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Equidistribution for weakly holomorphic sections of line bundles on algebraic curves

Dan Coman [(1)] and George Marinescu [(2)]

Dedicated to Professor Ahmed Zeriahi on the occasion of his retirement

Abstract. — We prove the convergence of the normalized Fubini–Study measures and the logarithms of the Bergman kernels of various Bergman spaces of holomorphic and weakly holomorphic sections associated to a singular Hermitian holomorphic line bundle on an algebraic curve. Using this, we study the asymptotic distribution of the zeros of random sequences of sections in these spaces.

Résumé. — Nous prouvons la convergence des mesures de Fubini–Study normalisées et des logarithmes des noyaux de Bergman de certains espaces de Bergman de sections holomorphes et faiblement holomorphes associées à un fibré holomorphe hermitien singulier sur une courbe algébrique. A l'aide de ce résultat, nous étudions la distribution asymptotique des zéros de suites aléatoires de sections dans ces espaces.

1. Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and (L, h) be a positive holomorphic line bundle on X such that $\omega = c_1(L, h)$. We let h_p be the metric induced by h on $L^p := L^{\otimes p}$ and denote by $H^0(X, L^p)$ the space of holomorphic sections of L^p. One can define a sequence of Fubini–Study forms γ_p on X by setting $\gamma_p = \Phi_p^\star(\omega_{\mathrm{FS}})$, where $\Phi_p : X \to \mathbb{P}(H^0(X, L^p)^\star)$ is the Kodaira map associated to (L^p, h_p) and ω_{FS} is the Fubini–Study form

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on a projective space. A theorem of Tian [31] states that $\frac{1}{p}\gamma_p \to c_1(L,h)$ as $p \to +\infty$, in the $\mathscr{C}^2$ topology on X. In [6] we proved the analogue of Tian's theorem in the case when h is a singular metric on L and its curvature is a Kähler current, i.e. $c_1(L,h) \geqslant \varepsilon\omega$ for some $\varepsilon > 0$. The above convergence is now in the weak sense of currents on X.

In [4] we generalized this further and studied the asymptotic behavior of the sequence of Fubini–Study currents associated to an arbitrary sequence of singular Hermitian holomorphic line bundles L_p on a compact normal Kähler space X. The normality of X was essential, in order to apply Riemann's second extension theorem for holomorphic functions [19, p. 143] and for plurisubharmonic functions [18, Satz 4] on a normal complex space. An interesting question is to analyze the general case when X is a compact Kähler space not necessarily normal.

In the present paper we study the one dimensional case. Note that any compact one dimensional complex space is projective and thus algebraic by [17, Satz 2, p. 343] (see also [27, Theorem 6.2]). We consider the following setting:

(A) $X \subset \mathbb{P}^N$ is an irreducible algebraic curve, $\Sigma = \{x_1, \dots, x_m\} \subset X$ is the set of singular points of X, and ω is a Hermitian form on X.
(B) L is a holomorphic line bundle on X with singular Hermitian metric h whose local weights are weakly subharmonic and such that $c_1(L,h) \geqslant \varepsilon\omega$ on $X \setminus \Sigma$ for some $\varepsilon > 0$.

Let us now introduce the normalization of X, which will be needed throughout the paper:

(C) $\sigma : \widetilde{X} \to \mathbb{P}^N$, where $\widetilde{X}$ is a compact Riemann surface, is the normalization of X and $\widetilde{\omega}$ is a Hermitian form on $\widetilde{X}$.

Here $c_1(L,h)$ denotes the curvature measure of h (see Section 3). A natural choice of the form ω is the restriction to X of the Fubini–Study form on $\mathbb{P}^N$, but ω can be any Hermitian form on X (see (2.3)). We denote by $H^0_w(X,L^p)$, respectively by $H^0_c(X,L^p)$, the space of weakly holomorphic sections, respectively continuous weakly holomorphic sections of L^p. Then

$$H^0(X,L^p) \subset H^0_c(X,L^p) \subset H^0_w(X,L^p) \subset H^0(X \setminus \Sigma, L^p),$$

where the latter is the space of holomorphic sections of $L^p|_{X\setminus\Sigma}$. We refer to Section 2 and Section 3 for the definitions of these notions.

We consider the corresponding Bergman subspaces of L^2-holomorphic sections with respect to the natural inner product induced by the metric h_p

and ω:

$$H^0_{c,(2)}(X,L^p)=\big\{S\in H^0_c(X,L^p):\|S\|_p<+\infty\big\},\tag{1.1}$$
$$H^0_{w,(2)}(X,L^p)=\big\{S\in H^0_w(X,L^p):\|S\|_p<+\infty\big\},\tag{1.2}$$
$$H^0_{(2)}(X\setminus\Sigma,L^p)=\big\{S\in H^0(X\setminus\Sigma,L^p):\|S\|_p<+\infty\big\},\tag{1.3}$$

where

$$\|S\|_p^2=\int_{X\setminus\Sigma}|S|^2_{h_p}\omega.$$

We show in Proposition 3.3 that the spaces $H^0_{(2)}(X\setminus\Sigma,L^p)$ are finite dimensional.

Let $P_{c,p},P_{w,p},P_p$ be the Bergman kernel functions and $\gamma_{c,p},\gamma_{w,p},\gamma_p$ be the Fubini–Study measures of the spaces $H^0_{c,(2)}(X,L^p)$, $H^0_{w,(2)}(X,L^p)$, $H^0_{(2)}(X\setminus\Sigma,L^p)$, respectively, induced by the above metric data. We refer to Section 3 for their definition and properties. In particular, we have that $\log P_{c,p}$, $\log P_{w,p}$, $\log P_p\in L^1(X,\omega)$ (see (3.10), (3.14)), and γ_p are signed measures with "small" negative variation (see Lemma 3.4). Our main result is the following:

Theorem 1.1. — *Let X,ω,L,h verify assumptions* (A) *and* (B)*. Then, as $p\to+\infty$, we have:*

(i) $\frac{1}{p}\log P_{c,p}\to 0$, $\frac{1}{p}\log P_{w,p}\to 0$, $\frac{1}{p}\log P_p\to 0$ *in* $L^1(X,\omega)$.

(ii) $\frac{1}{p}\gamma_{c,p}\to c_1(L,h)$, $\frac{1}{p}\gamma_{w,p}\to c_1(L,h)$, $\frac{1}{p}\gamma_p\to c_1(L,h)$, $\frac{1}{p}\gamma_p^+\to c_1(L,h)$ *in the weak sense of measures on X, where γ_p^+ is the positive variation of γ_p.*

In a series of papers starting with [29], Shiffman and Zelditch describe the asymptotic distribution of zeros of random sequences of holomorphic sections of powers of a positive line bundle L on a projective manifold (see also [14, 28, 30]). We showed in [6] that some of their results can be generalized to the setting of line bundles L with singular Hermitian metrics on compact Kähler manifolds, and also on compact Kähler orbifolds [5]. Further such equidistribution results with estimates on the speed of convergence are obtained in [7, 8, 13].

In [4] and [2] we prove equidistribution results for zeros of random sequences of sections in the case when the base space X is a compact normal Kähler space and the sequence of powers L^p is replaced by an arbitrary sequence of singular Hermitian holomorphic line bundles L_p satisfying certain assumptions. The results in [2] apply to very general probability measures on the Bergman spaces $H^0_{(2)}(X,L_p)$ of L^2-integrable holomorphic sections, and one of the key ingredients of the proof is the version of Theorem 1.1 in that setting (see [4, Theorem 1.1] and [2, Theorem 1.1]).

We conclude the paper by noting that [2, Theorem 1.1] holds in the present setting without any changes and with the same proof. Let $\{H^p\}_{p\geqslant 1}$ be any of the sequences of Bergman spaces defined in (1.1), (1.2), or (1.3). Given a section $S \in H^p$, we associate to S the measure $[\operatorname{div}(S)]$ defined in (3.5). Geometrically, $[\operatorname{div}(S)]$ is the sum of Dirac masses with multiplicities at the zeros of S in $X \setminus \Sigma$ plus a sum of Dirac masses at the singular points $x_j \in \Sigma$ with coefficients given in terms of the order of $\sigma^\star S$ at the points of $\sigma^{-1}(x_j)$ (see Proposition 3.3 and (3.6)).

We set $n_p = \dim H^p$ and let $S_1^p, \dots, S_{n_p}^p$ be an orthonormal basis of H^p. Using this basis we identify H^p to $\mathbb{C}^{n_p}$ and endow it with a probability measure μ_p which satisfies the moment condition (B) from [2]. Then the conclusions of [2, Theorem 1.1] hold with $A_p = p$ in the setting of Theorem 1.1 for the probability spaces (H^p, μ_p). Let us state here more precisely one particular case of this theorem.

We let μ_p be the normalized area measure on the unit sphere of $H^p \equiv \mathbb{C}^{n_p}$ (see [2, (4.13)]) and consider the product probability space $(\mathcal{H}, \mu) = (\prod_{p=1}^{\infty} H^p, \prod_{p=1}^{\infty} \mu_p)$. The expectation measure $\mathbb{E}[\operatorname{div}(s_p)]$ of the measure-valued random variable $H^p \ni s_p \mapsto [\operatorname{div}(s_p)]$ is defined by

$$\big\langle \mathbb{E}[\operatorname{div}(s_p)], \chi \big\rangle = \int_{H^p} \left(\int_X \chi \, \mathrm{d}[\operatorname{div}(s_p)] \right) \mathrm{d}\mu_p(s_p),$$

where χ is a continuous function on X. We have:

Theorem 1.2. — *Let X, ω, L, h verify assumptions* (A) *and* (B). *Then the following hold:*

(i) *The measure $\mathbb{E}[\operatorname{div}(s_p)]$ is well defined and $\frac{1}{p}\mathbb{E}[\operatorname{div}(s_p)] \to c_1(L,h)$, as $p \to +\infty$, in the weak sense of measures on X.*
(ii) *For μ-a.e. sequence $\{s_p\} \in \mathcal{H}$ we have $\frac{1}{p}\log|s_p|_{h_p} \to 0$ in $L^1(X,\omega)$ and $\frac{1}{p}[\operatorname{div}(s_p)] \to c_1(L,h)$, $\frac{1}{p}[\operatorname{div}(s_p)]^+ \to c_1(L,h)$, in the weak sense of measures on X, as $p \to +\infty$, where $[\operatorname{div}(s_p)]^+$ is the positive variation of $[\operatorname{div}(s_p)]$.*

The paper is organized as follows. In Section 2 we recall the notion of weakly subharmonic function on a complex curve and the definition of its Laplacian. In Section 3 we consider holomorphic line bundles endowed with singular metrics on an algebraic curve. We discuss the measures associated to divisors of holomorphic or weakly holomorphic sections and we define the Bergman kernel functions and Fubini–Study measures of the corresponding Bergman spaces. Theorems 1.1 and 1.2 are proved in Section 4. In Section 5 we give examples of algebraic curves in $\mathbb{P}^2$ for which we describe explicitly the Bergman spaces of sections considered in the paper. We also give a precise lower estimate of the Bergman kernel $P_{w,p}$ in the case of a smooth Hermitian metric on L.

2. Preliminaries

In this section we review the notions of (weakly) holomorphic and (weakly) subharmonic function on a complex curve.

Throughout the paper we denote by $\mathbb{D}_r \subset \mathbb{C}$ the open disc of radius $r > 0$ centered at 0, by $\mathbb{D} := \mathbb{D}_1$ the unit disc, and by λ the Lesbesgue measure on $\mathbb{C}$. We denote by $\operatorname{ord}(g, \zeta)$ the order of a meromorphic function g at $\zeta \in \mathbb{C}$.

Let X be a complex curve, i.e. a reduced complex space of dimension one, and let ω be a Hermitian form on X (see e.g. [4, Section 2.1] for the definition). Let Σ be the set of singular points of X.

Working locally near a singular point $x_j \in \Sigma$ and using a local embedding of X into $\mathbb{C}^N$ for some $N \geqslant 2$, we may assume that X is a complex curve in a polydisc $D_{x_j} \subset \mathbb{C}^N$ centered at x_j, and is the union of finitely many irreducible complex curves which intersect only at x_j. Moreover each such irreducible component Y of X at x_j has a local normalization (see [20, Theorem 5.7], [3, Section 6.1]):

$$\begin{aligned}&f = (f_1, \dots, f_N) : \mathbb{D} \to D_{x_j} \text{ holomorphic with } f(\mathbb{D}) = Y,\ f(0) = x_j,\\ &\text{and } f : \mathbb{D} \setminus \{0\} \to Y \setminus \{x_j\} \text{ is biholomorphic.}\end{aligned} \tag{2.1}$$

We denote by

$$\alpha = \alpha(x_j, Y) = \min\{\operatorname{ord}(f'_\ell, 0) : 1 \leqslant \ell \leqslant N\} \tag{2.2}$$

the ramification index of f at 0 (see e.g. [21, p. 264]).

By shrinking the polydisc D_{x_j}, we may assume that ω is the restriction to X of a Hermitian form on D_{x_j}. Hence

$$C_1^{-1}\beta_N|_X \leqslant \omega \leqslant C_1\beta_N|_X \tag{2.3}$$

for some constant $C_1 > 1$, where $\beta_N = \frac{i}{2}\sum_{\ell=1}^N \mathrm{d}z_\ell \wedge \mathrm{d}\bar{z}_\ell$ is the standard Kähler form on $\mathbb{C}^N$. By (2.3) we have $C_1^{-1} f^\star\beta_N \leqslant f^\star\omega \leqslant C_1 f^\star\beta_N$. Hence we may assume that

$$C_2^{-1}|\zeta|^{2\alpha}\frac{i}{2}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \leqslant f^\star\omega \leqslant C_2|\zeta|^{2\alpha}\frac{i}{2}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \leqslant C_2\frac{i}{2}\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \tag{2.4}$$

holds on $\mathbb{D}$, with some constant $C_2 > 1$.

If $U \subset X$ is an open set, a weakly holomorphic function on U is a holomorphic function on $U \setminus \Sigma$ which is locally bounded on U. A weakly holomorphic function on U that extends continuously at the points of $\Sigma \cap U$ is called a continuous weakly holomorphic function on U. Note that such a function is not necessarily holomorphic on U (see e.g. [22, p. 91]). Let

$$\mathscr{O}_X(U) \subset \mathscr{O}_{X,c}(U) \subset \mathscr{O}_{X,w}(U)$$

denote the set of holomorphic, continuous weakly holomorphic, respectively weakly holomorphic functions on U. We remark that if X is locally irreducible at any point then the sheaves $\mathscr{O}_{X,c}$ and $\mathscr{O}_{X,w}$ coincide.

A subharmonic function on U is a function which (using local embeddings $X \hookrightarrow \mathbb{C}^N$) is locally the restriction to X of a plurisubharmonic function in the ambient space $\mathbb{C}^N$, and which is not identically $-\infty$ on any irreducible component of U (see e.g. [4, Section 2.1]). A weakly subharmonic function on U is a subharmonic function on $U \setminus \Sigma$ which is locally upper bounded on U. Let $\mathrm{SH}(U) \subset \mathrm{WSH}(U)$ be the set of subharmonic, respectively weakly subharmonic functions on U.

Lemma 2.1. — *If $U \subset X$ is open then* $\mathrm{WSH}(U) \subset L^1_{\mathrm{loc}}(U, \omega)$.

Proof. — Let $u \in \mathrm{WSH}(U)$. Since u is subharmonic on $U \setminus \Sigma$ we have that $u \in L^1_{\mathrm{loc}}(U \setminus \Sigma, \omega)$. So we only need to show that u is integrable on each irreducible component Y of X at a point $x_j \in \Sigma \cap U$. Let $f : \mathbb{D} \to Y$ be a local normalization of Y and set $Y_r = f^{-1}(\mathbb{D}_r)$ for $r < 1$. Then $u \circ f$ is subharmonic on $\mathbb{D} \setminus \{0\}$ and upper bounded near 0, so it extends to a subharmonic function on $\mathbb{D}$. Hence by (2.4),

$$\int_{Y_r \setminus \{x_j\}} |u|\, \omega = \int_{\mathbb{D}_r \setminus \{0\}} |u \circ f| f^\star \omega \leqslant C_2 \int_{\mathbb{D}_r} |u \circ f| \mathrm{d}\lambda < +\infty.$$

This yields the conclusion. □

We refer to [10] (see also [4, Section 2.1]) for the definition of smooth forms on complex spaces. In our context let $\mathscr{C}^\infty_{X,0}(U)$ denote the set of smooth functions on X with compact support in U. Let $\mathrm{d} = \partial + \overline{\partial}$, $\mathrm{d}^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$, so $\mathrm{dd}^c = \frac{i}{\pi}\partial\overline{\partial}$. If $u \in \mathrm{WSH}(U)$ then $\mathrm{dd}^c u$ is a positive measure on $U \setminus \Sigma$ given in a local coordinate by $\frac{1}{2\pi}\Delta u$. Since $u \in L^1_{\mathrm{loc}}(U, \omega)$ we have that $\mathrm{dd}^c u$ is a distribution on U defined by

$$\langle \mathrm{dd}^c u, \chi \rangle = \int_{U \setminus \Sigma} u \, \mathrm{dd}^c \chi,\ \chi \in \mathscr{C}^\infty_{X,0}(U).$$

Lemma 2.2. — *If $u \in \mathrm{WSH}(U)$ then $\mathrm{dd}^c u$ is a positive measure on U. Moreover, assume that $x_j \in \Sigma \cap U$, $D_{x_j} \subset \mathbb{C}^N$ is a polydisc as in* (2.1), *and Y_ℓ are the irreducible components of X at x_j with local normalizations*

$f_\ell : \mathbb{D} \to Y_\ell$, $1 \leqslant \ell \leqslant k$. Then each function $v_\ell := u \circ f_\ell$ extends to a subharmonic function on $\mathbb{D}$ and $\mathrm{dd}^c u = \sum_{\ell=1}^k (f_\ell)_\star(\mathrm{dd}^c v_\ell)$.

Proof. — Since v_ℓ is subharmonic on $\mathbb{D}\setminus\{0\}$ and is upper bounded near 0, it extends to a subharmonic function on $\mathbb{D}$. If χ is a test function supported in D_{x_j} we have

$$\begin{aligned}\langle \mathrm{dd}^c u, \chi\rangle &= \sum_{\ell=1}^k \int_{Y_\ell\setminus\{x_j\}} u\, \mathrm{dd}^c\chi = \sum_{\ell=1}^k \int_{\mathbb{D}_\ell\setminus\{0\}} (u\circ f_\ell)\mathrm{dd}^c(\chi\circ f_\ell)\\ &= \sum_{\ell=1}^k \langle \mathrm{dd}^c v_\ell, \chi\circ f_\ell\rangle = \sum_{\ell=1}^k \langle (f_\ell)_\star(\mathrm{dd}^c v_\ell), \chi\rangle.\end{aligned}$$

This yields the conclusion. □

We conclude this section with the following lemma (see also [10, Theorem 1.7]):

Lemma 2.3. — *Let $u : U \to [-\infty, +\infty)$ be a function. Then $u \in \mathrm{SH}(U)$ if and only if $u \in \mathrm{WSH}(U)$ and, for every $x_j \in \Sigma \cap U$ and every irreducible component Y of X at x_j, we have that $u(x_j) = \limsup_{Y\ni x\to x_j} u(x)$.*

Proof. — One implication is obvious, so we assume that $u \in \mathrm{WSH}(U)$. Using [15, Theorem 5.3.1] we have to show that $u \circ g$ is subharmonic on $\mathbb{D}$, for any non-constant holomorphic function $g : \mathbb{D} \to U$. It suffices to assume that $g(0) = x_j \in \Sigma\cap U$ and to prove that $u\circ g$ is subharmonic on $\mathbb{D}_\varepsilon$ for some $\varepsilon > 0$. For $\varepsilon > 0$ sufficiently small we have that $Y = g(\mathbb{D}_\varepsilon)$ is an irreducible component of X at x_j such that $Y\setminus\{x_j\}$ is smooth and $g(\mathbb{D}_\varepsilon\setminus\{0\}) = Y\setminus\{x_j\}$. The function $v = u \circ g$ is subharmonic on $\mathbb{D}_\varepsilon \setminus \{0\}$ and

$$\limsup_{\zeta\to 0} v(\zeta) = \limsup_{Y\ni x\to x_j} u(x) = u(x_j) = v(0).$$

Hence v is subharmonic on $\mathbb{D}_\varepsilon$. □

We note that if X is locally irreducible at any point then the notions of subharmonic and weakly subharmonic function are the same.

3. Bergman kernels and Fubini–Study measures

We assume in this section that X, Σ, ω, $\sigma : \widetilde{X} \to \mathbb{P}^N$, $\widetilde{\omega}$, verify (A) and (C). We introduce and study the Bergman kernels and Fubini–Study measures for the various spaces of L^2-integrable holomorphic sections considered in this paper.

By the properties of the normalization (see e.g. [20]) we have that $\sigma(\widetilde{X}) = X$,

$$\sigma : \widetilde{X} \setminus \sigma^{-1}(\Sigma) \to X \setminus \Sigma$$

is biholomorphic, and the number of points in the preimage $\sigma^{-1}(x_j)$ of $x_j \in \Sigma$ is equal to the number of irreducible components of X at x_j. Moreover, for each such component Y there exists a unique point $y \in \sigma^{-1}(x_j)$ and a coordinate neighborhood of $y \equiv 0$ which contains $\mathbb{D}$, such that $\sigma|_{\mathbb{D}}$ is a local normalization of Y as in (2.1) with the polydisc $D_{x_j} \subset \mathbb{C}^N \hookrightarrow \mathbb{P}^N$. We denote by $\alpha(y) = \alpha(x_j, Y)$ the ramification index of σ at y defined in (2.2). Let

$$R_\sigma = \sum_{y \in \sigma^{-1}(\Sigma)} \alpha(y) y, \ [R_\sigma] = \sum_{y \in \sigma^{-1}(\Sigma)} \alpha(y) \delta_y, \tag{3.1}$$

be the ramification divisor of σ and its associated measure, where δ_y is the Dirac mass at y. Let $\mathscr{O}_{\widetilde{X}}(R_\sigma)$ denote the line bundle defined by R_σ.

We state the following version of Lemma 2.2 in the compact setting:

Lemma 3.1. — *If $u \in \mathrm{WSH}(U)$, where $U \subset X$ is open, then $v = u \circ \sigma$ extends to a subharmonic function on $\sigma^{-1}(U)$ and $\mathrm{dd}^c u = \sigma_\star(\mathrm{dd}^c v)$.*

Proof. — Note that $v \in \mathrm{SH}(\sigma^{-1}(U \setminus \Sigma))$ and v is upper bounded near each point of $\sigma^{-1}(\Sigma \cap U)$, so it extends to a subharmonic function on $\sigma^{-1}(U)$. If $\chi \in \mathscr{C}^\infty_{X,0}(U)$ we have

$$\langle \mathrm{dd}^c u, \chi \rangle = \int_{U \setminus \Sigma} u \, \mathrm{dd}^c \chi = \int_{\sigma^{-1}(U)} v \, \mathrm{dd}^c (\chi \circ \sigma) = \langle \mathrm{dd}^c v, \chi \circ \sigma \rangle. \qquad \square$$

Let $L \to X$ be a holomorphic line bundle and $\{U_\alpha\}$ be an open cover of X such that L has a holomorphic frame e_α on U_α. We define $H^0_w(X, L)$, respectively $H^0_c(X, L)$, by requiring that $S \in H^0_w(X, L)$, respectively $S \in H^0_c(X, L)$, if and only if $S \in H^0(X \setminus \Sigma, L)$ and for any α we have $s_\alpha \in \mathscr{O}_{X,w}(U_\alpha)$, respectively $s_\alpha \in \mathscr{O}_{X,c}(U_\alpha)$, where $S = s_\alpha e_\alpha$ on $U_\alpha \setminus \Sigma$.

The notion of singular Hermitian metric h on L is defined exactly as in the smooth case (see [11], [24, p. 97], [4, Section 2.2])). We have $|e_\alpha|^2_h = e^{-2\varphi_\alpha}$, where $\varphi_\alpha \in L^1_{\mathrm{loc}}(U_\alpha, \omega)$ are called the local weights of h. We assume in the sequel that h has weakly subharmonic weights, i.e. $\varphi_\alpha \in \mathrm{WSH}(U_\alpha)$. If $g_{\alpha\beta} = e_\beta / e_\alpha \in \mathscr{O}^*_X(U_\alpha \cap U_\beta)$ are the transition functions of L then $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$ holds on $(U_\alpha \cap U_\beta) \setminus \Sigma$. Set

$$c_1(L, h)|_{U_\alpha} = \mathrm{dd}^c \varphi_\alpha. \tag{3.2}$$

It follows from Lemma 2.2 that $c_1(L, h)$ is a well defined positive measure on X, called the curvature measure of h.

Let $\sigma^\star L \to \widetilde{X}$ be the pullback of the line bundle L, endowed with the pullback metric $\sigma^\star h$. Since $\sigma^\star h$ has weight $\varphi_\alpha \circ \sigma$ on $\sigma^{-1}(U_\alpha)$, we infer by Lemma 3.1 that $\sigma^\star h$ has subharmonic weights and

$$c_1(L,h) = \sigma_\star\big(c_1(\sigma^\star L, \sigma^\star h)\big).$$

Using the Riemann removable singularity theorem, it follows easily that the map

$$\sigma^\star : H^0_w(X, L^p) \to H^0\big(\widetilde{X}, \sigma^\star L^p\big) \tag{3.3}$$

is well defined and an isomorphism. Hence by the Riemann–Roch theorem [16, 16.9] or by Siegel's lemma (see [24, Lemma 2.2.6]), applied to $\sigma^\star L$, there exists a constant $C > 0$ such that

$$\dim H^0_w(X, L^p) \leqslant Cp \quad \text{for all } p \geqslant 1.$$

We show next that the Bergman spaces $H^0_{(2)}(X \setminus \Sigma, L^p)$ defined in (1.3) are finite dimensional, as they correspond to spaces of meromorphic section of $\sigma^\star L^p$ with poles in R_σ. We need the following simple lemma:

Lemma 3.2. — *Let $g(\zeta) = \sum_{j=-\infty}^{+\infty} a_j \zeta^j$ be a holomorphic function on $\mathbb{D} \setminus \{0\}$ such that $\int_{\mathbb{D}\setminus\{0\}} |g(\zeta)|^2 |\zeta|^{2n} \mathrm{d}\lambda < +\infty$ for some $n \in \mathbb{Z}$. Then $a_j = 0$ for all $j \leqslant -n-1$.*

Proof. — Let $\varepsilon \in (0,1)$. Using polar coordinates we obtain

$$\int_{\{\varepsilon<|\zeta|<1\}} |g(\zeta)|^2 |\zeta|^{2n} \mathrm{d}\lambda = 2\pi \sum_{j=-\infty}^{+\infty} |a_j|^2 \int_\varepsilon^1 r^{2j+2n+1} \mathrm{d}r.$$

The conclusion follows by letting $\varepsilon \to 0$. □

Proposition 3.3. — *The map $\sigma^\star : H^0_{(2)}(X \setminus \Sigma, L^p) \to H^0\big(\widetilde{X}, \sigma^\star L^p \otimes \mathscr{O}_{\widetilde{X}}(R_\sigma)\big)$ is well defined and injective. We have $\dim H^0_{(2)}(X \setminus \Sigma, L^p) \leqslant Cp$ for all $p \geqslant 1$, where $C > 0$ is a constant.*

Proof. — The holomorphic sections of $\sigma^\star L^p \otimes \mathscr{O}_{\widetilde{X}}(R_\sigma)$ can be identified to meromorphic sections of $\sigma^\star L^p$ with poles in R_σ, so we have to show that if $S \in H^0_{(2)}(X \setminus \Sigma, L^p)$ and $y \in \sigma^{-1}(\Sigma)$ then $\sigma^\star S$ has a pole of order at most $\alpha(y)$ at the isolated singularity y.

Let $x_j = \sigma(y)$ and $\mathbb{D}$ be the unit disc in a coordinate neighborhood of $y \equiv 0$ such that $\sigma|_{\mathbb{D}}$ is the local normalization of an irreducible component Y of X at x_j. We may assume that x_j has a neighborhood U_α on which L has a local holomorphic frame e_α such that $U_\alpha \cap \Sigma = \{x_j\}$ and $\mathbb{D} \subset \sigma^{-1}(U_\alpha)$. Then $S = s_\alpha e_\alpha^{\otimes p}$ for some $s_\alpha \in \mathscr{O}_X(U_\alpha \setminus \{x_j\})$, and $|e_\alpha|_h = e^{-\varphi_\alpha}$, where

$\varphi_\alpha \in \mathrm{WSH}(U_\alpha)$. We may assume that $\varphi_\alpha \leqslant M$ on U_α, for some constant M. Using (2.4) we obtain

$$\begin{aligned}\|S\|_p^2 &\geqslant \int_{Y\setminus\{x_j\}} |s_\alpha|^2 e^{-2p\varphi_\alpha}\omega \geqslant e^{-2pM}\int_{\mathbb{D}\setminus\{0\}} |s_\alpha\circ\sigma|^2\sigma^\star\omega \\ &\geqslant C_2^{-1}e^{-2pM}\int_{\mathbb{D}\setminus\{0\}} |s_\alpha(\sigma(\zeta))|^2|\zeta|^{2\alpha(y)}\mathrm{d}\lambda.\end{aligned}$$

By Lemma 3.2 we infer that the function $s_\alpha\circ\sigma$ has a pole of order at most $\alpha(y)$ at y.

The map $\sigma^\star$ is clearly injective, since $\sigma : \widetilde{X}\setminus\sigma^{-1}(\Sigma)\to X\setminus\Sigma$ is biholomorphic. The last assertion follows from the Riemann–Roch theorem [16, 16.9] or Siegel's lemma (see [24, Lemma 2.2.1] and its proof). □

Let $S\in H^0_{(2)}(X\setminus\Sigma, L)$, $S\neq 0$. It follows from Proposition 3.3 that $\sigma^\star S$ is a meromorphic section of $\sigma^\star L$, so it induces the divisor $\operatorname{div}(\sigma^\star S)$ and its associated signed measure $[\operatorname{div}(\sigma^\star S)]$ on $\widetilde{X}$, where

$$\operatorname{div}(\sigma^\star S) = \sum_{y\in\widetilde{X}} \operatorname{ord}(\sigma^\star S, y)y,\ [\operatorname{div}(\sigma^\star S)] = \sum_{y\in\widetilde{X}} \operatorname{ord}(\sigma^\star S, y)\delta_y.$$

Moreover, the divisor $\operatorname{div}(\sigma^\star S)+R_\sigma$ is effective. We define

$$\operatorname{ord}(S, x_j) = \sum_{y\in\sigma^{-1}(x_j)} \operatorname{ord}(\sigma^\star S, y),\ x_j\in\Sigma. \tag{3.4}$$

Writing $S = s_\alpha e_\alpha$, where $s_\alpha\in\mathscr{O}_X(U_\alpha\setminus\Sigma)$, we have that $s_\alpha\circ\sigma$ is meromorphic on $\sigma^{-1}(U_\alpha)$, hence $\log|s_\alpha\circ\sigma|$ is locally the difference of two subharmonic functions. We infer that $\log|s_\alpha|$ is locally the difference of two weakly subharmonic functions on U_α. By Lemma 2.1, this implies that $\log|s_\alpha|\in L^1_{\mathrm{loc}}(U_\alpha,\omega)$. Since $\log|s_\alpha| = \log|s_\beta|+\log|g_{\alpha\beta}|$ on $(U_\alpha\cap U_\beta)\setminus\Sigma$, we can define, using Lemma 2.2, the signed measure $[\operatorname{div}(S)]$ on X by setting

$$[\operatorname{div}(S)]|_{U_\alpha} = \mathrm{dd}^c\log|s_\alpha|. \tag{3.5}$$

Note that $[\operatorname{div}(\sigma^\star S)]|_{\sigma^{-1}(U_\alpha)} = \mathrm{dd}^c\log|s_\alpha\circ\sigma|$. Since $\sigma^\star S$ is a meromorphic section of $\sigma^\star L$, we see that S has finitely many zeros $z_1,\dots,z_k\in X\setminus\Sigma$. By Lemma 3.1 we obtain

$$[\operatorname{div}(S)] = \sigma_\star\big([\operatorname{div}(\sigma^\star S)]\big) = \sum_{j=1}^k \operatorname{ord}(S, z_j)\delta_{z_j} + \sum_{j=1}^m \operatorname{ord}(S, x_j)\delta_{x_j}, \tag{3.6}$$

where $\operatorname{ord}(S, x_j)$ is defined in (3.4). Let $[\operatorname{div}(S)]^\pm$ denote the positive and negative variations of the measure $[\operatorname{div}(S)]$, so

$$[\operatorname{div}(S)] = [\operatorname{div}(S)]^+ - [\operatorname{div}(S)]^-.$$

Then

$$[\operatorname{div}(S)]^+ = \sum_{j=1}^{k} \operatorname{ord}(S, z_j)\delta_{z_j} + \sum_{j=1}^{m} \operatorname{ord}(S, x_j)^+\delta_{x_j},$$
$$[\operatorname{div}(S)]^- = \sum_{j=1}^{m} \operatorname{ord}(S, x_j)^-\delta_{x_j},$$

where $\operatorname{ord}(S, x_j)^+ = \max\{\operatorname{ord}(S, x_j), 0\}$, $\operatorname{ord}(S, x_j)^- = \max\{-\operatorname{ord}(S, x_j), 0\}$.

Since $\operatorname{div}(\sigma^\star S) + R_\sigma$ is effective, we infer that

$$[\operatorname{div}(S)]^- \leqslant \sum_{j=1}^{m} \left(\sum_{y\in\sigma^{-1}(x_j)} \alpha(y) \right) \delta_{x_j} = \sigma_\star([R_\sigma]). \tag{3.7}$$

Note that the right hand side of (3.7) is independent of L.

Since $\log|S|_h = \log|s_\alpha| - \varphi_\alpha$ on U_α, the function $\log|S|_h \in L^1(X, \omega)$. Hence (3.2) and (3.5) yield the following version of the Lelong–Poincaré formula in this setting:

$$[\operatorname{div}(S)] = c_1(L, h) + \mathrm{dd}^c \log|S|_h. \tag{3.8}$$

The preceding discussion carries over for sections $S \in H^0_w(X, L)$, $S \neq 0$. As $\log|s_\alpha| \in \mathrm{WSH}(U_\alpha)$, the measure $[\operatorname{div}(S)]$ defined in (3.5) is now positive. Moreover, we have that $\sigma^\star S \in H^0(\widetilde{X}, \sigma^\star L)$, so the divisor $\operatorname{div}(\sigma^\star S)$ is effective and the measure $[\operatorname{div}(\sigma^\star S)]$ is positive. Formula (3.6) and the Lelong–Poincaré formula (3.8) hold for $[\operatorname{div}(S)]$.

We give now the definitions of the Bergman kernel functions $P_{w,p}, P_{c,p}, P_p$ and Fubini–Study measures $\gamma_{w,p}, \gamma_{c,p}, \gamma_p$ of the Bergman spaces considered in this paper. We start with the spaces $H^0_{w,(2)}(X, L^p)$ and $H^0_{c,(2)}(X, L^p)$ defined in (1.2), respectively (1.1).

Let $d_{w,p} = \dim H^0_{w,(2)}(X, L^p)$ and let S^p_j, $1 \leqslant j \leqslant d_{w,p}$, be an orthonormal basis of $H^0_{w,(2)}(X, L^p)$. We write $S^p_j = s^p_{j,\alpha} e_\alpha^{\otimes p}$, where $s^p_{j,\alpha} \in \mathscr{O}_{X,w}(U_\alpha)$. Then

$$P_{w,p}(x) = \sum_{j=1}^{d_{w,p}} |S^p_j(x)|^2_{h_p}\ , \quad \gamma_{w,p}|_{U_\alpha} = \mathrm{dd}^c u_{p,\alpha},$$
$$\text{where } u_{p,\alpha} = \frac{1}{2}\log\left(\sum_{j=1}^{d_{w,p}} |s^p_{j,\alpha}|^2\right). \tag{3.9}$$

Note that $P_{w,p}(x)$ is defined for all $x \in X \setminus \Sigma$ such that if $x \in U_\alpha$ then $\varphi_\alpha(x) > -\infty$. Moreover, $u_{p,\alpha} \in \mathrm{WSH}(U_\alpha)$ and $u_{p,\alpha} = u_{p,\beta} + \log|g_{\alpha\beta}|$ on

$(U_\alpha \cap U_\beta) \setminus \Sigma$, so by Lemma 2.2 $\gamma_{w,p}$ is a well defined positive measure on X. We have that $P_{w,p}$, $\gamma_{w,p}$ are independent of the choice of basis and

$$\log P_{w,p}|_{U_\alpha} = 2u_{p,\alpha} - 2p\varphi_\alpha. \tag{3.10}$$

By Lemma 2.1 we infer that $\log P_{w,p} \in L^1(X,\omega)$, and

$$\frac{1}{p}\gamma_{w,p} - c_1(L,h) = \frac{1}{2p}\mathrm{dd}^c \log P_{w,p}. \tag{3.11}$$

Moreover, as in [4, 5, 6], one has the following variational formula,

$$P_{w,p}(x) = \max\big\{|S(x)|^2_{h_p} : S \in H^0_{w,(2)}(X,L^p),\ \|S\|_p = 1\big\}, \tag{3.12}$$

for all $x \in X \setminus \Sigma$ where $P_{w,p}(x)$ is defined.

Let $d_{c,p} = \dim H^0_{c,(2)}(X,L^p)$. One defines the Bergman kernel function $P_{c,p}$ and Fubini–Study measure $\gamma_{c,p}$ of the space $H^0_{c,(2)}(X,L^p)$ in the same way as in (3.9). The formulas (3.10), (3.11), (3.12) hold in this case as well.

We next turn our attention to the Bergman kernel function P_p and Fubini–Study measure γ_p of the space $H^0_{(2)}(X \setminus \Sigma, L^p)$ defined in (1.3). Proceeding as above, let $d_p = \dim H^0_{(2)}(X \setminus \Sigma, L^p)$ and S^p_j, $1 \leqslant j \leqslant d_p$, be an orthonormal basis of $H^0_{(2)}(X \setminus \Sigma, L^p)$. We set $S^p_j = s^p_{j,\alpha} e^{\otimes p}_\alpha$, where $s^p_{j,\alpha} \in \mathscr{O}_X(U_\alpha \setminus \Sigma)$, and define for $x \in X \setminus \Sigma$,

$$P_p(x) = \sum_{j=1}^{d_p} |S^p_j(x)|^2_{h_p}\ , \quad \text{and } \gamma_p|_{U_\alpha} = \mathrm{dd}^c u_{p,\alpha},$$

$$\text{where } u_{p,\alpha} = \frac{1}{2}\log\left(\sum_{j=1}^{d_p} |s^p_{j,\alpha}|^2\right). \tag{3.13}$$

Proposition 3.3 shows that $u_{p,\alpha} \circ \sigma$ is locally the difference of two subharmonic functions on $\sigma^{-1}(U_\alpha)$. Hence $u_{p,\alpha}$ is locally the difference of two weakly subharmonic functions on U_α. By Lemma 2.1 and Lemma 2.2 we infer that $u_{p,\alpha} \in L^1_{\mathrm{loc}}(U_\alpha,\omega)$ and γ_p is a well defined signed measure on X. The analogue of (3.10) in this setting shows that

$$\log P_p \in L^1(X,\omega),\ \frac{1}{p}\gamma_p - c_1(L,h) = \frac{1}{2p}\mathrm{dd}^c \log P_p. \tag{3.14}$$

Moreover, the variational formula (3.12) also holds for P_p.

Lemma 3.4. — *Let γ_p^- be the negative variation of the measure γ_p. Then $\gamma_p^- \leqslant \sigma_\star([R_\sigma])$. In particular, γ_p^- is supported in Σ.*

Proof. — Since $u_{p,\alpha} \circ \sigma$ is locally the difference of two subharmonic functions on $\sigma^{-1}(U_\alpha)$, $\widetilde{\gamma}_p|_{\sigma^{-1}(U_\alpha)} := \mathrm{dd}^c(u_{p,\alpha} \circ \sigma)$ defines a signed measure $\widetilde{\gamma}_p$ on $\widetilde{X}$. By Lemma 3.1 we infer that $\gamma_p = \sigma_\star(\widetilde{\gamma}_p)$. Note that $\widetilde{X} \setminus \sigma^{-1}(\Sigma)$ is a positive set for $\widetilde{\gamma}_p$. Working locally near a point $y \equiv 0 \in \sigma^{-1}(\Sigma \cap U_\alpha)$ we have by Proposition 3.3 that $u_{p,\alpha}(\sigma(\zeta)) = v(\zeta) + n \log|\zeta|$, where v is a smooth subharmonic function and $n \geqslant -\alpha(y)$. Hence $\widetilde{\gamma}_p(\{y\}) = n \geqslant -\alpha(y)$. Therefore the measure $\widetilde{\gamma}_p + [R_\sigma]$ is positive, hence so is $\gamma_p + \sigma_\star([R_\sigma])$. The conclusion now follows. □

We conclude this section by noting that the spaces

$$H^0_{c,(2)}(X, L^p) \subset H^0_{w,(2)}(X, L^p) \subset H^0_{(2)}(X \setminus \Sigma, L^p), \tag{3.15}$$

are endowed with the same inner product, hence

$$P_{c,p}(x) \leqslant P_{w,p}(x) \leqslant P_p(x), \tag{3.16}$$

for every $x \in X \setminus \Sigma$ such that if $x \in U_\alpha$ then $\varphi_\alpha(x) > -\infty$.

4. Proof of Theorems 1.1 and 1.2

We start with some auxiliary results that are needed for the proof of Theorem 1.1.

Lemma 4.1. — *If v is subharmonic on $\mathbb{D}$, $A = \{\zeta \in \mathbb{C} : \frac{1}{8} \leqslant |\zeta| \leqslant \frac{5}{8}\}$ and $|x| \leqslant \frac{1}{8}$, then*

$$v(x) \leqslant 2 \int_A |v| \mathrm{d}\lambda.$$

Proof. — Let $A_x = \{\zeta \in \mathbb{C} : \frac{1}{4} \leqslant |\zeta - x| \leqslant \frac{1}{2}\}$. Since $A_x \subset A$ we obtain using the subaverage inequality that

$$\frac{3}{32} v(x) \leqslant \frac{1}{2\pi} \int_{\frac{1}{4}}^{\frac{1}{2}} \int_0^{2\pi} v(x + re^{it}) r \mathrm{d}t \mathrm{d}r \leqslant \frac{1}{2\pi} \int_{A_x} |v| \mathrm{d}\lambda \leqslant \frac{1}{2\pi} \int_A |v| \mathrm{d}\lambda. \quad \square$$

We state next a general result about the asymptotics of the logarithms of Bergman kernels. Let Y be a complex manifold of dimension n, ω be a Hermitian form on Y and (L, h) be a singular Hermitian holomorphic line bundle on Y. One defines the Bergman spaces $H^0_{(2)}(Y, L^p)$ in analogy to (1.3) using the metric h_p induced by h on L^p and the volume form $\frac{\omega^n}{n!}$ on Y. Let $V_p \leqslant H^0_{(2)}(Y, L^p)$ be a subspace of dimension $m_p > 0$ and define the Bergman kernel function Q_p of V_p as in (3.13). Note that $\log Q_p$ is locally the difference of integrable functions, so $\log Q_p \in L^1_{\mathrm{loc}}(Y, \omega^n)$.

Lemma 4.2. — *In the above setting, assume that:*

(i) *$m_p \leqslant cp^n$ holds for all $p \geqslant 1$, with some constant $c > 0$;*
(ii) *every $x \in Y$ has a neighborhood U_x such that $Q_p \geqslant \varepsilon_x$ holds ω^n-a.e. on U_x for all p sufficiently large, with some constant $\varepsilon_x > 0$.*

Then $\frac{1}{p}\log Q_p \to 0$ *as* $p \to +\infty$, *in* $L^1_{\mathrm{loc}}(Y, \omega^n)$.

Proof. — We may assume that $C_x := \int_{U_x} \omega^n < +\infty$. It suffices to show that $\frac{1}{p}\log Q_p \to 0$ in $L^1(U_x, \omega^n)$ for each $x \in Y$. Note that $\int_Y Q_p \frac{\omega^n}{n!} = m_p$. Using Jensen's inequality and hypothesis (i) we get

$$\int_{U_x} (\log Q_p)\omega^n \leqslant C_x \log\Big(\frac{1}{C_x}\int_{U_x} Q_p\,\omega^n\Big) \leqslant nC_x \log p + C'_x \tag{4.1}$$

for all $p \geqslant 1$, with some constant $C'_x > 0$. If $f_p := \frac{1}{p}(\log Q_p - \log\varepsilon_x)$ then by (ii), $f_p \geqslant 0$ ω^n-a.e. on U_x for all p sufficiently large. Hence $\int_{U_x} f_p\,\omega^n \to 0$ by (4.1). This implies the conclusion. $\square$

We now introduce the geometric setting and notation needed to describe the local structure of X near the singular points. Let $\sigma : \widetilde{X} \to \mathbb{P}^N$ be the normalization of X from (C). For each $x_j \in \Sigma$ we can find a neighborhood $V_j \subset X$ of x_j with the following properties:

- (P1) $\overline{V}_j \cap \overline{V}_k = \emptyset$ for $1 \leqslant j < k \leqslant m$.
- (P2) there exist coordinates $(z_1, \dots, z_N)$ on $\mathbb{C}^N \hookrightarrow \mathbb{P}^N$ and a polydisc $D_j \subset \mathbb{C}^N$ centered at 0 such that $x_j \equiv 0$, $V_j = X \cap D_j$, and $V_j \cap \{z_1 = 0\} = \{0\}$.
- (P3) $Y_1, \dots, Y_{k_j}$ denote the irreducible components of V_j.
- (P4) for $1 \leqslant \ell \leqslant k_j$, there exists $y_\ell \in \sigma^{-1}(x_j)$ and a coordinate neighborhood of $y_\ell \equiv 0$ containing $\mathbb{D}$, such that $\sigma_\ell := \sigma|_{\mathbb{D}} : \mathbb{D} \to D_j$ is a local normalization of Y_ℓ as in (2.1).

Note that by (P1), any two irreducible components of V_j intersect only at x_j. In the proof of Theorem 1.1 we will have to work with a different Hermitian form on $X \setminus \Sigma$, which is provided by the following lemma.

Lemma 4.3. — *Let* $\Omega := \sigma_\star\big(\widetilde{\omega}|_{\widetilde{X}\setminus\sigma^{-1}(\Sigma)}\big)$. *Then* Ω *is a Hermitian form which verifies* $\mathrm{Ric}_\Omega \geqslant -2\pi B\Omega$ *and* $\Omega \geqslant a\omega$ *on* $X \setminus \Sigma$, *with some constants* $a, B > 0$.

Proof. — As $\widetilde{X}$ is compact, there exists a constant $B > 0$ such that $\mathrm{Ric}_{\widetilde{\omega}} \geqslant -2\pi B\widetilde{\omega}$ on $\widetilde{X}$. Since $\sigma : \widetilde{X} \setminus \sigma^{-1}(\Sigma) \to X \setminus \Sigma$ is biholomorphic, Ω is a Hermitian form with $\mathrm{Ric}_\Omega \geqslant -2\pi B\Omega$ on $X \setminus \Sigma$. For $x_j \in \Sigma$, let Y_ℓ be a component of X at x_j as in (P3), and σ_ℓ be as in (P4). We infer from (2.4) that $\sigma_\ell^\star\omega \leqslant C\widetilde{\omega}$ holds on $\mathbb{D}$, with some constant $C > 0$. Hence $\omega \leqslant C\Omega$ on $Y_\ell \setminus \{x_j\}$. This yields the conclusion. $\square$

The function ϕ constructed in the next lemma will be used to obtain a suitable modified metric on L.

Lemma 4.4. — *Let* $X, \Sigma, \omega, \sigma$ *verify* (A), (C) *and* Ω *be as in Lemma 4.3. If* $U \supset \Sigma$ *is an open set, then there exists a continuous function* $\phi : X \to \mathbb{R}$

supported in U such that $\phi \leqslant 0$ on X, ϕ is smooth on $X \setminus \Sigma$, and $A\omega + \mathrm{dd}^c\phi \geqslant b\Omega$ on $X \setminus \Sigma$ for some constants $A, b > 0$.

Proof. — For $1 \leqslant j \leqslant m$, we choose a neighborhood $V_j \subset U$ of x_j such that properties (P1)–(P4) are satisfied. Fix now $x_j \in \Sigma$. Since $V_j \cap \{z_1 = 0\} = \{0\}$, by shrinking D_j we may assume that for each Y_ℓ we have

$$\sigma_\ell(\zeta) = (\zeta^{s_\ell}, f_{\ell,2}(\zeta), \dots, f_{\ell,N}(\zeta)),\ \zeta \in \mathbb{D},$$

where $s_\ell - 1 \geqslant \alpha(y_\ell)$ (see (2.2), (3.1)) and $f_{\ell,l}$ are holomorphic in $\mathbb{D}$. We consider the plurisubharmonic function $w_\ell(z_1, \dots, z_N) = \log\left(1 + |z_1|^{2/s_\ell}\right)$ on $\mathbb{C}^N$, and define

$$\rho_j = \sum_{\ell=1}^{k_j} w_\ell|_{V_j} - M_j,$$

where M_j is chosen so that $\rho_j < 0$ on V_j. Then ρ_j is continuous and subharmonic on V_j, and it is smooth on $V_j \setminus \{0\}$ since $z_1 \neq 0$ if $(z_1, \dots, z_N) \in V_j \setminus \{0\}$. Moreover,

$$\mathrm{dd}^c(\rho_j \circ \sigma_\ell) \geqslant \mathrm{dd}^c(w_\ell \circ \sigma_\ell) = \mathrm{dd}^c \log(1 + |\zeta|^2) \geqslant \frac{i}{4\pi} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \tag{4.2}$$

holds on $\mathbb{D}$, for each $\ell = 1, \dots, k_j$.

Let $\chi_j \in \mathscr{C}^\infty_{X,0}(V_j)$ be such that $\chi_j \geqslant 0$ on X and $\chi_j = 1$ on an open set W_j containing x_j. We define

$$\phi = \sum_{j=1}^{m} \chi_j \rho_j.$$

Then $\phi \leqslant 0$ is continuous on X, $\operatorname{supp} \phi \subset U$, and ϕ is smooth on $X \setminus \Sigma$. Since $\phi = \rho_j$ on W_j, ϕ is subharmonic in the neighborhood $W := \bigcup_{j=1}^m W_j$ of Σ. It follows that there exists a constant $M > 0$ such that $\mathrm{dd}^c\phi \geqslant -M\Omega$ on $X \setminus \Sigma$. Since $\phi = \rho_j$ on W_j we infer by (4.2) that $\mathrm{dd}^c(\phi \circ \sigma) \geqslant b\widetilde{\omega}$ holds on $\sigma^{-1}(W)$ for some constant $b > 0$, hence $\mathrm{dd}^c\phi \geqslant b\Omega$ on $W \setminus \Sigma$. Let $c > 0$ be a constant such that $\omega \geqslant c\Omega$ on $X \setminus W$. For $A > 0$ we obtain

$$\begin{aligned} &A\omega + \mathrm{dd}^c\phi \geqslant (Ac - M)\Omega \text{ on } X \setminus W, \\ &A\omega + \mathrm{dd}^c\phi \geqslant b\Omega \text{ on } W \setminus \Sigma. \end{aligned}$$

The conclusion follows by choosing $A = (M + b)/c$. □

Proof of Theorem 1.1. —

(i). — We show first that

$$\frac{1}{p} \log P_{c,p} \to 0,\ \frac{1}{p} \log P_{w,p} \to 0,\ \frac{1}{p} \log P_p \to 0, \text{ in } L^1_{\mathrm{loc}}(X \setminus \Sigma, \omega). \tag{4.3}$$

By Proposition 3.3, (3.15) and (3.16), this will follow from Lemma 4.2 once we show that hypothesis (ii) of Lemma 4.2 holds for $Q_p = P_{c,p}$. Let $x \in X \setminus \Sigma$ and U_α be a coordinate neighborhood of $x \equiv 0$ such that $\overline{U}_\alpha \subset X \setminus \Sigma$ and L has a holomorphic frame e_α on U_α. Set $|e_\alpha|_h = e^{-\varphi_\alpha}$, so $\varphi_\alpha \in \mathrm{SH}(U_\alpha)$ by (B).

For each $x_j \in \Sigma$ we choose a neighborhood $V_j \subset X \setminus \overline{U}_\alpha$ of x_j such that properties (P1)–(P4) are satisfied. We define the function ρ_j on V_j by setting

$$\rho_j = \big(\alpha(y_\ell) + 1\big) \log|\sigma_\ell^{-1}| \text{ on } Y_\ell \setminus \{x_j\},\ \ell = 1, \dots, k, \text{ and } \rho_j(x_j) = -\infty, \tag{4.4}$$

where $\alpha(y_\ell)$ is the ramification index of σ at y_ℓ. By Lemma 2.3 it follows that $\rho_j \in \mathrm{SH}(V_j)$. Moreover $\rho_j < 0$ and ρ_j is smooth on $V_j \setminus \{x_j\}$. Let $\chi_j \in \mathscr{C}^\infty_{X,0}(V_j)$ be such that $\chi_j \geqslant 0$ on X and $\chi_j = 1$ on an open set W_j containing x_j. We define $\eta = \sum_{j=1}^m \chi_j \rho_j$. Then η is smooth on $X \setminus \Sigma$, $\eta \leqslant 0$ on X, and $\eta = 0$ in a neighborhood of $\overline{U}_\alpha$. Since $\eta = \rho_j$ on W_j, η is subharmonic in a neighborhood of Σ. We infer that

$$dd^c \eta \geqslant -M\Omega \text{ on } X \setminus \Sigma, \tag{4.5}$$

for some constant $M > 0$, where Ω is as in Lemma 4.3.

Fix $r_0 > 0$ such that the disc $V := \mathbb{D}_{2r_0} \Subset U_\alpha$ and we set $U := \mathbb{D}_{r_0}$. We will show that there exist a constant $C > 0$ and $p_0 \in \mathbb{N}$ with the following property: if $p > p_0$, $z \in U$ and $\varphi_\alpha(z) > -\infty$, then there exists a section $S_{z,p} \in H^0_{c,(2)}(X, L^p)$ such that $S_{z,p}(z) \neq 0$ and

$$\|S_{z,p}\|_p^2 \leqslant C |S_{z,p}(z)|^2_{h_p}. \tag{4.6}$$

In view of (3.12) this implies that

$$P_{c,p}(z) \geqslant \frac{|S_{z,p}(z)|^2_{h_p}}{\|S_{z,p}\|_p^2} \geqslant C^{-1},$$

which shows that hypothesis (ii) of Lemma 4.2 folds for $P_{c,p}$.

For the proof of (4.6) we use techniques of Demailly [12, Section 9] (see also [6, Section 5]). By the Ohsawa–Takegoshi extension theorem [26] there exists a constant $C' > 0$ (depending only on x) such that for any $z \in U$ with $\varphi_\alpha(z) > -\infty$ and any p there exists a function $v_{z,p} \in \mathscr{O}_X(V)$ with $v_{z,p}(z) \neq 0$ and

$$\int_V |v_{z,p}|^2 e^{-2p\varphi_\alpha} \Omega \leqslant C' |v_{z,p}(z)|^2 e^{-2p\varphi_\alpha(z)}.$$

We will extend $v_{z,p}$ to a section $S_{z,p} \in H^0_{c,(2)}(X, L^p)$ by solving a $\overline{\partial}$-equation with L^2-estimates. Let $\chi : [0, +\infty) \to [0, 1]$ be a smooth function such that

$\chi = 1$ on $[0, \frac{1}{2}]$ and $\chi = 0$ on $[\frac{3}{4}, +\infty)$, and set

$$\theta_z(t) = \begin{cases} \chi\big(\frac{|t-z|}{r_0}\big) \log \frac{|t-z|}{r_0}, & \text{for } t \in U_\alpha, \\ 0, & \text{for } t \in X \setminus U_\alpha. \end{cases}$$

Let ϕ be the function constructed in Lemma 4.4 for the open set $X \setminus \overline{U}_\alpha \supset \Sigma$, so $\phi \leqslant 0$ on X, $\phi = 0$ in a neighborhood of $\overline{U}_\alpha$, and $dd^c\phi \geqslant b\Omega - A\omega$ on $X \setminus \Sigma$ for some constants $A, b > 0$. We consider the metric

$$\widetilde{h}_p = h_p e^{-2\delta p\phi - 2\eta - 2\theta_z} \text{ on } L^p|_{X \setminus \Sigma}, \text{ where } \delta = \frac{\varepsilon}{A}.$$

Since θ_z is subharmonic in a neighborhood of z, it follows that there exists a constant $M' > 0$ such that $dd^c\theta_z \geqslant -M'\Omega$ on $X \setminus \Sigma$, for all $z \in U$. Using Lemma 4.4, (4.5) and hypothesis (B), we obtain for all p sufficiently large that

$$\begin{aligned} c_1(L^p, \widetilde{h}_p) &= pc_1(L, h) + \delta p \, dd^c\phi + dd^c\eta + dd^c\theta_z \\ &\geqslant p\Big(c_1(L, h) + \frac{\varepsilon b}{A}\Omega - \varepsilon\omega\Big) - (M + M')\Omega \\ &\geqslant \Big(\frac{p\varepsilon b}{A} - M - M'\Big)\Omega \geqslant 2p\varepsilon'\Omega \end{aligned}$$

on $X \setminus \Sigma$, where $\varepsilon' = \frac{\varepsilon b}{4A}$. Note that the Riemann surface $X \setminus \Sigma$ is Stein (see e.g. [16, Corollary 26.8]), hence it carries a complete Kähler metric. By Lemma 4.3 we have $\mathrm{Ric}_\Omega \geqslant -2\pi B\Omega$ on $X \setminus \Sigma$. Let

$$g \in L^2_{0,1}(X \setminus \Sigma, L^p, \mathrm{loc}), \; g = \bar\partial\Big(v_{z,p}\chi\big(\tfrac{|t-z|}{r_0}\big)e_\alpha^{\otimes p}\Big).$$

By [9, Theorem 5.1] (see also [4, Theorem 2.5]) it follows that there exists $p_0 \in \mathbb{N}$ such that if $p > p_0$ there exists $u \in L^2_{0,0}(X \setminus \Sigma, L^p, \mathrm{loc})$ verifying $\bar\partial u = g$ and

$$\int_{X \setminus \Sigma} |u|^2_{h_p} e^{-2\delta p\phi - 2\eta - 2\theta_z}\Omega \leqslant \frac{1}{p\varepsilon'} \int_{X \setminus \Sigma} |g|^2_{h_p} e^{-2\delta p\phi - 2\eta - 2\theta_z}\Omega.$$

Since $\phi = \eta = 0$ on U_α we have that

$$\begin{aligned} &\int_{X \setminus \Sigma} |g|^2_{h_p} e^{-2\delta p\phi - 2\eta - 2\theta_z}\Omega \\ &\qquad = \int_{\{\frac{r_0}{2} < |t-z| < r_0\}} |v_{z,p}|^2 |\bar\partial\chi(\tfrac{|t-z|}{r_0})|^2 e^{-2p\varphi_\alpha - 2\theta_z}\Omega \\ &\qquad \leqslant C'' \int_V |v_{z,p}|^2 e^{-2p\varphi_\alpha}\Omega \leqslant C'C''|v_{z,p}(z)|^2 e^{-2p\varphi_\alpha(z)} < +\infty, \end{aligned}$$

where $C''>0$ is a constant depending only on x. Using $\phi\leqslant 0$, $\eta\leqslant 0$, $\theta_z\leqslant 0$, we get

$$\int_{X\setminus\Sigma}|u|^2_{h_p}\Omega\leqslant\int_{X\setminus\Sigma}|u|^2_{h_p}e^{-2\eta-2\theta_z}\Omega\leqslant\frac{C'C''}{p\varepsilon'}|v_{z,p}(z)|^2e^{-2p\varphi_\alpha(z)}.$$

Note that $u(z)=0$, as $e^{-2\theta_z(t)}=r_0^2|t-z|^{-2}$ is not integrable near z. Fix now $x_j\in\Sigma$. We have $\theta_z=0$ on V_j and $\eta=\rho_j$ on W_j. Let Y_ℓ be a component of V_j as in (P3) and $\sigma_\ell:\mathbb{D}\to Y_\ell$ be the normalization of Y_ℓ as in (P4). We may assume that L has a holomorphic frame e_L on V_j and that the corresponding local weight of h is upper bounded on V_j. Set $u=ve_L^{\otimes p}$. We infer that there exists a constant $c>0$ such that

$$\begin{aligned}\int_{Y_\ell\setminus\{x_j\}}|u|^2_{h_p}e^{-2\eta-2\theta_z}\Omega\geqslant c^p\int_{(Y_\ell\cap W_j)\setminus\{x_j\}}|v|^2e^{-2\rho_j}\Omega\\ \geqslant ac^p\int_{(Y_\ell\cap W_j)\setminus\{x_j\}}|v|^2e^{-2\rho_j}\omega,\end{aligned}$$

where the second inequality follows from Lemma 4.3. Using (4.4) and (2.4) we get

$$\int_{(Y_\ell\cap W_j)\setminus\{x_j\}}|v|^2e^{-2\rho_j}\omega\geqslant C_2^{-1}\int_{\mathbb{D}_r\setminus\{0\}}|v(\sigma_\ell(\zeta))|^2|\zeta|^{-2}d\lambda,$$

for some $r\in(0,1)$. Therefore $\int_{\mathbb{D}_r\setminus\{0\}}|v(\sigma_\ell(\zeta))|^2|\zeta|^{-2}d\lambda<+\infty$. Note that $\bar\partial u=g=0$ on $V_j\setminus\{x_j\}$, hence $v\circ\sigma_\ell$ is holomorphic on $\mathbb{D}\setminus\{0\}$. We infer from Lemma 3.2 that $v\circ\sigma_\ell$ extends holomorphically at 0 and $v\circ\sigma_\ell(0)=0$. This shows that u is a continuous weakly holomorphic section of L^p on V_j, for $j=1,\dots,m$.

Set $S_{z,p}:=v_{z,p}\chi\big(\frac{|t-z|}{r_0}\big)e_\alpha^{\otimes p}-u$. Then $\bar\partial S_{z,p}=0$, $S_{z,p}=-u$ on each V_j, so $S_{z,p}\in H^0_c(X,L^p)$. Moreover, $S_{z,p}(z)=v_{z,p}(z)e_\alpha^{\otimes p}(z)\neq 0$, as $u(z)=0$. Using Lemma 4.3 we obtain

$$\begin{aligned}\|S_{z,p}\|_p^2&\leqslant\frac{1}{a}\int_{X\setminus\Sigma}|S_{z,p}|^2_{h_p}\Omega\leqslant\frac{2}{a}\left(\int_V|v_{z,p}|^2e^{-2p\varphi_\alpha}\Omega+\int_{X\setminus\Sigma}|u|^2_{h_p}\Omega\right)\\&\leqslant\frac{2C'}{a}\left(1+\frac{C''}{p\varepsilon'}\right)|v_{z,p}(z)|^2e^{-2p\varphi_\alpha(z)}\leqslant C|S_{z,p}(z)|^2_{h_p},\end{aligned}$$

with a constant $C>0$ depending only on x. This concludes the proof of (4.6), and hence of (4.3).

We consider next the L^1-convergence near the singular points. Let V_j, $1\leqslant j\leqslant m$, verify properties (P1)–(P4), and fix $x_j\in\Sigma$. Let Y_ℓ be a component of V_j as in (P3) and σ_ℓ be as in (P4). We may assume that L has a holomorphic frame e_L on V_j and that the corresponding local weight φ of h is upper bounded on V_j. The proof of assertion (i) is complete if we show that

$$\frac{1}{p}\log P_{c,p} \to 0,\ \frac{1}{p}\log P_{w,p} \to 0,\ \frac{1}{p}\log P_p \to 0,\ \text{ in } L^1(Y_\ell,\omega). \tag{4.7}$$

Let Q_p denote either one of the Bergman kernels $P_{c,p}, P_{w,p}, P_p$, and let $S_1^p, \dots, S_{n_p}^p$ be an orthonormal basis of the corresponding Bergman space. We write

$$S_j^p = s_j^p e_L^{\otimes p},\ \text{ where } s_j^p \in \mathscr{O}_X(V_j \setminus \{x_j\}),\ u_p := \frac{1}{2}\log\left(\sum_{j=1}^{n_p} |s_j^p|^2\right).$$

Formula (3.10) implies that

$$\frac{1}{p}u_p \circ \sigma_\ell - \varphi \circ \sigma_\ell = \frac{1}{2p}\log(Q_p \circ \sigma_\ell) \text{ on } \mathbb{D} \setminus \{0\}.$$

By Proposition 3.3 and (3.15), we have that the functions $\zeta^{\alpha(y_\ell)} s_j^p(\zeta)$ extend holomorphically at $0 \in \mathbb{D}$. Hence the function

$$v_p(\zeta) := u_p \circ \sigma_\ell(\zeta) + \alpha(y_\ell)\log|\zeta|$$

extends to a subharmonic function on $\mathbb{D}$. Moreover, $\varphi \circ \sigma_\ell$ also extends to a subharmonic on $\mathbb{D}$.

We infer from (4.3) that $\frac{1}{p}\log(Q_p \circ \sigma_\ell) \to 0$, hence $\frac{1}{p}v_p \to \varphi \circ \sigma_\ell$, in $L^1_{\mathrm{loc}}(\mathbb{D} \setminus \{0\}, \lambda)$. Combined with Lemma 4.1 this implies that the sequence of subharmonic functions $\{\frac{1}{p}v_p\}$ is locally uniformly upper bounded in $\mathbb{D}$. Therefore [23, Theorem 3.2.12] yields that $\frac{1}{p}v_p \to \varphi \circ \sigma_\ell$, hence $\frac{1}{p}\log(Q_p \circ \sigma_\ell) \to 0$, in $L^1_{\mathrm{loc}}(\mathbb{D}, \lambda)$. Using (2.4) we get

$$\int_{Y_\ell \setminus x_j} |\log Q_p|\omega \leqslant C_2 \int_{\mathbb{D}} |\log(Q_p \circ \sigma_\ell)| \mathrm{d}\lambda.$$

Hence (4.7) follows, and the proof of assertion (i) is finished.

(ii). — In the case of $\gamma_{c,p}$ and $\gamma_{w,p}$, (ii) follows immediately from (i) by using (3.11). From (3.14) we see that $\frac{1}{p}\int_X \chi\, \mathrm{d}\gamma_p \to \int_X \chi\, \mathrm{d}c_1(L,h)$ for every smooth function χ on X. By Lemma 3.4 the measure $\gamma_p + \sigma_\star([R_\sigma])$ is positive, so we infer that $\frac{1}{p}\gamma_p \to c_1(L,h)$ in the weak sense of measures. Therefore $\frac{1}{p}\gamma_p^+ \to c_1(L,h)$, as $\frac{1}{p}\gamma_p^- \to 0$ in the weak sense of measures, by Lemma 3.4. □

Proof of Theorem 1.2. — Let Q_p, η_p be the Bergman kernel function and Fubini–Study measure of the space H^p. By Theorem 1.1 we have $\frac{1}{p}\log Q_p \to 0$ in $L^1(X,\omega)$ and $\frac{1}{p}\eta_p \to c_1(L,h)$ in the weak sense of measures on X.

Note that $c_1(L^p, h_p) = pc_1(L, h)$ and formulas (3.11), (3.14), which relate $\eta_p, Q_p, c_1(L, h)$, are valid. Moreover, we have the Lelong–Poincaré formula (3.8) relating $[\operatorname{div}(s_p)], |s_p|_{h_p}, c_1(L, h)$. Hence the proof of [2, Theorem 1.1] goes through with no change in our setting, and we can take $A_p = p$ (see also [2, Theorem 4.1] and its proof).

In our present situation μ_p is the normalized area measure on the unit sphere of H^p. By Proposition 3.3 we have $n_p = \dim H^p \leqslant Cp$ for some constant $C > 0$. Therefore Theorem 1.2 follows from [2, Theorem 4.12].

We have to consider further the case $H^p = H^0_{(2)}(X \setminus \Sigma, L^p)$, when the measures $\eta_p = \gamma_p$ and $[\operatorname{div}(s_p)]$ are not necessarily positive. Arguing as in the proof of [2, Theorem 1.1], we have that $\frac{1}{p}\log|s_p|_{h_p} \to 0$ in $L^1(X, \omega)$ for μ-a.e. sequence $\{s_p\} \in \mathcal{H}$. By (3.8), this implies that $\frac{1}{p}[\operatorname{div}(s_p)] \to c_1(L, h)$ in the sense of distributions. Since by (3.7), $[\operatorname{div}(s_p)]^- \leqslant \sigma_\star([R_\sigma])$, it follows that $\frac{1}{p}[\operatorname{div}(s_p)] \to c_1(L, h)$, $\frac{1}{p}[\operatorname{div}(s_p)]^+ \to c_1(L, h)$, in the weak sense of measures on X. This completes the proof of assertion (ii).

For (i), we have as in the proof of [2, Theorem 1.1] that $\mathbb{E}[\operatorname{div}(s_p)]$ is a well defined distribution on X and

$$\left\langle \frac{1}{p}\mathbb{E}[\operatorname{div}(s_p)], \chi \right\rangle \to \int_X \chi \, \mathrm{d}c_1(L, h)$$

for every smooth function χ. By (3.7) the measure $\nu := [\operatorname{div}(s_p)] + \sigma_\star([R_\sigma])$ is positive, hence the total variation $|[\operatorname{div}(s_p)]| \leqslant \nu + \sigma_\star([R_\sigma])$. Using (3.8) we get

$$\left| \int_X \chi \, \mathrm{d}[\operatorname{div}(s_p)] \right| \leqslant \int_X |\chi| \mathrm{d}(\nu + \sigma_\star([R_\sigma]) \leqslant \|\chi\|_\infty \big(pc_1(L, h) + 2\sigma_\star([R_\sigma])\big)(X).$$

We infer that $\mathbb{E}[\operatorname{div}(s_p)]$ is a well defined measure on X and its total variation verifies

$$\big|\mathbb{E}[\operatorname{div}(s_p)]\big|(X) \leqslant \big(pc_1(L, h) + 2\sigma_\star([R_\sigma])\big)(X).$$

This yields assertion (i). □

5. Examples

In this section we exemplify our results in the case of certain plane algebraic curves. We also give a precise lower estimate of the Bergman kernel $P_{w,p}$ in the case of a smooth Hermitian metric on L.

Example 5.1. — We consider a class of algebraic curves X in $\mathbb{P}^2$ which have one singular point and are normalized by $\mathbb{P}^1$. They are defined by graphs of polynomials and we will describe explicitly the spaces of holomorphic sections considered in the paper. We denote by $\mathbb{C}_n[\zeta]$ the space of polynomials of degree at most n in $\mathbb{C}$.

Let $[z_0 : z_1 : z_2]$ denote the homogeneous coordinates on $\mathbb{P}^2$, and consider the standard embedding $(z_1, z_2) \in \mathbb{C}^2 \hookrightarrow [1 : z_1 : z_2] \in \mathbb{P}^2$. Let

$$Q(z_0, z_1) = \sum_{j=0}^{d} a_j z_0^j z_1^{d-j}, \text{ where } a_0 \neq 0,$$

be a homogeneous polynomial of degree $d \geqslant 2$. Set $P(z_1) = Q(1, z_1)$, so $P \in \mathbb{C}[z_1]$ is a polynomial of degree d. Let $X = X_Q$ be the algebraic curve of degree d in $\mathbb{P}^2$ defined by

$$X = \left\{[z_0 : z_1 : z_2] \in \mathbb{P}^2 : z_0^{d-1} z_2 - Q(z_0, z_1) = 0\right\}.$$

Furthermore, let

$$\omega = \omega_{\mathrm{FS}}|_X, \ L = \mathscr{O}_{\mathbb{P}^2}(1)|_X,$$

where ω_{FS} is the Fubini–Study form and $\mathscr{O}_{\mathbb{P}^2}(1)$ is the hyperplane bundle on $\mathbb{P}^2$. Recall that $\pi^\star \omega_{\mathrm{FS}} = dd^c \log \|Z\|$, where $Z = (z_0, z_1, z_2) \in \mathbb{C}^3 \setminus \{0\}$ and $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ is the canonical projection. Moreover, if $U_j = \{z_j \neq 0\} \subset \mathbb{P}^2$ then the transition functions of $\mathscr{O}_{\mathbb{P}^2}(1)$ are given by $g_{jk}([z_0 : z_1 : z_2]) = z_k / z_j$ on $U_j \cap U_k$, $0 \leqslant j, k \leqslant 2$ (see e.g. [2, Example 4.4]).

Note that in $\mathbb{C}^2 \cong U_0$ we have $X \cap U_0 = \{(z_1, z_2) : z_2 = P(z_1)\}$, so $X \cap U_0$ is biholomorphic to $\mathbb{C}$ via the obvious map $\zeta \in \mathbb{C} \to (\zeta, P(\zeta)) \in X \cap U_0$. We also note that X has one point on the line at infinity, as $X \cap \{z_0 = 0\} = \{[0 : 0 : 1]\}$. If $d \geqslant 3$ then X is singular and locally irreducible at $x_1 := [0 : 0 : 1]$, so $\Sigma = \{x_1\}$. It follows that the normalization of X is $\mathbb{P}^1$. In fact we have the explicit formula for this:

$$\sigma : \mathbb{P}^1 \to \mathbb{P}^2, \ \sigma([t_0 : t_1]) = [t_0^d : t_0^{d-1} t_1 : Q(t_0, t_1)].$$

Here $[t_0 : t_1]$ denote the homogeneous coordinates on $\mathbb{P}^1$, and we consider the standard embedding $\zeta \in \mathbb{C} \hookrightarrow [1 : \zeta] \in \mathbb{P}^1$. Note that $\sigma([0 : 1]) = x_1$ and for $r > 0$ sufficiently small, the function

$$f(t) = \sigma[t : 1] = \left[\frac{t^d}{Q(t,1)} : \frac{t^{d-1}}{Q(t,1)} : 1\right], \ t \in \mathbb{D}_r,$$

is the local normalization of X at x_j as in (2.1). We infer that the ramification divisor of σ (see (3.1)) is

$$R_\sigma = (d-2)y, \text{ where } y = [0 : 1].$$

Since X is locally irreducible at x_1 we have that $\mathscr{O}_{X,c}(U) = \mathscr{O}_{X,w}(U)$, and, by Lemma 2.3, that $\mathrm{SH}(U) = \mathrm{WSH}(U)$, for any open set $U \subset X$. Let $S \in H^0_w(X, L^p)$ be represented by the holomorphic functions s_0 on $X \cap U_0$

and s_2 on $X \cap U_2$. Then there exists an entire function s such that $s_0([1 : \zeta : P(\zeta)]) = s(\zeta)$ for $\zeta \in \mathbb{C}$, and we have

$$s_0([1 : \zeta : P(\zeta)]) = P^p(\zeta) s_2([1 : \zeta : P(\zeta)])$$

for all ζ with $|\zeta|$ sufficiently large. We infer that s/P^p is bounded near ∞, hence s is a polynomial of degree $\leqslant dp$. We conclude that the space $H^0_w(X, L^p)$ is isomorphic to $\mathbb{C}_{dp}[\zeta]$, hence $\dim H^0_w(X, L^p) = dp + 1$. We can compare this to the subspace of restrictions to X of global holomorphic sections on $\mathbb{P}^2$,

$$V_p := \big\{ S|_X : S \in H^0(\mathbb{P}^2, L^p) \big\}.$$

If $H^0_X(\mathbb{P}^2, L^p) = \{S \in H^0(\mathbb{P}^2, L^p) : S = 0 \text{ on } X\}$ then $V_p \cong H^0(\mathbb{P}^2, L^p)/H^0_X(\mathbb{P}^2, L^p)$. Recall that sections in $H^0(\mathbb{P}^2, L^p)$ are given by homogeneous polynomials of degree p in z_0, z_1, z_2 so $\dim H^0(\mathbb{P}^2, L^p) = \frac{(p+1)(p+2)}{2}$. Therefore

$$\begin{aligned}\dim V_p = \frac{(p+1)(p+2)}{2} - \frac{(p-d+1)(p-d+2)}{2} &= dp - \frac{d(d-3)}{2} \\ &< \dim H^0_w(X, L^p),\end{aligned}$$

since $d \geqslant 3$. It is worth observing that $\sigma^\star : H^0_w(X, L^p) \to H^0(\mathbb{P}^1, \sigma^\star L^p)$ is an isomorphism and $\sigma^\star L \cong \mathscr{O}_{\mathbb{P}^1}(d)$.

We next describe the set of singular Hermitian metrics h on L that have subharmonic weights. Arguing as above, we infer that the weight of such h on $X \cap U_0$ is given by a subharmonic function φ on $\mathbb{C}$ such that $\varphi(\zeta) - \log|P(\zeta)|$ is bounded at infinity. Hence

$$\varphi(\zeta) \leqslant d \log^+ |\zeta| + C_\varphi, \ \forall\, \zeta \in \mathbb{C}, \tag{5.1}$$

with some constant C_φ. This shows that the set of singular Hermitian metrics on L is in one-to-one correspondence to the class $d\mathcal{L}(\mathbb{C})$, where $\mathcal{L}(\mathbb{C})$ is the Lelong class of subharmonic functions of logarithmic growth on $\mathbb{C}$.

We conclude this section by describing the Bergman space $H^0_{(2)}(X \setminus \Sigma, L^p)$ defined in (1.3), where h is the metric given by a function $\varphi \in d\mathcal{L}(\mathbb{C})$. In view of the above, this space consists of the entire functions s on $\mathbb{C}$ which verify

$$\int_{\mathbb{C}} |s|^2 e^{-2p\varphi} \sigma^\star(\omega) < +\infty.$$

We have

$$\begin{aligned}\sigma^\star(\omega) &= \frac{1}{2} \mathrm{dd}^c \log\big(1 + |\zeta|^2 + |P(\zeta)|^2\big) \\ &= \frac{1 + |P'(\zeta)|^2 + |\zeta P'(\zeta) - P(\zeta)|^2}{\pi\big(1 + |\zeta|^2 + |P(\zeta)|^2\big)^2} \frac{i}{2} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.\end{aligned}$$

Since $\zeta P' - P$ has degree d we infer that

$$\frac{c_1}{1+|\zeta|^{2d}}\frac{i}{2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta} \leqslant \sigma^\star(\omega) \leqslant \frac{c_2}{1+|\zeta|^{2d}}\frac{i}{2}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}$$

holds for $\zeta\in\mathbb{C}$, with some constants $c_1, c_2 > 0$. Using (5.1) we obtain that

$$\int_{\mathbb{C}\setminus\mathbb{D}} |s(\zeta)|^2|\zeta|^{-2d(p+1)}\mathrm{d}\lambda < +\infty,$$

which implies that s is a polynomial of degree $\leqslant dp+d-2$. In conclusion, $H^0_{(2)}(X\setminus\Sigma, L^p)$ is isomorphic to the space of polynomials $s\in\mathbb{C}_{dp+d-2}[\zeta]$ that verify

$$\int_{\mathbb{C}} \frac{|s|^2e^{-2p\varphi}}{1+|\zeta|^{2d}}\,\mathrm{d}\lambda < +\infty,$$

where $\varphi\in d\mathcal{L}(\mathbb{C})$ is the weight of h. We refer to the survey [1] for results about equidistribution of zeros of random polynomials.

Example 5.2. — Let X be an irreducible algebraic curve and $(L,h)\to X$ be a Hermitian holomorphic line bundle as in (A) and (B). We assume further that the Hermitian metric h is smooth. Since h is smooth, $H^0_{w,(2)}(X,L^p) = H^0_w(X,L^p)$ and $P_{w,p}(x)$ is defined for all $x\in X\setminus\Sigma$. Let $\sigma:\widetilde{X}\to X$ be the normalization of X, $(\sigma^*L,\sigma^*h)\to\widetilde{X}$ the pull-back of (L,h). The curvature $c_1(\sigma^*L,\sigma^*h)$ is semi-positive on $\widetilde{X}$, is positive on $\widetilde{X}\setminus\sigma^{-1}(\Sigma)$, and vanishes up to finite order at any point of $\sigma^{-1}(\Sigma)$. These are precisely the hypotheses of the results from [25] under which the Bergman kernel asymptotics hold for a semi-positive line bundle on a Riemann surface. There exists $C>0$ such that $\sigma^*\omega\leqslant C\widetilde{\omega}$ hence for any $S\in H^0_w(X,L^p)$ we have

$$\|S\|^2_p = \int_{X\setminus\Sigma}|S|^2_{h_p}\omega \leqslant C\int_{\tilde{X}\setminus\sigma^{-1}(\Sigma)}|\sigma^*S|^2_{\sigma^*h_p}\widetilde{\omega} = C\int_{\tilde{X}}|\sigma^*S|^2_{\sigma^*h_p}\widetilde{\omega}.$$

We consider the Bergman kernel function $\widetilde{P}_p$ of the space $H^0(\widetilde{X},\sigma^*L^p)$ with respect to the Hermitian metric σ^*h_p and volume form $\widetilde{\omega}$. By the isomorphism (3.3) and the variational principle for the Bergman kernel functions we have

$$P_{w,p}(x) \geqslant \frac{1}{C}\widetilde{P}_p(\sigma^{-1}(x)), \quad \text{for any } x\in X\setminus\Sigma. \tag{5.2}$$

By [25, Lemma 25] there exists $\widetilde{C}>0$ such that for p large enough we have

$$\widetilde{P}_p(\widetilde{x}) \geqslant \widetilde{C}p^{2/r}, \quad \text{for any } \widetilde{x}\in\widetilde{X}, \tag{5.3}$$

where r is the maximal normalized vanishing order of the curvature $c_1(\sigma^*L,\sigma^*h)$ on $\widetilde{X}$, namely, $r=\max\{r_{\tilde{x}} : \widetilde{x}\in\widetilde{X}\}$, $r_{\tilde{x}} = \operatorname{ord} c_1(\sigma^*L,\sigma^*h)_{\tilde{x}}+2$. Hence, (5.2) and (5.3) show that there exists $C'>0$ such that for p large enough we have

$$P_{w,p}(x) \geqslant C'p^{2/r}, \quad \text{for any } x\in X\setminus\Sigma. \tag{5.4}$$

Thus for h smooth the convergence $\frac{1}{p}\log P_{w,p} \to 0$, as $p \to \infty$ in $L^1(X,\omega)$ follows directly from Lemma 4.2 and its proof.

Note that (5.4) is a global lower bound for $P_{w,p}(x)$. By [25] we have $\widetilde{P}_p(\widetilde{x}) \sim p^{2/r_{\tilde{x}}}$ for any $\widetilde{x} \in \widetilde{X}$ and on any open set $\widetilde{U} \Subset \widetilde{X} \setminus \sigma^{-1}(\Sigma)$ we have $\widetilde{P}_p \sim p$ uniformly. Thus for any open set $U \Subset X \setminus \Sigma$ there exists $C_1 > 0$ such that $P_{w,p}(x) \geqslant C_1 p$ on U.

Bibliography

[1] T. Bayraktar, D. Coman, H. Herrmann & G. Marinescu, "A survey on zeros of random holomorphic sections", *Dolomites Res. Notes Approx.* **11** (2018), p. 1-19.

[2] T. Bayraktar, D. Coman & G. Marinescu, "Universality results for zeros of random holomorphic sections", *Trans. Am. Math. Soc.* **373** (2020), no. 6, p. 3765-3791.

[3] E. M. Chirka, *Complex analytic sets*, Mathematics and its Applications, vol. 46, Kluwer Academic Publishers, 1989, 372 pages.

[4] D. Coman, X. Ma & G. Marinescu, "Equidistribution for sequences of line bundles on normal Kähler spaces", *Geom. Topol.* **21** (2017), no. 2, p. 923-962.

[5] D. Coman & G. Marinescu, "Convergence of Fubini-Study currents for orbifold line bundles", *Int. J. Math.* **24** (2013), no. 7, article no. 1350051 (27 pages).

[6] ———, "Equidistribution results for singular metrics on line bundles", *Ann. Sci. Éc. Norm. Supér.* **48** (2015), no. 3, p. 497-536.

[7] D. Coman, G. Marinescu & V.-A. Nguyên, "Hölder singular metrics on big line bundles and equidistribution", *Int. Math. Res. Not.* **2016** (2016), no. 16, p. 5048-5075.

[8] ———, "Approximation and equidistribution results for pseudo-effective line bundles", *J. Math. Pures Appl.* **115** (2018), p. 218-236.

[9] J.-P. Demailly, "Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré holomorphe semipositif au–dessus d'une variété kählérienne complète", *Ann. Sci. Éc. Norm. Supér.* **15** (1982), p. 457-511.

[10] ———, "Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines", *Mém. Soc. Math. Fr., Nouv. Sér.* **19** (1985), p. 1-125.

[11] ———, "Singular Hermitian metrics on positive line bundles", in *Complex algebraic varieties (Bayreuth, 1990)*, Lecture Notes in Mathematics, vol. 1507, Springer, 1990, p. 87-104.

[12] ———, "A numerical criterion for very ample line bundles", *J. Differ. Geom.* **37** (1993), no. 2, p. 323-374.

[13] T.-C. Dinh, X. Ma & G. Marinescu, "Equidistribution and convergence speed for zeros of holomorphic sections of singular Hermitian line bundles", *J. Funct. Anal.* **271** (2016), no. 11, p. 3082-3110.

[14] T.-C. Dinh, G. Marinescu & V. Schmidt, "Asymptotic distribution of zeros of holomorphic sections in the non compact setting", *J. Stat. Phys.* **148** (2012), p. 113-136.

[15] J. E. Fornaess & R. Narasimhan, "The Levi problem on complex spaces with singularities", *Math. Ann.* **248** (1980), p. 47-72.

[16] O. Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer, 1981, 254 pages.

[17] H. Grauert, "Über Modifikationen und exzeptionelle analytische Mengen", *Math. Ann.* **146** (1962), p. 331-368.

[18] H. Grauert & R. Remmert, "Plurisubharmonische Funktionen in komplexen Räumen", *Math. Z.* **65** (1956), p. 175-194.

[19] ———, *Coherent Analytic Sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 265, Springer, 1984, 249 pages.

[20] P. A. Griffiths, *Introduction to algebraic curves*, Translations of Mathematical Monographs, vol. 76, American Mathematical Society, 1989, 221 pages.

[21] P. A. Griffiths & J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, 1994, 813 pages.

[22] R. C. Gunning, *Introduction to holomorphic functions of several variables. Vol. II. Local theory*, The Wadsworth & Brooks/Cole Mathematics Series, vol. 1990, Wadsworth & Brooks/Cole Advanced Books & Software, 1990, 218 pages.

[23] L. Hörmander, *Notions of convexity*, Progress in Mathematics, vol. 127, Birkhäuser, 1994, 414 pages.

[24] X. Ma & G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, vol. 254, Birkhäuser, 2007, 422 pages.

[25] G. Marinescu & N. Savale, "Bochner Laplacian and Bergman kernel expansion of semi-positive line bundles on a Riemann surface", `https://arxiv.org/abs/1811.00992`, 2018.

[26] T. Ohsawa & K. Takegoshi, "On the extension of L^2 holomorphic functions", *Math. Z.* **195** (1987), p. 97-204.

[27] J. Ruppenthal & M. Sera, "L^2-Riemann–Roch for singular complex curves", *J. Singul.* **11** (2015), p. 67-84.

[28] B. Shiffman, "Convergence of random zeros on complex manifolds", *Sci. China, Ser. A* **51** (2008), no. 4, p. 707-720.

[29] B. Shiffman & S. Zelditch, "Distribution of zeros of random and quantum chaotic sections of positive line bundles", *Commun. Math. Phys.* **200** (1999), no. 3, p. 661-683.

[30] ———, "Number variance of random zeros on complex manifolds", *Geom. Funct. Anal.* **18** (2008), p. 1422-1475.

[31] G. Tian, "On a set of polarized Kähler metrics on algebraic manifolds", *J. Differ. Geom.* **32** (1990), no. 1, p. 99-130.