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Tian's theorem for Grassmannian embeddings and degeneracy sets of random sections

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Abstract. Let (X,ω) be a compact Kähler manifold, (L,h^L) a positive line bundle, and (E,h^E) a Hermitian holomorphic vector bundle of rank r on X. We prove that the pullback by the Kodaira embedding associated to $L^p\otimes E$ of the k-th Chern form of the dual of the universal bundle over the Grassmannian converges as $p\to\infty$ to the k-th power of the Chern form $c_1(L,h^L)$ for $0\le k\le r$. If $c_1(L,h^L)=\omega$, we also determine the second term in the semi-classical expansion, which involves $c_1(E,h^E)$. As a consequence, we show that the limit distribution of zeros of random sequences of holomorphic sections of high powers $L^p\otimes E$ is $c_1(L,h^L)^r$. Furthermore, we compute the expectation of the currents of integration along degeneracy sets of random holomorphic sections.

1. Introduction

Let (X,ω) be a (connected) compact Kähler manifold of dimension n, let (L,h^L) be a positive line bundle on X, and let (E,h^E) be a Hermitian holomorphic vector bundle of rank $r\le n$ on X. For $p\ge 1$, we set

$$L^p:=L^{\otimes p},\quad V_p:=H^0(X,L^p\otimes E),\quad d_p:=\dim V_p-1.$$

We let h^{L^p} and $h^{L^p\otimes E}$ be the Hermitian metrics induced by h^L and h^E on L^p and on $L^p\otimes E$ respectively. We endow V_p with the L^2-inner product induced by this metric data, namely

$$(S,S')_p=\int_X\langle S,S'\rangle_{h^{L^p\otimes E}}\frac{\omega^n}{n!},\quad S,S'\in V_p. \tag{1.1}$$

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We endow its dual space $V_p^\star = H^0(X, L^p \otimes E)^\star$ with the inner product induced by the one on V_p.

Let $\mathbb{G}(r, V_p^\star)$ be the Grassmannian of r-dimensional complex subspaces of $V_p^\star$. The Kodaira map is defined by

$$\Phi_p : X \to \mathbb{G}(d_p + 1 - r, V_p) = \mathbb{G}(r, V_p^\star), \quad \Phi_p(x) = \{s \in V_p : s(x) = 0\}. \tag{1.2}$$

Since L is positive, the above map is a well-defined holomorphic embedding for all p sufficiently large (see [29, Theorem 5.1.18]).

Let $\mathscr{T}$ be the universal holomorphic vector bundle over $\mathbb{G}(r, V_p^\star)$ and let $h^{\mathscr{T}}$ be the Hermitian metric on it induced by the inner product on $V_p^\star$. Let $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ be the dual holomorphic vector bundle. We denote by $c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ the k-th Chern form of $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$. Our first main result is a generalization of Tian's theorem for Chern forms of any degree.

Theorem 1.1. *Let (X, ω) be a compact Kähler manifold of dimension n, let (L, h^L) be a positive line bundle on X, and let (E, h^E) be a Hermitian holomorphic vector bundle on X of rank $r \leq n$. Then, for every $0 \leq k \leq r$, we have in the $\mathscr{C}^\infty$ topology, as $p \to \infty$,*

$$\frac{1}{p^k} \Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) = \binom{r}{k} c_1(L, h^L)^k + O\Big(\frac{1}{p}\Big). \tag{1.3}$$

Assume further that the bundle (L, h^L) polarizes (X, ω), that is, $\omega = c_1(L, h^L)$. Then, for every $0 \leq k \leq r$, we have in the $\mathscr{C}^\infty$ topology, as $p \to \infty$,

$$\frac{1}{p^k} \Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) = \binom{r}{k} \omega^k + \frac{1}{p}\binom{r-1}{k-1} c_1(E, h^E) \wedge \omega^{k-1} + O\Big(\frac{1}{p^2}\Big). \tag{1.4}$$

The semi-classical asymptotics (1.3) in the $\mathscr{C}^\infty$ topology means that, for every $\ell \in \mathbb{N}$, there exists $C_\ell > 0$ such that, for p sufficiently large, we have

$$\Big\| \frac{1}{p^k} \Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) - \binom{r}{k} c_1(L, h^L)^k \Big\|_{\mathscr{C}^\ell(X)} \leq \frac{C_\ell}{p},$$

and similarly for (1.4).

Tian's theorem [37] (the case $r = 1$ and $\ell = 2$; see also [8, 29, 32, 38] for the $\mathscr{C}^\infty$ topology) follows from Theorem 1.1 by taking E to be the trivial line bundle $X \times \mathbb{C}$. The case $k = 1$ with E arbitrary is addressed in [29, Theorem 5.1.17], and is used to prove the Kodaira embedding theorem [29, Theorem 5.1.18]. In recent years, Tian's theorem was generalized to the case of singular Hermitian holomorphic line bundles, and moreover to the setting of compact normal analytic spaces X (see [10–15]).

Theorem 1.1 establishes that, for an arbitrary positive line bundle (L, h^L), we can determine the first-order approximation of the pullback of the k-th Chern form in the semi-classical limit. This approximation equals, up to a multiplicative factor, the k-th power of the Chern curvature $c_1(L, h^L)$. Furthermore, if $c_1(L, h^L) = \omega$, we compute the second-order approximation of the pullback of the k-th Chern form. In the semi-classical limit, we recover the first Chern form $c_1(E, h^E)$ of (E, h^E).

An important application of Tian's theorem [37] is the study of the asymptotic distribution of zeros of random sequences of sections of powers of a line bundle. This started with the work of Shiffman and Zelditch [34]. See [1, 4, 18–20, 22, 33] for interesting results in this direction. One motivation for this study comes from the particular case of random polynomials and the distributions of their zeros, which has a long history. We refer the reader to the papers [2, 3, 6] and the references therein.

As in the case of line bundles, Theorem 1.1 can be applied to prove the equidistribution of zeros of random sections in V_p. The setting is as follows. By Bertini's theorem, for the generic section $s \in V_p$, its zero set $Z_s = \{s = 0\}$ is a complex submanifold of X of codimension r (see Section 4). We denote by $[s = 0]$ the current of integration over Z_s. We endow the projective space $\mathbb{P}V_p$ with the probability measure $\Upsilon_p = \omega_{\mathrm{FS}}^{d_p}$, where ω_{FS} denotes the Fubini–Study form on $\mathbb{P}V_p$. We then consider the product probability space

$$(\mathscr{H}, \Upsilon) = \Big(\prod_{p=1}^{\infty} \mathbb{P}V_p, \prod_{p=1}^{\infty} \Upsilon_p \Big). \tag{1.5}$$

Our second main result describes the distribution of zeros of random sequences of sections of $L^p \otimes E$ and gives an estimate on the speed of convergence. An important ingredient of the proof is to consider the Kodaira map as a meromorphic transform in the sense of Dinh–Sibony [21].

Theorem 1.2. *Let (X, ω) be a compact Kähler manifold of dimension n, let (L, h^L) be a positive line bundle on X, and let (E, h^E) be a Hermitian holomorphic vector bundle of rank $r \leq n$ on X. Then there exist $C > 0$ and $p_0 \in \mathbb{N}$ such that the following holds: for any $\gamma > 1$, there exist subsets $\mathcal{E}_p = \mathcal{E}_p(\gamma) \subset \mathbb{P}V_p$ such that, for $p > p_0$, we have*

(i) *$\Upsilon_p(\mathcal{E}_p) \leq C p^{-\gamma}$;*

(ii) *if $s_p \in \mathbb{P}V_p \setminus \mathcal{E}_p$, then*

$$\Big| \Big\langle \frac{1}{p^r} [s_p = 0] - c_1(L, h^L)^r, \varphi \Big\rangle \Big| \leq C\gamma \frac{\log p}{p} \|\varphi\|_{\mathscr{C}^2(X)} \tag{1.6}$$

for any $(n-r, n-r)$ form φ of class $\mathscr{C}^2$ on X. Moreover, estimate (1.6) holds for Υ-a.e. sequence $\{s_p\}_{p\geq 1} \in \mathscr{H}$ provided that p is large enough. Hence $p^{-r}[s_p = 0] \to c_1(L, h^L)^r$ in the weak sense of currents as $p \to \infty$, almost surely.

By taking the Hermitian vector bundle (E, h^E) to be the trivial vector bundle $X \times \mathbb{C}^r$ endowed with the trivial metric, we recover the results by Shiffman–Zelditch [34] and Dinh–Sibony [21] that the normalized simultaneous zero currents $p^{-r}[s_{p,1} = \cdots = s_{p,r} = 0]$ of r independent random sections of $H^0(X, L^p)$ almost surely converge weakly to $c_1(L, h^L)^r$ for $1 \leq r \leq n$. We refer the reader to Section 7 for a more detailed treatment of this topic.

We now turn our attention to the expectation of the currents of integration along degeneracy sets of random holomorphic sections. This topic is closely related to the theorems stated above. We consider the following general setting.

(A) (X, ω) is a (connected) compact Kähler manifold of dimension n; (E, h^E) is a Hermitian holomorphic vector bundle of rank r on X, and

$$1 \leq r \leq n, \quad V := H^0(X, E), \quad N := \dim V - 1, \quad \mathbb{P}V = \mathbb{P}H^0(X, E).$$

(B) For every section $S \in V$, $Z_S := \{x \in X : S(x) = 0\} \neq \varnothing$.

(C) $N \geq r$, and for every $x \in X$, V spans the fiber E_x of E over x.

We show in Section 6 that assumptions (A)–(C) hold in our earlier setting of vector bundles of the form $L^p \otimes E$, where L is a positive line bundle and p is sufficiently large. Note that the set $Z_s = Z_S$ is well-defined for $s \in \mathbb{P}V$, as it is independent of the chosen representative $S \in V \setminus \{0\}$ of s. We endow V with the L^2-inner product

$$(S, S') = \int_X \langle S, S' \rangle_{h^E} \frac{\omega^n}{n!}, \quad S, S' \in V, \tag{1.7}$$

defined as in (1.1) using the Hermitian metric h^E. We consider the dual $V^\star$ with the inner product determined by that on V, and we let ω_{FS} be the induced Fubini–Study form on the N-dimensional projective space $\mathbb{P}V$.

Let $\mathbb{G}(k, V)$ denote the Grassmannian of (complex) k-planes in V. By (C), the Kodaira map

$$\Phi_E \colon X \to \mathbb{G}(N+1-r, V) = \mathbb{G}(r, V^\star), \quad \Phi_E(x) = \{s \in V : s(x) = 0\}, \tag{1.8}$$

is well-defined and holomorphic. Let $\mathscr{T} \to \mathbb{G}(r, V^\star)$ be the universal bundle endowed with the Hermitian metric $h^{\mathscr{T}}$ induced by the inner product on $V^\star$, and let $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ be the dual holomorphic vector bundle.

Let $1 \le k \le r$. Given $(s_1, \ldots, s_k) \in (\mathbb{P}V)^k$ (or $(s_1, \ldots, s_k) \in V^k$), the degeneracy set

$$D_k(s_1, \ldots, s_k) = \{x \in X : s_1(x) \wedge \cdots \wedge s_k(x) = 0\} \tag{1.9}$$

is the set of points x where $s_1(x), \ldots, s_k(x)$ are linearly dependent (see [24, p. 411]).

Let μ_k be a probability measure on $(\mathbb{P}V)^k$, and let $\mathbb{E}_k(\mu_k) = \mathbb{E}([D_k(s_1, \ldots, s_k)], \mu_k)$ be the expectation current of the current valued random variable

$$(\mathbb{P}V)^k \ni (s_1, \ldots, s_k) \mapsto [D_k(s_1, \ldots, s_k)],$$

defined by

$$\langle \mathbb{E}_k(\mu_k), \phi \rangle = \int_{(\mathbb{P}V)^k} \langle [D_k(s_1, \ldots, s_k)], \phi \rangle \, d\mu_k, \tag{1.10}$$

where ϕ is a smooth $(n+k-r-1, n+k-r-1)$ form on X. By using the fact that the Poincaré dual of $D_k(s_1, \ldots, s_k)$ is $c_{r+1-k}(E)$, we find the cohomology class of the expectation current for general probability measures.

Theorem 1.3. *Let (X, ω), (E, h^E) verify* (A)–(C). *For $1 \le k \le r$, let μ_k be a probability measure on $(\mathbb{P}V)^k$ such that $\mu_k(A) = 0$ for any proper analytic subset $A \subset (\mathbb{P}V)^k$. Then $\mathbb{E}_k(\mu_k)$ is a well-defined positive closed current of bidegree $(r+1-k, r+1-k)$ on X and it belongs to the Chern class $c_{r+1-k}(E)$.*

In the case of Gaussians or Fubini–Study volumes, we obtain a precise formula for the expectation currents.

Theorem 1.4. *Let (X, ω), (E, h^E) verify* (A)–(C). *Assume that the Kodaira map*

$$\Phi_E \colon X \to \mathbb{G}(r, V^\star)$$

defined in (1.8) *is an embedding. For* $1 \leq k \leq r$, *let* ν_k *be the Gaussian probability measure on* V^k *and let* μ_k *be the product measure on* $(\mathbb{P}V)^k$ *determined by the Fubini–Study volume* ω_{FS}^N *on* $\mathbb{P}V$. *Then the expectation currents* $\mathbb{E}_k(\mu_k), \mathbb{E}_k(\nu_k)$ *are given by*

$$\mathbb{E}_k(\mu_k) = \mathbb{E}_k(\nu_k) = \Phi_E^\star\big(c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})\big), \quad 1 \leq k \leq r.$$

Here, the expectation current $\mathbb{E}_k(\nu_k)$ is defined as in (1.10) for the probability space (V^k, ν_k). For $k = 1$, Theorem 1.4 was established using different methods by J. Sun [36, Theorem 4.2] (see Theorem 4.1). Combining Theorems 1.1 and 1.4, we obtain asymptotic equidistribution results for the degeneracy sets.

Corollary 1.5. *Let* (X, ω) *be a compact Kähler manifold of dimension* n, *let* (L, h^L) *be a positive line bundle on* X, *and let* (E, h^E) *be a Hermitian holomorphic vector bundle of rank* $r \leq n$ *on* X. *For* $p \in \mathbb{N}$ *and* $1 \leq k \leq r$, *let* ν_k^p *be the Gaussian probability measure on* V_p^k *and let* μ_k^p *be the product measure on* $(\mathbb{P}V_p)^k$ *determined by the Fubini–Study volume* Υ_p *on* $\mathbb{P}V_p$. *Then we have, as* $p \to \infty$,

$$\frac{1}{p^{r+1-k}}\mathbb{E}_k(\mu_k^p) = \frac{1}{p^{r+1-k}}\mathbb{E}_k(\nu_k^p) = \binom{r}{k-1} c_1(L, h^L)^{r+1-k} + O\Big(\frac{1}{p}\Big). \tag{1.11}$$

If we assume in addition that $\omega = c_1(L, h^L)$, *then*

$$\begin{aligned}\frac{1}{p^{r+1-k}}\mathbb{E}_k(\mu_k^p) &= \frac{1}{p^{r+1-k}}\mathbb{E}_k(\nu_k^p) \\ &= \binom{r}{k-1}\omega^{r+1-k} \\ &\qquad + \frac{1}{p}\binom{r-1}{r-k} c_1(E, h^E) \wedge \omega^{r-k} + O\Big(\frac{1}{p^2}\Big).\end{aligned} \tag{1.12}$$

We expect that an analog of Theorem 1.2 holds for higher degeneracy sets, namely that, after the natural normalization, the currents $[D_k(s_1, \ldots, s_k)]$ converge almost surely to a universal constant multiple of $c_1(L, h^L)^{r+1-k}$. We plan to address this problem in a subsequent paper.

Using developments in geometric quantization with L^2-norms defined by general measures and more singular weights, we expect that versions of our main results will extend beyond the smooth volume-form setting. In particular, under a suitable Bernstein–Markov condition on the measure, Theorem 1.1 should persist with weaker remainder control and weaker convergence of currents, with $c_1(L, h^L)$ replaced by the curvature current of the associated equilibrium envelope.

By Hsiao–Marinescu [25], the Bergman kernel admits a full local asymptotic expansion on the non-degenerate locus $X(0)$ where $c_1(L, h^L)$ is strictly positive, even when L is only semi-positive or big; accordingly, Theorem 1.1 should plausibly localize to compact subsets of $X(0)$. Likewise, for big line bundles endowed with singular Hermitian metrics, the asymptotics persist on $X(0)$ off the singular set, suggesting that the limiting degeneracy/zero currents should converge to the appropriate curvature current (in particular, $c_1(L, h^L)^r$). Finally, it would be natural to define the spectral Grassmannian map using the low-energy spectral space (eigenvalues $\leq p^{-N_0}$) as in [25], potentially yielding a Tian-type statement beyond the algebraic category.

This paper is organized as follows. After some preliminary material in Section 2, we present the proof of the approximation theorem for the pullback of Chern forms by the Kodaira map in Section 3. In Section 4, we show that the Kodaira map can be interpreted as a meromorphic transform in the sense of Dinh–Sibony. In Section 5, we compute the expectation of currents of integration along degeneracy sets of random holomorphic sections. In Section 6, we establish that the limit distribution of zeros of random sequences of holomorphic sections of high powers $L^p \otimes E$ is $c_1(L, h^L)^r$. Finally, in Section 7, we specialize our results to the case of simultaneous zeros of several random holomorphic sections of L^p.

2. Preliminaries

In this section, we gather the necessary materials regarding Chern forms, universal bundles on Grassmannians, and Bergman kernels.

2.1. Chern forms of Hermitian vector bundles. Let A be an $r \times r$ matrix of complex numbers. Recall that the *elementary invariant polynomials* P^k are defined by (see e.g. [24, p. 402])

$$\det(A + t \cdot \mathrm{Id}_r) = \sum_{k=0}^{r} P^{r-k}(A) t^k, \quad t \in \mathbb{C},$$

where Id_r is the $r \times r$ identity matrix. We have $P^k(U^{-1}AU) = P^k(A)$ for any invertible $r \times r$ matrix U and

$$P^k(A) = \sum_{\sharp I = k} \det A_{I,I}, \tag{2.1}$$

where $A_{I,J}$ denotes the (I, J)-th minor $(A_{ij})_{i \in I,\, j \in J}$ of A.

Since the wedge product is commutative on differential forms of degree 2, $P^k(A)$ is well-defined in the same way for any $r \times r$ matrix A of forms of degree 2, namely

$$\det(A + t \cdot \mathrm{Id}_r) = \sum_{k=0}^{r} P^{r-k}(A) \wedge t^k,$$

where t is a form of degree 2 or a number. Then

$$c_k(A) := P^k(A) \tag{2.2}$$

is a form of degree $2k$ given by (2.1).

Let (X, ω) be a compact Kähler manifold of dimension n, and let (E, h^E) be a holomorphic vector bundle on X of rank $r \geq 1$. Let ∇^E denote the associated Chern connection on E, and let $R^E = (\nabla^E)^2 \in \Omega^{1,1}(X, \mathrm{End}(E))$ be its curvature.

Definition 2.1. The total Chern form associated to a Hermitian holomorphic vector bundle (E, h^E) of rank r is defined by

$$c(E, h^E) = \det\Big(\frac{i}{2\pi} R^E + \mathrm{Id}_E\Big) \in \bigoplus_{k=0}^{r} \Omega^{(k,k)}(X, \mathbb{R}).$$

The differential form $c(E, h^E)$ is closed; thus it defines a de Rham cohomology class $c(E) \in H^{\bullet}_{\mathrm{dR}}(X, \mathbb{R})$, which is independent of the choice of h^E. The k-th Chern form $c_k(E, h^E)$ is the component of bidegree (k, k) of $c(E, h^E)$. Then

$$c(E, h^E) = \sum_{k=0}^{r} c_k(E, h^E), \quad c_k(E, h^E) = c_k\Big(\frac{i}{2\pi} R^E\Big). \tag{2.3}$$

In particular,

$$\begin{aligned}
c_0(E, h^E) &= 1,\\
c_1(E, h^E) &= \frac{i}{2\pi}\operatorname{Tr}^E[R^E],\\
c_2(E, h^E) &= \frac{1}{8\pi^2}(\operatorname{Tr}^E[R^{E,2}] - \operatorname{Tr}^E[R^E]^2),\\
c_r(E, h^E) &= \Big(\frac{i}{2\pi}\Big)^r \det R^E.
\end{aligned}$$

If (L, h^L) is a Hermitian holomorphic line bundle on X, then $\operatorname{rank}(L \otimes E) = r$ and

$$c(L \otimes E, h^{L\otimes E}) = \sum_{k=0}^{r} c_k(E, h^E) \wedge c(L, h^L)^{r-k},$$

where, by definition, $c(L, h^L) = 1 + c_1(L, h^L)$; cf. [7, (21.10)]. Then, for $k = 0, 1, \dots, r$, we have

$$c_k(L \otimes E, h^{L\otimes E}) = \sum_{j=0}^{k} \binom{r-j}{k-j} c_j(E, h^E) \wedge c_1(L, h^L)^{k-j}. \tag{2.4}$$

For every integer $p \geq 1$, let $(L^p, h^{L^p}) := (L^{\otimes p}, (h^L)^{\otimes p})$.

Lemma 2.2. *For every $\ell \in \mathbb{N}$, there exists $C_\ell > 0$ such that if $p \geq 1$ and $0 \leq k \leq r$, then*

$$\Big\| \frac{1}{p^k} c_k(L^p \otimes E, h^{L^p\otimes E}) - \binom{r}{k} c_1(L, h^L)^k \Big\|_{\mathscr{C}^\ell(X)} \leq \frac{C_\ell}{p}.$$

Proof. Since $c_1(L^p, h^{L^p}) = p c_1(L, h^L)$, we get, by (2.4),

$$\begin{aligned}
c_k(L^p \otimes E, h^{L^p\otimes E}) &= \sum_{j=0}^{k} \binom{r-j}{k-j} p^{k-j} c_j(E, h^E) \wedge c_1(L, h^L)^{k-j}\\
&= p^k \binom{r}{k} c_1(L, h^L)^k + O(p^{k-1}).
\end{aligned}$$

The lemma now follows directly from the above identity. □

2.2. Universal bundle on Grassmannian. The Grassmannian $\mathbb{G}(r, m)$ is the set of r-dimensional complex linear subspaces of $\mathbb{C}^m$ $(r \leq m)$. The universal (or tautological) vector bundle $\mathscr{T}$ on $\mathbb{G}(r, m)$ is the holomorphic vector bundle of rank r defined by

$$\mathscr{T} = \{(W, f) \in \mathbb{G}(r, m) \times \mathbb{C}^m : f \in W \subset \mathbb{C}^m\}.$$

Let $h^{\mathscr{T}}$ denote the Hermitian metric on $\mathscr{T}$ induced by the standard Hermitian metric $h^{\mathbb{C}^m}$ on $\mathbb{C}^m$. Let $\nabla^{\mathscr{T}}$ and $R^{\mathscr{T}}$ denote the Chern connection and the Chern curvature, respectively, of $(\mathscr{T}, h^{\mathscr{T}})$. We recall here the explicit formula for $R^{\mathscr{T}}$ in certain local charts.

Let $W_0 \in \mathbb{G}(r,m)$. We fix $\{e_j\}_{j=1}^r$ an orthonormal $\mathbb{C}$-basis of $(W_0, h^{\mathbb{C}^m}|_{W_0})$, and we extend it to an orthonormal basis $\{e_j\}_{j=1}^m$ of $\mathbb{C}^m$. The map

$$(z_{j\ell})_{1\leq j\leq r,\,1\leq \ell\leq m-r} \in \mathbb{C}^{r\times(m-r)} \mapsto \mathrm{Span}_{\mathbb{C}}\left\{e_j + \sum_{\ell=1}^{m-r} z_{j\ell}e_{\ell+r},\ j=1,\dots,r\right\} \subset \mathbb{C}^m$$

is a local chart of $\mathbb{G}(r,m)$ near the point W_0. Set $Z = (z_{j\ell})_{1\leq j\leq r,\,1\leq \ell\leq m-r}$ for the coordinate matrix of size $r\times(m-r)$. For $(W,f)\in\mathscr{T}$ and W near W_0, set

$$\eta_j(W) = e_j + \sum_{\ell=1}^{m-r} z_{j\ell}e_{\ell+r} \in \mathbb{C}^m, \quad j=1,\dots,r.$$

Then $\{\eta_j\}_{j=1}^r$ is a local holomorphic frame of $\mathscr{T}$ near W_0.

Let $K(W) = (K_{kj}(W))_{1\leq k,j\leq r}$ be the square matrix valued function near W_0 given by

$$K_{kj}(W) = h^{\mathscr{T}}(\eta_j, \eta_k).$$

Then

$$K(W) = I + ZZ^\star,$$

where $I = \mathrm{Id}_r$, $Z^\star = \bar{Z}^{\mathrm{T}}$, and A^{T} denotes the transpose of a matrix A.

Under this local holomorphic frame, the Chern connection of $(\mathscr{T}, h^{\mathscr{T}})$ is given by

$$\nabla^{\mathscr{T}} = d + K^{-1}\partial K.$$

Then the Chern curvature is

$$R^{\mathscr{T}} = K^{-1}\bar{\partial}\partial K - K^{-1}\bar{\partial}K\wedge K^{-1}\partial K.$$

With the notations $dZ = (dz_{j\ell})$, $dZ^\star = (d\bar{z}_{j\ell})^{\mathrm{T}}$, we have

$$R^{\mathscr{T}} = -(I+ZZ^\star)^{-1}\,dZ\wedge dZ^\star - (I+ZZ^\star)^{-1}Z\,dZ^\star\wedge(I+ZZ^\star)^{-1}\,dZZ^\star.$$

In particular, at W_0,

$$R^{\mathscr{T}}(W_0)_{kj} = -\sum_{\ell=1}^{m-r} dz_{k\ell}\wedge d\bar{z}_{j\ell}.$$

Let $\mathscr{T}^*$ be the dual holomorphic vector bundle of $\mathscr{T}$ on $\mathbb{G}(r,m)$, and let $h^{\mathscr{T}^*}$ be the induced Hermitian metric. For $j=1,\dots,r$, $(W,f)\in\mathscr{T}$, set $(\sigma_j(W), f) = (e^j, f)$. Then $\{\sigma_j\}_{j=1}^r$ is a local holomorphic frame of $\mathscr{T}^*$ near W_0. Note that $(\sigma_i, \eta_j) = \delta_{ij}$, i.e., $\{\sigma_j\}_{j=1}^r$ is exactly the dual frame of $\{\eta_j\}_{j=1}^r$.

Let $H(W) = (H_{kj}(W))_{1\leq k,j\leq r}$ be the square matrix valued function near W_0 given by

$$H_{kj}(W) = h^{\mathscr{T}^*}(\sigma_j, \sigma_k).$$

Then

$$H(W) = (K(W)^{-1})^{\mathrm{T}} = (I + \bar{Z}Z^{\mathrm{T}})^{-1}.$$

Under this local holomorphic frame of $\mathscr{T}^*$, the Chern connection of $(\mathscr{T}^*, h^{\mathscr{T}^*})$ is given by

$$\nabla^{\mathscr{T}^*} = d + H^{-1}\partial H. \tag{2.5}$$

Then the Chern curvature is

$$R^{\mathscr{T}^*} = H^{-1}\bar{\partial}\partial H - H^{-1}\bar{\partial}H \wedge H^{-1}\partial H. \tag{2.6}$$

In terms of the local coordinate Z (viewed as a matrix of size $r \times (m-r)$),

$$R^{\mathscr{T}^*} = -d\bar{Z} \wedge (dZ)^{\mathrm{T}}(I + \bar{Z}Z^{\mathrm{T}})^{-1} - \bar{Z}(dZ)^{T}(I + \bar{Z}Z^{\mathrm{T}})^{-1} \wedge d\bar{Z}Z^{\mathrm{T}}(1 + \bar{Z}Z^{\mathrm{T}})^{-1}.$$

Note that $R^{\mathscr{T}^*} = -(R^{\mathscr{T}})^{\mathrm{T}}$. In particular, at W_0,

$$R^{\mathscr{T}^*}(W_0)_{kj} = \sum_{\ell=1}^{m-r} dz_{j\ell} \wedge d\bar{z}_{k\ell}.$$

Now we can write the Chern forms for $\mathscr{T}^*$ at the point W_0 in the above coordinates. By (2.3) and (2.1), we have, for $k = 1, \ldots, r$,

$$c_k(\mathscr{T}^*, h^{\mathscr{T}^*})(W_0) = \frac{i^k}{(2\pi)^k} \sum_{\sharp I = k} \det R^{\mathscr{T}^*}(W_0)_{I,I},$$

where $R^{\mathscr{T}^*}(W_0)_{I,J}$ is the (I,J)-th minor $(R^{\mathscr{T}^*}(W_0)_{ij})_{i\in I,\, j\in J}$ of the matrix $R^{\mathscr{T}^*}(W_0)$.

2.3. Bergman kernel asymptotics. Let (X, ω) be a compact Hermitian manifold of dimension n, and let (L, h^L) be a positive holomorphic line bundle on X, that is, h^L is a smooth Hermitian metric on L such that the first Chern form $c_1(L, h^L)$ is a Kähler form on X. Moreover, $c_1(L, h^L)$ is a de Rham representative of the Chern class $c_1(L) \in H^2(X, \mathbb{R})$. We identify the 2-form R^L with the Hermitian matrix $\dot{R}^L \in \mathrm{End}(T^{(1,0)}X)$ such that, for $W, Y \in T^{(1,0)}X$, $R^L(W, \overline{Y}) = \langle \dot{R}^L W, \overline{Y}\rangle$.

We will call (L, h^L) a prequantum line bundle of (X, ω) if the following condition holds:

$$\omega = c_1(L, h^L) = \frac{i}{2\pi} R^L. \tag{2.7}$$

In particular, (X, ω) is a Kähler manifold in this case.

Let (E, h^E) be a holomorphic vector bundle on X of rank $r \geq 1$. Let ∇^E denote the associated Chern connection on E, and let $R^E = (\nabla^E)^2$ be its curvature.

Let $\mathscr{L}^2(X, L^p \otimes E)$ be the L^2-space corresponding to the L^2 inner product on X induced by g^{TX}, h^{L^p}, and h^E. Let $H^0(X, L^p \otimes E)$ denote the space of holomorphic sections of $L^p \otimes E$ over X, which is a finite-dimensional complex vector subspace of $\mathscr{L}^2(X, L^p \otimes E)$. Let

$$P_p \colon \mathscr{L}^2(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$$

be the orthogonal (Bergman) projection. The Schwartz kernel $P_p(\cdot,\cdot)$ of P_p with respect to the volume form $\frac{\omega^n}{n!}$ is called the Bergman kernel of $H^0(X, L^p \otimes E)$. It is a smooth section of $(L^p \otimes E) \boxtimes (L^p \otimes E)^\star$ over $X \times X$,

$$P_p(x, x') \in (L^p \otimes E)_x \otimes (L^p \otimes E)^\star_{x'},$$

especially,

$$P_p(x,x) \in \operatorname{End}(L^p \otimes E)_x = \operatorname{End} E_x,$$

where we use the canonical identification $\operatorname{End}(L^p) = \mathbb{C}$ for any line bundle L on X. Let $\{S_j^p\}_{j=0}^{d_p}$, $d_p := \dim H^0(X, L^p \otimes E) - 1$, be any orthonormal basis of $H^0(X, L^p \otimes E)$ with respect to the L^2 inner product. Then

$$P_p(x,x') = \sum_{j=0}^{d_p} S_j^p(x) \otimes S_j^p(x')^* \in (L^p \otimes E)_x \otimes (L^p \otimes E)^*_{x'},$$

$$P_p(x,x) = \sum_{j=0}^{d_p} S_j^p(x) \otimes S_j^p(x)^* \in \operatorname{End}(E_x).$$

We have the following diagonal asymptotic expansion of the Bergman kernel.

Theorem 2.3 ([29, Theorem 4.1.1]). *For every $m \in \mathbb{N}$, there exist smooth coefficients $\boldsymbol{b}_m(x) \in \operatorname{End} E_x$, which are polynomials in R^{TX}, R^E (and $d\omega$, R^L) and their derivatives of order at most $2m-2$ (resp. $2m-1$, $2m$) and reciprocals of linear combinations of eigenvalues of $\dot{R}^L$ at x, such that $\boldsymbol{b}_0 = \det(\dot{R}^L/(2\pi)) \operatorname{Id}_E$, and for any $k, \ell \in \mathbb{N}$, there exists $C_{k,\ell} > 0$ such that, for any $p \in \mathbb{N}$,*

$$\left\| P_p(x,x) - \sum_{m=0}^{k} \boldsymbol{b}_m(x) p^{n-m} \right\|_{\mathscr{C}^\ell(X)} \leq C_{k,\ell}\, p^{n-k-1}. \tag{2.8}$$

In [29, Theorem 4.1.1], additional information about the uniformity of the expansion with respect to g^{TX}, h^L, and h^E and their derivatives is given. We refer to [16,30] for further interesting results on the asymptotic expansion of the Bergman kernels for powers of a holomorphic line bundle, or more generally for an arbitrary sequence of holomorphic line bundles [9].

2.4. Dimension of the space of holomorphic sections. Let (X, ω) be a compact Kähler manifold of dimension n, and let (L, h^L) be a positive line bundle on X. Let (E, h^E) be a holomorphic vector bundle on X of rank $r \geq 1$. For p large enough, the Kodaira–Serre vanishing theorem [29, Theorem 1.5.6] says that $H^q(X, L^p \otimes E) = 0$ for every $q \geq 1$, and thus the Euler number

$$\chi(X, L^p \otimes E) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, L^p \otimes E)$$

equals $\dim H^0(X, L^p \otimes E)$. On the other hand, by the Riemann–Roch–Hirzebruch theorem [5, Theorem 4.9], we have

$$\chi(X, L^p \otimes E) = \int_X \operatorname{td}(T^{1,0}X) \operatorname{ch}(L^p \otimes E),$$

where $\operatorname{td}(T^{1,0}X)$ denotes the Todd class of $T^{1,0}X$, and $\operatorname{ch}(F)$ denotes the Chern character of a Hermitian vector bundle F (see e.g. [29, Appendix B.5]). Hence we have, for p large enough,

$$\begin{aligned} \dim H^0(X, L^p \otimes E) = \chi(X, L^p \otimes E) &= \int_X \operatorname{td}(T^{1,0}X) \operatorname{ch}(L^p \otimes E) \\ &= r \int_X \frac{c_1(L)^n}{n!} p^n + R_{n-1}(p), \end{aligned} \tag{2.9}$$

where $R_{n-1}(p)$ is a polynomial of degree $(n-1)$ in p. Consequently, $\dim H^0(X, L^p \otimes E)$ is a polynomial of degree n in p with a positive leading coefficient $r \int_X c_1(L)^n/n!$.

Furthermore, one can also determine the order of growth of $\dim H^0(X, L^p \otimes E)$ by utilizing the identity

$$\dim H^0(X, L^p \otimes E) = \int_X \operatorname{Tr}_E[P_p(x,x)]\, dv_X(x),$$

and integrating expansion (2.8) to obtain, for sufficiently large p,

$$\dim H^0(X, L^p \otimes E) = r \int_X \frac{c_1(L)^n}{n!} p^n + O(p^{n-1}).$$

3. Tian's theorem for Grassmannian embeddings

In this section, we provide the proof of Theorem 1.1. Recall from (1.2) the definition of the Kodaira map $\Phi_p \colon X \to \mathbb{G}(r, V_p^\star)$, where $V_p = H^0(X, L^p \otimes E)$. Since L is positive, this map is a holomorphic embedding for all p sufficiently large (see [29, Theorems 5.1.15–5.1.18]). Recall that $\mathscr{T}$ is the universal holomorphic vector bundle over $\mathbb{G}(r, V_p^\star)$ endowed with the Hermitian metric $h^{\mathscr{T}}$ induced by the inner product on $V_p^\star$. Let $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ be the corresponding dual holomorphic vector bundle.

If $s \in V_p$, the evaluation map $\varepsilon_s \colon V_p^\star \to \mathbb{C}$, $\varepsilon_s(f) = f(s)$ defines a holomorphic section $\sigma_s \in H^0(\mathbb{G}(r, V_p^\star), \mathscr{T}^\star) \equiv V_p$, where $\sigma_s(W) = \varepsilon_s|_W$ and $W \in \mathbb{G}(r, V_p^\star)$. For p large enough, we have the following isomorphism of holomorphic vector bundles on X (see [29, Theorem 5.1.16]):

$$\begin{aligned} &\Psi_p \colon \Phi_p^\star(\mathscr{T}^\star) \to L^p \otimes E, \\ &\Psi_p((\Phi_p^\star \sigma_s)(x)) = s(x) \quad \text{for any } s \in V_p,\ x \in X. \end{aligned} \tag{3.1}$$

Moreover, under this isomorphism, we have

$$h^{\Phi_p^\star(\mathscr{T}^\star)}(x) = h^{L^p \otimes E}(x) \circ P_p(x,x)^{-1}, \tag{3.2}$$

where $h^{\Phi_p^\star(\mathscr{T}^\star)}$ is the Hermitian metric on $\Phi_p^\star(\mathscr{T}^\star)$ induced by $h^{\mathscr{T}^\star}$.

Assertion (1.3) of Theorem 1.1 is a direct consequence of Theorem 3.1 and Lemma 2.2.

Theorem 3.1. *Let (X, ω) be a compact Hermitian manifold of dimension n, let (L, h^L) be a positive line bundle on X, and let (E, h^E) be a Hermitian holomorphic vector bundle on X of rank $r \le n$. Then, for every $\ell \in \mathbb{N}$, there exists $C_\ell > 0$ such that*

$$\| \Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) - c_k(L^p \otimes E, h^{L^p \otimes E}) \|_{\mathscr{C}^\ell(X)} \le C_\ell p^{k-1} \tag{3.3}$$

holds for $0 \le k \le r$, and for all p sufficiently large.

Proof. The proof of the statement proceeds as follows. The difference in Chern curvatures of two Hermitian structures can be expressed explicitly in terms of the derivatives of the endomorphism that relates the two metrics. In our specific case, (3.2) identifies this endomorphism with the diagonal Bergman kernel. Since the Bergman kernel expansion governs its derivatives, the desired estimate can be readily derived from standard curvature-comparison identities. For completeness, we provide the necessary details.

We assume throughout the proof that p is sufficiently large. By [29, (5.1.38)], the evaluation map $s \in V_p \mapsto s(x) \in (L^p \otimes E)_x$ is surjective at any point $x \in X$.

We fix a point $x_0 \in X$ and prove that (3.3) holds in a neighborhood of x_0. Let e_L be a holomorphic frame of L near x_0 and let $v_1, \dots, v_r$ be a holomorphic frame of E near x_0. Set

$$t_j = e_L^{\otimes p} \otimes v_j, \quad j = 1, \dots, r.$$

Then $t_1, \dots, t_r$ is a holomorphic frame of $L^p \otimes E$ near x_0. Using the isomorphism Ψ_p given in (3.1), we see that there is a holomorphic frame $\tau_1, \dots, \tau_r$ of $\Phi_p^\star(\mathscr{T}^\star)$ near x_0 such that

$$\Psi_p(\tau_j(x)) = t_j(x), \quad j = 1, \dots, r, \quad x \text{ near } x_0.$$

Using (3.2), we infer that

$$h^{\Phi_p^\star(\mathscr{T}^\star)}(\tau_m(x), \tau_j(x)) = h^{L^p \otimes E}\big(P_p(x,x)^{-1} t_m(x), t_j(x)\big). \tag{3.4}$$

To establish (3.3), we will connect the curvature $R^{\Phi_p^\star(\mathscr{T}^\star)}$ with the curvature $R^{L^p \otimes E}$ by means of the isomorphism described in (3.4). For sufficiently large p, $P_p(x,x) \in \operatorname{End} E_x$ is invertible. For x near x_0, we define the matrix valued function $Q_p(x)$ by

$$P_p(x,x)^{-1} t_j(x) = \sum_{k=1}^{r} (Q_p(x))_{kj} t_k(x).$$

This is the matrix representing the inverse of $P_p(x,x)$ in the frame $t_1, \dots, t_r$. For p sufficiently large, we obtain the asymptotics of $Q_p(x)$ from the asymptotics of $P_p(x,x)$ stated in Theorem 2.3. More precisely, we have the uniform expansion in any local $\mathscr{C}^\ell$-norm near x_0,

$$P_p(x,x) = Q_p(x)^{-1} = p^n b_0(x)(\mathrm{Id}_E + O(p^{-1})), \quad \text{where } b_0 = \det(\dot{R}^L/(2\pi)). \tag{3.5}$$

As a consequence, we have the analogous expansion in any local $\mathscr{C}^\ell$-norm

$$Q_p(x) = p^{-n}(b_0(x))^{-1}(\mathrm{Id}_E + O(p^{-1})). \tag{3.6}$$

Using this, we obtain

$$\begin{aligned} \partial Q_p(x) &= -p^{-n} \frac{\partial b_0(x)}{(b_0(x))^2} \mathrm{Id}_E + O(p^{-n-1}), \\ \bar\partial Q_p(x) &= -p^{-n} \frac{\bar\partial b_0(x)}{(b_0(x))^2} \mathrm{Id}_E + O(p^{-n-1}). \end{aligned} \tag{3.7}$$

Consider the $r \times r$-matrix valued functions

$$H^{\Phi_p^\star(\mathscr{T}^\star)} = (H_{jm}^{\Phi_p^\star(\mathscr{T}^\star)}), \quad H^{L^p \otimes E} = (H_{jm}^{L^p \otimes E})$$

defined near x_0 by

$$H_{jm}^{\Phi_p^\star(\mathscr{T}^\star)}(x) = h^{\Phi_p^\star(\mathscr{T}^\star)}(\tau_m(x), \tau_j(x)), \quad H_{jm}^{L^p \otimes E}(x) = h^{L^p \otimes E}(t_m(x), t_j(x)).$$

By (3.4), we obtain

$$H^{\Phi_p^\star(\mathscr{T}^\star)} = H^{L^p \otimes E} Q_p. \tag{3.8}$$

As in (2.5)–(2.6), the curvature tensor is given by

$$R^{\Phi_p^\star(\mathscr{T}^\star)} = -(H^{\Phi_p^\star(\mathscr{T}^\star)})^{-1}\bar\partial H^{\Phi_p^\star(\mathscr{T}^\star)} \wedge (H^{\Phi_p^\star(\mathscr{T}^\star)})^{-1}\partial H^{\Phi_p^\star(\mathscr{T}^\star)} + (H^{\Phi_p^\star(\mathscr{T}^\star)})^{-1}\bar\partial\partial H^{\Phi_p^\star(\mathscr{T}^\star)}.$$

Using (3.8), we obtain

$$\begin{aligned} R^{\Phi_p^\star(\mathscr{T}^\star)} = &-\big(Q_p^{-1}(H^{L^p\otimes E})^{-1}(\bar\partial H^{L^p\otimes E})Q_p + Q_p^{-1}\bar\partial Q_p\big)\\ &\wedge \big(Q_p^{-1}(H^{L^p\otimes E})^{-1}(\partial H^{L^p\otimes E})Q_p + Q_p^{-1}\partial Q_p\big)\\ &+ Q_p^{-1}(H^{L^p\otimes E})^{-1}(\bar\partial\partial H^{L^p\otimes E})Q_p - Q_p^{-1}(H^{L^p\otimes E})^{-1}\partial H^{L^p\otimes E}\wedge\bar\partial Q_p\\ &+ Q_p^{-1}(H^{L^p\otimes E})^{-1}\bar\partial H^{L^p\otimes E}\wedge\partial Q_p + Q_p^{-1}\bar\partial\partial Q_p. \end{aligned}$$

Hence

$$\begin{aligned} R^{\Phi_p^\star(\mathscr{T}^\star)} = &\,Q_p^{-1}R^{L^p\otimes E}Q_p - Q_p^{-1}\bar\partial Q_p\wedge Q_p^{-1}\partial Q_p + Q_p^{-1}\bar\partial\partial Q_p\\ &- Q_p^{-1}(H^{L^p\otimes E})^{-1}\partial H^{L^p\otimes E}\wedge\bar\partial Q_p\\ &- Q_p^{-1}(\bar\partial Q_p)Q_p^{-1}\wedge(H^{L^p\otimes E})^{-1}(\partial H^{L^p\otimes E})Q_p. \end{aligned} \tag{3.9}$$

Set $H_{jm}^E = h^E(v_m, v_j)$, $\varphi_L(x) = |e_L|^2_{h^L}(x)$. Then $H^{L^p\otimes E} = H^E\varphi_L^p$. A direct computation shows

$$(H^{L^p\otimes E})^{-1}\partial H^{L^p\otimes E} = (H^E)^{-1}\partial H^E + \frac{p}{\varphi_L}\partial\varphi_L \mathrm{Id}_E. \tag{3.10}$$

The presence of Id_E in formula (3.10) transforms (3.9) into the following:

$$\begin{aligned} R^{\Phi_p^\star(\mathscr{T}^\star)} = &\,Q_p^{-1}R^{L^p\otimes E}Q_p - Q_p^{-1}\bar\partial Q_p\wedge Q_p^{-1}\partial Q_p + Q_p^{-1}\bar\partial\partial Q_p\\ &- Q_p^{-1}(H^E)^{-1}\partial H^E\wedge\bar\partial Q_p - Q_p^{-1}(\bar\partial Q_p)Q_p^{-1}\wedge(H^E)^{-1}(\partial H^E)Q_p. \end{aligned}$$

Hence

$$\Phi_p^\star(R^{\mathscr{T}^\star}) = R^{\Phi_p^\star(\mathscr{T}^\star)} = Q_p^{-1}(R^{L^p\otimes E} + A_p)Q_p, \tag{3.11}$$

where

$$\begin{aligned} A_p = &-\bar\partial Q_p\wedge Q_p^{-1}(\partial Q_p)Q_p^{-1} + (\bar\partial\partial Q_p)Q_p^{-1}\\ &- (H^E)^{-1}\partial H^E\wedge(\bar\partial Q_p)Q_p^{-1} - (\bar\partial Q_p)Q_p^{-1}\wedge(H^E)^{-1}\partial H^E. \end{aligned}$$

Using (3.5)–(3.7), and exploiting the fact that the leading terms in these asymptotic expansions are diagonal matrices, we deduce that, for every ℓ, there exists a constant $C_\ell > 0$ such that, for all $p \geq 1$,

$$\|A_p\|_{\mathscr{C}^\ell(\text{near } x_0)} \leq C_\ell. \tag{3.12}$$

Since $R^{L^p\otimes E} = pR^L\otimes\mathrm{Id}_E + R^E$, it follows from (2.1), (2.2), (2.3), and (3.12) that, for every ℓ, there exists $C'_\ell > 0$ such that, for all $p \geq 1$ and $0 \leq k \leq r$, we have

$$\Big\|c_k\Big(\frac{i}{2\pi}(R^{L^p\otimes E} + A_p)\Big) - c_k(L^p\otimes E, h^{L^p\otimes E})\Big\|_{\mathscr{C}^\ell(\text{near } x_0)} \leq C'_\ell p^{k-1}. \tag{3.13}$$

Here we use the $\mathscr{C}^\ell$-norm on differential forms in local coordinates near x_0. Observe that the preceding computations are local in nature; however, by the compactness of the manifold X, the resulting estimates extend uniformly over the entire X. By (3.11), we have

$$\Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) = c_k\Big(\frac{i}{2\pi}(R^{L^p\otimes E} + A_p)\Big), \tag{3.14}$$

and (3.3) follows directly from (3.13) and (3.14). □

Remark 3.2. For $k = 1$, (1.3) coincides with [29, Theorem 5.1.17],

$$\frac{1}{p}\Phi_p^\star(\omega_{\mathrm{FS}}) - rc_1(L, h^L) = O\Big(\frac{1}{p}\Big), \tag{3.15}$$

where ω_{FS} is the Fubini–Study form on $\mathbb{G}(r, H^0(X, L^p \otimes E)^\star)$.

We now examine the situation in which (L, h^L) is a prequantum line bundle over (X, ω), that is, condition (2.7) is satisfied. In this case, if (E, h^E) is the trivial line bundle, Tian's estimate (3.15) of the speed of convergence of the induced Fubini–Study metric to ω can be improved as follows, as shown in [29, Remark 5.1.5]:

$$\frac{1}{p}\Phi_p^\star(\omega_{\mathrm{FS}}) - c_1(L, h^L) = O\Big(\frac{1}{p^2}\Big). \tag{3.16}$$

For general Chern forms, we obtain the following generalization of (3.16).

Theorem 3.3. *Let (X, ω) be a compact Kähler manifold of dimension n, let (L, h^L) be a positive line bundle on X, and let (E, h^E) be a Hermitian holomorphic vector bundle on X of rank $r \le n$. Assume that $\omega = c_1(L, h^L)$. Then, for every $\ell \in \mathbb{N}$, there exists $C_\ell > 0$ such that*

$$\|\Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) - c_k(L^p \otimes E, h^{L^p\otimes E})\|_{\mathscr{C}^\ell(X)} \le C_\ell p^{k-2} \tag{3.17}$$

holds for $0 \le k \le r$, and for all p sufficiently large.

Proof. Since $\omega = c_1(L, h^L)$, we have $b_0 = 1$ on X (see (3.5)). In the setting of the proof of Theorem 3.1 near $x_0 \in X$, since $b_0 = 1$, equations (3.7) become

$$\partial Q_p(x) = O(p^{-n-1}), \quad \bar\partial Q_p(x) = O(p^{-n-1}), \quad \partial\bar\partial Q_p(x) = O(p^{-n-1}).$$

Using this and (3.6), we deduce that (3.11) holds with estimate (3.12) replaced by

$$\|A_p\|_{\mathscr{C}^\ell(\text{near } x_0)} \le \frac{C_\ell}{p}.$$

Arguing as in (3.13), we obtain

$$\Big\|c_k\Big(\frac{i}{2\pi}(R^{L^p\otimes E} + A_p)\Big) - c_k(L^p \otimes E, h^{L^p\otimes E})\Big\|_{\mathscr{C}^\ell(\text{near } x_0)} \le C'_\ell p^{k-2}.$$

This directly implies estimate (3.17). □

As a consequence of Theorem 3.3, we compute the second-order approximation of the pullback of the k-th Chern form. In the semi-classical limit, we retrieve the first Chern form $c_1(E, h^E)$ of (E, h^E). This proves the semi-classical asymptotics (1.4) and completes the proof of Theorem 1.1.

Corollary 3.4. *Let (X, ω) be a compact Kähler manifold of dimension n, let (L, h^L) be a positive line bundle on X, and let (E, h^E) be a Hermitian holomorphic vector bundle on X of rank $r \leq n$. Assume that $\omega = c_1(L, h^L)$. Then we have, in the $\mathscr{C}^\infty$ topology on X,*

$$\Phi_p^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) = p^k \binom{r}{k} \omega^k + p^{k-1} \binom{r-1}{k-1} c_1(E, h^E) \wedge \omega^{k-1} + O(p^{k-2}).$$

Proof. This follows directly from Theorem 3.3 and (2.4). □

Consequently, if $\omega = c_1(L, h^L)$, we obtain the following refinement of (3.15) and generalization of (3.16):

$$\Phi_p^\star(\omega_{\mathrm{FS}}) = prc_1(L, h^L) + c_1(E, h^E) + O\Big(\frac{1}{p}\Big).$$

If (E, h^E) is trivial, this gives

$$\frac{1}{p} \Phi_p^\star(\omega_{\mathrm{FS}}) - rc_1(L, h^L) = O\Big(\frac{1}{p^2}\Big).$$

4. Kodaira maps as meromorphic transforms

In this section, we show that the Kodaira map can be interpreted as a meromorphic transform of codimension $n - r$ in the sense of Dinh–Sibony [21]. Furthermore, we establish that the intermediate degrees of this transform can be expressed in terms of the Chern classes of the vector bundle E.

Let (X, ω), (E, h^E), V, N verify assumptions (A)–(C), and let $\Phi_E : X \to \mathbb{G}(r, V^\star)$ be the Kodaira map defined in (1.8). It follows from (B) that $\dim Z_s \geq n - r$ for every $s \in \mathbb{P}V$. Moreover, Bertini's theorem for vector bundles [31, Theorem 1.2] implies that Z_s is a complex submanifold of X of dimension $n - r$ for all s outside a proper analytic subset of $\mathbb{P}V$. Hence the current of integration $[Z_s] = [s = 0]$ along Z_s is a well-defined positive closed current of bidegree (r, r) for such s. Locally, if $U \subset X$ is an open set such that $E|_U$ is trivial, then s is represented by an r-tuple of holomorphic functions $(f_1, \dots, f_r)$ on U and

$$[s = 0] = dd^c \log|f_1| \wedge \cdots \wedge dd^c \log|f_r|.$$

Consider the probability space $(\mathbb{P}V, \omega_{\mathrm{FS}}^N)$ and the current valued random variable

$$\mathbb{P}V \ni s \mapsto [Z_s].$$

The expectation of this random variable, also called the *expected zero current* of the ensemble $(\mathbb{P}V, \omega_{\mathrm{FS}}^N)$, is the current $\mathbb{E}[Z_s] = \mathbb{E}_1(\omega_{\mathrm{FS}}^N)$ defined in (1.10). Namely,

$$\langle \mathbb{E}[Z_s], \phi \rangle = \int_{\mathbb{P}V} \langle [s = 0], \phi \rangle \omega_{\mathrm{FS}}^N, \tag{4.1}$$

where ϕ is a smooth $(n-r, n-r)$ form on X. The following result of Sun [36, Theorem 4.2] provides a formula for $\mathbb{E}[Z_s]$ in terms of the r-th Chern form of $(\mathcal{T}^\star, h^{\mathcal{T}^\star})$. A novel proof is presented in Section 5.

Theorem 4.1 ([36]). *If* $(X,\omega), (E,h^E)$ *verify* (A)–(C) *and* Φ_E *is defined in* (1.8)*, then*

$$\mathbb{E}[Z_s] = \Phi_E^\star(c_r(\mathcal{T}^\star, h^{\mathcal{T}^\star})).$$

Let us recall in our setting the notion of meromorphic transform introduced by Dinh and Sibony [21]. If X_1, X_2 are connected compact Kähler manifolds of dimension k_1, k_2 and l is an integer such that $k_1 - k_2 \leq l \leq k_1$, a meromorphic transform F from X_1 to X_2 is an irreducible analytic subset $\Gamma \subset X_1 \times X_2$ of dimension $k_2 + l$ such that the restrictions of the canonical projections $X_1 \times X_2 \to X_j$ to Γ are surjective (see [21, Section 3.1] for the general definition). One calls Γ the graph of F and l the codimension of F.

We now consider the Kodaira map Φ_E as the meromorphic transform F_E from X to $\mathbb{P}V$ with graph

$$\Gamma_E = \{(x,s) \in X \times \mathbb{P}V : s(x) = 0\} \subset X \times \mathbb{P}V. \tag{4.2}$$

Let $\pi_1 : X \times \mathbb{P}V \to X$, $\pi_2 : X \times \mathbb{P}V \to \mathbb{P}V$ be the canonical projections.

Proposition 4.2. *F_E is a meromorphic transform of codimension $n-r$.*

Proof. We have to check that the restrictions of the canonical projections to Γ_E are surjective. It is clear by (B) that $\pi_2(\Gamma_E) = \mathbb{P}V$. By (C), the linear map $S \in V \to S(x) \in E_x$ has rank r for every $x \in X$, so $\{s \in \mathbb{P}V : s(x) = 0\}$ is a linear subspace of dimension $N-r$ of $\mathbb{P}V$. Hence $\pi_1(\Gamma_E) = X$ and, since X is connected, Γ_E is a connected complex submanifold of $X \times \mathbb{P}V$ of dimension $N + n - r$ (a projective bundle over X). $\square$

Let $I_2(F_E) = \{s \in \mathbb{P}V : \dim Z_s > n - r\}$ denote the second indeterminacy set of F_E. Then $\operatorname{codim} I_2(F_E) \geq 2$. For $s \in \mathbb{P}V \setminus I_2(F_E)$, we denote by $[Z_s] = [s = 0]$ the current of integration along Z_s. We also let δ_s be Dirac mass at $s \in \mathbb{P}V$, viewed as a probability measure on $\mathbb{P}V$.

By [21, Section 3.1], if $N - r \leq k \leq N$, the pullback of a current T on $\mathbb{P}V$ of bidegree (k,k) is defined by

$$F_E^\star(T) = (\pi_1)_\star(\pi_2^\star(T) \wedge [\Gamma_E]), \tag{4.3}$$

in the case when T is smooth, or when T is the current of integration over an analytic subset $H \subset \mathbb{P}V$ such that $\dim(\pi_2|_{\Gamma_E})^{-1}(H \cap I_2(F_E)) \leq N + n - r - k - 1$. Then $F_E^\star(T)$ is a current of bidegree $(k - N + r, k - N + r)$ on X. If T is smooth, then $F_E^\star(T)$ is given by a form with coefficients in $L^1(X)$. If $T = [H]$, where $H \subset \mathbb{P}V$ is an analytic set of pure dimension $N - k$, then $(\pi_2|_{\Gamma_E})^\star([H])$ is defined to be the current of integration $[(\pi_2|_{\Gamma_E})^{-1}(H)]$ over the analytic subset $(\pi_2|_{\Gamma_E})^{-1}(H) = \pi_2^{-1}(H) \cap \Gamma_E \subset \Gamma_E$ which has pure dimension $N + n - r - k$, and its push-forward $F_E^\star(T) = (\pi_1|_{\Gamma_E})_\star[(\pi_2|_{\Gamma_E})^{-1}(H)]$ is a current of bidegree $(k - N + r, k - N + r)$ on X. Similarly, if S is a smooth (k,k) form on X, where $n - r \leq k \leq n$, then the push-forward

$$(F_E)_\star(S) = (\pi_2)_\star(\pi_1^\star(S) \wedge [\Gamma_E])$$

is a well-defined current of bidegree $(k-n+r, k-n+r)$ on $\mathbb{P}V$ and it is given by a form with coefficients in $L^1(\mathbb{P}V)$.

We consider now the case $T = [H]$, where $H \subset \mathbb{P}V$ is a linear subspace of dimension k', $0 \le k' \le r$, so T is a current of bidegree (k,k), $k = N - k'$. For generic H, we have $\dim(\pi_2|_{\Gamma_E})^{-1}(H \cap I_2(F_E)) \le k' + n - r - 1$. Moreover, since $k' \le r$, we have for generic H that $\dim \pi_1(\pi_2^{-1}(H) \cap \Gamma_E) = k' + n - r$ and $\pi_1 \colon \pi_2^{-1}(H) \cap \Gamma_E \to \pi_1(\pi_2^{-1}(H) \cap \Gamma_E)$ is bimeromorphic. It follows that, for the generic linear subspace $H \subset \mathbb{P}V$ of dimension $k' \le r$, $F_E^\star([H])$ is the current of integration along the analytic set $F_E^{-1}(H)$,

$$F_E^\star([H]) = [F_E^{-1}(H)], \quad F_E^{-1}(H) := \pi_1(\pi_2^{-1}(H) \cap \Gamma_E). \tag{4.4}$$

Let

$$J = I_2(F_E) \cup \{s \in \mathbb{P}V : Z_s \text{ is not smooth}\}. \tag{4.5}$$

By [31, Theorem 1.2], J is an analytic subset of $\mathbb{P}V$. We infer from (4.4) that if $s \in \mathbb{P}V \setminus J$, then Z_s is a complex submanifold of X of dimension $n - r$ and

$$F_E^\star(\delta_s) = [s = 0]. \tag{4.6}$$

Indeed, in this case, $\pi_2^{-1}(s) \cap \Gamma_E = Z_s \times \{s\}$ and $\pi_1 \colon Z_s \times \{s\} \to Z_s$ is biholomorphic.

The intermediate degree of F_E of order k, $N - r \le k \le N$, is defined by

$$\begin{aligned} \lambda_k(F_E) &= \int_X F_E^\star(\omega_{\mathrm{FS}}^k) \wedge \omega^{N+n-r-k} = \int_{\mathbb{P}V} \omega_{\mathrm{FS}}^k \wedge (F_E)_\star(\omega^{N+n-r-k}) \\ &= \int_{\Gamma_E} \pi_1^\star(\omega^{N+n-r-k}) \wedge \pi_2^\star(\omega_{\mathrm{FS}}^k) \end{aligned} \tag{4.7}$$

(see [21, (3.1)]). We show here that they can be expressed in terms of the Chern classes $c_{k-N+r}(E)$ of E.

Theorem 4.3. *Let* (X,ω), (E,h^E) *verify assumptions* (A)–(C). *If* F_E *is the meromorphic transform defined in* (4.2)*, then*

$$\lambda_k(F_E) = \int_X c_{k-N+r}(E) \wedge \omega^{N+n-r-k}, \quad N - r \le k \le N, \tag{4.8}$$

$$F_E^\star(\omega_{\mathrm{FS}}^N) = \Phi_E^\star(c_r(\mathscr{T}^\star, h^{\mathscr{T}^\star})). \tag{4.9}$$

Proof. Let ϕ be a smooth $(n-r, n-r)$ form on X. We have

$$\langle F_E^\star(\omega_{\mathrm{FS}}^N), \phi \rangle = \int_{\Gamma_E} \pi_1^\star(\phi) \wedge \pi_2^\star(\omega_{\mathrm{FS}}^N) = \int_{\mathbb{P}V} (F_E)_\star(\phi) \wedge \omega_{\mathrm{FS}}^N,$$

where $(F_E)_\star(\phi) \in L^1(\mathbb{P}V)$. If $s \notin J$, where J is defined in (4.5), then

$$\pi_2 \colon \pi_2^{-1}(U) \cap \Gamma_E \to U$$

is a submersion for some neighborhood $U \subset \mathbb{P}V$ of s. We infer that

$$\begin{aligned} (F_E)_\star(\phi)(s) = (\pi_2|_{\Gamma_E})_\star\big((\pi_1|_{\Gamma_E})^\star\phi\big)(s) &= \int_{\pi_2^{-1}(s)\cap\Gamma_E} (\pi_1|_{\Gamma_E})^\star\phi \\ &= \int_{Z_s} \phi = \langle [s=0], \phi \rangle, \quad s \notin J, \end{aligned}$$

where the third equality follows since $\pi_1|_{\Gamma_E} : \pi_2^{-1}(s) \cap \Gamma_E = Z_s \times \{s\} \to Z_s$ is a biholomorphism. This yields that

$$\langle F_E^\star(\omega_{\mathrm{FS}}^N), \phi\rangle = \int_{\mathbb{P}V} \langle [s=0], \phi\rangle \omega_{\mathrm{FS}}^N = \langle \mathbb{E}[Z_s], \phi\rangle, \tag{4.10}$$

where $\mathbb{E}[Z_s]$ is the expectation current from (4.1), and (4.9) follows from Theorem 4.1. The Chern class $c_r(E)$ is the Poincaré dual of Z_s for generic $s \in \mathbb{P}V$ (see e.g. [24, p. 413]), so

$$\int_{Z_s} \omega^{n-r} = \int_X c_r(E) \wedge \omega^{n-r}.$$

Hence (4.10) applied with $\phi = \omega^{n-r}$ yields (4.8) for $k = N$.

To prove (4.8) for $N - r \le k \le N$, we use a cohomological argument. Let $H_k \subset \mathbb{P}V$ be a linear subspace of dimension $N - k$ such that

$$\dim(\pi_2|_{\Gamma_E})^{-1}(H_k \cap I_2(F_E)) \le N + n - r - k - 1,$$

and let $s_0, \dots, s_{N-k} \in \mathbb{P}V$ be linearly independent points such that

$$H_k = \{a_0 s_0 + \dots + a_{N-k} s_{N-k} : (a_0, \dots, a_{N-k}) \in \mathbb{C}^{N-k+1} \setminus \{0\}\}.$$

If $[H_k]$ denotes the current of integration along H_k, then $\omega_{\mathrm{FS}}^k \sim [H_k]$, so by (4.7),

$$\lambda_k(F_E) = \int_X F_E^\star(\omega_{\mathrm{FS}}^k) \wedge \omega^{N+n-r-k} = \int_X F_E^\star([H_k]) \wedge \omega^{N+n-r-k}.$$

By (4.4), $F_E^\star([H_k])$ is the current of integration along $F_E^{-1}(H_k)$. Note that $F_E^{-1}(H_k)$ is the degeneracy set $D_{N-k+1}(s_0, \dots, s_{N-k})$ defined in (1.9), i.e., the set of points $x \in X$ where $s_0(x), \dots, s_{N-k}(x)$ are linearly dependent (see (5.1)). By [24, p. 413], the Poincaré dual of $D_{N-k+1}(s_0, \dots, s_{N-k})$ is $c_{k-N+r}(E)$. Therefore, since ω is closed,

$$\begin{aligned}\lambda_k(F_E) = \int_X F_E^\star([H_k]) \wedge \omega^{N+n-r-k} &= \int_{F_E^{-1}(H_k)} \omega^{N+n-r-k} \\ &= \int_X c_{k-N+r}(E) \wedge \omega^{N+n-r-k}. \qquad \square\end{aligned}$$

5. Expectation currents of degeneracy sets

In this section, we investigate the expectation currents $\mathbb{E}_k(\mu_k)$ defined in (1.10) and we establish Theorems 1.3 and 1.4. When $k = 1$, our results recover those obtained by Sun [36] (as presented in Theorem 4.1), provided that the Kodaira map Φ_E is an embedding.

5.1. Cohomology class of expectation currents. Let $1 \le k \le r$. We start by recalling some facts about the degeneracy sets $D_k(s_1, \dots, s_k)$. We will denote by $\Lambda^k E$ the k-th exterior power of E, so in particular, $\Lambda^r E$ is the determinant bundle of E. If $s_1, \dots, s_k \in V$, then $s_1 \wedge \dots \wedge s_k \in H^0(X, \Lambda^k E)$.

For

$$s = (s_1, \dots, s_k) \in (\mathbb{P}V)^k,$$

let $H(s) \subset \mathbb{P}V$ be the linear subspace spanned by $s_1, \dots, s_k$. We have

$$F_E^{-1}(H(s)) = \pi_1\big(\pi_2^{-1}(H(s)) \cap \Gamma_E\big) = \bigcup_{\sigma \in H(s)} Z_\sigma = D_k(s_1, \dots, s_k). \tag{5.1}$$

Recall that, by assumption (B), $Z_\sigma \neq \varnothing$, so Z_σ has codimension at most r for each $\sigma \in H(s)$. If s is generic (i.e., outside an analytic subset of $(\mathbb{P}V)^k$), then $H(s)$ has dimension $k-1$ and $\dim(\pi_2|_{\Gamma_E})^{-1}(H \cap I_2(F_E)) \leq n - r + k - 2$, where $I_2(F_E)$ is the second indeterminacy set of the meromorphic transform F_E defined in (4.2). So, by (4.4) and (5.1),

$$[D_k(s_1, \dots, s_k)] = F_E^\star([H(s)])$$

is a positive closed current of bidegree $(r+1-k, r+1-k)$ on X.

For generic $(s_1, \dots, s_k) \in (\mathbb{P}V)^k$, the following property holds: each degeneracy set $D_j(s_1, \dots, s_j)$, where $2 \leq j \leq k$, has the expected codimension $r - j + 1$ and is smooth away from $D_{j-1}(s_1, \dots, s_{j-1})$; moreover, $D_1(s_1) = Z_{s_1}$ is a complex submanifold of X of codimension r (see [24, pp. 411–412]). Thus the current of integration $[D_k(s_1, \dots, s_k)]$ on $D_k(s_1, \dots, s_k)$ is equal to the current of integration $[s_1 \wedge \dots \wedge s_k = 0]$, since the latter has multiplicity one along each irreducible component of the zero set of the holomorphic section $s_1 \wedge \dots \wedge s_k$.

We now provide the proof of Theorem 1.3.

Proof of Theorem 1.3. If $S_0, \dots, S_N$ is a basis of V, we have a bijective map

$$\mathbb{P}^N \ni [x_0 : \dots : x_N] \mapsto x_0 S_0 + \dots + x_N S_N \in \mathbb{P}V.$$

Let $U \subset X$ be an open set such that $E|_U$ has a holomorphic frame $e_1, \dots, e_r$, and write

$$S_j = \sum_{\ell=1}^{r} h_{j\ell} e_\ell, \quad \text{where } h_{j\ell} \in \mathscr{O}_X(U).$$

If $s = (s_1, \dots, s_k) \in (\mathbb{P}V)^k$, there exist $a_m = [a_{m0} : \dots : a_{mN}] \in \mathbb{P}^N$ such that

$$s_m = \sum_{j=0}^{N} a_{mj} S_j = \sum_{\ell=1}^{r} \Big(\sum_{j=0}^{N} a_{mj} h_{j\ell} \Big) e_\ell, \quad 1 \leq m \leq k.$$

Therefore, $x \in D_k(s) \cap U$, where

$$D_k(s) := D_k(s_1, \dots, s_k),$$

if and only if $\operatorname{rank} A(s, x) \leq k - 1$, where

$$A(s, x) = A(a_1, \dots, a_k; x) = \Big[\sum_{j=0}^{N} a_{mj} h_{j\ell}(x) \Big]_{1 \leq m \leq k,\, 1 \leq \ell \leq r}$$

is a $k \times r$ matrix of holomorphic functions on $(\mathbb{P}^N)^k \times U$. Thus $D_k(s) \cap U$ is defined by $\binom{r}{k}$ equations, given by the vanishing of all the $k \times k$ minors of $A(s, x)$.

By the considerations preceding the proof, there exists an analytic set $\mathscr{S} \subsetneq (\mathbb{P}V)^k$ such that, for $s = (s_1, \dots, s_k) \in (\mathbb{P}V)^k \setminus \mathscr{S}$, $D_k(s)$ has pure dimension $n + k - r - 1$ and it is smooth away from its singular locus $D_{k-1}(s_1,\dots,s_{k-1})$. Let $x_0 \in U \setminus D_{k-1}(s_1,\dots,s_{k-1})$. We may assume that the $(k-1)\times(k-1)$ minor of $A(s,x)$ determined by its first $(k-1)$ rows and columns does not vanish in an open set W, $x_0 \in W \subset U \setminus D_{k-1}(s_1,\dots,s_{k-1})$. Let $g_j(s,x)$ be the $k\times k$ minor of A determined by its first $k-1$ columns together with the column $j+k-1$, $1 \le j \le r+1-k$. Then g_j are holomorphic functions on $(\mathbb{P}^N)^k \times U$, and $g_j(s,\cdot)$ are defining functions for the complex manifold $D_k(s)\cap W$. Moreover, by shrinking W, we may assume that if $\sigma \in (\mathbb{P}V)^k \setminus \mathscr{S}$ is close to s, the corresponding functions $g_j(\sigma,\cdot)$, $1 \le j \le r+1-k$, are defining functions for the complex manifold $D_k(\sigma)\cap W$. From this, we infer that, as $\sigma \to s \in (\mathbb{P}V)^k \setminus \mathscr{S}$, the currents $[D_k(\sigma)]$ converge weakly to $[D_k(s)]$ in $X \setminus D_{k-1}(s_1,\dots,s_{k-1})$. Since $c_{r+1-k}(E)$ is the Poincaré dual of $D_k(\sigma)$ (see [24, p. 413]) and ω is Kähler, we have

$$\int_{D_k(\sigma)} \omega^{n+k-r-1} = \int_X c_{r+1-k}(E)\wedge\omega^{n+k-r-1}, \tag{5.2}$$

so the currents $[D_k(\sigma)]$ have the same mass. By the support theorem, this implies that

$$[D_k(\sigma)] \to [D_k(s)] \quad \text{weakly on } X \text{ as } \sigma \to s \in (\mathbb{P}V)^k \setminus \mathscr{S}.$$

Therefore, the function $f(s) := \langle [D_k(s)], \phi\rangle$ is continuous on $(\mathbb{P}V)^k \setminus \mathscr{S}$, where ϕ is a fixed $(n+k-r-1, n+k-r-1)$ form on X. We now observe that f is bounded. Indeed, we can assume that ϕ is real, so there exists $C_\phi > 0$ such that

$$-C_\phi\omega^{n+k-r-1} \le \phi \le C_\phi\omega^{n+k-r-1}.$$

Using (5.2), we obtain for $s \in (\mathbb{P}V)^k \setminus \mathscr{S}$ that

$$|f(s)| \le C_\phi \int_X c_{r+1-k}(E)\wedge\omega^{n+k-r-1}.$$

We conclude that $\mathbb{E}_k(\mu_k)$ is a well-defined current of bidegree $(r+1-k, r+1-k)$ on X, and it is clearly positive and closed. Moreover, if ϕ is a closed $(n+k-r-1, n+k-r-1)$ form on X, then (5.2) holds for $s \in (\mathbb{P}V)^k \setminus \mathscr{S}$ with ω replaced by ϕ, so

$$\langle \mathbb{E}_k(\mu_k), \phi\rangle = \int_{(\mathbb{P}V)^k\setminus\mathscr{S}} \langle [D_k(s)], \phi\rangle\, d\mu_k(s) = \int_X c_{r+1-k}(E)\wedge\phi.$$

This shows that $\mathbb{E}_k(\mu_k)$ belongs to the Chern class $c_{r+1-k}(E)$. □

Remark 5.1. We can also consider the expectation current

$$\mathbb{E}_k(\mu_k) = \int_{V^k} [D_k(s_1,\dots,s_k)]\, d\mu_k$$

with respect to a probability measure μ_k on V^k that does not charge proper analytic subsets. Arguing as in the proof of Theorem 1.3 we see that $\mathbb{E}_k(\mu_k)$ is a well-defined positive closed current of bidegree $(r+1-k, r+1-k)$ on X in the Chern class $c_{r+1-k}(E)$.

If μ_1 is a continuous probability measure on $\mathbb{P}V$, i.e., of the form $\mu_1 = \chi\omega_{\mathrm{FS}}^N$, where χ is a continuous function on $\mathbb{P}V$, we remark that the case $k=1$ of Theorem 1.3 follows from [36, Theorem 4.2].

5.2. Invariance properties under group actions. In this section, we establish Theorem 1.4 in the case of the dual universal bundle on a Grassmannian (see Theorem 5.4 below).

Let (X,ω), (E,h^E), V, N verify (A)–(C) and let V be endowed with the L^2-inner product (1.7). Assume that there exist holomorphic automorphisms $g \in \mathrm{Aut}(X)$ and $G \in \mathrm{Aut}(E)$ such that, for every $x \in X$, $G\colon E|_x \to E|_{g(x)}$ is a linear isometry, i.e.,

$$h^E_{g(x)}(G(u),G(v)) = h^E_x(u,v) \quad \text{for all } u,v \in E_x. \tag{5.3}$$

Then

$$A_1\colon V \to V, \quad A_1(S) = G \circ S \circ g^{-1}, \tag{5.4}$$

is a linear isomorphism, and we define

$$A_k\colon V^k \to V^k, \quad A_k(S_1,\dots,S_k) = (A_1(S_1),\dots,A_1(S_k)). \tag{5.5}$$

We denote again by A_k the induced maps $A_k\colon (\mathbb{P}V)^k \to (\mathbb{P}V)^k$.

Lemma 5.2. *In the above setting, the following hold.*

(i) $g^\star(c_k(E,h^E)) = c_k(E,h^E)$, $0 \le k \le r$.

(ii) *If* $(A_k)_\star\mu_k = \mu_k$, *then* $g_\star(\mathbb{E}_k(\mu_k)) = \mathbb{E}_k(\mu_k)$.

Proof. (i) Let $e_1(x),\dots,e_r(x)$ be a holomorphic frame for E in a neighborhood of a point $x_0 \in X$. Then $A_1(e_1),\dots,A_1(e_r)$ is a holomorphic frame for E near $g(x_0)$. Consider the matrices

$$H_1 = [h^E_x(e_k,e_j)]_{1\le j\le r,\,1\le k\le r}, \quad H_2 = [h^E_{g(x)}(A_1(e_k),A_1(e_j))]_{1\le j\le r,\,1\le k\le r}$$

that represent h^E near x_0 and near $g(x_0)$ respectively. By the isometry property (5.3) of G, we infer that

$$H_2 = [h^E_x(e_k\circ g^{-1}, e_j\circ g^{-1})]_{j,k};$$

hence

$$g^\star H_2 = [g^\star h^E_x(e_k\circ g^{-1}, e_j\circ g^{-1})]_{j,k} = H_1.$$

Therefore,

$$g^\star R^E_{g(x)} = g^\star\big(\bar\partial(H_2^{-1}\partial H_2)\big) = \bar\partial\big((g^\star H_2)^{-1}\partial(g^\star H_2)\big) = \bar\partial(H_1^{-1}\partial H_1) = R^E_x,$$

which yields (i).

(ii) Let ϕ be a smooth $(n+k-r-1, n+k-r-1)$ form on X. Then

$$\langle g_\star(\mathbb{E}_k(\mu_k)),\phi\rangle = \int_{(\mathbb{P}V)^k}\int_{D_k(s_1,\dots,s_k)} g^\star\phi\, d\mu_k = \int_{(\mathbb{P}V)^k}\int_{g(D_k(s_1,\dots,s_k))} \phi\, d\mu_k.$$

Since $G\colon E|_x \to E|_{g(x)}$ is an isomorphism, we have, by (5.4),

$$\begin{aligned} g(D_k(s_1,\dots,s_k)) &= \{g(x) : s_1(x)\wedge\cdots\wedge s_k(x) = 0\} \\ &= \{y : s_1(g^{-1}(y))\wedge\cdots\wedge s_k(g^{-1}(y)) = 0\} \\ &= \{y : A_1(s_1)(y)\wedge\cdots\wedge A_1(s_k)(y) = 0\} \\ &= D_k(A_1(s_1),\dots,A_1(s_k)). \end{aligned}$$

Hence, by the invariance assumption on μ_k, we obtain

$$\begin{aligned}\langle g_\star(\mathbb{E}_k(\mu_k)), \phi\rangle &= \int_{(\mathbb{P}V)^k}\int_{D_k(A_1(s_1),\dots,A_1(s_k))}\phi\, d\mu_k \\ &= \int_{(\mathbb{P}V)^k}\int_{D_k(s_1,\dots,s_k)}\phi\, d((A_k)_\star\mu_k) = \langle \mathbb{E}_k(\mu_k), \phi\rangle. \qquad \square\end{aligned}$$

We now apply these invariance properties to the case of the universal bundle on a Grassmannian. Let W be a complex vector space of dimension m endowed with an inner product $\langle\cdot,\cdot\rangle$, and let $\mathbb{G}(r, W)$ be the Grassmannian of r-planes in W. Let $\mathscr{T}\to\mathbb{G}(r, W)$ be the universal bundle endowed with the Hermitian metric $h^{\mathscr{T}}$ induced by the inner product on W, and let $(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ be the dual holomorphic vector bundle. The unitary group $\mathrm{U}(W)$ of $(W,\langle\cdot,\cdot\rangle)$ acts on $\mathbb{G}(r, W)$ in the obvious way: if $g\in\mathrm{U}(W)$ and $\{H\}\in\mathbb{G}(r, W)$, where H is an r-plane in W, then $g\{H\} = \{g(H)\}$. This action lifts to $\mathscr{T}^\star$ as follows: if $\ell\in H^\star$, we define $G(\ell) = \ell\circ g^{-1}\in g(H)^\star$. Since $H^0(\mathbb{G}(r, W), \mathscr{T}^\star) = W^\star$, the map A_1 defined in (5.4) is given by

$$A_1\colon W^\star\to W^\star,\quad A_1(\ell) = \ell\circ g^{-1}. \tag{5.6}$$

We consider $W^\star$ with the inner product $\langle\cdot,\cdot\rangle$ induced by that on W. Let ω_{FS} be the Fubini–Study form on $\mathbb{P}W^\star$ and let ν the Gaussian probability measure on $W^\star$ given in coordinates with respect to an orthonormal basis of $W^\star$.

Lemma 5.3. *If $g\in\mathrm{U}(W)$, then $G\in\mathrm{Aut}(\mathscr{T}^\star)$ verifies* (5.3). *Moreover, the map A_1 in* (5.6) *is an isometry of $W^\star$; hence $A_1^\star\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$ and $(A_1)_\star\nu = \nu$.*

Proof. For each $\ell\in W^\star$, there is a unique $w_\ell\in W$ such that $\ell(w) = \langle w, w_\ell\rangle$, $w\in W$. We have $\langle\ell,\ell'\rangle = \langle w_\ell, w_{\ell'}\rangle$, and if $g\in\mathrm{U}(W)$, then

$$A_1(\ell)(w) = \langle g^{-1}(w), w_\ell\rangle = \langle w, g(w_\ell)\rangle,\quad \langle A_1(\ell), A_1(\ell')\rangle = \langle g(w_\ell), g(w_{\ell'})\rangle = \langle w_\ell, w_{\ell'}\rangle.$$

The assertion about G follows in a similar way. $\square$

Theorem 5.4. *In the above setting, let ν_k be the Gaussian probability measure on $(W^\star)^k$ and let μ_k be the product measure on $(\mathbb{P}W^\star)^k$ determined by the Fubini–Study volume $(\omega_{\mathrm{FS}})^{m-1}$ on $\mathbb{P}W^\star$. Then*

$$\mathbb{E}_k(\mu_k) = \mathbb{E}_k(\nu_k) = c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star}),\quad 1\le k\le r.$$

Proof. We have $\mu_k = \omega_{\mathrm{FS}}(z_1)^{m-1}\wedge\dots\wedge\omega_{\mathrm{FS}}(z_k)^{m-1}$, where $\omega_{\mathrm{FS}}(z_j) = \pi_j^\star\omega_{\mathrm{FS}}$ and $\pi_j\colon(\mathbb{P}W^\star)^k\to\mathbb{P}W^\star_{z_j}$ is the projection on the j-th factor. If A_k is the map defined in (5.5), then by Lemma 5.3, we infer that $(A_k)_\star\nu_k = \nu_k$, $(A_k)_\star\mu_k = \mu_k$. Hence, by Lemma 5.2, we have $g_\star\mathbb{E}_k = \mathbb{E}_k$ for all $g\in\mathrm{U}(W)$, where $\mathbb{E}_k$ denotes either one of $\mathbb{E}_k(\mu_k)$ or $\mathbb{E}_k(\nu_k)$.

Let dg denote the Haar measure on $\mathrm{U}(W)$. Then

$$\int_{\mathrm{U}(W)}(g^{-1})^\star\mathbb{E}_k\, dg = \int_{\mathrm{U}(W)} g_\star\mathbb{E}_k\, dg = \mathbb{E}_k,$$

so the current $\mathbb{E}_k$ is represented by a smooth $\mathrm{U}(W)$-invariant form [26, Proposition 1.2.1]. Since $\mathbb{G}(r, W)\cong\mathrm{U}(m)/\mathrm{U}(r)\times\mathrm{U}(m-r)$ is a symmetric space, the $\mathrm{U}(W)$-invariant forms coincide with the harmonic forms (see e.g. [28, p. 26]), so each cohomology class contains a unique invariant form. By Lemma 5.2, $c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})$ is $\mathrm{U}(W)$-invariant, and by Theorem 1.3, $\mathbb{E}_k \in c_{r+1-k}(\mathscr{T}^\star)$. Therefore, $\mathbb{E}_k = c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})$. □

The case $k = 1$ of Theorem 5.4 was obtained via different methods in [36, Theorem 3.1].

5.3. Formula for the expectation current. We give here the proof of Theorem 1.4. This will be done by using the Kodaira map (1.8) and Theorem 5.4. We will consider the case of the product measure μ_k on $(\mathbb{P}V)^k$. The proof in the case of the Gaussian probability measure on V^k is identical. Let us write for simplicity

$$Y = \mathbb{G}(r, V^\star), \quad m = \dim Y = r(N+1-r), \quad \Phi = \Phi_E \colon X \to Y,$$

where Φ_E is the Kodaira map defined in (1.8). Recall that every $s \in V$ induces, by evaluation, a linear functional $\sigma_s \colon V^\star \to \mathbb{C}$, $\sigma_s(v^\star) = v^\star(s)$. By restriction to the fibers, σ_s determines a global section $\sigma_s \in H^0(Y, \mathscr{T}^\star)$, given by $\sigma_s(y) = \sigma_s|_{\mathscr{T}_y}$, and we have a linear isometry

$$V \to H^0(Y, \mathscr{T}^\star), \quad s \mapsto \sigma_s.$$

For $\sigma_{s_1}, \dots, \sigma_{s_k} \in H^0(Y, \mathscr{T}^\star)$, we consider the degeneracy set

$$D_k(\sigma_{s_1}, \dots, \sigma_{s_k}) = \{y \in Y : \sigma_{s_1}(y) \wedge \dots \wedge \sigma_{s_k}(y) = 0\}.$$

Since Φ is holomorphic, we have as in (3.1) an isomorphism of holomorphic vector bundles on X,

$$\Psi \colon \Phi^\star(\mathscr{T}^\star) \to E, \quad \Psi((\Phi^\star\sigma_s)(x)) = s(x), \quad s \in V,\ x \in X.$$

Moreover, if $s_1, \dots, s_k \in \mathbb{P}V$, then

$$D_k(s_1, \dots, s_k) = \Phi^{-1}(D_k(\sigma_{s_1}, \dots, \sigma_{s_k})), \quad 1 \le k \le r. \tag{5.7}$$

Recall from Section 5.1 that, for generic $(s_1, \dots, s_k) \in (\mathbb{P}V)^k$, we have

$$\operatorname{codim}_X D_k(s_1, \dots, s_k) = r+1-k, \quad \operatorname{codim}_Y D_k(\sigma_{s_1}, \dots, \sigma_{s_k}) = r+1-k. \tag{5.8}$$

Lemma 5.5. *The following statements hold.*

(i) *If* $(s_1, \dots, s_k) \in (\mathbb{P}V)^k$ *verifies* (5.8), *then*

$$\Phi_\star[D_k(s_1, \dots, s_k)] = [D_k(\sigma_{s_1}, \dots, \sigma_{s_k}) \cap \Phi(X)].$$

(ii) *We have*

$$\Phi_\star \mathbb{E}_k(\mu_k) = \int_{(\mathbb{P}V)^k} [D_k(\sigma_{s_1}, \dots, \sigma_{s_k}) \cap \Phi(X)]\, d\mu_k(s_1, \dots, s_k).$$

Proof. Let χ be a smooth $(n+k-r-1, n+k-r-1)$ form on Y and set

$$Z = D_k(s_1, \dots, s_k).$$

Since Φ is an embedding, it follows by (5.7) and (5.8) that

$$\begin{gathered} \Phi(Z) = D_k(\sigma_{s_1}, \dots, \sigma_{s_k}) \cap \Phi(X), \\ \operatorname{codim}_X Z = \operatorname{codim}_{\Phi(X)} \Phi(Z) = r+1-k. \end{gathered} \tag{5.9}$$

Hence $\Phi(Z) \subset \Phi(X) \subset Y$ is an analytic subset of pure dimension $n+k-r-1$. We have

$$\langle \Phi_\star[D_k(s_1,\dots,s_k)], \chi\rangle = \int_Z \Phi^\star\chi = \int_{\Phi(Z)} \chi = \langle [D_k(\sigma_{s_1},\dots,\sigma_{s_k}) \cap \Phi(X)], \chi\rangle,$$

which proves (i). Using (i), we obtain

$$\begin{aligned}\langle \Phi_\star \mathbb{E}_k(\mu_k), \chi\rangle &= \int_{(\mathbb{P}V)^k} \langle [D_k(s_1,\dots,s_k)], \Phi^\star\chi\rangle\, d\mu_k \\ &= \int_{(\mathbb{P}V)^k} \langle [D_k(\sigma_{s_1},\dots,\sigma_{s_k}) \cap \Phi(X)], \chi\rangle\, d\mu_k. \qquad \square\end{aligned}$$

Lemma 5.6. *The current $T = \Phi^\star(c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})) - \mathbb{E}_k(\mu_k)$ is strongly positive.*

Proof. The statement is local. Let $x_0 \in X$. Since Φ is an embedding, we can find a coordinate polydisc $U \subset Y$ centered at $y_0 = \Phi(x_0)$ such that if $z = (z', z'') \in \mathbb{C}^{m-n} \times \mathbb{C}^n$ are the coordinates on U, then $\Phi(X) \cap U = \{z' = 0\}$.

Let χ be a (weakly) positive $(n+k-r-1, n+k-r-1)$ form supported in $\Phi^{-1}(U)$. It is easy to see that there exists a positive $(n+k-r-1, n+k-r-1)$ form $\widetilde{\chi}$ supported in U such that $\Phi^\star\widetilde{\chi} = \chi$.

Assume that $(s_1,\dots,s_k) \in (\mathbb{P}V)^k$ verifies (5.8) and let $Z := D_k(s_1,\dots,s_k)$. We have

$$[\Phi(X) \cap U] = (dd^c v)^{m-n}, \quad \text{where } v(z) = \log\|z'\|, \quad z \in U. \tag{5.10}$$

Note that, by (5.9), the unbounded locus $\{z' = 0\}$ of v intersects the support of the current $[D_k(\sigma_{s_1},\dots,\sigma_{s_k})]$ in the analytic set $\Phi(Z) \cap U$, which has dimension $n+k-r-1$. Hence $R := [D_k(\sigma_{s_1},\dots,\sigma_{s_k})] \wedge (dd^c v)^{m-n}$ is a well-defined positive closed current of bidimension $(n+k-r-1, n+k-r-1)$ in U by [17, Theorem 2.5] (see also [23, Corollary 3.6]). Moreover, if $v_\varepsilon(z) = \frac{1}{2}\log(\|z'\|^2 + \varepsilon)$, then by the continuity of the complex Monge–Ampère operator along decreasing sequences ([17, Proposition 2.9], [23, Corollary 3.6]), we have

$$[D_k(\sigma_{s_1},\dots,\sigma_{s_k})] \wedge (dd^c v_\varepsilon)^{m-n} \to [D_k(\sigma_{s_1},\dots,\sigma_{s_k})] \wedge (dd^c v)^{m-n} \tag{5.11}$$

as $\varepsilon \searrow 0$, in the weak sense of currents on U. By (5.10), the support of R is equal to $\Phi(Z) \cap U$. The support theorem for positive closed currents implies that $R = \sum a_j [A_j]$, where A_j are the irreducible components of $\Phi(Z) \cap U$ and $a_j \geq 1$ are integers. Hence

$$[D_k(\sigma_{s_1},\dots,\sigma_{s_k}) \cap \Phi(X)] \leq [D_k(\sigma_{s_1},\dots,\sigma_{s_k})] \wedge (dd^c v)^{m-n} \tag{5.12}$$

holds on U. Using Lemma 5.5 and (5.12), we obtain

$$\begin{aligned}\langle \mathbb{E}_k(\mu_k), \chi\rangle = \langle \Phi_\star\mathbb{E}_k(\mu_k), \widetilde{\chi}\rangle &= \int_{(\mathbb{P}V)^k} \langle [D_k(\sigma_{s_1},\dots,\sigma_{s_k}) \cap \Phi(X)], \widetilde{\chi}\rangle\, d\mu_k \\ &\leq \int_{(\mathbb{P}V)^k} \langle [D_k(\sigma_{s_1},\dots,\sigma_{s_k})] \wedge (dd^c v)^{m-n}, \widetilde{\chi}\rangle\, d\mu_k.\end{aligned}$$

Applying (5.11) and Fatou's lemma, we infer that

$$\langle \mathbb{E}_k(\mu_k), \chi\rangle \leq \liminf_{\varepsilon\searrow 0} \int_{(\mathbb{P}V)^k} \langle [D_k(\sigma_{s_1},\dots,\sigma_{s_k})], (dd^c v_\varepsilon)^{m-n} \wedge \widetilde{\chi}\rangle\, d\mu_k.$$

By Theorem 5.4, we have

$$\int_{(\mathbb{P}V)^k} \langle [D_k(\sigma_{s_1},\dots,\sigma_{s_k})], (dd^c v_\varepsilon)^{m-n} \wedge \widetilde{\chi}\rangle \, d\mu_k = \int_Y c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star}) \wedge (dd^c v_\varepsilon)^{m-n} \wedge \widetilde{\chi}.$$

As $\varepsilon \searrow 0$, it follows that

$$\langle \mathbb{E}_k(\mu_k), \chi\rangle \le \int_{\Phi(X)} c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star}) \wedge \widetilde{\chi} = \int_X \Phi^\star\big(c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})\big) \wedge \chi.$$

This concludes the proof of the lemma. □

Proof of Theorem 1.4. By Lemma 5.6,

$$T = \Phi^\star(c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})) - \mathbb{E}_k(\mu_k)$$

is a positive closed current of bidegree $(r+1-k, r+1-k)$ on X which is null-cohomologous. Indeed, by Theorem 1.3, $\mathbb{E}_k(\mu_k)$ belongs to the Chern class $c_{r+1-k}(E)$, and so does the form $\Phi^\star(c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star}))$ since the vector bundles $\Phi^\star(\mathscr{T}^\star)$ and E are isomorphic. Since ω is closed, we have $\int_X T \wedge \omega^{n+k-r-1} = 0$; hence $T = 0$. □

Proof of Corollary 1.5. Under the hypothesis of Corollary 1.5, for all sufficiently large p, the Kodaira map Φ_p in (1.2) is a holomorphic embedding. Moreover, by the isomorphism in (3.1), assumptions (A)–(C) hold for (X,ω) and $(L^p \otimes E, h^{L^p\otimes E})$. So Theorem 1.4 applies, and we have

$$\mathbb{E}_k(\mu_k^p) = \mathbb{E}_k(\nu_k^p) = \Phi_p^\star\big(c_{r+1-k}(\mathscr{T}^\star, h^{\mathscr{T}^\star})\big), \quad 1 \le k \le r, \quad p \gg 1.$$

Then (1.11) and (1.12) follow directly from (1.3) and (1.4) respectively. □

5.4. Degeneracy set of a tuple of sections. Let us consider a special case of Theorem 1.4 for $E = L^{\oplus r}$ $(1 \le r \le n)$, where (L, h^L) is a positive line bundle on X such that the Kodaira map $\Phi_L \colon X \to \mathbb{P}V_L^\star$ is an embedding, where $V_L := H^0(X, L)$. We also assume that every r sections in V_L have at least one common zero in X. Then (X,ω) and

$$(E, h^E) := (L^{\oplus r}, (h^L)^{\oplus r})$$

satisfy assumptions (A)–(C). Note that $V := H^0(X, L^{\oplus r}) = V_L^{\oplus r}$, and let ν_L be the Gaussian probability measure on V_L.

We define a canonical embedding $\Theta_r \colon \mathbb{P}V_L^\star \to \mathbb{G}(r, V^\star)$ that sends the complex line $[\xi]$ to the r-subspace $(\mathbb{C}\xi)^{\oplus r}$. Moreover, we have the following isomorphism of the Hermitian bundles on $\mathbb{P}V_L^\star$:

$$\Theta_r^\star \mathscr{T}^\star \simeq \mathscr{O}(1)^{\oplus r}.$$

As a consequence, on $\mathbb{P}V_L^\star$, for $0 \le k \le r$,

$$\Theta_r^\star c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star}) = \binom{r}{k} \omega_{\mathrm{FS}}^k,$$

where ω_{FS} is the Fubini–Study form on $\mathbb{P}V_L^\star$ associated to the L^2-inner product on V_L.

Let $\Phi_E\colon X \to \mathbb{G}(r, V^\star)$ be the Kodaira map (1.2). Then we have $\Phi_E = \Theta_r \circ \Phi_L$. Hence Φ_E is an embedding, and

$$\Phi_E^\star(c_k(\mathscr{T}^\star, h^{\mathscr{T}^\star})) = \binom{r}{k} \Phi_L^\star(\omega_{\mathrm{FS}})^k. \tag{5.13}$$

When $k = r$, combining Theorem 1.4 and (5.13), we recover [34, Lemma 4.3] for the expectation current of simultaneous zeros of r independent random holomorphic sections of L.

Consider the random square matrix $(s_{j\ell})_{1\le j,\ell\le r} \in M_{r\times r}(V_L) \simeq V_L^{r^2}$, where the coefficients $s_{j\ell}$ are independent and identically distributed random elements in V_L with the law ν_L. Define the determinant section

$$\det(s_{j\ell})_{1\le j,\ell\le r} \in H^0(X, L^r). \tag{5.14}$$

Then the zero set of $\det(s_{j\ell})_{1\le j,\ell\le r}$ equals the degeneracy set of $(s_1, \dots, s_r) \in V^r$, where $s_\ell = (s_{j\ell})_{1\le j\le r} \in V$. We endow V^r with the Gaussian probability measure $\nu_r = \nu_L^{r^2}$. By Theorem 1.4 and (5.13), the expectation current for the zeros of the determinant section (5.14) is given by

$$\mathbb{E}[\det(s_{j\ell})_{1\le j,\ell\le r} = 0] = r\Phi_L^\star(\omega_{\mathrm{FS}}). \tag{5.15}$$

We will discuss further the case $E = L^{\oplus r}$ in Section 7.

6. Distribution of zeros of random sections

In this section, we give the proof of Theorem 1.2. To this end, we will use Theorem 1.1, together with an equidistribution theorem of Dinh–Sibony for meromorphic transforms [21]. We will apply their result to the meromorphic transforms determined by the Kodaira maps into Grassmannians as in Section 4.

We start by recalling a few notions that we will need. Let $\mathbb{P}^\ell$ be the complex projective space of dimension ℓ and let ω_{FS} be the Fubini–Study form. We denote by $\mathrm{PSH}(\mathbb{P}^\ell, \omega_{\mathrm{FS}})$ the class of ω_{FS}-plurisubharmonic functions. These are the functions φ on $\mathbb{P}^\ell$ which are locally the sum of a plurisubharmonic (psh) function and a smooth one, such that $\omega_{\mathrm{FS}} + dd^c\varphi \ge 0$ in the sense of currents. Let

$$R(\mathbb{P}^\ell) = \sup_\varphi \Big\{ \max_{\mathbb{P}^\ell} \varphi : \varphi \in \mathrm{PSH}(\mathbb{P}^\ell, \omega_{\mathrm{FS}}),\ \int_{\mathbb{P}^\ell} \varphi\, \omega_{\mathrm{FS}}^\ell = 0 \Big\}, \tag{6.1}$$

$$\Delta(\mathbb{P}^\ell, t) = \sup_\varphi \Big\{ \int_{\{\varphi < -t\}} \omega_{\mathrm{FS}}^\ell : \varphi \in \mathrm{PSH}(\mathbb{P}^\ell, \omega_{\mathrm{FS}}),\ \int_{\mathbb{P}^\ell} \varphi\, \omega_{\mathrm{FS}}^\ell = 0 \Big\}, \quad t \in \mathbb{R}. \tag{6.2}$$

Then

$$R(\mathbb{P}^\ell) \le \frac{1}{2}(1 + \log \ell), \quad \Delta(\mathbb{P}^\ell, t) \le c\ell e^{-\alpha t} \text{ for all } t \in \mathbb{R}, \tag{6.3}$$

where $c > 0$, $\alpha > 0$ are constants independent of ℓ; cf. [21, Proposition A.3, Corollary A.5].

Theorem 6.1. *Let (X, ω) be a compact Kähler manifold of dimension n, let (L, h^L) be a positive line bundle on X, and let (E, h^E) be a Hermitian holomorphic vector bundle of rank $r \le n$ on X. Let $(\mathscr{H}, \Upsilon)$ be the probability space defined in* (1.5). *Then there exist $C > 0$ and*

$p_0 \in \mathbb{N}$ such that the following holds: for any $\gamma > 1$, there exist subsets $\mathcal{E}_p = \mathcal{E}_p(\gamma) \subset \mathbb{P}V_p$ such that, for $p > p_0$, we have

(i) *$\Upsilon_p(\mathcal{E}_p) \leq Cp^{-\gamma}$;*

(ii) *if $s_p \in \mathbb{P}V_p \setminus \mathcal{E}_p$, then*

$$\Big|\frac{1}{p^r}\langle [s_p = 0] - \Phi_p^\star(c_r(\mathscr{T}^\star, h^{\mathscr{T}^\star})), \phi\rangle\Big| \leq C\gamma\frac{\log p}{p}\|\phi\|_{\mathscr{C}^2(X)}$$

for any $(n-r, n-r)$ form ϕ of class $\mathscr{C}^2$ on X. Moreover, estimate (ii) *holds for Υ-a.e. sequence $\{s_p\}_{p\geq 1} \in \mathcal{H}$ provided that p is large enough.*

Proof. We consider the Kodaira maps Φ_p from (1.2) as meromorphic transforms as in Section 4 (see (1.8), (4.2)). Since (L, h^L) is positive, we have by [29, Theorem 5.1.15] that the vector bundles $L^p \otimes E$ verify (C) for all p sufficiently large.

Let $\Gamma_p \subset X \times \mathbb{P}V_p$ be the set defined by Φ_p as in (4.2), where $\mathbb{P}V_p$ is endowed with the Fubini–Study form ω_{FS}. It follows from [27, Corollary 3.6] that, for all p sufficiently large and for generic $s \in V_p$, we have $Z_s \neq \varnothing$, so the image of the projection $\pi_2(\Gamma_p) \subset \mathbb{P}V_p$ is dense. Thus π_2 is surjective since $\pi_2(\Gamma_p)$ is an analytic subset of $\mathbb{P}V_p$, so (B) holds as well. By Proposition 4.2, Φ_p defines a meromorphic transform F_p of codimension $n-r$, with graph Γ_p. Let $d(F_p)$, $\delta(F_p)$ denote its intermediate degrees of order d_p and $d_p - 1$ respectively; see (4.7). Using Theorem 4.3 and Lemma 2.2, we get

$$\begin{aligned} d(F_p) &= \int_X c_r(L^p \otimes E, h^{L^p\otimes E}) \wedge \omega^{n-r} \\ &= p^r \int_X c_1(L, h^L)^r \wedge \omega^{n-r} + O(p^{r-1}), \\ \delta(F_p) &= \int_X c_{r-1}(L^p \otimes E, h^{L^p\otimes E}) \wedge \omega^{n-r+1} \\ &= rp^{r-1} \int_X c_1(L, h^L)^{r-1} \wedge \omega^{n-r+1} + O(p^{r-2}). \end{aligned}$$

Using these, (2.9), and (6.3), we infer that there exist $C_1 > 1$ and $p_0 \in \mathbb{N}$ such that, for all $p > p_0$, we have

$$\begin{aligned} &d_p \leq C_1 p^n, \quad d(F_p) \leq C_1 p^r, \quad \frac{d(F_p)}{\delta(F_p)} \geq \frac{p}{C_1}, \\ &R(\mathbb{P}V_p) \leq C_1 + \frac{n}{2}\log p, \quad \Delta(\mathbb{P}V_p, t) \leq C_1 p^n e^{-\alpha t} \quad \text{for all } t \in \mathbb{R}, \end{aligned} \tag{6.4}$$

where $R(\mathbb{P}V_p)$, $\Delta(\mathbb{P}V_p, t)$ are defined in (6.1) and (6.2), and $\alpha > 0$ is the constant from (6.3). For $\varepsilon > 0$, let

$$\begin{aligned} \mathcal{E}'_p(\varepsilon) &:= \bigcup_{\|\phi\|_{\mathscr{C}^2(X)}\leq 1} \{s_p \in \mathbb{P}V_p : |\langle F_p^\star(\delta_{s_p}) - F_p^\star(\Upsilon_p), \phi\rangle| \geq d(F_p)\varepsilon\}, \\ \mathcal{E}''_p(\varepsilon) &:= \bigcup_{\|\phi\|_{\mathscr{C}^2(X)}\leq 1} \{s_p \in \mathbb{P}V_p : |\langle F_p^\star(\delta_{s_p}) - F_p^\star(\Upsilon_p), \phi\rangle| \geq C_1 p^r\varepsilon\}, \end{aligned}$$

where ϕ is an $(n-r, n-r)$ form on X of class $\mathscr{C}^2$. Recall that $F_p^\star(\Upsilon_p)$ is defined by (4.3), since Υ_p is smooth. Lemma 4.2 (d) in [21] states that, for all $\varepsilon > 0$, we have

$$\Upsilon_p(\mathcal{E}''_p(\varepsilon)) \leq \Upsilon_p(\mathcal{E}'_p(\varepsilon)) \leq \Delta(\mathbb{P}V_p, t_{\varepsilon,p}), \quad \text{where } t_{\varepsilon,p} = \varepsilon\frac{d(F_p)}{\delta(F_p)} - 3R(\mathbb{P}V_p). \tag{6.5}$$

Recall by (4.6) that $F_p^\star(\delta_{s_p}) = [s_p = 0]$ for s_p outside an analytic subset of $\mathbb{P}V_p$. Moreover, by Theorem 4.3, $F_p^\star(\Upsilon_p) = \Phi_p^\star(c_r(\mathscr{T}^\star, h^{\mathscr{T}^\star}))$. Using (6.4) and (6.5), we get for all $p > p_0$ and $\varepsilon > 0$ that

$$t_{\varepsilon,p} \geq \frac{\varepsilon p}{C_1} - 3\Big(C_1 + \frac{n}{2}\log p\Big), \quad \Upsilon_p(\mathcal{E}_p''(\varepsilon)) \leq C_1 p^n \exp\Big(-\frac{\alpha\varepsilon p}{C_1} + 3\alpha C_1 + \frac{3\alpha n}{2}\log p\Big).$$

We now take $\gamma > 1$ and choose $\varepsilon = \varepsilon_{p,\gamma}$ such that

$$-\frac{\alpha p \varepsilon}{C_1} + 3\alpha C_1 + \frac{3\alpha n}{2}\log p = -(n+\gamma)\log p.$$

Let $\mathcal{E}_p = \mathcal{E}_p(\gamma) := \mathcal{E}_p''(\varepsilon_{p,\gamma})$. Then, for all $p > p_0$, we have

$$\Upsilon_p(\mathcal{E}_p(\gamma)) \leq C_1 p^{-\gamma}.$$

Moreover, if $s_p \in \mathbb{P}V_p \setminus \mathcal{E}_p$ and ϕ is an $(n-r, n-r)$ form on X of class $\mathscr{C}^2$, we infer by the definition of $\mathcal{E}_p$ that

$$\Big|\frac{1}{p^r}\langle [s_p = 0] - \Phi_p^\star(c_r(\mathscr{T}^\star, h^{\mathscr{T}^\star})), \phi\rangle\Big| \leq C_1 \varepsilon_{p,\gamma}\|\phi\|_{\mathscr{C}^2(X)}.$$

Note that

$$\varepsilon_{p,\gamma} = \frac{C_1}{\alpha p}\Big(\Big(n + \gamma + \frac{3\alpha n}{2}\Big)\log p + 3\alpha C_1\Big) \leq C_2\gamma\frac{\log p}{p}$$

holds for $p > p_0$, where $C_2 > 0$ is a constant independent of p and γ. This yields assertion (ii) of Theorem 6.1. Finally, the last assertion of Theorem 6.1 follows from the Borel–Cantelli lemma since $\sum_{p=1}^{\infty} \Upsilon_p(\mathcal{E}_p) < \infty$. □

Proof of Theorem 1.2. The proof is a direct application of Theorems 6.1 and 1.1. □

Remark 6.2. Note that, in Theorem 1.2, we may always replace $(\mathbb{P}V_p, \Upsilon_p)$ by $(V_p, \mathscr{G}_p)$, where $\mathscr{G}_p$ denotes the Gaussian probability measure on V_p induced by the L^2-inner product. Indeed, if $\pi: V_p \setminus \{0\} \to \mathbb{P}V_p$ is the canonical projection and $\widetilde{\mathcal{E}_p} = \pi^{-1}(\mathcal{E}_p)$, then

$$\mathscr{G}_p(\widetilde{\mathcal{E}_p}) = \Upsilon_p(\mathcal{E}_p),$$

so Theorem 1.2 holds for the sets $\widetilde{\mathcal{E}_p}$.

7. Simultaneous zero sets of holomorphic sections

Let (X, ω) be a compact Kähler manifold of dimension n and let (L, h^L) be a positive line bundle on X. Let $E_0 = X \times \mathbb{C}^r$ be the trivial vector bundle on X, where $r \leq n$. In this section, we specialize the results of Theorem 1.2 and Corollary 1.5 to the case $E = E_0$. In particular, we recover the results of Shiffman–Zelditch [34] and Dinh–Sibony [21] for the simultaneous zeros of several independent random sections of a line bundle.

If $V_p := H^0(X, L^p \otimes E_0)$, we have a natural isomorphism

$$V_p \cong H^0(X, L^p)^{\oplus r}, \tag{7.1}$$

which identifies sections of $L^p \otimes E_0$ to r-tuples of sections of L^p. For $j = 1, \dots, r$, let $\Pi_j^p: V_p \to H^0(X, L^p)$ be the projection onto the j-th component in the direct sum of (7.1).

Let $h_0^{E_0}$ denote the trivial Hermitian metric on E_0. Then (7.1) is also an isometry of the L^2-inner products induced by h^L and $h_0^{E_0}$. If we equip the trivial bundle E_0 with an arbitrary smooth Hermitian metric h^{E_0}, then the identification (7.1) is generally no longer isometric, and the direct sum is not orthogonal with respect to the L^2-inner product induced by h^L and h^{E_0}.

Let $M_+(\mathbb{C}^r)$ be the set of positive definite Hermitian matrices of rank r, which carries the usual smooth structure. Then the set of smooth Hermitian metrics h^{E_0} on E_0 is given by the set $\mathscr{C}^\infty(X, M_+(\mathbb{C}^r))$ of all smooth functions on X with values in $M_+(\mathbb{C}^r)$.

If $h_H^{E_0}$ on E_0 is given by an element $H \in \mathscr{C}^\infty(X, M_+(\mathbb{C}^r))$, that is,

$$h_H^{E_0}(\cdot,\cdot)(x) = h_0^{E_0}(H(x)\cdot,\cdot), \quad x \in X,$$

then the Chern curvature associated with $h_H^{E_0}$ is given by the following matrix of $(1,1)$-forms (see (2.5)):

$$R^E = H^{-1}\bar\partial\partial H - H^{-1}\bar\partial H \wedge H^{-1}\partial H.$$

We can write the formula for $c_1(E_0, h_H^{E_0})$ in terms of the matrix H,

$$\begin{aligned} c_1(E_0, h_H^{E_0}) &= \frac{i}{2\pi}\operatorname{Tr}[H^{-1}\bar\partial\partial H - H^{-1}\bar\partial H \wedge H^{-1}\partial H] \\ &= -\frac{i}{2\pi}\partial\bar\partial \log\det H(x) \in \Omega^{(1,1)}(X). \end{aligned}$$

We equip V_p with the inner product induced by h^L and $h_H^{E_0}$, and we denote by $\mathscr{G}_p^k = \mathscr{G}_{p,H}^k$ the corresponding Gaussian probability measure on V_p^k, where $1 \le k \le r$ (it corresponds to the measure ν_k^p in Corollary 1.5). If $(s_p^1, \dots, s_p^k) \in V_p^k$, then $s_p^1 \wedge \cdots \wedge s_p^k$ defines a random holomorphic section of $L^{pk} \otimes \Lambda^k E_0$ over X, where $\Lambda^k E_0 = X \times \Lambda^k \mathbb{C}^r$. Let

$$\mathbb{E}_k(\mathscr{G}_p^k) = \mathbb{E}[D_k(s_p^1, \dots, s_p^k)]$$

be the expectation current of the degeneracy set of $s_p^1, \dots, s_p^k$ defined in (1.10). Then Theorem 1.2 and Corollary 1.5 hold in this setting (see also Remark 6.2). In particular, we have the following proposition.

Proposition 7.1. *In the above setting, where $h_H^{E_0}$ is the Hermitian metric associated to $H \in \mathscr{C}^\infty(X, M_+(\mathbb{C}^r))$, the following hold.*

(i) *If $s_p = (s_{p,1}, \dots, s_{p,r}) \in V_p$, then, as $p \to \infty$, the normalized current of integration*

$$p^{-r}[s_p = 0] = p^{-r}[s_{p,1} = \cdots = s_{p,r} = 0]$$

over the simultaneous zeros converges weakly almost surely to $c_1(L, h^L)^r$, with convergence speed $O(p^{-1}\log p)$.

(ii) *If $\omega = c_1(L, h^L)$, then, as $p \to \infty$,*

$$\tag{7.2} \begin{aligned} \frac{1}{p^{r+1-k}}\mathbb{E}_k(\mathscr{G}_p^k) = &\binom{r}{k-1}\omega^{r+1-k} \\ &- \frac{i}{2\pi p}\binom{r-1}{r-k}\partial\bar\partial\log\det H \wedge \omega^{r-k} + O\Big(\frac{1}{p^2}\Big). \end{aligned}$$

The sub-leading term (coefficient of p^{-1}) vanishes identically if $\det H$ *is constant on X.*

In particular, when $k = 1$ and $\omega = c_1(L, h^L)$, we obtain the following asymptotics for the expectation of the currents of integration along simultaneous zeros as $p \to \infty$:

$$\begin{aligned}\frac{1}{p^r}\mathbb{E}[s_p = 0] &= \frac{1}{p^r}\mathbb{E}[s_{p,1} = \cdots = s_{p,r} = 0] \\ &= \omega^r - \frac{i}{2\pi p}\partial\bar{\partial}\log\det H \wedge \omega^{r-1} + O\Big(\frac{1}{p^2}\Big),\end{aligned} \tag{7.3}$$

where $s_p = (s_{p,1}, \dots, s_{p,r}) \in V_p$.

If we assume that the function H is the constant identity matrix $H \equiv \mathrm{Id}_r$, then the random sections $s_{p,1}, \dots, s_{p,r}$ are mutually independent identically distributed Gaussian variables. Consequently, Proposition 7.1 (i) and (7.3) precisely replicate the results obtained by Shiffman–Zelditch [34, Propositions 4.4 and 4.5] and Dinh–Sibony [21, Section 7].

Letting $k = r$ in (7.2), we obtain

$$\frac{1}{p}\mathbb{E}[D_r(s_p^1, \dots, s_p^r)] = r\omega - \frac{i}{2\pi p}\partial\bar{\partial}\log\det H + O\Big(\frac{1}{p^2}\Big), \quad p \to \infty. \tag{7.4}$$

Let $\det(s_p^1, \dots, s_p^r)$ be the determinant of $(s_p^1, \dots, s_p^r)$, which is a holomorphic section of $L^{pr} \otimes \Lambda^r E_0$ (see Section 5.4). Note that $\Lambda^r E_0 = X \times \mathbb{C}$ is the trivial line bundle on X. If we equip it with the Hermitian metric h_H such that $|1|^2_{h_H}(x) := \det H(x)$, then we can rewrite (7.4) as

$$\frac{1}{p}\mathbb{E}[D_r(s_p^1, \dots, s_p^r)] = c_1(L^r \otimes \Lambda^r E_0, h^{L^r} \otimes h_H^{1/p}) + O\Big(\frac{1}{p^2}\Big). \tag{7.5}$$

If we take $H \equiv \mathrm{Id}_r$, then we are in the setting of Section 5.4, that is, $s_p^1, \dots, s_p^r$ are independent and identically distributed Gaussian variables. By (5.15) and under the assumption that $\omega = c_1(L, h^L)$, we get a special case of (7.5),

$$\frac{1}{p}\mathbb{E}[D_r(s_p^1, \dots, s_p^r)]_{H=\mathrm{Id}_r} = \frac{r}{p}\Phi^\star_{L^p}(\omega_{\mathrm{FS}}) = c_1(L^r, h^{L^r}) + O\Big(\frac{1}{p^2}\Big),$$

where the last identity is deduced from Tian's original theorem for positive line bundles.

We conclude the paper by considering constant metrics on E_0. In this instance, we compute the covariance between the components of a random vector $s_p = (s_{p,1}, \dots, s_{p,r}) \in V_p$.

Proposition 7.2. *Given $A = (a_{j\ell})_{1\le j,\ell\le r} \in M_+(\mathbb{C}^r)$, we equip E_0 with the Hermitian metric $h_A^{E_0}(u, v) := h_0^{E_0}(A^{-1}u, v)$. Let $\mathscr{G}_p$ denote the Gaussian probability measure on V_p induced by h^L and $h_A^{E_0}$. Let $s_p = (s_{p,1}, \dots, s_{p,r})$ be a random element in V_p, where each $s_{p,j} = \Pi_j^p(s_p)$ is a random variable valued in $H^0(X, L^p)$. Then, for all sufficiently large $p \in \mathbb{N}$, we have the following.*

(i) *Each $s_{p,j}$ is a Gaussian random variable valued in $H^0(X, L^p)$ with the same distribution as $\sqrt{a_{jj}}\, s_{p,0}$ (note that $a_{jj} > 0$). Here $s_{p,0}$ denotes the random element in $H^0(X, L^p)$ with the Gaussian probability measure induced by the L^2-inner product associated with h^L.*

(ii) *The covariance matrix is given by*

$$\mathbb{E}[(s_{p,j}, s_{p,\ell})_p] = n_p a_{j\ell} \quad \textit{for } 1 \le j, \ell \le r,$$

where $n_p := \dim H^0(X, L^p)$.

Proof. Let $(\cdot,\cdot)_{p,L^2(h_0^{E_0})}$, $(\cdot,\cdot)_{p,L^2(h_A^{E_0})}$ denote the L^2-inner products on V_p induced by $h_0^{E_0}$ and $h_A^{E_0}$ respectively. Let $A_p \in \operatorname{End} V_p$ be defined by

$$A_p s_p = \Big(\sum_{j=1}^{r} a_{1j} s_{p,j}, \dots, \sum_{j=1}^{r} a_{rj} s_{p,j}\Big)$$

with respect to the splitting (7.1). Then, for $s_p, s'_p \in V_p$, we have

$$(s_p, s'_p)_{p,L^2(h_0^{E_0})} = (A_p s_p, s'_p)_{p,L^2(h_A^{E_0})}. \tag{7.6}$$

For $j \in \{1,\dots,r\}$, let $E_j^p \colon H^0(X,L^p) \to V_p$ be the linear embedding that gives the j-th component in the splitting (7.1). Let $\{e_{p,k}\}_{k=1}^{n_p}$ be an orthonormal basis of $H^0(X,L^p)$ with respect to the L^2-inner product $(\cdot,\cdot)_p$ induced by h^L. Then we have

$$s_{p,0} = \sum_{k=1}^{n_p} \eta_k e_{p,k},$$

where $\{\eta_k\}_k$ is a vector of independent and identically distributed standard complex Gaussian variables (with complex variance one).

Let s_p be the random element in V_p equipped with the Gaussian probability measure $\mathscr{G}_p$ (induced by $h_A^{E_0}$). We proceed to prove (i) and (ii). Since $s_{p,j} = \Pi_j^p(s_p)$ is a linear transformation of a Gaussian vector, $s_{p,j}$ is a Gaussian random variable valued in $H^0(X,L^p)$. That is, for any $v \in H^0(X,L^p)$, $(s_{p,j}, v)_p$ is a complex Gaussian variable. To determine the distribution of $s_{p,j}$, it suffices to study the Gaussian variables $(s_{p,j}, e_{p,k})_p$ for $k = 1,\dots,n_p$. By the definitions of $h_A^{E_0}$ and A_p, we have the following identity:

$$(A_p E_j^p e_{p,k'}, A_p E_{j'}^p e_{p,k})_{p,L^2(h_A^{E_0})} = \delta_{k'k} a_{j'j}. \tag{7.7}$$

Using (7.6) and the fact that A is Hermitian, we obtain

$$\begin{aligned}(s_{p,j}, e_{p,k})_p &= (\Pi_j^p(s_p), e_{p,k})_p = (s_p, E_j^p e_{p,k})_{p,L^2(h_0^{E_0})} \\ &= (A_p s_p, E_j^p e_{p,k})_{p,L^2(h_A^{E_0})} = (s_p, A_p E_j^p e_{p,k})_{p,L^2(h_A^{E_0})}.\end{aligned}$$

Therefore,

$$s_{p,j} = \sum_{k=1}^{n_p} (s_p, A_p E_j^p e_{p,k})_{p,L^2(h_A^{E_0})} e_{p,k}. \tag{7.8}$$

By (7.7), for a fixed $j \in \{1,\dots,r\}$, the set

$$\Big\{\frac{1}{\sqrt{a_{jj}}} A_p E_j^p e_{p,k}\Big\}_{k=1}^{n_p}$$

forms an orthonormal set in V_p with respect to the inner product $(\cdot,\cdot)_{p,L^2(h_A^{E_0})}$. By the fact that s_p is Gaussian with respect to this inner product, we obtain that

$$\Big\{\frac{1}{\sqrt{a_{jj}}} (s_p, A_p E_j^p e_{p,k})_{p,L^2(h_A^{E_0})}\Big\}_{k=1}^{n_p}$$

is a vector of independent and identically distributed standard complex Gaussian variables; see [35, Lemma 2.4.2 and Theorem 2.4.3]. Thus (i) follows from the definition of $s_{p,0}$ and (7.8). Moreover, we have

$$(s_{p,j}, s_{p,\ell})_p = \sum_{k=1}^{n_p} (s_p, A_p E_j^p e_{p,k})_{p,L^2(h_A^{E_0})} \overline{(s_p, A_p E_\ell^p e_{p,k})}_{p,L^2(h_A^{E_0})}.$$

Consequently, (7.7) and the fact that s_p is Gaussian directly imply (ii). □

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