ASYMPTOTICS OF SPECTRAL FUNCTION OF LOWER ENERGY FORMS AND BERGMAN KERNEL OF SEMI-POSITIVE AND BIG LINE BUNDLES

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ABSTRACT. In this paper we study the asymptotic behaviour of the spectral function corresponding to the lower part of the spectrum of the Kodaira Laplacian on high tensor powers of a holomorphic line bundle. This implies a full asymptotic expansion of this function on the set where the curvature of the line bundle is non-degenerate. As application we obtain the Bergman kernel asymptotics for adjoint semi-positive line bundles over complete Kähler manifolds, on the set where the curvature is positive. We also prove the asymptotics for big line bundles endowed with singular Hermitian metrics with strictly positive curvature current. In this case the full asymptotics holds outside the singular locus of the metric.

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1. Introduction and statement of the main results

Let L be a holomorphic line bundle over a complex manifold M and let L^k be the k-th tensor power of L. The Bergman kernel is the smooth kernel of the orthogonal projection onto the space of L^2-integrable holomorphic sections of L^k. The study of the large k behavior of the Bergman kernel is an active research subject in complex geometry and is closely related to topics like the existence of canonical Kähler metrics (e. g. [19], [23], [24], [47]), Berezin-Toeplitz quantization (e. g. [5], [22], [38], [39]), equidistribution of zeros of holomorphic sections (e. g. [11], [18], [44]), quantum chaos and mathematical physics [21]. We refer the reader to the book [36] for a comprehensive study of the Bergman kernel and its applications and also to the survey [33].

In the case of a positive line bundle L over a compact base manifold M, D. Catlin [10] and S. Zelditch [51] established the asymptotic expansion of the Bergman kernel on the diagonal by using a result by Boutet de Monvel-Sjöstrand [8] about the asymptotics of the Szegö kernel on a strictly pseudoconvex boundary, together with a reduction idea of Boutet de Monvel-Guillemin [9].

X. Dai, K. Liu and X. Ma [12], [13] obtained the full off-diagonal asymptotic expansion and Agmon estimates of the Bergman kernel for a high power of positive line bundle on a compact complex manifold by using the heat kernel method. Their result holds actually for the more general Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on a compact symplectic manifold. In [34], [36], [37], X. Ma and the second-named author proved the asymptotic expansion for yet another generalization of the Kodaira Laplacian, namely the renormalized Bochner-Laplacian on a symplectic manifold and also showed the existence of the estimate on a large class of non-compact manifolds. The main analytic tool in [12], [13], [34], [36], [37] is the analytic localization technique in local index theory developed by Bismut-Lebeau [36].

Another proof of the existence of the complete asymptotic expansion for the Bergman kernel for a high power of a positive line bundle on a compact complex manifold was obtained by B. Berndtsson, R. Berman and J. Sjöstrand [2].

A natural generalization is the asymptotic of the kernel of the projection on the harmonic forms in the case of a line bundle with non-degenerate curvature. R. Berman and J. Sjöstrand [3] obtained this result building on the heat equation method of Menikoff-Sjöstrand [42]. More generally, the expansion in the non-degenerate case was obtained independently by X. Ma and the second-named author [35] for the kernel of the projection on the kernel of the spin^c Dirac operator on symplectic manifolds.

When the Hermitian fiber metric on L is a strictly positive singular Hermitian metric, X. Ma and the second-named author introduced a complete base metric on the smooth part of L and modified the fiber metric on L and then they obtained a full asymptotic expansion for the associated Bergman kernel [36], [37]. As a corollary, they could reprove the Shiffman conjecture.

The first main result of this paper is a local asymptotic expansion of the spectral function of the Kodaira Laplacian on L^k on a Hermitian manifold M for forms of energy less than k^{-N_0}, for $N_0 \in \mathbb{N}$ fixed, on the non-degenerate part of L, cf. Theorem 1.1. This is a very general result since it holds without global assumptions on the manifold or the line bundle. However, the estimates obtained do not apply directly to the Bergman kernel, which is obtained by formally letting $N_0 \to \infty$ in (1.7). We then impose a very mild semi-classical local condition on the Kodaira Laplacian, namely the $O(k^{-n_0})$ small spectral gap on an open set $D \Subset M$ (see Definition 1.5). We prove that the Bergman kernel admits an asymptotic expansion on D if the Kodaira Laplacian has $O(k^{-n_0})$ small spectral gap on D, cf. Theorem 1.6.

We apply further these results to study the Bergman kernel of semi-positive or positive but singular Hermitian line bundles. We prove that if M is a complete Kähler manifold and L is semi-positive on M, then the Bergman kernel of $L^k \otimes K_M$ admits a full asymptotic expansion on the non-degenerate part of L, cf. Theorem 1.7. Moreover, we show in Theorem 1.10 that if M is any compact complex manifold and L semi-positive and positive at some point, then the Bergman kernel of L^k admits a full asymptotic expansion on the set where L is positive, with the possible exception of a proper analytic variety $\Sigma \subset M$.

We also consider the case of a singular Hermitian fiber metric on L. The holomorphic sections which are L^2 with respect to the singular metric turn out to be sections of L twisted with a multiplier ideal sheaf. One can naturally define the orthogonal projection on this space of sections and consider its kernel on the regular locus of the metric. We show that this kernel has an asymptotic expansion on the regular locus, if the curvature current is strictly positive and smooth outside a proper analytic set (Theorem 1.8). This yields yet another proof of the Shiffman conjecture.

We further give formulas for the first top leading terms of the asymptotic expansion of the spectral function and recover the top leading coefficients of the Bergman kernel expansion. These coefficients recently attracted a lot of attention, see the comments after Theorem 1.6.

Other applications of the main results are local and global holomorphic Morse inequalities, a local form of the expansion of the Bergman kernel on forms and examples of manifolds having very small spectral gap.

We now start to formulate the main results. For the precise meanings of some standard notations and terminology, see Section 2.

1.1. **Statement of main results.** Let (M, Θ, J) be a Hermitian manifold of complex dimension n, where Θ is a smooth positive $(1,1)$-form and J is the complex structure. Let $g^{TM}_{\Theta}(\cdot,\cdot) = \Theta(\cdot, J\cdot)$ be the Riemannian metric on TM induced by Θ and J and let $\langle\,\cdot\,,\cdot\,\rangle$ be the Hermitian metric on $\mathbb{C}TM$ induced by g^{TM}_{Θ}. The Riemannian volume form dv_M of (M, Θ) has the form $dv_M = \Theta^n/n!$.

Let (L, h^L) be a holomorphic Hermitian line bundle on M and set $L^k := L^{\otimes k}$. Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature $R^L = (\nabla^L)^2$. We will identify the curvature form R^L with the Hermitian matrix $\dot{R}^L \in \mathscr{C}^\infty(M, \mathrm{End}(T^{(1,0)}M))$

satisfying for every $U, V \in T_x^{(1,0)}M$, $x \in M$,

$$\langle R^L(x), U \wedge \overline{V}\rangle = \langle \dot{R}^L(x)U, V\rangle. \tag{1.1}$$

Let $\det \dot{R}^L(x) := \mu_1(x)\dots\mu_n(x)$, where $\mu_j(x), j = 1, \dots, n$, are the eigenvalues of $\dot{R}^L$ with respect to $\langle\,\cdot\,,\cdot\,\rangle$. For $q \in \{0, 1, \dots, n\}$, let

$$\begin{aligned} M(q) = \Big\{x \in M;\, \dot{R}^L(x) \in \operatorname{End}(T_x^{(1,0)}M) &\text{ is non–degenerate} \\ &\text{and has exactly } q \text{ negative eigenvalues}\Big\}. \end{aligned} \tag{1.2}$$

We denote by W the subbundle of rank q of $T^{(1,0)}M|_{M(q)}$ generated by the eigenvectors corresponding to negative eigenvalues of $\dot{R}^L$. Then $\det \overline{W}^* := \Lambda^q\overline{W}^* \subset \Lambda^q T^{*(0,1)}M|_{M(q)}$ is a rank one subbundle, where $\Lambda^q T^{*(0,1)}M$ is the bundle of $(0,q)$ forms, $\overline{W}^*$ is the dual bundle of the complex conjugate bundle of W and $\Lambda^q\overline{W}^*$ is the vector space of all finite sums of $v_1 \wedge \cdots \wedge v_q$, $v_1, \dots, v_q \in \overline{W}^*$. We denote by $I_{\det \overline{W}^*} \in \operatorname{End}(\Lambda^q T^{*(0,1)}M))$ the orthogonal projection from $\Lambda^q T^{*(0,1)}M$ onto $\det \overline{W}^*$.

Let (L^k, h^{L^k}) be the k-th tensor power of (L, h^L) and let $(\cdot\,,\cdot)_k$ be the inner product on $\Omega_0^{0,q}(M, L^k)$ induced by g_Θ^{TM} and h^{L^k} (see (2.3)). Let $\|\cdot\|$ be the corresponding norm and let $L^2_{(0,q)}(M, L^k)$ be the completion of $\Omega_0^{0,q}(M, L^k)$ with respect to $\|\cdot\|$.

Let $\Box_k^{(q)}$ be the Kodaira Laplacian acting on $(0,q)$–forms with values in L^k, cf. (2.6). We denote by the same symbol $\Box_k^{(q)}$ the Gaffney extension of the Kodaira Laplacian, cf. (2.9). It is well-known that $\Box_k^{(q)}$ is self-adjoint and the spectrum of $\Box_k^{(q)}$ is contained in $\overline{\mathbb{R}}_+$ (see [36, Prop. 3.1.2]). For a Borel set $B \subset \mathbb{R}$ we denote by $E(B)$ the spectral projection of $\Box_k^{(q)}$ corresponding to the set B, where E is the spectral measure of $\Box_k^{(q)}$ (see Section 2 in Davies [14]) and for $\lambda \in \mathbb{R}$ we set $E_\lambda = E\big((-\infty, \lambda]\big)$ and

$$\mathscr{E}^q_\lambda(M, L^k) = \operatorname{Range} E_\lambda \subset L^2_{(0,q)}(M, L^k)\,. \tag{1.3}$$

If $\lambda = 0$, then $\mathscr{E}^q_0(M, L^k) = \operatorname{Ker} \Box_k^{(q)} =: \mathscr{H}^q(M, L^k)$ is the space of global harmonic sections. For a holomorphic vector bundle over M we have

$$H^0_{(2)}(M, E) := \Big\{s \in L^2(M, E);\, \overline{\partial}_E s = 0\Big\} = \operatorname{Ker} \Box^E\,,$$

where $\overline{\partial}_E$ is the Cauchy-Riemann operator with values in E and $\Box^E$ is the Kodaira Laplacian with values in E (see Section 2.3). The *spectral projection* of $\Box_k^{(q)}$ is the orthogonal projection

$$P^{(q)}_{k,\lambda} : L^2_{(0,q)}(M, L^k) \to \mathscr{E}^q_\lambda(M, L^k)\,. \tag{1.4}$$

The *spectral function* $P^{(q)}_{k,\lambda}(\,\cdot\,,\cdot\,) = P^{(q)}_k(\,\cdot\,,\cdot, \lambda)$ is the Schwartz kernel of $P^{(q)}_{k,\lambda}$, see (4.16) and (4.17). Since $\Box_k^{(q)}$ is elliptic, it is not difficult to see that

$$P^{(q)}_k(\,\cdot\,,\cdot\,, \lambda) \in \mathscr{C}^\infty\Big(M \times M, L^k \otimes (\Lambda^q T^{*(0,1)}M \boxtimes \Lambda^q T^{*(0,1)}M) \otimes (L^k)^*\Big)$$

and $\mathscr{E}^q_\lambda(M, L^k) \subset \Omega^{0,q}(M, L^k)$. Since $L^k_x \otimes (L^k_x)^* \cong \mathbb{C}$ we can identify $P^{(q)}_k(x, x, \lambda)$ to an element of $\operatorname{End}(\Lambda^q T_x^{*(0,1)}M)$. Then

$$X \ni x \longmapsto P^{(q)}_k(x, x, \lambda) = P^{(q)}_{k,\lambda}(x, x) \in \operatorname{End}(\Lambda^q T_x^{*(0,1)}M) \tag{1.5}$$

is a smooth section of $\mathrm{End}(\Lambda^q T^{*(0,1)}M)$, called *local density of states* of $\mathscr{E}^q_\lambda(M, L^k)$. The trace of $P^{(q)}_k(x, x, \lambda)$ is given by

$$\operatorname{Tr} P^{(q)}_{k,\lambda}(x, x) = \operatorname{Tr} P^{(q)}_k(x, x, \lambda) := \sum_{j=1}^{d} \Big\langle\, P^{(q)}_k(x, x, \lambda)\, e_{J_j}(x)\, , e_{J_j}(x)\, \Big\rangle,$$

where $e_{J_1}, \dots, e_{J_d}$ is a local orthonormal basis of $\Lambda^q T^{*(0,1)}M$ with respect to $\langle\cdot,\cdot\rangle$. The projection

$$P^{(q)}_k := P^{(q)}_{k,0} : L^2_{(0,q)}(M, L^k) \to \operatorname{Ker} \Box^{(q)}_k \tag{1.6}$$

on the lowest energy level $\lambda = 0$ is called the *Bergman projection*, its kernel $P^{(q)}_k(\,\cdot\,,\cdot)$ is called the *Bergman kernel*. The restriction to the diagonal of $P^{(q)}_k(\,\cdot\,,\cdot)$ is denoted $P^{(q)}_k(\cdot)$ and is called the *Bergman kernel form*. The function $\operatorname{Tr} P^{(q)}_{k,0}(x, x) := \operatorname{Tr} P^{(q)}_k(x)$ is called the *Bergman kernel function*. We notice that $\operatorname{Tr} P^{(0)}_k(x) = P^{(0)}_k(x)$.

We introduce now the notion of asymptotic expansion (see Definition 3.9). Let $D \subset M$ be an open set and $a(x, k), a_j(x) \in \mathscr{C}^\infty(M, \mathrm{End}(\Lambda^q T^{*(0,1)}M))$, $j = 0, 1, \dots$ and $m \in \mathbb{Z}$. We say that $a(x, k)$ has an asymptotic expansion

$$a(x, k) \sim \sum_{j=0}^{\infty} a_j(x) k^{m-j} \text{ locally uniformly on } D,$$

if for every $N \in \mathbb{N}_0$, $\ell \in \mathbb{N}_0$ and every compact set $K \subset D$, there exists a constant $C_{N,\ell,K} > 0$ independent of k, such that for k sufficiently large

$$\Big| a(x, k) - \sum_{j=0}^{N} k^{m-j} a_j(x) \Big|_{\mathscr{C}^\ell(K)} \leq C_{N,\ell,K}\, k^{m-N-1}\, .$$

We say that $a(x, k) = O(k^{-\infty})$ locally uniformly on D if $a(x, k) \sim 0$ locally uniformly on D.

The following theorem is one of the main results. It expresses the fact that the Kodaira Laplacian acting on $\Omega^{\bullet,\bullet}(M, L^k)$ admits a *local* semi-classical Hodge decomposition. Note that there are neither global assumptions on the positivity of the bundle nor on the base manifold.

Theorem 1.1. *Let (M, Θ) be a Hermitian manifold, (L, h^L) be a holomorphic Hermitian line bundle on M. Fix $q \in \{0, 1, \dots, n\}$ and $N_0 \geqslant 1$. Then for every $m \in \{0, 1, \dots, n\}$ there exists a k-dependent section $b^{(m)}(x, k) \in \mathscr{C}^\infty(M(q), \mathrm{End}(\Lambda^m T^{*(0,1)}M))$ with the following property: for every $D \Subset M(q)$, $\ell \in \mathbb{N}_0$, there exists a constant $C_{D,\ell} > 0$ independent of k with*

$$\Big| P^{(m)}_k(x, x, k^{-N_0}) - b^{(m)}(x, k) \Big|_{\mathscr{C}^\ell(D)} \leqslant C_{D,\ell}\, k^{3n+\ell-N_0}\, . \tag{1.7}$$

Moreover, $b^{(m)}(x, k) = 0$ for $m \neq q$ and $b^{(q)}(x, k)$ has an asymptotic expansion

$$b^{(q)}(x, k) \sim \sum_{j=0}^{\infty} b^{(q)}_j(x) k^{n-j} \text{ locally uniformly on } M(q), \tag{1.8}$$

for some $b^{(q)}_j \in \mathscr{C}^\infty(M(q), \mathrm{End}(\Lambda^q T^{*(0,1)}M))$, $j = 0, 1, \dots$. On $M(q)$ we have

$$b^{(q)}_0 = (2\pi)^{-n} \Big| \det \dot{R}^L \Big| I_{\det \overline{W}^*}\, . \tag{1.9}$$

We wish to give formulas for the top coefficients of the expansion in the case $q = 0$. We introduce the geometric objects used in Theorem 1.2 and Theorem 1.7 below. Put

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.10}$$

On the set $M(0)$ the $(1,1)$-form ω is positive and induces a Riemannian metric $g^{TM}_\omega(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Let ∇^{TM}_ω be the Levi-Civita connection on (M, g^{TM}_ω), $R^{TM}_\omega = (\nabla^{TM}_\omega)^2$ its curvature (cf. (4.72)), Ric its Ricci curvature and r the scalar curvature of g^{TM}_ω (see (4.70)). We denote by $\mathrm{Ric}_\omega = \mathrm{Ric}\,(J\cdot,\cdot)$ the $(1,1)$-form associated to Ric (cf. (4.74)) and by $\triangle_\omega$ be the complex Laplacian with respect to ω (see (4.68)). We also denote by $\langle\,\cdot\,,\cdot\,\rangle_\omega$ the pointwise Hermitian metric induced by g^{TM}_ω on (p,q)-forms on M and by $|\cdot|_\omega$ the corresponding norm.

Let $R^{\det}_\Theta$ denote the curvature of the canonical line bundle $K_M = \det T^{*(1,0)}M$ with respect to the metric induced by Θ (see (4.71)). Put

$$\widehat{r} = \triangle_\omega \log V_\Theta\,, \quad V_\Theta = \det\,(\Theta_{j,k})^n_{j,k=1}$$

where $\Theta = \sqrt{-1}\sum_{j,k=1}^n \Theta_{j,k} dz_j \wedge d\overline{z}_k$ in local coordinates $z = (z_1, \dots, z_n)$.

Theorem 1.2. *Let (M,Θ) and (L,h^L) be as in Theorem 1.1. The coefficients $b^{(0)}_1$ and $b^{(0)}_2$ in the expansion(1.8) for $q = 0$ have the following form:*

$$b^{(0)}_1 = (2\pi)^{-n}\det \dot{R}^L\Big(\frac{1}{8\pi}r - \frac{1}{4\pi}\Delta_\omega \log\det \dot{R}^L\Big) = (2\pi)^{-n}\det \dot{R}^L(x)\Big(\frac{1}{4\pi}\widehat{r} - \frac{1}{8\pi}r\Big)(x), \tag{1.11}$$

$$\begin{aligned} b^{(0)}_2 = (2\pi)^{-n}\det \dot{R}^L\Big(&\frac{1}{128\pi^2}r^2 - \frac{1}{32\pi^2}r\widehat{r} + \frac{1}{32\pi^2}(\widehat{r})^2 - \frac{1}{32\pi^2}\triangle_\omega\widehat{r} - \frac{1}{8\pi^2}\left|R^{\det}_\Theta\right|^2_\omega \\ &+ \frac{1}{8\pi^2}\langle\,\mathrm{Ric}_\omega\,, R^{\det}_\Theta\,\rangle_\omega + \frac{1}{96\pi^2}\triangle_\omega r - \frac{1}{24\pi^2}\left|\mathrm{Ric}_\omega\right|^2_\omega + \frac{1}{96\pi^2}\left|R^{TM}_\omega\right|^2_\omega\Big), \end{aligned} \tag{1.12}$$

where $\left|R^{TM}_\omega\right|^2_\omega$ is given by (4.73).

On the set where the curvature of L is degenerate we have the following behaviour.

Theorem 1.3. *Let (M,Θ) and (L,h^L) be as in Theorem 1.1. Set*

$$M_{\deg} = \Big\{x \in M;\ \dot{R}^L \text{ is degenerate at } x \in M\Big\}\,.$$

Then for every $x_0 \in M_{\deg}$, $\varepsilon > 0$ and every $q \in \{0,1,\dots,n\}$, there exist a neighborhood U of x_0 and $k_0 > 0$, such that for all $k \geq k_0$ we have

$$\mathrm{Tr}\, P^{(q)}_k(x,x,k^{-N_0}) \leq \varepsilon k^n, \quad x \in U. \tag{1.13}$$

As a Corollary of Theorem 1.1 and Theorem 1.3, we obtain

Corollary 1.4 (Local holomorphic Morse inequalities). *Let (M,Θ) and (L,h^L) be as in Theorem 1.1. Let $N_0 \geq 2n+1$. Then the spectral function of the Kodaira Laplacian has the following asymptotic bahaviour:*

$$\mathrm{Tr}\, P^{(q)}_k(x,x,k^{-N_0}) = k^n(2\pi)^{-n}\left|\det \dot{R}^L(x)\right| + O(k^{n-1}), \tag{1.14}$$

locally uniformly on $M(q)$, and

$$\lim_{k\to\infty} k^{-n}\mathrm{Tr}\, P^{(q)}_k(x,x,k^{-N_0}) = (2\pi)^{-n}\left|\det \dot{R}^L(x)\right| 1_{M(q)}(x), \quad \forall x \in M. \tag{1.15}$$

Moreover, for every $\varepsilon > 0$, there exists a $k_0 > 0$, such that for all $k \geq k_0$, we have

$$\operatorname{Tr} P_k^{(q)}(x,x,k^{-N_0}) \leq \Big(\varepsilon + (2\pi)^{-n}\left|\det \dot{R}^L(x)\right| 1_{M(q)}(x)\Big)k^n, \quad \forall x \in M, \tag{1.16}$$

and when $q = 0$ and $N_0 \geq 2n+3$, we have

$$P_k^{(0)}(x,x,k^{-N_0}) \leq k^n(2\pi)^{-n}\det \dot{R}^L(x) + k^{n-1}b_1^{(0)}(x) + k^{n-2}b_2^{(0)}(x) + O(k^{n-3}), \tag{1.17}$$

locally uniformly on $M(0)$, where $b_1^{(0)}(x)$ and $b_2^{(0)}(x)$ are as in (1.11) *and* (1.12) *respectively.*

The term local holomorphic Morse inequalities is motivated by the fact that integration of the inequalities from Corollary 1.4 yields the holomorphic Morse inequalities of Demailly, see Section 10.5. Berman [1] proved that

$$\limsup_{k\to\infty} k^{-n}\operatorname{Tr} P_k^{(q)}(x) \leq (2\pi)^{-n}\left|\det \dot{R}^L(x)\right| 1_{M(q)}(x), \quad x \in M,$$

and when M is compact, there exists a sequence $\mu_k \to 0$, as $k \to \infty$, such that

$$\lim_{k\to\infty} k^{-n}\operatorname{Tr} P_k^{(q)}(x,x,\mu_k) = (2\pi)^{-n}\left|\det \dot{R}^L(x)\right| 1_{M(q)}(x), \quad x \in M.$$

Corollary 1.4 refines and generalizes Berman's results.

In order to obtain precise asymptotics we combine the local asymptotics from Theorem 1.1 with a mild condition on the semiclassical behaviour of the spectrum of the Kodaira Laplacian $\Box_k^{(q)}$ for k large, which we call (local) $O(k^{-n_0})$ small spectral gap.

Definition 1.5. Let $D \subset M$. We say that $\Box_k^{(q)}$ has $O(k^{-n_0})$ *small spectral gap on D* if there exist constants $C_D > 0$, $n_0 \in \mathbb{N}$, $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ and $u \in \Omega_0^{0,q}(D, L^k)$, we have

$$\left\|(I - P_k^{(q)})u\right\| \leq C_D\, k^{n_0}\left\|\Box_k^{(q)}u\right\|.$$

To explain this condition, assume that M is a complete Hermitian manifold. Then the operator $\Box_k^{(q)}$ is essentially self-adjoint and $\Omega_0^{0,q}(D, L^k)$ is dense with respect to the graph-norm in the domain of the quadratic form of $\Box_k^{(q)}$ (see e. g. [36, § 3.3]). If $\Box_k^{(q)}$ has $O(k^{-n_0})$ small spectral gap on M then $\inf\left\{\lambda \in \operatorname{Spec}(\Box_k^{(q)});\ \lambda \neq 0\right\} \geq Ck^{-n_0}$.

From Theorem 1.1, Definition 1.5 and some simple arguments (see Section 4.4), we deduce:

Theorem 1.6. *Let (M,Θ) be a Hermitian manifold, (L,h^L) be a holomorphic Hermitian line bundle on M. Fix $q \in \{0,1,\ldots,n\}$ and $N_0 \geqslant 1$. Let $D \Subset M(q)$. If $\Box_k^{(q)}$ has $O(k^{-n_0})$ small spectral gap on D, then for every $D' \Subset D$, $\ell \in \mathbb{N}_0$, there exists a constant $C_{D',\ell} > 0$ independent of k with*

$$\left|P_k^{(q)}(x,x,k^{-N_0}) - P_k^{(q)}(x)\right|_{\mathscr{C}^\ell(D')} \leqslant C_{D',\ell}\, k^{3n+\ell-N_0}.$$

In particular,

$$P_k^{(q)}(x) \sim \sum_{j=0}^{\infty} b_j^{(q)}(x)k^{n-j} \ \text{ locally uniformly on } D, \tag{1.18}$$

where $b_j^{(q)} \in \mathscr{C}^\infty(D, \operatorname{End}(\Lambda^q T^{*(0,1)}M))$, $j = 0,1,\ldots$, are as in (1.8) *and $b_0^{(q)}$, $b_1^{(0)}$, $b_2^{(0)}$ are given by* (1.9), (1.11), (1.12).

Note that if L is a positive line bundle on a compact manifold M, or more generally L is uniformly positive on a complete manifold (M, Θ) with $\sqrt{-1}R^{K^*_M}$ and $\partial\Theta$ bounded below, then the Kodaira Laplacian $\square^{(0)}_k$ has a "large" spectral gap on M, i.e. there exists a constant $C > 0$ such that for all k we have $\inf\left\{\lambda \in \operatorname{Spec}(\square^{(0)}_k);\, \lambda \neq 0\right\} \geq Ck$ (see [36, Th. 1.5.5], [36, Th. 6.1.1, (6.1.8)]). Therefore the Bergman kernel $P^{(q)}_k$ has the asymptotic expansion (1.18) and we recover from Theorem 1.6 the asymptotic expansion of the Bergman kernel for compact manifolds [10], [51] (cf. also [36, Th. 4.1.1]) for $q = 0$, [3], [35] [36, Th. 8.2.4] for arbitrary q and for complete manifolds [37, Th. 3.11], [36, Th. 6.1.1].

In the case $q = 0$ the precise formulas (1.11), (1.12) for the coefficients of the Bergman kernel expansion (1.18) play an important role in the investigations about the relation between canonical metrics in Kähler geometry and stability in algebraic geometry see e.g. [19], [23], [24], [25], [47], [49], [50] (cf. also [36, § 5.2]).

The coefficients $b^{(0)}_1$, $b^{(0)}_2$ were computed by Lu [32], L. Wang [49], X. Wang [50], in various degrees of generality. The method of these authors is to construct appropriate peak sections as in [47], using Hörmander's L^2 $\overline{\partial}$-method.

In [12, §5.1], Dai-Liu-Ma computed $b^{(0)}_1$ by using the heat kernel, and in [37, §2], [35, §2] (cf. also [36, §4.1.8, §8.3.4]), $b^{(0)}_1$ was computed in the symplectic case. The coefficient $b^{(0)}_2$ was calculated in [40, Th. 0.1] (these results include a twisting Hermitian vector bundle E). Recently, a combinatorial formula for the coefficients $b^{(0)}_j$ was obtained in [48] and the formula for $b^{(0)}_2$ was rederived in [25].

In the above mentioned results it was supposed that the curvature $\omega = \frac{\sqrt{-1}}{2\pi}R^L$ equals the underlying metric Θ. If $\omega \neq \Theta$ formulas for $b^{(0)}_1$, $b^{(0)}_2$ were given in [36, Th. 4.1.3], [40, Remark 0.5], [30, Th. 1.4]. Since we allow a local $O(k^{-n_0})$ small spectral gap, we can obtain the Bergman kernel expansion under weak conditions, such as semi-positivity of the line bundle. In this case we have to twist L^k with the canonical line bundle K_M, which we endow with the natural Hermitian metric induced by Θ. We denote by P_{k,K_M} the orthogonal projection from $L^2(M, L^k \otimes K_M)$ on $H^0_{(2)}(M, L^k \otimes K_M) = \mathscr{H}^0(M, L^k \otimes K_M)$.

Theorem 1.7. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a semi-positive line bundle over M. Then the Bergman kernel function $P_{k,K_M}(\,\cdot\,)$ of $H^0_{(2)}(M, L^k \otimes K_M)$ has the asymptotic expansion*

$$P_{k,K_M}(x) \sim \sum_{j=0}^{\infty} k^{n-j} b^{(0)}_{j,K_M}(x) \;\; \text{locally uniformly on } M(0), \tag{1.19}$$

where $b^{(0)}_{j,K_M} \in \mathscr{C}^\infty(M(0), \operatorname{End}(K_M))$, $j = 0, 1, \ldots$, are given by

(1.20)

$$\begin{aligned}
b^{(0)}_{0,K_M} &= (2\pi)^{-n} \det \dot{R}^L \operatorname{Id}_{K_M}, \\
b^{(0)}_{1,K_M} &= (2\pi)^{-n} \det \dot{R}^L \Big(-\frac{1}{8\pi} r \Big) \operatorname{Id}_{K_M}, \\
b^{(0)}_{2,K_M} &= (2\pi)^{-n} \det \dot{R}^L \Big(\frac{1}{128\pi^2} r^2 + \frac{1}{96\pi^2} \triangle_\omega r - \frac{1}{24\pi^2} \left|\operatorname{Ric}_\omega\right|^2_\omega + \frac{1}{96\pi^2} \left|R^{TM}_\omega\right|^2_\omega \Big) \operatorname{Id}_{K_M},
\end{aligned}$$

where $\left|R^{TM}_\omega\right|^2_\omega$ is given by (4.73) and Id_{K_M} is the identity map on K_M.

Let us consider now a singular Hermitian holomorphic line bundle $(L,h^L)\to M$ (see e. g. [36, Def. 2.3.1]). We assume that h^L is smooth outside a proper analytic set Σ and the curvature current of h^L is strictly positive. The metric h^L induces singular Hermitian metrics h^{L^k} on L^k. We denote by $\mathscr{I}(h^{L^k})$ the Nadel ideal multiplier sheaf associated to h^{L^k} and by $H^0(M,L^k\otimes\mathscr{I}(h^{L^k}))$ the space of global sections of the sheaf $\mathscr{O}(L^k)\otimes\mathscr{I}(h^{L^k})$ (see (9.2)). We denote by $(\cdot\,,\cdot)_k$ the natural inner products on $\mathscr{C}^\infty(M,L^k)$ induced by h^L and the volume form dv_M on M (see (9.1)) Let $\{S^k_j\}$ be an orthonormal basis of $H^0(X,L^k\otimes\mathscr{I}(h^{L^k}))$ with respect to the inner product induced $(\cdot\,,\cdot)_k$. The (multiplier ideal) Bergman kernel function is defined by

$$P_k(x) := \sum_{j=1}^{d_k}\left|S^k_j(x)\right|^2_{h^{L^k}}\,,\quad x\in M\setminus\Sigma\,. \tag{1.21}$$

Theorem 1.8. *Let (L,h^L) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold (M,Θ). We assume that h^L is smooth outside a proper analytic set Σ and the curvature current of h^L is strictly positive. Then the Bergman kernel function $P_k(\cdot)$ of $H^0(M,L^k\otimes\mathscr{I}(h^{L^k}))$ has the asymptotic expansion*

$$P_k(x)\sim\sum_{j=0}^{\infty}k^{n-j}b^{(0)}_j(x)\ \textit{ locally uniformly on } M\setminus\Sigma,$$

where $b^{(0)}_j\in\mathscr{C}^\infty(M\setminus\Sigma)$, $j=0,1,\ldots$, $b^{(0)}_0=(2\pi)^{-n}\det\dot{R}^L$ and $b^{(0)}_1$ and $b^{(0)}_2$ are given by (1.11) *and* (1.12)*, respectively.*

We obtain in this way another proof of the Shiffman-Ji-Bonavero-Takayama criterion (cf. [36, Th. 2.3.28, 2.3.30]).

Corollary 1.9. *Under the assumptions in Theorem 1.8, we have*

$$\dim H^0(M,L^k\otimes\mathscr{I}(h^{L^k}))\geq ck^n$$

for k large, where $c>0$ is independent of k. Therefore, L is big and M is Moishezon.

We assume that (M,Θ) is compact and we set

$$\mathrm{Herm}(L)=\Big\{\text{singular Hermitian metrics on } L\Big\}\,,$$

$$\begin{aligned}\mathcal{M}(L)=&\Big\{h^L\in\mathrm{Herm}(L);\ h^L \text{ is smooth outside a proper analytic set}\\ &\text{and the curvature current of } h^L \text{ is strictly positive}\Big\}\,.\end{aligned}$$

Note that by Siu's criterion [36, Th. 2.2.27], L is big under the hypotheses of Theorem 1.10 below. By [36, Lemma 2.3.6], $\mathcal{M}(L)\neq\emptyset$. Set

$$M' := \Big\{p\in M\,;\ \exists\ h^L\in\mathcal{M}(L) \text{ with } h^L \text{ smooth near } p\Big\}\,. \tag{1.22}$$

Theorem 1.10. *Let (M,Θ) be a compact Hermitian manifold. Let $(L,h^L)\to M$ be a Hermitian holomorphic line bundle with smooth Hermitian metric h^L having semi-positive curvature and with $M(0)\neq\emptyset$. Then the Bergman kernel function $P_k(\cdot)$ of $\mathscr{H}^0(M,L^k)$ has the asymptotic expansion*

$$P_k(x)\sim\sum_{j=0}^{\infty}k^{n-j}b^{(0)}_j(x)\ \textit{ locally uniformly on } M(0)\cap M',$$

where $b_j^{(0)} \in \mathscr{C}^\infty(M(0))$, $b_0^{(0)} = (2\pi)^{-n}\det \dot{R}^L$ *and* $b_1^{(0)}$ *and* $b_2^{(0)}$ *are given in* (1.11) *and* (1.12)*, respectively.*

The existence of the asymptotic expansion from Theorem 1.10 was obtained by Berman [4] in the case of a projective manifold M.

Remark 1.11. (I) In Theorems 1.1, 1.6, we obtain the diagonal expansion of the kernels $P^{(q)}_{k,k^{-N_0}}(\cdot,\cdot)$. We will prove actually more, namely the off-diagonal asymptotic expansion for $P^{(q)}_{k,k^{-N_0}}(x,y)$ on the non-degenerate part of L, see Theorem 4.11, Theorem 4.12 and Theorem 4.14 for the details. In the same vein, the diagonal expansions of the Bergman kernels from Theorems 1.6, 1.7, 1.8, 1.10 have off-diagonal counterparts. See Theorem 6.4, Theorem 9.1 and Theorem 8.3 for the details.
(II) Let E be a holomorphic vector bundle over M. Theorem 1.1, Theorem 1.3, Theorem 1.6, Theorem 1.8 and Theorem 1.10 and their off-diagonal counterparts can be generalized to the situation when L^k is replaced by $L^k \otimes E$. See Remark 4.13 and the discussions in the end of Section 4.4 and Section 5, for the details.

The layout of this paper is as follows. In Section 2 we collect some notations, definitions and statements we use throughout (geometric set-up, self-adjoint extension of the Kodaira Laplacian, Schwartz kernel theorem). In Section 3 we exhibit a microlocal Hodge decomposition for the Kohn Laplacian on a non-degenerate CR manifold and apply this to obtain the semiclassical Hodge decomposition for the Kodaira Laplacian on a complex manifold. In Section 4 we prove the existence of the asymptotic expansion of the spectral function associated to forms of energy less that k^{-N_0}. As a consequence we obtain the expansion of the Bergman kernel if the local $O(k^{-n_0})$ spectral gap exists. In Section 5 we get an asymptotic upper bound near the degeneracy set of the curvature of L. In Section 6 we prove the expansion of the Bergman kernel on the positivity set of an adjoint semi-positive line bundle over a complete Kähler manifold. In Section 7 we prove an L^2-estimate for the $\overline{\partial}$ for singular metrics. We use this estimate in Sections 8 and 9 to prove the existence of the Bergman kernel expansion for semi-positive line bundles and bundles endowed with a strictly positively-curved singular Hermitian metric. In Section 10 we apply the previous methods to obtain miscellaneous results, such as Bergman kernel expansion under various conditions and holomorphic Morse inequalities.

2. Preliminaries

2.1. **Some standard notations.** We shall use the following notations: $\mathbb{R}$ is the set of real numbers, $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$. An element $\alpha = (\alpha_1,\ldots,\alpha_n)$ of $\mathbb{N}_0^n$ will be called a multiindex and the length of α is: $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $x = (x_1,\ldots,x_n)$ be coordinates of $\mathbb{R}^n$. We write $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$, $D_x = \frac{1}{i}\partial_x$, $D_{x_j} = \frac{1}{i}\partial_{x_j}$. Let $z = (z_1,\ldots,z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1,\ldots,n$, be coordinates of $\mathbb{C}^n$. We write $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, $\overline{z}^\alpha = \overline{z}_1^{\alpha_1}\cdots\overline{z}_n^{\alpha_n}$, $\frac{\partial^{|\alpha|}}{\partial z^\alpha} = \partial_z^\alpha = \partial_{z_1}^{\alpha_1}\cdots\partial_{z_n}^{\alpha_n}$, $\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}})$, $j = 1,\ldots,n$. $\frac{\partial^{|\alpha|}}{\partial \overline{z}^\alpha} = \partial_{\overline{z}}^\alpha = \partial_{\overline{z}_1}^{\alpha_1}\cdots\partial_{\overline{z}_n}^{\alpha_n}$, $\partial_{\overline{z}_j} = \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}})$, $j = 1,\ldots,n$.

Let M be a complex manifold of dimension n. We always assume that M is paracompact. We denote charts on M by (D,z), where $z = (z_1,\ldots,z_n) : D \to \mathbb{C}^n$ are local coordinates. We set $z_j = x_{2j-1} + ix_{2j}$, $j = 1,\ldots,n$, where $x = (x_1,\ldots,x_{2n})$ are real coordinates on M. We write $z=(z_1,\ldots,z_n)=(x_1,\ldots,x_{2n})=x$. For a multiindex $J=(j_1,\ldots,j_q)\in\{1,\ldots,n\}^q$ we set $|J|=q$. We say that J is strictly increasing if $1\leqslant j_1<j_2<\cdots<j_q\leqslant n$. We put $d\overline{z}^J=d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}$. A $(0,q)$-form f on M has the local representation

$$f|_D=\sum_{|J|=q}{}' f_J(z)d\overline{z}^J\,,$$

where $\sum'$ means that the summation is performed only over strictly increasing multiindices.

Let Ω be a $\mathscr{C}^\infty$ paracompact manifold equipped with a smooth density of integration. We let $T\Omega$ and $T^*\Omega$ denote the tangent bundle of Ω and the cotangent bundle of Ω respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T\Omega$ and $\mathbb{C}T^*\Omega$ respectively. We write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between $T\Omega$ and $T^*\Omega$. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}T\Omega\times\mathbb{C}T^*\Omega$. Let E be a $\mathscr{C}^\infty$ vector bundle over Ω. We write E^* to denote the dual bundle of E. The fiber of E at $x\in\Omega$ will be denoted by E_x. Let F be another vector bundle over Ω. We write $E\boxtimes F$ to denote the vector bundle over $\Omega\times\Omega$ with fiber over $(x,y)\in\Omega\times\Omega$ consisting of the linear maps from E_x to F_y. Let $Y\subset\Omega$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $\mathscr{C}^\infty(Y,E)$ and $\mathscr{D}'(Y,E)$ respectively. Let $\mathscr{E}'(Y,E)$ be the subspace of $\mathscr{D}'(Y,E)$ whose elements have compact support in Y. For $s\in\mathbb{R}$, we let $H^s(Y,E)$ denote the Sobolev space of order s of sections of E over Y. Put

$$H^s_{\rm loc}(Y,E)=\{u\in\mathscr{D}'(Y,E);\ \varphi u\in H^s(Y,E),\forall\varphi\in\mathscr{C}^\infty_0(Y)\}$$

and $H^s_{\rm comp}(Y,E)=H^s_{\rm loc}(Y,E)\bigcap\mathscr{E}'(Y,E)$.

2.2. **Metric data.** Let (M,Θ) be a complex manifold of dimension n, where Θ is a smooth positive $(1,1)$ form, which induces a Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle$ on the holomorphic tangent bundle $T^{(1,0)}M$. In local holomorphic coordinates $z=(z_1,\ldots,z_n)$, if $\Theta=\sqrt{-1}\sum_{j,k=1}^n\Theta_{j,k}dz_j\wedge d\overline{z}_k$, then $\langle\,\frac{\partial}{\partial z_j}\,,\frac{\partial}{\partial z_k}\,\rangle=\Theta_{j,k}, j,k=1,\ldots,n$. Let $T^{(0,1)}M$ be the anti-holomorphic tangent bundle of M. We extend the Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle$ to $\mathbb{C}TM$ in a natural way by requiring $T^{(1,0)}M$ to be orthogonal to $T^{(0,1)}M$ and $\overline{\langle\,u\,,v\,\rangle}=\langle\,\overline{u}\,,\overline{v}\,\rangle$, $u,v\in T^{(0,1)}M$. Let $T^{*(1,0)}M$ be the holomorphic cotangent bundle of M and let $T^{*(0,1)}M$ be the anti-holomorphic cotangent bundle of M. For $p,q\in\mathbb{N}_0$, let $\Lambda^{p,q}T^*M=\Lambda^pT^{*(1,0)}M\otimes\Lambda^qT^{*(0,1)}M$ be the bundle of (p,q) forms of M. We write $\Lambda^{0,q}T^*M=\Lambda^qT^{*(0,1)}M$. The Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathbb{C}TM$ induces a Hermitian metric on $\Lambda^{p,q}T^*M$ also denoted by $\langle\,\cdot\,,\cdot\,\rangle$. Let $D\subset M$ be an open set. Let $\Omega^{p,q}(D)$ denote the space of smooth sections of $\Lambda^{p,q}T^*M$ over D. Similarly, if E is a vector bundle over D, then we let $\Omega^{p,q}(D,E)$ denote the space of smooth sections of $(\Lambda^{p,q}T^*M)\otimes E$ over D. Let $\Omega^{p,q}_0(D,E)$ be the subspace of $\Omega^{p,q}(D,E)$ whose elements have compact support in D.

If $w\in\Lambda^rT^{*(0,1)}_zM$, $r\in\mathbb{N}$, let $(w\wedge)^*:\Lambda^{q+r}T^{*(0,1)}_zM\to\Lambda^qT^{*(0,1)}_zM$, $q\geq 0$, be the adjoint of left exterior multiplication $w\wedge:\Lambda^qT^{*(0,1)}_zM\to\Lambda^{q+r}T^{*(0,1)}_zM$. That is,

$$\langle\,w\wedge u\,,v\,\rangle=\langle\,u\,,(w\wedge)^*v\,\rangle,\tag{2.1}$$

for all $u\in\Lambda^qT^{*(0,1)}_zM$, $v\in\Lambda^{q+r}T^{*(0,1)}_zM$. Notice that $(w\wedge)^*$ depends anti-linearly on w.

In this paper, let (L,h^L) be a holomorphic line bundle over M, where the Hermitian fiber metric on L is denoted by h^L. Until further notice, we assume that h^L is smooth. Let R^L be the canonical curvature two form induced by h^L. Let ϕ denote the local weights of the Hermitian metric such that $2\partial\overline{\partial}\phi = R^L$. More precisely, if $s(z)$ is a local trivializing section of L on an open subset $D \subset M$, then the pointwise norm of s is

$$|s(x)|^2 = |s(x)|^2_{h^L} = e^{-2\phi(x)}, \quad \phi \in \mathscr{C}^\infty(D, \mathbb{R}). \tag{2.2}$$

Let L^k, $k > 0$, be the k-th tensor power of the line bundle L. The Hermitian fiber metric on L induces a Hermitian fiber metric on L^k that we shall denote by h^{L^k}. If s is a local trivializing section of L then s^k is a local trivializing section of L^k. For $p, q \in \mathbb{N}_0$, the Hermitian metric $\langle \cdot , \cdot \rangle$ on $\Lambda^{p,q}T^*M$ and h^{L^k} induce a Hermitian metric on $\Lambda^{p,q}T^*M \otimes L^k$, also denoted by $\langle \cdot , \cdot \rangle$. For $f \in \Omega^{p,q}(M, L^k)$, we denote the pointwise norm $|f(x)|^2 := |f(x)|^2_{h^{L^k}} = \langle f, f\rangle(x)$. We take $dv_M = dv_M(x)$ as the induced volume form on M. The L^2–Hermitian inner products on the spaces $\Omega^{p,q}_0(M, L^k)$ and $\Omega^{p,q}_0(M)$ are given by

$$\begin{aligned} (s_1, s_2)_k &= \int_M \langle s_1, s_2\rangle(x)\, dv_M(x)\,, \quad s_1, s_2 \in \Omega^{p,q}_0(M, L^k),\\ (f_1, f_2) &= \int_M \langle f_1, f_2\rangle(x)\, dv_M(x)\,, \quad f_1, f_2 \in \Omega^{p,q}_0(M). \end{aligned} \tag{2.3}$$

We write $\|f\|^2 := \|f\|^2_{h^{L^k}} = (f, f)_k$, $f \in \Omega^{p,q}_0(M, L^k)$. For $g \in \Omega^{p,q}_0(M)$, we also write $\|g\|^2 := (g, g)$. Let $L^2_{(p,q)}(M, L^k)$ be the completion of $\Omega^{p,q}_0(M, L^k)$ with respect to $\|\cdot\|$.

2.3. **A selfadjoint extension of the Kodaira Laplacian.** We denote by

$$\overline{\partial}_k : \Omega^{0,r}(M, L^k) \to \Omega^{0,r+1}(M, L^k) \tag{2.4}$$

the Cauchy-Riemann operator with values in L^k and let

$$\overline{\partial}^*_k : \Omega^{0,r+1}(M, L^k) \to \Omega^{0,r}(M, L^k) \tag{2.5}$$

denote the formal adjoint of $\overline{\partial}_k$ with respect to $(\cdot, \cdot)_k$. Let

$$\Box^{(q)}_k := \overline{\partial}_k\overline{\partial}^*_k + \overline{\partial}^*_k\overline{\partial}_k : \Omega^{0,q}(M, L^k) \to \Omega^{0,q}(M, L^k) \tag{2.6}$$

be the Kodaira Laplacian acting on $(0, q)$–forms with values in L^k. We extend $\overline{\partial}_k$ to $L^2_{(0,r)}(M, L^k)$ by

$$\overline{\partial}_k : \mathrm{Dom}\,\overline{\partial}_k \subset L^2_{(0,r)}(M, L^k) \to L^2_{(0,r+1)}(M, L^k)\,, \tag{2.7}$$

where $\mathrm{Dom}\,\overline{\partial}_k := \{u \in L^2_{(0,r)}(M, L^k);\ \overline{\partial}_k u \in L^2_{(0,r+1)}(M, L^k)\}$, where for any $u \in L^2_{(0,r)}(M, L^k)$, $\overline{\partial}_k u$ is defined in the sense of distributions. We also write

$$\overline{\partial}^*_k : \mathrm{Dom}\,\overline{\partial}^*_k \subset L^2_{(0,r+1)}(M, L^k) \to L^2_{(0,r)}(M, L^k) \tag{2.8}$$

to denote the Hilbert space adjoint of $\overline{\partial}_k$ in the L^2 space with respect to $(\cdot, \cdot)_k$. Let $\Box^{(q)}_k$ denote the Gaffney extension of the Kodaira Laplacian given by

$$\mathrm{Dom}\,\Box^{(q)}_k = \left\{ s \in L^2_{(0,q)}(M, L^k);\ s \in \mathrm{Dom}\,\overline{\partial}_k \cap \mathrm{Dom}\,\overline{\partial}^*_k,\ \overline{\partial}_k u \in \mathrm{Dom}\,\overline{\partial}^*_k,\ \overline{\partial}^*_k u \in \mathrm{Dom}\,\overline{\partial}_k \right\}, \tag{2.9}$$

and $\Box^{(q)}_k s = \overline{\partial}_k\overline{\partial}^*_k s + \overline{\partial}^*_k\overline{\partial}_k s$ for $s \in \mathrm{Dom}\,\Box^{(q)}_k$. By a result of Gaffney [36, Prop. 3.1.2], $\Box^{(q)}_k$ is a positive self-adjoint operator. Note that if M is complete, the Kodaira Laplacian $\Box^{(q)}_k$ is essentially selfadjoint [36, Cor. 3.3.4] and the Gaffney extension coincides with the unique selfadjoint extension of $\Box^{(q)}_k$.

2.4. **Schwartz kernel theorem.** We recall the Schwartz kernel theorem [28, Th. 5.2.1, 5.2.6], [46, p. 296]. Let Ω be a $\mathscr{C}^\infty$ paracompact manifold equipped with a smooth density of integration. Let E and F be smooth vector bundles over Ω. Then any continuous linear operator $A : \mathscr{C}^\infty_0(\Omega, E) \to \mathscr{D}'(\Omega, F)$ has a Schwartz distribution kernel, denoted $K_A(x, y)$ or $A(x, y)$. Moreover, the following two statements are equivalent

(I) A is continuous: $\mathscr{E}'(\Omega, E) \to \mathscr{C}^\infty(\Omega, F)$,

(II) $K_A(x, y) \in \mathscr{C}^\infty(\Omega \times \Omega, E_y \boxtimes F_x)$.

If A satisfies (I) or (II), we say that A is a *smoothing operator*. Furthermore, A is smoothing if and only if $A : H^s_{\rm comp}(\Omega, E) \to H^{s+N}_{\rm loc}(\Omega, F)$ is continuous, for all $N \geq 0$, $s \in \mathbb{R}$. We say that A is properly supported if $\operatorname{Supp} K_A \subset \Omega \times \Omega$ is proper. That is, the two projections: $t_x : (x, y) \in \operatorname{Supp} K_A \to x \in \Omega$, $t_y : (x, y) \in \operatorname{Supp} K_A \to y \in \Omega$ are proper. We say that A is smoothing away the diagonal if $\chi_1 A \chi_2$ is smoothing, for all $\chi_1, \chi_2 \in \mathscr{C}^\infty_0(\Omega)$ with $\operatorname{Supp} \chi_1 \bigcap \operatorname{Supp} \chi_2 = \emptyset$.

Let $H(x, y) \in \mathscr{D}'(\Omega \times \Omega, E_y \boxtimes F_x)$. We write H to denote the continuous operator $H : \mathscr{C}^\infty_0(\Omega, E) \to \mathscr{D}'(\Omega, F)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$. Let $A, B : \mathscr{C}^\infty_0(\Omega, E) \to \mathscr{D}'(\Omega, F)$ be continuous operators. We write $A \equiv B$ or $A(x, y) \equiv B(x, y)$ if $A - B$ is a smoothing operator.

3. Szegö kernels and semi-classical Hodge decomposition

The goal of this Section is to prove the semiclassical Hodge decomposition for the Kodaira Laplacian, i.e. to find a semi-classical partial inverse and an approximate kernel for $\square^{(q)}_k$, cf. Theorem 3.10. For this purpose we reduce the analysis of the Kodaira Laplacian to the analysis of the Kohn Laplacian on the Grauert tube of the line bundle L. In Section 3.1 we recall the construction of these two objects. Section 3.2 contains a detailed study of the microlocal Hodge decomposition of the Kohn Laplacian on a non-degenerate CR manifold and especially on the Grauert tube, by following [29]. Finally, in Section 3.3 we apply this results in order to obtain the semi-classical Hodge decomposition for the Kodaira Laplacian.

3.1. **The associated CR manifold.** Let (M, Θ) be a Hermitian manifold and (L, h^L) be a holomorphic Hermitian line bundle on M. Let (L^*, h^{L^*}) be the dual bundle of L. We denote

$$G := \{v \in L^*;\ |v|_{h^{L^*}} < 1\}\ ,\quad X := \partial G = \{v \in L^*;\ |v|_{h^{L^*}} = 1\}\ . \tag{3.1}$$

The domain G is called *Grauert tube* associated to L. We denote

$$T^{(1,0)}X := T^{(1,0)}L^* \cap \mathbb{C}TX\ ,\quad T^{(0,1)}X := T^{(0,1)}L^* \cap \mathbb{C}TX\ .$$

Then $(X, T^{(1,0)}X)$ is a CR manifold of dimension $2n + 1$ and the bundle $T^{(1,0)}X$ is called the holomorphic tangent bundle of X. The manifold X is equipped with a natural S^1 action. Locally X can be represented in local holomorphic coordinates (z, λ), where λ is the fiber coordinate, as the set of all (z, λ) such that $|\lambda|^2 e^{2\phi(z)} = 1$, where ϕ is a local weight of h^L. The S^1 action on X is given by $e^{i\theta} \circ (z, \lambda) = (z, e^{i\theta}\lambda)$, $e^{i\theta} \in S^1$, $(z, \lambda) \in X$. Let Y be the global real vector field on X determined by

$$Yu(x) = \frac{\partial}{\partial\theta} u(e^{i\theta} \circ x)|_{\theta=0} \quad \text{for all } u \in \mathscr{C}^\infty(X)\,.$$

Let $\pi : X \to M$ be the natural projection. We have the bijective map:

$$\pi^* : T^{(1,0)}X \oplus T^{(0,1)}X \to T^{(1,0)}M \oplus T^{(0,1)}M\,, \quad W \to \pi^* W,$$

where $(\pi^*W)f = W(f \circ \pi)$, for all $f \in \mathscr{C}^\infty(M)$. We take the Hermitian metric $\langle \cdot\,,\cdot \rangle$ on $\mathbb{C}TX$ so that $Y \perp \left(T^{(1,0)}X \oplus T^{(0,1)}X\right)$, $\langle\, Y\,, Y\,\rangle = 1$ and $\langle\, Z\,, W\,\rangle = \langle\, \pi^*Z\,, \pi^*W\,\rangle$, $Z, W \in T^{(1,0)}X \oplus T^{(0,1)}X$. The Hermitian metric $\langle \cdot\,,\cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on the complexified cotangent bundle $\mathbb{C}T^*X$ that we shall also denote by $\langle \cdot\,,\cdot \rangle$. Define $T^{*(1,0)}X := \left(T^{(0,1)}X \oplus \mathbb{C}Y\right)^\perp \subset \mathbb{C}T^*X$, $T^{*(0,1)}X := \left(T^{(1,0)}X \oplus \mathbb{C}Y\right)^\perp \subset \mathbb{C}T^*X$. For $q \in \mathbb{N}$, the bundle of $(0,q)$ forms of X is given by $\Lambda^q T^{*(0,1)}X := \Lambda^q\left(T^{*(0,1)}X\right)$. The Hermitian metric $\langle \cdot\,,\cdot \rangle$ on $\mathbb{C}T^*X$ induces a Hermitian metric on $\Lambda^q T^{*(0,1)}X$ also denoted by $\langle \cdot\,,\cdot \rangle$.

Locally there is a real one form ω_0 of length one which is pointwise orthogonal to $T^{*(0,1)}X \oplus T^{*(1,0)}X$. ω_0 is unique up to the choice of sign. We take ω_0 so that $\langle\, \omega_0\,, Y\,\rangle = -1$. Therefore ω_0, so chosen, is globally defined.

The *Levi form* $\mathcal{L}_p$ of X at $p \in X$ is the Hermitian quadratic form on $T_p^{(1,0)}X$ defined as follows:

$$\mathcal{L}_p(U, \overline{V}) = \frac{1}{2i}\langle\, [\mathcal{U}\,, \overline{\mathcal{V}}](p)\,, \omega_0(p)\,\rangle\,, \quad U, V \in T_p^{(1,0)}X \tag{3.2}$$

where $\mathcal{U}, \mathcal{V} \in \mathscr{C}^\infty(X, T^{(1,0)}X)$ that satisfy $\mathcal{U}(p) = U$, $\mathcal{V}(p) = V$ and $[\mathcal{U}\,, \overline{\mathcal{V}}] = \mathcal{U}\overline{\mathcal{V}} - \overline{\mathcal{V}}\mathcal{U}$ denotes the commutator of $\mathcal{U}$ and $\overline{\mathcal{V}}$.

Let $B \subset X$ be an open set. Let $\Omega^{0,q}(B)$ denote the space of smooth sections of $\Lambda^q T^{*(0,1)}X$ over B. Let $\Omega_0^{0,q}(B)$ be the subspace of $\Omega^{0,q}(B)$ whose elements have compact support in B. Let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. We take $dv_X = dv_X(x)$ as the induced volume form on X. Then, we get natural inner product $(\,,)$ on $\Omega^{0,q}(X)$. Let $\overline{\partial}_b^* : \Omega^{0,q+1}(X) \to \Omega^{0,q}(X)$ be the formal adjoint of $\overline{\partial}_b$ with respect to $(\,,)$. The *Kohn Laplacian* on $(0,q)$ forms is given by

$$\Box_b^{(q)} := \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q}(X)\,.$$

We introduce now a local holomorphic frame and local coordinates in terms of which we shall write down the operators explicitly. Let $s(z)$ be a local trivializing section of L on an open set $D \Subset M$. We have $|s(z)|^2_{h^L} = e^{-2\phi(z)}$. Then, $s^*(z) := s^{-1}(z)$ is a local trivializing section of L^* on $D \Subset M$. We have $|s^*(z)|^2_{h^{L^*}} = e^{2\phi(z)}$. Let $z = (z_1, \ldots, z_n)$ be holomorphic local coordinates of D. We identify D with an open set of $\mathbb{C}^n$. We have the local diffeomorphism:

$$\tau : (z, \theta) \in D\times]-\varepsilon_0, \varepsilon_0[\to e^{-\phi(z)}s^*(z)e^{-i\theta} \in X, \quad 0 < \varepsilon_0 \leq \pi. \tag{3.3}$$

It is convenient to work with the local coordinates (z, θ). In terms of the coordinates (z, θ), it is straightforward to see that $Y = -\frac{\partial}{\partial\theta}$. Moreover, the fiber $T_v^{*(1,0)}X$, for $v = e^{-\phi(z)}s^*(z)e^{-i\theta} \in X$, is the vector space spanned by $\frac{\partial}{\partial z_j} - i\frac{\partial\phi}{\partial z_j}(z)\frac{\partial}{\partial\theta}$, $j = 1, \ldots, n$. Further, let $\overline{Z}_j$, $j = 1, \ldots, n$, be an orthonormal basis for the holomorphic tangent bundle $T^{(1,0)}M$ and let $\overline{e}_j$, $j = 1, \ldots, n$, be an orthonormal basis of $T^{*(1,0)}M$ which is dual to $\overline{Z}_j$, $j = 1, \ldots, n$. Then, $\overline{Z}_j - i\overline{Z}_j(\phi)\frac{\partial}{\partial\theta}$, $j = 1, \ldots, n$, is an orthonormal basis for $T^{(1,0)}X$ and $\overline{e}_j$, $j = 1, \ldots, n$, is an orthonormal basis for $T^{*(1,0)}X$ which is dual to $\overline{Z}_j - i\overline{Z}_j(\phi)\frac{\partial}{\partial\theta}$,

$j=1,\ldots,n$. Furthermore, we can check that

$$\omega_0 = d\theta + \sum_{j=1}^{n}(-iZ_j(\phi)e_j + i\overline{Z}_j(\phi)\overline{e}_j). \tag{3.4}$$

From this, we can compute $\mathcal{L}_p(\frac{\partial}{\partial z_j} - i\frac{\partial\phi}{\partial z_j}(\pi(p))\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\overline{z}_k} + i\frac{\partial\phi}{\partial\overline{z}_k}(\pi(p))\frac{\partial}{\partial\theta}) = \frac{\partial^2\phi}{\partial z_j\partial\overline{z}_k}(\pi(p))$, $j,k=1,\ldots,n$, $p\in X$. Thus, for a given point $p\in X$, we have

$$\begin{aligned}\mathcal{L}_p(U,\overline{V}) &= \langle\, \partial\overline{\partial}\phi(\pi(p))\,,\pi^*U\wedge\pi^*\overline{V}\,\rangle\\ &= \langle\, \frac{1}{2}R^L(\pi(p))\,,\pi^*U\wedge\pi^*\overline{V}\,\rangle = \langle\, \frac{1}{2}\dot{R}^L(\pi(p))\pi^*U\,,\pi^*V\,\rangle,\ \ \forall\, U,V\in T_p^{(1,0)}X.\end{aligned} \tag{3.5}$$

We deduce the following:

Proposition 3.1. *Let (L,h^L) be a Hermitian holomorphic line bundle over a complex manifold M and let $X\subset L^*$ be the boundary of the Grauert tube. Let $p\in X$. If the curvature R^L has signature (n_-,n_+) at $\pi(p)$, then $\mathcal{L}_p$ has signature (n_-,n_+).*

We define the operators $\overline{\partial}_s$, $\overline{\partial}_s^*$, $\Box_s^{(q)}$, which are the local versions of the operators $\overline{\partial}_k$, $\overline{\partial}_k^*$, $\Box_k^{(q)}$ (see (2.4)-(2.9)), by the following equations:

$$\begin{aligned}&\overline{\partial}_s = \overline{\partial} + k(\overline{\partial}\phi)\wedge : \Omega^{0,q}(D)\to\Omega^{0,q+1}(D),\\ &\overline{\partial}_s^* = \overline{\partial}^* + k\big((\overline{\partial}\phi)\wedge\big)^* : \Omega^{0,q+1}(D)\to\Omega^{0,q}(D),\\ &\Box_s^{(q)} = \overline{\partial}_s\overline{\partial}_s^* + \overline{\partial}_s^*\overline{\partial}_s : \Omega^{0,q}(D)\to\Omega^{0,q}(D).\end{aligned} \tag{3.6}$$

Here $\overline{\partial}^* : \Omega^{0,q+1}(D)\to\Omega^{0,q}(D)$ is the formal adjoint of $\overline{\partial}$ with respect to $(\ ,\ )$. We have the unitary identifications:

$$\begin{aligned}&\Omega^{0,q}(D,L^k)\longleftrightarrow\Omega^{0,q}(D)\\ &f = s^k g\longleftrightarrow \widehat{f}(z) = e^{-k\phi}s^{-k}f = g(z)e^{-k\phi(z)},\ \ g\in\Omega^{0,q}(D),\\ &\overline{\partial}_k\longleftrightarrow\overline{\partial}_s,\ \ \overline{\partial}_k f = s^k e^{k\phi}\overline{\partial}_s\widehat{f},\\ &\overline{\partial}_k^*\longleftrightarrow\overline{\partial}_s^*,\ \ \overline{\partial}_k^* f = s^k e^{k\phi}\overline{\partial}_s^*\widehat{f},\\ &\Box_k^{(q)}\longleftrightarrow\Box_s^{(q)},\ \ \Box_k^{(q)} f = s^k e^{k\phi}\Box_s^{(q)}\widehat{f}\,.\end{aligned} \tag{3.7}$$

We continue to work with the local coordinates (z,θ). As above, let $(Z_j)_{j=1}^n$ be an orthonormal basis for $T^{(0,1)}M$ and let $(e_j)_{j=1}^n$ be an orthonormal basis for $T^{*(0,1)}M$ which is dual to $(Z_j)_{j=1}^n$. We can check that

$$\overline{\partial}_b = \sum_{j=1}^{n}(e_j\wedge)\circ\Big(Z_j + iZ_j(\phi)\frac{\partial}{\partial\theta}\Big) + \sum_{j=1}^{n}\Big((\overline{\partial}e_j)\wedge\Big)\circ(e_j\wedge)^* \tag{3.8}$$

and correspondingly

$$\overline{\partial}_b^* = \sum_{j=1}^{n}\Big((e_j\wedge)^*\Big)\circ\Big(Z_j^* + i\overline{Z}_j(\phi)\frac{\partial}{\partial\theta}\Big) + \sum_{j=1}^{n}(e_j\wedge)\circ\Big((\overline{\partial}e_j)\wedge\Big)^*, \tag{3.9}$$

where Z_j^* is the formal adjoint of Z_j with respect to $(\ ,\ )$.

Let $\overline{\partial}_s$ and $\overline{\partial}_s^*$ be as in (3.7) and (3.6). We can check that

$$\begin{aligned}\overline{\partial}_s &= \sum_{j=1}^{n}(e_j\wedge)\circ(Z_j+kZ_j(\phi))+\sum_{j=1}^{n}\Big((\overline{\partial}e_j)\wedge\Big)\circ(e_j\wedge)^*,\\ \overline{\partial}_s^* &= \sum_{j=1}^{n}\Big((e_j\wedge)^*\Big)\circ(Z_j^*+k\overline{Z}_j(\phi))+\sum_{j=1}^{n}(e_j\wedge)\circ\Big((\overline{\partial}e_j)\wedge\Big)^*.\end{aligned}\tag{3.10}$$

From now on, we identify $\Lambda^qT^{*(0,1)}M$ with $\Lambda^qT^{*(0,1)}X$. From (3.7), (3.6), explicit formulas of $\overline{\partial}_s$, $\overline{\partial}_s^*$ and (3.8), (3.9), we get

$$\Box_k^{(q)}f=s^ke^{k\phi}e^{ik\theta}\Box_b^{(q)}(\widehat{f}e^{-ik\theta}),\tag{3.11}$$

for all $f\in\Omega^{0,q}(D,L^k)$, where $\widehat{f}$ is given by (3.7).

Let $u(z,\theta)\in\Omega_0^{0,q}(D\times(-\varepsilon_0,\varepsilon_0))$. Note that

$$k\int e^{i\theta k}u(z,\theta)d\theta=\int(-i)\frac{\partial}{\partial\theta}(e^{i\theta k})u(z,\theta)d\theta=\int e^{i\theta k}i\frac{\partial u}{\partial\theta}(z,\theta)d\theta.$$

From this observation and explicit formulas of $\overline{\partial}_b$, $\overline{\partial}_b^*$, $\overline{\partial}_s$ and $\overline{\partial}_s^*$ (see (3.8), (3.9) and (3.10)), we conclude that

$$\Box_s^{(q)}\Big(\int e^{i\theta k}u(z,\theta)d\theta\Big)=\int e^{i\theta k}(\Box_b^{(q)}u)(z,\theta)d\theta,\tag{3.12}$$

for all $u(z,\theta)\in\Omega_0^{0,q}(D\times(-\varepsilon_0,\varepsilon_0))$.

3.2. **Approximate Szegö kernels.** In this Section we establish the existence of a microlocal Hodge decomposition of the Kohn Laplacian on an open set of a CR manifold where the Levi form is non-degenerate. The approximate Szegö kernel is a Fourier integral operator with complex phase in the sense of Melin-Sjöstrand [41]. We then specialize to the case of the Grauert tube of a line bundle and give a useful formula for the phase function of the approximate Szegö kernel in Theorem 3.8.

Theorems 3.2-3.4 are consequences of the analysis from chapter 6 and chapter 7 of part I in [29]. In [29] the existence of the microlocal Hodge decomposition is stated for compact CR manifolds, but the construction and arguments used are essentially local.

Theorem 3.2. *Let X be an orientable CR manifold whose Levi form $\mathcal{L}$ is non-degenerate of constant signature (n_-,n_+) at each point of an open set $B\Subset X$. Let $q\neq n_-,n_+$. There exists a properly supported continuous operator*

$$A:\begin{cases}H^s_{\rm loc}(B,\Lambda^qT^{*(0,1)}X)\to H^{s+1}_{\rm loc}(B,\Lambda^qT^{*(0,1)}X),\\ H^s_{\rm comp}(B,\Lambda^qT^{*(0,1)}X)\to H^{s+1}_{\rm comp}(B,\Lambda^qT^{*(0,1)}X)\end{cases}\tag{3.13}$$

for all $s\geq 0$, such that A is smoothing away the diagonal and $\Box_b^{(q)}A\equiv I$.

For $m\in\mathbb{R}$ let $S^m_{1,0}$ be the Hörmander symbol space (see Grigis-Sjöstrand [26, Def. 1.1] for the precise meaning of $S^m_{1,0}$). Let $p_0(x,\xi)\in\mathscr{C}^\infty(T^*X)$ be the principal symbol of $\Box_b^{(q)}$. Note that $p_0(x,\xi)$ is a polynomial of degree 2 in ξ. The characteristic manifold of $\Box_b^{(q)}$ is given by $\Sigma=\Sigma^+\bigcup\Sigma^-$, where

$$\begin{aligned}\Sigma^+&=\{(x,\lambda\omega_0(x))\in T^*X;\ \lambda>0\},\\ \Sigma^-&=\{(x,\lambda\omega_0(x))\in T^*X;\ \lambda<0\}.\end{aligned}$$

Theorem 3.3. *Let X, B and (n_-, n_+) as in Theorem 3.2. Let $q = n_-$ or n_+. Then there exist properly supported continuous operators*

$$\begin{aligned}&A:\begin{cases}H^s_{\rm loc}(B,\Lambda^qT^{*(0,1)}X)\to H^{s+1}_{\rm loc}(B,\Lambda^qT^{*(0,1)}X),\\ H^s_{\rm comp}(B,\Lambda^qT^{*(0,1)}X)\to H^{s+1}_{\rm comp}(B,\Lambda^qT^{*(0,1)}X)\,,\end{cases}\\ &S_-,S_+:\begin{cases}:H^s_{\rm loc}(B,\Lambda^qT^{*(0,1)}X)\to H^s_{\rm loc}(B,\Lambda^qT^{*(0,1)}X),\\ H^s_{\rm comp}(B,\Lambda^qT^{*(0,1)}X)\to H^s_{\rm comp}(B,\Lambda^qT^{*(0,1)}X)\,,\end{cases}\end{aligned}\tag{3.14}$$

for all $s\geq 0$, such that A, S_-, S_+ are smoothing away the diagonal and

$$\Box^{(q)}_bA+S_-+S_+=I,\,,\quad \Box^{(q)}_bS_-\equiv 0,\ \Box^{(q)}_bS_+\equiv 0,$$
$$A\equiv A^*\,,\quad S_-\equiv S_-^*\equiv S_-^2\,,\quad S_+\equiv S_+^*\equiv S_+^2\,,\quad S_-S_+\equiv S_+S_-\equiv 0,$$

where A^*, S_-^* and S_+^* are the formal adjoints of A, S_- and S_+ with respect to $(\ ,\ )$ respectively and $K_{S_-}(x,y)$ satisfies

$$K_{S_-}(x,y)\equiv\int_0^\infty e^{i\varphi_-(x,y)t}s_-(x,y,t)dt$$

with a symbol

$$\begin{aligned}&s_-(x,y,t)\in S^n_{1,0}\Big(B\times B\times]0,\infty[,\Lambda^qT^{*(0,1)}_yX\boxtimes\Lambda^qT^{*(0,1)}_xX\Big),\\ &s_-(x,y,t)\sim\sum_{j=0}^\infty s^j_-(x,y)t^{n-j}\ \ in\ S^n_{1,0}\Big(B\times B\times]0,\infty[,\Lambda^qT^{*(0,1)}_yX\boxtimes\Lambda^qT^{*(0,1)}_xX\Big)\,,\\ &s^j_-(x,y)\in\mathscr{C}^\infty\Big(B\times B,\Lambda^qT^{*(0,1)}_yX\boxtimes\Lambda^qT^{*(0,1)}_xX\Big),\ \ j=0,1,\ldots,\end{aligned}\tag{3.15}$$

and phase function

$$\varphi_-\in\mathscr{C}^\infty(B\times B),\ \ {\rm Im}\,\varphi_-(x,y)\geq 0\,,\ \varphi_-(x,x)=0,\ \ \varphi_-(x,y)\neq 0\ \ if\ \ x\neq y,\tag{3.16}$$
$$d_x\varphi_-\neq 0,\ \ d_y\varphi_-\neq 0\ \ where\ \ {\rm Im}\,\varphi_-=0,\tag{3.17}$$
$$d_x\varphi_-(x,y)|_{x=y}=-\omega_0(x),\ \ d_y\varphi_-(x,y)|_{x=y}=\omega_0(x),\tag{3.18}$$
$$\varphi_-(x,y)=-\overline{\varphi}_-(y,x).\tag{3.19}$$

Moreover, there is a function $f\in\mathscr{C}^\infty(B\times B)$ such that

$$p_0(x,(\varphi_-)'_x(x,y))-f(x,y)\varphi_-(x,y)\tag{3.20}$$

vanishes to infinite order at $x=y$.

Similarly,

$$K_{S_+}(x,y)\equiv\int_0^\infty e^{i\varphi_+(x,y)t}s_+(x,y,t)dt$$

with $s_+(x,y,t)\in S^n_{1,0}\Big(B\times B\times]0,\infty[,\Lambda^qT^{*(0,1)}_yX\boxtimes\Lambda^qT^{*(0,1)}_xX\Big)$,

$$s_+(x,y,t)\sim\sum_{j=0}^\infty s^j_+(x,y)t^{n-j}$$

in $S^n_{1,0}\Big(B\times B\times]0,\infty[,\Lambda^qT^{*(0,1)}_yX\boxtimes\Lambda^qT^{*(0,1)}_xX\Big)$, where

$$s^j_+(x,y)\in\mathscr{C}^\infty\Big(B\times B,\Lambda^qT^{*(0,1)}_yX\boxtimes\Lambda^qT^{*(0,1)}_xX\Big),\ \ j=0,1,\ldots,$$

and $-\overline{\varphi}_+(x,y)$ satifies (3.16)–(3.20). Moreover, if $q\neq n_+$, then $s_+(x,y,t)$ vanishes to infinite order at $x=y$. If $q\neq n_-$, then $s_-(x,y,t)$ vanishes to infinite order at $x=y$.

The operators S_+, S_- are called *approximate Szegö kernels*.

Proof. We only sketch the proof. For all the details, we refer the reader to Part I in [29]. We will use the heat equation method. We work with some real local coordinates $x = (x_1, \ldots, x_{2n+1})$ defined on B. We will say that $a \in \mathscr{C}^\infty(\overline{\mathbb{R}}_+ \times B \times \mathbb{R}^{2n+1})$ is quasi-homogeneous of degree j if $a(t, x, \lambda\eta) = \lambda^j a(\lambda t, x, \eta)$ for all $\lambda > 0$. We consider the problem

$$\begin{cases} (\partial_t + \Box_b^{(q)})u(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times B, \\ u(0,x) = v(x). \end{cases} \tag{3.21}$$

We start by a formal construction. We look for an approximate solution of (3.21) of the form $u(t, x) = A(t)v(x)$,

$$A(t)v(x) = \frac{1}{(2\pi)^{2n+1}} \int e^{i(\psi(t,x,\eta) - \langle y,\eta\rangle)} a(t,x,\eta)v(y)dyd\eta \tag{3.22}$$

where formally

$$a(t,x,\eta) \sim \sum_{j=0}^{\infty} a_j(t,x,\eta),$$

$a_j(t, x, \eta)$ is a matrix-valued quasi-homogeneous function of degree $-j$.

The full symbol of $\Box_b^{(q)}$ equals $\sum_{j=0}^{2} p_j(x, \xi)$, where $p_j(x, \xi)$ is positively homogeneous of order $2 - j$ in the sense that

$$p_j(x, \lambda\eta) = \lambda^{2-j} p_j(x,\eta), \ |\eta| \geq 1, \ \lambda \geq 1.$$

We apply $\partial_t + \Box_b^{(q)}$ formally inside the integral in (3.22) and then introduce the asymptotic expansion of $\Box_b^{(q)}(ae^{i\psi})$. Set $(\partial_t + \Box_b^{(q)})(ae^{i\psi}) \sim 0$ and regroup the terms according to the degree of quasi-homogeneity. The phase $\psi(t, x, \eta)$ should solve

$$\begin{cases} \dfrac{\partial\psi}{\partial t} - ip_0(x, \psi'_x) = O(|\operatorname{Im}\psi|^N), \quad \forall N \geq 0, \\ \psi|_{t=0} = \langle x, \eta\rangle. \end{cases} \tag{3.23}$$

This equation can be solved with $\operatorname{Im}\psi(t, x, \eta) \geq 0$ and the phase $\psi(t, x, \eta)$ is quasi-homogeneous of degree 1. Moreover,

$$\psi(t,x,\eta) = \langle x,\eta\rangle \text{ on } \Sigma, \ d_{x,\eta}(\psi - \langle x,\eta\rangle) = 0 \text{ on } \Sigma,$$

$$\operatorname{Im}\psi(t,x,\eta) \asymp \Big(|\eta| \frac{t\,|\eta|}{1+t\,|\eta|}\Big)\Big(\operatorname{dist}\big((x,\frac{\eta}{|\eta|}),\Sigma\big)\Big)^2, \ \ |\eta| \geq 1.$$

Furthermore, there exists $\psi(\infty, x, \eta) \in \mathscr{C}^\infty(B \times \dot{\mathbb{R}}^{2n+1})$ with a uniquely determined Taylor expansion at each point of Σ such that for every compact set $K \subset B \times \dot{\mathbb{R}}^{2n+1}$ there is a constant $c_K > 0$ such that

$$\operatorname{Im}\psi(\infty,x,\eta) \geq c_K\,|\eta|\,\Big(\operatorname{dist}\big((x,\frac{\eta}{|\eta|}),\Sigma\big)\Big)^2, \ \ |\eta| \geq 1.$$

If $\lambda \in \mathscr{C}(T^*B \setminus 0)$, $\lambda > 0$ is positively homogeneous of degree 1 and $\lambda|_\Sigma < \min\lambda_j$, $\lambda_j > 0$, where $\pm i\lambda_j$ are the non-vanishing eigenvalues of the fundamental matrix of $\Box_b^{(q)}$, then the solution $\psi(t, x, \eta)$ of (3.23) can be chosen so that for every compact set $K \subset B \times \dot{\mathbb{R}}^{2n+1}$ and all indices α, β, γ, there is a constant $c_{\alpha,\beta,\gamma,K}$ such that

$$\Big|\partial_x^\alpha \partial_\eta^\beta \partial_t^\gamma(\psi(t,x,\eta) - \psi(\infty,x,\eta))\Big| \leq c_{\alpha,\beta,\gamma,K}\, e^{-\lambda(x,\eta)t} \text{ on } \overline{\mathbb{R}}_+ \times K.$$

We obtain the transport equations

$$\begin{cases} T(t,x,\eta,\partial_t,\partial_x)a_0 = O(|\mathrm{Im}\,\psi|^N),\ \forall N, \\ T(t,x,\eta,\partial_t,\partial_x)a_j + l_j(t,x,\eta,a_0,\ldots,a_{j-1}) = O(|\mathrm{Im}\,\psi|^N),\ \forall N. \end{cases} \tag{3.24}$$

Following the method of Menikoff-Sjöstrand [42], we see that we can solve (3.24). Moreover, a_j decay exponentially fast in t when $q \neq n_-$, n_+, and has subexponential growth in general. We assume that $q = n_-$ or n_+. We use $\overline{\partial}_b \Box_b^{(q)} = \Box_b^{(q+1)}\overline{\partial}_b$, $\overline{\partial}_b^* \Box_b^{(q)} = \Box_b^{(q-1)}\overline{\partial}_b^*$ and get

$$\partial_t(\overline{\partial}_b(e^{i\psi}a)) + \Box_b^{(q+1)}(\overline{\partial}_b(e^{i\psi}a)) \sim 0,$$
$$\partial_t(\overline{\partial}_b^*(e^{i\psi}a)) + \Box_b^{(q-1)}(\overline{\partial}_b^*(e^{i\psi}a)) \sim 0.$$

Put

$$\overline{\partial}_b(e^{i\psi}a) = e^{i\psi}\widehat{a},\ \overline{\partial}_b^*(e^{i\psi}a) = e^{i\psi}\widetilde{a}.$$

We have

$$(\partial_t + \Box_b^{(q+1)})(e^{i\psi}\widehat{a}) \sim 0,$$
$$(\partial_t + \Box_b^{(q-1)})(e^{i\psi}\widetilde{a}) \sim 0.$$

The corresponding degrees of $\widehat{a}$ and $\widetilde{a}$ are $q+1$ and $q-1$. We deduce as above that $\widehat{a}$ and $\widetilde{a}$ decay exponentially fast in t. This also applies to

$$\Box_b^{(q)}(ae^{i\psi}) = \overline{\partial}_b(\overline{\partial}_b^* ae^{i\psi}) + \overline{\partial}_b^*(\overline{\partial}_b ae^{i\psi}) = \overline{\partial}_b(e^{i\psi}\widetilde{a}) + \overline{\partial}_b^*(e^{i\psi}\widehat{a}).$$

Thus, $\partial_t(ae^{i\psi})$ decay exponentially fast in t. Since $\partial_t\psi$ decay exponetially fast in t so does $\partial_t a$. Hence, there exist

$$a_j(\infty,x,\eta) \in C^\infty\Big(T^*B, \Lambda^q T^{*(0,1)}X \boxtimes \Lambda^q T^{*(0,1)}X\Big),\ \ j = 0,1,\ldots,$$

positively homogeneous of degree $-j$ such that $a_j(t,x,\eta)$ converges exponentially fast to $a_j(\infty,x,\eta)$, for all $j \in \mathbb{N}_0$.

Choose $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^{2n+1})$ so that $\chi(\eta) = 1$ when $|\eta| < 1$ and $\chi(\eta) = 0$ when $|\eta| > 2$. We formally set

$$A = \frac{1}{(2\pi)^{2n+1}} \int\int_0^\infty \Big(e^{i(\psi(t,x,\eta)-\langle y,\eta\rangle)}a(t,x,\eta) - e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)(1-\chi(\eta))\,dt\,d\eta$$

and

$$S = \frac{1}{(2\pi)^{2n+1}} \int \Big(e^{i(\psi(\infty,x,\eta)-\langle y,\eta\rangle)}a(\infty,x,\eta)\Big)d\eta.$$

We can show that A is a pseudodifferential operator of order -1 type $(\frac{1}{2},\frac{1}{2})$ satisfying

$$S + \Box_b^{(q)} \circ A \equiv I,\ \ \Box_b^{(q)} \circ S \equiv 0.$$

Moreover, from stationary phase formula of Melin-Sjöstrand [41], we can show that $S \equiv S_- + S_+$, where S_-, S_+ are as in Theorem 3.3. $\square$

The following result describes the phase function in local coordinates.

Theorem 3.4. *Let X, B and (n_-, n_+) as in Theorem 3.2. For a given point $x_0 \in B$, let $W_1, \ldots, W_n$ be an orthonormal frame of $T^{(1,0)}X$ in a neighborhood of x_0, such that the Levi form is diagonal at x_0, i.e. $\mathcal{L}_{x_0}(W_j, \overline{W}_j) = \mu_j$, $j = 1, \ldots, n$. We take local coordinates $x = (x_1, \ldots, x_{2n+1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, defined on some neighborhood of x_0 such that $\omega_0(x_0) = dx_{2n+1}$, $x(x_0) = 0$, and for some $c_j \in \mathbb{C}$,*

$$W_j = \frac{\partial}{\partial z_j} - i\mu_j \overline{z}_j \frac{\partial}{\partial x_{2n+1}} - c_j x_{2n+1} \frac{\partial}{\partial x_{2n+1}} + O(|x|^2),\ j = 1, \ldots, n\,.$$

Set $y = (y_1, \ldots, y_{2n+1})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n$. Then, for φ_- in Theorem 3.3, we have

$$\operatorname{Im}\varphi_-(x,y) \geq c\sum_{j=1}^{2n} |x_j - y_j|^2\,,\quad c > 0, \tag{3.25}$$

in some neighborhood of $(0,0)$ and

$$\begin{aligned}\varphi_-(x,y) &= -x_{2n+1} + y_{2n+1} + i\sum_{j=1}^{n-1} |\mu_j|\,|z_j - w_j|^2 \\ &+ \sum_{j=1}^{n-1}\Big(i\mu_j(\overline{z}_j w_j - z_j\overline{w}_j) + c_j(-z_j x_{2n+1} + w_j y_{2n+1}) \\ &+ \overline{c}_j(-\overline{z}_j x_{2n+1} + \overline{w}_j y_{2n+1})\Big) + (x_{2n+1} - y_{2n+1})f(x,y) \\ &+ O(|(x,y)|^3),\end{aligned} \tag{3.26}$$

where $f \in \mathscr{C}^\infty$, $f(0,0) = 0$, $f(x,y) = \overline{f}(y,x)$.

Remark 3.5. If we go through the proofs of Theorem 3.2 and Theorem 3.3 (see [29]), it is not difficult to see that Theorem 3.2 and Theorem 3.3 have straightforward generalizations to the case when the functions take values in $\Lambda^q T^{*(0,1)}X \otimes F$, for a given smooth CR vector bundle F over X. We recall that F is a CR vector bundle if its transition functions are CR.

Remark 3.6. Let $\widehat{\varphi} \in \mathscr{C}^\infty(B \times B)$. If $\widehat{\varphi}$ satisfies (3.16)–(3.18), (3.20) and (3.25), (3.26), then it is well-known that (see Section 3 and 7 of [29] and Menikoff-Sjöstrand [42]) $\widehat{\varphi}(x,y)t$, $t > 0$, and $\varphi_-(x,y)t$, $t > 0$, are equivalent at each point of $\operatorname{diag}\Big((\Sigma^- \bigcap T^*B) \times (\Sigma^- \bigcap T^*B)\Big)$ in the sense of Melin-Sjöstrand (see Melin-Sjöstrand [41, p. 172]). We recall briefly that $\widehat{\varphi}(x,y)t$, $t > 0$, and $\varphi_-(x,y)t$, $t > 0$, are equivalent at each point of

$$\operatorname{diag}\Big((\Sigma^- \cap T^*B) \times (\Sigma^- \cap T^*B)\Big)$$

if for every

$$(x_0, -\lambda_0\omega_0) = (x_0, \lambda_0 d_x\varphi_-(x_0,x_0)) = (x_0, \lambda_0 d_x\widehat{\varphi}(x_0,x_0)) \in \Sigma^- \bigcap T^*B,$$

there is a conic neighborhood Γ of (x_0, x_0, λ_0), such that for every $a(x,y,t) \in S^m_{\rm cl}(B \times B \times \mathbb{R}_+)$, $m \in \mathbb{Z}$, with support in Γ, we can find $\widehat{a}(x,y,t) \in S^m_{\rm cl}(B \times B \times \mathbb{R}_+)$ with support in Γ, such that

$$\int_0^\infty e^{i\varphi_-(x,y)t} a(x,y,t)dt \equiv \int_0^\infty e^{i\widehat{\varphi}(x,y)t}\widehat{a}(x,y,t)dt$$

and vise versa, where $S^m_{\rm cl}$ denotes the classical symbol of order m (see page 38 in [26], for the precise meaning of $S^m_{\rm cl}$).

If $\omega \in T^{*(0,1)}_x X$, as (2.1), we let $(\omega\wedge)^* : \Lambda^{q+1}T^{*(0,1)}_x X \to \Lambda^q T^{*(0,1)}_x X$, $q \geq 0$, denote the adjoint of left exterior multiplication $\omega\wedge : \Lambda^q T^{*(0,1)}_x X \to \Lambda^{q+1}T^{*(0,1)}_x X$.

The following formula for the principal symbol s^0_- on the diagonal follows from [29, §8], its calculation being local in nature.

Theorem 3.7. *Let $q = n_-$. For a given point $x_0 \in X$, let $W_1(x), \ldots, W_n(x)$ be an orthonormal frame of $T^{(1,0)}_x X$, for which the Levi form is diagonalized at x_0. Let $T_j(x)$, $j = 1, \ldots, n$, denote the basis of $T^{*(0,1)}_x X$, which is dual to $\overline{W}_j(x)$, $j = 1, \ldots, n$. Let $\mu_j(x)$, $j = 1, \ldots, n$, be the eigenvalues of the Levi form L_x with respect to $\langle\,\cdot\,,\cdot\,\rangle$. We assume that $\mu_j(x_0) < 0$ if $1 \leq j \leq n_-$. Then, for $s^0_-(x,y)$ in (3.15), we have*

$$s^0_-(x_0,x_0) = \frac{1}{2}\,|\mu_1(x_0)|\cdots|\mu_n(x_0)|\,\pi^{-n-1}\prod_{j=1}^{j=n_-}(T_j(x_0)\wedge)\circ(T_j(x_0)\wedge)^*.$$

We return now to the situation where X is the Grauert tube of a line bundle L as in Section 3.1 and use the notations introduced there. Let (z,θ) be the coordinates as in (3.3). We assume that $B = D\times]-\varepsilon_0,\varepsilon_0[$, $\varepsilon_0 > 0$, $D \Subset M$. We write $(z,\theta) = (x', x_{2n+1}) = x$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, $\theta = x_{2n+1}$, $x' = (x_1, \ldots, x_{2n})$. We also write $x' = (x', 0)$. Until further notice, we work with the local coordinates $(z,\theta) = (x', x_{2n+1})$. Let ξ be the dual variables of x. From (3.8) and (3.9), we can check that the principal symbol of $\Box^{(q)}_b$ satisfies

$$p_0(x,\xi) = p_0(x',\xi). \tag{3.27}$$

That is, the principal symbol of $\Box^{(q)}_b$ is independent of x_{2n+1}.

From (3.18) and recall the form ω_0 (see (3.4)), we have

$$d_x\varphi_-(x,x) = -dx_{2n+1} + a(x')dx', \ \ a \in \mathscr{C}^\infty.$$

Thus, near a given point $(x_0,x_0) \in B \times B$, we have $\frac{\partial\varphi_-}{\partial x_{2n+1}} \neq 0$. From Malgrange preparation theorem(see Theorem 7.57 in Hörmander [28]), we have

$$\varphi_-(x,y) = g(x,y)(-x_{2n+1} + h(x',y)) \tag{3.28}$$

in some neighborhood of (x_0,x_0), where $g, h \in \mathscr{C}^\infty$, $g(x,x) = 1$, $h(x',x) = x_{2n+1}$. Since $\operatorname{Im}\varphi_- \geq 0$, from this, it is not difficult to see that $\operatorname{Im} h \geq 0$ in some neighborhood of (x_0,x_0). We may take B small enough so that (3.28) holds and $\operatorname{Im} h \geq 0$ on $B \times B$. From global theory of Fourier integral operators (see Theorem 4.2 of [41]), we see that $\varphi_-(x,y)t$ and $(-x_{2n+1} + h(x',y))t$ are equivalent in the sense of Melin-Sjöstrand. We can replace the phase φ_- by $-x_{2n+1} + h(x',y)$. Again, from (3.18), we have

$$\frac{\partial h}{\partial x'}(x',x)dx' - dx_{2n+1} = -\omega_0(x) = -dx_{2n+1} + a(x')dx'.$$

Thus, $\frac{\partial h}{\partial x'}(x',x)$ is independent of x_{2n+1}. We conclude that

$$\frac{\partial h}{\partial x'}(x',x)dx' - dx_{2n+1} = \frac{\partial h}{\partial x'}(x',x')dx' - dx_{2n+1} = -\omega_0(x). \tag{3.29}$$

Similarly, we have

$$\frac{\partial h}{\partial y}(y',y)dy = dy_{2n+1} + \frac{\partial h}{\partial y'}(y',y')dy' = \omega_0(y). \tag{3.30}$$

Put

$$\widehat{\varphi} = -x_{2n+1} + y_{2n+1} + h(x', y').$$

Note that $-x_{2n+1} + h(x', y)$ satisfies (3.20). From this and (3.27), we have

$$p_0\Big(x, (\frac{\partial h}{\partial x'}(x', y), -1)\Big) = p_0\Big(x', (\frac{\partial h}{\partial x'}(x', y), -1)\Big) = f(x, y)(-x_{2n+1}+h(x', y))+O(|x-y|^N)$$

for all $N \in \mathbb{N}$, for some $f \in \mathscr{C}^\infty$. Hence,

$$p_0(x, \widehat{\varphi}'_x) = p_0(x', \widehat{\varphi}'_x) = f(x, y')(-x_{2n+1} + h(x', y')) + O(|x'-y'|^N + |x_{2n+1}|^N), \tag{3.31}$$

for all $N \in \mathbb{N}$. We replace x_{2n+1} by $x_{2n+1} - y_{2n+1}$ in (3.31) and get

$$p_0(x, \widehat{\varphi}'_x) = p_0(x', \widehat{\varphi}'_x) = \widehat{f}(x, y)\widehat{\varphi} + O(|x-y|^N), \tag{3.32}$$

for all $N \in \mathbb{N}$, for some $\widehat{f} \in \mathscr{C}^\infty$. Thus, $\widehat{\varphi}$ satisfies (3.20). Note that $p_0(x, \widehat{\varphi}'_x)$ is independent of x_{2n+1}. Take $x_{2n+1} = y_{2n+1} + h(x', y')$ in (3.32) and notice that $h(x', x') = 0$, we conclude that

$$p_0(x, \widehat{\varphi}'_x) = O(|x'-y'|^N), \ \ \forall N \in \mathbb{N}. \tag{3.33}$$

Furthermore, from (3.29) and (3.30), we see that $\widehat{\varphi}$ satisfies (3.18). Moreover, for a given point $p \in D$, we may take local coordinates $z = (z_1, \ldots, z_n) = (x_1, \ldots, x_{2n}) = x$ centered at p such that the metric on $T^{(1,0)}M$ is $\sum_{j=1}^n dz_j \otimes d\overline{z}_j$ at p and

$$\phi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + O(|z|^3), \tag{3.34}$$

in some neighborhood of p, where $\lambda_j \neq 0$, $j = 1, \ldots, n$. From (3.26) and (3.28), it is not difficult to see that

$$h(x', y') = i\sum_{j=1}^n |\lambda_j| |z_j - w_j|^2 + i\sum_{j=1}^n \lambda_j(\overline{z}_j w_j - z_j\overline{w}_j) + O(|(x', y')|^3). \tag{3.35}$$

Thus $\widehat{\varphi}$ satisfies (3.26). Furthermore, from (3.35) and consider Taylor expansion of $h(x', y')$ at $x' = y'$, we see that

$$\operatorname{Im} h(x', y') \geq c\,|x'-y'|^2, \ \ c > 0.$$

Thus, $\widehat{\varphi} = 0$ if and only if $x = y$. We conclude that $\widehat{\varphi}$ satisfies (3.16)–(3.18), (3.20) and (3.25), (3.26). In view of Remark 3.6, we see that $t\varphi_-$ and $t\widehat{\varphi}$ are equivalent at each point of $\operatorname{diag}\Big((\Sigma^- \cap T^*B) \times (\Sigma^- \cap T^*B)\Big)$ in the sense of Melin-Sjöstrand. Since $\varphi_-(x, y) = -\overline{\varphi}_-(y, x)$, we can replace φ_- by

$$\frac{1}{2}(\widehat{\varphi}(x, y) - \overline{\widehat{\varphi}(y, x)}) = (-x_{2n+1} + y_{2n+1}) + \frac{1}{2}(h(x', y') - \overline{h}(y', x')).$$

Summing up, we get the following.

Theorem 3.8. *Let (L, h^L) be a holomorphic Hermitian line bundle whose curvature R^L is non-degenerate of constant signature (n_-, n_+) at each point of an open set $D \Subset M$. Let $\pi : X \to M$ be the Grauert tube of L (cf. (3.1)) and let $B = \pi^{-1}(D)$. With the notations used before, we can take the phase $\varphi_-(x, y)$ in Theorem 3.3 so that*

$$\begin{aligned} &\varphi_-(x, y) = -x_{2n+1} + y_{2n+1} + \Psi(z, w), \ \ \Psi(x', y') = \Psi(z, w) \in \mathscr{C}^\infty, \\ &p_0(x, \varphi'_-(x, y)) = O(|x'-y'|^N), \text{ locally uniformly on } B \times B, \text{ for all } N \in \mathbb{N}, \end{aligned} \tag{3.36}$$

where $p_0(x,\xi)$ *is the principal symbol of* $\Box_b^{(q)}$ *and* Ψ *satisfies*

$$\Psi(z,w) = -\overline{\Psi}(w,z)\,,\ \exists\, c>0:\ \operatorname{Im}\Psi \geq c\,|z-w|^2\ ,\ \Psi(z,w)=0 \Leftrightarrow z=w\,.$$

Moreover, for a given point $p \in D$*, we may take local holomorphic coordinates* $z = (z_1,\ldots,z_n)$ *centered at* p *such that the metric on* $T^{(1,0)}M$ *equals* $\sum_{j=1}^{n} dz_j \otimes d\overline{z}_j$ *at* p *and* (3.34) *holds. Then, near* $(0,0)$*, we have*

$$\Psi(z,w) = i\sum_{j=1}^{n}|\lambda_j|\,|z_j-w_j|^2 + i\sum_{j=1}^{n}\lambda_j(\overline{z}_jw_j - z_j\overline{w}_j) + O(|(z,w)|^3). \tag{3.37}$$

From now on, we assume that φ_- has the form (3.36).

3.3. **Semi-classical Hodge decomposition for the Kodaira Laplacian.** In this Section we apply the results about the Szegö kernel previously deduced in order to describe the semiclassical behavior of the spectrum of the Kodaira Laplacian $\Box_k^{(q)}$. We first introduce some notations. Let $D \Subset M$ be an open set. For any k-dependent continuous function

$$F_k : H^s_{\rm comp}(D,\Lambda^qT^{*(0,1)}M) \to H^{s'}_{\rm loc}(D,\Lambda^qT^{*(0,1)}M),\quad s,s'\in\mathbb{R},$$

we write

$$F_k = O(k^{n_0}) : H^s_{\rm comp}(D,\Lambda^qT^{*(0,1)}M) \to H^{s'}_{\rm loc}(D,\Lambda^qT^{*(0,1)}M),\quad n_0\in\mathbb{Z},$$

if for any $\chi_0,\chi_1 \in \mathscr{C}_0^\infty(D)$, there is a positive constant c, c is independent of k, such that

$$\|(\chi_0F_k\chi_1)u\|_{s'} \leq ck^{n_0}\|u\|_s\,,\ \ \forall u\in H^s_{\rm loc}(D,\Lambda^qT^{*(0,1)}M), \tag{3.38}$$

where $\|u\|_s$ is the usual Sobolev norm of order s.

A k-dependent smoothing operator $A_k : \Omega_0^{0,q}(D) \to \Omega^{0,q}(D)$ is called k-negligible if the kernel $A_k(x,y)$ of A_k satisfies $\left|\partial_x^\alpha\partial_y^\beta A_k(x,y)\right| = O(k^{-N})$ locally uniformly on every compact set in $D\times D$, for all multiindices α, β and all $N\in\mathbb{N}$. A_k is k-negligible if and only if

$$A_k = O(k^{-N'}) : H^s_{\rm comp}(D,\Lambda^qT^{*(0,1)}M) \to H^{s+N}_{\rm loc}(D,\Lambda^qT^{*(0,1)}M)\,,\quad \text{for all } N,N'\geq 0,\ s\in\mathbb{R}.$$

Let $C_k : \Omega_0^{0,q}(D) \to \Omega^{0,q}(D)$ be another k-dependent smoothing operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ or $A_k(x,y)\equiv C_k(x,y) \mod O(k^{-\infty})$ if $A_k - C_k$ is k-negligible.

We recall the definition of semi-classical Hörmander symbol spaces:

Definition 3.9. Let U be an open set in $\mathbb{R}^N$. Let $S(1;U)=S(1)$ be the set of $a\in\mathscr{C}^\infty(U)$ such that for every $\alpha\in\mathbb{N}^N$, there exists $C_\alpha>0$, such that $|\partial_x^\alpha a(x)|\leq C_\alpha$ on U. If $a=a(x,k)$ depends on $k\in]1,\infty[$, we say that $a(x,k)\in S_{\rm loc}(1)$ if $\chi(x)a(x,k)$ uniformly bounded in $S(1)$ when k varies in $]1,\infty[$, for any $\chi\in\mathscr{C}_0^\infty(U)$. For $m\in\mathbb{R}$, we put $S^m_{\rm loc}(1)=k^mS_{\rm loc}(1)$. If $a_j\in S^{m_j}_{\rm loc}(1)$, $m_j\searrow-\infty$, we say that $a\sim\sum_{j=0}^\infty a_j$ in $S^{m_0}_{\rm loc}(1)$ if $a-\sum_{j=0}^{N_0}a_j\in S^{m_{N_0+1}}_{\rm loc}(1)$ for every N_0. From this, we form $S^m_{\rm loc}(1;Y,E)$ in the natural way, where Y is a smooth paracompact manifold and E is a vector bundle over Y.

We return to our situation. Let $s(z)$ be a local trivializing section of L on an open subset $D\Subset M$ and $|s|^2_{h^L}=e^{-2\phi}$. We assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-,n_+) at each point of D. Let (z,θ) be the local coordinates as in (3.3) defined on $D\times]-\varepsilon_0,\varepsilon_0[$, $\varepsilon_0>0$. We identify $D\times]-\varepsilon_0,\varepsilon_0[$ with some open set of X and until further notice we work with the local coordinates (z,θ). Let $\Box_s^{(q)}$ be the operator as in (3.7) and (3.6). Since $\partial\overline{\partial}\phi$ has constant signature (n_-,n_+) at each point of D, from

(3.5), we know that the Levi form $\mathcal{L}$ has constant signature (n_-, n_+) at each point of $D\times]-\varepsilon_0, \varepsilon_0[$.

Let $q = n_-$ or n_+ and let S_-, S_+ be the approximate Szegö kernels defined in Theorem 3.3. Define also the approximate Szegö kernel

$$S = S_- + S_+ \,. \tag{3.39}$$

Let $\chi(\theta), \chi_1(\theta) \in \mathscr{C}_0^\infty(]-\varepsilon_0, \varepsilon_0[)$, $\chi, \chi_1 \geq 0$. We assume that $\chi_1 = 1$ on $\operatorname{Supp}\chi$. We take χ so that $\int \chi(\theta)d\theta = 1$. Put

$$\chi_k(\theta) = e^{-ik\theta}\chi(\theta). \tag{3.40}$$

The approxiamte Szegö kernel was introduced in (3.39). We introduce the *localized approximate Szegö kernel* $\mathcal{S}_k$ by

$$\begin{aligned}&\mathcal{S}_k : H^s_{\rm loc}(D, \Lambda^q T^{*(0,1)}M) \to H^s_{\rm loc}(D, \Lambda^q T^{*(0,1)}M), \quad \forall s \in \mathbb{N}_0,\\ &u(z) \to \int e^{i\theta k}\chi_1(\theta)S(\chi_k u)(z,\theta)d\theta.\end{aligned} \tag{3.41}$$

Let $u(z) \in H^s_{\rm loc}(D, \Lambda^q T^{*(0,1)}M)$, $s \in \mathbb{N}_0$. We have $\chi_k(\theta)u(z) \in H^s_{\rm loc}(D\times]-\varepsilon_0, \varepsilon_0[, \Lambda^q T^{*(0,1)}X)$. From Theorem 3.3, we know that

$$S(\chi_k u) \in H^s_{\rm loc}(D\times]-\varepsilon_0, \varepsilon_0[, \Lambda^q T^{*(0,1)}X).$$

From this, it is not difficult to see that $\int e^{i\theta k}\chi_1(\theta)S(\chi_k u)(z,\theta)d\theta \in H^s_{\rm loc}(D, \Lambda^q T^{*(0,1)}M)$. Thus, the localization $\mathcal{S}_k$ is well-defined. Since S is properly supported, $\mathcal{S}_k$ is properly supported, too. Moreover, from (3.14) and (3.41), we can check that

$$\mathcal{S}_k = O(k^s) : H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M) \to H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M), \tag{3.42}$$

for all $s \in \mathbb{N}_0$.

Let $\mathcal{S}^*_k : \mathscr{D}'(D, \Lambda^q T^{*(0,1)}M) \to \mathscr{D}'(D, \Lambda^q T^{*(0,1)}M)$ be the formal adjoint of $\mathcal{S}_k$ with respect to $(\ ,\ )$. Then $\mathcal{S}^*_k$ is also properly supported and we have

$$\mathcal{S}^*_k : \mathscr{E}'(D, \Lambda^q T^{*(0,1)}M) \to \mathscr{E}'(D, \Lambda^q T^{*(0,1)}M). \tag{3.43}$$

From (3.12), we have

$$\begin{aligned}\Box^{(q)}_s \circ \Big(\int e^{i\theta k}\chi_1(\theta)S(\chi_k u)d\theta\Big) &= \int e^{i\theta k}\Big(\Box^{(q)}_b(\chi_1 S)\Big)(\chi_k u)(z,\theta)d\theta\\ &= \int e^{i\theta k}\Big(\Box^{(q)}_b(\chi_1 S\tilde{\chi})\Big)(\chi_k u)(z,\theta)d\theta,\end{aligned} \tag{3.44}$$

where $\tilde{\chi} \in \mathscr{C}_0^\infty(]-\varepsilon_0, \varepsilon_0[)$, $\tilde{\chi} = 1$ on $\operatorname{Supp}\chi$ and $\chi_1 = 1$ on $\operatorname{Supp}\tilde{\chi}$ and $u \in \Omega^{0,q}_0(D)$. Note that $\Box^{(q)}_b(\chi_1 S\tilde{\chi}) = \Box^{(q)}_b(S\tilde{\chi}) - \Box^{(q)}_b((1-\chi_1)S\tilde{\chi})$. From Theorem 3.3, we know that $\Box^{(q)}_b S$ is smoothing and the kernel of S is smoothing away the diagonal. Thus, $(1-\chi_1)S\tilde{\chi}$ is smoothing. It follows that $\Box^{(q)}_b((1-\chi_1)S\tilde{\chi})$ is smoothing. We conclude that $\Box^{(q)}_b(\chi_1 S\tilde{\chi})$ is smoothing. Let $K((z,\theta),(w,\eta)) \in \mathscr{C}^\infty$ be the distribution kernel of $\Box^{(q)}_b(\chi_1 S\tilde{\chi})$, where $w = (w_1, \ldots, w_n)$ are the local holomorphic cooordinates of D and η is the coordinate of $]-\varepsilon_0, \varepsilon_0[$. From (3.44) and recall the form χ_k (see (3.40)), we see that the distribution kernel of $\Box^{(q)}_s\mathcal{S}_k$ is given by

$$(\Box^{(q)}_s\mathcal{S}_k)(z,w) = \int e^{i(\theta-\eta)k}K((z,\theta),(w,\eta))\chi(\eta)d\eta d\theta. \tag{3.45}$$

For $N\in\mathbb{N}$, we have

$$\begin{aligned}(3.46)\quad \left|k^N\Box_s^{(q)}\mathcal{S}_k(z,w)\right| &= \left|\int\Big((i\frac{\partial}{\partial\eta})^N e^{i(\theta-\eta)k}\Big)K((z,\theta),(w,\eta))\chi(\eta)d\eta d\theta\right| \\ &= \left|\int e^{i(\theta-\eta)k}(-i\frac{\partial}{\partial\eta})^N\Big(K((z,\theta),(w,\eta))\chi(\eta)\Big)d\eta d\theta\right|.\end{aligned}$$

Thus, $\Box_s^{(q)}\mathcal{S}_k(z,w)=O(k^{-N})$, locally uniformly for all $N\in\mathbb{N}$, and similarly for the derivatives. We deduce that

$$(3.47)\qquad \Box_s^{(q)}\mathcal{S}_k\equiv 0\mod O(k^{-\infty}).$$

Thus,

$$(3.48)\qquad \mathcal{S}_k^*\Box_s^{(q)}\equiv 0\mod O(k^{-\infty}).$$

Let A be the partial parametrix of $\Box_b^{(q)}$ described in Theorem 3.3. Define the *localized partial parametrix* $\mathcal{A}_k$ by

$$\begin{aligned}(3.49)\quad &\mathcal{A}_k: H^s_{\rm loc}(D,\Lambda^qT^{*(0,1)}M)\to H^{s+1}_{\rm loc}(D,\Lambda^qT^{*(0,1)}M),\ \ \forall s\in\mathbb{N}_0,\\ &u(z)\to\int e^{i\theta k}\chi_1A(\chi_ku)(z,\theta)d\theta.\end{aligned}$$

As above, we can show that $\mathcal{A}_k$ is well-defined. Since A is properly supported, $\mathcal{A}_k$ is properly supported, too. Moreover, from (3.14) and (3.49), we can check that

$$(3.50)\qquad \mathcal{A}_k=O(k^s): H^s_{\rm comp}(D,\Lambda^qT^{*(0,1)}M)\to H^{s+1}_{\rm comp}(D,\Lambda^qT^{*(0,1)}M),$$

for all $s\in\mathbb{N}_0$.

Let $\mathcal{A}_k^*:\mathscr{D}'(D,\Lambda^qT^{*(0,1)}M)\to\mathscr{D}'(D,\Lambda^qT^{*(0,1)}M)$ be the formal adjoint of $\mathcal{A}_k$ with respect to $(\,,\,)$. We can check that

$$(\mathcal{A}_k^*v)(z)=\int\overline{\chi_k(\theta)}A^*(ve^{-i\theta k}\chi_1)(z,\theta)d\theta\in\Omega_0^{0,q}(D),$$

for all $v\in\Omega_0^{0,q}(D)$. Thus, $\mathcal{A}_k^*:\Omega_0^{0,q}(D)\to\Omega_0^{0,q}(D)$. Moreover, as before, we can show that

$$(3.51)\quad \mathcal{A}_k^*=O(k^s): H^s_{\rm comp}(D,\Lambda^qT^{*(0,1)}M)\to H^{s+1}_{\rm comp}(D,\Lambda^qT^{*(0,1)}M),\quad\text{for all } s\in\mathbb{N}_0.$$

Let $u\in\Omega_0^{0,q}(D)$. From (3.12), we have

$$\Box_s^{(q)}(\mathcal{A}_ku)=\Box_s^{(q)}\circ\Big(\int e^{i\theta k}\chi_1A(\chi_ku)d\theta\Big)=\int e^{i\theta k}\Big(\Box_b^{(q)}\chi_1A\tilde{\chi})(\chi_ku)(z,\theta)d\theta,$$

where $\tilde{\chi}$ is as in (3.44). Note that $\Box_b^{(q)}(\chi_1A\tilde{\chi})=\Box_b^{(q)}(A\tilde{\chi})-\Box_b^{(q)}((1-\chi_1)A\tilde{\chi})$. From Theorem 3.3, we know that $\Box_b^{(q)}A+S=I$ and the kernel of A is smoothing away the diagonal. Thus, $(1-\chi_1)A\tilde{\chi}$ is smoothing. It follows that $\Box_b^{(q)}((1-\chi_1)A\tilde{\chi})$ is smoothing. We conclude that $\Box_b^{(q)}(\chi_1A\tilde{\chi})\equiv(I-S)\tilde{\chi}$. From this, we get

$$\begin{aligned}\Box_s^{(q)}(\mathcal{A}_ku)&=\int e^{i\theta k}(I-S)(\chi_ku)(z,\theta)d\theta+\int e^{i\theta k}F(\chi_ku)(z,\theta)d\theta\\ (3.52)\quad &=u-\int e^{i\theta k}S(\chi_ku)(z,\theta)d\theta+\int e^{i\theta k}F(\chi_ku)(z,\theta)d\theta\\ &=(I-\mathcal{S}_k)u-\int e^{i\theta k}(1-\chi_1)S(\chi_ku)(z,\theta)d\theta+\int e^{i\theta k}F(\chi_ku)(z,\theta)d\theta,\end{aligned}$$

where F is a smoothing operator. We can repeat the procedure as in (3.46) and conclude that the operator

$$u \to \int e^{i\theta k} F(\chi_k u)(z,\theta)d\theta, \ \ u \in \Omega^{0,q}_0(D),$$

is k-negligible. Similarly, since $(1-\chi_1)S\chi$ is smoothing, the operator

$$u \to \int e^{i\theta k}(1-\chi_1)S(\chi_k u)(z,\theta)d\theta, \ \ u \in \Omega^{0,q}_0(D),$$

is also k-negligible. Summing up, we obtain

$$\Box^{(q)}_s \mathcal{A}_k + \mathcal{S}_k \equiv I \mod O(k^{-\infty}). \tag{3.53}$$

We may replace $\mathcal{S}_k$ by $I - \Box^{(q)}_s \mathcal{A}_k$ and we have $\Box^{(q)}_s \mathcal{A}_k + \mathcal{S}_k = I$ and hence $\mathcal{A}^*_k \Box^{(q)}_s + \mathcal{S}^*_k = I$. Thus,

$$\mathcal{S}_k = (\mathcal{A}^*_k \Box^{(q)}_s + \mathcal{S}^*_k)\mathcal{S}_k = \mathcal{A}^*_k \Box^{(q)}_s \mathcal{S}_k + \mathcal{S}^*_k \mathcal{S}_k. \tag{3.54}$$

From (3.47) and (3.51), we see that

$$\mathcal{A}^*_k \Box^{(q)}_s \mathcal{S}_k = O(k^{-N'}) : H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M) \to H^{s+N}_{\rm comp}(D, \Lambda^q T^{*(0,1)}M'),$$

for all $s \in \mathbb{R}$ and $N', N \in \mathbb{N}$. Thus, $\mathcal{A}^*_k \Box^{(q)}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$. From this and (3.54), we get

$$\mathcal{S}^*_k \mathcal{S}_k \equiv \mathcal{S}_k \mod O(k^{-\infty}). \tag{3.55}$$

It follows that

$$\mathcal{S}_k \equiv \mathcal{S}^*_k \mod O(k^{-\infty}),\ \mathcal{S}^2_k \equiv \mathcal{S}_k \mod O(k^{-\infty}). \tag{3.56}$$

From (3.42), (3.47), (3.48), (3.50), (3.51), (3.53), (3.55) and (3.56), we get our main technical result:

Theorem 3.10. *Let $s(z)$ be a local trivializing section of L on an open subset $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that $2\partial\overline{\partial}\phi = R^L$ is non-degenerate of constant signature (n_-, n_+) at each point of D. Let $q = n_-$ or n_+ and let $\mathcal{S}_k$ be the localized approximate Szegö kernel* (3.41) *and $\mathcal{A}_k$ the localized partial parametrix* (3.49). *Then,*

$$\begin{aligned} &\mathcal{S}^*_k, \mathcal{S}_k = O(k^s) : H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M) \to H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M),\ \ \forall s \in \mathbb{N}_0, \\ &\mathcal{A}^*_k, \mathcal{A}_k = O(k^s) : H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M) \to H^{s+1}_{\rm comp}(D, \Lambda^q T^{*(0,1)}M),\ \ \forall s \in \mathbb{N}_0, \end{aligned} \tag{3.57}$$

and we have

$$\Box^{(q)}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty}),\ \ \mathcal{S}^*_k \Box^{(q)}_s \equiv 0 \mod O(k^{-\infty}), \tag{3.58}$$

$$\mathcal{S}_k \equiv \mathcal{S}^*_k \mod O(k^{-\infty}),\ \mathcal{S}_k \equiv \mathcal{S}^2_k \mod O(k^{-\infty}),\ \mathcal{S}_k \equiv \mathcal{S}^*_k \mathcal{S}_k \mod O(k^{-\infty}), \tag{3.59}$$

$$\mathcal{S}^*_k + \mathcal{A}^*_k \Box^{(q)}_s \equiv I \mod O(k^{-\infty}),\ \ \mathcal{S}_k + \Box^{(q)}_s \mathcal{A}_k \equiv I \mod O(k^{-\infty}), \tag{3.60}$$

where $\mathcal{S}^*_k$ and $\mathcal{A}^*_k$ are the formal adjoints of $\mathcal{S}_k$ and $\mathcal{A}_k$ with respect to $(\,,\,)$ respectively and $\Box^{(q)}_s$ is given by (3.7) *and* (3.6).

We notice that $\mathcal{S}_k$, $\mathcal{S}^*_k$, $\mathcal{A}_k$, $\mathcal{A}^*_k$, are all properly supported. We need

Theorem 3.11. *The localized approximate Szegö kernel $\mathcal{S}_k$ given by* (3.41) *is a smoothing operator. Moreover, if $q = n_-$, then the kernel of $\mathcal{S}_k$ satisfies*

$$\mathcal{S}_k(z,w) \equiv e^{ik\Psi(z,w)} b(z,w,k) \mod O(k^{-\infty}), \tag{3.61}$$

with

$$\begin{aligned} &b(z,w,k) \in S^n_{\mathrm{loc}}\Big(1; D\times D, \Lambda^q T^{*(0,1)}_w M \boxtimes \Lambda^q T^{*(0,1)}_z M\Big),\\ &b(z,w,k) \sim \sum_{j=0}^{\infty} b_j(z,w) k^{n-j} \text{ in } S^n_{\mathrm{loc}}\Big(1; D\times D, \Lambda^q T^{*(0,1)}_w M \boxtimes \Lambda^q T^{*(0,1)}_z M\Big),\\ &b_j(z,w) \in \mathscr{C}^\infty\Big(D\times D, \Lambda^q T^{*(0,1)}_w M \boxtimes \Lambda^q T^{*(0,1)}_z M\Big),\ \ j = 0,1,\ldots \end{aligned} \tag{3.62}$$

and $\Psi(z,w)$ is as in Theorem 3.8.

If $q = n_+$, $n_- \neq n_+$, then

$$\mathcal{S}_k(z,w) \equiv 0 \mod O(k^{-\infty}). \tag{3.63}$$

Proof. Theorem 3.11 essentially follows from the stationary phase formula of Melin-Sjöstrand [41]. Let $s(z)$ be a local trivializing section of L on an open subset $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. Assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point of D. Let $q = n_-$ or n_+. Let (z,θ) be the local coordinates as in (3.3) defined on $D\times]-\varepsilon_0,\varepsilon_0[$. We identify $D\times]-\varepsilon_0,\varepsilon_0[$ with some open set of X and until further notice we work with the local coordinates (z,θ). We write $(z,\theta) = (x', x_{2n+1}) = x$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1,\ldots,n$, $\theta = x_{2n+1}$, $x' = (x_1,\ldots,x_{2n})$. We identify x' with $(x',0)$. We also write $(w,\eta) = (y', y_{2n+1}) = y$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1,\ldots,n$, $\eta = y_{2n+1}$, $y' = (y_1,\ldots,y_{2n})$, where $w = (w_1,\ldots,w_n)$ are the local holomorphic coordinates of D and η is the coordinate of $]-\varepsilon_0,\varepsilon_0[$. From the definition (3.41) of $\mathcal{S}_k$ and Theorem 3.3, we see that the distribution kernel of $\mathcal{S}_k$ is given by

(3.64)

$$\begin{aligned} &\mathcal{S}_k(x',y') \equiv \int_{t\geq 0} e^{i\varphi_-(x,y)t + ix_{2n+1}k - iy_{2n+1}k} s_-(x,y,t)\chi_1(x_{2n+1})\chi(y_{2n+1}) dx_{2n+1}dtdy_{2n+1}\\ &+ \int_{t\geq 0} e^{i\varphi_+(x,y)t + ix_{2n+1}k - iy_{2n+1}k} s_+(x,y,t)\chi_1(x_{2n+1})\chi(y_{2n+1}) dx_{2n+1}dtdy_{2n+1} \mod O(k^{-\infty})\\ &\equiv I_0(x',y') + I_1(x',y') \mod O(k^{-\infty}), \end{aligned}$$

where the integrals above are defined as oscillatory integrals. First, we study the kernel

$$I_1(x',y') = \int_{t\geq 0} e^{i\varphi_+(x,y)t + ix_{2n+1}k - iy_{2n+1}k} s_+(x,y,t)\chi_1(x_{2n+1})\chi(y_{2n+1}) dx_{2n+1}dtdy_{2n+1}.$$

By the change of variables $t = k\sigma$ we get

$$I_1(x',y') = \int_{\sigma\geq 0} e^{ik\left(\varphi_+(x,y)\sigma + x_{2n+1} - y_{2n+1}\right)} k s_+(x,y,k\sigma)\chi_1(x_{2n+1})\chi(y_{2n+1}) dx_{2n+1}d\sigma dy_{2n+1}.$$

Note that $d_x\varphi_+(x,x) = \omega_0(x)$. Taking into account the form $\omega_0(x)$ (cf. (3.4)), we see that $\frac{\partial\varphi_+}{\partial x_{2n+1}}(x,x) = 1$. In view of Theorem 3.3, we see that $\varphi_+(x,y) = 0$ if and only if $x = y$. We conclude that

$$\Big(d_\sigma(\varphi_+(x,y)\sigma + x_{2n+1} - y_{2n+1}), d_{x_{2n+1}}(\varphi_+(x,y)\sigma + x_{2n+1} - y_{2n+1})\Big) \neq \Big(0,0\Big),\ \ \sigma \geq 0.$$

Thus, we can integrate by parts in σ and x_{2n+1} and conclude that I_1 is smoothing and

$$I_1 \equiv 0 \mod O(k^{-\infty}). \tag{3.65}$$

Now, we study the kernel

$$I_0(x',y')=\int_{t\geq 0}e^{i\varphi_-(x,y)t+ix_{2n+1}k-iy_{2n+1}k}s_-(x,y,t)\chi_1(x_{2n+1})\chi(y_{2n+1})dx_{2n+1}dtdy_{2n+1}.$$

As before, by letting $t=k\sigma$, we get

(3.66)

$$I_0(x',y')=\int_{\sigma\geq 0}e^{ik\Big(\varphi_-(x,y)\sigma+x_{2n+1}-y_{2n+1}\Big)}ks_-(x,y,k\sigma)\chi_1(x_{2n+1})\chi(y_{2n+1})dx_{2n+1}d\sigma dy_{2n+1}.$$

In view of (3.36), we see that

$$\varphi_-(x,y)=\Psi(x',y')+y_{2n+1}-x_{2n+1},\quad \operatorname{Im}\Psi(x',y')\geq 0. \tag{3.67}$$

Put

$$\Psi(x,y,\sigma)=(\Psi(x',y')+y_{2n+1}-x_{2n+1})\sigma+x_{2n+1}-y_{2n+1}. \tag{3.68}$$

Let $\varphi(\sigma)\in\mathscr{C}_0^\infty(\mathbb{R}_+)$ with $\varphi(\sigma)=1$ in some small neighborhood of 1. We introduce the cut-off functions $\varphi(\sigma)$ and $1-\varphi(\sigma)$ in the integral (3.66):

$$I_0^0(x',y'):=\int_{\sigma\geq 0}e^{ik\Psi(x,y,\sigma)}\varphi(\sigma)ks_-(x,y,k\sigma)\chi_1(x_{2n+1})\chi(y_{2n+1})dx_{2n+1}d\sigma dy_{2n+1}, \tag{3.69}$$

(3.70)

$$I_0^1(x',y'):=\int_{\sigma\geq 0}e^{ik\Psi(x,y,\sigma)}(1-\varphi(\sigma))ks_-(x,y,k\sigma)\chi_1(x_{2n+1})\chi(y_{2n+1})dx_{2n+1}d\sigma dy_{2n+1}\,,$$

so that

$$I_0(x',y')=I_0^0(x',y')+I_0^1(x',y')\,.$$

First, we study $I_0^1(x',y')$. Note that when $\sigma\neq 1$, $d_{x_{2n+1}}\Psi(x,y,\sigma)=1-\sigma\neq 0$. Thus, we can integrate by parts and get that I_0^1 is smoothing and $I_0^1(x',y')\equiv 0 \mod O(k^{-\infty})$.

Next, we study the kernel $I_0^0(x',y')$. First, we assume that $q=n_+$, $n_+\neq n_-$. In view of Theorem 3.3, we see that $s_-(x,y,t)$ vanishes to infinite order at $x=y$. Thus, $I_0^0\equiv 0 \mod O(k^{-\infty})$. Therefore, we get (3.63).

Now, we assume that $q=n_-$. Since the integral (3.69) converges, we have

$$\begin{aligned}&I_0^0(x',y')=\int H(x',y)\chi(y_{2n+1})dy_{2n+1},\\&H(x',y)=\int_{\sigma\geq 0}e^{ik\Psi(x,y,\sigma)}\varphi(\sigma)ks_-(x,y,k\sigma)\chi_1(x_{2n+1})dx_{2n+1}d\sigma.\end{aligned} \tag{3.71}$$

Recalling the form of $\Psi(x,y,\sigma)$, we have $\operatorname{Im}\Psi(x,y,\sigma)\geq 0$, $d_\sigma\Psi(x,y,\sigma)=0$ if and only if $x=y$ and $d_{x_{2n+1}}\Psi(x,y,\sigma)|_{x=y}=1-\sigma$. Thus, $x=y$ and $\sigma=1$ are real critical points. Moreover, we can check that the Hessian of $\Psi(x,y,\sigma)$ at $x=y$, $\sigma=1$, is given by

$$\begin{pmatrix}\Psi''_{\sigma\sigma}(x,x,1) & \Psi''_{x_{2n+1}\sigma}(x,x,1)\\ \Psi''_{\sigma x_{2n+1}}(x,x,1) & \Psi''_{x_{2n+1}x_{2n+1}}(x,x,1)\end{pmatrix}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}.$$

Thus, $\Psi(x,y,\sigma)$ is a non-degenerate complex valued phase function in the sense of Melin-Sjöstrand [41]. Let

$$\widetilde{\Psi}(\widetilde{x},\widetilde{y},\widetilde{\sigma}):=\Big(\widetilde{\Psi}(\widetilde{x}',\widetilde{y}')+(\widetilde{y}_{2n+1}-\widetilde{x}_{2n+1})\Big)\widetilde{\sigma}+\widetilde{x}_{2n+1}-\widetilde{y}_{2n+1}$$

be an almost analytic extension of $\Psi(x,y,\sigma)$, where $\widetilde{\Psi}(\widetilde{x}',\widetilde{y}')$ is an almost analytic extension of $\Psi(x',y')$ (with $\Psi(x',y')$ as in (3.67)) and similarly for $\widetilde{y}_{2n+1}$, $\widetilde{x}_{2n+1}$ and $\widetilde{\sigma}$ (see [41, §2] for the precise meaning of the almost analytic extension). We can check that given

y_{2n+1} and (x', y'), $\tilde{x}_{2n+1} = y_{2n+1} + \Psi(x', y')$, $\tilde{\sigma} = 1$ are the solutions of $\frac{\partial \tilde{\Psi}}{\partial \tilde{\sigma}} = 0$, $\frac{\partial \tilde{\Psi}}{\partial \tilde{x}_{2n+1}} = 0$. From this and by the stationary phase formula of Melin-Sjöstrand [41], we get

$$H(x', y) \equiv e^{ik\Psi(x',y')} a(x', y, k) \mod O(k^{-\infty}), \tag{3.72}$$

where $a(x', y, k) \in \mathscr{C}^\infty\Big(D \times (D \times]-\varepsilon_0, \varepsilon_0[), \Lambda^q T^{*(0,1)}M \boxtimes \Lambda^q T^{*(0,1)}M\Big)$,

$$a(x', y, k) \sim \sum_{j=0}^{\infty} k^{n-j} a_j(x', y) \text{ in } S^n_{\rm loc}\Big(1; D \times (D \times]-\varepsilon_0, \varepsilon_0[), \Lambda^q T^{*(0,1)}M \boxtimes \Lambda^q T^{*(0,1)}M\Big),$$

$$a_j(x', y) \in \mathscr{C}^\infty\Big(D \times (D \times]-\varepsilon_0, \varepsilon_0[), \Lambda^q T^{*(0,1)}M \boxtimes \Lambda^q T^{*(0,1)}M\Big), \ \ j = 0, 1, \ldots$$

and

$$a_0(x', y) = 2\pi \tilde{s}^0_-\Big((x', y_{2n+1} + \Psi(x', y')), y\Big), \tag{3.73}$$

where $\tilde{s}^0_-$ is an almost analytic extension of s^0_-, s^0_- is as in (3.15).

From (3.71) and (3.72), we get,

$$I^0_0(x', y') \equiv e^{ik\Psi(x',y')} b(x', y', k) \mod O(k^{-\infty}), \tag{3.74}$$

where

$$b(x', y', k) \sim \sum_{j=0}^{\infty} k^{n-j} b_j(x', y') \text{ in } S^n_{\rm loc}(1; D \times D, \Lambda^q T^{*(0,1)}M \boxtimes \Lambda^q T^{*(0,1)}M),$$

$$b_j(x', y') = \int a_j(x', y)\chi(y_{2n+1}) dy_{2n+1} \in \mathscr{C}^\infty(D \times D, \Lambda^q T^{*(0,1)}M \boxtimes \Lambda^q T^{*(0,1)}M), \tag{3.75}$$

$j = 0, 1, \ldots$. Theorem 3.11 follows. □

As before, let s be a local trivializing section of L on an open set $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point D and let $q \neq n_-$. We first assume that $q = n_+$, $n_+ \neq n_-$. In view of Theorem 3.10 and (3.63), we see that when $q = n_+$, $n_+ \neq n_-$, we have

$$\Box^{(q)}_s \mathcal{A}_k \equiv I \mod O(k^{-\infty}), \tag{3.76}$$

where $\mathcal{A}_k$ is as in Theorem 3.10.

Now, we assume that $q \neq n_-, n_+$. From Theorem 3.2, we can repeat the proof of Theorem 3.10 and conclude that there exists a properly supported continuous operator

$$\mathcal{A}_k = O(k^s) : H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M) \to H^{s+1}_{\rm comp}(D, \Lambda^q T^{*(0,1)}M), \ \ \forall s \geq 0,$$

such that

$$\Box^{(q)}_s \mathcal{A}_k \equiv I \mod O(k^{-\infty}). \tag{3.77}$$

Summing up, we obtain

Theorem 3.12. *Let $s(z)$ be a local trivializing section of L on an open subset $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that $2\partial\overline{\partial}\phi = R^L$ is non-degenerate of constant signature (n_-, n_+) at each point of D. Let $q \neq n_-$. Then, there exists a properly supported continuous operator*

$$\mathcal{A}_k = O(k^s) : H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M) \to H^{s+1}_{\rm comp}(D, \Lambda^q T^{*(0,1)}M), \ \ \forall s \geq 0,$$

such that

$$\Box^{(q)}_s \mathcal{A}_k \equiv I \mod O(k^{-\infty}).$$

Remark 3.13. From Remark 3.5, we can generalize Theorem 3.10 and Theorem 3.12 with essentially the same proofs to the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M.

We have the following

Theorem 3.14. *Let $s(z)$ be a local trivializing section of L on an open subset $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that $2\partial\overline{\partial}\phi = R^L$ is non-degenerate of constant signature (n_-, n_+) at each point of D. Let $q = n_-$. For a given point $p \in D$, let $V_1, \ldots, V_n$ be an orthonormal frame of $T^{(1,0)}M$ in a neighborhood of p, for which $\dot{R}^L$ is diagonalized at p, namely,*

$$\begin{aligned}
&\dot{R}^L(p)V_j(p) = \lambda_j(p)V_j(p)\,, \quad j = 1, \ldots, n\,,\\
&\lambda_j(p) < 0\,, \quad j = 1, \ldots, q\,,\\
&\lambda_j(p) > 0\,, \quad j = q+1, \ldots, n.
\end{aligned}$$

Let $(T_j)_{j=1}^n$ denote the basis of $T^{*(0,1)}M$, which is dual to $(\overline{V}_j)_{j=1}^n$. Then, for $b_0(z,w)$ in (3.62)*, we have*

$$\begin{aligned}
b_0(p,p) &= (2\pi)^{-n}\left|\det \dot{R}^L(p)\right| \prod_{j=1}^{q} (T_j(p)\wedge) \circ (T_j(p)\wedge)^* \\
&= (2\pi)^{-n}\left|\det \dot{R}^L(p)\right| I_{\det \overline{W}^*}\,,
\end{aligned} \tag{3.78}$$

where $I_{\det \overline{W}^*} \in \operatorname{End}(\Lambda^q T^{*(0,1)}M))$ is as in the discussion after (1.2).

Proof. From (3.73) and (3.75), we have

$$b_0(x', x') = 2\pi \int s^0_-\Big((x', y_{2n+1}), (x', y_{2n+1})\Big)\chi(y_{2n+1})dy_{2n+1}. \tag{3.79}$$

In view of Theorem 3.7, we know that

$$s^0_-\Big((x', y_{2n+1}), (x', y_{2n+1})\Big) = \frac{1}{2}\,|\mu_1(x')| \cdots |\mu_n(x')|\,\pi^{-n-1} \prod_{j=1}^{j=n_-} (T_j(x')\wedge) \circ (T_j(x')\wedge)^*,$$

where $\mu_j(x')$, $j = 1, \ldots, n$, are the eigenvalues of $L_{x'}$ and $T_1(x'), \ldots, T_n(x')$, are as in Theorem 3.7. From this, (3.79) and the identification of the Levi form and R^L (see (3.5)) and notice that $|\mu_1(p)| \cdots |\mu_n(p)| = 2^{-n}\,|\lambda_1(p)| \cdots |\lambda_n(p)| = 2^{-n}\left|\det \dot{R}^L(p)\right|$, (3.78) follows. ∎

4. Asymptotic expansion of the spectral function for lower energy forms

Let (M, Θ) be a Hermitian manifold and let (L, h^L) be a Hermitian holomorphic line bundel on M. We recall that (cf. (1.3)) $\mathscr{E}^q_{k^{-N_0}}(M, L^k)$ denote the spectral space of $\Box^{(q)}_k$ corresponding to energy less than k^{-N_0}. In the present Section we study the asymptotic expansion of the spectral function associated to $\mathscr{E}^q_{k^{-N_0}}(M, L^k)$. In Section 4.1 we prove pointwise upper bounds for the eigenforms of the spectral spaces $\mathscr{E}^q_{k^{-N_0}}(M, L^k)$ in terms of their L^2-norm (Theorem 4.3). In Section 4.2 we compare the localized spectral projection with the localized approximate Szegö projection $\mathcal{S}_k$. In Section4.3 we apply this results to prove the asympotic expansion of the spectral function and thus give the proof of Theorem 1.1. In Section 4.4 we exhibit the asympotic expansion of the Bergman kernel and prove Theorem 1.6. Finally, in Section 4.5 we calculate the coefficients b^0_1 and b^0_2 and thus prove Theorem 1.2.

4.1. **Asymptotic upper bounds.** Fix $N_0 \geq 1$. In this Section we will give pointwise upper bounds for u and $\partial^\alpha u$, where $u \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$.

Let $D \Subset M$ be a chart domain such that $L|_D$ is trivial. Let s be a local trivializing section of L on D and set $|s|^2_{h^L} = e^{-2\phi}$. Let $(\,,)_{k\phi}$ be the inner product on the space $\Omega^{0,q}_0(D)$ defined as follows:

$$(f, g)_{k\phi} = \int_D \langle\, f\,, g\,\rangle e^{-2k\phi} dv_M(x)\,, \quad f, g \in \Omega^{0,q}_0(D)\,.$$

Let $\overline{\partial}^{*,k\phi} : \Omega^{0,q+1}(D) \to \Omega^{0,q}(D)$ be the formal adjoint of $\overline{\partial}$ with respect to $(\,,)_{k\phi}$. Put $\Box^{(q)}_{k\phi} = \overline{\partial}\,\overline{\partial}^{*,k\phi} + \overline{\partial}^{*,k\phi}\overline{\partial} : \Omega^{0,q}(D) \to \Omega^{0,q}(D)$. Let $u \in \Omega^{0,q}(D, L^k)$. On D, we write $u = s^k\widetilde{u}$, $\widetilde{u} \in \Omega^{0,q}(D)$. We have

$$\Box^{(q)}_k u = s^k \Box^{(q)}_{k\phi}\widetilde{u}. \tag{4.1}$$

Fix $p \in D$ and consider local coordinates $z = (z_1, \ldots, z_n) = (x_1, \ldots, x_{2n}) = x$ on D, such that $x(p) = z(p) = 0$ and $\phi(z) = O(|z|^2)$ near p. Let $F_k(z) := \frac{z}{\sqrt{k}}$ be the scaling map. For $r > 0$, let $D_r = \{x;\ |x_j| < r, j = 1, \ldots, 2n\}$. Let $f \in \Omega^{0,q}(D_{\frac{\log k}{\sqrt{k}}})$, $f = \sum'_{|J|=q} f_J d\overline{z}^J$. We define the scaled form $F^*_k f \in \Omega^{0,q}(D_{\log k})$ by

$$F^*_k f = \sum_{|J|=q}{}' f_J\Big(k^{-1/2}z\Big) d\overline{z}^J \in \Omega^{0,q}(D_{\log k}).$$

Let $\Box^{(q)}_{k\phi,(k)} : \Omega^{0,q}(D_{\log k}) \to \Omega^{0,q}(D_{\log k})$ be the scaled Laplacian defined by

$$\Box^{(q)}_{k\phi,(k)}(F^*_k u) = \frac{1}{k} F^*_k(\Box^{(q)}_{k\phi} u), \quad u \in \Omega^{0,q}(D_{\frac{\log k}{\sqrt{k}}}). \tag{4.2}$$

By Berman [1, § 2] and Hsiao-Marinescu [31, § 2] it is known all the derivatives of the coefficients of the operator $\Box^{(q)}_{k\phi,(k)}$ are uniformly bounded in k on $D_{\log k}$. Let $D_r \subset D_{\log k}$ and let $W^s_{kF^*_k\phi}(D_r, \Lambda^q T^{*(0,1)}M)$, $s \in \mathbb{N}_0$, denote the Sobolev space of order s of sections of $\Lambda^q T^{*(0,1)}M$ over D_r with respect to the weight $e^{-2kF^*_k\phi}$. The Sobolev norm on this space is given by

$$\|u\|^2_{kF^*_k\phi,s,D_r} = \sum_{\alpha\in\mathbb{N}_0^{2n}, |\alpha|\leq s, |J|=q}{}' \int_{D_r} |\partial^\alpha_x u_J|^2\, e^{-2kF^*_k\phi}(F^*m)(x)dx,$$

where $u = \sum'_{|J|=q} u_J d\overline{z}^J \in W^s_{kF^*_k\phi}(D_r, \Lambda^q T^{*(0,1)}M)$ and $m(x)dx$ is the volume form. If $s = 0$, we write $\|\cdot\|_{kF^*_k\phi,D_r}$ to denote $\|\cdot\|_{kF^*_k\phi,0,D_r}$.

Lemma 4.1. *For every $r > 0$ with $D_{2r} \subset D_{\log k}$ and $s \in \mathbb{N}_0$, there is a constant $C_{r,s} > 0$ independent of k, such that*

$$\|u\|^2_{kF^*_k\phi,2s,D_r} \leqslant C_{r,s}\bigg(\|u\|^2_{kF^*_k\phi,D_{2r}} + \sum_{m=1}^{s}\Big\|(\Box^{(q)}_{k\phi,(k)})^m u\Big\|^2_{kF^*_k\phi,D_{2r}}\bigg)\,,\ u \in \Omega^{0,q}(D_{\log k})\,. \tag{4.3}$$

Proof. Since $\Box^{(q)}_{k\phi,(k)}$ is elliptic, we conclude from Gårding's inequality that for every $r > 0$, $D_{2r} \subset D_{\log k}$ and $s \in \mathbb{N}_0$, we have

$$\|u\|^2_{kF^*_k\phi,s+2,D_r} \leqslant \widetilde{C}_{r',s}\bigg(\|u\|^2_{kF^*_k\phi,D_{r'}} + \Big\|\Box^{(q)}_{k\phi,(k)} u\Big\|^2_{kF^*_k\phi,s,D_{r'}}\bigg)\,,\ u \in \Omega^{0,q}(D_{\log k})\,, \tag{4.4}$$

for some $r' > r$. Since all the derivatives of the coefficients of the operator $\Box^{(q)}_{k\phi,(k)}$ are uniformly bounded in k, it is straightforward to see that $\widetilde{C}_{r',s}$ can be taken to be independent of k. (See Proposition 2.4 and Remark 2.5 in Hsiao-Marinescu [31].) From (4.4) and using induction, we get (4.3). $\square$

Lemma 4.2. *For k large and for every $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_\alpha > 0$ independent of k, such that*

$$|(\partial_x^\alpha u)(0)| \leq C_\alpha, \tag{4.5}$$

where $u \in \Omega^{0,q}(D_{\log k})$, $\|u\|_{kF_k^*\phi, D_{\log k}} \leq 1$, $\left\|(\Box^{(q)}_{k\phi,(k)})^m u\right\|_{kF_k^*\phi, D_{\log k}} \leq k^{-m}$, $m \in \mathbb{N}_0$.

Proof. Let $u \in \Omega^{0,q}(D_{\log k})$, $\|u\|_{kF_k^*\phi, D_{\log k}} \leq 1$, $\left\|(\Box^{(q)}_{k\phi,(k)})^m u\right\|_{kF_k^*\phi, D_{\log k}} \leq k^{-m}$, $m \in \mathbb{N}_0$. By using Fourier transform, it is easy to see that (cf. Lemma 2.6 in [31])

$$|(\partial_x^\alpha u)(0)| \leq C \|u\|_{kF_k^*\phi, n+1+|\alpha|, D_r}, \tag{4.6}$$

for some $r > 0$, where $C > 0$ only depends on the dimension and the length of α. From (4.3), we see that

(4.7)

$$\begin{aligned}\|u\|^2_{kF_k^*\phi, n+1+|\alpha|, D_r} &\leq C_{r,\alpha}\Big(\|u\|^2_{kF_k^*\phi, D_{2r}} + \sum_{m=1}^{N} \left\|(\Box^{(q)}_{k\phi,(k)})^m u\right\|_{kF_k^*\phi, D_{2r}}\Big), \quad 2N \geq n+1+|\alpha|, \\ &\leq C_{r,\alpha}\Big(1 + \sum_{m=1}^{\infty} k^{-m}\Big) \leq \widetilde{C}_\alpha\end{aligned}$$

if k large, where $\widetilde{C}_\alpha > 0$ is independent of k. Combining (4.6) with (4.7), (4.5) follows. $\square$

Now, we can prove

Theorem 4.3. *For k large and for every $\alpha \in \mathbb{N}_0^{2n}$, $D' \Subset D$, there is a constant $C_{\alpha,D'} > 0$ independent of k, such that*

$$\left|(\partial_x^\alpha(\tilde{u}e^{-k\phi}))(x)\right| \leq C_{\alpha,D'} k^{\frac{n}{2}+|\alpha|} \|u\|, \quad \forall x \in D', \tag{4.8}$$

where $u \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$, $N_0 \geq 1$, $u|_D = s^k\tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$.

Remark 4.4. Let s_1 be another local trivializing section of L on D, $|s_1|^2 = e^{-2\phi_1}$. We have $s_1 = gs$ for some holomorphic function $g \in \mathscr{C}^\infty(D)$, $g \neq 0$ on D. Let $u \in \Omega^{0,q}(D, L^k)$. On D, we write $u = s^k\tilde{u} = s_1^k\tilde{v}$. Then, we can check that

$$\tilde{v}e^{-k\phi_1} = \tilde{u}(\overline{g}^{1/2}g^{-1/2})^k e^{-k\phi}. \tag{4.9}$$

From (4.9), it is easy to see that if $\tilde{u}$ satisfies (4.8), then $\tilde{v}$ also satisfies (4.8). Thus, the conclusion of Theorem 4.3 makes sense.

Proof of Theorem 4.3. We may assume that $0 \in D'$. Let $u \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$, $N_0 \geq 1$, $u|_D = s^k\tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$. We may assume that $D_{\frac{\log k}{\sqrt{k}}} \subset D$ and consider $\tilde{u}|_{D_{\frac{\log k}{\sqrt{k}}}}$. Set $\beta_k := k^{-\frac{n}{2}}F_k^*\tilde{u} = k^{-\frac{n}{2}}\tilde{u}(\frac{x}{\sqrt{k}}) \in \Omega^{0,q}(D_{\log k})$. We can check that

$$\|\beta_k\|_{kF_k^*\phi, D_{\log k}} \leq \|u\|. \tag{4.10}$$

Since $u \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$, we have $\left\|(\Box^{(q)}_k)^m u\right\| \leq k^{-mN_0}\left\|u\right\|$, $m = 1, 2, \ldots$. From this observation and (4.2), we have

$$\begin{aligned}\left\|(\Box^{(q)}_{k\phi,(k)})^m \beta_k\right\|_{kF^*_k\phi, D_{\log k}} &= \frac{1}{k^{m+\frac{n}{2}}}\left\|F^*_k\Big((\Box^{(q)}_{k\phi})^m \tilde{u}\Big)\right\|_{kF^*_k\phi, D_{\log k}} \\ &\leq \frac{1}{k^m}\left\|(\Box^{(q)}_k)^m u\right\| \leq k^{-mN_0-m}\left\|u\right\|.\end{aligned} \tag{4.11}$$

From (4.10), (4.11) and Lemma 4.2, we conclude that for every $\alpha \in \mathbb{N}^{2n}_0$, there is a constant $\tilde{C}_\alpha > 0$ independent of k, such that

$$\left|k^{-\frac{n}{2}-\frac{|\alpha|}{2}}(\partial^\alpha_x \tilde{u})(0)\right| = \left|(\partial^\alpha_x \beta_k)(0)\right| \leq \tilde{C}_\alpha \left\|u\right\|.$$

Thus, for every $\alpha \in \mathbb{N}^{2n}_0$, there is a constant $C_\alpha > 0$ independent of k, such that

$$\left|(\partial^\alpha_x(\tilde{u}e^{-k\phi}))(0)\right| \leq C_\alpha k^{\frac{n}{2}+|\alpha|}\left\|u\right\|.$$

Let x_0 be another point of D'. We can repeat the procedure above and conclude that for every $\alpha \in \mathbb{N}^{2n}_0$, there is a $C_\alpha(x_0) > 0$ independent of k, such that

$$\left|(\partial^\alpha_x(\tilde{u}e^{-k\phi}))(x_0)\right| \leq C_\alpha(x_0) k^{\frac{n}{2}+|\alpha|}\left\|u\right\|.$$

It is straightforward to see that the constant $C_\alpha(x_0)$ depends continuously on ϕ and the coeffceients of $\Box^{(q)}_{k\phi,(k)}$ in $\mathscr{C}^m(D)$ topology, for some $m \in \mathbb{N}_0$. (See Remark 2.5 and Theorem 2.7 in [31], for the details.) Since $\overline{D}' \subset D$ is compact, $C_\alpha(x_0)$ can be taken to be independent of the point x_0. The theorem follows. ☐

4.2. **Kernel of the spectral function.** As (1.4), let

$$P^{(q)}_{k,k^{-N_0}} : L^2_{(0,q)}(M, L^k) \to \mathscr{E}^q_{k^{-N_0}}(M, L^k)$$

be the spectral projection on the spectral space of $\Box^{(q)}_k$ corresponding to energy less than k^{-N_0}. The goal of this Section is to compare the localized spectral projection $\hat{P}^{(q)}_{k,k^{-N_0},s}$ (see (4.18)) to the localized approximate Szegö projection $\mathcal{S}_k$ defined in (3.41). This will be achieved in Proposition 4.10.

We introduce some notations. Let $(e_1, \ldots, e_n)$ be a local orthonormal frame of $T^{*(0,1)}_x M$ over an open set $D \Subset M$. Then $(e^J := e_{j_1} \wedge \cdots \wedge e_{j_q})_{1 \leqslant j_1 < j_2 < \cdots < j_q \leqslant n}$ is an orthonormal frame of $\Lambda^q T^{*(0,1)}_x M$ over D. For $f \in \Omega^{0,q}(D)$, we may write $f = \sum'_{|J|=q} f_J e^J$, with $f_J = \langle\, f\,, e^J\,\rangle \in \mathscr{C}^\infty(D)$. We call f_J the component of f along e^J. Let $A : \Omega^{0,q}_0(D) \to \Omega^{0,q}(D)$ be an operator with smooth kernel. We write

$$A(x, y) = \sideset{}{'}\sum_{|I|=q,|J|=q} e^I(x) A_{I,J}(x, y) e^J(y), \tag{4.12}$$

where $A_{I,J}(x, y) \in \mathscr{C}^\infty(D \times D)$, for all strictly increasing I, J, with $|I| = |J| = q$. We have

$$(Au)(x) = \sideset{}{'}\sum_{|I|=q,|J|=q} e^I(x) \int_D A_{I,J}(x, y) u_J(y) dv_M(y), \tag{4.13}$$

for all $u = \sum'_{|J|=q} u_J e^J \in \Omega^{0,q}_0(D)$. Let A^* be the formal adjoint of A with respect to $(\,,\,)$. We also write $A^*(x, y) = \sum'_{|I|=q,|J|=q} e^I(x) A^*_{I,J}(x, y) e^J(y)$. We can check that

$$A^*_{I,J}(x, y) = \overline{A_{J,I}(y, x)}, \tag{4.14}$$

for all strictly increasing I, J, with $|I|=|J|=q$. Let

$$B:\Omega^{0,q}(D)\to\Omega^{0,q}(D),\ \ \Omega^{0,q}_0(D)\to\Omega^{0,q}_0(D),$$
$$B(x,y)=\sum\nolimits'_{|I|=q,|J|=q}e^I(x)B_{I,J}(x,y)e^J(y),$$

be a properly supported smoothing operator. We write

$$(B\circ A)(x,y)=\sum\nolimits'_{|I|=q,|J|=q}e^I(x)(B\circ A)_{I,J}(x,y)e^J(y)$$

in the sense of (4.13). It is not difficult to see that

$$(B\circ A)_{I,J}(x,y)=\sum\nolimits'_{|K|=q}\int_D B_{I,K}(x,z)A_{K,J}(z,y)dv_M(z),\tag{4.15}$$

for all strictly increasing I, J, with $|I|=|J|=q$.

Now, we return to our situation. Let

$$P^{(q)}_{k,\lambda}(x,y)\in C^\infty\Big(M\times M,(\Lambda^qT^{*(0,1)}_yM\otimes L^k_y)\boxtimes(\Lambda^qT^{*(0,1)}_xM\otimes L^k_x)\Big)$$

be the spectral function, i. e., the Schwartz kernel of $P^{(q)}_{k,\lambda}$:

$$(P^{(q)}_{k,\lambda}u)(x)=\int_M P^{(q)}_{k,\lambda}(x,y)u(y)dv_M(y),\ \ u\in L^2_{(0,q)}(M,L^k).\tag{4.16}$$

Let s be a local section of L over D, where $D\subset M$. Then on $D\times D$ we can write

$$P^{(q)}_{k,\lambda}(x,y)=s(x)^kP^{(q)}_{k,\lambda,s}(x,y)s^*(y)^k,$$

where $P^{(q)}_{k,\lambda,s}(x,y)$ is smooth on $D\times D$, so that for $x\in D$, $u\in\Omega^{0,q}_0(D,L^k)$,

$$\begin{aligned}(P^{(q)}_{k,\lambda}u)(x)&=s(x)^k\int_M P^{(q)}_{k,\lambda,s}(x,y)\langle\,u(y)\,,s^*(y)^k\rangle\,dv_M(y)\\&=s(x)^k\int_M P^{(q)}_{k,\lambda,s}(x,y)\tilde u(y)dv_M(y),\ \ u=s^k\tilde u,\ \ \tilde u\in\Omega^{0,q}_0(D).\end{aligned}\tag{4.17}$$

For $x=y$, we can check that the function $P^{(q)}_{k,\lambda,s}(x,x)\in\mathscr{C}^\infty(D,\mathrm{End}(\Lambda^qT^{*(0,1)}M))$ is independent of the choices of local section s.

Let $D\Subset M$ be an open set, s be a local trivializing section of L on D and $|s|^2_{h^L}=e^{-2\phi}$. Let $z=(z_1,\ldots,z_n)=(x_1,\ldots,x_{2n})=x$ be local coordinates of D. Fix $N_0\geq1$. We define the *localized spectral projection* (with respect to the trivializing section s) by

$$\begin{aligned}&\widehat P^{(q)}_{k,k^{-N_0},s}:L^2_{(0,q)}(D)\cap\mathscr{E}'(D,\Lambda^qT^{*(0,1)}M)\to\Omega^{0,q}(D),\\&\qquad\qquad u\to e^{-k\phi}s^{-k}P^{(q)}_{k,k^{-N_0}}(s^ke^{k\phi}u).\end{aligned}\tag{4.18}$$

That is, if $P^{(q)}_{k,k^{-N_0}}(s^ke^{k\phi}u)=s^kv$ on D, then $\widehat P^{(q)}_{k,k^{-N_0},s}u=e^{-k\phi}v$. We notice that

$$\widehat P^{(q)}_{k,k^{-N_0},s}(x,y)=e^{-k\phi(x)}P^{(q)}_{k,k^{-N_0},s}(x,y)e^{k\phi(y)},\tag{4.19}$$

where $\widehat P^{(q)}_{k,k^{-N_0},s}(x,y)$ is the kernel of $\widehat P^{(q)}_{k,k^{-N_0},s}$ with respect to $(\,,\,)$ and $P^{(q)}_{k,k^{-N_0},s}(x,y)$ is as in (4.17). We write

$$\widehat P^{(q)}_{k,k^{-N_0},s}(x,y)=\sum\nolimits'_{|I|=q,|J|=q}e^I(x)\widehat P^{(q)}_{k,k^{-N_0},s,I,J}(x,y)e^J(y)\tag{4.20}$$

in the sense of (4.13). We remind that $\widehat{P}^{(q)}_{k,k^{-N_0},s,I,J}(x,y)$ is smooth for all $|I|=q$, $|J|=q$, I, J are strictly increasing. Since $\widehat{P}^{(q)}_{k,k^{-N_0},s}$ is self-adjoint, we have

$$\widehat{P}^{(q)}_{k,k^{-N_0},s,I,J}(x,y)=\overline{\widehat{P}^{(q)}_{k,k^{-N_0},s,J,I}(y,x)}, \tag{4.21}$$

for all $|I|=q$, $|J|=q$, I, J, are strictly increasing. Let $f_j\in\Omega^{0,q}(M,L^k), j=1,\ldots,d_k$, be an orthonormal frame for $\mathscr{E}^q_{k^{-N_0}}(M,L^k)$, where $d_k\in\mathbb{N}_0\bigcup\{\infty\}$. On D, we may write $f_j|_D=\sum'_{|J|=q}f_{j,J}(x)e^J(x)$, $f_{j,J}\in\mathscr{C}^\infty(D,L^k)$, $j=1,\ldots,d_k$, $|J|=q$, J is strictly increasing. Put

$$\begin{aligned}&f_{j,J}=s^k\widetilde{f}_{j,J},\ \ \widetilde{f}_{j,J}\in\mathscr{C}^\infty(D),\ \ j=1,\ldots,d_k,\ \ |J|=q,\ J\text{ is strictly increasing},\\&\widetilde{f}_j=\sum_{|J|=q}{}'\,\widetilde{f}_{j,J}(x)e^J(x)\in\Omega^{0,q}(D),\ \ j=1,\ldots,d_k.\end{aligned}$$

Then, $f_j|_D=s^k\widetilde{f}_j$, $j=1,\ldots,d_k$, and it is not difficult to see that

$$\widehat{P}^{(q)}_{k,k^{-N_0},s,I,J}(x,y)=\sum_{j=1}^{d_k}\widetilde{f}_{j,I}(x)\overline{\widetilde{f}_{j,J}(y)}e^{-k(\phi(x)+\phi(y))}, \tag{4.22}$$

for all $|I|=q$, $|J|=q$, I, J, are strictly increasing. Since $\widehat{P}^{(q)}_{k,k^{-N_0},s,I,J}(x,y)\in\mathscr{C}^\infty$ for all $|I|=q$, $|J|=q$, I, J, are strictly increasing, we conclude that for all $\alpha\in\mathbb{N}_0^{2n}$,

$$\textstyle\sum_{j=1}^{d_k}\left|(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(x)\right|^2 \text{ converges at each point of } x. \tag{4.23}$$

Similarly, if $F:\mathscr{E}'(D,\Lambda^qT^{*(0,1)}M)\rightarrow\mathscr{E}'(D,\Lambda^qT^{*(0,1)}M)$ is a properly supported continuous operator such that for all $s\in\mathbb{R}$, $F:H^s_{\rm comp}(D,\Lambda^qT^{*(0,1)}M)\rightarrow H^{s+s_0}_{\rm comp}(D,\Lambda^qT^{*(0,1)}M)$ is continuous, for some $s_0\in\mathbb{R}$. Then, we can check that

$$\textstyle\sum_{j=1}^{d_k}\left|(F(\widetilde{f}_je^{-k\phi}))(x)\right|^2 \text{ converges at each point of } x. \tag{4.24}$$

First, we need

Proposition 4.5. *With the notations used above, for every $\alpha\in\mathbb{N}_0^{2n}$, $D'\Subset D$, there is a constant $C_{\alpha,D'}>0$ independent of k, such that*

$$\sum_{j=1}^{d_k}\left|(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(x)\right|^2\leq C_{\alpha,D'}k^{n+2|\alpha|},\ \ \forall x\in D'. \tag{4.25}$$

Proof. Fix $\alpha\in\mathbb{N}_0^{2n}$ and $p\in D'$. We may assume that $\sum_{j=1}^{d_k}\left|(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(p)\right|^2\neq0$. Set

$$u(x)=\frac{1}{\sqrt{\sum_{j=1}^{d_k}\left|(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(p)\right|^2}}\sum_{j=1}^{d_k}f_j(x)\overline{(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(p)}.$$

Since $\sum_{j=1}^{d_k}\left|(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(p)\right|^2$ converges, we can check that $u\in\mathscr{E}^q_{k^{-N_0}}(M,L^k)$, $||u||=1$. On D, we write $u=s^k\widetilde{u}$, $\widetilde{u}\in\Omega^{0,q}(D)$. We can check that

$$\widetilde{u}=\frac{1}{\sqrt{\sum_{j=1}^{d_k}\left|(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(p)\right|^2}}\sum_{j=1}^{d_k}\widetilde{f}_j(x)\overline{(\partial^\alpha_x(\widetilde{f}_je^{-k\phi}))(p)}. \tag{4.26}$$

In view of Theorem 4.3, we see that $\left|(\partial_x^\alpha(\tilde{u}e^{-k\phi}))(p)\right| \leq C_\alpha k^{\frac{n}{2}+|\alpha|}$, $C_\alpha > 0$ is independent of k and the point p. From (4.26), it is straightforward to see that

$$\left|(\partial_x^\alpha(\tilde{u}e^{-k\phi}))(p)\right| = \sqrt{\sum_{j=1}^{d_k}\left|(\partial_x^\alpha(\tilde{f}_j e^{-k\phi}))(p)\right|^2} \leq C_\alpha k^{\frac{n}{2}+|\alpha|}.$$

The proposition follows. □

Now, we assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point of D and let $q = n_-$. Let $\mathcal{S}_k$, $\mathcal{A}_k$ be as in Theorem 3.10 and let $\Box_s^{(q)}$ be as in (3.7), (3.6). If we replace $\mathcal{S}_k$ by $I - \Box_s^{(q)}\mathcal{A}_k$, then $\Box_s^{(q)}\mathcal{A}_k + \mathcal{S}_k = I = \mathcal{A}_k^*\Box_s^{(q)} + \mathcal{S}_k^*$ on $\mathscr{D}'(D, \Lambda^q T^{*(0,1)}M)$. Now,

$$\widehat{P}^{(q)}_{k,k^{-N_0},s} = (\mathcal{A}_k^*\Box_s^{(q)} + \mathcal{S}_k^*)\widehat{P}^{(q)}_{k,k^{-N_0},s} = R + \mathcal{S}_k^*\widehat{P}^{(q)}_{k,k^{-N_0},s} \ \text{ on } \mathscr{E}'(D, \Lambda^q T^{*(0,1)}M), \tag{4.27}$$

where we denote

$$R = \mathcal{A}_k^*\Box_s^{(q)}\widehat{P}^{(q)}_{k,k^{-N_0},s}.$$

We write

$$R(x,y) = \sum\nolimits'_{|I|=q,|J|=q} e^I(x)R_{I,J}(x,y)e^J(y)$$

in the sense of (4.13), where $R_{I,J}(x,y) \in \mathscr{C}^\infty(D\times D)$, for all $|I| = q$, $|J| = q$, I, J, are strictly increasing. From (4.22), it is straightforward to see that

$$\begin{aligned} &R_{I,J}(x,y) = \sum_{j=1}^{d_k}\tilde{g}_{j,I}(x)\tilde{f}_{j,J}(y)e^{-k\phi(y)}, \\ &\tilde{g}_j = \mathcal{A}_k^*\Box_s^{(q)}(\tilde{f}_j e^{-k\phi})(x), \ \ \tilde{g}_j(x) = \sum\nolimits'_{|I|=q}\tilde{g}_{j,I}(x)e^I(x), \ \ j = 1,\ldots,d_k, \end{aligned} \tag{4.28}$$

for all $|I| = q$, $|J| = q$, I, J are strictly increasing. To estimate $R_{I,J}(x,y)$, we first need

Lemma 4.6. *With the notations used above, for every $D' \Subset D$, $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha,D'} > 0$ independent of k, such that for all $u \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$, $\|u\| = 1$, $u|_D = s^k\tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$, if we set $\tilde{v}(x) = \mathcal{A}_k^*\Box_s^{(q)}(\tilde{u}e^{-k\phi})$, then*

$$|(\partial_x^\alpha\tilde{v})(x)| \leq C_{\alpha,D'}k^{\frac{5n}{2}+2|\alpha|-N_0}, \ \ \forall x \in D'.$$

Proof. Let $u \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$, $\|u\| = 1$, $u|_D = s^k\tilde{u}$, $\tilde{u} \in \Omega^{0,q}(D)$. We set $\tilde{v}(x) = \mathcal{A}_k^*\Box_s^{(q)}(\tilde{u}e^{-k\phi})$. We recall that

$$\mathcal{A}_k^* : O(k^s) : H^s_{\rm comp}(D, \Lambda^q T^{*(0,1)}M) \to H^{s+1}_{\rm comp}(D, \Lambda^q T^{*(0,1)}M), \ \ \forall s \in \mathbb{N}_0. \tag{4.29}$$

Let $D' \Subset D'' \Subset D$. By using Fourier transforms, we see that for all $x \in D'$, we have

$$|(\partial_x^\alpha\tilde{v})(x)| \leq C_\alpha\|\tilde{v}\|_{n+1+|\alpha|,D''},$$

where C_α only depends on the dimension and the length of α and $\|.\|_{s,D''}$ denotes the usual Sobolev norm of order s on D''. From this observation and (4.29), we see that

$$|(\partial_x^\alpha\tilde{v})(x)| \leq C_\alpha\|\tilde{v}\|_{n+1+|\alpha|,D''} \leq C'_\alpha k^{n+|\alpha|}\left\|\Box_s^{(q)}(\tilde{u}e^{-k\phi})\right\|_{n+|\alpha|,D''}, \tag{4.30}$$

where $C'_\alpha > 0$ is independent of k. Let $\Box_k^{(q)}u = f$, $f|_D = s^k\tilde{f}$, $\tilde{f} \in \Omega^{0,q}(D)$. We can check that $f \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$ and $\|f\| \leq k^{-N_0}$. From (3.7), we see that

$$\Box_s^{(q)}(e^{-k\phi}\tilde{u}) = e^{-k\phi}\tilde{f}. \tag{4.31}$$

In view of Theorem 4.3, we know that for all $\beta \in \mathbb{N}_0^{2n}$,

$$\left|\partial_x^\beta(\Box_s^{(q)}(e^{-k\phi}\tilde{u}))\right| = \left|\partial_x^\beta(e^{-k\phi}\tilde{f})\right| \leq C_\beta k^{\frac{n}{2}+|\beta|}\|f\| \leq C_\beta k^{\frac{n}{2}+|\beta|-N_0} \text{ on } D,$$

where $C_\beta > 0$ is independent of k. Thus,

$$\left\|\Box_s^{(q)}(e^{-k\phi}\tilde{u})\right\|_{n+|\alpha|,D''} \leq \tilde{C}_\alpha k^{\frac{3n}{2}+|\alpha|-N_0}, \tag{4.32}$$

where $\tilde{C}_\alpha > 0$ is independent of k. Combining (4.32) with (4.30), the lemma follows. □

We also need

Lemma 4.7. *Let $\tilde{g}_j(x) \in \Omega^{0,q}(D)$, $j = 1, \ldots, d_k$, be as in* (4.28). *For every $D' \Subset D$, $\alpha \in \mathbb{N}_0^{2n}$, there is a constant $C_\alpha > 0$ independent of k, such that*

$$\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(x)|^2 \leq C_\alpha k^{5n+4|\alpha|-2N_0}, \ \ \forall x \in D'.$$

Proof. Fix $\alpha \in \mathbb{N}_0^{2n}$ and $p \in D'$. We may assume that $\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2 \neq 0$. Set

$$h(x) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2}} \sum_{j=1}^{d_k} f_j(x)\overline{(\partial_x^\alpha \tilde{g}_j)(p)}.$$

Since $\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2$ converges, we can check that $h \in \mathscr{E}^q_{k^{-N_0}}(M, L^k)$, $\|h\| = 1$. On D, we write $h = s^k\tilde{h}$. We can check that

$$\mathcal{A}_k^*\Box_s^{(q)}(\tilde{h}e^{-k\phi}) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2}} \sum_{j=1}^{d_k} \tilde{g}_j(x)\overline{(\partial_x^\alpha \tilde{g}_j)(p)}.$$

In view of Lemma 4.6, we see that

$$\left|\partial_x^\alpha(\mathcal{A}_k^*\Box_s^{(q)}(\tilde{h}e^{-k\phi}))(p)\right| = \sqrt{\sum_{j=1}^{d_k} |(\partial_x^\alpha \tilde{g}_j)(p)|^2} \leq C_\alpha k^{\frac{5n}{2}+2|\alpha|-N_0},$$

where $C_\alpha > 0$ is independent of k and the point p. The lemma follows. □

Now, we can prove

Proposition 4.8. *With the notations used above, for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha,\beta} > 0$ independent of k, such that*

$$\left|(\partial_x^\alpha \partial_y^\beta R_{I,J})(x, y)\right| \leq C_{\alpha,\beta} k^{3n+2|\alpha|+|\beta|-N_0}, \ \ \forall (x, y) \in D' \times D', \tag{4.33}$$

for all $|I| = q$, $|J| = q$, I, J are strictly increasing, where $R_{I,J}(x, y)$ is as in (4.28).

Proof. Fix $p \in D'$ and $|J| = q$, J is strictly increasing. Let $\alpha, \beta \in \mathbb{N}_0^{2n}$. We may assume that $\sum_{j=1}^{d_k} \left|(\partial_y^\beta(\tilde{f}_{j,J}e^{-k\phi}))(p)\right|^2 \neq 0$. Put

$$u(x) = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left|(\partial_y^\beta(\tilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}} \sum_{j=1}^{d_k} f_j(x)\overline{(\partial_y^\beta(\tilde{f}_{j,J}e^{-k\phi}))(p)}. \tag{4.34}$$

Then, $u \in \mathscr{E}^q_{k-N_0}(M, L^k)$, $\|u\| = 1$. On D, we write $u = s^k\tilde{u}$, $\tilde{u} = \sum'_{|I|=q} \tilde{u}_I e^I$. Put $\tilde{v} = \mathcal{A}^*_k \Box^{(q)}_s(\tilde{u}e^{-k\phi}) = \sum'_{|I|=q} \tilde{v}_I e^I \in \Omega^{0,q}(D)$. It is not difficult to check that

$$\tilde{v} = \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left|(\partial^\beta_y(\tilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}} \sum_{j=1}^{d_k} \tilde{g}_j \overline{(\partial^\beta_y(\tilde{f}_{j,J}e^{-k\phi}))(p)},$$

where $\tilde{g}_j$, $j = 1, \ldots, d_k$, are as in (4.28). In view of Lemma 4.6, we know that $|(\partial^\alpha_x \tilde{v})(x)| \leq C_\alpha k^{\frac{5n}{2}+2|\alpha|-N_0}$, for all $x \in D'$, $C_\alpha > 0$ is independent of k and the point p. In particular,

$$\begin{split}|(\partial^\alpha_x \tilde{v}_I)(x)| &= \frac{1}{\sqrt{\sum_{j=1}^{d_k} \left|(\partial^\beta_y(\tilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}} \left|\sum_{j=1}^{d_k} (\partial^\alpha_x \tilde{g}_{j,I})(x) \overline{(\partial^\beta_y(\tilde{f}_{j,J}e^{-k\phi}))(p)}\right| \\ &\leq C_\alpha k^{\frac{5n}{2}+2|\alpha|-N_0}, \ \ \forall x \in D',\end{split} \tag{4.35}$$

for all $|I| = q$, I is strictly increasing. In view of Proposition 4.5, we see that

$$\sum_{j=1}^{d_k} \left|(\partial^\beta_y(\tilde{f}_j e^{-k\phi}))(p)\right|^2 \leq C_\beta k^{n+2|\beta|},$$

where $C_\beta > 0$ is independent of k and the point p. From this and (4.35), we conclude that

$$\left|(\partial^\alpha_x \partial^\beta_y R_{I,J})(x,p)\right| = \sqrt{\sum_{j=1}^{d_k} \left|(\partial^\beta_y(\tilde{f}_{j,J}e^{-k\phi}))(p)\right|^2}\, |(\partial^\alpha_x \tilde{v}_I)(x)| \leq C_{\alpha,\beta} k^{3n+2|\alpha|+|\beta|-N_0},$$

for all $x \in D'$, $|I| = q$, I is strictly increasing, $C_{\alpha,\beta} > 0$ is independent of k and the point p. The proposition follows. $\square$

From (4.27) and Proposition 4.8, we know that

$$\widehat{P}^{(q)}_{k,k^{-N_0},s} = R + \mathcal{S}^*_k \widehat{P}^{(q)}_{k,k^{-N_0},s},$$

where $R(x,y)$ satisfies (4.33). We have

$$\widehat{P}^{(q)}_{k,k^{-N_0},s}\mathcal{S}_k = (R + \mathcal{S}^*_k \widehat{P}^{(q)}_{k,k^{-N_0},s})\mathcal{S}_k = R\mathcal{S}_k + \mathcal{S}^*_k \widehat{P}^{(q)}_{k,k^{-N_0},s}\mathcal{S}_k. \tag{4.36}$$

Let R^* be the formal adjoint R with respect to $(\,,\,)$. Then,

$$\widehat{P}^{(q)}_{k,k^{-N_0},s} = R^* + \widehat{P}^{(q)}_{k,k^{-N_0},s}\mathcal{S}_k. \tag{4.37}$$

From (4.37) and (4.36), we get

$$\widehat{P}^{(q)}_{k,k^{-N_0},s} = R^* + R\mathcal{S}_k + \mathcal{S}^*_k \widehat{P}^{(q)}_{k,k^{-N_0},s}\mathcal{S}_k. \tag{4.38}$$

We also write

$$R^*(x,y) = \sum_{|I|=q,|J|=q}{}' e^I(x) R^*_{I,J}(x,y) e^J(y).$$

Since $R^*_{I,J}(x,y) = \overline{R_{J,I}(y,x)}$, $R^*(x,y)$ also satisfies (4.33).

Now, we study the kernel of $R\mathcal{S}_k$. We write

$$(R\mathcal{S}_k)(x,y) = \sum_{|I|=q,|J|=q}{}' e^I(x)(R\mathcal{S}_k)_{I,J}(x,y) e^J(y).$$

From (4.15), we know that

$$(RS_k)_{I,J}(x,y)=\sideset{}{'}\sum_{|K|=q}\int_D R_{I,K}(x,z)S_{kK,J}(z,y)dv_M(z), \tag{4.39}$$

for all $|I|=q$, $|J|=q$, I, J are strictly increasing. First, we need

Lemma 4.9. *For every $D'\Subset D$, $\alpha\in\mathbb{N}_0^{2n}$, there is a constant $C_\alpha>0$ independent of k, such that*

$$\sideset{}{'}\sum_{|K|=q}\int_D\left|(\partial_x^\alpha R_{I,K})(x,z)\right|^2dv_M(z)\leq C_\alpha k^{5n+4|\alpha|-2N_0},\ \ x\in D', \tag{4.40}$$

for all $|I|=q$, I is strictly increasing.

Proof. From (4.28), we see that

$$(\partial_x^\alpha R_{I,K})(x,y)=\sum_{j=1}^{d_k}(\partial_x^\alpha\tilde{g}_{j,I})(x)\tilde{f}_{j,K}(y)e^{-k\phi(y)}, \tag{4.41}$$

$\alpha\in\mathbb{N}_0^{2n}$, $|I|=q$, $|K|=q$, I, K are strictly increasing. We claim that

$$\sideset{}{'}\sum_{|K|=q}\int_D\left|(\partial_x^\alpha R_{I,K})(x,y)\right|^2dv_M(y)\leq\sum_{j=1}^{d_k}\left|(\partial_x^\alpha\tilde{g}_{j,I})(x)\right|^2, \tag{4.42}$$

for all $x\in D$, $|I|=q$, I is strictly increasing. Fix $p\in D$ and $|I|=q$, I is strictly increasing. We may assume that $\sum_{j=1}^{d_k}\left|(\partial_x^\alpha\tilde{g}_{j,I})(p)\right|^2\neq0$. Put

$$u(x)=\frac{1}{\sqrt{\sum_{j=1}^{d_k}\left|(\partial_x^\alpha\tilde{g}_{j,I})(p)\right|^2}}\sum_{j=1}^{d_k}(\partial_x^\alpha\tilde{g}_{j,I})(p)f_j(x)\in\mathscr{E}^q_{k^{-N_0}}(M,L^k).$$

We see that $\|u\|=1$. Thus, $\int_D|u|^2\leq1$. On D, we can check that

$$\int_D|u|^2=\frac{1}{\sum_{j=1}^{d_k}\left|(\partial_x^\alpha\tilde{g}_{j,I})(p)\right|^2}\sideset{}{'}\sum_{|K|=q}\int_D\left|\sum_{j=1}^{d_k}(\partial_x^\alpha\tilde{g}_{j,I})(p)\tilde{f}_{j,K}(y)\right|^2e^{-2k\phi(y)}dv_M(y)\leq1. \tag{4.43}$$

From (4.41) and (4.43), we see that

$$\sideset{}{'}\sum_{|K|=q}\int_D\left|(\partial_x^\alpha R_{I,K})(p,y)\right|^2dv_M(y)\leq\sum_{j=1}^{d_k}\left|(\partial_x^\alpha\tilde{g}_{j,I})(p)\right|^2.$$

(4.42) follows. From (4.42) and Lemma 4.7, the lemma follows. □

Fix $\alpha,\beta\in\mathbb{N}_0^{2n}$, $|I|=q$, $|J|=q$, I, J are strictly increasing. From (4.39), we see that

$$\begin{aligned}&\left|\partial_x^\alpha\partial_y^\beta((RS_k)_{I,J})(x,y)\right|\\&=\left|\sideset{}{'}\sum_{|K|=q}\int_D(\partial_x^\alpha R_{I,K})(x,z)(\partial_y^\beta S_{kK,J})(z,y)dv_M(z)\right|\\&\leq\sideset{}{'}\sum_{|K|=q}\Big(\int_D\left|(\partial_x^\alpha R_{I,K})(x,z)\right|^2dv_M(z)\Big)^{\frac12}\Big(\int_D\left|(\partial_y^\beta S_{kK,J})(z,y)\right|^2dv_M(z)\Big)^{\frac12}\\&\leq\Big(\sideset{}{'}\sum_{|K|=q}\int_D\left|(\partial_x^\alpha R_{I,K})(x,z)\right|^2dv_M(z)\Big)^{\frac12}\Big(\sideset{}{'}\sum_{|K|=q}\int_D\left|(\partial_y^\beta S_{kK,J})(z,y)\right|^2dv_M(z)\Big)^{\frac12}.\end{aligned} \tag{4.44}$$

Note that

$$
\begin{aligned}
&\sum_{|K|=q}' \int_D \left|(\partial_y^\beta \mathcal{S}_{kK,J})(z,y)\right|^2 dv_M(z) \\
&= \sum_{|K|=q}' \int_D (\partial_x^\beta \mathcal{S}^*_{kJ,K})(y,z)(\partial_y^\beta \mathcal{S}_{kK,J})(z,y) dv_M(z) \\
&= (\partial_x^\beta \partial_y^\beta (\mathcal{S}_k^* \mathcal{S}_k)_{J,J})(y,y).
\end{aligned}
\tag{4.45}
$$

We notice that $\mathcal{S}_k^*\mathcal{S}_k \equiv \mathcal{S}_k \mod O(k^{-\infty})$. From this observation and the explicit formula of the kernel of $\mathcal{S}_k$ (see (3.61)), we conclude that

$$
\left|(\partial_x^\beta \partial_y^\beta (\mathcal{S}_k^* \mathcal{S}_k)_{J,J})(y,y)\right| \leq C_\beta k^{n+2|\beta|}, \tag{4.46}
$$

locally uniformly on D, for all $|J| = q$, J is strictly increasing, where $C_\beta > 0$ is independent of k. From (4.46), (4.45), (4.44) and Lemma 4.9, we conclude that

$$
\left|(\partial_x^\alpha \partial_y^\beta (R\mathcal{S}_k)_{I,J})(x,y)\right| \leq C_{\alpha,\beta} k^{3n+2|\alpha|+|\beta|-N_0},
$$

locally uniformly on D, where $C_{\alpha,\beta} > 0$ is independent of k. Put

$$
T = R^* + R\mathcal{S}_k.
$$

We write

$$
T(x,y) = \sum_{|I|=q,|J|=q}' e^I(x) T_{I,J}(x,y) e^J(y)
$$

in the sense of (4.13). From (4.38), we know that

$$
\hat{P}^{(q)}_{k,k^{-N_0},s} = T + \mathcal{S}_k^* \hat{P}^{(q)}_{k,k^{-N_0},s} \mathcal{S}_k. \tag{4.47}
$$

From the discussion above, we know that for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha,\beta} > 0$ independent of k such that

$$
\left|(\partial_x^\alpha \partial_y^\beta T_{I,J})(x,y)\right| \leq C_{\alpha,\beta} k^{3n+2|\alpha|+|\beta|-N_0}, \quad \forall (x,y) \in D' \times D', \tag{4.48}
$$

for all $|I| = q$, $|J| = q$, I, J are strictly increasing.

Let T^* be the formal adjoint of T. From (4.47), we see that $T^* = T$. Thus,

$$
\left|(\partial_x^\alpha \partial_y^\beta T_{I,J})(x,y)\right| = \left|\overline{(\partial_x^\alpha \partial_y^\beta T_{I,J})(x,y)}\right| = \left|(\partial_y^\alpha \partial_x^\beta T_{J,I})(y,x)\right| \leq C_{\beta,\alpha} k^{3n+2|\beta|+|\alpha|-N_0}.
$$

Combining this with (4.48), we conclude that for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha,\beta} > 0$ independent of k such that

$$
\left|(\partial_x^\alpha \partial_y^\beta T_{I,J})(x,y)\right| \leq C_{\alpha,\beta} \min\left\{k^{3n+2|\alpha|+|\beta|-N_0}, k^{3n+|\alpha|+2|\beta|-N_0}\right\}, \ \forall (x,y) \in D' \times D', \tag{4.49}
$$

for all $|I| = q$, $|J| = q$, I, J are strictly increasing. Summing up, we get the following.

Proposition 4.10. *Let s be a local trivializing section of L on an open set $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point of D and let $q = n_-$. Fix $N_0 \geq 1$. Let $\mathcal{S}_k$ be the localized approximate Szegö kernel* (3.41) *and let $\hat{P}^{(q)}_{k,k^{-N_0},s}$ be the localized spectral projection* (4.18). *Then,*

$$
\hat{P}^{(q)}_{k,k^{-N_0},s} = T + \mathcal{S}_k^* \hat{P}^{(q)}_{k,k^{-N_0},s} \mathcal{S}_k,
$$

where the distribution kernel of T satisfies (4.49).

4.3. **Asymptotic expansion of the spectral function. Proof of Theorem 1.1.** As before, let s be a local trivializing section of L on an open set $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. Let $z = (z_1, \ldots, z_n) = (x_1, \ldots, x_{2n}) = x$ be local coordinates of D. We assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point of D and until further notice, we assume that $q = n_-$.

For $\lambda \geqslant 0$ we denote by $\mathscr{E}^q_{>\lambda}(M, L^k) \subset L^2_{(0,q)}(M, L^k)$ the spectral space given by the range of $E((\lambda, \infty))$, where E is the spectral measure of $\Box^{(q)}_k$. Let

$$P^{(q)}_{k,>\lambda} : L^2_{(0,q)}(M, L^k) \to \mathscr{E}^q_{>\lambda}(M, L^k)$$

be the orthogonal projection. Consider the localization

$$\begin{aligned}\widehat{P}^{(q)}_{k,>\lambda,s} : L^2_{(0,q)}(D) \cap \mathscr{E}'(D, \Lambda^q T^{*(0,1)} M) &\to L^2_{(0,q)}(D),\\ u &\to e^{-k\phi} s^{-k} P^{(q)}_{k,>\lambda}(s^k e^{k\phi} u).\end{aligned} \tag{4.50}$$

Fix $N_0 \geq 1$. It is well-known that (see Section 2 in Davies [14])

$$L^2_{(0,q)}(M) = \mathscr{E}^q_{k^{-N_0}}(M, L^k) \oplus \mathscr{E}^q_{>k^{-N_0}}(M, L^k)$$

and

$$\|u\| \leq k^{N_0} \left\|\Box^{(q)}_k u\right\|, \ \ \forall u \in \mathscr{E}^q_{>k^{-N_0}}(M, L^k) \cap \operatorname{Dom} \Box^{(q)}_k. \tag{4.51}$$

We have the decomposition

$$u = \widehat{P}^{(q)}_{k,k^{-N_0},s} u + \widehat{P}^{(q)}_{k,>k^{-N_0},s} u, \ \ \forall u \in \Omega^{0,q}_0(D). \tag{4.52}$$

Let $\mathcal{S}_k$ be the localized approximate Szegö kernel (3.41). From the explicit formula of the kernel of $\mathcal{S}_k$ (see (3.61)), we can check that

$$\mathcal{S}^*_k, \mathcal{S}_k = O(k^{n+|s_1|+|s|}) : H^{s_1}_{\mathrm{loc}}(D, \Lambda^q T^{*(0,1)} M) \to H^{s}_{\mathrm{loc}}(D, \Lambda^q T^{*(0,1)} M), \tag{4.53}$$

locally uniformly on D, for all $s, s_1 \in \mathbb{Z}$, $s_1 \leq 0$, $s \geq 0$.

Let $u \in H^{s_1}_{\mathrm{comp}}(D, \Lambda^q T^{*(0,1)} M)$, $s_1 \leq 0$, $s_1 \in \mathbb{Z}$. From (4.52), we have

$$\mathcal{S}_k u = \widehat{P}^{(q)}_{k,k^{-N_0},s} \mathcal{S}_k u + \widehat{P}^{(q)}_{k,>k^{-N_0},s} \mathcal{S}_k u. \tag{4.54}$$

From (4.50) and (4.51), we can check that

$$\begin{aligned}\left\|\widehat{P}^{(q)}_{k,>k^{-N_0},s} \mathcal{S}_k u\right\|_D &\leq \left\|P^{(q)}_{k,>k^{-N_0}}(s^k e^{k\phi}(\mathcal{S}_k u))\right\| \leq k^{N_0} \left\|\Box^{(q)}_k P^{(q)}_{k,>k^{-N_0}}(s^k e^{k\phi}(\mathcal{S}_k u))\right\|\\ &\leq k^{N_0} \left\|\Box^{(q)}_k (s^k e^{k\phi}(\mathcal{S}_k u))\right\| = k^{N_0} \left\|\Box^{(q)}_s (\mathcal{S}_k u)\right\|.\end{aligned} \tag{4.55}$$

Here we used (3.7). In view of Theorem 3.10, we see that $\Box^{(q)}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$. From this observation and (4.55), we conclude that

$$\widehat{P}^{(q)}_{k,>k^{-N_0},s} \mathcal{S}_k = O(k^{-N}) : H^{s_1}_{\mathrm{comp}}(D, \Lambda^q T^{*(0,1)} M) \to H^{0}_{\mathrm{loc}}(D, \Lambda^q T^{*(0,1)} M), \tag{4.56}$$

locally uniformly on D, for all $N \geq 0$, $s_1 \leq 0$, $s_1 \in \mathbb{Z}$. From (4.53) and (4.56), we conclude that

$$\mathcal{S}^*_k \widehat{P}^{(q)}_{k,>k^{-N_0},s} \mathcal{S}_k \equiv 0 \mod O(k^{-\infty}). \tag{4.57}$$

Combining (4.57) with (4.54) and note that $\mathcal{S}^*_k \mathcal{S}_k \equiv \mathcal{S}_k \mod O(k^{-\infty})$, we get

$$\mathcal{S}_k \equiv \mathcal{S}^*_k \widehat{P}^{(q)}_{k,k^{-N_0},s} \mathcal{S}_k \mod O(k^{-\infty}). \tag{4.58}$$

From (4.58) and Proposition 4.10, Theorem 3.11 and Theorem 3.14, we get one of the main results of this work:

Theorem 4.11. *Let s be a local trivializing section of L on an open set $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. We assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) at each point D and let $q = n_-$. Let $z = (z_1, \ldots, z_n)$ be local holomorphic coordinates of D. We write $z = x = (x_1, \ldots, x_{2n})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$. Fix $N_0 \geq 1$. Let $\widehat{P}^{(q)}_{k,k^{-N_0},s}$ be the localized spectral projection (4.18) and let $\widehat{P}^{(q)}_{k,k^{-N_0},s}(x, y) \in \mathscr{C}^\infty(D \times D, \Lambda^q T^{*(0,1)}_y M \boxtimes \Lambda^q T^{*(0,1)}_x M)$ be the distribution kernel of $\widehat{P}^{(q)}_{k,k^{-N_0},s}$. Then, for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha,\beta} > 0$ independent of k, such that*

(4.59)
$$\left|\partial^\alpha_x \partial^\beta_y (\widehat{P}^{(q)}_{k,k^{-N_0},s}(x, y) - \mathcal{S}_k(x, y))\right| \leq C_{\alpha,\beta}\min\left\{k^{3n+2|\alpha|+|\beta|-N_0}, k^{3n+|\alpha|+2|\beta|-N_0}\right\} \text{ on } D' \times D',$$

where

$$\mathcal{S}_k(x, y) = \mathcal{S}_k(z, w) \equiv e^{ik\Psi(z,w)} b(z, w, k) \mod O(k^{-\infty}),$$

with

$$\begin{aligned}
&b(z, w, k) \in S^n_{\rm loc}\left(1; D \times D, \Lambda^q T^{*(0,1)}_w M \boxtimes \Lambda^q T^{*(0,1)}_z M\right),\\
&b(z, w, k) \sim \sum_{j=0}^{\infty} b_j(z, w) k^{n-j} \text{ in } S^n_{\rm loc}\left(1; D \times D, \Lambda^q T^{*(0,1)}_w M \boxtimes \Lambda^q T^{*(0,1)}_z M\right),\\
&b_j(z, w) \in \mathscr{C}^\infty\left(D \times D, \Lambda^q T^{*(0,1)}_w M \boxtimes \Lambda^q T^{*(0,1)}_z M\right), \quad j = 0, 1, \ldots,\\
&b_0(z, z) \text{ is given by } (3.78),
\end{aligned}$$

and $\Psi(z, w) \in \mathscr{C}^\infty(D \times D)$, $\Psi(z, w) = -\overline{\Psi}(w, z)$, $\operatorname{Im}\Psi \geq c\,|z - w|^2$, $c > 0$, $\Psi = 0$ if and only if $z = w$ and for a given point $p \in D$, we may take local holomorphic coordinates $z = (z_1, \ldots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, vanishing at p such that the metric on $T^{(1,0)}M$ is $\sum_{j=1}^n dz_j \otimes d\overline{z}_j$ at p and $\phi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + O(|z|^3)$, in some neighborhood of p, where $\lambda_j \neq 0$, $j = 1, \ldots, n$. Then, near $(0, 0)$, we have

$$\Psi(z, w) = i\sum_{j=1}^n |\lambda_j|\,|z_j - w_j|^2 + i\sum_{j=1}^n \lambda_j(\overline{z}_j w_j - z_j\overline{w}_j) + O(|(z, w)|^3).$$

Moreover, let $Z_j(z) \in \mathscr{C}^\infty$, $j = 1, \ldots, n$, be an orthonormal basis for $T^{(0,1)}_z M$. Then,

$$\sum_{j=1}^n \left(\left(iZ_j\Psi\right)(z, w) + \left(Z_j\phi\right)(z)\right)\left(\left(-i\overline{Z}_j\Psi\right)(z, w) + \left(\overline{Z}_j\phi\right)(z)\right) = O(|z - w|^N), \tag{4.60}$$

locally uniformly on $D \times D$, for all $N \in \mathbb{N}$.

When $q \neq n_-$, from Theorem 3.12, we can repeat the proof of Theorem 4.11 and conclude that

Theorem 4.12. *With the notations used in Theorem 4.11, let $q \neq n_-$. Then, for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}_0^{2n}$, there is a constant $C_{\alpha,\beta} > 0$ independent of k, such that*

$$\left|\partial^\alpha_x \partial^\beta_y (\widehat{P}^{(q)}_{k,k^{-N_0},s}(x, y))\right| \leq C_{\alpha,\beta}\min\left\{k^{3n+2|\alpha|+|\beta|-N_0}, k^{3n+|\alpha|+2|\beta|-N_0}\right\} \text{ on } D' \times D'. \tag{4.61}$$

Proof of Theorem 1.1. Combining Theorem 4.11 and Theorem 4.12, we get (1.7), (1.8) and (1.9). ☐

Remark 4.13. In view of Remark 3.13, we can generalize Theorem 4.11 and Theorem 4.12 with essentially the same proofs to the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M.

4.4. **Asymptotic expansion of the Bergman kernel. Proof of Theorem 1.6.** We are now ready to prove Theorem 1.6. Let $D \Subset M(q)$. Let s be a local trivializing section of L on D and $|s|^2_{h^L} = e^{-2\phi}$. Let $z = (z_1, \ldots, z_n) = (x_1, \ldots, x_{2n}) = x$ be local coordinates of D. Define the *localized Bergman projection* (with respect to s) by

$$\begin{aligned} \widehat{P}^{(q)}_{k,s} : L^2_{(0,q)}(D) \cap \mathscr{E}'(D, \Lambda^q T^{*(0,1)} M) &\to \Omega^{0,q}(D), \\ u &\to e^{-k\phi} s^{-k} P^{(q)}_k (s^k e^{k\phi} u). \end{aligned} \tag{4.62}$$

Let $\widehat{P}^{(q)}_{k,s}(x, y)$ be the distribution kernel of $\widehat{P}^{(q)}_{k,s}$. We have the following

Theorem 4.14. *With the assumptions and notations above, fix $N_0 \geq 1$ and assume that $\Box^{(q)}_k$ has $O(k^{-n_0})$ small spectral gap on D, then for every $D' \Subset D$, $\alpha, \beta \in \mathbb{N}^{2n}_0$, there is a constant $C_{\alpha,\beta} > 0$ independent of k, such that*

$$\begin{aligned} &\left|\partial^\alpha_x \partial^\beta_y (\widehat{P}^{(q)}_{k,k^{-N_0},s}(x, y) - \widehat{P}^{(q)}_{k,s}(x, y))\right| \\ &\quad \leq C_{\alpha,\beta} \min\left\{k^{3n+2|\alpha|+|\beta|-N_0}, k^{3n+|\alpha|+2|\beta|-N_0}\right\} \quad \text{on } D' \times D', \end{aligned} \tag{4.63}$$

where $\widehat{P}^{(q)}_{k,k^{-N_0},s}$ is as in Theorem 4.11.

In particular,

$$\widehat{P}^{(q)}_{k,s} \equiv \mathcal{S}_k \mod O(k^{-\infty})$$

locally uniformly on D, where $\mathcal{S}_k$ is as in Theorem 4.11.

Proof. Let $\mathcal{S}_k$ be as in Theorem 4.11. We can repeat the proof of Proposition 4.10 and conclude that

$$\widehat{P}^{(q)}_{k,k^{-N_0},s} - \widehat{P}^{(q)}_{k,s} = T + \mathcal{S}^*_k \left(\widehat{P}^{(q)}_{k,k^{-N_0},s} - \widehat{P}^{(q)}_{k,s}\right)\mathcal{S}_k, \tag{4.64}$$

where $T \in \mathscr{C}^\infty$ and the distribution kernel $T(x, y)$ of T satisfies (4.49). Let

$$u \in H^m_{\mathrm{comp}}(D, \Lambda^q T^{*(0,1)} M), \ \ m \leq 0.$$

We consider

$$v = s^k e^{k\phi} \mathcal{S}_k u - P^{(q)}_k (s^k e^{k\phi} \mathcal{S}_k u).$$

Since $\mathcal{S}_k$ is a smoothing operator, $v \in \mathscr{C}^\infty(M, L^k)$. Moreover, it is easy to see that $v \perp \mathscr{H}^0(M, L^k)$. We have

$$\Box^{(q)}_k v = s^k e^{k\phi} \Box^{(q)}_s \mathcal{S}_k u. \tag{4.65}$$

From Theorem 3.10, we see that $\Box^{(q)}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$. Combining this with (4.65), we obtain

$$\left\|\Box^{(q)}_k v\right\| \leq C_N k^{-N} \|u\|_m ,$$

for every $N > 0$, where $C_N > 0$ is independent of k. Since $v \perp \mathscr{H}^0(M, L^k)$, from Definition 1.5 we conclude that

$$\|v\| \leq \widetilde{C}_N k^{-N} \|u\|_m ,$$

for every $N > 0$, where $\widetilde{C}_N > 0$ is independent of k. Thus,

$$\mathcal{S}_k - \widehat{P}^{(q)}_{k,s} \mathcal{S}_k = O(k^{-N}) : H^m_{\mathrm{comp}}(D, \Lambda^q T^{*(0,1)} M) \to L^2(D, \Lambda^q T^{*(0,1)} M),$$

for all $N>0$, $s\in\mathbb{R}$, and hence

$$\mathcal{S}_k^*\mathcal{S}_k-\mathcal{S}_k^*\widehat{P}^{(q)}_{k,s}\mathcal{S}_k=O(k^{-N}):H^m_{\rm comp}(D,\Lambda^qT^{*(0,1)}M)\to H^{m+N_1}_{\rm loc}(D,\Lambda^qT^{*(0,1)}M),$$

for all $N,N_1>0$, $s\in\mathbb{R}$. We conclude that

$$\mathcal{S}_k^*\mathcal{S}_k\equiv\mathcal{S}_k^*\widehat{P}^{(q)}_{k,s}\mathcal{S}_k\mod O(k^{-\infty}).$$

From this, (3.59) and (4.59), we obtain

$$\widehat{P}^{(q)}_{k,k^{-N_0},s}=\widetilde{T}+\mathcal{S}_k^*\widehat{P}^{(q)}_{k,s}\mathcal{S}_k,$$

where $\widetilde{T}\in\mathscr{C}^\infty$ and the distribution kernel $\widetilde{T}(x,y)$ of $\widetilde{T}$ satisfies (4.49). From this and Proposition 4.10, we conclude that the distribution kernel of $\mathcal{S}_k^*\Big(\widehat{P}^{(q)}_{k,k^{-N_0},s}-\widehat{P}^{(q)}_{k,s}\Big)\mathcal{S}_k$ satisfies (4.49). Combining this with (4.64), (4.63) follows. ☐

Since Theorem 3.10 and Theorem 4.11 hold in the case when the forms take values in $L^k\otimes E$, for a given holomorphic vector bundle E over M, we can generalize Theorem 4.14 with the same proof to the case when the forms take values $L^k\otimes E$.

4.5. **Calculation of the leading coefficients. Proof of Theorem 1.2.** Now, we prove (1.11) and (1.12). In this Section we assume that $q=0$. First let us review the necessary definitions from Riemannian geometry.

Let ω be as in (1.10). The real two form ω induces a Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle_\omega$ on $\mathbb{C}TM$. The Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle_\omega$ on $\mathbb{C}TM$ induces a Hermitian metric on $\Lambda^{p,q}T^*M$ the bundle of (p,q) forms of M, also denoted by $\langle\,\cdot\,,\cdot\,\rangle_\omega$. For $u\in\Lambda^{p,q}T^*M$, we denote $|u|^2_\omega:=\langle\,u,u\,\rangle_\omega$. In local holomorphic coordinates $z=(z_1,\ldots,z_n)$, put

$$\begin{aligned}\omega&=\sqrt{-1}\sum_{j,k=1}^n\omega_{j,k}dz_j\wedge d\overline{z}_k,\\ \Theta&=\sqrt{-1}\sum_{j,k=1}^n\Theta_{j,k}dz_j\wedge d\overline{z}_k.\end{aligned}\tag{4.66}$$

We notice that $\Theta_{j,k}=\langle\,\frac{\partial}{\partial z_j}\,,\frac{\partial}{\partial z_k}\,\rangle$, $\omega_{j,k}=\langle\,\frac{\partial}{\partial z_j}\,,\frac{\partial}{\partial z_k}\,\rangle_\omega$, $j,k=1,\ldots,n$. Put

$$h=\left(h_{j,k}\right)^n_{j,k=1},\quad h_{j,k}=\omega_{k,j},\quad j,k=1,\ldots,n,\tag{4.67}$$

and $h^{-1}=\left(h^{j,k}\right)^n_{j,k=1}$, h^{-1} is the inverse matrix of h. The complex Laplacian with respect to ω is given by

$$\triangle_\omega=(-2)\sum_{j,k=1}^nh^{j,k}\frac{\partial^2}{\partial z_j\partial\overline{z}_k}.\tag{4.68}$$

We notice that $h^{j,k}=\langle\,dz_j\,,dz_k\,\rangle_\omega$, $j,k=1,\ldots,n$. Put

$$\begin{aligned}V_\omega&:=\det\left(\omega_{j,k}\right)^n_{j,k=1},\\ V_\Theta&:=\det\left(\Theta_{j,k}\right)^n_{j,k=1}\end{aligned}\tag{4.69}$$

and set

$$\begin{aligned}r&=\triangle_\omega\log V_\omega,\\ \widehat{r}&=\triangle_\omega\log V_\Theta.\end{aligned}\tag{4.70}$$

r is called the scalar curvature with respect to ω. Let $R^{\det}_{\Theta}$ be the curvature of the canonical line bundle $K_M = \det T^{*(1,0)}M$ with respect to the real two form Θ. We recall that

$$R^{\det}_{\Theta} = -\overline{\partial}\partial \log V_{\Theta}. \tag{4.71}$$

Let h be as in (4.67). Put $\theta = h^{-1}\partial h = (\theta_{j,k})^n_{j,k=1}$, $\theta_{j,k} \in T^{*(1,0)}M$, $j,k = 1,\ldots,n$. θ is the Chern connection matrix with respect to ω. The Chern curvature with respect to ω is given by

$$\begin{aligned}
&R^{TM}_{\omega} = \overline{\partial}\theta = \left(\overline{\partial}\theta_{j,k}\right)^n_{j,k=1} = (\mathcal{R}_{j,k})^n_{j,k=1} \in \mathscr{C}^{\infty}\Big(M, \Lambda^{1,1}T^*M \otimes \operatorname{End}(T^{(1,0)}M)\Big),\\
&R^{TM}_{\omega}(\overline{U}, V) \in \operatorname{End}(T^{(1,0}M), \ \ \forall U, V \in T^{(1,0)}M,\\
&R^{TM}_{\omega}(\overline{U}, V)\xi = \sum_{j,k=1}^{n} \langle\, \mathcal{R}_{j,k}\,, \overline{U} \wedge V\,\rangle \xi_k \frac{\partial}{\partial z_j}, \ \ \xi = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial z_j}, \ \ U, V \in T^{(1,0)}M.
\end{aligned} \tag{4.72}$$

Set

$$\left|R^{TM}_{\omega}\right|^2_{\omega} := \sum_{j,k,s,t=1}^{n} \left|\langle\, R^{TM}_{\omega}(\overline{e}_j, e_k)e_s\,, e_t\,\rangle_{\omega}\right|^2, \tag{4.73}$$

where $e_1,\ldots,e_n$ is an orthonormal frame for $T^{(1,0)}X$ with respect to $\langle\,\cdot\,,\cdot\,\rangle_{\omega}$. It is straightforward to see that the definition of $\left|R^{TM}_{\omega}\right|^2_{\omega}$ is independent of the choices of orthonormal frames. Thus, $\left|R^{TM}_{\omega}\right|^2_{\omega}$ is globally defined. The Ricci curvature with respect to ω is given by

$$\operatorname{Ric}_{\omega} := -\sum_{j=1}^{n} \langle\, R^{TM}_{\omega}(\cdot, e_j)\cdot\,, e_j\,\rangle_{\omega}, \tag{4.74}$$

where $e_1,\ldots,e_n$ is an orthonormal frame for $T^{(1,0)}M$ with respect to $\langle\,\cdot\,,\cdot\,\rangle_{\omega}$. That is,

$$\langle\, \operatorname{Ric}_{\omega}\,, U \wedge V\,\rangle = -\sum_{j=1}^{n} \langle\, R^{TX}_{\omega}(U, e_j)V\,, e_j\,\rangle_{\omega}, \ \ U, V \in \mathbb{C}TM.$$

$\operatorname{Ric}_{\omega}$ is a global $(1,1)$ form. We can check that

$$\operatorname{Ric}_{\omega} = -\overline{\partial}\partial \log V_{\omega},$$

where V_{ω} is as in (4.69).

Le s be a local trivializing section of L on an open set $D \Subset M(0)$, $|s|^2_{h^L} = e^{-2\phi}$. Let $\mathcal{S}_k$, $b(z,w,k)$, $b_j(z,w)$, $j = 0,1,\ldots$, be as in Theorem 4.11. Fix $p \in D$, we will calculate $b_1(p,p)$ and $b_2(p,p)$. We take local coordinates $z = (z_1,\ldots,z_n) = (x_1,\ldots,x_{2n}) = x$ defined in some neighborhood of p such that

$$\begin{aligned}
&z(p) = 0,\\
&\phi(z) = \sum_{j=1}^{n} \lambda_j\,|z_j|^2 + \phi_1(z),\\
&\phi_1(z) = O(|z|)^4), \ \ \frac{\partial^{|\alpha|+|\beta|}\phi_1}{\partial z^{\alpha}\partial \overline{z}^{\beta}}(0) = 0 \ \text{ if } |\alpha| \leq 1 \text{ or } |\beta| \leq 1, \ \ \alpha, \beta \in \mathbb{N}^n_0,\\
&\Theta(z) = \sqrt{-1}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j + O(|z|).
\end{aligned} \tag{4.75}$$

This is always possible, see Ruan [43]. First, we claim that

$$\overline{\partial}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty}), \tag{4.76}$$

where $\overline{\partial}_s$ is as in (3.6). We notice that $\Box_s^{(0)} \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$. Thus, $\Box_s^{(1)} \overline{\partial}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$. From Theorem 3.12, we know that $\Box_s^{(1)}$ has semi-classical parametrices. Thus, $\overline{\partial}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty})$ so (4.76) follows. Now, we claim that

$$\overline{\partial}_z \Big(i\Psi(z,w) + \phi(z) \Big) \text{ vanishes to infinite order at } z = w. \tag{4.77}$$

We write $w = (w_1, \ldots, w_n) = y = (y_1, \ldots, y_{2n})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n$. We assume that there exist $\alpha_0, \beta_0 \in N_0^{2n}$, $|\alpha_0| + |\beta_0| \geq 1$ and $(z_0, z_0) \in D \times D$, such that

$$\partial_x^{\alpha_0} \partial_y^{\beta_0} \left(\overline{\partial}_z \Big(i\Psi(z,w) + \phi(z) \Big) \right) \Big|_{(z_0,z_0)} = C_{\alpha_0,\beta_0} \neq 0, \tag{4.78}$$

and

$$\partial_x^{\alpha} \partial_y^{\beta} \left(\overline{\partial}_z \Big(i\Psi(z,w) + \phi(z) \Big) \right) \Big|_{(z_0,z_0)} = 0, \text{ if } |\alpha| + |\beta| < |\alpha_0| + |\beta_0|, \ \ \alpha, \beta \in \mathbb{N}_0^{2n}. \tag{4.79}$$

From (4.78), (4.79) and since $b_0(z_0, z_0) \neq 0$, $\Psi(z_0, z_0) = 0$, we can check that

$$\lim_{k \to \infty} k^{-n-1} \, \partial_x^{\alpha_0} \partial_y^{\beta_0} \left(\overline{\partial}_s \Big(e^{ik\Psi(z,w)} b(z,w,k) \Big) \right) \Big|_{(z_0,z_0)} = C_{\alpha_0,\beta_0} b_0(z_0,z_0) \neq 0. \tag{4.80}$$

On the other hand, since $\overline{\partial}_s (e^{ik\Psi(z,w)} b(z,w,k)) \equiv 0 \mod O(k^{-\infty})$, we can check that

$$\lim_{k \to \infty} k^{-n-1} \, \partial_x^{\alpha_0} \partial_y^{\beta_0} \left(\overline{\partial}_s \Big(e^{ik\Psi(z,w)} b(z,w,k) \Big) \right) \Big|_{(z_0,z_0)} = 0. \tag{4.81}$$

We get a contradiction. The claim (4.77) follows. Similarly, we have

$$\partial_w \Big(i\Psi(z,w) + \phi(w) \Big) \text{ vanishes to infinite order at } z = w. \tag{4.82}$$

In particular, we have

$$\overline{\partial}_z \Big(i\Psi(z,0) + \phi(z) \Big) \text{ and } \partial_z \Big(i\Psi(0,z) + \phi(z) \Big) \text{ vanish to infinite order at } z = 0. \tag{4.83}$$

Combining (4.77), (4.82), (4.83) with $\Psi(z,z) = 0$, it is easy to check that for all $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned} i \frac{\partial^{|\alpha|} \Psi(z,0)}{\partial z^\alpha} \Big|_{z=0} &= -\, i \frac{\partial^{|\alpha|} \Psi(0,z)}{\partial z^\alpha} \Big|_{z=0} = \frac{\partial^{|\alpha|} \phi}{\partial z^\alpha}(0) = 0 \ \text{ here we used (4.75)}, \\ i \frac{\partial^{|\alpha|} \Psi(0,z)}{\partial \overline{z}^\alpha} \Big|_{z=0} &= -\, i \frac{\partial^{|\alpha|} \Psi(z,0)}{\partial \overline{z}^\alpha} \Big|_{z=0} = \frac{\partial^{|\alpha|} \phi}{\partial \overline{z}^\alpha}(0) = 0 \ \text{ here we used (4.75)}. \end{aligned} \tag{4.84}$$

From (4.83) and (4.84), we deduce that

$$\begin{aligned} \Psi(z,0) &= i\phi(z) + O(|z|^N), \\ \Psi(0,z) &= i\phi(z) + O(|z|^N), \end{aligned} \tag{4.85}$$

for every $N \in \mathbb{N}_0$.

We claim that

$$\overline{\partial}_z b_j(z,w) \text{ and } \partial_w b_j(z,w) \text{ vanish to infinite order at } z = w, \text{ for all } j = 0, 1, \ldots. \tag{4.86}$$

In view of (4.77), we see that $\overline{\partial}_z (i\Psi(z,w) + \phi(z))$ vanishes to infinite order at $z = w$. From this observation and (4.76), we conclude that

$$e^{ik\Psi(z,w)} \overline{\partial}_z b(z,w,k) = H_k(z,w), \tag{4.87}$$

where $H_k(z,w) \equiv 0 \mod O(k^{-\infty})$. We assume that there exist $\gamma_0, \delta_0 \in \mathbb{N}_0^{2n}$, $|\gamma_0|+|\delta_0| \geq 1$ and $(z_1,z_1) \in D \times D$, such that

$$\partial_x^{\gamma_0}\partial_y^{\delta_0}(\overline{\partial}_z b_0(z,w))\Big|_{(z_1,z_1)} = D_{\gamma_0,\delta_0} \neq 0,$$

and

$$\partial_x^{\gamma}\partial_y^{\delta}(\overline{\partial}_z b_0(z,w))\Big|_{(z_1,z_1)} = 0 \text{ if } |\gamma|+|\delta| < |\gamma_0|+|\delta_0|, \quad \gamma,\delta \in \mathbb{N}_0^{2n}.$$

From (4.87), we have

$$\partial_x^{\gamma_0}\partial_y^{\delta_0}\left(\overline{\partial}_z b(z,w,k)\right)\Big|_{(z_1,z_1)} = \partial_x^{\gamma_0}\partial_y^{\delta_0}\left(e^{-ik\Psi(z,w)}H_k(z,w)\right)\Big|_{(z_1,z_1)}. \tag{4.88}$$

Since $\Psi(z_1,z_1)=0$, we have

$$\lim_{k\to\infty} k^{-n}\ \partial_x^{\gamma_0}\partial_y^{\delta_0}\left(e^{-ik\Psi(z,w)}H_k(z,w)\right)\Big|_{(z_1,z_1)} = 0. \tag{4.89}$$

On the other hand, we can check that

$$\lim_{k\to\infty} k^{-n}\ \partial_x^{\gamma_0}\partial_y^{\delta_0}\left(\overline{\partial}_z b(z,w,k)\right)\Big|_{(z_1,z_1)} = D_{\gamma_0,\delta_0} \neq 0. \tag{4.90}$$

From (4.90), (4.89) and (4.88), we get a contradiction. Thus, $\overline{\partial}_z b_0(z,w)$ vanishes to infinite order at $z=w$. Similarly, we can repeat the procedure above and conclude that $\overline{\partial}_z b_j(z,w)$ and $\partial_w b_j(z,w)$ vanish to infinite order at $z=w$, $j=0,1,\ldots$. The claim (4.86) follows.

Now, we are ready to calculate $b_1(0,0)$ and $b_2(0,0)$. We notice that

$$b_0(z,z) = (2\pi)^{-n}\det \dot{R}^L(z).$$

From this and (4.86), it is easy to see that for all $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned} \frac{\partial^{|\alpha|} b_0(z,0)}{\partial z^\alpha}\Big|_{z=0} &= (2\pi)^{-n}\ \frac{\partial^{|\alpha|}(\det \dot{R}^L(z))}{\partial z^\alpha}\Big|_{z=0}, \\ \frac{\partial^{|\alpha|} b_0(z,0)}{\partial \overline{z}^\alpha}\Big|_{z=0} &= 0. \end{aligned} \tag{4.91}$$

Since $\mathcal{S}_k \circ \mathcal{S}_k \equiv \mathcal{S}_k \mod O(k^{-\infty})$, we have

$$b(0,0,k) = \int_D e^{ik(\Psi(0,z)+\Psi(z,0))} b(0,z,k)b(z,0,k)V_\Theta(z)d\lambda(z) + r_k, \tag{4.92}$$

where $d\lambda(z) = 2^n dx_1 dx_2 \cdots dx_{2n}$, V_Θ is given by (4.69) and

$$\lim_{k\to\infty} \frac{r_k}{k^N} = 0, \quad \forall N \geq 0.$$

We notice that since $b(z,w,k)$ is properly supported, we have

$$b(0,z,k) \in C_0^\infty(D), \quad b(z,0,k) \in C_0^\infty(D).$$

We apply the stationary phase formula of Hörmander (see Theorem 7.7.5 in [28]) to the integral in (4.92) and obtain (see Section 4 in Hsiao [30], for the details)

Theorem 4.15. *We have*

$$b_1(0,0) = (2\pi)^n(\det \dot{R}^L(0))^{-1}\Big(2b_0(0,0)b_1(0,0) + \frac{1}{2}\triangle_0\Big(V_\Theta b_0(0,z)b_0(z,0)\Big)(0) - \frac{1}{4}\triangle_0^2\Big(\phi_1 V_\Theta b_0(0,z)b_0(z,0)\Big)(0)\Big) \tag{4.93}$$

and

$$\begin{aligned} b_2(0,0) = (2\pi)^n(\det \dot{R}^L(0))^{-1}\Big(&2b_0(0,0)b_2(0,0) + b_1(0,0)^2 \\ &+ \frac{1}{2}\triangle_0\Big(V_\Theta(b_0(0,z)b_1(z,0) + b_1(0,z)b_0(z,0))\Big)(0) \\ &- \frac{1}{4}\triangle_0^2\Big(\phi_1 V_\Theta(b_0(0,z)b_1(z,0) + b_1(0,z)b_0(z,0))\Big)(0) \\ &+ \frac{1}{8}\triangle_0^2\Big(V_\Theta b_0(0,z)b_0(z,0)\Big)(0) - \frac{1}{24}\triangle_0^3\Big(\phi_1 V_\Theta b_0(0,z)b_0(z,0)\Big)(0) \\ &+ \frac{1}{192}\triangle_0^4\Big(\phi_1^2 V_\Theta b_0(0,z)b_0(z,0)\Big)(0)\Big), \end{aligned} \tag{4.94}$$

where $\triangle_0 = \sum_{j=1}^n \frac{1}{\lambda_j}\frac{\partial^2}{\partial \overline{z}_j \partial z_j}$, ϕ_1 *is as in* (4.75) *and* V_Θ *is as in* (4.69).

From (4.91) and (4.93), it is straightforward to see that (see Section 4.2 in [30], for the details)

$$\begin{aligned} b_1(0,0) &= (2\pi)^{-n}\det \dot{R}^L(0)\Big(\frac{1}{4\pi}\widehat{r}(0) - \frac{1}{8\pi}r(0)\Big) \\ &= \frac{V_\omega(0)}{V_\Theta(0)}\Big(\frac{1}{4\pi}\Big(\triangle_\omega \log V_\Theta\Big)(0) - \frac{1}{8\pi}\Big(\triangle_\omega \log V_\omega\Big)(0)\Big), \end{aligned} \tag{4.95}$$

where $\widehat{r}$ and r are as in (4.70) and V_ω is as in (4.69). From this, (1.11) follows.

Similarly, from (1.11) and (4.86), it is easy to see that for all $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned} &\frac{\partial^{|\alpha|}b_1(z,0)}{\partial z^\alpha}\Big|_{z=0} = (2\pi)^{-n}\,\frac{\partial^{|\alpha|}\Big(\det \dot{R}^L(z)\big(\frac{1}{4\pi}\widehat{r}(z) - \frac{1}{8\pi}r(z)\big)\Big)}{\partial z^\alpha}\Bigg|_{z=0}, \\ &\frac{\partial^{|\alpha|}b_1(z,0)}{\partial \overline{z}^\alpha}\Big|_{z=0} = 0. \end{aligned} \tag{4.96}$$

From (4.96) and (4.94), it is straightforward to see that (see Section 4.3 in [30], for the details)

$$\begin{aligned} b_2(0,0) = (2\pi)^{-n}\det \dot{R}^L(0)\Big(&\frac{1}{128\pi^2}r^2 - \frac{1}{32\pi^2}r\widehat{r} + \frac{1}{32\pi^2}(\widehat{r})^2 - \frac{1}{32\pi^2}\triangle_\omega\widehat{r} - \frac{1}{8\pi^2}\left|R_\Theta^{\det}\right|_\omega^2 \\ &+ \frac{1}{8\pi^2}\langle\, \mathrm{Ric}_\omega\,, R_\Theta^{\det}\,\rangle_\omega + \frac{1}{96\pi^2}\triangle_\omega r - \frac{1}{24\pi^2}\left|\mathrm{Ric}_\omega\right|_\omega^2 + \frac{1}{96\pi^2}\left|R_\omega^{TM}\right|_\omega^2\Big)(0), \end{aligned} \tag{4.97}$$

where $\triangle_\omega$, $R_\Theta^{\det}$, Ric_ω and R_ω^{TM} are as in (4.68), (4.71), (4.74) and (4.72) respectively, and $\langle\,\cdot,\cdot\,\rangle_\omega$ is as in the discussion before (4.66). From (4.97), (1.12) follows.

5. Asymptotic upper bounds near the degeneracy set

In this Section, we will use the heat equation expansion for $\Box_k^{(q)}$ of Ma-Marinescu [36, § 1.6] to get an asymptotic upper bound near the degenerate part of L. The goal of this Section is to prove (1.13).

By the spectral theorem (see Davies [14, Th. 2.5.1]), there exists a finite measure μ on $\mathbb{S}\times\mathbb{N}$, where $\mathbb{S}$ denotes the spectrum of $\Box_k^{(q)}$, and a unitary operator

$$U : L^2_{(0,q)}(M, L^k) \to L^2(\mathbb{S}\times\mathbb{N}, d\mu) \tag{5.1}$$

with the following properties. If $h : \mathbb{S}\times\mathbb{N} \to \mathbb{R}$ is the function $h(s,n) = s$, then the element ξ of $L^2_{(0,q)}(M, L^k)$ lies in $\operatorname{Dom}\Box_k^{(q)}$ if and only if $hU(\xi) \in L^2$. We have $U\Box_k^{(q)}U^{-1}\varphi = h\varphi$ for all $\varphi \in U(\operatorname{Dom}\Box_k^{(q)})$.

We identify $L^2_{(0,q)}(M, L^k)$ with $L^2(\mathbb{S}\times\mathbb{N}, d\mu)$. Then the heat operator $e^{-t\Box_k^{(q)}}$, $t > 0$, is the operator on $L^2(\mathbb{S}\times\mathbb{N}, d\mu)$ given by

$$\begin{aligned} e^{-t\Box_k^{(q)}} &: L^2(\mathbb{S}\times\mathbb{N}, d\mu) \to L^2(\mathbb{S}\times\mathbb{N}, d\mu) \\ u(s,n) &\in L^2(\mathbb{S}\times\mathbb{N}, d\mu) \to e^{-st}u(s,n). \end{aligned}$$

Since $\Box_k^{(q)}$ is elliptic, the distribution kernel of $e^{-t\Box_k^{(q)}}$ is smooth (see [36, Th. D.1.2]). Let

$$\exp(-t\,\Box_k^{(q)})(x,y) \in \mathscr{C}^\infty(M\times M, L^k_y\otimes\Lambda^qT^{*(0,1)}_yM \boxtimes L^k_x\otimes\Lambda^qT^{*(0,1)}_xM)$$

be the distribution kernel of $e^{-t\Box_k^{(q)}}$ with respect to $(\cdot\,,\cdot)_k$. That is,

$$(e^{-t\Box_k^{(q)}}u)(x) = \int_M \exp(-t\,\Box_k^{(q)})(x,y)u(y)dv_M(y), \quad u\in L^2_{(0,q)}(M,L^k).$$

Let s be a local section of L over $\widetilde{X}$, where $\widetilde{X}\subset M$. Then on $\widetilde{X}\times\widetilde{X}$ we can write

$$\exp(-t\,\Box_k^{(q)})(x,y) = \exp(-t\,\Box_k^{(q)})_s(x,y)s(x)^k s^*(y)^k,$$

where $\exp(-t\,\Box_k^{(q)})_s(x,y) \in \mathscr{C}^\infty(\widetilde{X}\times\widetilde{X}, \Lambda^qT^*_yM\boxtimes\Lambda^qT^*_xM)$ so that for $x\in\widetilde{X}$, $u\in\Omega_0^{0,q}(\widetilde{X}, L^k)$,

$$\begin{aligned} (e^{-t\Box_k^{(q)}}u)(x) &= s(x)^k\int_M \exp(-t\,\Box_k^{(q)})_s(x,y)\langle\, u(y)\,, s^*(y)^k\,\rangle dv_M(y) \\ &= s(x)^k\int_M \exp(-t\,\Box_k^{(q)})_s(x,y)\tilde{u}(y)dv_M(y), \quad u = s^k\tilde{u},\ \ \tilde{u}\in\Omega_0^{0,q}(\widetilde{X}). \end{aligned} \tag{5.2}$$

For $x = y$, we can check that the function

$$\exp(-t\,\Box_k^{(q)})_s(x,x) \in \mathscr{C}^\infty(\widetilde{X}, \Lambda^qT^*_xM\boxtimes\Lambda^qT^*_xM)$$

is independent of the choices of local section s. The trace of $\exp(-t\,\Box_k^{(q)})(x,x)$ is given by

$$\operatorname{Tr}\,\exp(-t\,\Box_k^{(q)})(x,x) := \sum_{j=1}^d\langle\, \exp(-t\,\Box_k^{(q)})(x,x)e_{J_j}(x)\,, e_{J_j}(x)\,\rangle,$$

where $e_{J_1}(x),\ldots,e_{J_d}(x)$, is an orthonormal basis of $\Lambda^qT^{*(0,1)}_xM$, $\dim\Lambda^qT^{*(0,1)}_xM = d$. First, we need

Proposition 5.1. *Fix $t > 0$ and $N_0 \geq 1$. We have for k large,*

$$\mathrm{Tr}\,\exp(-\frac{t}{k}\Box_k^{(q)})(x,x) \geq (1-k^{-N_0})\mathrm{Tr}\,P^{(q)}_{k,k^{-N_0}}(x,x),\quad \forall x\in M, \tag{5.3}$$

where $P^{(q)}_{k,k^{-N_0}}(x,x)$ is as in (1.5).

Proof. First, we claim that

$$(e^{-\frac{t}{k}\Box_k^{(q)}}u,u)_k \geq (1-k^{-N_0})(P^{(q)}_{k,k^{-N_0}}u,u)_k,\quad \forall u\in\Omega^{0,q}_0(M,L^k). \tag{5.4}$$

We identify $L^2_{(0,q)}(M,L^k)$ with $L^2(\mathbb{S}\times\mathbb{N},d\mu)$. Then

$$e^{-\frac{t}{k}\Box_k^{(q)}} : u(s,n)\in L^2(\mathbb{S}\times\mathbb{N},d\mu)\to e^{-s\frac{t}{k}}u(s,n)$$

and

$$P^{(q)}_{k,k^{-N_0}} : u(s,n)\in L^2(\mathbb{S}\times\mathbb{N},d\mu)\to u(s,n)1_{[0,k^{-N_0}]}(s).$$

For $u(s,n)\in L^2(\mathbb{S}\times\mathbb{N},d\mu)$, we have

$$\begin{aligned}(e^{-\frac{t}{k}\Box_k^{(q)}}u,u)_k &= \int_{\mathbb{S}\times\mathbb{N}} e^{-s\frac{t}{k}}\left|u(s,n)\right|^2 d\mu\\ &\geq \int_{\mathbb{S}\times\mathbb{N}} e^{-s\frac{t}{k}}\left|u(s,n)\right|^2 1_{[0,k^{-N_0}]}(s)d\mu\\ &\geq \int_{\mathbb{S}\times\mathbb{N}} \left|u(s,n)\right|^2 1_{[0,k^{-N_0}]}(s)d\mu\\ &\quad -\int_{\mathbb{S}\times\mathbb{N}}\left|e^{-s\frac{t}{k}}-1\right|\left|u(s,n)\right|^2 1_{[0,k^{-N_0}]}(s)d\mu\\ &\geq\Big(1-\sup_{s\in[0,k^{-N_0}]}(1-e^{-s\frac{t}{k}})\Big)(P^{(q)}_{k,k^{-N_0}}u,u)_k.\end{aligned} \tag{5.5}$$

It is easy to see that fix $t>0$, we have $\sup_{s\in[0,k^{-N_0}]}(1-e^{-s\frac{t}{k}})\leq k^{-N_0}$ if k large. From this observation and (5.5), the claim (5.4) follows.

Now, fix $p\in M$ and let s be a local section of L defined in some open neighborhood D of p, $|s|^2_{h^L}=e^{-2\phi}$. Let $e_{J_1}(p),\ldots,e_{J_d}(p)$, be an orthonormal basis of $\Lambda^qT^{*(0,1)}_pM$. Fix $u\in\{1,\ldots,d\}$. Take $\chi_j\in\Omega^{0,q}_0(D,L^k)$, $j=1,2,\ldots$, so that for every continuous operator $F:\mathscr{C}^\infty(D,L^k\otimes\Lambda^qT^{*(0,1)}M)\to\mathscr{C}^\infty(D,L^k\otimes\Lambda^qT^{*(0,1)}M)$ with smooth kernel $F(x,y)\in\mathscr{C}^\infty(M\times M,L^k_y\otimes\Lambda^qT^{*(0,1)}_yM\boxtimes L^k_x\otimes\Lambda^qT^{*(0,1)}_xM)$, we have

$$(F\chi_j,\chi_j)_k\to\langle F(p,p)e_{J_u}(p),e_{J_u}(p)\rangle,\quad j\to\infty.$$

Then, we have

$$\begin{aligned}&(e^{-\frac{t}{k}\Box_k^{(q)}}\chi_j,\chi_j)_k\to\langle\exp(-\frac{t}{k}\Box_k^{(q)})(p,p)e_{J_u}(p),e_{J_u}(p)\rangle,\quad j\to\infty,\\ &(P^{(q)}_{k,k^{-N_0}}\chi_j,\chi_j)_k\to\langle P^{(q)}_{k,k^{-N_0}}(p,p)e_{J_u}(p),e_{J_u}(p)\rangle,\quad j\to\infty.\end{aligned}$$

Combining this with (5.4), we conclude that

$$\langle\exp(-\frac{t}{k}\Box_k^{(q)})(p,p)e_{J_u}(p),e_{J_u}(p)\rangle\geq(1-k^{-N_0})\langle P^{(q)}_{k,k^{-N_0}}(p,p)e_{J_u}(p),e_{J_u}(p)\rangle.$$

Thus,

$$\mathrm{Tr}\,\exp(-\frac{t}{k}\Box_k^{(q)})(p,p)\geq(1-k^{-N_0})\mathrm{Tr}\,P^{(q)}_{k,k^{-N_0}}(p,p).$$

(5.3) follows. ∎

The following is well-known (see Theorem 1.6.1 in Ma-Marinescu [36])

Theorem 5.2. *For each $t > 0$ fixed and any $D \Subset M$, $m \in \mathbb{N}$, we have as $k \to \infty$,*

(5.6)
$$\mathrm{Tr}\,\exp(-\frac{t}{k}\Box_k^{(q)})(x,x) = k^n(2\pi)^{-n}\Big(\sum_{j_1<j_2<\cdots<j_q}\exp(-t\sum_{i=1}^q a_{j_i}(x))\Big)\prod_{j=1}^n\frac{a_j(x)}{1-e^{-ta_j(x)}} + o(k^n),$$

in the $\mathscr{C}^m$ norm on $\mathscr{C}^\infty(D, \Lambda^q T^{*(0,1)}M \boxtimes \Lambda^q T^{*(0,1)}M)$, where $a_1(x), \ldots, a_n(x)$ are the eigenvalues of $\dot{R}^L(x)$. Here we use the convention that if $a_j(x) = 0$, then $\frac{a_j(x)}{1-e^{-ta_j(x)}} := \frac{1}{t}$.

From (5.3) and (5.6), we know that

(5.7)
$$(1-k^{-N_0})\mathrm{Tr}\,P^{(q)}_{k,k^{-N_0}}(x,x) \leq k^n(2\pi)^{-n}\Big(\sum_{j_1<j_2<\cdots<j_q}\exp(-t\sum_{i=1}^q a_{j_i}(x))\Big)\prod_{j=1}^n\frac{a_j(x)}{1-e^{-ta_j(x)}} + o(k^n),$$

locally uniformly on M.

Now, let $M_{\deg}$ be as in Theorem 1.3. Fix $t > 1$, t large and $x_0 \in M_{\deg}$ and let U be a small neighborhood of x_0 such that for every point $x \in U$, there is an eigenvalue $a_0(x)$ of $\dot{R}^L(x)$ such that $|ta_0(x)| < 1$. Fix $p \in U$. Set

$$\iota(p) = \Big\{j \in \{1,\ldots,n\};\ |a_j(p)t| < 1,\ \ \text{where } a_1(p),\ldots,a_n(p) \text{ are the eigenvalues of } \dot{R}^L(p)\Big\}.$$

Fix $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. We have

$$\begin{aligned}\exp(-t\sum_{i=1}^q a_{j_i}(p))\prod_{j=1}^n\frac{a_j(p)}{1-e^{-ta_j(p)}} &= \prod_{j_i\in\iota(p)}\frac{e^{-ta_{j_i}(p)}a_{j_i}(p)}{1-e^{-ta_{j_i}(p)}}\prod_{j_i\notin\iota(p)}\frac{e^{-ta_{j_i}(p)}a_{j_i}(p)}{1-e^{-ta_{j_i}(p)}}\\ &\times\prod_{j\in\iota(p),j\notin\{j_1,\ldots,j_q\}}\frac{a_j(p)}{1-e^{-ta_j(p)}}\prod_{j\notin\iota(p),j\notin\{j_1,\ldots,j_q\}}\frac{a_j(p)}{1-e^{-ta_j(p)}}.\end{aligned}\tag{5.8}$$

We observe that there is a constant $C > 0$ such that

$$\begin{aligned}&\left|\frac{x}{1-e^x}\right| \leq C,\quad \left|\frac{xe^x}{1-e^x}\right| \leq C,\quad \forall x \in \mathbb{R},\ |x| \leq 1,\\ &\left|\frac{1}{1-e^x}\right| \leq C,\quad \left|\frac{e^x}{1-e^x}\right| \leq C,\quad \forall x \in \mathbb{R},\ |x| > 1.\end{aligned}\tag{5.9}$$

From (5.9) and (5.8), it is straightforward to see that

$$\exp(-t\sum_{i=1}^q a_{j_i}(p))\prod_{j=1}^n\frac{a_j(p)}{1-e^{-ta_j(p)}} \leq \prod_{j\in\iota(p)}\frac{C}{t}\prod_{j\notin\iota(p)}C\,|a_j(p)|\,,\tag{5.10}$$

where C is the constant as in (5.9).

The proof of (1.13). Let $\varepsilon > 0$. Let $W \Subset M$ be any open set of x_0. Take $t > \max\{C, 1\}$ large enough so that

$$(2\pi)^{-n}d\frac{C}{t}\Big(1 + C\sup\Big\{|a(x)|\,;\ a(x)\text{: eigenvalue of } \dot{R}^L(x),\ x \in W\Big\}\Big)^n < \frac{\varepsilon}{2},\tag{5.11}$$

where C is the constant as in (5.9) and $d = \dim\Lambda^q T_x^{*(0,1)}M$. Let $U \Subset W$ be a small neighborhood of x_0 such that for every point $x \in U$, there is an eigenvalue $a_0(x)$ of $\dot{R}^L(x)$ such that $|ta_0(x)| < 1$. From (5.10), (5.7) and (5.11), we see that

$$\mathrm{Tr}\,P^{(q)}_{k,k^{-N_0}}(x,x) \leq \frac{1}{1-k^{-N_0}}\frac{\varepsilon}{2}k^n + o(k^n),\quad \forall x \in U.$$

(1.13) follows. □

Theorem 5.2 also holds on the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M. In this case the right side of (5.6) gets multiplied by $\mathrm{rank}(E)$. (See Theorem 1.61 in [36]). From this observation, (1.13) remains true with the same proof on the case when the forms take values in $L^k \otimes E$, for a given holomorphic vector bundle E over M.

6. Bergman kernel asymptotics for adjoint semi-positive line bundles

In this Section we prove the asymptotic expansion of the Bergman kernel of $L^k \otimes K_M$ where L is a semi-positive line bundle over a complete Kähler manifold and K_M is its canonical line bundle, cf. Theorem 1.7. The existence of the expansion (1.19) follows immediately from Theorem 6.4, while the calculation of the coefficients is given at the end of this Section.

We assume that (M, Θ) is a complete Kähler manifold. Let K_M be the canonical line bundle over M. Then, $\Omega^{n,q}(M, L^k) = \Omega^{0,q}(M, L^k \otimes K_M)$. Let $\Box^{(0)}_{k,K_M}$ be the Gaffney extension of the Kodaira Laplacian acting on $L^k \otimes K_M$. Then

$$\operatorname{Ker} \Box^{(0)}_{k,K_M} = \mathscr{H}^0(M, L^k \otimes K_M) = \left\{ u \in L^2(M, L^k \otimes K_M);\ \overline{\partial}_k u = 0 \right\}.$$

Set

$$P^{(0)}_{k,K_M} : L^2(M, L^k \otimes K_M) \to \mathscr{H}^0(M, L^k \otimes K_M)$$

be the orthogonal projection with respect to $(\cdot\,,\cdot)_k$. The goal of this Section is to prove that the kernel of $P^{(0)}_{k,K_M}$ admits a full asymptotic expansion on the non-degenerate part of L. We recall the following form of the L^2-estimates for $\overline{\partial}$ for semi-positive line bundles. Assume that (L, h) is a semi-positive Hermitian line bundle over a complex manifold M. Let $g \in \Lambda^{n,0} T^*M \otimes L$. For $x \in M$, we denote by $|g|_{R^L}(x) \in [0, \infty]$ the smallest constant such that $\langle g, g' \rangle^2(x) \leq |g|^2_{R^L}(x) \langle \sqrt{-1} R^L \wedge (\Theta \wedge)^* g', g' \rangle(x)$ for all $g' \in \Lambda^{n,0} T^*M \otimes L$.

Theorem 6.1 ([15, Th. 4.1]). *Let (M, Θ) be a complete Kähler manifold, (L, h^L) be a semi-positive Hermitian line bundle over M. Then for any form $g \in L^2_{(0,1)}(M, L \otimes K_M)$ satisfying $\overline{\partial} g = 0$ and $\int_M |g|^2_{R^L}(x)\, dv_M(x) < \infty$ there exists $f \in L^2_{(0,0)}(M, L \otimes K_M)$ with $\overline{\partial} f = g$ and*

$$\int_M |f|^2_{h^L}(x)\, dv_M(x) \leq \int_M |g|^2_{R^L}(x)\, dv_M\,.$$

Denote by $\gamma(x)$ the smallest eigenvalue of the curvature $\sqrt{-1} R^L_x$ with respect to Θ_x, for $x \in M$; the function $\gamma : M \to [0, \infty)$ is continuous. Moreover, $|g|^2_{R^L}(x) \leq \gamma^{-1}(x) |g|^2_{h^L}(x)$, for any $x \in M$ and $g \in \Lambda^{n,1} T^*M \otimes L$ (where $\gamma^{-1} := \infty$ if $\gamma = 0$). Therefore we deduce the following.

Theorem 6.2. *Let (M, Θ) be a complete Kähler manifold and (L, h^L) be a smooth semi-positive line bundle over M. Let $D \Subset M(0)$ be a relatively compact open set. There exists a constant $C_D > 0$ such that for any $k > 0$ and any $g \in \Omega^{0,1}_0(D, L^k \otimes K_M)$ satisfying $\overline{\partial}_k g = 0$ there exists $f \in \mathscr{C}^\infty(M, L^k \otimes K_M)$ such that $\overline{\partial}_k f = g$ and*

$$\|f\|^2 \leq \frac{C_D}{k} \|g\|^2\,. \tag{6.1}$$

We can actually take $C_D = \sup_D \gamma^{-1}$. We need

Lemma 6.3. *Let (M,Θ) be a complete Kähler manifold and (L,h^L) be a smooth semi-positive line bundle over M. Let $D \Subset M(0)$ be a relatively compact open set. Then $\Box^{(0)}_{k,K_M}$ has $O(k^{-n_0})$ small spectral on D.*

Proof. Let $u \in \mathscr{C}^\infty_0(D, L^k \otimes K_M)$. We consider $\overline{\partial}_k u \in \Omega^{0,1}(D, L^k \otimes K_M)$. From Theorem 6.2, we know that there exists $f \in \mathscr{C}^\infty(M, L^k \otimes K_M)$ such that $\overline{\partial}_k f = \overline{\partial}_k u$ and

$$\|f\|^2 \leq \frac{C_D}{k}\left\|\overline{\partial}_k u\right\|^2, \tag{6.2}$$

where $C_D > 0$ is independent of k and u. We notice that $(I - P^{(0)}_{k,K_M})u$ has minimal L^2 norm of the set $\left\{f \in \mathscr{C}^\infty(D, L^k \otimes K_M) \bigcap L^2(M, L^k \otimes K_M);\ \overline{\partial}_k f = \overline{\partial}_k u\right\}$. From this observation and (6.2), we conclude that

$$\left\|(I - P^{(0)}_{k,K_M})u\right\|^2 \leq \frac{C_D}{k}\left\|\overline{\partial}_k u\right\|^2. \tag{6.3}$$

It is easy to check that

$$\left\|\overline{\partial}_k u\right\|^2 \leq \left\|\Box^{(0)}_{k,K_M} u\right\| \left\|(I - P^{(0)}_{k,K_M})u\right\|.$$

Combining this with (6.3), we get $\left\|(I - P^{(0)}_{k,K_M})u\right\| \leq \frac{C_D}{k}\left\|\Box^{(0)}_{k,K_M} u\right\|$. Thus, $\Box^{(0)}_{k,K_M}$ has $O(k^{-n_0})$ small spectral on D. The lemma follows. ☐

Let s be a local trivializing section of L on an open set $D \Subset M(0)$ and $|s|^2_{h^L} = e^{-2\phi}$. As in (4.62), we consider the localized Bergman projection

$$\begin{aligned} \hat{P}^{(0)}_{k,s,K_M} : L^2(D, K_M) \cap \mathscr{E}'(D, K_M) &\to L^2(D, K_M), \\ u &\to e^{-k\phi} s^{-k} P^{(0)}_{k,K_M}(s^k e^{k\phi} u). \end{aligned} \tag{6.4}$$

From Lemma 6.3 and Theorem 4.14, we get one of the main results of this work

Theorem 6.4. *Let (M,Θ) be a complete Kähler manifold and (L,h^L) be a smooth semi-positive line bundle over M. Let $D \Subset M(0)$ be a relatively compact open set and s be a local trivializing section of L on D. Then the localized Bergman projection $\hat{P}^{(0)}_{k,s,K_M}$ satisfies*

$$\hat{P}^{(0)}_{k,s,K_M} \equiv \mathcal{S}_k \mod O(k^{-\infty})$$

on D, where $\mathcal{S}_k : \mathscr{E}'(D, K_M) \to \mathscr{C}^\infty_0(D, K_M)$ is a smoothing operator and the distribution kernel $\mathcal{S}_k(z,w) \in \mathscr{C}^\infty(D \times D, K_M \boxtimes K_M)$ of $\mathcal{S}_k$ satisfies

$$\mathcal{S}_k(z,w) \equiv e^{ik\Psi(z,w)} b(z,w,k) \mod O(k^{-\infty}),$$

with

$$\begin{aligned} & b(z,w,k) \in S^n_{\mathrm{loc}}\left(1; D \times D, K_M \boxtimes K_M\right), \\ & b(z,w,k) \sim \sum_{j=0}^{\infty} b_j(z,w) k^{n-j} \text{ in } S^n_{\mathrm{loc}}\left(1; D \times D, K_M \boxtimes K_M\right), \\ & b_j(z,w) \in \mathscr{C}^\infty\left(D \times D, K_M \boxtimes K_M\right), \ \ j = 0, 1, \ldots, \\ & b_0(z,z) = (2\pi)^{-n} \det \dot{R}^L(z) \otimes \mathrm{Id}_{K_M}(z), \ \ \mathrm{Id}_{K_M} \text{ is the identity map on } K_M, \end{aligned}$$

and $\Psi(z,w)$ is as in Theorem 3.8.

From Theorem 6.4, the existence of the asymptotic expansion (1.19) for $L^k \otimes K_M$ follows immediately.

We prove now the formulas (1.20) for the coefficients. Le s be a local trivializing section of L on an open set $D \Subset M(0)$. We take local coordinates $z = (z_1, \ldots, z_n) = (x_1, \ldots, x_{2n}) = x$ defined in D. We also write $y = w = (w_1, \ldots, w_n)$, $y = (y_1, \ldots, y_{2n})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \ldots, n$. Let $\mathcal{S}_k$ and $\mathcal{S}_k(z, w) \in \mathscr{C}^\infty(D \times D, K_M \boxtimes K_M)$ be as in Theorem 6.4. We may replace $\mathcal{S}_k$ by $\frac{1}{2}(\mathcal{S}_k + \mathcal{S}_k^*)$, where $\mathcal{S}_k^*$ is the formal adjoint of $\mathcal{S}_k$ with respect to $(\,,\,)$. Then,

$$\mathcal{S}_k^* = \mathcal{S}_k. \tag{6.5}$$

Let $e(z)$ be a local section of K_M so that $|e(z)|^2 = (V_\Theta(z))^{-1}$, where $V_\Theta(z)$ is given by (4.69). Define the smooth kernels $\mathcal{S}_k(\cdot, \cdot), \widehat{\mathcal{S}}_k(\cdot, \cdot) \in \mathscr{C}^\infty(D \times D)$ by

$$\mathcal{S}_k(z, w) = e(z)\tilde{\mathcal{S}}_k(z, w)e^*(w), \;\; \widehat{\mathcal{S}}_k(z, w) = \tilde{\mathcal{S}}_k(z, w)V_\Theta(w)\,. \tag{6.6}$$

From Theorem 6.4, we have

$$\begin{aligned} &\widehat{\mathcal{S}}_k(z, w) \equiv e^{ik\Psi(z,w)}\widehat{b}(z, w, k) \quad \mathrm{mod}\ O(k^{-\infty}),\\ &\widehat{b}(z, w, k) \in S^n_{\mathrm{loc}}\Big(1; D \times D\Big),\\ &\widehat{b}(z, w, k) \sim \sum_{j=0}^{\infty} \widehat{b}_j(z, w)k^{n-j} \text{ in } S^n_{\mathrm{loc}}\Big(1; D \times D\Big),\\ &\widehat{b}_j(z, w) \in \mathscr{C}^\infty\Big(D \times D\Big), \;\; j = 0, 1, \ldots,\\ &\widehat{b}_0(z, z) = (2\pi)^{-n}V_\Theta(z) \det \dot{R}^L(z). \end{aligned} \tag{6.7}$$

Let $(\,,\,)_{d\lambda}$ be the inner product on $\mathscr{C}_0^\infty(D)$ given by

$$(u, v)_{d\lambda} = \int u(z)\overline{v(z)}d\lambda(z), \;\; u, v \in \mathscr{C}_0^\infty(D),$$

where $d\lambda(z) = 2^n dx_1 dx_2 \cdots dx_{2n}$. Let $\widehat{\mathcal{S}_k}$ be the continuous operator given by

$$\begin{aligned} \widehat{\mathcal{S}_k} : \mathscr{C}_0^\infty(D) &\to \mathscr{C}_0^\infty(D),\\ u &\to \int \widehat{\mathcal{S}_k}(z, w)u(w)d\lambda(w). \end{aligned}$$

Let $\widehat{\mathcal{S}_k}^{*,d\lambda}$ be the formal adjoint of $\widehat{\mathcal{S}_k}$ with respect to $(\,,\,)_{d\lambda}$. From (6.5), (6.6) we can check that

$$\widehat{\mathcal{S}_k}^{*,d\lambda} = \widehat{\mathcal{S}_k}. \tag{6.8}$$

Since $\mathcal{S}_k^2 \equiv \mathcal{S}_k \mod O(k^{-\infty})$, we can check that

$$(\widehat{\mathcal{S}_k})^2 \equiv \widehat{\mathcal{S}}_k \quad \mathrm{mod}\ O(k^{-\infty}). \tag{6.9}$$

Moreover, it is obviously that

$$\overline{\partial}_s\widehat{\mathcal{S}_k} \equiv 0 \quad \mathrm{mod}\ O(k^{-\infty}). \tag{6.10}$$

We recall that $\overline{\partial}_s = \overline{\partial} + k(\overline{\partial}\phi)\wedge$.

From (6.8), (6.9) and (6.10), we can repeat the procedure in Section 3.5 and conclude that (see (4.95) and (4.97))

$$
\begin{aligned}
&\widehat{b}_1(0,0) = V_\omega(0)\Big(-\frac{1}{8\pi}r(0)\Big),\\
(6.11)\qquad &\widehat{b}_2(0,0) = V_\omega(0)\Big(\frac{1}{128\pi^2}r^2 + \frac{1}{96\pi^2}\triangle_\omega r - \frac{1}{24\pi^2}\left|\mathrm{Ric}_{\,\omega}\right|^2_\omega + \frac{1}{96\pi^2}\left|R^{TM}_\omega\right|^2_\omega\Big)(0),
\end{aligned}
$$

where V_ω, $\triangle_\omega$, $\mathrm{Ric}_{\,\omega}$ and R^{TM}_ω are as in (4.69), (4.68), (4.74) and (4.72) respectively, and $\langle\,\cdot,\cdot\,\rangle_\omega$ is as in the discussion before (4.66). From (6.6) and (6.7), we can check that for $b^{(0)}_{1,K_M}(z)$, $b^{(0)}_{2,K_M}(z)$ in (1.19), we have

$$
b^{(0)}_{1,K_M}(0) = \frac{1}{V_\Theta(0)}\widehat{b}_1(0,0)\,\mathrm{Id}_{K_M}(0),\quad b^{(0)}_{2,K_M}(0) = \frac{1}{V_\Theta(0)}\widehat{b}_2(0,0)\,\mathrm{Id}_{K_M}(0).
$$

Combining this with (6.11) and notice that

$$
\frac{1}{V_\Theta(0)}V_\omega(0) = (2\pi)^{-n}\det\dot{R}^L(0),
$$

(1.20) follows.

7. Singular L^2-estimates

In the rest of this work, we need a singular version of L^2 estimates. We assume that (M,Θ) is a compact Hermitian manifold and (L,h^L) is a holomorphic line bundle over M, endowed with a singular Hermitian metric h^L. We solve the $\overline{\partial}$-equation $\overline{\partial}_k f = g$ for forms with values in L^k with a rough estimate L^2-estimate, namely $\left\|f\right\|^2 \leq C_D k^N \left\|g\right\|^2$ with $N > 0$, instead of the estimate $\left\|f\right\|^2 \leq \frac{C_D}{k}\left\|g\right\|^2$ from (6.1).

For a singular Hermitian metric h^L on L (see e.g. [36, Def. 2.3.1]) the local weight with respect to a holomorphic frame $s : D \to L$ is a function $\phi \in L^1_{\mathrm{loc}}(D)$, ϕ is bounded above on D, defined by

$$
\left|s\right|^2_{h^L} = e^{-2\phi} \in [0,\infty].
$$

The curvature current R^L is given locally by $R^L := 2\partial\overline{\partial}\phi$ and does not depend on the choice of local frame s, is thus well-defined as a $(1,1)$ current on M.

We say that $\sqrt{-1}R^L$ is strictly positive if there exists $\varepsilon > 0$ such that $\sqrt{-1}R^L \geq \varepsilon\Theta$, that is, $\sqrt{-1}R^L - \varepsilon\Theta$ is a positive current in the sense of Lelong (see e.g. [36, Def. B.2.11]).

The goal of this Section is to prove the following.

Theorem 7.1. *Let (L,h^L) be a singular Hermitian holomorphic line bundle over a compact Hermitian manifold (M,Θ). We assume that h^L is smooth outside a proper analytic set Σ and*

$$
(7.1)\qquad \sqrt{-1}R^L \geq \varepsilon\Theta,\quad \varepsilon > 0.
$$

Let $D \Subset M\setminus\Sigma$. Then, there exist $k_0 > 0$, $N > 0$ and $C_D > 0$, such that for all $k \geq k_0$, and $g \in \Omega^{0,1}_0(D,L^k)$ with $\overline{\partial}_k g = 0$, there is $u \in \mathscr{C}^\infty(M,L^k)$ such that $\overline{\partial}_k u = g$ and

$$
(7.2)\qquad \left\|u\right\|^2_{h^{L^k},\Theta} \leq k^N C_D \left\|g\right\|^2_{h^{L^k},\Theta},
$$

where $\left\|u\right\|^2_{h^{L^k},\Theta} := \int_M \left|u\right|^2_{h^{L^k}} dv_M$, $dv_M := \frac{\Theta^n}{n!}$.

Proof. Let Θ_{ϵ_0} be the generalized Poincaré metric on $M \setminus \Sigma$ (see [36, p. 276]). Let $\mathcal{T}_{\epsilon_0}$ be the Hermitian torsion induced by Θ_{ϵ_0}. We recall that $\mathcal{T}_{\epsilon_0} := [(\Theta_{\epsilon_0}\wedge)^*, \partial\Theta_{\epsilon_0}]$. Let $R^{\det}_{\Theta_{\epsilon_0}}$ denote the curvature of the holomorphic line bindle $\Lambda^n T^{*(1,0)}M$ induced by Θ_{ϵ_0}. By [36, Lemma 6.2.1] we have

$$\begin{aligned}&\Theta_{\epsilon_0} \text{ is a complete Hermitian metric of finite volume on } M\setminus\Sigma,\\ &\Theta_{\epsilon_0} \geq c_0\Theta \text{ for some } c_0>0,\\ &-C\Theta_{\epsilon_0} < \sqrt{-1}R^{\det}_{\Theta_{\epsilon_0}} < C\Theta_{\epsilon_0},\ |\mathcal{T}_{\epsilon_0}|_{\Theta_{\epsilon_0}} < C,\end{aligned} \tag{7.3}$$

where $C>0$ is a constant and $|\mathcal{T}_{\epsilon_0}|_{\Theta_{\epsilon_0}}$ is the norm with respect to Θ_{ϵ_0}. Moreover, by [36, §. 6.2] there is a Hermitian metric $h^L_{\epsilon_0}$ of L on $M\setminus\Sigma$ such that $h^L_{\epsilon_0}$ is smooth on $M\setminus\Sigma$ and

$$h^L_{\epsilon_0} > h^L,\ \ \sqrt{-1}R^L_{\epsilon_0} > c\Theta_{\epsilon_0}, \tag{7.4}$$

where $c>0$ is a constant and $R^L_{\epsilon_0}$ is the curvature of L induced by $h^L_{\epsilon_0}$.

Let s is a local frame of L and define local weights ϕ_{ϵ_0} and ϕ for $h^L_{\epsilon_0}$ and h^L by $|s|^2_{h^L_{\epsilon_0}} = e^{-2\phi_{\epsilon_0}}$, $|s|^2_{h^L} = e^{-2\phi}$. Let $\widehat{h}^{L^k}$ be the Hermitian metric on L^k locally given by

$$|s|^2_{\widehat{h}^{L^k}} := \exp(-2(\log k)\phi_{\epsilon_0} - 2(k-\log k)\phi).$$

Since $h^L_{\epsilon_0} > h^L$, we have $\widehat{h}^{L^k} > h^{L^k}$. Moreover, from (7.1) and (7.4), we can check that

$$\sqrt{-1}\widehat{R}^{L^k} > c(\log k)\Theta_{\epsilon_0}, \tag{7.5}$$

where $\widehat{R}^{L^k}$ denotes the curvature of L^k associated to $\widehat{h}^{L^k}$ and $c>0$ is the constant as in (7.4). Let $(\,,\,)_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}$ denote the L^2 inner product on $\Omega^{0,q}_0(M\setminus\Sigma, L^k)$ with respect to $\widehat{h}^{L^k}$ and Θ_{ϵ_0} as (2.3). For $f\in\Omega^{0,q}_0(M\setminus\Sigma,L^k)$, we write $\|f\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}} := (f,f)_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}$. Let $\widehat{L}^2_{(0,q)}(M\setminus\Sigma,L^k)$ be the completion of $\Omega^{0,q}_0(M\setminus\Sigma,L^k)$ with respect to $\|\cdot\|_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}$. Let

$$\widehat{\Box}^{(1)}_k = \overline{\partial}_k\overline{\partial}^*_k + \overline{\partial}^*_k\overline{\partial}_k : \mathrm{Dom}\,\widehat{\Box}^{(1)}_k \subset \widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k) \to \widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k)$$

be the Gaffney extension of the Kodaira Laplacian with respect to $\widehat{h}^{L^k}$ and Θ_{ϵ_0}(see (2.9)). Here $\overline{\partial}^*_k$ is the Hilbert space adjoint of $\overline{\partial}_k$ with respect to $(\,,\,)_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}$. From (7.3) and (7.5), we can repeat the procedure in [36, p. 272–273] and conclude that for k large, we have

$$\|g\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}} \leq \frac{1}{c(\log k)}\left\|\widehat{\Box}^{(1)}_k g\right\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}, \tag{7.6}$$

for all $g\in\Omega^{0,1}_0(M\setminus\Sigma,L^k)$, where $c>0$ is a positive constant. From this, we can repeat the method in [36, p. 272–273] and conclude that $\widehat{\Box}^{(1)}_k$ has closed range in $\widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k)$, $\mathrm{Ker}\,\widehat{\Box}^{(1)}_k \bigcap \widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k) = \{0\}$ and there is a bounded operator $G_k : \widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k) \to \mathrm{Dom}\,\widehat{\Box}^{(1)}_k$ such that $\widehat{\Box}^{(1)}_k G_k = I$ on $\widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k)$, $G_k\widehat{\Box}^{(1)}_k = I$ on $\mathrm{Dom}\,\widehat{\Box}^{(1)}_k$ and

$$\|G_k g\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}} \leq \frac{1}{c(\log k)}\|g\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}} \tag{7.7}$$

for k large, for all $g\in\widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k)$, where $c>0$ is independent of g and k, and

$$G_k : \Omega^{0,1}(M\setminus\Sigma,L^k) \to \Omega^{0,1}(M\setminus\Sigma,L^k), \tag{7.8}$$

$$g = \widehat{\Box}^{(1)}_k G_k g = \overline{\partial}_k\overline{\partial}^*_k G_k g,\ \ \text{if } \overline{\partial}_k g = 0,\ g\in\widehat{L}^2_{(0,1)}(M\setminus\Sigma,L^k). \tag{7.9}$$

Now let $g\in\Omega^{0,1}_0(D,L^k)$ with $\overline{\partial}_kg=0$ and set

$$u=\overline{\partial}^*_kG_kg\in\Omega^{0,1}(M\setminus\Sigma,L^k)\bigcap\widehat{L}^2_{(0,0)}(M\setminus\Sigma,L^k).$$

From (7.9) and (7.7), it is not difficult to see that

$$\begin{aligned}&\overline{\partial}_ku=g\ \text{ on } M\setminus\Sigma,\\ &\|u\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}\leq\frac{1}{c_1\sqrt{\log k}}\|g\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}\,,\end{aligned}\tag{7.10}$$

where $c_1>0$ is a constant independent of g and k. Now, let's compare the norms $\|\cdot\|_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}$ and $\|\cdot\|_{h^{L^k},\Theta}$. Let s be a local section of L on D and $|s|^2_{h^L_{\epsilon_0}}=e^{-2\phi_{\epsilon_0}}$, $|s|^2_{h^L}=e^{-2\phi}$. Then,

$$|s|^2_{\widehat{h}^{L^k}}=e^{-2k\phi}e^{2\log k(\phi-\phi_{\epsilon_0})}=|s|^2_{h^{L^k}}\,e^{2\log k(\phi-\phi_{\epsilon_0})}.$$

Thus, on D, we have

$$|s|^2_{\widehat{h}^{L^k}}<k^N\,|s|^2_{h^{L^k}}\,,\tag{7.11}$$

where $N>\sup_{x\in D}|2\phi(x)-2\phi_{\epsilon_0}(x)|$. Thus,

$$\|g\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}<\widetilde{C}_Dk^N\,\|g\|^2_{h^{L^k},\Theta}\,,\tag{7.12}$$

where $\widetilde{C}_D>0$ is a constant independent of g and k. From $\widehat{h}^{L^k}>h^{L^k}$ and the second property in (7.3), we have $\|u\|^2_{h^{L^k},\Theta}<\tilde{c}\,\|u\|^2_{\widehat{h}^{L^k},\Theta_{\epsilon_0}}$, where $\tilde{c}>0$ is a constant independent of k and u. Combiming this with (7.12) and (7.10), we obtain

$$\|u\|^2_{h^{L^k},\Theta}\leq C_Dk^N\,\|g\|^2_{h^{L^k},\Theta}\,,\tag{7.13}$$

where $C_D>0$ is a constant independent of k and g. Note that h^L is bounded away from zero and Σ has Lebesgue measure zero. From this observation and (7.13), we see that u is L^2 integrable with respect to some smooth metric of L over M. Combining this with Skoda's Lemma (see Lemma 7.2 below), we get $\overline{\partial}_ku=g$ on M and $u\in\Omega^{0,1}(M,L^k)$. The theorem follows. ☐

We recall the following result of Skoda (see Lemma 7.3 of Chapter VIII in Demailly [17])

Lemma 7.2. *Let $u\in\mathscr{D}'(M,L^k)$, $g\in\mathscr{D}'(M,L^k\otimes T^{*(0,1)}M)$. We assume that u and g are L^2 integrable with respect to some smooth metric of h^L and Θ over M. If $\overline{\partial}_ku=g$ on $M\setminus\Sigma$ in the sense of distribution, then, $\overline{\partial}_ku=g$ on M in the sense of distribution.*

8. Bergman kernel asymptotics for semi-positive line bundles

In this Section we prove Theorem 1.10. Let (M,Θ) a compact Hermitian manifold. Assume that $(L,h^L)\to M$ is a smooth semi-positive line bundle which is positive at some point of M. By Siu's criterion [36, Th. 2.2.27] (see also Corollary 10.8) we know that L is big and M is Moishezon. By [36, Lemma 2.3.6], L admits a singular Hermitian metric $h^L_{\rm sing}$, smooth outside a proper analytic set Σ, and with strictly positive curvature current.

Lemma 8.1. *With the assumptions and notations above, let $D\Subset M\setminus\Sigma$ be an open set. Then, there exist $k_0>0$, $N>0$ and $C_D>0$, such that for all $k\geq k_0$, and $g\in\Omega^{0,1}_0(D,L^k)$ with $\overline{\partial}_kg=0$, there is $u\in\mathscr{C}^\infty(M,L^k)$ such that $\overline{\partial}_ku=g$ and*

$$\|u\|^2\leq k^NC_D\,\|g\|^2\,.$$

Proof. Let ϕ and $\widehat{\phi}$ denote local weights for h^L and $h^L_{\rm sing}$ respectively. Then, $\widehat{\phi}$ is smooth on $M \setminus \Sigma$ and bounded above. We may assume that

$$\widehat{\phi} \leq \phi.$$

Let $\widetilde{h}^{L^k}$ be the Hermitian metric on L^k induced by the local weight

$$\widetilde{\phi} := (\log k)\widehat{\phi} + (k - \log k)\phi.$$

We can check that $\widetilde{h}^{L^k}$ is a strictly positive singular Hermitian metric, smooth outside a proper analytic set Σ. We can repeat the proof of Theorem 7.1 and conclude that there is $u \in \mathscr{C}^\infty(M, L^k)$ such that $\overline{\partial}_k u = g$ and

$$\|u\|^2_{\widetilde{h}^{L^k}} \leq \frac{1}{c\sqrt{\log k}} \|g\|^2_{\widetilde{h}^{L^k}}, \tag{8.1}$$

where $c > 0$ is independent of k and g. Since $\widehat{\phi} \leq \phi$, we have

$$\|u\|_{h^{L^k}} \leq \|u\|_{\widetilde{h}^{L^k}}. \tag{8.2}$$

On the other hand, we have

$$\begin{aligned} \|g\|^2_{\widetilde{h}^{L^k}} &= \int_D |g|^2 e^{-2(\log k)\widehat{\phi} - 2(k - \log k)\phi} dv_M(x) \\ &\leq (\sup_{x \in D} e^{2(\log k)(\phi(x) - \widehat{\phi}(x))}) \int_D |g|^2 e^{-2k\phi} dv_M(x) \\ &\leq k^N \|g\|^2_{h^{L^k}}, \end{aligned} \tag{8.3}$$

where $N = \sup_{x \in D} 2(\phi(x) - \widehat{\phi}(x))$. From (8.2) and (8.3), the lemma follows. $\square$

For a holomorphic line bundle L over a compact Hermitian manifold (M, Θ) we set $\mathrm{Herm}(L) = \left\{\text{singular Hermitian metrics on } L\right\}$,

$$\mathcal{M}(L) = \Big\{h^L \in \mathrm{Herm}(L);\ h^L \text{ is smooth outside a proper analytic set,} \\ \sqrt{-1}R^L > \varepsilon\Theta,\ \varepsilon > 0\Big\}.$$

By [36, Lemma 2.3.6], $\mathcal{M}(L) \neq \emptyset$ under the hypotheses of Theorem 8.2 below. Set

$$M' := \left\{p \in M;\ \exists\ h^L \in \mathcal{M}(L) \text{ with } h^L \text{ smooth near } p\right\}. \tag{8.4}$$

From Lemma 8.1, we can repeat the proof of Lemma 6.3 with minor changes and conclude the following.

Theorem 8.2. *Let (M, Θ) be a compact Hermitian manifold. Let $(L, h^L) \to M$ be a Hermitian holomorphic line bundle with smooth Hermitian metric h^L having semi-positive curvature and with $M(0) \neq \emptyset$. Let $D \Subset M' \cap M(0)$ be an open set, where M' is given by* (8.4). *Then, $\Box^{(0)}_k$ has $O(k^{-n_0})$ small spectral gap on D.*

Let s be a local trivializing section of L on an open set $D \Subset M$ and $|s|^2_{h^L} = e^{-2\phi}$. We define the *localized Bergman projection* (with respect to s) by

$$\begin{aligned} \widehat{P}^{(0)}_{k,s} : L^2(D) \cap \mathscr{E}'(D) &\to \mathscr{C}^\infty_0(D), \\ u &\to e^{-k\phi} s^{-k} P^{(0)}_k (s^k e^{k\phi} u). \end{aligned} \tag{8.5}$$

That is, if $P^{(0)}_k(s^k e^{k\phi} u) = s^k v$ on D, then $\widehat{P}^{(0)}_{k,s} u = e^{-k\phi} v$.

From Theorem 8.2 and Theorem 4.14, we get the following result-

Theorem 8.3. *Let (M,Θ) be a compact Hermitian manifold. Let $(L,h^L)\to M$ be a Hermitian holomorphic line bundle with smooth Hermitian metric h^L having semi-positive curvature and with $M(0)\neq\emptyset$. Let s be a local trivializing section of L on an open set $D\Subset M'\bigcap M(0)$. Then the localized Bergman projection $\widehat{P}^{(0)}_{k,s}$ satisfies*

$$\widehat{P}^{(0)}_{k,s}\equiv\mathcal{S}_k \mod O(k^{-\infty})$$

on D, where $\mathcal{S}_k$ is as in Theorem 4.11.

Theorem 8.3 immediately implies Theorem 1.10.

9. Multiplier ideal Bergman kernel asymptotics. Proof of Theorem 1.8

Let us first recall the notion of multiplier ideal sheaf. Let $\varphi\in L^1_{loc}(M,\mathbb{R})$. The *Nadel multiplier ideal sheaf* $\mathscr{I}(\varphi)$ is the ideal subsheaf of germs of holomorphic functions $f\in\mathscr{O}_{M,x}$ such that $|f|^2e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in local coordinates near x.

Consider now a singular Hermitian metric h^L on a holomorphic line bundle L over M. We will assume that h^L is smooth outside a proper analytic set Σ. If h^L_0 is a smooth Hermitian metric on L then $h^L=h^L_0e^{-2\varphi}$ for some function $\varphi\in L^1_{loc}(M,\mathbb{R})$. The Nadel multiplier ideal sheaf of h^L is defined by $\mathscr{I}(h^L)=\mathscr{I}(\varphi)$; the definition does not depend on the choice of h^L_0. With the help of h^L and the volume form dv_M we can define an L^2 inner product on $\mathscr{C}^\infty(M,L)$:

$$(S,S')=\int_M\langle S,S'\rangle_{h^L_0}e^{-2\varphi}dv_M\,,\quad S,S'\in\mathscr{C}^\infty(M,L)\,. \tag{9.1}$$

The singular Hermitian metric h^L induces a singular Hermitian metric $h^{L^k}=h^{L^k}_0e^{-2k\varphi}$ on L^k, $k>0$. We denote by $(\cdot\,,\cdot)_k$ the natural inner products on $\mathscr{C}^\infty(M,L^k)$ defined as in (9.1). The space of global sections in the sheaf $\mathscr{O}(L^k)\otimes\mathscr{I}(h^{L^k})$ is given by

$$\begin{aligned}&H^0(M,L^k\otimes\mathscr{I}(h^{L^k}))\\&=\Big\{s\in\mathscr{C}^\infty(M,L^k);\,\overline{\partial}_ks=0,\int_M\Big|s\Big|^2_{h^{L^k}}dv_M=\int_M\Big|s\Big|^2_{h^{L^k}_0}e^{-2\varphi}dv_M<\infty\Big\}\,.\end{aligned} \tag{9.2}$$

Let $\{S^k_j\}$ be an orthonormal basis of $H^0(X,L^k\otimes\mathscr{I}(h^{L^k}))$ with respect to the inner product induced $(\cdot\,,\cdot)_k$. The (multiplier ideal) Bergman kernel function is defined by (1.21).

We assume that h^L is a strictly positive singular Hermitian metric on L, smooth outside a proper analytic set Σ of M.

Let

$$P^{(0)}_{k,\mathscr{I}}:L^2(M,L^k)\to H^0(M,L^k\otimes\mathscr{I}(h^{L^k}))$$

be the orthogonal projection. Let s be a local trivializing section of L on an open set $D\Subset M\setminus\Sigma$ and $|s|^2_{h^L}=e^{-2\phi}$. Then, ϕ is smooth on D and $\partial\overline{\partial}\phi$ is positive defined at each point of D. Let us denote by

$$\widehat{P}^{(0)}_{k,s,\mathscr{I}}:L^2(D)\cap\mathscr{E}'(D)\longrightarrow L^2(D)\,,\quad u\longmapsto e^{-k\phi}s^{-k}P^{(0)}_{k,\mathscr{I}}(s^ke^{k\phi}u). \tag{9.3}$$

the localized (multiplier ideal) Bergman projection.

Theorem 9.1. *Let (L,h^L) be a singular Hermitian holomorphic line bundle with strictly positive curvature current over a compact Hermitian manifold (M,Θ). We assume that h^L*

is smooth outside a proper analytic set Σ. Let s be a local trivializing section of L on an open set $D \Subset M \setminus \Sigma$. Then the localized multiplier ideal Bergman projection $\widehat{P}^{(0)}_{k,s,\mathscr{I}}$ satisfies

$$\widehat{P}^{(0)}_{k,s,\mathscr{I}} \equiv \mathcal{S}_k \mod O(k^{-\infty})$$

on D, where $\mathcal{S}_k$ is as in Theorem 4.11

Proof. Let s be a local trivializing section of L on an open set $D \Subset M \setminus \Sigma$ and $|s|^2_{h^L} = e^{-2\phi}$. Then, ϕ is smooth on D and $\partial\overline{\partial}\phi$ is positive defined at each point of D. Let $\Box^{(0)}_s$ be the operator as in (3.7), (3.6). Let $\mathcal{S}_k$ and $\mathcal{A}_k$ be the operators as in Theorem 3.10. We recall that $\mathcal{S}_k$ and $\mathcal{A}_k$ are properly supported and

$$\mathcal{S}_k + \Box^{(0)}_s \mathcal{A}_k \equiv I \mod O(k^{-\infty}).$$

We now replace $\mathcal{S}_k$ by $I - \Box^{(0)}_s \mathcal{A}_k$ and we have

$$\begin{aligned} \Box^{(0)}_s \mathcal{A}_k + \mathcal{S}_k &= I, \\ \mathcal{A}^*_k \Box^{(0)}_s + \mathcal{S}^*_k &= I, \end{aligned} \tag{9.4}$$

where $\mathcal{S}^*_k$ and $\mathcal{A}^*_k$ are adjoints of $\mathcal{S}_k$ and $\mathcal{A}_k$ with respect to $(\,,)$ respectively. From (3.7), (3.6), we can check that $\Box^{(0)}_s \widehat{P}^{(0)}_{k,s,\mathscr{I}} = 0$. From this and (9.4), we have

$$\widehat{P}^{(0)}_{k,s,\mathscr{I}} = (\mathcal{A}^*_k \Box^{(0)}_s + \mathcal{S}^*_k)\widehat{P}^{(0)}_{k,s,\mathscr{I}} = \mathcal{S}^*_k \widehat{P}^{(0)}_{k,s,\mathscr{I}} \text{ on } L^2(D) \cap \mathscr{E}'(D).$$

Thus,

$$\widehat{P}^{(0)}_{k,s,\mathscr{I}} = \widehat{P}^{(0)}_{k,s,\mathscr{I}} \mathcal{S}_k \text{ on } L^2(D) \cap \mathscr{E}'(D)$$

and hence,

$$\mathcal{S}^*_k \widehat{P}^{(0)}_{k,s,\mathscr{I}} = \mathcal{S}^*_k \widehat{P}^{(0)}_{k,s,\mathscr{I}} \mathcal{S}_k = \widehat{P}^{(0)}_{k,s,\mathscr{I}} \text{ on } L^2(D) \cap \mathscr{E}'(D). \tag{9.5}$$

Let $u \in H^m_{\rm comp}(D)$, $m \in \mathbb{R}$. We consider

$$v = s^k e^{k\phi} \mathcal{S}_k u - P^{(0)}_{k,s,\mathscr{I}}(s^k e^{k\phi} \mathcal{S}_k u).$$

Since $\mathcal{S}_k$ is a smoothing operator, $v \in \mathscr{C}^\infty(M, L^k)$. Moreover, it is easy to see that $v \perp H^0(M, L^k \otimes \mathscr{I}(h^{L^k}))$. In view of (3.7), we have

$$\overline{\partial}_k v = s^k e^{k\phi} \overline{\partial}_s \mathcal{S}_k u. \tag{9.6}$$

As in (4.76), we have

$$\overline{\partial}_s \mathcal{S}_k \equiv 0 \mod O(k^{-\infty}). \tag{9.7}$$

Combining (9.6) with (9.7), we obtain

$$\left\|\overline{\partial}_k v\right\| \leq C_N k^{-N} \|u\|_m,$$

for every $N > 0$, where $C_N > 0$ is independent of k. Since $v \perp H^0(M, L^k \otimes \mathscr{I}(h^{L^k}))$, v is the element of $\left\{u \in \mathscr{C}^\infty(M, L^k) \cap L^2(M, L^k);\, \overline{\partial}_k u = \overline{\partial}_k v\right\}$ with minimal L^2 norm. From this observation and Theorem 7.1, we conclude that

$$\|v\| \leq \widetilde{C}_N k^{-N} \|u\|_m,$$

for every $N > 0$, where $\widetilde{C}_N > 0$ is independent of k. Thus,

$$\mathcal{S}_k - \widehat{P}^{(0)}_{k,s,\mathscr{I}} \mathcal{S}_k = O(k^{-N}) : H^m_{\rm comp}(D) \to L^2(D),$$

for all $N > 0$, $m \in \mathbb{R}$, and hence

$$\mathcal{S}_k^*\mathcal{S}_k - \mathcal{S}_k^*\widehat{P}^{(0)}_{k,s,\mathscr{I}}\mathcal{S}_k = O(k^{-N}) : H^m_{\mathrm{comp}}(D) \to H^{m+N_1}_{\mathrm{loc}}(D),$$

for all $N, N_1 > 0$, $m \in \mathbb{R}$. We conclude that

$$\mathcal{S}_k^*\mathcal{S}_k \equiv \mathcal{S}_k^*\widehat{P}^{(0)}_{k,s,\mathscr{I}}\mathcal{S}_k \mod O(k^{-\infty}).$$

From this and (9.5), (3.59), the theorem follows. □

From Theorem 9.1, we get Theorem 1.8.

10. Further applications

In this Section we collect further applications of the methods developed here. In Section 10.1 we show the existence of manifolds and line bundles whose Kodaira-Laplace operator has no $O(k^{-n_0})$ small spectral gap. In Section10.2 we show that under an integral condition (due to Bouche) on the first eigenvalue of the curvature, the asymptotic expansion of the Bergman kernel of a semi-positive line bundle holds. In Section 10.3 we apply our results to prove a result of Berman about the Bergman kernel associated to an arbitrary semi-positive Hermitian metric on an ample line bundle. In Section 10.4 we give a local version of the Bergman kernel expansion for q-forms. In Section 10.5 we obtain prcise semiclassical estimates for the dimension of the spectral spaces of the Kodaira Laplacian. Using them one obtains immediately the holomorphic Morse inequalities of Demailly.

10.1. **Existence of "small" eigenvalues of the Kodaira Laplacian.** The hypothesis on the existence of a $O(k^{-n_0})$ small spectral gap was of central importance in our approach. We are therefore interested in if there is a compact complex manifold M and a holomorphic line bundle L over M such that the associated Kodaira Laplacian has very small eigenvalues. We will construct a compact manifold and a holomorphic line bundle L over M such that the associated Kodaira Laplacian $\Box^{(q)}_k$ has non-vanishing eigenvalues of order $O(k^{-\infty})$.

Theorem 10.1. *Let $0 \le q \le n$. There exists a compact complex manifold M and a holomorphic line bundle L over M such that for*

$$\lambda_k := \inf\left\{\lambda;\ \lambda:\ \text{non-zero eigenvalues of } \Box^{(q)}_k\right\}$$

we have for every $N > 0$

$$\lim_{k\to\infty} k^N\lambda_k = 0\,.$$

Let S be a compact Riemann surface with a smooth Hermitian metric. Let (L_0, h^{L_0}) be a holomorphic line bundle over S. We assume that $\sqrt{-1}R^{L_0}$ is positive. It is not difficult to see that L_0 admits another smooth Hermitian fiber metric $\tilde{h}^{L_0}$ such that the associated curvature form $\sqrt{-1}\tilde{R}^{L_0}$ is positive on $S_+ \subset S$, negative on $S_- \subset S$ and degenerate on $S_0 \subset S$, where $S = S_+ \bigcup S_- \bigcup S_0$, S_+, S_- contain non-empty open subsets of S.

Let M_1 be a compact complex manifold of dimension $n-1$ with a smooth Hermitian metric and let (L_1, h^{L_1}) be a holomorphic line bundle over M_1. We assume that $\sqrt{-1}R^{L_1}$ is non-degenerate of constant signature (n_-, n_+), $n_- + n_+ = n-1$, at each point of M_1. Put

$$M := M_1 \times S,\quad L := L_1 \otimes L_0.$$

Then, M is a compact complex manifold of dimension n and L is a holomorphic line bundle over M. The Hermitian metrics on M_1 and S induce a Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle$ on M. Consider the metric $h^L = h^{L_0} \otimes h^{L_1}$ on L; then the associated curvature $\sqrt{-1}R^L$ is non-degenerate of constant signature $(n_-, n_+ + 1)$ at each point of M. Similarly, setting $\tilde{h}^L = \tilde{h}^{L_0} \times h^{L_1}$, the associated curvature $\sqrt{-1}\tilde{R}^L$ is non-degenerate of constant signature $(n_-, n_+ + 1)$ on $M_+ \subset M$, non-degenerate of constant signature $(n_- + 1, n_+)$ on $M_- \subset M$ and degenerate on $M_0 \subset M$, where $M = M_- \bigcup M_+ \bigcup M_0$, M_-, M_+ contain non-empty open subsets of M. First, we need

Lemma 10.2. *Under the notations above let $q = n_-$. Then*

$$\dim \mathscr{H}^q(M, L^k) = \frac{k^n}{n!}\Big(\int_{M_+}\left(\tfrac{\sqrt{-1}}{2\pi}\tilde{R}^L\right)^n - \int_{M_-}\left(\tfrac{\sqrt{-1}}{2\pi}\tilde{R}^L\right)^n\Big) + o(k^n)\,, \quad k \to \infty\,.$$

Proof. Note that L admits a smooth Hermitian fiber metric such that the induced curvature is non-degenerate of constant signature $(n_-, n_+ + 1)$ at each point of M. From this observation and Andreotti-Grauert vanishing theorem, we know that if k large, then

$$\mathscr{H}^j(M, L^k) = 0 \ \text{ if } j \neq n_-. \tag{10.1}$$

From the Riemann-Roch-Hirzebruch theorem (see e. g. [36, (4.1.10)]), we see that

$$\sum_{j=0}^{n}(-1)^j \dim \mathscr{H}^j(M, L^k) = \frac{k^n}{n!}\int_M c_1(L)^n + O(k^{n-1}), \tag{10.2}$$

where $c_1(L)$ is the first Chern class. Combining (10.2) with (10.1), we have for k large enough

$$\dim \mathscr{H}^q(M, L^k) = (-1)^q \frac{k^n}{n!}\int_M c_1(L)^n + O(k^{n-1}). \tag{10.3}$$

But $\frac{\sqrt{-1}}{2\pi}\tilde{R}^L$ represents the Chern class so

$$\int_M c_1(L)^n = \int_M \left(\tfrac{\sqrt{-1}}{2\pi}\tilde{R}^L\right)^n.$$

The lemma follows from (10.3). □

The Hermitian fiber metric $\tilde{h}^L$ induces a Hermitian fiber metric $\tilde{h}^{L^k}$ on the k-th tensor power of L. As before, let $\Box_k^{(q)}$ be the Kodaira Laplacian with values in L^k associated to $\tilde{h}^{L^k}$.

Theorem 10.3. *Under the notations above let $q = n_-$. Then, for any $N > 2n$, we have*

$$\lim_{k\to\infty} k^N \lambda_k = 0.$$

Proof. Fix $N_0 > 2n$. From Corollary 10.7 below and Lemma 10.2, we know that

$$\begin{aligned}\dim \mathscr{E}^q_{k^{-N_0}}(M, L^k) &= \frac{k^n}{n!}\int_{M_+}\left(\tfrac{\sqrt{-1}}{2\pi}\tilde{R}^L\right)^n + o(k^n)\\ &> \frac{k^n}{n!}\Big(\int_{M_+}\left(\tfrac{\sqrt{-1}}{2\pi}\tilde{R}^L\right)^n - \int_{M_-}\left(\tfrac{\sqrt{-1}}{2\pi}\tilde{R}^L\right)^n\Big) + o(k^n)\\ &> \dim \mathscr{H}^q(M, L^k) + o(k^n).\end{aligned}$$

Thus, for k large, we have

$$\dim \mathscr{E}^q_{0<\lambda\leq k^{-N_0}}(M, L^k) > 0,$$

where $\mathscr{E}^q_{0<\lambda\leq k^{-N_0}}(M,L^k)$ denotes the spectral space spanned by the eigenforms of $\Box^{(q)}_k$ whose eigenvalues are bounded by k^{-N_0} and >0. We notice that since M is compact, $\Box^{(q)}_k$ has a discrete spectrum, each eigenvalues occurs with finite multiplicity. Thus, $\lambda_k \leq k^{-N_0}$ for k large. The theorem follows. □

From Theorem 10.3, we get Theorem 10.1.

10.2. **Bouche integral condition.** Let (L,h^L) be a semi-positive holomorphic line bundle over a compact Hermitian manifold (M,Θ). Let $0\leq\lambda_1(x)\leq\lambda_2(x)\leq\cdots\leq\lambda_n(x)$ be the eigenvalues of $\dot{R}^L(x)$. We say that (L,h^L) satisfies the Bouche integral condition [6] if

$$\int_M \lambda_1^{-6n} < \infty\,. \tag{10.4}$$

Let (L,h^L) be a semi-positive holomorphic line bundle over a compact Hermitian manifold (M,Θ). If (L,h^L) satisfies (10.4) then Bouche [6] proved that

$$\inf\Big\{\lambda\in\operatorname{Spec}(\Box^{(q)}_k);\ \lambda\neq0\Big\}\geq k^{\frac{10n+1}{12n+1}},$$

for k large. From this and Theorem 1.6, we deduce

Corollary 10.4. *Let (L,h^L) be a semi-positive holomorphic line bundle over a compact Hermitian manifold (M,Θ). If (L,h^L) satisfies* (10.4) *then*

$$P^{(0)}_k(x)\sim\sum_{j=0}^{\infty}k^{n-j}b^{(0)}_j(x)\ \text{ locally uniformly on } M(0),$$

where $b^{(0)}_j(x)\in\mathscr{C}^\infty(M(0))$, $j\in\mathbb{N}_0$, are as in (1.8).

10.3. **Asymptotics for arbitrary semi-positive metrics on ample line bundles.** We consider now the Bergman kernel of a metric with semi-positive curvature on an ample line bundle and recover the following result of Berman [4].

Corollary 10.5. *Let L be an ample line bundle over a projective manifold M. We endow M with a Hermitian metric Θ and L with a Hermitian metric h^L with semi-positive curvature. Then the Bergman kernel function associated to these metric data admits an asymptotic expansion*

$$P^{(0)}_k(x)\sim\sum_{j=0}^{\infty}k^{n-j}b^{(0)}_j(x)\ \text{ locally uniformly on } M(0),$$

where $b^{(0)}_j(x)\in\mathscr{C}^\infty(M(0))$, $j\in\mathbb{N}_0$, are as in (1.8).

Proof. By a result due to Donnelly [20] there exist $C>0$ and $k_0\in\mathbb{N}$ such that for all $k\geq k_0$

$$\inf\Big\{\lambda\in\operatorname{Spec}(\Box^{(0)}_k);\ \lambda\neq0\Big\}\geq C\,.$$

In particular, $\Box^{(0)}_k$ has an $O(k^{-n_0})$ small spectral gap. By applying Theorem 1.6 we immediately deduce the result. □

10.4. **Expansion for Bergman kernel on forms.** Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ). Given q, $0 \leq q \leq n$, $\dot{R}^L$ is said to satisfy condition $Z(q)$ at $p \in M$ if $\dot{R}^L(p)$ has at least $n+1-q$ positive eigenvalues or at least $q+1$ negative eigenvalues. If $\dot{R}^L(p)$ is non-degenerate of signature (n_-, n_+), then $Z(q)$ holds at p if and only if $q \neq n_-$. It is well-known that if $Z(q-1)$ and $Z(q+1)$ hold at each point of M, then $\Box_k^{(q)}$ has a "large" spectral gap, i.e. there exists a constant $C > 0$ such that for all k we have

$$\inf\left\{\lambda \in \mathrm{Spec}(\Box_k^{(q)});\ \lambda \neq 0\right\} \geq Ck\,. \tag{10.5}$$

This fact essentially follows from the L^2 method for $\overline{\partial}$ of Hörmander (see Hörmander [27] for the classical case and Sjöstrand [45, Appendix] for the semi-classical case). From this and Theorem 1.6, we deduce the following local version of the results due to Catlin [10], Zelditch [51], Dai-Liu-Ma [12] (for $q = 0$) and Berman-Sjöstrand [3], Ma-Marinescu [35] (for $q > 0$):

Corollary 10.6. *Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ). We assume that $Z(q-1)$ and $Z(q+1)$ hold at each point of M. If $\dot{R}^L$ is non-degenerate of constant signature (n_-, n_+) on an open set D of $p \in M$, where $q = n_-$, then we have*

$$P_k^{(q)}(x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j^{(q)}(x) \ \text{ locally uniformly on } D,$$

where $b_j^{(q)}(x) \in \mathscr{C}^\infty(D, \mathrm{End}(\Lambda^q T^{*(0,1)}M))$, $j = 0, 1, \ldots$, are as in (1.8).

Let us illustrate Corollary 10.6 in the case $q = 0$: if the curvature R^L has either positive eigenvalues or at least two negative eigenvalues at each point, then the Bergman kernel of the sections of L^k has an asymptotic expansion as $k \to \infty$.

10.5. **Holomorphic Morse inequalities.** Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ). Since M is compact, $\Box_k^{(q)}$ has a discrete spectrum, each eigenvalues occurs with finite multiplicity. From (1.15), (1.16) and the Lebesgue dominated convergence theorem, we deduce the following.

Corollary 10.7. *Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n. If $N_0 \geq 2n+1$, then*

$$\dim \mathscr{E}^q_{k^{-N_0}}(M, L^k) = k^n (2\pi)^{-n} \int_{M(q)} \left|\det \dot{R}^L(x)\right| dv_M(x) + o(k^n).$$

Fix $N_0 \geq 1$. Let $\mathscr{E}^q_{0<\lambda\leq k^{-N_0}}(M, L^k)$ denote the spectral space spanned by the eigenforms of $\Box_k^{(q)}$ whose eigenvalues are bounded by k^{-N_0} and > 0. Since the operator $\overline{\partial}_k + \overline{\partial}_k^*$ maps $\mathscr{E}^q_{0<\lambda\leq k^{-N_0}}(M, L^k)$ injectively into $\mathscr{E}^{q+1}_{0<\lambda\leq k^{-N_0}}(M, L^k) \oplus \mathscr{E}^{q-1}_{0<\lambda\leq k^{-N_0}}(M, L^k)$. Thus,

$$\dim \mathscr{E}^q_{0<\lambda\leq k^{-N_0}}(M, L^k) \leq \dim \mathscr{E}^{q+1}_{0<\lambda\leq k^{-N_0}}(M, L^k) + \dim \mathscr{E}^{q-1}_{0<\lambda\leq k^{-N_0}}(M, L^k).$$

From this observation and Corollary 10.7, we deduce:

Corollary 10.8. *Let (L, h^L) be a holomorphic line bundle over a compact Hermitian manifold (M, Θ) of dimension n. If $N_0 \geq 2n+1$, then*

$$\begin{aligned}&\dim \mathscr{H}^q(M, L^k) + \dim \mathscr{E}^{q-1}_{0<\lambda\leq k^{-N_0}}(M, L^k) + \dim \mathscr{E}^{q+1}_{0<\lambda\leq k^{-N_0}}(M, L^k)\\ &\quad\geq k^n (2\pi)^{-n} \int_{M(q)} \left|\det \dot{R}^L(x)\right| dv_M(x) + o(k^n).\end{aligned}$$

In particular, we have

$$\begin{aligned}&\dim \mathscr{H}^q(M, L^k)\\ &\geq k^n(2\pi)^{-n}\Big(\int_{M(q)} \left|\det \dot{R}^L(x)\right| dv_M(x) - \int_{M(q-1)} \left|\det \dot{R}^L(x)\right| dv_M(x)\\ &\quad - \int_{M(q+1)} \left|\det \dot{R}^L(x)\right| dv_M(x)\Big) + o(k^n).\end{aligned} \tag{10.6}$$

Hence, if $M(q-1) = \emptyset$, $M(q+1) = \emptyset$, then

$$\dim \mathscr{H}^q(M, L^k) = k^n(2\pi)^{-n}\Big(\int_{M(q)} \left|\det \dot{R}^L(x)\right| dv_M(x)\Big) + o(k^n). \tag{10.7}$$

By Corollary 10.7 and a straightforward application of the linear algebra result from Demailly [16, Lemma 4.2] and [36, Lemma 3.2.12] to the complex $(\mathscr{E}^{\bullet}_{k^{-N_0}}(M, L^k), \overline{\partial}_k)$, we obtain Demailly's strong holomorphic Morse inequalities [16, Th. 0.1] (see also [36, Th. 1.7.1]): for any $q \in \{0, 1, \ldots, n\}$ we have for $k \to \infty$

$$\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(M, L^k) \leq k^n(2\pi)^{-n} \sum_{j=0}^{q} (-1)^{q-j} \int_{M(j)} \left|\det \dot{R}^L(x)\right| dv_M(x) + o(k^n)\,. \tag{10.8}$$

Let us close with an amusing by-product of Theorem 1.1. Assume that $\dot{R}^L$ is non-degenerate of constant signature (n_-, n_+) at each point of M. From Theorem 1.1, we see that if $q \neq n_-$, then $P^{(q)}_k(x) = O(k^{-N})$, for every $N \geq 0$. Thus,

$$\dim \mathscr{H}^q(M, L^k) = O(k^{-N}), \quad \forall N \geq 0.$$

Since $\dim \mathscr{H}^q(M, L^k)$ is an integer, we obtain the Andreotti-Grauert coarse vanishing theorem (see [35, Th. 1.5], [36, Rem. 8.2.6]):

$$\dim \mathscr{H}^q(M, L^k) = 0\,, \quad \text{for } k \text{ large enough.} \tag{10.9}$$

This proof uses just estimates of the spectral spaces. The original proof of Andreotti-Grauert was based on cohomology finiteness theorems for the disch bundle L^*. Ph. Griffiths gave a proof using the Bochner-Kodaira-Nakano formula. For a proof using Lichnerowicz formula and a comparison of methods see [35, Th. 1.5], [35, Rem. 1.6]. Note that the above proof of (10.9) provides a positive answer to a question of Bouche [7] whether one could get vanishing theorems by just using (heat or Bergman) kernel methods.

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