Annali di Matematica 184, 1–16 (2005)

Digital Object Identifier (DOI) 10.1007/s10231-003-0087-z

George Marinescu⋆

A criterion for Moishezon spaces with isolated singularities

Received: March 6, 2001; new version received: April 23, 2003
Published online: April 5, 2004 –

Abstract. We give a criterion for a compact complex space with isolated singularities to be Moishezon in the spirit of Siu–Demailly's solution to the Grauert–Riemenschneider conjecture. It refines a previous work by Nadel and Tsuji, and another one by Takayama, in a more specific situation.

1. Introduction

The object of this paper is the study of compact complex spaces which carry a maximal number of independent meromorphic functions. Such spaces are called Moishezon. Let us recall the solution of the Grauert–Riemenschneider [10] conjecture as given by Siu [25] and Demailly [9]. Let X be a compact complex manifold of dimension n and E a line bundle over X. Assume that either E is semi-positive and positive at one point (Siu's condition), or

$$\int_{X(\leqslant 1)} \big(\sqrt{-1}\Theta(E)\big)^n > 0 \tag{D}$$

(Demailly's condition). Then X is Moishezon. Here $X(\leqslant 1)$ is the open set where the curvature $\sqrt{-1}\Theta(E)$ is non-degenerate and has at most one negative eigenvalue.

Our purpose is to prove the following:

Main Theorem. *Let X be a compact complex space of dimension $n \geqslant 2$ and with isolated singularities. Suppose that we have one the following conditions.*

(i) *There exists a holomorphic hermitian line bundle E on* $\operatorname{Reg}(X)$ *which is semi-positive in a deleted neighbourhood of* $\operatorname{Sing}(X)$ *and satisfies condition* (D) *on* $\operatorname{Reg}(X)$.

G. Marinescu: Institut für Mathematik, Humboldt–Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Deutschland, e-mail: george@mathematik.hu-berlin.de and Institute of Mathematics of the Romanian Academy, PO Box 1–764, RO–70700, Bucharest, Romania

⋆ Supported by a DFG Stipendium at the Graduiertenkolleg "Geometrie und Nicht-lineare Analysis" at Humboldt–Universität zu Berlin and SFB 288.

(ii) *There exists a hermitian holomorphic line bundle on X, with possibly singular hermitian metric at* Sing (X)*, such that the curvature current* $\sqrt{-1}\Theta(E)$ *is dominated by the euclidian metric near* Sing (X) *and condition* (D) *is fulfilled on* Reg (X).

Then X is Moishezon.

In the case of a compact manifold the proof is based on asymptotic Morse (or Riemann–Roch) inequalities, namely $\dim H^0(X, E^k) \geqslant k^n \int_{X(\leqslant 1)} \left(\frac{\sqrt{-1}}{2\pi}\Theta(E)\right)^n + o(k^n)$ (see the notations below). Moreover, Ji and Shiffman [14] and Bonavero [5] found the following characterization: a compact manifold X is Moishezon if and only if it carries a strictly positive closed integral (1, 1)-current.

Nadel and Tsuji [19] extended the asymptotic Morse inequality for complete Kähler manifolds for which the Kähler metric satisfies $\operatorname{Ric}\eta \leqslant -\eta$. Elaborating on the proof of Nadel and Tsuji, Takayama [26] proved the following version of the theorem of Ji–Shiffman. Let X be a compact complex space and E a line bundle over X endowed with a singular hermitian metric which is smooth outside a proper analytic set $Z \supset \operatorname{Sing}(X)$. Assume that the curvature current of the metric is strictly positive in a neighbourhood of Z and that condition (D) is fulfilled on $X \smallsetminus Z$. Then X is Moishezon.

Our sufficient condition (i) shows that in the case of isolated singularities we can significantly weaken the hypothesis of the previous result. The bundle E may be defined only on Reg (X) and we may ask just semipositivity in a deleted neighbourhood of Sing (X). If E is defined over all X the hypothesis about the curvature near Sing (X) can be further weakened, as shown in (ii).

The Main Theorem is proved in Section 2. We extend the proof of Demailly and Nadel–Tsuji to our case by using a strictly plurisubharmonic exhaustion function from below for Reg (X). The existence of such a function expresses the strong concavity of Reg (X) for spaces with isolated singularities. That is why we cannot state our result for arbitrary compact complex spaces.

In Section 3 we apply the inequalities obtained in the proof of the Main Theorem. As already observed in [19] the asymptotic Morse inequality produces upper bounds for the volume of the manifold. In a similar manner, Corollary 2.1 below gives estimates for the volume of Reg (X) in the metric $\sqrt{-1}\Theta(E)$ (assuming that E is positive). This is linked to the following question of Ph. A. Griffiths [11]. Assume that a quasiprojective manifold M is the quotient of a bounded domain D in $\mathbb{C}^n$ by discrete group action. Has M finite volume with respect to the induced Bergman metric? Mok [17, Proposition 1, p. 168] gave a positive answer, when $\operatorname{codim}(\overline{M} \smallsetminus M) \geqslant 2$, where $\overline{M}$ is a projective compactificatin of M.

We precise this result if M can be compactified by adding finitely many points, that is, $M = X \smallsetminus S$, where X is a compact complex space and $S \supset \operatorname{Sing}(X)$ is finite. Then the volume of M in the induced Bergman metric is less than the growth of the canonical bundle K_M of M. An other question in [11] is whether the finiteness of the volume of the induced Bergman metric on the quotient M implies the completeness of the Bergman metric on the covering D. Using our arguments we can show that the answer to the latter question is negative.

I would like to thank the referee for suggestions concerning the style of the paper.

1.1. Notations and terminology

We denote by Reg (X) and Sing (X) the regular and singular parts of the complex space X of dimension n. The complex spaces are assumed to be irreducible and reduced. The space X is called Moishezon if the transcendence degree of its meromorphic function field equals its complex dimension.

Let us remind that by the definition of Andreotti and Grauert [2] a manifold Z is called 1-concave if there exists a smooth function $\varphi : Z \longrightarrow (a, b\,]$ where $a \in \{-\infty\} \cup \mathbb{R}$, $b \in \mathbb{R}$, such that $\{\varphi > c\} \Subset Z$ for all $c \in (a, b\,]$ and φ is strictly plurisubharmonic outside a compact set. A manifold Z is called pseudoconcave in the sense of Andreotti [1], if it contains a relatively open set Y with pseudoconcave boundary (this is true if, for example, the boundary is smooth and the Levi form has at least one negative eigenvalue). It is easy to see that Reg (X) is 1-concave (since X has just isolated singularities).

Let Z be a pseudoconcave manifold in the sense of Andreotti. By [1] the space of global holomorphic sections in any holomorphic vector bundle over Z is finite dimensional. Let $E \longrightarrow Z$ be a holomorphic line bundle on Z. The growth of E is defined by

$$\mathrm{gw}(E) = \liminf_{k\to\infty} k^{-n} \dim H^0\big(Z, E^k \otimes K_Z\big)\,,$$

where K_Z is the canonical bundle of Z.

If (E, h) is a line bundle over Reg (X) endowed with a smooth hermitian metric h we denote by Reg $(X)(0)$ the open set where $\sqrt{-1}\Theta(E)$ is positive definite and by Reg $(X)(1)$ the open set where $\sqrt{-1}\Theta(E)$ is non-degenerate and has exactly one negative eigenvalue. Note that the integral of the (n, n)-form $\big(\sqrt{-1}\Theta(E)\big)^n$ over Reg $(X)(0)$ is positive while the integral over Reg $(X)(1)$ is negative. For the notions of hermitian metric on a singular space and singular hermitian metric see Section 2.2.

We denote by K_X the canonical bundle of Reg (X). The abbreviation psh means plurisubharmonic. We denote the characteristic function of a set Ω by $\mathbf{1}_\Omega$.

2. Proof of the Main Theorem

We will work on the open manifold Reg (X) and prove that it posseses a lot of meromorphic functions which extend to X by the Levi extension theorem.

In order to perform analysis on Reg (X) we introduce first a good exhaustion function and a complete metric. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities of X. Let us denote by D_i the components of the exceptional divisor. Then there exist positive integers n_i such that $D := \sum n_i\, D_i$ admits a smooth hermitian metric such that the induced line bundle $[D]$ is negative in a neighbourhood $\widetilde{U}$ of D (cf. [24]). Let us consider a canonical section s of $[D]$, i.e. $D = (s)$, and denote by $|s|^2$ the poinwise norm of s with respect to the above metric. By Lelong-

Poincaré

$$\frac{\sqrt{-1}}{2\pi}\,\partial\bar\partial\log|s| = (\text{the current of integration on } D) - \frac{\sqrt{-1}}{2\pi}\,\Theta([D])\,.$$

Hence $\varphi = \log|s|^2$ is strictly plurisubharmonic on $\widetilde{U}\smallsetminus D$. By using a smooth function on $\widetilde{X}$ with compact support in $\widetilde{U}$ which equals one near D we construct a smooth function χ on $\widetilde{X}\smallsetminus D \simeq \operatorname{Reg}(X)$ such that

$$\chi = -\log\big(-\log|s|^2\big) \qquad \text{on} \quad \widetilde{U}\smallsetminus D\,.$$

Since $\log|s|^2$ goes to $-\infty$ on D, χ has the same property. Moreover, near D,

$$\partial\bar\partial\chi = \frac{\partial\varphi\wedge\bar\partial\varphi}{\varphi^2} + \frac{\partial\bar\partial\varphi}{-\varphi}$$

so that $\sqrt{-1}\partial\bar\partial\chi$ is positive definite and

$$\sqrt{-1}\partial\chi\wedge\bar\partial\chi \leqslant \sqrt{-1}\partial\bar\partial\chi \tag{2.1}$$

outside a compact set of $\operatorname{Reg}(X)$.

By patching $\partial\bar\partial\chi$ with an arbitrary metric on $\operatorname{Reg}(X)$ with the use of a partition of unity we obtain a metric ω_0 on $\operatorname{Reg}(X)$ such that

$$\omega_0 = \partial\bar\partial\chi \quad \text{outside a compact set} \tag{2.2}$$

It is easily seen that ω_0 is complete by relation (2.1) since the function $-\chi$ is an exhaustion function and $d(-\chi)$ is bounded in the metric ω_0. Also ω_0 is obviously Kähler near $\operatorname{Sing}(X)$.

2.1. *Proof of part (i) of the Main Theorem*

We prove now that condition (i) in Main Theorem entails that X is Moishezon. We introduce a hermitian metric on the line bundle which approximates well the initial metric on compact sets. So let us consider a holomorphic hermitian line bundle E on $\operatorname{Reg}(X)$ endowed with a metric h such that $\sqrt{-1}\Theta(E,h)\geqslant 0$ on $\operatorname{Reg}(X)\smallsetminus K$ where K is a compact set. By streching K we may also assume that ω_0 is Kähler outside K. We equip E with the metric $h_\varepsilon = h\exp(-\varepsilon\chi)$.

Our strategy of finding meromorphic functions is that of Siu and Demailly, namely to show that there are a lot of holomorphic sections in the high powers E^k, since by taking quotients of holomorphic sections we get meromorphic functions. By following Demailly [9] we reduce the problem to estimating the size of certain spectral spaces of the $\bar\partial$-laplacian.

Let us consider the operator $\frac{1}{k}\Delta''_{k,\varepsilon}$ where $\Delta''_{k,\varepsilon} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is the Laplace–Beltrami operator acting on (n,j)-forms with values in E^k over $\operatorname{Reg}(X)$. The metrics used to construct the adjoint $\bar\partial^*$ are ω_0 and h_ε. Let $Q_{k,\varepsilon}$ the quadratic form associated to $\frac{1}{k}\Delta''_{k,\varepsilon}$, that is, $Q_{k,\varepsilon}(u,u) = \frac{1}{k}(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2)$. We denote by E_λ the spectral projectors and by $L^j_k(\lambda) = \operatorname{Ran} E_\lambda(\frac{1}{k}\Delta''_{k,\varepsilon})$ the spectral space of $\frac{1}{k}\Delta''_{k,\varepsilon}$ on

(n, j)-forms. Let $N^j\big(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}\big) = \dim L^j_k(\lambda)$ the spectrum distribution functions of the above operator.

As Lemma 2.2 below shows we have to estimate $N^1(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon})$ from above and then $N^0(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon})$ from below. We do this thanks to a remark of Witten (see [28, p. 666] and [12, Lemma 2.1]): the L^2 norm of the eigenforms of $\frac{1}{k}\Delta''_{k,\varepsilon}$ on (n, j)-forms concentrates asymptotically for $k \longrightarrow \infty$ on the critical set $\mathrm{Reg}\,(X)(j)$. In the original setting of classical Morse theory the rôle of the curvature is played by the hessian of a Morse function f and the eigenforms of the modified laplacian $\Delta_t = (d_t + d_t^*)^2$ where $d_t = e^{-tf} d e^{tf}$ and $t > 0$, concentrate near the critical points of f as $t \longrightarrow \infty$. For the complex geometry setting see [9], [19] and [6].

Let us fix an open, relatively compact neighbourhood Ω of K. We show that the essential spectrum of $\frac{1}{k}\Delta''_{k,\varepsilon}$ does not contain the open interval $(0, \varepsilon/12)$. Then we can compare the counting function on this interval with the counting function of the same operator considered with Dirichlet boundary conditions on Ω and denoted $\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega$. Let $L^j_{k,\Omega}(\mu) = \mathrm{Ran}\, E_\mu(\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega)$, the spectral spaces of $\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega$. Let $N^j\big(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega\big) = \dim L^j_{k,\Omega}(\lambda)$ be the spectrum distribution functions of the above operators acting on (n, j)-forms. For the following lemma compare [12, Lemma 2.1] and [6, Théorème 2.1].

Lemma 2.1. *For k sufficiently large the operator $\frac{1}{k}\Delta''_{k,\varepsilon}$ on $(n, 1)$ forms has discrete spectrum in $(0, \varepsilon/12)$ and $N^1(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}) \leqslant N^1(12\,\lambda + C_1\, k^{-1}, \frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega)$, for λ in $(0, \varepsilon/12)$, where C_1 is a constant independent of k and ε.*

Proof. The curvature relative to the metric h_ε satisfies

$$\sqrt{-1}\Theta(E, h_\varepsilon) = \sqrt{-1}\mathbf{c}(E, h) + \sqrt{-1}\varepsilon\partial\bar\partial\chi + \sqrt{-1}\varepsilon\partial\chi \wedge \bar\partial\chi \geqslant \sqrt{-1}\varepsilon\partial\bar\partial\chi$$

which shows that

$$\sqrt{-1}\Theta(E, h_\varepsilon) \geqslant \varepsilon\,\omega_0 \quad \text{on} \quad X \smallsetminus K\,. \tag{2.3}$$

We use now the Bochner–Kodaira–Nakano inequality for smooth $(n, 1)$-forms compactly supported outside K (where the metric is Kähler):

$$\|\bar\partial\, u\|^2 + \|\bar\partial^*\, u\|^2 \geqslant \Big(\big[\sqrt{-1}\Theta\big(E^k, h^k_\varepsilon\big), \Lambda\big]\, u,\, u\Big)\,, \quad u \in \mathcal{D}^{n,1}\big(X \smallsetminus K, E^k\big)\,.$$

Then (2.3) shows that the curvature term satisfies $\big(\big[\sqrt{-1}\Theta\big(E^k, h^k_\varepsilon\big), \Lambda\big]\, u,\, u\big) \geqslant k\,\varepsilon\,\|u\|^2$ and therefore

$$Q_{k,\varepsilon}(u, u) \geqslant k\,\varepsilon\,\|u\|^2\,, \quad u \in \mathcal{D}^{n,1}\big(X \smallsetminus K, E^k\big)\,. \tag{2.4}$$

Let $\rho \in \mathcal{C}^\infty(\mathrm{Reg}\,(X))$ such that $\rho = 0$ on L and $\rho = 1$ on $\mathrm{Reg}\,(X) \smallsetminus \Omega$, where L is a neighbourhood of K in Ω. Let $u \in \mathcal{D}^{n,1}(\mathrm{Reg}\,(X), E^k)$, so that $\rho\, u$ has support outside K. Denote $C = 6 \sup |d\rho|^2 < \infty$. By the elementary estimate: $Q_{k,\varepsilon}(\rho\, u, \rho\, u) \leqslant \frac{3}{2} Q_{k,\varepsilon}(u, u) + C\, k^{-1} \|u\|^2$, we obtain

$$\|u\|^2 \leqslant 6\,\varepsilon^{-1}\, Q_{k,\varepsilon}(u, u) + 4 \int_\Omega \big|(1-\rho)u\big|^2\,, \quad k \geqslant 4\, C\, \varepsilon^{-1} \tag{2.5}$$

for any compactly supported u. Since the metric ω_0 is complete, the density lemma of Andreotti and Vesentini [3, Proposition 5, p. 93] shows that $\frac{1}{k}\Delta''_{k,\varepsilon}$ is essentially self-adjoint. Thus (2.5) is true for any u in the domain of the quadratic form $Q_{k,\varepsilon}$. From relation (2.5) we infer that the spectral spaces corresponding to the lower part of the spectrum of $\frac{1}{k}\Delta''_{k,\varepsilon}$ on $(n,1)$-forms can be injected into the spectral spaces of $\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega$ which correspond to the Dirichlet problem on Ω. Namely, for $\lambda < \varepsilon/12$, the morphism $L^1_k(\lambda) \longrightarrow L^1_{k,\Omega}(12\lambda + C_1 k^{-1})$, $u \longmapsto E_{12\lambda+C_1k^{-1}}(\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega)(1-\rho)u$ is injective, where $C_1 = 8\,C$. In order to prove the injectivity we choose $u \in L^1_k(\lambda)$, $\lambda < \varepsilon/12$ to the effect that $Q_k(u) \leqslant \lambda\|u\|^2 \leqslant (\varepsilon/12)\|u\|^2$. Plugging this relation in (2.5) we get

$$\|u\|^2 \leqslant 8\int_\Omega \left|(1-\rho)u\right|^2, \quad u \in L^1_k(\lambda), \quad \lambda < \varepsilon/12. \tag{2.6}$$

Let us denote by $Q_{k,\Omega}$ the quadratic form of $\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega$. Then by the elementary estimate above and (2.6),

$$Q_{k,\Omega}\big((1-\rho)u\big) \leqslant \frac{3}{2}\,Q_{k,\varepsilon}(u) + \frac{C}{k}\|u\|^2 \leqslant \left(12\,\lambda + \frac{8\,C}{k}\right)\int_\Omega \left|(1-\rho)u\right|^2$$

which shows that $E_{12\lambda+C_1k^{-1}}(\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega)\,(1-\rho)u = 0$ entails $(1-\rho)u = 0$ so that $u = 0$ by (2.6). □

Lemma 2.2. *For $\lambda < \varepsilon/12$ and sufficiently large k we have*

$$\dim H^0\big(\mathrm{Reg}\,(X),\, E^k \otimes K_X\big) \geqslant N^0\big(\lambda,\, \tfrac{1}{k}\Delta''_{k,\varepsilon}\big) - N^1\big(\lambda,\, \tfrac{1}{k}\Delta''_{k,\varepsilon}\big).$$

Proof. Since $\frac{1}{k}\Delta''_{k,\varepsilon}$ commutes with $\bar\partial$ it follows that the spectral projections of $\frac{1}{k}\Delta''_{k,\varepsilon}$ commute with $\bar\partial$ too, showing thus $\bar\partial L^0_k(\lambda) \subset L^1_k(\lambda)$ and therefore we have the bounded operator $\bar\partial_\lambda : L^0_k(\lambda) \longrightarrow L^1_k(\lambda)$ where $\bar\partial_\lambda$ denotes the restriction of $\bar\partial$ (by the definition of $L^0_k(\lambda)$, $\bar\partial_\lambda$ is bounded by $k\lambda$). Since the kernel of $\bar\partial_\lambda$ consits of holomorphic sections we infer: $\dim H^0(\mathrm{Reg}\,(X), E^k \otimes K_X) \geqslant \dim\ker\bar\partial_\lambda$. But $N^0(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}) = \dim\ker\bar\partial_\lambda + \dim\mathrm{Ran}\,\bar\partial_\lambda$. By Lemma 2.1, $N^1(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon})$ is finite dimensional. Obviously $\dim\mathrm{Ran}\,\bar\partial_\lambda \leqslant N^1(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon})$. Moreover, since $\mathrm{Reg}\,(X)$ is a 1-concave manifold $\dim H^0(\mathrm{Reg}\,(X), E^k \otimes K_X) < \infty$ so $\dim\ker\bar\partial_\lambda < \infty$. Thus all spaces appearing in the statement are finite dimensional and we have the desired inequality. □

We obtain now a lower estimate for $N^0\big(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}\big)$.

Lemma 2.3. $N^0\big(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}\big) \geqslant N^0\big(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega\big)$.

Proof. This is an immediate consequence of the following form of the variational principle (called Glazman lemma, see [23]). Let P be a self-adjoint positive operator on a Hilbert space $\mathcal{H}$. Then the spectrum distribution function $N(\lambda, P) := \dim\mathrm{Ran}\,E_\lambda(P)$ satisfies:

$$N(\lambda, P) = \sup\Big\{\dim L \mid L \text{ closed } \subset \mathrm{Dom}(Q),\ Q(f,f) \leqslant \lambda\|f\|^2,\ \forall f \in L\Big\}$$

where Q is the quadratic form of P. The lemma follows by the variational principle and the simple remark that $\mathrm{Dom}(Q_{k,\varepsilon}) \supset \mathrm{Dom}(Q_{k,\Omega})$. Indeed, let us denote by $\lambda_0 \leqslant \lambda_1 \leqslant \dots$ the spectrum of $\frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega$ acting on $(n,0)$-forms. Let $\{e_i\}_i$ be an orthonormal basis which consists of eigenforms corresponding to the eigenvalues $\{\lambda_i\}_i$; if we let $\widetilde{e}_i = 0$ on $\mathrm{Reg}\,(X) \setminus \Omega$ and $\widetilde{e}_i = e_i$ on Ω, $\widetilde{e}_i \in \mathrm{Dom}(Q_{k,\varepsilon})$ and $Q_{k,\varepsilon}(\widetilde{e}_i, \widetilde{e}_j) = \delta_{i,j}\lambda_i$. Let Φ^0_λ be the subspace spanned by $\{e_i : \lambda_i \leqslant \lambda\}$ in $L^2_{n,0}(\Omega, E^k)$ and Φ_λ the closed subspace spanned by $\{\widetilde{e}_i : \lambda_i \leqslant \lambda\}$ in $L^2_{n,0}(\mathrm{Reg}\,(X), E^k)$. Then $\dim \Phi_\lambda = \dim \Phi^0_\lambda = N\big(\lambda, \frac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega\big)$. If f is a linear combination of $\{\widetilde{e}_i : \lambda_i \leqslant \lambda\}$, $Q_{k,\varepsilon}(f,f) \leqslant \lambda\|f\|^2$ and, as $\mathrm{Dom}(Q_{k,\varepsilon})$ is complete in the graph norm, we obtain $\Phi_\lambda \subset \mathrm{Dom}(Q_{k,\varepsilon})$ and $Q_{k,\varepsilon}(f,f) \leqslant \lambda\|f\|^2$, $f \in \Phi_\lambda$. The variational principle implies now the lemma. □

Remark. By using the Perturbation lemma of Takayama [26] we can actually show the spectrum of $\frac{1}{k}\Delta''_{k,\varepsilon}$ acting on $(n,0)$-forms is discrete near zero for large k. Then we can apply the min–max principle in the usual form in order to obtain Lemma 2.3.

The benefit of Lemmas 2.1–2.3 is that the asymptotic behaviour of the spectrum distribution function for the Dirichlet problem on Ω has been determined explicitely by Demailly [9]. Assume that $\partial\Omega$ has measure zero. Then there exists a function $\nu^j_\varepsilon(\lambda, x)$ depending on the eigenvalues of the curvature of (E, h_ε), bounded on compact sets of $\mathrm{Reg}\,(X)$ and right continuous in μ, such that for any $\mu \in \mathbb{R}$

$$\limsup_{k\longrightarrow\infty} k^{-n} N^j\big(\lambda, \tfrac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega\big) \leqslant \frac{1}{n!}\int_\Omega \nu^j_\varepsilon(\lambda, x)\, dV(x)\,. \tag{2.7}$$

Moreover there exists an at most countable set $\mathcal{D}_\varepsilon \subset \mathbb{R}$ such that for μ outside $\mathcal{D}_\varepsilon$ the limit of the left-hand side expression exists and we have equality in (2.7). We do not need the explicit form of $\nu^j_\varepsilon(\lambda, x)$.

For $\lambda < \varepsilon/12$ and λ outside $\mathcal{D}_\varepsilon$ we apply Demailly's theorem and Lemma 2.3:

$$\lim_{k\longrightarrow\infty} k^{-n} N^0\big(\lambda, \tfrac{1}{k}\Delta''_{k,\varepsilon}\big) \geqslant \int_\Omega \nu^0_\varepsilon\big(\lambda, x\big)\, dV(x)\,.$$

For $j = 1$ we remark that given $\delta > 0$ we have

$$\begin{aligned}\limsup_{k\longrightarrow\infty} k^{-n} N^1\big(12\lambda + C_1 k^{-1}, \tfrac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega\big) &\leqslant \limsup_{k\longrightarrow\infty} k^{-n} N^1\big(12\lambda + \delta, \tfrac{1}{k}\Delta''_{k,\varepsilon}\restriction_\Omega\big)\\ &= \int_\Omega \nu^1_{(E,h_\varepsilon)}(12\lambda + \delta, x)\, dV(x)\end{aligned}$$

and that after letting k go to infinity we can also let δ go to zero. Using these remarks we see that for all but a countable set of λ we have

$$\liminf_{k\longrightarrow\infty} k^{-n} \dim H^0\big(\mathrm{Reg}\,(X), E^k \otimes K_X\big) \geqslant \int_\Omega \big[\nu^0_{(E,h_\varepsilon)}(\lambda, x) - \nu^1_{(E,h_\varepsilon)}(12\lambda, x)\big] dV(x).$$

In the latter estimate we may let $\lambda \longrightarrow 0$ (by avoiding the exeptional countable set) and this yields, by the formulas in [9, p. 224] for the right-hand side

$$\liminf_{k\longrightarrow\infty} k^{-n} \dim H^0\big(\mathrm{Reg}\,(X), E^k \otimes K_X\big) \geqslant \frac{1}{n!(2\pi)^n}\int_{\Omega(\leqslant 1, h_\varepsilon)} \big(\sqrt{-1}\Theta(E, h_\varepsilon)\big)^n\,.$$

Since on Ω the metric h_ε converges uniformly to h together with all its derivatives, we may let $\varepsilon \longrightarrow 0$. Thus we get rid of ε in the right-hand side. Moreover the fact that E is semipositive outside a compact set shows that we may let Ω exhaust $\mathrm{Reg}\,(X)$. Indeed, for $\Omega \Supset K$ we have $\int_{\Omega(1,h)} \big(\sqrt{-1}\Theta(E,h)\big)^n = \int_{\mathrm{Reg}\,(X)(1,h)} \big(\sqrt{-1}\Theta(E,h)\big)^n$ and obviously

$$\int_{\Omega(0,h)} \big(\sqrt{-1}\Theta(E,h)\big)^n \longrightarrow \int_{\mathrm{Reg}\,(X)(0,h)} \big(\sqrt{-1}\Theta(E,h)\big)^n$$

for $\Omega \longrightarrow \mathrm{Reg}\,(X)$, by the monotone convergence theorem. Therefore,

$$\liminf_{k\longrightarrow\infty} k^{-n} \dim H^0(\mathrm{Reg}\,(X), E^k \otimes K_X) \geqslant \frac{1}{n!(2\pi)^n} \int_{\mathrm{Reg}\,(X)(\leqslant 1,h)} \big(\sqrt{-1}\Theta(E,h)\big)^n \tag{2.8}$$

and the last integral is positive by hypothesis.

Moreover the method of proving the finiteness in [1] gives us an upper bound for $\dim H^0(\mathrm{Reg}\,(X), E^k \otimes K_X)$. Let Z be a pseudoconcave manifold in the sense of Andreotti and E a line bundle over Z. Denote by ϱ_k the generic rank of the canonical meromorphic mapping

$$\Phi_{E^k\otimes K_Z} : Z \dashrightarrow \mathbb{P}\big(H^0(Z, E^k \otimes K_Z)^*\big)\,.$$

Lemma 2.4 (Siegel–Serre). $\dim H^0(Z, E^k \otimes K_Z) \leqslant C\,k^{\varrho_k}$ *for some* $C > 0$. *In particular* $gw(E) < \infty$.

Proof. The proof follows the Siegel–Serre idea to apply the Schwarz lemma in order to show that a section vanishes identically if it vanishes to order ϱ_k at some points of a fine net. In our case we follow Andreotti [1] to choose the points of the net as the centers of polydiscs which cover a relatively compact open set Y with pseudoconcave boundary such that their Shilov boundaries lie in Y. For details see [16, Proposition 5.7]. □

Let $\varkappa(E)$ is the supremum over k of ϱ_k. By the Siegel–Serre lemma,

$$\dim H^0(\mathrm{Reg}\,(X), E^k \otimes K_X) \leqslant C\,k^{\varkappa(E)}\,, \quad k > 0\,. \tag{2.9}$$

From (2.8), (2.9) and condition (D) (the positivity of the integral in (2.8)) we obtain $\varkappa(E) = n$, that is, the line bundle $E^k \otimes K_X$ gives local coordinates on an open dense set of $\mathrm{Reg}\,(X)$ for sufficiently large k. This clearly implies X Moishezon and thereby concludes the proof of the first half of Main Theorem. □

From the proof we infer the following.

Corollary 2.1. *Let E is a line bundle over* $\mathrm{Reg}\,(X)$*, where X is a compact complex space with only isolated singularities.*

(i) *If E is semipositive outside a compact set,*

$$\int_{\mathrm{Reg}\,(X)(0)} \Big(\tfrac{\sqrt{-1}}{2\pi}\Theta(E)\Big)^n \leqslant gw(E) - \int_{\mathrm{Reg}\,(X)(1)} \Big(\tfrac{\sqrt{-1}}{2\pi}\Theta(E)\Big)^n < \infty.$$

(ii) *If E is positive on* Reg (X)

$$\int_{\mathrm{Reg}\,(X)(0)} \left(\frac{\sqrt{-1}}{2\pi}\Theta(E)\right)^n \leqslant gw(E) < \infty.$$

(iii) *If* $\psi : \mathrm{Reg}\,(X) \longrightarrow \mathbb{R}$ *is a smooth function which is psh outside a compact set,*

$$\int_{\mathrm{Reg}\,(X)(0)} (\sqrt{-1}\partial\bar{\partial}\psi)^n \leqslant -\int_{\mathrm{Reg}\,(X)(1)} (\sqrt{-1}\partial\bar{\partial}\psi)^n < \infty$$

where Reg $(X)(0)$ *is the open set where* ψ *is strictly psh.*

Proof. Relation (2.8) shows the left-hand side inequality in (i), since the integral in (2.8) is the sum of two corresponding integrals taken over the sets Reg $(X)(0)$ and Reg $(X)(1)$. The latter is finite since Reg $(X)(1)$ is relatively compact by the hypothesis on the semipositivity of E. By the Serre-Siegel lemma we get also the finiteness in (i).

From (i) we infer immediately (ii).

To prove (iii) we apply (i) to the trivial bundle E endowed with the metric $\exp(-\psi)$ and we use the obvious fact that $\mathrm{gw}(E) = 0$. □

2.2. *Proof of part (ii) of the Main Theorem*

We show now the second hypothesis of Main Theorem implies X is Moishezon. We will consider a variation of the complete metric used hitherto. First we recall the notion of hermitian metric on a singular space. Let us consider a covering $\{U_\alpha\}$ of X and embeddings $\iota_\alpha : U_\alpha \hookrightarrow \mathbb{C}^{N_\alpha}$. A metric on X is a metric ω on Reg (X) which on every open set U_α as above is the pullback of a hermitian metric on the ambient space $\mathbb{C}^{N_\alpha}$, $\omega = \iota_\alpha^* \omega_\alpha$. It is constructed as usual by a partition of unity argument. Since the singularities are isolated we can assume that the metric is *distinguished*, that is, in the neighbourhood of the singular points ω_α is the euclidian metric. In particular ω is Kähler near Sing (X). We consider then the metric $\omega_0 = A\omega + \partial\bar{\partial}\chi$ where $A > 0$ is chosen sufficiently large (to ensure that ω_0 is a metric away from the open set where $\partial\bar{\partial}\chi$ is positive definite). ω_0 is complete by the same argument as in the previous proof (see relation (2.1)). Note that by Corollary 2.1 the metric ω_0 has finite volume. This follows from the fact that, near Sing (X), χ is strictly psh and ω is given by the euclidian potential.

Assume now that $E|_{U_\alpha}$ is the inverse image by ι_α of the trivial line bundle $\mathbb{C}_\alpha$ on $\mathbb{C}^{N_\alpha}$. Moreover we consider hermitian metrics $h_\alpha = e^{-\varphi_\alpha}$ on $\mathbb{C}_\alpha$ such that $\iota_\alpha^* h_\alpha = \iota_\beta^* h_\beta$ on $U_\alpha \cap U_\beta \cap \mathrm{Reg}\,(X)$. The system $h = \left\{\iota_\alpha^* h_\alpha\right\}$ is called a hermitian metric on X. It clearly induces a hermitian metric on Reg (X). We shall allow our metrics to be singular at the singular points, that is, $\varphi_\alpha \in L^1_{loc}(\mathbb{C}^{N_\alpha})$ and φ_α is smooth outside $\iota_\alpha(\mathrm{Sing}\,(X))$. The curvature current $\sqrt{-1}\Theta(E)$ is given in U_α by $\iota_\alpha^*(\sqrt{-1}\partial\bar{\partial}\varphi_\alpha)$ which on Reg (X) agrees with the curvature of the induced metric. We shall suppose in the sequel that the curvature current is dominated by the euclidian metric i.e. $\sqrt{-1}\partial\bar{\partial}\varphi_\alpha$ is bounded above and below by constant times $\omega_E = \sqrt{-1}\sum dz_j \wedge d\bar{z}_j$.

Let us consider now a neighbourhood U of the singular set. We assume that U is small enough so that there are well defined on U a potential ρ for ω and a potential φ for the curvature $\sqrt{-1}\Theta(E)$ (they are restrictions from ambient spaces). By suitably cutting-off we may define a function $\psi \in \mathcal{C}^\infty(\mathrm{Reg}\,(X))$ such that

$$\psi = \chi - \varphi + A\,\rho \tag{2.10}$$

near $\mathrm{Sing}\,(X)$. Remark that, since $\sqrt{-1}\Theta(E)$ is bounded above by a continuous $(1,1)$ form near $\mathrm{Sing}\,(X)$, the potential $-\varphi$ is bounded above near the singular set. This holds true for ρ too (it is smooth) so that ψ tends to $-\infty$ at the singular set $\mathrm{Sing}\,(X)$. Let us consider a smooth function $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\gamma(t) = \begin{cases} 0 & \text{if} \quad t \geqslant 0\,, \\ t & \text{if} \quad t \leqslant -1\,. \end{cases}$$

and the functions $\gamma_\nu : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\gamma_\nu(t) = \gamma(t-\nu)$ for all positive integers ν. Let us denote the hermitian metric on E by h and let us consider the metric $h_\nu = h\,\exp\big(-\gamma_\nu(\psi)\big)$ with curvature $\sqrt{-1}\Theta(E,h_\nu) = \sqrt{-1}\mathbf{c}(E,h) + \gamma_\nu'(\psi)\partial\bar\partial\psi + \gamma_\nu''(\psi)\partial\psi\wedge\bar\partial\psi$. On the set $\{\psi \leqslant -\nu-1\}$ we have $\gamma_\nu(\psi) = \psi - \nu$ so that $\gamma_\nu'(\psi) = 1$ and $\gamma_\nu''(\psi) = 0$ and therefore $\sqrt{-1}\Theta(E,h_\nu) = \sqrt{-1}\mathbf{c}(E,h) + \partial\bar\partial\psi$. Since ψ goes to $-\infty$ when we approach the singular set we may choose ν_0 such that for $\nu \geqslant \nu_0$ we have $\{\psi \leqslant -\nu-1\} \subset U$ where U is the neighbourhood of $\mathrm{Sing}\,(X)$ where ψ has the form (2.10). Bearing in mind the meaning of φ and ρ together with the definition of ω_0 it is straightforward that $\sqrt{-1}\Theta(E,h_\nu) = \omega_0$ on $\{\psi \leqslant -\nu-1\}$. This relation is analogous to (2.3). Therefore we may apply now the same argument as in the proof of the first part in order to obtain as in (2.8), for $k \longrightarrow \infty$,

$$\dim H^0\big(\mathrm{Reg}\,(X),\, E^k \otimes K_X\big) \geqslant \frac{k^n}{n!(2\pi)^n}\int_{\Omega_\nu(\leqslant 1,h_\nu)} \big(\sqrt{-1}\Theta(E,h_\nu)\big)^n + o(k^n).$$

We have denoted Ω_ν the compact set $\{\psi \geq -\nu-2\}$. We decompose this set in $\Omega_\nu' = \{\psi \geqslant -\nu\}$ and $\Omega_\nu'' = \{-\nu-2 \leqslant \psi \leqslant -\nu\}$ since on Ω_ν' we have $\gamma_\nu(\psi) = 0$ and $\sqrt{-1}\Theta(E,h_\nu) = \sqrt{-1}\mathbf{c}(E,h)$. We infer that

$$\int_{\Omega_\nu'(\leqslant 1,h_\nu)} \big(\sqrt{-1}\Theta(E,h_\nu)\big)^n = \int_{\mathrm{Reg}\,(X)(\leqslant 1,h)} \mathbf{1}_{\Omega_\nu'}\alpha_1\cdots\alpha_n\,dV_0 \tag{2.11}$$

where $\alpha_1,\ldots,\alpha_n$ are the eigenvalues of $\sqrt{-1}\Theta(E,h)$ with respect to ω_0 and dV_0 is the volume form of the same metric. Our hypothesis on the domination of $\sqrt{-1}\Theta(E,h)$ by the euclidian metric implies that $\sqrt{-1}\Theta(E,h)$ is dominated by ω and by ω_0. Hence the product $\alpha_1\cdots\alpha_n$ is bounded on $\mathrm{Reg}\,(X)$. Since $\mathrm{Reg}\,(X)(\leqslant 1)$ has finite volume with respect to ω_0 the functions $|\mathbf{1}_{\Omega_\nu'}\alpha_1\cdots\alpha_n|$ are bounded by an integrable function. On the other hand $\mathbf{1}_{\Omega_\nu'} \longrightarrow 1$ when $\nu \longrightarrow \infty$ so that the integrals in (2.11) tend to $\int_{\mathrm{Reg}\,(X)(\leqslant 1,h)}\big(\sqrt{-1}\Theta(E,h)\big)^n$ which is assumed to be positive.

Thus it suffices to show that the integral on the set Ω''_ν i.e.

$$\int_{\Omega''_\nu(\leqslant 1, h_\nu)} \left(\sqrt{-1}\Theta(E, h_\nu)\right)^n$$

tends to zero as $\nu \longrightarrow \infty$. For this purpose we use the obvious bound

$$\int_{\Omega''_\nu(\leqslant 1, h_\nu)} \left(\sqrt{-1}\Theta(E, h_\nu)\right)^n \leqslant \sup|\delta_1 \cdots \delta_n| \cdot \mathrm{vol}\,(\Omega''_\nu)$$

where $\delta_1, \ldots, \delta_n$ are the eigenvalues of $\sqrt{-1}\Theta(E, h_\nu)$ with respect to ω_0 and the volume is taken in the same metric. We use now the minimum-maximum principle to see that δ_1 is bounded bellow and $\delta_2, \ldots, \delta_n$ are bounded above on the set of integration $\Omega''_\nu(\leqslant 1, h_\nu)$. For this we need the domination of $\sqrt{-1}\Theta(E, h)$ by ω and the boundedness of γ'_ν and γ''_ν. Since $\mathrm{vol}\,(\Omega''_\nu) \longrightarrow 0$ as $\nu \longrightarrow \infty$ our contention follows. Hence $\dim H^0(\mathrm{Reg}\,(X), E^k \otimes K_X) \gtrsim k^n$ so that Reg (X) has n independent meromorphic functions which can be extended to X by the Levi extension theorem. □

The proof of the Main Theorem is based on the existence of the exhaustion function from below χ and of the complete metric ω_0 with the properties (2.1) and (2.2). These objects are specific to the case of isolated singularities. If X is a compact complex space with $\dim \mathrm{Sing}\,(X) \geqslant 1$, Reg (X) does not generally possess a strictly psh exhaustion function from below. That is why for general complex spaces we need stronger hypothesis in order to obtain the crucial L^2 estimate (2.4) for $(n, 1)$-forms. For example if X is a compact complex Kähler space, Reg (X) admits complete Kähler metric (Ohsawa [21]). Therefore, if Reg (X) admits a semi-positive line bundle which is positive at a point p, standard L^2 estimates for $\bar\partial$ show that $E^k \otimes K_{\mathrm{Reg}\,(X)}$ gives local coordinates at p. Assuming that $\mathrm{codimSing}\,(X) \geqslant 2$ it follows first that Reg (X) has a maximal number of meromorphic functions (since Reg (X) is pseudoconcave in the sense of Andreotti) and then that X is Moishezon (by the Levi extension theorem).

In the non-Kähler case we need a sort of uniform positivity condition on E near Sing (X) in order to absorb the torsion of a complete metric on Reg (X). In this respect the hypothesis in Takayama's theorem seem appropriate. If we want the line bundle E to be defined only on Reg (X) we can introduce the following alternative condition.

Let ω be a hermitian metric on Reg (X) induced from a resolution of singularities $\overline{X}$ of X. Assume that $\sqrt{-1}\Theta(E) \geqslant \omega$ outside a compact set of Reg (X) and that E satisfies condition (D). Suppose moreover that $\mathrm{codimSing}\,(X) \geqslant 2$. Then X is Moishezon. Indeed, the condition $\sqrt{-1}\Theta(E) \geqslant \omega$ shows that we can argue as in [26] and use a generalized Poincaré metric to deduce the L^2 estimate as in (2.4).

It would be interesting to know whether criteria as the Main Theorem carry over to general complex spaces.

3. The volume of the Bergman metric on Zariski open sets

In [11, Question 8.7] Griffiths raised the following question. Assume that D is a bounded open set in $\mathbb{C}^n$ having a discrete group of automorphisms Γ such that D/Γ is quasiprojective. Then (i) is D a domain of holomorphy and (ii) has D/Γ finite volume with respect to the induced Bergman metric? In connection to (i), Mok and Wong [18, Main Theorem] showed that if D/Γ is a Zariski open set in a compact manifold, D is obtained by removing an analytic set from a domain of holomorphy. Moreover, by [18, Theorem, p. 1482], if D is a domain of holomorphy, the hypothesis that D/Γ is a Zariski open set implies that D/Γ is quasiprojective.

In connection to (ii), Mok [17, Proposition 1, p. 168] proved the following using techniques of extending positive currents. Let M be a complex manifold admitting a projective-algebraic compactification $\overline{M}$ such that $\operatorname{codim}(\overline{M} \setminus M) \geqslant 2$. Then any Kähler metric on M has finite volume.

On the other hand, Nadel and Tsuji [19] used the Riemann–Roch inequalities to show that that the volume of a complete Kähler metric with $\operatorname{Ric}\eta \leqslant -\eta$ on a pseudoconcave manifold is finite.

In this section we use the same idea to give an upper bound of the volume of D/Γ in the Bergman metric.

Theorem 3.1. *Let D be a bounded open set in a Stein manifold of dimension $\geqslant 2$. Suppose $\Gamma \subset \operatorname{Aut} D$ is a properly discontinous group without fixed points. Assume D/Γ is a Zariski open set which can be compactified to a complex space by adding finitely many points. Then $vol(D/\Gamma) \leqslant gw(K_{D/\Gamma})$.*

The hypothesis means there exists a compact complex space X with $D/\Gamma \subset \operatorname{Reg} X$ and $D/\Gamma = X \setminus S$, where S is a finite set.

Proof. By hypothesis D possesses a Bergman metric ω which is invariant under analytic automorphisms. It descends to a Kähler metric on any quotient of the domain by a properly discontinuous discrete group $\Gamma \subset \operatorname{Aut}(D)$. We denote $M = D/\Gamma$ and ω_* the induced Bergman metric on $M = D/\Gamma$. If we denote by $B(z, \overline{z})$ the Bergman kernel of D we know that B^{-1} can be considered as a hermitian Γ-invariant metric on the canonical bundle K_D. Since $\omega = \partial\overline{\partial} \log B(z, \overline{z})$ there exists a hermitian metric h_* on K_M such that $\Theta(K_M) = \omega_*$. In other words the canonical bundle K_M is positive and its curvature is given by the induced Bergman metric. This is the observation of Kodaira [15] which permits him to use his embedding theorem if M is merely compact. Our assumption is that $M = X \setminus S$, where X is a compact complex space and where S is a finite set containing the singularities of X. We are therefore in the hypothesis of Corollary 2.1 (ii) for $E = K_M$ and $\sqrt{-1}\Theta(E, h_*) = \omega_*$. □

We can still prove the estimate even if the singular set of a compactification of D/Γ is not of dimension 0. But we need to strengthen the hypothesis on D.

Proposition 3.1. *Let D be a bounded domain of holomorphy in $\mathbb{C}^n$ with complete Bergman metric. Let $\Gamma \subset \operatorname{Aut} D$ which acts freely and properly discontinuously such that D/Γ is Zariski open set in a compact complex space X and* $\operatorname{codim}_X D/\Gamma \geqslant 2$. *Then $vol(D/\Gamma) \leqslant gw(K_{D/\Gamma})$.*

The strong hypothesis here is that the Bergman metric is complete. Thus we can dispense ourselves of the existence of a good exhaustion function as in the proof of Main Theorem. The class of bounded holomorphy domains in $\mathbf{C}^n$ which admit a complete Bergman metric has been intensively studied. If either the domain has $\mathcal{C}^1$ boundary (Ohsawa [20]), or the domain is hyperconvex (Blocki–Pflug [4] and Herbort [13]) then the Bergman metric is complete. For another sufficient condition see Diederich–Ohsawa [8].

Proof. In distinction to the previous case, we have the additional information that the induced Bergman metric ω_* is a *complete* Kähler metric on $M := D/\Gamma$. Then the Riemann–Roch inequality we need is essentialy due to Nadel and Tsuji (loc. cit.). The only difference is that in their case we are given a complete Kähler metric η on M such that the curvature of the induced metric on K_M, i.e. the Ricci curvature, satisfies $\operatorname{Ric}\eta \leqslant -\eta$. In the present case there exists a hermitian metric on K_M whose curvature equals a complete Kähler metric. Then the proof of Nadel and Tsuji goes through to show that

$$\liminf_{k\longrightarrow\infty} k^{-n} \dim H^0\big(M, K_M^k\big) \geqslant \frac{1}{n!(2\pi)^n}\int_M \big(\sqrt{-1}\Theta(K_M)\big)^n = \frac{1}{(2\pi)^n}\operatorname{Vol}_{\omega_*}(M)\,. \tag{3.1}$$

For the sake of completeness, let us say that we can use the proof of Main Theorem in order to get (3.1) (our proof of Main Theorem is actually a generalization of the Riemann–Roch inequality of Nadel and Tsuji). Namely, we work with the metrics ω_* on M and h_* on K_M (no approximation $(h_*)_\varepsilon$ is needed) and we replace (2.3) with the equality $\sqrt{-1}\Theta(E, h_*) = \omega_*$. The proof goes through with the obvious simplifications. In fact we see that the laplacians Δ''_k have no spectrum at all in an interval $(0, a)$, for some $a > 0$.

To conclude we remark that the hypothesis about codimension shows that M is pseudoconcave in the sense of Andreotti and so the left-hand side of (3.1) is finite. □

Let us remark that if D is a bounded symmetric domain and Γ is a torsion-free arithmetic group, the compactification theorem of Satake–Baily–Borel shows the existence of a projective compactification X of D/Γ with $\operatorname{codim}_X D/\Gamma \geqslant 2$. In this case the finiteness of $\operatorname{vol}(D/\Gamma)$ was known by Raghunatan [22].

Finally let us answer negatively to the following complement of Griffiths' [11, Question 8.7]. Namely, does the finite volume assumption for a quasiprojective quotient D/Γ force the Bergman metric to be complete?

Proposition 3.2. *There exists a quasiprojective manifold covered by a bounded domain in* $\mathbb{C}^n$ $(n \geqslant 2)$ *such that the induced Bergman metric is not complete but it has finite volume.*

Proof. Let us consider D a bounded domain in $\mathbb{C}^n$ $(n \geqslant 2)$ having a group Γ of automorphism with $X = D/\Gamma$ a compact manifold. Then, by Kodaira's embedding theorem X is projective. By a theorem of Siegel D has to be a domain of holomorphy. Consider now a point $p \in X$ and the quasiprojective manifold $M = X \setminus \{p\}$. Let us denote by $\pi : D \longrightarrow X$ the covering map. Consider the domain $D_p = \pi^{-1}(M) = D \smallsetminus \pi^{-1}(p)$. Since D_p is obtained by removing a discrete set (the orbit of one point) from D, D_p is not a domain of holomorphy. The domain D_p covers the manifold M. We consider the Bergman metric ω_* on M induced by the Bergman metric ω on D_p. If ω_* were complete then ω would have to be complete. But then, by a theorem of Bremermann [7] this would force D_p to be a domain of holomorphy. On the other hand, ω_* has finite volume by Theorem 3.1. □

4. Generalizations

We shall consider the following setup. Let X be a compact complex manifold and let Z be a complete pluripolar set. This means that there exists a neighbourhood V of Z and a psh function $\varphi : V \longrightarrow [-\infty, \infty)$ on V such that $Z = \varphi^{-1}(-\infty)$. We shall assume that φ is smooth outside Z. Then we say that Z is a complete pluripolar set defined by a smooth function outside Z.

Theorem 4.1. *Let X be a compact manifold and $Z \subset X$ be a complete pluripolar set defined by a smooth function outside Z. Assume that E is a holomorphic line bundle on $M := X \smallsetminus Z$ which is positive outside a compact set in M. Then*

$$\liminf_{k \longrightarrow \infty} k^{-n} \dim H^0\big(M, E^k \otimes K_M\big) \geqslant \frac{1}{n!(2\pi)^n} \int_{M(\leqslant 1)} \big(\sqrt{-1}\Theta(E)\big)^n . \tag{4.1}$$

Proof. We show what are the modifications in the proof of the Main Theorem. Let us extend the function φ to a smooth function on M with values in $(-\infty, 0)$. We consider then $\chi \in \mathcal{C}^\infty(M)$ defined by $\chi = -\log(-\varphi)$ so that relation (2.1) is still true: $\sqrt{-1}\partial\chi \wedge \bar\partial\chi \leqslant \sqrt{-1}\partial\bar\partial\chi$. We take a hermitian metric ω on M such that $\omega = \sqrt{-1}\Theta(E)$ outside a compact set of M. Define the metric

$$\omega_0 = A\omega + \partial\bar\partial\chi$$

for a sufficiently large constant $A > 0$. Then ω_0 is complete and if we endow E with the metric $h_\varepsilon = h\exp(-\varepsilon\chi)$ then

$$\sqrt{-1}\Theta(E, h_\varepsilon) = \sqrt{-1}\mathbf{c}(E, h) + \sqrt{-1}\varepsilon\partial\bar\partial\chi + \sqrt{-1}\varepsilon\partial\chi \wedge \bar\partial\chi \geqslant \varepsilon\omega_0$$

outside a compact set and for ε small enough. Thus we have an analogous inequality to (2.3). Whith these modifications the proof of the Main Theorem applies word by word. □

We are interested in the case when the group $H^0(M, E^k \otimes K_M)$ is finite dimensional so we can formulate also a variant of the Corollary 2.1. We need the notion of very strongly q-convex function. Let $\varphi : X \longrightarrow \mathbb{R}$ be a smooth function on a complex manifold of dimension n. We say that φ is very strongly q-convex if φ is psh and $\partial\bar\partial\varphi$ has at least $n - q + 1$ positive eigenvalues.

Corollary 4.1. *Let X be a compact manifold of dimension $n \geqslant 2$ and let Z be a complete pluripolar set defined by a smooth very $(n-1)$-convex function. Let $M := X \smallsetminus Z$.*

(i) *If E is a line bundle on M positive outside a compact set,*

$$\int_{M(0)} \big(\sqrt{-1}\Theta(E)\big)^n \leqslant gw(E) < \infty\,.$$

(ii) *If $\psi : M \longrightarrow \mathbb{R}$ is a smooth function which is strictly psh outside a compact set,*

$$\int_{M(0)} \big(\sqrt{-1}\partial\bar\partial\psi\big)^n \leqslant -\int_{M(1)} \big(\sqrt{-1}\partial\bar\partial\psi\big)^n < \infty\,.$$

Proof. Since φ is $(n-1)$-convex we see that M is $(n-1)$-concave (and so Andreotti-pseudoconcave). The proof of Corollary 2.1 applies without change. □

The following result was proved in [27] in a more general setting.

Corollary 4.2. *Let X be a compact manifold of dimension $n \geqslant 2$ and let Z be a complete pluripolar set defined by a smooth strictly psh function. Assume that E is a line bundle on $X \smallsetminus Z$ which is semipositive outside a compact set and satisfies* (D). *Then X is Moishezon.*

Proof. Since φ is strictly psh we need just the semipositivity of E near Z. Namely we can construct a metric ω_0 such that $\omega_0 = -\partial\bar\partial\varphi$ near Z. This metric is complete and the proof of Theorem 4.1 applies to get $\dim H^0(M, E^k \otimes K_M) = O(k^n)$. The hypothesis on φ implies that $M = X \smallsetminus Z$ is 1-concave in the sense of Andreotti–Grauert. By Lemma 2.4 there exist n independent meromorphic functions on M. They extend across the pluripolar set Z (to independent meromorphic functions) since Z has a strongly pseudoconvex neighbourhood. □

Remark. In [19] the following class of manifolds was introduced. We call the complex manifold M very strongly q-concave if there exists a smooth function $\varphi : M \longrightarrow \mathbb{R}$ such that $\{\varphi > c\} \Subset M$ for any $c \in \mathbb{R}$ and which is strongly q-convex outside a compact set of M.

It is easy to see that Corollary 4.1 holds for very strongly q-concave manifolds. Similarly, we can show that a very strongly 1-concave manifold M with a semipositive line bundle outside a compact set which satisfies (D) has a maximal number of meromorphic functions (whithout the assumption that M can be compactified; note that the compactification is always possible if $\dim M \geqslant 3$). In [27] very strongly 1-concave manifolds were called hyper 1-concave (at the suggestion of M. Colţoiu).

References

1. Andreotti, A.: Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves. Bull. Soc. Math. Fr. **91**, 1–38 (1963)
2. Andreotti, A., Grauert, H.: Théorème de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. Fr. **90**, 193–259 (1962)
3. Andreotti, A., Vesentini, E.: Carleman estimates for the Laplace–Beltrami equation on complex manifolds. Publ. Math., Inst. Hautes Étud. Sci. **25**, 81–130 (1965)

4. Blocki, Z., Pflug, P.: Hyperconvexity and Bergman completeness. Nagoya Math. J. **151**, 221–225 (1998)
5. Bonavero, L.: Inégalités de Morse holomorphes singulières. J. Geom. Anal. **8**, 409–425 (1998)
6. Bouche, T.: Inégalités de Morse pour la d''-cohomologie sur une variété non-compacte. Ann. Sci. Éc. Norm. Supér., IV Sér. **22**, 501–513 (1989)
7. Bremermann, H.J.: Holomorphic continuation of the kernel function and the Bergman metric in several complex variables. In: Lectures on functions of a complex variable, pp. 349–383. Ann Arbor: University of Michigan Press 1955
8. Diederich, K., Ohsawa, T.: An estimate for the Bergman distance on pseudoconvex domains. Ann. Math. (2) **141**, 181–190 (1995)
9. Demailly, J.P.: Champs magnétiques et inegalités de Morse pour la d''-cohomologie. Ann. Inst. Fourier **35**, 189–229 (1985)
10. Grauert, H., Riemenschneider, O.: Verschwindungssätze für analytische Kohomologiegruppen auf Komplexen Räume. Invent. Math. **11**, 263–292 (1970)
11. Griffiths, Ph.A.: Complex-analytic properties of certain Zariski open sets on algebraic varieties. Ann. Math. **94**, 21–51 (1971)
12. Henniart, G.: Les inégalités de Morse (d'après E. Witten). Astérisque **121/122**, 43–61 (1985)
13. Herbort, G.: The Bergman kernel on hyperconvex domains. Math. Z. **232**, 183–196 (1999)
14. Ji, S., Shiffman, B.: Properties of compact complex manifolds carrying closed positive currents. J. Geom. Anal. **3**, 37–61 (1993)
15. Kodaira, K.: On Kähler varieties of restricted type (An intrinsic characterization of algebraic varieties). Ann. Math. **60**, 28–48 (1954)
16. Marinescu, G.: Asymptotic Morse Inequalities for Pseudoconcave Manifolds. Ann. Sc. Norm. Sup. Pisa, Cl. Sci., IV Ser. **23**, 27–55 (1996)
17. Mok, N.: Metric rigidity theorems on Hermitian locally symmetric manifolds. Singapore: World Scientific 1989
18. Mok, N., Wong, B.: Characterisation of bounded domains covering Zariski dense subsets of compact complex spaces. Am. J. Math. **105**, 1481–1487 (1983)
19. Nadel, A., Tsuji, H.: Compactification of complete Kähler manifolds of negative Ricci curvature. J. Differ. Geom. **28**, 503–512 (1988)
20. Ohsawa, T.: A remark on the completeness of the Bergman metric. Proc. Japan Acad., Ser. A **57**, 238–240 (1981)
21. Ohsawa, T.: Hodge spectral sequence on compact Kähler spaces. Publ. Res. Inst. Math. Sci. **23**, 613–625 (1987)
22. Raghunathan, M.S.: Discrete subgroups of Lie groups. In: Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. New York: Springer 1972
23. Rozenblum, G.V., Shubin, M.A., Solomyak, M.Z.: Spectral theory of differential operators. In: Partial differential equations VII. Encycl. Math. Sci. **64**. Berlin: Springer 1994
24. Saper, L.: L^2-cohomology and intersection homology of certain algebraic varieties with isolated singularities. Invent. Math. **82**, 207–255 (1985)
25. Siu, Y. T.: A vanishing theorem for semipositive line bundles over non-Kähler manifolds. J. Differ. Geom. **20**, 431–452 (1984)
26. Takayama, S.: A differential geometric property of big line bundles. Tôhoku Math. J., II Ser. **46**, 281–291 (1994)
27. Todor, R., Chiose, I., Marinescu, G.: Asymptotic Morse inequalities for covering manifolds. Nagoya Math. J. **163**, 145–165 (2001)
28. Witten, E.: Supersymmetry and Morse theory. J. Differ. Geom. **17**, 661–692 (1982)