,1)}X)$ which preserves its \mathbb{Z} -grading, and with the connections ∇^L, ∇^E , we get a connection ∇^{-B, E_p} on $\Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$. Let $(\nabla^{-B, E_p})^*$ be the formal adjoint of ∇^{-B, E_p} . Let $C(TX)$ be the Clifford bundle of TX . We define a map $c : \Lambda(T^*X) \rightarrow C(TX)$, by sending $e^{i_1} \wedge \dots \wedge e^{i_j}$ to $c(e_{i_1}) \cdots c(e_{i_j})$ for $i_1 < \dots < i_j$. For $B \in \Lambda^3(T^*X)$, set $|B|^2 = \sum_{i < j < k} |B(e_i, e_j, e_k)|^2$. Then we can formulate [7, Theorem 2.3] as following:

$$\begin{aligned} D_p^2 &= (\nabla^{-B, E_p})^* \nabla^{-B, E_p} + \frac{r^X}{4} + c \left(R^E + pR^L + \frac{1}{2}R^{\det} \right) \\ &\quad + \frac{\sqrt{-1}}{2} c(\bar{\partial}\partial\Theta) - \frac{1}{8} |(\partial - \bar{\partial})\Theta|^2. \end{aligned} \tag{1.32}$$

((1.32) can be seen as a Bochner–Kodaira–Nakano type formula.) By using (1.8), (1.11) and (1.32), as in Theorems 1.1 and 1.2, we see that the conclusions of these theorems still hold for the operator D_p defined in (1.27). In particular (1.9) holds. Now from (1.9) and (1.28), we get (1.29). □

Remark 1.6. The vanishing result (1.29) is Andreotti–Grauert’s coarse vanishing theorem [1, Sec. 23] (where it is proved by using the cohomology finiteness theorem for the disc bundle of L^*). It can also be deduced, as shown by Griffiths [19, p. 432],

from the usual Bochner–Kodaira–Nakano formula [16, 19, 26]. The latter implies the following inequality for $u \in \Omega^{m,k}(X, L^p \otimes E)$:

$$\frac{3}{2}(\|\bar{\partial}^{L^p \otimes E} u\|^2 + \|\bar{\partial}^{L^p \otimes E, * } u\|^2) \geq \int_X \langle [\sqrt{-1}(pR^L + R^E), \Lambda]u, u \rangle dv_X - \frac{1}{2}(\|Tu\|^2 + \|T^*u\|^2 + \|\bar{T}u\|^2 + \|\bar{T}^*u\|^2) \tag{1.33}$$

where $\Lambda = i(\Theta)$ denotes the interior product with Θ and $T = [\Lambda, \partial\Theta]$ is the torsion of the metric g^{TX} . We have pointwise

$$\langle [\sqrt{-1}R^L, \Lambda]u, u \rangle \geq (a_1 + \dots + a_k - a_{m+1} - \dots - a_n)|u|^2, \tag{1.34}$$

where $a_1 \leq a_2 \leq \dots \leq a_n$ are the eigenvalues of R^L with respect to g^{TX} as in Sec. 1.2. If $m = 0$, the right-hand side becomes $(-a_{k+1} - \dots - a_n)|u|^2$. As in [26, Lemma 4.3], we can *restrict* ourselves to those metrics g^{TX} such that the negative eigenvalues a_1, \dots, a_q are very large and the positive ones a_{q+1}, \dots, a_n are very small in absolute value. Therefore, for $k < q$ there exists a constant $\mu_1 > 0$ such that $-a_{k+1} - \dots - a_n \geq \mu_1$ on X . By (1.33), we obtain (1.9) for $u \in \Omega^{0, < q}(X, L^p \otimes E) = \oplus_{k < q} \Omega^{0,k}(X, L^p \otimes E)$.

In order to consider the case $k > q$, we apply again (1.33) and (1.34) for (n, q) $L^p \otimes K_X^* \otimes E$ -valued forms, which involves another change of metric, for which a_1, \dots, a_q are small and a_{q+1}, \dots, a_n are large in absolute value. Thus we get (1.9) also for $u \in \Omega^{0, > q}(X, L^p \otimes E) = \oplus_{k > q} \Omega^{0,k}(X, L^p \otimes E)$, but for yet another class of metrics g^{TX} . Of course, the estimates just obtained entail immediately (1.29).

We see however that by using (1.33), the essential estimate (1.9) for a fixed metric g^{TX} seems out of reach, as well as the existence of the spectral gap (1.13).

By Theorem 1.5, the kernel of D_p^2 is concentrated in degree q , for p large enough. We consider thus the Bergman kernel of D_p^2 in this particular degree. Let $P_p^{0,q}(x, x')$ be the smooth kernel with respect to $dv_X(x')$ of the orthogonal projection from $\Omega^{0,q}(X, L^p \otimes E)$ on $\text{Ker } D_p^2$.

Theorem 1.7. *The Bergman kernel $P_p^{0,q}(x, x')$ has a full off-diagonal asymptotic expansion analogous to (1.20) with $J_0 = \det_{\mathbb{C}}(|\mathbf{J}|)I_{\det(\overline{W^*}) \otimes E}$ as $p \rightarrow \infty$.*

Proof. We use now the connection ∇^{-B, E_p} instead of ∇^{E_p} in [15, Sec. 3]. Then by (1.9) and (1.32), everything goes through perfectly well and as in [15, Theorem 4.18], so we can directly apply the result from [15] to get the *full off-diagonal* asymptotic expansion of the Bergman kernel. As the above construction preserves the \mathbb{Z} -grading on $\Omega^{0, \bullet}(X, L^p \otimes E)$, we can directly work on $\Omega^{0,q}(X, L^p \otimes E)$. \square

Remark 1.8. From the arguments here and [23], [24, Sec. 3.5], we get naturally the covering version and orbifold version of Secs. 1.2–1.4.

1.5. Relation to the tangential Cauchy–Riemann complex

If $q = 0$, i.e. \mathbf{J} has only positive eigenvalues, Theorem 1.7 boils down to [24, Theorem 3.9]. Theorem 1.7 for $x = x'$ is first due to Zelditch [28] and Catlin [14]

and is based on the Boutet de Monvel–Sjöstrand parametrix [12] for the Szegö projector on CR functions on the boundary of the “Grauert tube” associated to L . For general $q \neq 0$, Berman and Sjöstrand [5] recently studied the asymptotic expansion $P_p(x, x')$, too. They use an approach of Melin–Sjöstrand originating in the theory of Fourier integral operators with complex phase.

In this section, we briefly discuss the link between our analysis for $q \neq 0$ and the the kernel of the Szegö projector on $(0, k)$ forms on the boundary of the Grauert tube. We use the notations and assumptions from Sec. 1.4.

Let $Y = \{u \in L^*, |u|_{hL^*} = 1\}$ be the unit circle bundle in L^* . Y is a real hypersurface in the complex manifold L^* which the boundary of the disc bundle $D = \{u \in L^*, |u|_{hL^*} < 1\}$, with defining function $\varrho = |u|_{hL^*} - 1$. The Levi form of ϱ restricted to the complex tangent plane of Y coincides with the pull-back of ω through the canonical projection $\pi : Y \rightarrow X$. Hence it has q negative and $n - q$ positive eigenvalues. We denote by $T^{*(0,1)}(Y) = T^{*(0,1)}L^* \cap (T^*Y \otimes_{\mathbb{R}} \mathbb{C})$ the bundle of $(0, 1)$ -forms tangential to Y and by $\Omega^{0,k}(Y)$ the space of smooth sections of $\Lambda^k(T^{*(0,1)}Y)$. The $\bar{\partial}$ operator on the ambient manifold L^* induces as usual a tangential Cauchy–Riemann complex on the hypersurface Y [21, 2, 3].

$$0 \rightarrow \Omega^{0,0}(Y) \xrightarrow{\bar{\partial}_b} \Omega^{0,1}(Y) \xrightarrow{\bar{\partial}_b} \dots \xrightarrow{\bar{\partial}_b} \Omega^{0,n}(Y) \rightarrow 0. \tag{1.35}$$

The $\bar{\partial}_b$ operator commutes with the action of S^1 on Y .

The connection ∇^L on L induces a connection on the S^1 -principal bundle $\pi : Y \rightarrow X$, and let $T^HY \subset TY$ be the corresponding horizontal bundle. Let us introduce the Riemannian metric $g^{TY} = \pi^*(g^{TX}) \oplus d\vartheta^2$ on $TY = T^HY \oplus TS^1$. We will denote by $\bar{\partial}_b^*$ the formal adjoint of $\bar{\partial}_b$ with respect to this metric and form the Kohn–Laplacian

$$\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b. \tag{1.36}$$

The operators $\bar{\partial}_b^*$ and \square_b also commute with the action of S^1 on Y . Consider the space $\mathcal{C}^\infty(Y)_p$ of smooth functions f on Y which transform under the action $(y, \vartheta) \mapsto e^{i\vartheta}y$ of S^1 according to the law

$$f(e^{i\vartheta}y) = e^{ip\vartheta} f(y). \tag{1.37}$$

This space of functions can be identified naturally with the space of smooth sections $\Omega^{0,0}(X, L^p)$. More generally, the space of sections $\Omega^{0,k}(Y)_p$ which transform under the action of S^1 according to the law (1.37) can be naturally identified with the space $\Omega^{0,k}(X, L^p)$. Therefore, for each integer p , we get a subcomplex $(\Omega^{0,\bullet}(Y)_p, \bar{\partial}_b)$ of the tangential Cauchy–Riemann complex (1.35), isomorphic to the Dolbeault complex $(\Omega^{0,\bullet}(X, L^p), \bar{\partial}^{L^p})$. Moreover, the action of \square_b on $\Omega^{0,\bullet}(Y)_p$ is identical to the action of the Kodaira–Laplacian \square^{L^p} on $\Omega^{0,\bullet}(X, L^p)$, via the the complex isomorphism just mentioned.

Let us consider the spaces of \square_b -harmonic spaces

$$\mathcal{H}^{0,k}(Y) = \ker \square_b|_{\Omega^{0,k}(Y)}, \quad \mathcal{H}^{0,k}(Y)_p = \ker \square_b|_{\Omega^{0,k}(Y)_p}. \tag{1.38}$$

Then

$$\mathcal{H}^{0,k}(Y) = \bigoplus_{p \in \mathbb{Z}} \mathcal{H}^{0,k}(Y)_p \cong \bigoplus_{p \in \mathbb{Z}} H^{0,k}(X, L^p) \tag{1.39}$$

where $H^{0,k}(X, L^p) := \ker \square^{L^p}|_{\Omega^{0,k}(X, L^p)}$ is the space of harmonic forms (1.28).

The Szegő projector $\Pi^{0,k}$ is by definition the orthogonal projection from $\Omega^{0,k}(Y)$ to $\mathcal{H}^{0,k}(Y)$. The Szegő projector $\Pi^{0,k}$ has finite-dimensional range for the degrees $k \neq q, n - q$. This follows from the decomposition (1.39) and the vanishing theorem of Andreotti–Grauert (1.29) applied for both L and L^* . It shows that there exists $p_0 \in \mathbb{N}$ such that $H^{0,k}(X, L^p) = 0$ for all $p \in \mathbb{Z}$ with $|p| \geq p_0$ and all $k \neq q, n - q$ (note that $R^{L^*} = -R^L$ has $n - q$ negative and q positive eigenvalues).

On the other hand, the Szegő projector $\Pi^{0,k}$ has infinite-dimensional range for the degrees $k = q, n - q$ as shown by (1.29) combined with the Riemann–Roch–Hirzebruch formula, which in turn is a consequence of the integration of the asymptotic expansion from Theorem 1.7 over the manifold X . To obtain the result for $k = n - q$ we have to replace L by L^* in the above mentioned results.

The description of the dimension of the harmonic spaces is consistent with the general geometric information from [2, 3] and [11, p. 626], where general hypersurfaces Y are considered, with non-degenerate Levi form of signature $(q, n - q)$.

The relation between the Bergman kernels $P_p^{0,q}$ considered in Theorem 1.7 and the Szegő kernel $\Pi^{0,q}$ is given by

$$P_p^{0,q}(x, x) = \frac{1}{2\pi} \int_{S^1} \Pi^{0,q}(e^{i\vartheta} y, y) e^{-ip\vartheta} d\vartheta \tag{1.40}$$

where $x \in X$ and $y \in Y$ satisfy $\pi(y) = x$. This means that the $P_p^{0,q}(x, x)$ represent the Fourier coefficients of the distribution $\Pi^{0,q}(y, y)$. Since we know the asymptotic expansion of $P_p^{0,q}(x, x)$ as $p \rightarrow \infty$ given in Theorem 1.7, we can recover from (1.40) the restriction on the diagonal of the Szegő kernel $\Pi^{0,q}$.

It could also be possible to work in the opposite direction and start with the Szegő kernel. Namely, using a similar analysis as the one of Boutet de Monvel and Sjöstrand [12], one can find the parametrix of the Szegő kernel $\Pi^{0,q}$ and determine its singularity on the diagonal. Then, working as Zelditch [28] (where the case $q = 0$ is considered), one can deduce the asymptotic of $P_p^{0,q}(x, x)$ for $p \rightarrow \infty$.

The same discussion applies to $\Pi^{0,n-q}$ and the Bergman kernels $P_p^{0,n-q}$ associated to L^* .

2. The Coefficient b_1

This section is organized as follows. In Sec. 2.1, we state our main result, the formula for the coefficient b_1 for the spin^c Dirac operator. In Sec. 2.2, we obtain an

asymptotic expansion of the rescaled spin^c Dirac operator L_2^t (cf. (2.4)) in normal coordinates. In Sec. 2.3, we finally compute the coefficient \mathbf{b}_1 .

2.1. Main result

We use the notation in Sec. 1.1. We denote by $I_{\mathbb{C} \otimes E}$ the projection from $\Lambda(T^{*(0,1)}X) \otimes E$ onto $\mathbb{C} \otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X) = \mathbb{C} \oplus \Lambda^{>0}(T^{*(0,1)}X)$. For any tensor ψ on X , we denote by $\nabla^X \psi$ the covariant derivative of ψ induced by ∇^{TX} . Thus $\nabla^X J \in T^*X \otimes \text{End}(TX)$, $\nabla^X \nabla^X J \in T^*X \otimes T^*X \otimes \text{End}(TX)$. Let $\{w_j\}$ be an orthonormal basis of $(T^{(1,0)}X, g^{TX})$, and its dual basis $\{w^j\}$. Let $\{e_i\}_{1 \leq i \leq 2n}$ be an orthonormal basis of (TX, g^{TX}) . Then we denote by $|\nabla^X J|^2 := \sum_{ij} |(\nabla_{e_i}^X J)e_j|^2$.

The following result is the main result of this paper.

Theorem 2.1. *If $\mathbf{J} = J$, then for \mathbf{b}_1 in (1.25), we have*

$$\begin{aligned} \mathbf{b}_1(x) &= \frac{1}{8\pi} \left[r^X + \frac{1}{4} |\nabla^X J|^2 + 4R^E(w_j, \bar{w}_j) \right] I_{\mathbb{C} \otimes E} - \frac{1}{144\pi} \sum_{kl} |(\nabla_{w_k}^X J)w_l|^2 I_{\mathbb{C} \otimes E} \\ &\quad + \frac{1}{288\pi} \langle (\nabla_{\bar{w}_k}^X J) \bar{w}_l, \bar{w}_m \rangle \langle (\nabla_{w_k}^X J)w_i, w_j \rangle \bar{w}^l \wedge \bar{w}^m I_{\mathbb{C} \otimes E} i \bar{w}_j \wedge i \bar{w}_i \\ &\quad - \frac{1}{8\pi} \left(\frac{1}{3} \langle R^{TX} w_i, \bar{w}_i \rangle + R^E \right) (\bar{w}_l, \bar{w}_m) \bar{w}^l \wedge \bar{w}^m I_{\mathbb{C} \otimes E} \\ &\quad + \frac{1}{8\pi} \left(\frac{1}{3} \langle R^{TX} w_i, \bar{w}_i \rangle + R^E \right) (w_l, w_m) I_{\mathbb{C} \otimes E} i \bar{w}_m \wedge i \bar{w}_l. \end{aligned} \tag{2.1}$$

Especially,

$$\text{Tr} |_{\Lambda(T^{*(0,1)}X)} [\mathbf{b}_1(x)] = \frac{1}{8\pi} \left[r^X + \frac{1}{4} |\nabla^X J|^2 + 4R^E(w_j, \bar{w}_j) \right]. \tag{2.2}$$

The term $r^X + \frac{1}{4} |\nabla^X J|^2$ in (2.2) is called the Hermitian scalar curvature in the literature [17], [18, Chap. 10], [24] and is a natural substitute for the Riemannian scalar curvature in the almost-Kähler case. It was used by Donaldson [17] to define the moment map on the space of compatible almost-complex structures.

2.2. Taylor expansion of the operator L_2^t

To compare with [24, Sec. 1.2], in this part, we assume that $\omega(\cdot, J\cdot)$ is positive, i.e. $q = 0$ in (1.6).

We fix $x_0 \in X$. From now on, we identify $B^{T_{x_0}X}(0, \varepsilon)$ with $B^X(x_0, \varepsilon)$ by the exponential map $T_{x_0}X \ni Z \rightarrow \exp_{x_0}^X(Z) \in X$. We identify L_Z, E_Z and $(E_p)_Z$ for $Z \in B^{T_{x_0}X}(0, \varepsilon)$ to L_{x_0}, E_{x_0} and $(E_p)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and ∇^{E_p} along the curve $\gamma_Z : [0, 1] \ni u \rightarrow uZ$. Let $\{e_i\}_i$ be an oriented orthonormal basis of $T_{x_0}X$. We also denote by $\{e^i\}_i$ the dual basis of

$\{e_i\}$. Let $\tilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TX} along the above curve.

Let S_L be an unit vector of L_{x_0} . Using S_L and the above discussion, we get an isometry $E_p \simeq (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0} =: \mathbf{E}_{x_0}$ on $B^{T_{x_0}X}(0, \varepsilon)$. Under our identification, h^{E_p} is $h^{\mathbf{E}_{x_0}}$ on $B^{T_{x_0}X}(0, \varepsilon)$.

For $s \in \mathcal{C}^\infty(T_{x_0}X, \mathbf{E}_{x_0})$, set

$$\|s\|_{0,0}^2 = \int_{\mathbb{R}^{2n}} |s(Z)|_{h_{x_0}^{\Lambda(T^{*(0,1)}X) \otimes E}}^2 dv_{TX}(Z). \tag{2.3}$$

Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U . If $\alpha = (\alpha_1, \dots, \alpha_{2n})$ is a multi-index, set $Z^\alpha = Z_1^{\alpha_1} \dots Z_{2n}^{\alpha_{2n}}$. Let $(\partial^\alpha R^L)_{x_0}$ be the tensor $(\partial^\alpha R^L)_{x_0}(e_i, e_j) = \partial^\alpha (R^L(e_i, e_j))_{x_0}$. We denote by $\mathcal{R} = \sum_i Z_i e_i = Z$ the radial vector field on \mathbb{R}^{2n} . Recall that the function κ was defined in (1.16). For $s \in \mathcal{C}^\infty(B^{T_{x_0}X}(0, \varepsilon), \mathbf{E}_{x_0})$ and $Z \in B^{T_{x_0}X}(0, \varepsilon)$, for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned} (S_t s)(Z) &= s(Z/t), & \nabla_t &= S_t^{-1} t \kappa^{\frac{1}{2}} \nabla^{E_p} \kappa^{-\frac{1}{2}} S_t, \\ \nabla_{0,\cdot} &= \nabla + \frac{1}{2} R_{x_0}^L(\mathcal{R}, \cdot), & L_2^t &= S_t^{-1} t^2 \kappa^{\frac{1}{2}} D_p^2 \kappa^{-\frac{1}{2}} S_t. \end{aligned} \tag{2.4}$$

By our trivialization, L_2^t is self-adjoint with respect to $\|\cdot\|_{0,0}$ on $\mathcal{C}_0^\infty(B^{T_{x_0}X}(0, \varepsilon/t), \mathbf{E}_{x_0})$. Note that comparing with [15, (4.37)], we conjugate with $\kappa^{1/2}$ in (2.4), which simplifies the computation of the coefficient \mathbf{b}_1 .

We adopt the convention that all tensors will be evaluated at the base point $x_0 \in X$, and most of the time, we will omit the subscript x_0 . Let $\mathcal{L}_0, \mathcal{O}_1, \mathcal{O}_2$ be the operators defined in [24, Theorem 1.4] associated the renormalized Bochner-Laplacian $\Delta_{p,0}$. Recall that $\tau = \sum_j R^L(w_j, \bar{w}_j)$, $\omega_d = -R^L(w_l, \bar{w}_m) \bar{w}^m \wedge i \bar{w}_l$. Thus we have

$$\begin{aligned} \mathcal{L}_0 &= - \sum_j (\nabla_{0,e_j})^2 - \tau_{x_0}, \\ \mathcal{O}_1(Z) &= -\frac{2}{3} (\partial_j R^L)_{x_0}(\mathcal{R}, e_i) Z_j \nabla_{0,e_i} - \frac{1}{3} (\partial_i R^L)_{x_0}(\mathcal{R}, e_i) - (\partial \mathcal{R} \tau)_{x_0}, \\ \mathcal{O}_2(Z) &= \frac{1}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_i) \mathcal{R}, e_j \rangle_{x_0} \nabla_{0,e_i} \nabla_{0,e_j} + \left[\frac{2}{3} \langle R_{x_0}^{TX}(\mathcal{R}, e_j) e_j, e_i \rangle \right. \\ &\quad \left. - \left(\frac{1}{2} \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} + R_{x_0}^E \right) (\mathcal{R}, e_i) \right] \nabla_{0,e_i} \\ &\quad - \frac{1}{4} \nabla_{e_i} \left(\sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} (\mathcal{R}, e_i) \right) - \frac{1}{9} \sum_i \left[\sum_j (\partial_j R^L)_{x_0}(\mathcal{R}, e_i) Z_j \right]^2 \\ &\quad - \frac{1}{12} [\mathcal{L}_0, \langle R_{x_0}^{TX}(\mathcal{R}, e_i) \mathcal{R}, e_i \rangle_{x_0}] - \sum_{|\alpha|=2} (\partial^\alpha \tau)_{x_0} \frac{Z^\alpha}{\alpha!}. \end{aligned} \tag{2.5}$$

Theorem 2.2. *There are second order differential operators $L_2^0, \underline{Q}_r (r \geq 1)$ which are self-adjoint with respect to $\|\cdot\|_{0,0}$ on $\mathcal{C}_0^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$, and*

$$\begin{aligned} L_2^0 &= \mathcal{L}_0 - 2\omega_{d,x_0}, \\ \underline{Q}_1 &= \mathcal{O}_1 - \pi\sqrt{-1}\langle (\nabla_{\mathcal{R}}^X \mathbf{J})_{x_0} e_l, e_m \rangle c(e_l) c(e_m) + (\partial_{\mathcal{R}}\tau)_{x_0}, \\ \underline{Q}_2 &= \mathcal{O}_2 - R_{x_0}^{\text{Cliff}}(\mathcal{R}, e_l)\nabla_{0,e_l} - \frac{\pi}{2}\sqrt{-1}\langle (\nabla^X \nabla^X \mathbf{J})_{(\mathcal{R}, \mathcal{R}), x_0} e_l, e_m \rangle c(e_l) c(e_m) \\ &\quad + \frac{1}{2}\left(R_{x_0}^E + \frac{1}{2}\text{Tr}[R_{x_0}^{T(1,0)X}]\right)(e_l, e_m) c(e_l) c(e_m) + \sum_{|\alpha|=2} (\partial^\alpha \tau)_{x_0} \frac{Z^\alpha}{\alpha!} + \frac{1}{4}r_{x_0}^X \end{aligned} \tag{2.6}$$

such that

$$L_2^t = L_2^0 + \sum_{r=1}^\infty \underline{Q}_r t^r. \tag{2.7}$$

Proof. Set $g_{ij}(Z) = g^{TX}(e_i, e_j)(Z) = \langle e_i, e_j \rangle_Z$ and let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z))$. By [4, Proposition 1.28] (cf. [15, Lemma 4.5]), we have

$$\begin{aligned} g_{ij}(Z) &= \delta_{ij} + \frac{1}{3}\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j \rangle_{x_0} + \mathcal{O}(|Z|^3), \\ \kappa(Z) &= |\det(g_{ij}(Z))|^{1/2} = 1 + \frac{1}{6}\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_i \rangle_{x_0} + \mathcal{O}(|Z|^3). \end{aligned} \tag{2.8}$$

If Γ_{ij}^l is the connection form of ∇^{TX} with respect to the basis $\{e_i\}$, we have $(\nabla_{e_i}^{TX} e_j)(Z) = \Gamma_{ij}^l(Z)e_l$. Owing to (2.8) (cf. [24, (1.32)]),

$$\Gamma_{ij}^l(Z) = \frac{1}{3}\langle R_{x_0}^{TX}(\mathcal{R}, e_j)e_i + R_{x_0}^{TX}(\mathcal{R}, e_i)e_j, e_l \rangle_{x_0} + \mathcal{O}(|Z|^2). \tag{2.9}$$

Let $\Gamma^E, \Gamma^L, \Gamma^{\text{Cliff}}$ be the connection forms of $\nabla^E, \nabla^L, \nabla^{\text{Cliff}}$ with respect to any fixed frames for $E, L, \Lambda(T^{*(0,1)}X)$ which are parallel along the curve γ_Z under our trivializations on $B^{T_{x_0}X}(0, \varepsilon)$. By [4, Proposition 1.18], the Taylor coefficients of $\Gamma^\bullet(e_j)(Z)$ at x_0 to order r are only determined by those of R^\bullet to order $r - 1$, and

$$\sum_{|\alpha|=r} (\partial^\alpha \Gamma^\bullet)_{x_0}(e_j) \frac{Z^\alpha}{\alpha!} = \frac{1}{r+1} \sum_{|\alpha|=r-1} (\partial^\alpha R^\bullet)_{x_0}(\mathcal{R}, e_j) \frac{Z^\alpha}{\alpha!}. \tag{2.10}$$

Equations (2.4) and (2.10) yield on $B^{T_{x_0}X}(0, \varepsilon/t)$,

$$\begin{aligned} \nabla_{t,e_i}|_Z &= \kappa^{\frac{1}{2}}(tZ) \left[\nabla_{e_i} + \left(\frac{1}{t}\Gamma^L(e_i) + t\Gamma^E(e_i) + t\Gamma^{\text{Cliff}}(e_i) \right) (tZ) \right] \kappa^{-\frac{1}{2}}(tZ) \\ &= \kappa^{\frac{1}{2}}(tZ) \left[\nabla_{e_i} + \left(\frac{1}{2}R_{x_0}^L + \frac{t}{3}(\partial_k R^L)_{x_0} Z_k \right. \right. \\ &\quad \left. \left. + \frac{t^2}{4} \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}R_{x_0}^E + \frac{t^2}{2}R_{x_0}^{\text{Cliff}} \right) (\mathcal{R}, e_i) + \mathcal{O}(t^3) \right] \kappa^{-\frac{1}{2}}(tZ). \end{aligned} \tag{2.11}$$

By the definition of ∇^{Cliff} , for $Y, U \in \mathcal{C}^\infty(X, TX)$,

$$[\nabla_U^{\text{Cliff}}, c(Y)] = c(\nabla_U^{TX} Y). \tag{2.12}$$

For $\psi \in T^*X \otimes \text{End}(\Lambda(T^{*(0,1)}X)) \simeq T^*X \otimes (C(TX) \otimes_{\mathbb{R}} \mathbb{C})$, where $C(TX)$ is the Clifford algebra bundle of TX , we still denote by $\nabla^X \psi$ the covariant derivative of ψ induced by ∇^{TX} . By using (2.12), we observe that

$$\begin{aligned} \nabla_Y^X(\psi(\tilde{e}_j)c(\tilde{e}_j)) &= (\nabla_Y^X \psi)(\tilde{e}_j)c(\tilde{e}_j) + \psi(\nabla_Y^{TX} \tilde{e}_j)c(\tilde{e}_j) + \psi(\tilde{e}_j)c(\nabla_Y^{TX} \tilde{e}_j) \\ &= (\nabla_Y^X \psi)(\tilde{e}_j)c(\tilde{e}_j). \end{aligned} \tag{2.13}$$

Thus for $k \geq 2$,

$$\begin{aligned} &(R^L(\tilde{e}_l, \tilde{e}_m) c(\tilde{e}_l) c(\tilde{e}_m))(tZ) \\ &= \sum_{r=0}^k \frac{\partial^r}{\partial t^r} [(R^L(\tilde{e}_l, \tilde{e}_m) c(\tilde{e}_l) c(\tilde{e}_m))(tZ)]|_{t=0} \frac{t^r}{r!} + \mathcal{O}(t^{k+1}) \\ &= (R_{x_0}^L + t(\nabla_{\mathcal{R}}^X R^L)_{x_0} + \frac{t^2}{2}(\nabla^X \nabla^X R^L)_{(\mathcal{R}, \mathcal{R}), x_0})(e_l, e_m) c(e_l) c(e_m) + \mathcal{O}(t^3). \end{aligned} \tag{2.14}$$

Owing to (1.4), (2.4), (2.11) and (2.13)–(2.14),

$$\begin{aligned} L_2^t &= -g^{ij}(tZ)(\nabla_{t, e_i} \nabla_{t, e_j} - t\Gamma_{ij}^l(tZ)\nabla_{t, e_l}) + \frac{t^2}{4}r^X(tZ) \\ &\quad + \frac{1}{2} \left\{ \left[R^L + t^2 \left(R^E + \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}] \right) \right] (\tilde{e}_l, \tilde{e}_m) c(\tilde{e}_l) c(\tilde{e}_m) \right\} (tZ) \\ &= -g^{ij}(tZ)(\nabla_{t, e_i} \nabla_{t, e_j} - t\Gamma_{ij}^l(tZ)\nabla_{t, e_l}) \\ &\quad + \frac{1}{2} \left(R_{x_0}^L + t(\nabla_{\mathcal{R}}^X R^L)_{x_0} + \frac{t^2}{2}(\nabla^X \nabla^X R^L)_{(\mathcal{R}, \mathcal{R}), x_0} \right) (e_l, e_m) c(e_l) c(e_m) \\ &\quad + \frac{t^2}{2} \left(R_{x_0}^E + \frac{1}{2} \text{Tr} [R_{x_0}^{T^{(1,0)}X}] \right) (e_l, e_m) c(e_l) c(e_m) + \frac{t^2}{4}r_{x_0}^X + \mathcal{O}(t^3). \end{aligned} \tag{2.15}$$

Note that $\nabla^X g^{TX} = 0$. Comparing (1.5), (1.8), (2.8), (2.11), (2.15), with [24, (1.37)], we get (2.7).

To prove the self-adjointness of L_2^0 , and $\underline{Q}_r (r \geq 1)$ with respect to $\|\cdot\|_{0,0}$ on $\mathcal{C}_0^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$, we observe that it follows from the fact that L_2^t is self-adjoint with respect to $\|\cdot\|_{0,0}$ on $\mathcal{C}_0^\infty(B^{T_{x_0}X}(0, \varepsilon/t), \mathbf{E}_{x_0})$. \square

Let P^N, \underline{P}^N be the orthogonal projections from $(L^2(\mathbb{R}^{2n}, \mathbf{E}_{x_0}), \|\cdot\|_{0,0})$ onto $N = \text{Ker } \mathcal{L}_0, \text{Ker } L_2^0$, and let $P^N(Z, Z'), \underline{P}^N(Z, Z')$ be the smooth kernel of P^N, \underline{P}^N with respect to $dv_{TX}(Z)$. Set $P^{N^\perp} = \text{Id} - P^N, \underline{P}^{N^\perp} = \text{Id} - \underline{P}^N$. Recall that $q = 0$ in (1.6), thus $\omega_d \leq -\mu_0$ on $\Lambda^{>0}(T^{*(0,1)}X)$, by (2.6),

$$\underline{P}^N(Z, Z') = P^N(Z, Z')I_{\mathbb{C} \otimes E}. \tag{2.16}$$

Theorem 2.3. *We have the relation*

$$\underline{P}^N \underline{Q}_1 \underline{P}^N = 0. \tag{2.17}$$

Proof. By [24, (1.92), (1.94), (1.96)], we have

$$\begin{aligned} (\nabla_U^X R^L)(V, W) &= -2\pi\sqrt{-1}\langle (\nabla_U^X \mathbf{J})V, W \rangle, \\ (\nabla_{\mathcal{R}}\tau)_{x_0} &= -2\pi\sqrt{-1}\langle (\nabla_{\mathcal{R}}^X \mathbf{J})w_i, \bar{w}_i \rangle, \\ P^N \mathcal{O}_1 P^N &= 0. \end{aligned} \tag{2.18}$$

By (2.6), (2.16) and (2.18), we get (2.17). □

Set

$$\begin{aligned} F_2 &= (L_2^0)^{-1} \underline{P}^{N^\perp} \underline{Q}_1 (L_2^0)^{-1} \underline{P}^{N^\perp} \underline{Q}_1 \underline{P}^N - (L_2^0)^{-1} \underline{P}^{N^\perp} \underline{Q}_2 \underline{P}^N \\ &\quad + \underline{P}^N \underline{Q}_1 (L_2^0)^{-1} \underline{P}^{N^\perp} \underline{Q}_1 (L_2^0)^{-1} \underline{P}^{N^\perp} - \underline{P}^N \underline{Q}_2 (L_2^0)^{-1} \underline{P}^{N^\perp} \\ &\quad + \underline{P}^{N^\perp} (L_2^0)^{-1} \underline{Q}_1 \underline{P}^N \underline{Q}_1 (L_2^0)^{-1} \underline{P}^{N^\perp} - \underline{P}^N \underline{Q}_1 \underline{P}^{N^\perp} (L_2^0)^{-2} \underline{Q}_1 \underline{P}^N. \end{aligned} \tag{2.19}$$

Then by Theorem 2.3 and the same argument as in [24, Sec. 1.5, 1.6], we get

$$\mathbf{b}_1(x_0) = F_2(0, 0). \tag{2.20}$$

By Theorem 2.2, the third and fourth terms in (2.19) are adjoint of the first two terms, thus we only need to compute the first two terms and the last two terms in (2.19).

2.3. Computing the coefficient \mathbf{b}_1

From now on, we assume that $\mathbf{J} = J$. By [24, (2.13)],

$$\begin{aligned} \nabla_U^X J \text{ is skew-adjoint and the tensor } \langle (\nabla_U^X J)\cdot, \cdot \rangle \text{ is of the type} \\ (T^{*(1,0)}X)^{\otimes 3} \oplus (T^{*(0,1)}X)^{\otimes 3}. \end{aligned} \tag{2.21}$$

In what follows, we will use the complex coordinates $z = (z_1, \dots, z_n)$, such that $w_i = \sqrt{2} \frac{\partial}{\partial z_i}$ is an orthonormal basis of $T_{x_0}^{(1,0)}X$. Then $Z = z + \bar{z}$ and will also identify z to $\sum_i z_i \frac{\partial}{\partial z_i}$ and \bar{z} to $\sum_i \bar{z}_i \frac{\partial}{\partial \bar{z}_i}$ when we consider z and \bar{z} as vector fields. Remark that

$$\left| \frac{\partial}{\partial z_i} \right|^2 = \left| \frac{\partial}{\partial \bar{z}_i} \right|^2 = \frac{1}{2}, \quad \text{so that } |z|^2 = |\bar{z}|^2 = \frac{1}{2}|Z|^2. \tag{2.22}$$

It is very useful to rewrite \mathcal{L}_0 by using the creation and annihilation operators. Set

$$b_i = -2\nabla_{0, \frac{\partial}{\partial \bar{z}_i}} = -2\frac{\partial}{\partial z_i} + \pi\bar{z}_i, \quad b_i^+ = 2\nabla_{0, \frac{\partial}{\partial z_i}} = 2\frac{\partial}{\partial \bar{z}_i} + \pi z_i, \quad b = (b_1, \dots, b_n). \tag{2.23}$$

Then for any polynomial $g(z, \bar{z})$ on z and \bar{z} ,

$$\begin{aligned} [b_i, b_j^+] &= b_i b_j^+ - b_j^+ b_i = -4\pi\delta_{ij}, \\ [b_i, b_j] &= [b_i^+, b_j^+] = 0, \\ [g(z, \bar{z}), b_j] &= 2\frac{\partial}{\partial z_j}g(z, \bar{z}), \\ [g(z, \bar{z}), b_j^+] &= -2\frac{\partial}{\partial \bar{z}_j}g(z, \bar{z}). \end{aligned} \tag{2.24}$$

As $\mathbf{J} = J$, $a_j = 2\pi$ in (1.7). By (2.5) and (2.6) (cf. [24, (2.23)]),

$$\begin{aligned} \mathcal{O}_1 &= -\frac{4\pi\sqrt{-1}}{3}b_i \left\langle (\nabla_{\bar{z}}^X J)\bar{z}, \frac{\partial}{\partial \bar{z}_i} \right\rangle + \frac{4\pi\sqrt{-1}}{3} \left\langle (\nabla_z^X J)z, \frac{\partial}{\partial z_i} \right\rangle b_i^+, \\ \mathcal{L}_0 &= b_j b_j^+, \quad L_2^0 = b_j b_j^+ + 4\pi\bar{w}^j \wedge i\bar{w}_j. \end{aligned} \tag{2.25}$$

We found the following result in [24, Theorem 1.15].

Theorem 2.4. *The spectrum of the restriction of \mathcal{L}_0 on $L^2(\mathbb{R}^{2n})$ is given by*

$$\text{Spec } \mathcal{L}_0|_{L^2(\mathbb{R}^{2n})} = \left\{ 4\pi \sum_{i=1}^n \alpha_i : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n \right\} \tag{2.26}$$

and an orthogonal basis of the eigenspace of $4\pi \sum_{i=1}^n \alpha_i$ is given by

$$b^\alpha \left(z^\beta \exp\left(-\frac{\pi}{2} \sum_i |z_i|^2\right) \right), \quad \text{with } \beta \in \mathbb{N}^n. \tag{2.27}$$

From (2.27), we get

$$P^N(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_i (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i)\right). \tag{2.28}$$

As $\mathbf{J} = J$ here, we know the function τ therein is $2\pi n$. By (2.6), (2.21)–(2.23),

$$\begin{aligned} \underline{\mathcal{Q}}_1 &= \mathcal{O}_1 - 2\pi\sqrt{-1} \left[\left\langle (\nabla_{\bar{z}}^X J)\bar{z}, \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m \right. \\ &\quad \left. + 4 \left\langle (\nabla_z^X J)z, \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle i_{\frac{\partial}{\partial \bar{z}_l}} i_{\frac{\partial}{\partial \bar{z}_m}} \right], \\ \underline{\mathcal{Q}}_2 &= \mathcal{O}_2 + R_{x_0}^{\text{Cliff}} \left(\mathcal{R}, \frac{\partial}{\partial \bar{z}_i} \right) b_i - R_{x_0}^{\text{Cliff}} \left(\mathcal{R}, \frac{\partial}{\partial z_i} \right) b_i^+ \\ &\quad - \frac{\pi}{2}\sqrt{-1} \langle (\nabla^X \nabla^X J)_{(\mathcal{R}, \mathcal{R})} e_l, e_m \rangle c(e_l) c(e_m) \\ &\quad + \frac{1}{2} \left(R_{x_0}^E + \frac{1}{2} \text{Tr} [R_{x_0}^{T(1,0)X}] \right) (e_l, e_m) c(e_l) c(e_m) + \frac{1}{4} r_{x_0}^X. \end{aligned} \tag{2.29}$$

Recall that by [24, (1.98), (2.24)] and (2.23),

$$b_i^+ P^N = 0, \quad (b_i P^N)(Z, Z') = 2\pi(\bar{z}_i - \bar{z}'_i)P^N(Z, Z'), \quad (\mathcal{O}_1 P^N)(Z, 0) = 0. \tag{2.30}$$

By (2.16), (2.21), (2.25), (2.28)–(2.30), as in [24, (2.24)],

$$\begin{aligned}
 & (\underline{Q}_1 \underline{P}^N)(Z, Z') \\
 &= \left[-\frac{2\sqrt{-1}}{3} b_i b_j \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_j}}^X J \right) \bar{z}', \frac{\partial}{\partial \bar{z}_i} \right\rangle - \frac{4\pi\sqrt{-1}}{3} b_i \left\langle \left(\nabla_{\bar{z}'}^X J \right) \bar{z}', \frac{\partial}{\partial \bar{z}_i} \right\rangle \right. \\
 & \quad \left. - \sqrt{-1} \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m (b_k + 2\pi \bar{z}'_k) \right] P^N(Z, Z') I_{\mathbb{C} \otimes E}. \quad (2.31)
 \end{aligned}$$

Thus by Theorem 2.4 and relations (2.21), (2.25) and (2.31),

$$\begin{aligned}
 & ((L_2^0)^{-1} \underline{P}^{N^\perp} \underline{Q}_1 \underline{P}^N)(Z, Z') \\
 &= -\sqrt{-1} \left[\frac{b_i b_j}{12\pi} \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_j}}^X J \right) \bar{z}', \frac{\partial}{\partial \bar{z}_i} \right\rangle + \frac{b_i}{3} \left\langle \left(\nabla_{\bar{z}'}^X J \right) \bar{z}', \frac{\partial}{\partial \bar{z}_i} \right\rangle \right. \\
 & \quad \left. + \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m \left(\frac{b_k}{12\pi} + \frac{\bar{z}'_k}{4} \right) \right] P^N(Z, Z') I_{\mathbb{C} \otimes E}. \quad (2.32)
 \end{aligned}$$

By (2.21), (2.30) and (2.32),

$$\begin{aligned}
 (\underline{P}^{N^\perp} (L_2^0)^{-1} \underline{Q}_1 \underline{P}^N)(0, Z') &= -\frac{\sqrt{-1}}{12} \left\langle \left(\nabla_{\bar{z}'}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m P^N(0, Z') I_{\mathbb{C} \otimes E}, \\
 (\underline{P}^{N^\perp} (L_2^0)^{-1} \underline{Q}_1 \underline{P}^N)(Z, 0) &= -\frac{\sqrt{-1}}{6} \left\langle \left(\nabla_{\bar{z}}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m P^N(Z, 0) I_{\mathbb{C} \otimes E}.
 \end{aligned} \quad (2.33)$$

Recall that L_2^0, \underline{Q}_1 are self-adjoint with respect to $\| \cdot \|_{0,0}$ on $\mathcal{C}_0^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$. After taking the adjoint of (2.33), by (2.22) and (2.28), we get

$$\begin{aligned}
 (\underline{P}^N \underline{Q}_1 (L_2^0)^{-1} \underline{P}^{N^\perp})(Z', 0) &= \frac{\sqrt{-1}}{3} \left\langle \left(\nabla_{z'}^X J \right) \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle I_{\mathbb{C} \otimes E} i_{\frac{\partial}{\partial \bar{z}_m}} i_{\frac{\partial}{\partial \bar{z}_l}} P^N(Z', 0), \\
 (\underline{P}^N \underline{Q}_1 (L_2^0)^{-1} \underline{P}^{N^\perp})(0, Z) &= \frac{2\sqrt{-1}}{3} \left\langle \left(\nabla_z^X J \right) \frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right\rangle I_{\mathbb{C} \otimes E} i_{\frac{\partial}{\partial \bar{z}_m}} i_{\frac{\partial}{\partial \bar{z}_l}} P^N(0, Z).
 \end{aligned} \quad (2.34)$$

Note that $\int_{\mathbb{C}} |z|^2 e^{-\pi|z|^2} = 1/\pi$. By (2.21), (2.22), (2.28), (2.33) and (2.34),

$$\begin{aligned}
 & (\underline{P}^{N^\perp} (L_2^0)^{-1} \underline{Q}_1 \underline{P}^N \underline{Q}_1 (L_2^0)^{-1} \underline{P}^{N^\perp})(0, 0) \\
 &= \frac{1}{36\pi} \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J \right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle d\bar{z}_l d\bar{z}_m I_{\mathbb{C} \otimes E} i_{\frac{\partial}{\partial \bar{z}_j}} i_{\frac{\partial}{\partial \bar{z}_i}},
 \end{aligned} \quad (2.35)$$

$$\begin{aligned}
 & - (\underline{P}^N \underline{Q}_1 \underline{P}^{N^\perp} (L_2^0)^{-2} \underline{Q}_1 \underline{P}^N)(0, 0) \\
 = & -\frac{1}{9\pi} \left\langle \left(\nabla_{\frac{\partial}{\partial z_k}}^X J \right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle I_{\mathbb{C} \otimes E} i_{\frac{\partial}{\partial \bar{z}_j}} i_{\frac{\partial}{\partial \bar{z}_i}} d\bar{z}_l d\bar{z}_m I_{\mathbb{C} \otimes E} \\
 = & -\frac{1}{9\pi} \left\langle \left(\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \left(\nabla_{\frac{\partial}{\partial z_k}}^X J \right) \frac{\partial}{\partial z_l} \right\rangle. \tag{2.36}
 \end{aligned}$$

Let $h_i(Z)$ (respectively, $F(Z)$) be polynomials in Z with degree 1 (respectively, 2), then by Theorem 2.4, (2.24), (2.25) and (2.30),

$$\begin{aligned}
 (\mathcal{L}_0^{-1} P^{N^\perp} h_i b_i P^N)(0, 0) &= (\mathcal{L}_0^{-1} P^{N^\perp} b_i h_i P^N)(0, 0) = -\frac{1}{2\pi} \frac{\partial h_i}{\partial z_i}, \\
 (\mathcal{L}_0^{-1} P^{N^\perp} F P^N)(0, 0) &= -\frac{1}{4\pi^2} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_i}, \\
 ((L_2^0)^{-1} F d\bar{z}_l d\bar{z}_m \underline{P}^N)(0, 0) &= \frac{1}{2\pi} \left((L_2^0)^{-1} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_i} (b_i z_i + 2) d\bar{z}_l d\bar{z}_m \underline{P}^N \right) (0, 0) \tag{2.37} \\
 &= \frac{1}{8\pi^2} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_i} \left(\left(\frac{b_i z_i}{3} + 1 \right) d\bar{z}_l d\bar{z}_m \underline{P}^N \right) (0, 0) \\
 &= \frac{1}{24\pi^2} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_i} d\bar{z}_l d\bar{z}_m I_{\mathbb{C} \otimes E}.
 \end{aligned}$$

Observe that for a monomial Q in $b_i, z_i, b_i^+, \bar{z}_i$, if the total degree of b_i, z_i is not the same as the total degree of b_i^+, \bar{z}_i , then $(Q \underline{P}^N)(0, 0) = 0$. Theorem 2.4 and relations (2.22), (2.25), (2.29), (2.30), (2.33) and (2.37) imply that

$$\begin{aligned}
 & ((L_2^0)^{-1} \underline{P}^{N^\perp} \underline{Q}_1 (L_2^0)^{-1} \underline{Q}_1 \underline{P}^N)(0, 0) \\
 = & \left((L_2^0)^{-1} \underline{P}^{N^\perp} (-8\pi\sqrt{-1}) \left\langle \left(\nabla_z^X J \right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle \right. \\
 & \quad \times i_{\frac{\partial}{\partial \bar{z}_i}} i_{\frac{\partial}{\partial \bar{z}_j}} \times \frac{-\sqrt{-1}}{6} \left\langle \left(\nabla_{\bar{z}}^X J \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m \underline{P}^N \left. \right) (0, 0) I_{\mathbb{C} \otimes E} \tag{2.38} \\
 = & \frac{4\pi}{3} \left((L_2^0)^{-1} \underline{P}^{N^\perp} \left\langle \left(\nabla_z^X J \right) \frac{\partial}{\partial z_l}, \left(\nabla_{\bar{z}}^X J \right) \frac{\partial}{\partial \bar{z}_l} \right\rangle \underline{P}^N \right) (0, 0) I_{\mathbb{C} \otimes E} \\
 = & -\frac{1}{3\pi} \left\langle \left(\nabla_{\frac{\partial}{\partial z_k}}^X J \right) \frac{\partial}{\partial z_l}, \left(\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J \right) \frac{\partial}{\partial \bar{z}_l} \right\rangle I_{\mathbb{C} \otimes E}.
 \end{aligned}$$

Now we will compute $((L_2^0)^{-1} \underline{P}^{N^\perp} \underline{Q}_2 \underline{P}^N)(0, 0)$. By (1.1) and (2.21),

$$\begin{aligned}
 \text{Tr} [R^{T(1,0)X}] &= 2 \left\langle R^{TX} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right\rangle \\
 & \quad + \frac{1}{2} \left\langle \left(\nabla_{\frac{\partial}{\partial z_l}}^X J \right) \frac{\partial}{\partial z_j}, \left(\nabla_{\frac{\partial}{\partial \bar{z}_m}}^X J \right) \frac{\partial}{\partial \bar{z}_j} \right\rangle d\bar{z}_m \wedge dz_l. \tag{2.39}
 \end{aligned}$$

By definition, for $U, V \in TX$, we have

$$(\nabla^X \nabla^X J)_{(U,V)} - (\nabla^X \nabla^X J)_{(V,U)} = [R^{TX}(U, V), J]. \tag{2.40}$$

By [24, (2.19)], for $u_1, u_2, u_3 \in T^{(1,0)}X$, $\bar{v}_1, \bar{v}_2 \in T^{(0,1)}X$,

$$\begin{aligned} &(\nabla^X \nabla^X J)_{(u_1, u_2)} u_3, (\nabla^X \nabla^X J)_{(\bar{u}_1, \bar{u}_2)} u_3 \in T^{(0,1)}X, \\ &(\nabla^X \nabla^X J)_{(\bar{u}_1, u_2)} \bar{u}_3 \in T^{(0,1)}X, \\ &2\sqrt{-1} \langle (\nabla^X \nabla^X J)_{(u_1, \bar{v}_1)} u_2, \bar{v}_2 \rangle = \langle (\nabla^X_{u_1} J) u_2, (\nabla^X_{\bar{v}_1} J) \bar{v}_2 \rangle. \end{aligned} \tag{2.41}$$

By (1.1), (2.16), (2.28)–(2.30) and (2.39),

$$\begin{aligned} &\underline{P}^{N^\perp} (\underline{\mathcal{Q}}_2 - \mathcal{O}_2) \underline{P}^N \\ &= \underline{P}^{N^\perp} \left\{ \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}] \left(\mathcal{R}, \frac{\partial}{\partial \bar{z}_i} \right) b_i \right. \\ &\quad - \left\langle \left(R^{TX} \left(\mathcal{R}, \frac{\partial}{\partial \bar{z}_i} \right) b_i - 2\pi\sqrt{-1} (\nabla^X \nabla^X J)_{(\mathcal{R}, \mathcal{R})} \right) \frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_l} \right\rangle \\ &\quad + \left\langle \left(\frac{1}{2} R^{TX} \left(\mathcal{R}, \frac{\partial}{\partial \bar{z}_i} \right) b_i - \pi\sqrt{-1} (\nabla^X \nabla^X J)_{(\mathcal{R}, \mathcal{R})} \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m \\ &\quad \left. + \left(R^E + \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}] \right) \left(\frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right) d\bar{z}_l d\bar{z}_m \right\} \underline{P}^N. \end{aligned} \tag{2.42}$$

Thus by (2.30), (2.37), (2.40) and (2.42),

$$\begin{aligned} &-((L_2^0)^{-1} \underline{P}^{N^\perp} (\underline{\mathcal{Q}}_2 - \mathcal{O}_2) \underline{P}^N)(0, 0) \\ &= \left\{ \frac{1}{4\pi} \text{Tr} [R^{T^{(1,0)}X}] \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right) \right. \\ &\quad - \frac{1}{2\pi} \left\langle R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right) - \sqrt{-1} \left(2(\nabla^X \nabla^X J)_{\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right)} \right. \right. \\ &\quad \left. \left. - \left[R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right), J \right] \right) \frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_l} \right\rangle \\ &\quad - \frac{1}{24\pi} \left\langle R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right) - \sqrt{-1} \left(2(\nabla^X \nabla^X J)_{\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_i}\right)} \right. \right. \\ &\quad \left. \left. + \left[R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right), J \right] \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle d\bar{z}_l d\bar{z}_m \\ &\quad \left. - \frac{1}{8\pi} \left(R^E + \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}] \right) \left(\frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right) d\bar{z}_l d\bar{z}_m \right\} I_{\mathbb{C} \otimes E}. \end{aligned} \tag{2.43}$$

From (2.39), (2.41) and (2.43), we have

$$\begin{aligned} &-((L_2^0)^{-1} \underline{P}^{N^\perp} (\underline{\mathcal{Q}}_2 - \mathcal{O}_2) \underline{P}^N)(0, 0) \\ &= \left\{ \frac{1}{4\pi} \text{Tr} [R^{T^{(1,0)}X}] \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right) \right. \\ &\quad \left. - \frac{1}{2\pi} \left\langle \left(R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right) - 2\sqrt{-1} (\nabla^X \nabla^X J)_{\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right)} \right) \frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar{z}_l} \right\rangle \right\} \end{aligned}$$

$$\begin{aligned}
 & + \left[\frac{1}{24\pi} \left\langle R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle \right. \\
 & \left. - \frac{1}{8\pi} \left(R^E + \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}] \right) \left(\frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right) \right] d\bar{z}_l d\bar{z}_m \Big\} I_{\mathbb{C} \otimes E} \\
 & = \left\{ \frac{3}{8\pi} \left\langle (\nabla_{\frac{\partial}{\partial z_l}}^X J) \frac{\partial}{\partial z_j}, (\nabla_{\frac{\partial}{\partial \bar{z}_i}}^X J) \frac{\partial}{\partial \bar{z}_j} \right\rangle \right. \\
 & \left. - \frac{1}{12\pi} \left\langle R^{TX} \left(\frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\rangle d\bar{z}_l d\bar{z}_m \right. \\
 & \left. - \frac{1}{8\pi} R^E \left(\frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right) d\bar{z}_l d\bar{z}_m \right\} I_{\mathbb{C} \otimes E}. \tag{2.44}
 \end{aligned}$$

By [24, (2.39)],

$$\begin{aligned}
 & -(\mathcal{L}_0^{-1} P^{N^+} \mathcal{O}_2 P^N)(0, 0) \\
 & = \frac{1}{2\pi} \left\{ \left\langle R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i} \right\rangle + R^E \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right) \right\} I_{\mathbb{C} \otimes E}. \tag{2.45}
 \end{aligned}$$

By (2.19), (2.20), (2.35), (2.36), (2.38), (2.44), (2.45), and the discussion after (2.20), we get

$$\begin{aligned}
 \mathbf{b}_1(x) & = \frac{1}{\pi} \left[\left\langle R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i} \right\rangle + R^E \left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right) \right. \\
 & \left. - \frac{1}{36} \sum_{kl} \left| \left\langle (\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J) \frac{\partial}{\partial \bar{z}_l} \right| \right|^2 \right] I_{\mathbb{C} \otimes E} + \frac{1}{36\pi} \left\langle (\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right\rangle \\
 & \times \left\langle (\nabla_{\frac{\partial}{\partial \bar{z}_k}}^X J) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right\rangle d\bar{z}_l \wedge d\bar{z}_m I_{\mathbb{C} \otimes E} i_{\frac{\partial}{\partial \bar{z}_j}} \wedge i_{\frac{\partial}{\partial \bar{z}_i}} \\
 & - \frac{1}{8\pi} \left(\frac{2}{3} \left\langle R^{TX} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\rangle + R^E \right) \left(\frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial \bar{z}_m} \right) d\bar{z}_l \wedge d\bar{z}_m I_{\mathbb{C} \otimes E} \\
 & + \frac{1}{2\pi} \left(\frac{2}{3} \left\langle R^{TX} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\rangle + R^E \right) \left(\frac{\partial}{\partial z_l}, \frac{\partial}{\partial z_m} \right) I_{\mathbb{C} \otimes E} i_{\frac{\partial}{\partial \bar{z}_m}} \wedge i_{\frac{\partial}{\partial \bar{z}_l}}. \tag{2.46}
 \end{aligned}$$

Moreover, we learn from [24, Lemma 2.2] that

$$r^X = -\langle R^{TX}(e_i, e_j)e_i, e_j \rangle = 8 \left\langle R^{TX} \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_i} \right\rangle - \frac{1}{4} |\nabla^X J|^2. \tag{2.47}$$

We are now ready to conclude. By (2.46) and (2.47), we get the formula (2.1) for \mathbf{b}_1 . We obtain then (2.2) by taking the trace of (2.1). The proof of Theorem 2.1 is complete.

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