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Equidistribution results for singular metrics on line bundles

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EQUIDISTRIBUTION RESULTS FOR SINGULAR METRICS ON LINE BUNDLES

BY DAN COMAN AND GEORGE MARINESCU

ABSTRACT. – Let (L, h) be a holomorphic line bundle with a positively curved singular Hermitian metric over a complex manifold X . One can define naturally the sequence of Fubini-Study currents γ_p associated to the space of L^2 -holomorphic sections of $L^{\otimes p}$. Assuming that the singular set of the metric is contained in a compact analytic subset Σ of X and that the logarithm of the Bergman density function of $L^{\otimes p}|_{X \setminus \Sigma}$ grows like $o(p)$ as $p \rightarrow \infty$, we prove the following:

1) the currents γ_p^k converge weakly on the whole X to $c_1(L, h)^k$, where $c_1(L, h)$ is the curvature current of h .

2) the expectations of the common zeros of a random k -tuple of L^2 -holomorphic sections converge weakly in the sense of currents to $c_1(L, h)^k$.

Here k is so that $\text{codim } \Sigma \geq k$. Our weak asymptotic condition on the Bergman density function is known to hold in many cases, as it is a consequence of its asymptotic expansion. We also prove it here in a quite general setting. We then show that many important geometric situations (singular metrics on big line bundles, Kähler-Einstein metrics on Zariski-open sets, arithmetic quotients) fit into our framework.

RÉSUMÉ. – Considérons un fibré holomorphe en droites L muni d'une métrique singulière h au-dessus d'une variété complexe X . Soit γ_p le courant de Fubini-Study associé naturellement à l'espace des sections holomorphes de carré intégrable de $L^{\otimes p}$. En supposant que le lieu singulier de la métrique h est contenu dans un ensemble analytique compact $\Sigma \subset X$ tel que $\text{codim } \Sigma \geq k$ et que le logarithme du noyau de Bergman associé à $L^{\otimes p}|_{X \setminus \Sigma}$ a l'ordre de croissance $o(p)$, $p \rightarrow \infty$, nous prouvons que :

1) Les courants γ_p^k convergent faiblement sur X vers $c_1(L, h)^k$, où $c_1(L, h)$ est le courant de courbure de h .

2) Les moyennes des zéros communs d'un k -vecteur aléatoire de sections holomorphes L^2 -intégrables convergent faiblement dans le sens des courants vers $c_1(L, h)^k$.

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L'hypothèse de croissance du noyau de Bergman est la conséquence de son développement asymptotique dans le cas d'une métrique lisse h . Nous la démontrons ici sous des conditions assez générales. Nous montrons ensuite que nos résultats s'appliquent à nombre de situations géométriques (métriques singulières sur un fibré gros, métriques de Kähler-Einstein sur des ouverts de Zariski, quotients arithmétiques...).

1. Introduction

Let X be a compact complex manifold of dimension n , $L \rightarrow X$ be a positive holomorphic line bundle, and h be a smooth Hermitian metric on L whose curvature $c_1(L, h)$ is a positive (1,1) form on X . Let $\Phi_p : X \rightarrow \mathbb{P}^{d_p-1}$ be the Kodaira map defined by an orthonormal basis of $H^0(X, L^p)$ with respect to the inner product given by the metric induced by h on $L^p := L^{\otimes p}$ and a fixed volume form on X , where $d_p = \dim H^0(X, L^p)$. The pull-back $\Phi_p^*(\omega_{FS})$ of the Fubini-Study form ω_{FS} is a smooth (1,1) form for all p sufficiently large, since Φ_p is an embedding by Kodaira's embedding theorem. A theorem of Tian [59] (with improvements by Ruan [50]) asserts that $\frac{1}{p} \Phi_p^*(\omega_{FS}) \rightarrow c_1(L, h)$ as $p \rightarrow \infty$, in the \mathcal{C}^∞ topology on X .

Tian's theorem is a consequence of the asymptotic expansion of the Bergman density function associated to the inner product on $H^0(X, L^p)$ mentioned above. In the context of positive line bundles this asymptotic expansion is proved in various forms in [59, 11, 62, 17, 42, 43, 44, 6]. For line bundles endowed with arbitrary smooth Hermitian metrics the Bergman density function behavior and important consequences are studied in [5] and [7].

In the case of holomorphic Hermitian line bundles over complete Hermitian manifolds the asymptotic expansion of the Bergman density function associated to the corresponding spaces of L^2 -holomorphic sections was proved in [44] (see also [42, 43]). In particular, a version of Tian's theorem was obtained for a big line bundle L over a (compact) manifold X . Such a line bundle admits a singular Hermitian metric h , smooth outside a proper analytic subvariety $\Sigma \subset X$, and whose curvature current $c_1(L, h)$ is strictly positive. It is shown in [43, Section 6.2] that there exist a smooth positively curved Hermitian metric h_ε on $L|_{X \setminus \Sigma}$, which is a small perturbation of h , and a smooth positive (1,1) form Θ defining a generalized Poincaré metric on $X \setminus \Sigma$, so that the following hold. If $H_{(2)}^0(X \setminus \Sigma, L^p)$ is the space of L^2 -holomorphic sections of $L^p|_{X \setminus \Sigma}$ relative to the metrics h_ε and Θ then $H_{(2)}^0(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$, so a Kodaira map $\Phi_p : X \dashrightarrow \mathbb{P}^{d_p-1}$ can be defined by using an orthonormal basis of $H_{(2)}^0(X \setminus \Sigma, L^p)$. Let $\gamma_p = \Phi_p^*(\omega_{FS})$ and $\omega = c_1(L|_{X \setminus \Sigma}, h_\varepsilon)$. Then $\frac{1}{p} \gamma_p \rightarrow \omega$ as $p \rightarrow \infty$, locally uniformly in the \mathcal{C}^∞ topology on $X \setminus \Sigma$.

Since γ_p are currents on X it is natural to try and study the weak convergence of the sequence $\{\gamma_p/p\}$, and to ask whether a global version of Tian's theorem holds in this setting. We will show that this is indeed the case.

Let us work in the following more general setting:

- (A) X is a complex manifold of dimension n (not necessarily compact), Σ is a compact analytic subvariety of X , and Ω is a smooth positive (1, 1) form on X .

- (B) (L, h) is a holomorphic line bundle on X with a singular (semi)positively curved Hermitian metric h which is continuous on $X \setminus \Sigma$. We denote by h_p the Hermitian metric induced by h on $L^p := L^{\otimes p}$.
- (C) The volume form on $X \setminus \Sigma$ is $f\Omega^n$, where $f \in L^1_{\text{loc}}(X \setminus \Sigma, \Omega^n)$ verifies $f \geq c_x > 0$ Ω^n -a.e. in a neighborhood U_x of each $x \in (X \setminus \Sigma) \cup \Sigma_{\text{reg}}^{n-1}$. Here $\Sigma_{\text{reg}}^{n-1}$ is the set of regular points y where $\dim_y \Sigma = n - 1$.

We denote the curvature current of h by $\gamma = c_1(L, h)$ and consider the space $H^0_{(2)}(X \setminus \Sigma, L^p)$ of L^2 -holomorphic sections of $L^p|_{X \setminus \Sigma}$ relative to the metric h_p on L^p and the volume form $f\Omega^n$ on $X \setminus \Sigma$, endowed with the inner product

$$(S, S')_p = \int_{X \setminus \Sigma} \langle S, S' \rangle_{h_p} f\Omega^n, \text{ where } \langle S, S' \rangle_{h_p} = h_p(S, S'), S, S' \in H^0_{(2)}(X \setminus \Sigma, L^p).$$

We let $\|S\|_p^2 = (S, S)_p$. Since $H^0_{(2)}(X \setminus \Sigma, L^p)$ is separable, let $\{S_j^p\}_{j \geq 1}$ be an orthonormal basis and denote by P_p the Bergman density function defined by

$$(1) \quad P_p(x) = \sum_{j=1}^{\infty} |S_j^p(x)|_{h_p}^2, \quad |S_j^p(x)|_{h_p}^2 := \langle S_j^p(x), S_j^p(x) \rangle_{h_p}, \quad x \in X \setminus \Sigma.$$

Note that this definition is independent of the choice of basis, and the function P_p is continuous on $X \setminus \Sigma$ (see Section 3).

Next we define the Fubini-Study currents γ_p on $X \setminus \Sigma$ by

$$(2) \quad \gamma_p|_U = \frac{1}{2} dd^c \log \left(\sum_{j=1}^{\infty} |s_j^p|^2 \right), \quad U \subset X \setminus \Sigma \text{ open},$$

where $d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})$, $S_j^p = s_j^p e^{\otimes p}$, and e is a local holomorphic frame for L on U .

One of our main results is the following:

THEOREM 1.1. – *If $X, \Sigma, (L, h), f, \Omega$ verify assumptions (A)-(C) then $H^0_{(2)}(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$ and γ_p extends to a positive closed current on X defined locally by Formula (2) and which is independent of the choice of basis $\{S_j^p\}_{j \geq 1}$. Assume further that*

$$(3) \quad \lim_{p \rightarrow \infty} \frac{1}{p} \log P_p(x) = 0, \text{ locally uniformly on } X \setminus \Sigma.$$

Then $\frac{1}{p} \gamma_p \rightarrow \gamma$ weakly on X . If, in addition, $\dim \Sigma \leq n - k$ for some $2 \leq k \leq n$, then the currents γ^k and γ_p^k are well defined on X , respectively on each relatively compact neighborhood of Σ , for all p sufficiently large. Moreover, $\frac{1}{p^k} \gamma_p^k \rightarrow \gamma^k$ weakly on X .

This theorem is proved in Section 3. The proof relies on a local continuity property of the complex Monge-Ampère operator which is of independent interest (see Theorem 3.4). Some background material about singular Hermitian metrics and pluripotential theory needed in the paper is recalled in Section 2. We note here that if $\text{codim } \Sigma < k$ the current γ^k cannot be defined (see [9, 10]), so the assumption on the dimension in Theorem 1.1 is optimal.

We examine in Section 6 a series of important situations where condition (3) of Theorem 1.1 holds, as it is an immediate consequence of deep results regarding the asymptotic expansion of the Bergman density function $P_p(x) \sim b_0(x)p^n + b_1(x)p^{n-1} + \dots$. Especially, Theorem 1.1 yields equidistribution results for singular metrics on big line bundles

(Sections 6.1, 6.2), on Zariski-open sets of bounded negative Ricci curvature (Section 6.3), on toroidal compactifications of arithmetic quotients (Section 6.4), and finally on 1-convex manifolds (Sections 6.5, 6.6).

The point of view adopted in Theorem 1.1 is that once some information is known on the asymptotic behavior of P_p on the set where the metric is continuous, then the global weak convergence on X of the currents γ_p/p and their powers follows. Hypothesis (3) is obviously a much weaker condition than the asymptotic expansion of P_p mentioned above. Indeed, in Section 5 we give a simple proof that (3) holds in the case of line bundles over compact Kähler manifolds endowed with metrics that are assumed to be only continuous outside of Σ (see Theorems 5.3 and 5.4). In this case the asymptotic expansion of P_p is not known.

We also prove in Theorem 5.1 that Tian's theorem [59] holds for any singular metric with strictly positive curvature. Namely, let (X, Ω) be a compact Kähler manifold and (L, h) be a holomorphic line bundle on X with a singular metric h so that $c_1(L, h)$ is a strictly positive current. If γ_p are the Fubini-Study currents defined by (2) for the spaces of L^2 -holomorphic sections of L^p relative to the metric induced by h and the volume form Ω^n , then $\frac{1}{p} \gamma_p \rightarrow c_1(L, h)$ in the weak sense of currents on X . The proofs of Theorems 5.1 and 5.3 rely on techniques developed by Demailly [19, 22].

In a series of papers including [54, 55, 56, 53], Shiffman and Zelditch describe the asymptotic distribution of zeros of random holomorphic sections of a positive line bundle over a projective manifold endowed with a smooth positively curved metric. They also study the distribution of zeros of quantum ergodic eigenfunctions. To prove these results they develop interesting new techniques, based in part on methods in complex dynamics from [29].

Later, using different methods, Dinh and Sibony [24] obtain sharper estimates for the speed of convergence in the asymptotic distribution of zeros of random holomorphic sections. In [23] these results are generalized to the case of complete Hermitian manifolds. The problem of the distribution of zeros of random sections of line bundles appears in other contexts as well. For example, the case of canonical line bundles over towers of covers is studied in [60].

We show here how some of the important results of Shiffman and Zelditch can be obtained in our setting from Theorem 1.1, assuming in addition that X is compact. More precisely, following the framework in [54], we let λ_p be the normalized surface measure on the unit sphere \mathcal{S}^p of $H_{(2)}^0(X \setminus \Sigma, L^p)$, defined in the natural way by using a fixed orthonormal basis (see Section 4). We denote by λ_p^k the product measure on $(\mathcal{S}^p)^k$, and by $[S = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{S = 0\}$ determined by a nontrivial section $S \in H^0(X, L^p)$. We prove in Section 4 the following generalization of some results of Shiffman and Zelditch [54, 56] to our situation:

THEOREM 1.2. – *In the setting of Theorem 1.1, assume that X is compact, $\dim \Sigma \leq n - k$ for some $1 \leq k \leq n$, and that (3) holds. Then, for all p sufficiently large:*

- (i) $[\sigma = 0] := [\sigma_1 = 0] \wedge \cdots \wedge [\sigma_k = 0]$ is a well defined positive closed current of bidegree (k, k) on X , for λ_p^k -a.e. $\sigma = (\sigma_1, \dots, \sigma_k) \in (\mathcal{S}^p)^k$.
- (ii) The expectation $E_p^k[\sigma = 0]$ of the current-valued random variable $\sigma \rightarrow [\sigma = 0]$, given by

$$\langle E_p^k[\sigma = 0], \varphi \rangle = \int_{(\mathcal{S}^p)^k} \langle [\sigma = 0], \varphi \rangle d\lambda_p^k,$$

where φ is a test form on X , is a well defined current and $E_p^k[\sigma = 0] = \gamma_p^k$.

(iii) We have $\frac{1}{p^k} E_p^k[\sigma = 0] \rightarrow \gamma^k$ as $p \rightarrow \infty$, weakly in the sense of currents on X .

In particular, this theorem together with [54, Lemma 3.3] or [43, Section 5.3], yields an equidistribution result for the zeros of a random sequence of sections $\{\sigma_p\}_{p \geq 1} \in \prod_{p=1}^{\infty} \mathcal{F}^p$, i.e., $\frac{1}{p} [\sigma_p = 0] \rightarrow \gamma$ as $p \rightarrow \infty$, in the weak sense of currents on X (see Theorem 4.3).

The kind of results proved in the present paper can be extended to the case of orbifolds [15]. Theorem 1.2 has applications to the approximation of γ^k by currents of integration on analytic varieties [16].

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2. Preliminaries

We recall here a few of the notions that we will need. We start with the notion of singular Hermitian metric in Section 2.1 and some necessary notions about desingularization in Section 2.2. In Section 2.3 we introduce the generalized Poincaré metric on a manifold and an associated metric on a line bundle with strictly positive curvature current. In Section 2.4 we recall a few facts regarding the definition of complex Monge-Ampère operators.

2.1. Singular Hermitian metrics on line bundles

Let $L \rightarrow X$ be a holomorphic line bundle over a complex manifold X and fix an open cover $X = \bigcup U_\alpha$ for which there exist local holomorphic frames $e_\alpha : U_\alpha \rightarrow L$. The transition functions $g_{\alpha\beta} = e_\beta/e_\alpha \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ determine the Čech 1-cocycle $\{(U_\alpha, g_{\alpha\beta})\}$.

Let h be a smooth Hermitian metric on L . If $|e_\alpha(x)|_h^2 = h(e_\alpha(x), e_\alpha(x))$ for $x \in U_\alpha$, we recall that the curvature form $c_1(L, h)$ of h is defined by

$$c_1(L, h)|_{U_\alpha} = -dd^c \log |e_\alpha|_h = \frac{i}{2\pi} R^L,$$

where R^L is the curvature of the holomorphic Hermitian connection ∇^L on (L, h) .

If h is a singular Hermitian metric on L then (see [20], also [43, p. 97]) $h(e_\alpha, e_\alpha) = e^{-2\varphi_\alpha}$, where the functions $\varphi_\alpha \in L_{\text{loc}}^1(U_\alpha)$ are called the local weights of the metric h . One has $\varphi_\alpha = \varphi_\beta + \log |g_{\alpha\beta}|$ on $U_\alpha \cap U_\beta$, and the curvature of h ,

$$c_1(L, h)|_{U_\alpha} = dd^c \varphi_\alpha,$$

is a well defined closed (1,1) current on X . We say that the metric h is (semi)positively curved if $c_1(L, h)$ is a positive current. Equivalently, the weights φ_α can be chosen to be plurisubharmonic (psh) functions.

Let $L' \rightarrow X$ be a holomorphic line bundle isomorphic to L . A metric h^L on L induces a metric $h^{L'}$ on L' with curvature current $c_1(L, h^L) = c_1(L', h^{L'})$.

Suppose now that M is a complex manifold and $f : M \rightarrow X$ is a locally biholomorphic map. A metric h^L on L induces a metric f^*h^L on f^*L whose curvature current is $c_1(f^*L, f^*h^L) = f^*(c_1(L, h^L))$.

2.2. Desingularization

We recall here Hironaka's embedded resolution of singularities theorem (see e.g., [8], [43, Theorem 2.1.13]). Let X be a complex manifold and $\Sigma \subset X$ be a compact analytic subvariety of X . Then there exists a finite sequence of blow up maps $\sigma_{j+1} : X_{j+1} \rightarrow X_j$ with smooth centers Y_j ,

$$\begin{array}{cccccccc} X_m & \xrightarrow{\sigma_m} & X_{m-1} & \longrightarrow & \cdots & \longrightarrow & X_{j+1} & \xrightarrow{\sigma_{j+1}} & X_j & \longrightarrow & \cdots & \longrightarrow & X_1 & \xrightarrow{\sigma_1} & X_0 = X \\ \Sigma_m & & \Sigma_{m-1} & & & & \Sigma_{j+1} & & \Sigma_j & & & & \Sigma_1 & & \Sigma_0 = \Sigma \\ E_m & & E_{m-1} & & & & E_{j+1} & & E_j & & & & E_1 & & E_0 = \emptyset, \end{array}$$

such that:

- (i) Y_j is a compact submanifold of X_j with $\dim Y_j \leq \dim X - 2$ and $Y_j \subset \Sigma_j$,
- (ii) $\Sigma_{j+1} = \Sigma'_j$ is the strict transform of Σ_j by σ_{j+1} ,
- (iii) $E_{j+1} = E'_j \cup \sigma_{j+1}^{-1}(Y_j)$ is the set of exceptional hypersurfaces in X_{j+1} ,
- (iv) Σ_m is a smooth hypersurface and $\Sigma_m \cup E_m$ is a divisor with normal crossings.

Let $\tau_j = \sigma_1 \circ \cdots \circ \sigma_j : X_j \rightarrow X$. Since $\sigma_{j+1} : X_{j+1} \setminus E_{j+1} \rightarrow X_j \setminus (E_j \cup Y_j)$ is a biholomorphism, it follows that

$$\tau_m : X_m \setminus E_m \rightarrow \tau_m(X_m \setminus E_m) = X \setminus Y$$

is a biholomorphism, where

$$Y = Y_0 \cup \tau_1(Y_1) \cup \tau_2(Y_2) \cup \cdots \cup \tau_{m-1}(Y_{m-1}).$$

As $Y_j \subset \Sigma_j$ and $\sigma_j(\Sigma_j) \subset \Sigma_{j-1}$, we have $\tau_j(Y_j) \subset \Sigma$ for every $j = 1, \dots, m-1$. Since Y_j is compact $\tau_j : Y_j \rightarrow X$ is proper, so $\tau_j(Y_j)$ is an analytic subvariety of X of dimension $\leq \dim Y_j$. Hence Y is an analytic subvariety of X , $Y \subset \Sigma$ and $\dim Y \leq \dim X - 2$.

In conclusion, setting $\tilde{X} = X_m$, $E = E_m$, and $\pi = \tau_m : \tilde{X} \rightarrow X$, we have:

THEOREM 2.1 (Hironaka). – *Let X be a complex manifold and $\Sigma \subset X$ be a compact analytic subvariety of X . Then there exist a complex manifold \tilde{X} , an analytic subvariety $Y \subset \Sigma$ with $\dim Y \leq \dim X - 2$, and a proper surjective holomorphic map $\pi : \tilde{X} \rightarrow X$ with the following properties:*

- (i) $\pi : \tilde{X} \setminus E \rightarrow X \setminus Y$ is a biholomorphism, where $E = \pi^{-1}(Y)$;
- (ii) the strict transform $\Sigma' = \overline{\pi^{-1}(\Sigma \setminus Y)}$ is smooth and $\pi^{-1}(\Sigma) = \Sigma' \cup E$ is a divisor with normal crossings.

2.3. Special metrics

Let X be a complex manifold of dimension n . Assume that $L \rightarrow X$ is a holomorphic line bundle with a singular Hermitian metric h which is continuous outside a proper compact analytic subvariety $\Sigma \subset X$, and whose curvature $\gamma = c_1(L, h)$ is a *strictly positive* closed $(1,1)$ current on X (i.e., it dominates a smooth positive $(1,1)$ form on X). We write

$$\Sigma = Z_1 \cup Z_2,$$

where Z_1, Z_2 are analytic subvarieties of X , Z_1 has pure dimension $n-1$, and $\dim Z_2 \leq n-2$. Let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities of Σ as in Theorem 2.1. Then $\pi : \tilde{X} \setminus E \rightarrow X \setminus Y$

is a biholomorphism, where $Y \subset \Sigma$ is an analytic subvariety with $\dim Y \leq n - 2$, $E = \pi^{-1}(Y)$, $Z_2 \subset Y$, $\Sigma' = Z'_1$ is smooth, and $\pi^{-1}(\Sigma) = Z'_1 \cup E$ is a divisor with normal crossings.

2.3.1. *The metric Θ .* – We recall here the construction and properties of the generalized Poincaré metric on $X \setminus \Sigma$ (cf. [43, Lemma 6.2.1]). Let $\tilde{\Omega}$ be a smooth positive $(1,1)$ form on \tilde{X} . When X is not compact we take $\tilde{\Omega}$ so that the associated metric is complete on \tilde{X} .

Let $\Sigma_1, \dots, \Sigma_N$ be the irreducible components of $\pi^{-1}(\Sigma)$, so Σ_j is a smooth hypersurface in \tilde{X} . Let σ_j be a holomorphic section of the associated holomorphic line bundle $\mathcal{O}_{\tilde{X}}(\Sigma_j)$ vanishing to first order on Σ_j and let $|\cdot|_j$ be a smooth Hermitian metric on $\mathcal{O}_{\tilde{X}}(\Sigma_j)$ so that $|\sigma_j|_j < 1$ and $|\sigma_j|_j = 1/e$ outside a relatively compact open set containing $\pi^{-1}(\Sigma)$. We define

$$(4) \quad \tilde{\Theta}_\delta = \tilde{\Omega} + \delta dd^c F, \text{ where } \delta > 0, F = -\frac{1}{2} \sum_{j=1}^N \log(-\log |\sigma_j|_j).$$

If δ is small enough, $\tilde{\Theta}_\delta$ defines a complete Hermitian metric on $\tilde{X} \setminus \pi^{-1}(\Sigma)$ and we have $\tilde{\Theta}_\delta \geq \tilde{\Omega}/2$ in the sense of currents on \tilde{X} . Moreover, if X is compact then so is \tilde{X} and we have that $\tilde{\Theta}_\delta$ has finite volume (see [43, Lemma 6.2.1]). Fixing such a δ , we define the Poincaré metric on $X \setminus \Sigma$ as the metric associated to the $(1, 1)$ form

$$\Theta = (\pi^{-1})^* \tilde{\Theta}_\delta.$$

This metric has the same properties on $X \setminus \Sigma$ as $\tilde{\Theta}_\delta$ does on $\tilde{X} \setminus \pi^{-1}(\Sigma)$.

Let now $x \in \Sigma_{\text{reg}}^{n-1}$ and local coordinates z_1, \dots, z_n be chosen so that $x = 0, \Sigma = \{z_1 = 0\}$. Then $\Theta^n \sim (|z_1| \log |z_1|)^{-2} d\lambda$ near x , where λ is the Lebesgue measure in coordinates (see [43, (6.2.11)]). In particular, we have that $\Theta^n = f\Omega^n$, where the function f verifies assumption (C) stated in the introduction.

2.3.2. *The metric h_ε .* – It is necessary to perturb the original metric h of L in order to obtain a metric on $L|_{X \setminus \Sigma}$ whose curvature current dominates a small multiple of Θ . By [43, Lemma 6.2.2] there exists a holomorphic line bundle $\tilde{L} \rightarrow \tilde{X}$ which has a singular Hermitian metric $h^{\tilde{L}}$, continuous on $\tilde{X} \setminus \pi^{-1}(\Sigma)$, and such that $\tilde{L}|_{\tilde{X} \setminus E}$ is isomorphic to $\pi^*(L^k|_{X \setminus Y})$, for some $k \in \mathbb{N}$. Moreover, $c_1(\tilde{L}, h^{\tilde{L}}) = k\pi^*\gamma + \theta$ is a strictly positive current on \tilde{X} , where θ is a smooth real closed $(1, 1)$ form supported in a neighborhood of E and strictly positive along E .

Since $\tilde{L}|_{\tilde{X} \setminus E} \cong \pi^*(L^k|_{X \setminus Y})$ the metric $h^{\tilde{L}}$ induces a singular Hermitian metric $h^{L'}$ on $L' = \pi^*(L|_{X \setminus Y})$ with curvature current $\gamma' = \pi^*\gamma + \theta'$, where $\theta' = \theta/k$. For $\varepsilon > 0$,

$$h_\varepsilon^{L'} = h^{L'} \prod_{j=1}^N (-\log |\sigma_j|_j)^\varepsilon$$

is a singular Hermitian metric on L' with curvature current

$$\gamma'_\varepsilon = \gamma' + \varepsilon dd^c F = \pi^*\gamma + \theta' + \varepsilon dd^c F,$$

where F is given in (4). Since γ' is a strictly positive current it follows by the above choice of $|\sigma_j|_j$ that γ'_ε is a strictly positive current on \tilde{X} , provided that ε is sufficiently small (cf. [43, Lemma 6.2.1]). We fix such an ε and note that, as $\pi : \tilde{X} \setminus E \rightarrow X \setminus Y$ is a biholomorphism,

the metric $h_\varepsilon^{L'}$ on L' induces a singular metric h_ε on $L|_{X \setminus Y}$ which is continuous on $X \setminus \Sigma$. When X is compact the curvature current of h_ε dominates a small multiple of Θ on $X \setminus \Sigma$.

2.4. Wedge products of singular currents

We recall here a few facts that we need regarding the definition of complex Monge-Ampère operators. Let T be a positive closed current of bidimension (l, l) , $l > 0$, on an open set U in \mathbb{C}^n . The coefficients of T are complex Radon measures and their total variations are locally dominated, up to multiplicative constants, by the trace measure of T , $|T| = T \wedge \Omega^l$, where Ω is any fixed smooth positive $(1, 1)$ form on U . If u is a psh function on U so that $u \in L_{\text{loc}}^1(U, |T|)$ we say that the wedge product $dd^c u \wedge T$ is well defined. This is the positive closed current of bidimension $(l-1, l-1)$ defined by $dd^c u \wedge T = dd^c(uT)$.

If u_1, \dots, u_q are psh functions on U we say that $dd^c u_1 \wedge \dots \wedge dd^c u_q$ is well defined if one can define inductively as above all intermediate currents

$$dd^c u_{j_1} \wedge \dots \wedge dd^c u_{j_l} = dd^c(u_{j_1} dd^c u_{j_2} \wedge \dots \wedge dd^c u_{j_l}), \quad 1 \leq j_1 < \dots < j_l \leq q.$$

The wedge product is well defined for locally bounded psh functions [3, 4], for psh functions that are locally bounded outside a compact subset of a pseudoconvex open set U , or when the mutual intersection of their unbounded loci is small in a certain sense [57, 21, 30]. We recall here one such situation [21, Corollary 2.11]: if u_1, \dots, u_q are psh functions on U so that u_j is locally bounded outside an analytic subset A_j of U and $\text{codim}(A_{j_1} \cap \dots \cap A_{j_l}) \geq l$ for each l , $1 \leq j_1 < \dots < j_l \leq q$, then $dd^c u_1 \wedge \dots \wedge dd^c u_q$ is well defined. We also note that the natural domain of definition of the Monge-Ampère operator $u \rightarrow (dd^c u)^n$ is completely described in [9, 10, 12].

If T is a positive closed current of bidegree $(1, 1)$ on a complex manifold X then locally $T = dd^c u$ for a psh function u [41, Theorem 2.28]. Hence defining $T_1 \wedge \dots \wedge T_q$ for such currents T_j amounts to verifying locally one of the conditions mentioned above for their psh potentials u_j . We conclude this brief overview by noting that when X is compact the class of currents for which the wedge product can be globally defined so that it has good continuity properties is larger than the one for which it is well defined by local considerations as above (see e.g., [32, 33, 14]).

3. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. We start with a rather elementary property of the Bergman density function P_p in Lemma 3.1 and show in Lemma 3.2 that $\log P_p$ is, locally on X , the difference of two psh functions. Moreover, the Fubini-Study currents γ_p are well defined, and if the codimension of Σ is bigger than $k \geq 2$, then also the wedge products γ_p^k are well defined (Lemma 3.3). We continue with the crucial Theorem 3.4 about the local continuity properties of the Monge-Ampère operator. This result is of independent interest. With these preparations we can then prove Theorem 1.1.

For the convenience of the reader, we include a proof of the following properties of the function P_p in our setting.

LEMMA 3.1. – *If P_p is the Bergman density function defined in (1) then the definition is independent of the basis $\{S_j^p\}_{j \geq 1}$ and the function P_p is continuous on $X \setminus \Sigma$.*

Proof. – By the Riesz-Fischer theorem we have that $S \in H_{(2)}^0(X \setminus \Sigma, L^p)$ if and only if there exists a sequence $a = \{a_j\} \in l^2$ so that $S = S_a$, where $S_a = \sum_{j=1}^{\infty} a_j S_j^p$ and $\|S_a\|_p = \|a\|_2$.

Fix $x \in X \setminus \Sigma$ and a neighborhood $U_\alpha \Subset X \setminus \Sigma$ of x with a holomorphic frame e_α of L over U_α and so that $f \geq c > 0$ on U_α . We write $S_a = s_a e_\alpha^{\otimes p}$, $S_j^p = s_j^p e_\alpha^{\otimes p}$, and we let ψ_α be a continuous psh weight of h on U_α . It follows that $s_a = \sum_{j=1}^{\infty} a_j s_j^p$ and the series converges locally uniformly on U_α . As this holds for every sequence $a \in l^2$ we have that $\{s_j^p(z)\} \in l^2$ for all $z \in U_\alpha$.

We fix compacts K_i so that $x \in \text{int } K_1$, $K_1 \Subset K_2 \Subset K_3 \subset U_\alpha$. For $z \in K_2$ consider the sections $S_z = \sum_{j=1}^{\infty} \overline{s_j^p(z)} S_j^p \in H_{(2)}^0(X \setminus \Sigma, L^p)$, and write $S_z = s_z e_\alpha^{\otimes p}$. Then

$$\left(\sum_{j=1}^{\infty} |s_j^p(z)|^2 \right)^2 = |s_z(z)|^2 \leq C_1 \int_{K_3} |s_z|^2 e^{-2p\psi_\alpha} f \Omega^n \leq C_1 \|S_z\|_p^2 = C_1 \sum_{j=1}^{\infty} |s_j^p(z)|^2,$$

for some constant C_1 . This implies that

$$\sum_{j=1}^{\infty} |s_j^p(z)|^2 \leq C_1, \quad \forall z \in K_2.$$

We have

$$|s_j^p(y)|^2 \leq C_2 \int_{K_2} |s_j^p|^2 \Omega^n, \quad \forall y \in K_1,$$

where C_2 is a constant. Therefore

$$(5) \quad \sum_{j=1}^{\infty} \max_{K_1} |s_j^p|^2 \leq C_2 \int_{K_2} \left(\sum_{j=1}^{\infty} |s_j^p|^2 \right) \Omega^n \leq C_1 C_2 \int_{K_2} \Omega^n,$$

so the series $\sum_{j=1}^{\infty} |s_j^p|^2$ converges uniformly on K_1 . This shows that the function $P_p = \sum_{j=1}^{\infty} |s_j^p|^2 e^{-2p\psi_\alpha}$ is continuous near x .

To see that P_p does not depend on the choice of basis, observe that

$$P_p(x) = \max\{|S(x)|_{h_p}^2 : S \in H_{(2)}^0(X \setminus \Sigma, L^p), \|S\|_p = 1\}.$$

Indeed, using the above notations we have for $a \in l^2$ with $\|a\|_2 = 1$,

$$|S_a(x)|_{h_p}^2 = \left| \sum_{j=1}^{\infty} a_j s_j^p(x) \right|^2 e^{-2p\psi_\alpha(x)} \leq \left(\sum_{j=1}^{\infty} |s_j^p(x)|^2 \right) e^{-2p\psi_\alpha(x)} = P_p(x).$$

Moreover, if

$$a = \left\{ c^{-1} \overline{s_j^p(x)} \right\}_{j \geq 1}, \quad c := \left(\sum_{j=1}^{\infty} |s_j^p(x)|^2 \right)^{1/2},$$

then $\|a\|_2 = 1$, $S_a(x) = c e_\alpha^{\otimes p}$, so $|S_a(x)|_{h_p}^2 = P_p(x)$. \square

We start the proof of Theorem 1.1 with two lemmas.

LEMMA 3.2. – *If $X, \Sigma, (L, h), f, \Omega$ are as in Theorem 1.1 then:*

(i) $H_{(2)}^0(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$.

- (ii) γ_p extends to a positive closed current of bidegree $(1, 1)$ on X defined locally by Formula (2) and which is independent of the choice of basis $\{S_j^p\}$.
- (iii) $\log P_p \in L_{\text{loc}}^1(X, \Omega^n)$ and $dd^c \log P_p = 2\gamma_p - 2p\psi$ as currents on X .

Proof. – (i) Let $x \in \Sigma_{\text{reg}}^{n-1}$ and let e_α be a holomorphic frame of L on a neighborhood U_α of x . A section $S \in H_{(2)}^0(X \setminus \Sigma, L^p)$ can be written on U_α as $S = se_\alpha^{\otimes p}$ where s is a holomorphic function on $U_\alpha \setminus \Sigma$. We may assume that h has a psh weight ψ_α which is bounded above on U_α and that $f \geq c > 0$ on U_α for some constant c . Then

$$\int_{U_\alpha \setminus \Sigma} |s|^2 \Omega^n \leq C \int_{U_\alpha \setminus \Sigma} |s|^2 e^{-2p\psi_\alpha} f \Omega^n \leq C \|S\|_p^2 < \infty.$$

By Skoda’s lemma [43, Lemma 2.3.22], this implies that S extends holomorphically near x .

Thus any section $S \in H_{(2)}^0(X \setminus \Sigma, L^p)$ extends holomorphically to a section of L^p over $X \setminus Y$, where $Y = \Sigma \setminus \Sigma_{\text{reg}}^{n-1}$, and hence to a holomorphic section of L^p since Y is an analytic subvariety of X of codimension ≥ 2 .

(ii) Let U_α be an open set in X on which there exists a holomorphic frame e_α of L and set $S_j^p = s_j^p e_\alpha^{\otimes p}$, where by (i) s_j^p are holomorphic on U_α . Let

$$u_p := \log \left(\sum_{j=1}^\infty |s_j^p|^2 \right) \text{ on } U_\alpha.$$

It follows by (5) that the series $\sum_{j=1}^\infty |s_j^p|^2$ converges locally uniformly on $U_\alpha \setminus \Sigma$. If $y \in \Sigma \cap U_\alpha$ we fix local coordinates (z_1, \dots, z_n) so that $y = 0$ and $U_\alpha \cap \Sigma$ is contained in the cone $\{|z_n| \leq \max(|z_1|, \dots, |z_{n-1}|)\}$. Applying the maximum principle on complex lines parallel to the z_n axis, we see that there exist a neighborhood $V \subset U_\alpha$ of y and a compact set $K \subset U_\alpha \setminus \Sigma$ so that $\sup_{z \in V} |s_j^p|^2 \leq \max_{z \in K} |s_j^p|^2$. By (5),

$$\sum_{j=1}^\infty \sup_V |s_j^p|^2 \leq \sum_{j=1}^\infty \max_K |s_j^p|^2 < +\infty.$$

We conclude that the series $\sum_{j=1}^\infty |s_j^p|^2$ converges locally uniformly on U_α , hence u_p is a psh function and γ_p is a positive closed current on X defined locally by (2). Since the function P_p is independent on the choice of basis $\{S_j^p\}$, so is the current γ_p .

(iii) If U_α is as above, then by (ii) u_p is psh on U_α and $\log P_p = u_p - 2p\psi_\alpha$, Ω^n -a.e. on U_α . Since psh functions are locally integrable (see e.g., [37, Corollary 3.2.8]), this implies that $\log P_p \in L_{\text{loc}}^1(X, \Omega^n)$. □

LEMMA 3.3. – *If $\dim \Sigma \leq n - k$ for some $2 \leq k \leq n$ and Hypothesis (3) holds then the currents γ^k and γ_p^k are well defined on X , respectively on each relatively compact neighborhood of Σ , for all p sufficiently large.*

Proof. – The current γ^k is well defined by [21, Corollary 2.11], since $\dim \Sigma \leq n - k$.

Let $A_p = \{x \in X : S_j^p(x) = 0, \forall j \geq 1\}$. Lemma 3.2 shows that the current γ_p has local psh potentials which are continuous away from $A_p \cup \Sigma$. By [21, Corollary 2.11], it suffices to show that given any relatively compact neighborhood U of Σ we have $\dim(A_p \cap U) \leq n - k$ for all p sufficiently large.

Assuming the contrary, there exist $m > n - k$ and a sequence $p_j \rightarrow \infty$ so that each analytic set $A_{p_j} \cap U$ has an irreducible component Y_j of dimension m . It follows from (3) that, given any ε -neighborhood V_ε of Σ , $Y_j \subset A_{p_j} \cap U \subset V_\varepsilon$ for all j sufficiently large, hence Y_j are compact. Let $|Y_j| = \int_{Y_j} \Omega^m$ and $T_j = [Y_j]/|Y_j|$, where $[Y_j]$ denotes the current of integration on Y_j . Since T_j have unit mass, we may assume by passing to a subsequence that T_j converges weakly to a positive closed current T of bidimension (m, m) . But T is supported by Σ , so $T = 0$ by Federer's support theorem as $\dim \Sigma \leq n - k < m$ (see e.g., [35, Theorem 1.7] and references therein). On the other hand $\langle T, \Omega^m \rangle = \lim_{j \rightarrow \infty} \langle T_j, \Omega^m \rangle = 1$, a contradiction. \square

We will need the following local property of the complex Monge-Ampère operator:

THEOREM 3.4. – *Let U be an open set in \mathbb{C}^n , Σ be a proper analytic subvariety of U , and v be a psh function on U which is continuous on $U \setminus \Sigma$. Assume that $v_p, p \geq 1$, are psh functions on U so that $v_p \rightarrow v$ locally uniformly on $U \setminus \Sigma$. Then:*

- (i) *The sequence $\{v_p\}$ is locally uniformly upper bounded in U .*
- (ii) *Assume that $\dim \Sigma \leq n - k$ and the currents $(dd^c v_p)^k$ are well defined on U for some $k \geq 1$. Then $(dd^c v_p)^k \rightarrow (dd^c v)^k$ weakly in the sense of currents on U .*

Proof. – (i) The sequence $\{v_p\}$ is clearly locally uniformly upper bounded in $U \setminus \Sigma$. If $x \in \Sigma$ we may assume that there exist coordinates (z_1, \dots, z_n) on some neighborhood V of $x = 0$ so that $V \cap \Sigma$ is contained in the cone $\{|z_n| \leq \max(|z_1|, \dots, |z_{n-1}|)\}$. Applying the maximum principle on complex lines parallel to the z_n axis, we see that there exist a neighborhood $V_1 \subset V$ of x and a compact set $K \subset V \setminus \Sigma$ so that $\sup_{V_1} v_p \leq \sup_K v_p$. Hence $\{v_p\}$ is uniformly upper bounded on V_1 .

(ii) Recall that the current $(dd^c v)^k$ is well defined on U since $\dim \Sigma \leq n - k$ [21, Corollary 2.11]. Since $v_p \rightarrow v$ locally uniformly on $U \setminus \Sigma$ and v is continuous there we have that $(dd^c v_p)^k \rightarrow (dd^c v)^k$ weakly in the sense of currents on $U \setminus \Sigma$ (see e.g., [3, 4], also [21, Corollary 1.6]). We divide the proof into three steps.

Step 1. – We prove here assertion (ii) when $k = n$. Then Σ consists of isolated points of U . Let $x \in \Sigma$ and $\chi \geq 0$ be a smooth function with compact support in U so that $\chi = 1$ near x and $\text{supp } \chi \cap \Sigma = \{x\}$. Then

$$\int \chi (dd^c v_p)^n = \int v_p (dd^c v_p)^{n-1} \wedge dd^c \chi \rightarrow \int v (dd^c v)^{n-1} \wedge dd^c \chi = \int \chi (dd^c v)^n,$$

since $v_p \rightarrow v$ locally uniformly in a neighborhood of $\text{supp } dd^c \chi$ and v is continuous there [21, Corollary 1.6]. This shows that the sequence of positive measures $(dd^c v_p)^n$ has locally bounded mass on U and that if ν is any weak limit point of this sequence then $\nu(\{x\}) = (dd^c v)^n(\{x\})$ for each $x \in \Sigma$. It follows that $(dd^c v_p)^n \rightarrow (dd^c v)^n$ weakly in the sense of measures on U .

We assume in the sequel that $1 \leq k \leq n - 1$.

Step 2. – We show that the currents $(dd^c v_p)^k$ have locally uniformly bounded mass in U (see e.g., [30] for the definition of mass of positive, or negative, currents). Note that we only have to show this near points $x \in \Sigma$. The proof is quite standard in the case $k = 1$ and when $k > 1$ it follows from Oka’s inequality for currents due to Fornæss and Sibony [30].

Consider first the case $k = 1$. Fix $V \subset U$ a relatively compact neighborhood of x and compacts $K_j \subset V$ so that $x \in \text{int } K_1$, $K_1 \subset \text{int } K_2$, and $K_3 \subset V \setminus \Sigma$ is any compact set of positive Lebesgue measure. Subtracting a constant we may assume that $v_p, v < 0$ on V . There exists a constant $C(K_1, K_2)$ so that $\|dd^c v_p\|_{K_1} \leq C(K_1, K_2) \int_{K_2} |v_p|$ for every p (see e.g., [21, Remark 1.4]). Here $\|dd^c v_p\|_{K_1}$ denotes the mass of the current $dd^c v_p$ on the compact set K_1 . By [37, Theorem 3.2.12], the family of psh functions u on V so that $u < 0$ and $\int_{K_3} |u| = 1$ is compact in $L^1_{\text{loc}}(V)$. Hence there exists a constant $C(K_2, K_3)$ so that $\int_{K_2} |v_p| \leq C(K_2, K_3) \int_{K_3} |v_p|$ for every p . We conclude that the currents $dd^c v_p$ have uniformly bounded mass on K_1 .

Assume next that $2 \leq k \leq n - 1$. Let x be a regular point of Σ so that $\dim_x \Sigma = n - k$. By a change of coordinates near x we may assume that

$$x = (1/2, \dots, 1/2) \in \bar{\Delta}^n \subset U, \quad \Sigma \cap \bar{\Delta}^n = \{z = (z_1, \dots, z_n) : z_1 = \dots = z_k = 1/2\},$$

where Δ is the unit disc in \mathbb{C} . We may also assume that $v_p, v < 0$ near $\bar{\Delta}^n$. Consider the $(k - 1, n - k + 1)$ Hartogs figure

$$H = \{(z', z'') \in \mathbb{C}^{k-1} \times \mathbb{C}^{n-k+1} : \|z'\| \leq 1, \|z''\| \leq 1/4\} \\ \cup \{(z', z'') \in \mathbb{C}^{k-1} \times \mathbb{C}^{n-k+1} : 3/4 \leq \|z'\| \leq 1, \|z''\| \leq 1\},$$

where $\|z'\| = \max(|z_1|, \dots, |z_{k-1}|)$. The current $T = v_p(dd^c v_p)^{k-1}$ is a negative current near $\bar{\Delta}^n$ of bidegree $(k - 1, k - 1)$ and $dd^c T = (dd^c v_p)^k \geq 0$. By Oka’s inequality applied to T [30, Theorem 2.4],

$$\|v_p(dd^c v_p)^{k-1}\|_K + \|(dd^c v_p)^k\|_K \leq C \|v_p(dd^c v_p)^{k-1}\|_H$$

for some absolute constant C , where $K = \bar{\Delta}_{3/4}^n$ is the polydisc of radius $3/4$. Note that $x \in \text{int } K$. Since $H \cap \Sigma = \emptyset$ we have $v_p(dd^c v_p)^{k-1} \rightarrow v(dd^c v)^{k-1}$ near H [21, Corollary 1.6]. It follows that $\|(dd^c v_p)^k\|_K$ are uniformly bounded.

Therefore we showed that the currents $(dd^c v_p)^k$ have locally bounded mass on $U \setminus Y$, where $Y \subset \Sigma$ is an analytic set of codimension $\geq k + 1$. Oka’s inequality applied to the currents $(dd^c v_p)^k$ implies that they have locally uniformly bounded mass near each $y \in Y$ (see also [30, Corollary 2.6]).

Step 3. – We now prove that $(dd^c v_p)^k \rightarrow (dd^c v)^k$ weakly on U . Since the currents $(dd^c v_p)^k$ have locally uniformly bounded mass on U , it suffices to prove that any weak limit point T of $(dd^c v_p)^k$ is equal to $(dd^c v)^k$. Let us write

$$\Sigma = Y \cup \bigcup_{j \geq 1} Y_j,$$

where Y_j are the irreducible components of dimension $n - k$ and $\dim Y \leq n - k - 1$. Recall that $T = (dd^c v)^k$ on $U \setminus \Sigma$. Hence by Federer’s support theorem ([26], see also [35, Theorem 1.7]), $T = (dd^c v)^k$ on $D = U \setminus \bigcup_{j \geq 1} Y_j$, since Y is an analytic subvariety of D of dimension $\leq n - k - 1$.

By Siu's decomposition formula ([58], see also [21, Theorem 6.19]) we write

$$T = R + \sum_{j \geq 1} c_j [Y_j], \quad (dd^c v)^k = S + \sum_{j \geq 1} d_j [Y_j],$$

where $[Y_j]$ denotes the current of integration on Y_j , $c_j, d_j \geq 0$, and R, S are positive closed currents of bidegree (k, k) on U which do not charge any Y_j (i.e., the trace measure of R is 0 on Y_j). It follows by above that $R = S$. To conclude the proof we have to show that $c_j = d_j$ for each j . This will be done using slicing.

Without loss of generality, let $j = 1$ and $x \in Y_1$ be a regular point of Σ . By a change of coordinates $z = (z', z'')$ near x we may assume that $x = 0 \in \overline{\Delta^n} \subset U$ and $\Sigma \cap \Delta^n = Y_1 \cap \Delta^n = \{z' = 0\}$, where $z' = (z_1, \dots, z_k)$, $z'' = (z_{k+1}, \dots, z_n)$. Since $v_p \rightarrow v$ locally uniformly on $U \setminus \Sigma$ and v is continuous there, it follows that for each $z'' \in \Delta^{n-k}$ the functions $v_{p, z''}(z') = v_p(z', z'')$, $v_{z''}(z') = v(z', z'')$, are locally bounded near the boundary of Δ^k , so their Monge-Ampère measures $(dd^c v_{p, z''})^k$, $(dd^c v_{z''})^k$ are well defined on Δ^k (see [21, Corollary 2.3]). Arguing as in the proof of Step 1, it follows that $(dd^c v_{p, z''})^k \rightarrow (dd^c v_{z''})^k$ weakly on Δ^k as $p \rightarrow \infty$, for each $z'' \in \Delta^{n-k}$.

Let $\chi_1(z') \geq 0$ (resp. $\chi_2(z'') \geq 0$) be a smooth function with compact support in Δ^k (resp. Δ^{n-k}) so that $\chi_1 = 1$ near $0 \in \mathbb{C}^k$ (resp. $\chi_2 = 1$ near $0 \in \mathbb{C}^{n-k}$). Let $\beta = i/2 \sum_{j=k+1}^n dz_j \wedge d\bar{z}_j$ be the standard Kähler form in \mathbb{C}^{n-k} . One has the slicing formula (see e.g., [25, formula (2.1)])

$$\int_{\Delta^n} \chi_1(z') \chi_2(z'') (dd^c v_p)^k \wedge \beta^{n-k} = \int_{\Delta^{n-k}} \left(\int_{\Delta^k} \chi_1(z') (dd^c v_{p, z''})^k \right) \chi_2(z'') \beta^{n-k},$$

and similarly for $(dd^c v)^k$. Note that

$$\int \chi_1 (dd^c v_{p, z''})^k = \int v_{p, z''} (dd^c v_{p, z''})^{k-1} \wedge dd^c \chi_1.$$

Since $dd^c \chi_1$ is supported away from Σ , the Chern-Levine-Nirenberg estimates [21, (1.3)] imply that this integral is locally uniformly bounded as a function of z'' . Letting $p \rightarrow \infty$ we infer by above that

$$\int_{\Delta^n} \chi_1(z') \chi_2(z'') T \wedge \beta^{n-k} = \int_{\Delta^n} \chi_1(z') \chi_2(z'') (dd^c v)^k \wedge \beta^{n-k}.$$

By Siu's decomposition formulas of T and $(dd^c v)^k$, and since $R = S$, this implies that

$$c_1 \int_{\{z'=0\}} \chi_2(z'') \beta^{n-k} = d_1 \int_{\{z'=0\}} \chi_2(z'') \beta^{n-k}.$$

As $\int_{\{z'=0\}} \chi_2(z'') \beta^{n-k} > 0$ we see that $c_1 = d_1$, and the proof is complete. \square

We finish now the proof of Theorem 1.1 by showing that $\frac{1}{p^k} \gamma_p^k \rightarrow \gamma^k$ weakly on X . Since this is local, we fix $x \in X$ and let U_α be a relatively compact neighborhood of x such that there exists a holomorphic frame e_α of L over U_α . Let ψ_α be a psh weight of h on U_α and let

$$v_p = \frac{1}{2p} \log \left(\sum_{j=1}^{\infty} |s_j^p|^2 \right), \quad \text{where } S_j^p = s_j^p e_\alpha^{\otimes p}, \quad s_j^p \in \mathcal{O}_X(U_\alpha).$$

By Lemma 3.2 the function v_p is psh on U_α and we have $\frac{1}{p} \gamma_p = dd^c v_p$, $\gamma = dd^c \psi_\alpha$. Moreover, Lemma 3.3 shows that the currents $(dd^c v_p)^k$ are well defined on U_α for all p

sufficiently large. Note that ψ_α is continuous on $U_\alpha \setminus \Sigma$. Since $\frac{1}{2p} \log P_p = v_p - \psi_\alpha$, Hypothesis (3) implies that $v_p \rightarrow \psi_\alpha$ locally uniformly on $U_\alpha \setminus \Sigma$. It follows by Theorem 3.4 that $\frac{1}{p^k} \gamma_p^k = (dd^c v_p)^k \rightarrow (dd^c \psi_\alpha)^k = \gamma^k$ weakly on U_α . This concludes the proof of Theorem 1.1.

REMARK 3.5. – *In the setting of Theorem 1.1, assume that $\dim \Sigma \leq n - k$ and that (3) holds. The proof of Lemma 3.3 shows that all currents $\gamma_p^j \wedge \gamma^l$, $j + l \leq k$ are well defined positive closed currents on X . By Lemma 3.2 $\log P_p \in L_{\text{loc}}^1(X, \Omega^n)$ and $dd^c \log P_p = 2\gamma_p - 2p\gamma$ is a current of order 0 on X . It follows that $(dd^c \log P_p)^j$, $j \leq k$, are currents of order 0 on X which can be defined inductively by*

$$(dd^c \log P_p)^{j+1} = dd^c (\log P_p (dd^c \log P_p)^j), \quad j < k,$$

since locally, $\log P_p$ is integrable with respect to the measure coefficients of $(dd^c \log P_p)^j$. Moreover, we have

$$\left(\frac{1}{2p} dd^c \log P_p \right)^k = \left(\frac{1}{p} \gamma_p - \gamma \right)^k = \sum_{j=0}^k \binom{k}{j} \frac{(-1)^{k-j}}{p^j} \gamma_p^j \wedge \gamma^{k-j}.$$

A straightforward adaptation of the proof of Theorems 1.1 and 3.4 shows that

$$p^{-j} \gamma_p^j \wedge \gamma^{k-j} \rightarrow \gamma^k, \quad \text{as } p \rightarrow \infty,$$

weakly on X . Hence $p^{-j} (dd^c \log P_p)^j \rightarrow 0$ as $p \rightarrow \infty$ in the weak sense of currents of order 0 on X , for all $1 \leq j \leq k$.

REMARK 3.6. – *Observe that the hypothesis $f \geq c_x > 0$ Ω^n -a. e. in a neighborhood U_x of each $x \in \Sigma_{\text{reg}}^{n-1}$ was only needed in the proof of Lemma 3.2 (i), i.e., to show that $H_{(2)}^0(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$. Therefore, Theorem 1.1 also holds provided that $X, \Sigma, (L, h), f, \Omega$ verify assumptions (A), (B), (C') and (D), where:*

- (C') *The volume form on $X \setminus \Sigma$ is $f \Omega^n$, where $f \in L_{\text{loc}}^1(X \setminus \Sigma, \Omega^n)$ verifies $f \geq c_x > 0$ Ω^n -a. e. in a neighborhood U_x of each $x \in X \setminus \Sigma$.*
- (D) *$H_{(2)}^0(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$ for every $p \geq 1$.*

This variant of Theorem 1.1 will be useful to us for some applications in Section 6, where the fact that the sections in $H_{(2)}^0(X \setminus \Sigma, L^p)$ extend holomorphically to sections of L^p over X is known to hold by other considerations (see Sections 6.3 and 6.4).

4. Distribution of zeros of random sections

The purpose of this section is to give the proof of Theorem 1.2. As a consequence we show in Theorem 4.3 that zeros of random holomorphic sections are equidistributed with respect to the curvature current.

Let $X, \Sigma, (L, h), f, \Omega$ verify assumptions (A)-(C) stated in the introduction. In addition, we assume in this section that X is compact. By Lemma 3.2, $H_{(2)}^0(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$. Let

$$d_p = \dim H_{(2)}^0(X \setminus \Sigma, L^p), \quad \{S_j^p\}_{1 \leq j \leq d_p} \text{ a fixed orthonormal basis of } H_{(2)}^0(X \setminus \Sigma, L^p).$$

The currents γ_p can now be described as pullbacks $\gamma_p = \Phi_p^*(\omega_{FS})$, where $\Phi_p : X \dashrightarrow \mathbb{P}^{d_p-1}$ is the Kodaira map defined by $\{S_j^p\}$ and ω_{FS} is the Fubini-Study form on \mathbb{P}^{d_p-1} . Recall that if $S_j^p = s_j^p e_\alpha^{\otimes p}$ where e_α is a holomorphic frame for L on an open set U_α then

$$\Phi_p(x) = [s_1^p(x) : \dots : s_{d_p}^p(x)], \quad x \in U_\alpha.$$

Following the framework in [54], we identify the unit sphere ϕ^p of $H_{(2)}^0(X \setminus \Sigma, L^p)$ to the unit sphere \mathbf{S}^{2d_p-1} in \mathbb{C}^{d_p} by

$$a = (a_1, \dots, a_{d_p}) \in \mathbf{S}^{2d_p-1} \longrightarrow S_a = \sum_{j=1}^{d_p} a_j S_j^p \in \phi^p,$$

and we let λ_p be the probability measure on ϕ^p induced by the normalized surface measure on \mathbf{S}^{2d_p-1} , denoted also by λ_p (i.e., $\lambda_p(\mathbf{S}^{2d_p-1}) = 1$). We let λ_p^k denote the product measure on $(\phi^p)^k$ determined by λ_p . Given a nontrivial section $S \in H^0(X, L^p)$ we denote by $[S = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{S = 0\}$ of X .

We give now the proof of Theorem 1.2. Let us note that some of the main ideas involved in proving this theorem are similar to those in [54, 56], however special attention has to be given as we have to work with currents rather than smooth forms and the subspaces of sections we consider have nonempty base locus. To prove assertion (i) we will need the following version of Bertini’s theorem:

PROPOSITION 4.1. – *Let $L \longrightarrow X$ be a holomorphic line bundle over a compact complex manifold X of dimension n . Assume that:*

- (i) V is a vector subspace of $H^0(X, L)$ with basis S_1, \dots, S_d , and with base locus $\text{Bs}(V) = \{S_1 = \dots = S_d = 0\} \subset X$ so that $\dim \text{Bs}(V) \leq n - k$.
- (ii) $Z(h) := \{x \in X : \sum_{j=1}^d h_j S_j(x) = 0\}$, where $h = [h_1 : \dots : h_d] \in \mathbb{P}^{d-1}$.
- (iii) ν_l is the product measure on $(\mathbb{P}^{d-1})^l$ induced by the Fubini-Study volume μ_{d-1} on \mathbb{P}^{d-1} .

Then for ν_k -a.e. $(h^1, \dots, h^k) \in (\mathbb{P}^{d-1})^k$ we have that $\dim Z(h^{i_1}) \cap \dots \cap Z(h^{i_l}) \leq n - l$ for every $l \leq k$ and $i_1 < \dots < i_l$ in $\{1, \dots, k\}$.

The proof is included at the end of this section for the convenience of the reader, since we could not find it in the literature. Assertion (ii) of Theorem 1.2 is proved by repeated application of the following proposition:

PROPOSITION 4.2. – *Let $L \longrightarrow X, V, S_1, \dots, S_d$, be as in Proposition 4.1. Assume that:*

- (i) $\phi := \left\{ \sum_{j=1}^d a_j S_j : \sum_{j=1}^d |a_j|^2 = 1 \right\}$ is endowed with the probability measure λ induced via the natural identification by the normalized surface measure on \mathbf{S}^{2d-1} .
- (ii) $\beta := \Phi^*(\omega_{FS})$, where $\Phi : X \dashrightarrow \mathbb{P}^{d-1}$ is the Kodaira map defined by $\{S_j\}$.
- (iii) T is a positive closed current on X of bidimension (l, l) , $l > 0$, such that the current $[S = 0] \wedge T$ is well defined for λ -a.e. $S \in \phi$.

Then the current $\beta \wedge T$ is well defined on X . Moreover, if φ is a smooth $(l-1, l-1)$ form on X the function $S \rightarrow \langle [S = 0] \wedge T, \varphi \rangle$ is in $L^1(\phi, \lambda)$ and

$$\int_{\phi} \langle [S = 0] \wedge T, \varphi \rangle d\lambda(S) = \langle \beta \wedge T, \varphi \rangle.$$

We postpone for the time being the proof of Proposition 4.2, and we continue with the proof of Theorem 1.2.

Proof of Theorem 1.2. – (i) Lemma 3.3 (and its proof) show that

$$\dim \text{Bs}(H_{(2)}^0(X \setminus \Sigma, L^p)) \leq n - k,$$

for all p sufficiently large. It follows from Prop. 4.1 that, for λ_p^k -a.e. $\sigma = (\sigma_1, \dots, \sigma_k) \in (\varphi^p)^k$, the analytic subset $\{\sigma_{i_1} = 0\} \cap \dots \cap \{\sigma_{i_l} = 0\}$ has dimension $\leq n - l$ for every $l \leq k$ and $i_1 < \dots < i_l$ in $\{1, \dots, k\}$. Therefore the current $[\sigma = 0]$ is well defined [21, Corollary 2.11], and it equals the current of integration with multiplicities along $\{\sigma_1 = 0\} \cap \dots \cap \{\sigma_k = 0\}$ [21, Proposition 2.12].

(ii) One can show that the function $\sigma \in (\varphi^p)^k \rightarrow \langle [\sigma = 0], \varphi \rangle$ is measurable by an approximation argument similar to that in the proof of Proposition 4.2.

Let $\sigma = (\sigma_1, \dots, \sigma_k) \in (\varphi^p)^k$ be so that the analytic subset $\{\sigma_{i_1} = 0\} \cap \dots \cap \{\sigma_{i_l} = 0\}$ has dimension $\leq n - l$ for every $l \leq k$ and $i_1 < \dots < i_l$ in $\{1, \dots, k\}$. Corollary 2.11 in [21] and the considerations from (i) show that $[\sigma_{i_1} = 0] \wedge \dots \wedge [\sigma_{i_l} = 0] \wedge \gamma_p$ is a well defined positive closed current of bidegree $(l + 1, l + 1)$ on X , for every $i_1 < \dots < i_l$ in $\{1, \dots, k\}$, $l < k$.

By adding to φ a large multiple of Ω^{n-k} we may assume that φ is a strongly positive $(n - k, n - k)$ test form on X . Hence the integral in (ii) can be evaluated as an iterated integral by Tonelli's theorem (see e.g., [27, Theorem 2.37]). We apply Proposition 4.2 with

$$V = H_{(2)}^0(X \setminus \Sigma, L^p), \quad T = [\sigma_2 = 0] \wedge \dots \wedge [\sigma_k = 0].$$

Then for λ_p^{k-1} -a.e. $(\sigma_2, \dots, \sigma_k) \in (\varphi^p)^{k-1}$,

$$\int_{\varphi^p} \langle [\sigma = 0], \varphi \rangle d\lambda_p(\sigma_1) = \langle T \wedge \gamma_p, \varphi \rangle = \langle [\sigma_2 = 0] \wedge \dots \wedge [\sigma_k = 0] \wedge \gamma_p, \varphi \rangle,$$

since $[\sigma = 0] = [\sigma_1 = 0] \wedge T$. Proposition 4.1 shows that Proposition 4.2 can be applied again for λ_p^{k-2} -a.e. $(\sigma_3, \dots, \sigma_k) \in (\varphi^p)^{k-2}$ with $T = [\sigma_3 = 0] \wedge \dots \wedge [\sigma_k = 0] \wedge \gamma_p$, so

$$\begin{aligned} \int_{\varphi^p} \int_{\varphi^p} \langle [\sigma = 0], \varphi \rangle d\lambda_p(\sigma_1) d\lambda_p(\sigma_2) &= \int_{\varphi^p} \langle [\sigma_2 = 0] \wedge \dots \wedge [\sigma_k = 0] \wedge \gamma_p, \varphi \rangle d\lambda_p(\sigma_2) \\ &= \langle [\sigma_3 = 0] \wedge \dots \wedge [\sigma_k = 0] \wedge \gamma_p^2, \varphi \rangle. \end{aligned}$$

Continuing like this we obtain that the k th iterated integral in (ii) equals $\langle \gamma_p^k, \varphi \rangle$. This proves (ii), and then (iii) follows at once from Theorem 1.1. \square

Let us now consider the probability space $\mathcal{I}_\infty = \prod_{p=1}^\infty \varphi^p$ endowed with the probability measure $\lambda_\infty = \prod_{p=1}^\infty \lambda_p$. The proof of the variance estimate from Lemma 3.3 in [54] goes through with no change. Combined with Theorem 1.2, it implies that Theorem 1.1 of [54] holds in our setting. Namely, we have the following:

THEOREM 4.3. – *In the setting of Theorem 1.1, assume that X is compact and that (3) holds. Then, in the weak sense of currents on X ,*

$$\lim_{p \rightarrow \infty} \frac{1}{p} [\sigma_p = 0] = \gamma, \text{ for } \lambda_\infty\text{-a.e. sequence } \{\sigma_p\}_{p \geq 1} \in \mathcal{I}_\infty.$$

Note that X is not assumed to be Kähler in Theorem 4.3. Since X is compact, there exists a Gauduchon form on X , i.e., a positive $(n - 1, n - 1)$ form $\tilde{\Omega}$ such that $dd^c\tilde{\Omega} = 0$, cf. [31]. Hence

$$\int_X [\sigma_p = 0] \wedge \tilde{\Omega} = \int_X \gamma_p \wedge \tilde{\Omega} = p \int_X \gamma \wedge \tilde{\Omega}.$$

Then Theorem 4.3 can be proved by the same arguments as those in the proof of [54, Theorem 1.1].

Proof of Proposition 4.2. – We fix a holomorphic frame e_α of L over an open set U_α , and write $S = se_\alpha$, $S_j = s_j e_\alpha$, where $S = \sum_{j=1}^d a_j S_j \in \mathcal{J}$ is chosen so that the current $[S = 0] \wedge T$ is well defined, hence $\log |s| \in L^1_{\text{loc}}(U_\alpha, |T|)$ (see Section 2.4). Then

$$\log |s| = \log \left| \sum_{j=1}^d a_j s_j \right| \leq \frac{1}{2} \log \left(\sum_{j=1}^d |s_j|^2 \right).$$

Since the latter function is locally bounded above on U_α and $\log |s| \in L^1_{\text{loc}}(U_\alpha, |T|)$, we conclude that

$$(6) \quad \log \left(\sum_{j=1}^d |s_j|^2 \right) \in L^1_{\text{loc}}(U_\alpha, |T|),$$

so $\beta \wedge T$ is well defined.

To show that the function $\mathcal{J} \ni S \mapsto \langle [S = 0] \wedge T, \varphi \rangle$ is measurable, we may assume by using a partition of unity that φ is supported in U_α . Since $[S = 0] \wedge T$ is well defined for λ -a.e. $S \in \mathcal{J}$ and the sequence of smooth psh functions $\log(|s|^2 + 1/m)^{1/2}$ decreases to $\log |s|$ as $m \rightarrow \infty$, we have

$$\langle [S = 0] \wedge T, \varphi \rangle = \langle \log |s| \cdot T, dd^c \varphi \rangle = \lim_{m \rightarrow \infty} \langle T, \log(|s|^2 + 1/m)^{1/2} dd^c \varphi \rangle.$$

Our claim follows as $S \in \mathcal{J} \rightarrow \langle T, \log(|s|^2 + 1/m)^{1/2} dd^c \varphi \rangle$ are continuous functions.

For $S \in \mathcal{J}$ we define the function $N(S)$ on X by

$$N(S)|_{U_\alpha} = \log \frac{|s|}{\sqrt{|s_1|^2 + \dots + |s_d|^2}}.$$

Note that $N(S) \in L^1(X, \Omega^n)$, where Ω is a smooth positive $(1, 1)$ form on X , since it is locally the difference of psh functions. Moreover, Hypothesis (iii) in the statement of Proposition 4.2 and (6) imply that $N(S) \in L^1(X, |T|)$ for λ -a.e. $S \in \mathcal{J}$. Therefore we have

$$[S = 0] = \beta + dd^c N(S), \quad [S = 0] \wedge T = \beta \wedge T + dd^c(N(S)T).$$

Indeed, the first formula follows from the definition of the function $N(S)$, while for the second, working locally on U_α , we have

$$\begin{aligned} [S = 0] \wedge T &= dd^c(\log |s| T) = dd^c \left(\log \sqrt{|s_1|^2 + \dots + |s_d|^2} T \right) + dd^c(N(S)T) \\ &= \beta \wedge T + dd^c(N(S)T). \end{aligned}$$

Thus, for λ -a.e. $S \in \mathcal{J}$,

$$\langle [S = 0] \wedge T, \varphi \rangle = \langle \beta \wedge T, \varphi \rangle + \int_X N(S) T \wedge dd^c \varphi,$$

and the proof is finished if we show that the function $S \rightarrow \int_X N(S) T \wedge dd^c \varphi$ belongs to $L^1(\mathcal{J}, \lambda)$ and

$$\int_{\mathcal{J}} \left(\int_X N(S) T \wedge dd^c \varphi \right) d\lambda(S) = 0.$$

We may assume that φ is real, so $dd^c \varphi$ is a real (l, l) form on X . There exists a constant M so that $dd^c \varphi + M\Omega^l$ is a strongly positive (l, l) form, so $T \wedge (dd^c \varphi + M\Omega^l)$ is a positive measure. It follows that we can write

$$T \wedge dd^c \varphi = \mu_1 - \mu_2, \text{ where } \mu_1 = T \wedge (dd^c \varphi + M\Omega^l), \mu_2 = M T \wedge \Omega^l.$$

Note that μ_j are positive measures dominated up to multiplicative constants by $|T|$. Since $N(S) \in L^1(X, |T|)$ for λ -a.e. $S \in \mathcal{J}$ we see that $N(S) \in L^1(X, \mu_j)$ for λ -a.e. $S \in \mathcal{J}$. Note also that $N(S) \leq 0$ on X . Hence by Tonelli's theorem,

$$\int_{\mathcal{J}} \left(\int_X N(S) d\mu_j \right) d\lambda(S) = \int_X \left(\int_{\mathcal{J}} N(S) d\lambda(S) \right) d\mu_j.$$

Recall that on U_α the function $\log(|s_1|^2 + \dots + |s_d|^2)$ is locally integrable with respect to $|T|$. Hence, by the above considerations, it is locally integrable with respect to μ_j . Thus $|s_1|^2 + \dots + |s_d|^2 > 0$ μ_j -a.e. on U_α . So

$$u_\alpha := \left(\frac{s_1}{\sqrt{|s_1|^2 + \dots + |s_d|^2}}, \dots, \frac{s_d}{\sqrt{|s_1|^2 + \dots + |s_d|^2}} \right)$$

is a well defined function μ_j -a.e. on U_α with values in the unit sphere \mathbf{S}^{2d-1} in \mathbb{C}^d . We have

$$N(S) = N(a_1 S_1 + \dots + a_d S_d) = \log |a \cdot u_\alpha| \text{ on } U_\alpha,$$

where $a = (a_1, \dots, a_d)$ and $a \cdot u = a_1 u_1 + \dots + a_d u_d$. Therefore

$$\int_{\mathcal{J}} N(S)(x) d\lambda(S) = \int_{\mathbf{S}^{2d-1}} \log |a \cdot u_\alpha(x)| d\lambda(a) = C_d,$$

for μ_j -a.e. $x \in U_\alpha$, where $C_d < 0$ is a dimensional constant. It follows that

$$\int_{\mathcal{J}} \left(\int_X N(S) d\mu_j \right) d\lambda(S) = C_d \mu_j(X) > -\infty,$$

so the function $S \rightarrow \int_X N(S) d\mu_j$ is in $L^1(\mathcal{J}, \lambda)$, hence so is the function

$$S \rightarrow \int_X N(S) T \wedge dd^c \varphi = \int_X N(S) d\mu_1 - \int_X N(S) d\mu_2.$$

Finally, since T is closed we have

$$\int_{\mathcal{J}} \left(\int_X N(S) T \wedge dd^c \varphi \right) d\lambda(S) = C_d (\mu_1(X) - \mu_2(X)) = C_d \int_X T \wedge dd^c \varphi = 0.$$

This concludes the proof. □

Proof of Proposition 4.1. – We divide the proof into four steps.

Step 1. – We show that for ν_{k-1} -a.e. $(h^1, \dots, h^{k-1}) \in (\mathbb{P}^{d-1})^{k-1}$ the analytic set $Z(h^1) \cap \dots \cap Z(h^{k-1})$ has dimension at most $n - k + 1$.

Consider the set $\mathcal{J} \subset (X \setminus \text{Bs}(V)) \times (\mathbb{P}^{d-1})^{k-1}$ defined by

$$(x, h^1, \dots, h^{k-1}) \in \mathcal{J} \iff \sum_{j=1}^d h_j^i S_j(x) = 0, \quad 1 \leq i \leq k-1,$$

where $h^i = [h_1^i : \dots : h_d^i]$. If $z = (x, h^1, \dots, h^{k-1}) \in \mathcal{J}$ then $x \notin \text{Bs}(V)$, and we may assume that for each i , $h_{j_i}^i = 1$ for some $1 \leq j_i \leq d$. For each i there exists $l_i \neq j_i$ so that $S_{l_i}(x) \neq 0$. Indeed, otherwise $S_l(x) = 0$ for all $l \neq j_i$, so

$$S_{j_i}(x) = \sum_{l \neq j_i} h_l^i S_l(x) + S_{j_i}(x) = 0,$$

hence $x \in \text{Bs}(V)$, a contradiction. We obtain that for $z' = (x', \zeta^1, \dots, \zeta^{k-1})$ near z , \mathcal{J} is the graph

$$\zeta_{l_i}^i = -\frac{S_{j_i}(x')}{S_{l_i}(x')} - \sum_{l \neq l_i, j_i} \frac{S_l(x')}{S_{l_i}(x')} \zeta_{l_i}^i, \quad 1 \leq i \leq k-1.$$

Thus \mathcal{J} is a submanifold of $(X \setminus \text{Bs}(V)) \times (\mathbb{P}^{d-1})^{k-1}$ of dimension $n + (k-1)(d-2)$.

Consider the projection

$$\pi_2 : \mathcal{J} \longrightarrow (\mathbb{P}^{d-1})^{k-1}, \quad \pi_2(x, h^1, \dots, h^{k-1}) = (h^1, \dots, h^{k-1}).$$

By Sard's theorem, for ν_{k-1} -a.e. $(h^1, \dots, h^{k-1}) \in (\mathbb{P}^{d-1})^{k-1}$ the set

$$\pi_2^{-1}(h^1, \dots, h^{k-1}) = \{(x, h^1, \dots, h^{k-1}) : x \in Z(h^1) \cap \dots \cap Z(h^{k-1}) \cap (X \setminus \text{Bs}(V))\}$$

is either empty or a submanifold of \mathcal{J} of dimension $\dim \mathcal{J} - (k-1)(d-1) = n - k + 1$. Since $\pi_1 : (X \setminus \text{Bs}(V)) \times \{(h^1, \dots, h^{k-1})\} \longrightarrow X \setminus \text{Bs}(V)$ is a biholomorphism, we conclude that $Z(h^1) \cap \dots \cap Z(h^{k-1}) \cap (X \setminus \text{Bs}(V))$ is either empty or a submanifold of $X \setminus \text{Bs}(V)$ of dimension $n - k + 1$. Hence $Z(h^1) \cap \dots \cap Z(h^{k-1})$ is an analytic subset of X which is either empty or is of pure dimension $n - k + 1$ and smooth away from $\text{Bs}(V)$.

Step 2. – We show that the set G_k is open, where

$$G_k = \{(h^1, \dots, h^k) \in (\mathbb{P}^{d-1})^k : \dim Z(h^1) \cap \dots \cap Z(h^k) \leq n - k\}.$$

Indeed, assume for a contradiction that $(h^1, \dots, h^k) \in G_k$ but there exist sequences $h_N^i \rightarrow h^i$ in \mathbb{P}^{d-1} , as $N \rightarrow \infty$, so that the set $Z(h_N^1) \cap \dots \cap Z(h_N^k)$ has an irreducible component A_N of dimension m , for some $m > n - k$. Consider the currents $T_N = (\text{vol } A_N)^{-1} [A_N]$, where $[A_N]$ is the current of integration on A_N . Since T_N have unit mass, we may assume by passing to a subsequence that T_N converge weakly to a positive closed current T of unit mass and bidimension (m, m) . Note that the sets A_N cluster to the analytic set $A = Z(h^1) \cap \dots \cap Z(h^k)$, so T is supported on A . Since $\dim A \leq n - k < m$, $T = 0$ by the support theorem, a contradiction.

Step 3. – We show that the complement $G_k^c = (\mathbb{P}^{d-1})^k \setminus G_k$ has ν_k measure 0. Let

$$G_{k-1} = \{(h^1, \dots, h^{k-1}) \in (\mathbb{P}^{d-1})^{k-1} : \dim Z(h^1) \cap \dots \cap Z(h^{k-1}) \leq n - k + 1\}.$$

By steps 1 and 2, the set G_{k-1} is open and $\nu_{k-1}(G_{k-1}^c) = 0$. We have

$$G_k^c \subset (G_{k-1}^c \times \mathbb{P}^{d-1}) \cup (G_k^c \cap (G_{k-1} \times \mathbb{P}^{d-1})).$$

Note that $\nu_k(G_{k-1}^c \times \mathbb{P}^{d-1}) = 0$ and the set $F_k = G_k^c \cap (G_{k-1} \times \mathbb{P}^{d-1})$ is ν_k measurable.

For $(h^1, \dots, h^{k-1}) \in G_{k-1}$ consider the slice

$$\begin{aligned} F_k(h^1, \dots, h^{k-1}) &= \{h \in \mathbb{P}^{d-1} : (h^1, \dots, h^{k-1}, h) \in F_k\} \\ &= \{h \in \mathbb{P}^{d-1} : (h^1, \dots, h^{k-1}, h) \in G_k^c\}. \end{aligned}$$

Since G_k^c is closed, the above slices are closed. We are done if we show that they have μ_{d-1} measure 0. Indeed, since F_k is measurable this will imply that $\nu_k(F_k) = 0$.

To this end, observe that if $Z(h^1) \cap \dots \cap Z(h^{k-1}) = \emptyset$ then $F_k(h^1, \dots, h^{k-1}) = \emptyset$. Otherwise, we let $Y := Z(h^1) \cap \dots \cap Z(h^{k-1}) = Y_1 \cup \dots \cup Y_N$, where Y_l are the irreducible components of Y . Since all of them have dimension $n - k + 1$ it follows that

$$F_k(h^1, \dots, h^{k-1}) = \bigcup_{j=1}^N E_j, \quad E_j := \{h \in \mathbb{P}^{d-1} : Y_j \subset Z(h)\}.$$

Note that the sets E_j are closed. We will be done if we show that $\mu_{d-1}(E_j) = 0$.

Let us fix j . The basis sections S_i cannot all vanish identically on Y_j , since $\dim Y_j = n - k + 1$ and $\dim \text{Bs}(V) \leq n - k$. We may assume that $S_d \neq 0$ on Y_j . So

$$E_j \subset \{\zeta_1 = 0\} \cup H_j, \quad H_j := \{(\zeta_2, \dots, \zeta_d) \in \mathbb{C}^{d-1} : [1 : \zeta_2 : \dots : \zeta_d] \in E_j\}.$$

Note that H_j is closed in \mathbb{C}^{d-1} , and we are done if we show that it has Lebesgue measure 0. This follows since for each $(\zeta_2, \dots, \zeta_{d-1}) \in \mathbb{C}^{d-2}$ the slice

$$H_j(\zeta_2, \dots, \zeta_{d-1}) = \{\zeta \in \mathbb{C} : (\zeta_2, \dots, \zeta_{d-1}, \zeta) \in H_j\}$$

contains at most one element. Indeed, if $\zeta \neq \zeta' \in H_j(\zeta_2, \dots, \zeta_{d-1})$ then

$$S_1 + \zeta_2 S_2 + \dots + \zeta_{d-1} S_{d-1} + \zeta S_d \equiv 0, \quad S_1 + \zeta_2 S_2 + \dots + \zeta_{d-1} S_{d-1} + \zeta' S_d \equiv 0$$

on Y_j , hence $S_d \equiv 0$ on Y_j , a contradiction.

Step 4. – For $l \leq k - 1$ let

$$G_l = \{(h^1, \dots, h^l) \in (\mathbb{P}^{d-1})^l : \dim Z(h^1) \cap \dots \cap Z(h^l) \leq n - l\}.$$

The arguments in steps 1 and 2 show that, for every $l \leq k - 1$, the set G_l is open and $\nu_l(G_l^c) = 0$. Hence if $l \leq k - 1$ and $i_1 < \dots < i_l$ in $\{1, \dots, k\}$ the set

$$\tilde{F}_{i_1 \dots i_l} = \{(h^1, \dots, h^k) \in (\mathbb{P}^{d-1})^k : \dim Z(h^{i_1}) \cap \dots \cap Z(h^{i_l}) > n - l\}$$

is closed and has ν_k measure 0. Therefore, by step 3, the set D is open and has full ν_k measure, where $D = G_k \setminus \bigcup \tilde{F}_{i_1 \dots i_l}$ and the union is over all $l \leq k - 1$ and $i_1 < \dots < i_l$ in $\{1, \dots, k\}$. Note that $(h^1, \dots, h^k) \in D$ if and only if $\dim Z(h^{i_1}) \cap \dots \cap Z(h^{i_l}) \leq n - l$ for every $l \leq k$ and $i_1 < \dots < i_l$ in $\{1, \dots, k\}$, and the proof of Proposition 4.1 is complete. \square

5. Asymptotic behavior of the Bergman density function

Using techniques of Demailly from [19, Proposition 3.1], [22, Section 9] we prove here two theorems about the asymptotic behavior of the Bergman density function. The first one hereafter holds for arbitrary singular metrics with strictly positive curvature, while the second one, Theorem 5.3, shows that our Hypothesis (3) is satisfied in a quite general setting.

THEOREM 5.1. – *Let (X, Ω) be a compact n -dimensional Kähler manifold and (L, h) be a holomorphic line bundle on X with a singular Hermitian metric h so that $c_1(L, h)$ is a strictly positive current. If P_p, γ_p are the Bergman density function, resp. the Fubini-Study currents, defined by (1)-(2) for the spaces $H_{(2)}^0(X, L^p)$ of L^2 -holomorphic sections of L^p relative to the metric induced by h and the volume form Ω^n , then as $p \rightarrow \infty$,*

$$\frac{1}{p} \log P_p \rightarrow 0 \text{ in } L^1(X, \Omega^n), \quad \frac{1}{p} \gamma_p \rightarrow c_1(L, h), \quad \frac{1}{p} [\sigma_p = 0] \rightarrow c_1(L, h)$$

for λ_∞ -a.e. sequence $\{\sigma_p\}_{p \geq 1} \in \mathcal{J}_\infty$, in the weak sense of currents on X , where $\mathcal{J}_\infty, \lambda_\infty$ are as in Theorem 4.3.

We will need the following existence theorem for $\bar{\partial}$ in the case of singular Hermitian metrics. The smooth case goes back to Andreotti-Vesentini and Hörmander, while the singular case was first observed by Bombieri and Skoda and proved in generality by Demailly [18, Theorem 5.1].

THEOREM 5.2 (L^2 -estimates for $\bar{\partial}$). – *Let (M, Θ) be a complete Kähler manifold, (L, h) be a singular Hermitian line bundle and φ a quasi-psh function on M . Assume that there exist constants $a > 0, C > 0$ such that*

$$c_1(L, h) > 2a\Theta, \quad dd^c \varphi > -C\Theta, \quad c_1(K_M, h^{K_M}) < C\Theta,$$

where h^{K_M} is the metric induced on K_M by Θ . Then there exists $p_0 = p_0(a, C)$ such that for any $p \geq p_0$ and for any form $g \in L_{0,1}^2(M, L^p)$ satisfying $\bar{\partial}g = 0$ there exists $u \in L_{0,0}^2(M, L^p)$ with $\bar{\partial}u = g$ and

$$\int_M |u|_{h^p}^2 e^{-\varphi} dv_M \leq \frac{1}{ap} \int_M |g|_{h^p}^2 e^{-\varphi} dv_M.$$

Proof of Theorem 5.1. – Let $x \in X$ and $U_\alpha \subset X$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_α of L . Let ψ_α be a psh weight of h on U_α . Fix $r_0 > 0$ so that the (closed) ball $V := B(x, 2r_0) \Subset U_\alpha$ and let $U := B(x, r_0)$.

We show that there exist constants $C_1 > 0, p_0 \in \mathbb{N}$ so that

$$(7) \quad -\frac{\log C_1}{p} \leq \frac{1}{p} \log P_p(z) \leq \frac{\log(C_1 r^{-2n})}{p} + 2 \left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z) \right)$$

holds for all $p > p_0, 0 < r < r_0$ and $z \in U$ with $\psi_\alpha(z) > -\infty$.

For the upper estimate, fix $z \in U$ with $\psi_\alpha(z) > -\infty$ and $r < r_0$. Let $S \in H_{(2)}^0(X, L^p)$ with $\|S\|_p = 1$ and write $S = se_\alpha^{\otimes p}$. Repeating an argument of Demailly we obtain

$$\begin{aligned} |S(z)|_{h_p}^2 &= |s(z)|^2 e^{-2p\psi_\alpha(z)} \leq e^{-2p\psi_\alpha(z)} \frac{C_1}{r^{2n}} \int_{B(z,r)} |s|^2 \Omega^n \\ &\leq \frac{C_1}{r^{2n}} \exp\left(2p \left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right)\right) \int_{B(z,r)} |s|^2 e^{-2p\psi_\alpha} \Omega^n \\ &\leq \frac{C_1}{r^{2n}} \exp\left(2p \left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right)\right), \end{aligned}$$

where C_1 is a constant that depends only on x . Hence

$$\frac{1}{p} \log P_p(z) = \frac{1}{p} \max_{\|S\|_p=1} \log |S(z)|_{h_p}^2 \leq \frac{\log(C_1 r^{-2n})}{p} + 2 \left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right).$$

Note that this estimate holds for all p and it does not require the strict positivity of the current $c_1(L, h)$, nor the hypotheses that X is compact or Ω is a Kähler form.

For the lower estimate in (7), we proceed as in [22, Section 9] to show that there exist a constant $C_1 > 0$ and $p_0 \in \mathbb{N}$ such that for all $p > p_0$ and all $z \in U$ with $\psi_\alpha(z) > -\infty$ there is a section $S_{z,p} \in H_{(2)}^0(X, L^p)$ with $S_{z,p}(z) \neq 0$ and

$$\|S_{z,p}\|_p^2 \leq C_1 |S_{z,p}(z)|_{h_p}^2.$$

Observe that this implies that

$$\frac{1}{p} \log P_p(z) = \frac{1}{p} \max_{\|S\|_p=1} \log |S(z)|_{h_p}^2 \geq -\frac{\log C_1}{p}.$$

Let us prove the existence of $S_{z,p}$ as above. By the Ohsawa-Takegoshi extension theorem [49] there exists a constant $C' > 0$ (depending only on x) such that for any $z \in U$ and any p there exists a function $v_{z,p} \in \mathcal{O}_X(V)$ with $v_{z,p}(z) \neq 0$ and

$$\int_V |v_{z,p}|^2 e^{-2p\psi_\alpha} \Omega^n \leq C' |v_{z,p}(z)|^2 e^{-2p\psi_\alpha(z)}.$$

We shall now solve the $\bar{\partial}$ -equation with L^2 -estimates in order to extend $v_{z,p}$ to a section of L^p over X . We apply Theorem 5.2 for (X, Ω) and (L, h) . Let $\theta \in \mathcal{C}^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \theta \leq 1$, $\theta(t) = 1$ for $|t| \leq \frac{1}{2}$, $\theta(t) = 0$ for $|t| \geq 1$. Define the quasi-psh function φ_z on X by

$$\varphi_z(y) = \begin{cases} 2n\theta\left(\frac{|y-z|}{r_0}\right) \log \frac{|y-z|}{r_0}, & \text{for } y \in U_\alpha, \\ 0, & \text{for } y \in X \setminus B(z, r_0). \end{cases}$$

Note that the function φ_z is psh, hence $dd^c \varphi_z \geq 0$, on $\{y : |y-z| < r_0/2\}$. Since $V \Subset U_\alpha$ it follows that there exists a constant $c > 0$ so that for all $z \in U$ we have $dd^c \varphi_z \geq -c\Omega$ on X . Therefore there exist $a > 0$, $C > 0$ such that the hypotheses of Theorem 5.2 are satisfied for (X, Ω) , (L, h) and all φ_z , $z \in U$. Let p_0 be as in Theorem 5.2. Consider the form

$$g \in L_{0,1}^2(X, L^p), \quad g = \bar{\partial}(v_{z,p}\theta\left(\frac{|y-z|}{r_0}\right)e_\alpha^{\otimes p}).$$

By Theorem 5.2, for each $p > p_0$ there exists $u \in L_{0,0}^2(X, L^p)$ such that $\bar{\partial}u = g$ and

$$\int_X |u|_{h_p}^2 e^{-\varphi_z} \Omega^n \leq \frac{1}{ap} \int_X |g|_{h_p}^2 e^{-\varphi_z} \Omega^n < \infty.$$

Here the second integral is finite since $\psi_\alpha(z) > -\infty$ and

$$\int_X |g|_{h_p}^2 e^{-\varphi_z} \Omega^n = \int_V |v_{z,p}|^2 |\bar{\partial}\theta(\frac{|y-z|}{r_0})|^2 e^{-2p\psi_\alpha} e^{-\varphi_z} \Omega^n \leq C'' \int_V |v_{z,p}|^2 e^{-2p\psi_\alpha} \Omega^n,$$

where $C'' > 0$ is a constant that depends only on x . Near z , $e^{-\varphi_z(y)} = r_0^{2n} |y - z|^{-2n}$ is not integrable, thus $u(z) = 0$. Define

$$S_{z,p} := v_{z,p} \theta(\frac{|y-z|}{r_0}) e_\alpha^{\otimes p} - u.$$

Then $\bar{\partial} S_{z,p} = 0$, $S_{z,p}(z) = v_{z,p}(z) e_\alpha^{\otimes p}(z) \neq 0$, $S_{z,p} \in H_{(2)}^0(X, L^p)$. Since $\varphi_z \leq 0$ on X ,

$$\begin{aligned} \|S_{z,p}\|_p^2 &\leq 2 \left(\int_V |v_{z,p}|^2 e^{-2p\psi_\alpha} \Omega^n + \int_X |u|_{h_p}^2 e^{-\varphi_z} \Omega^n \right) \\ &\leq 2C'' \left(1 + \frac{C''}{ap} \right) |v_{z,p}(z)|^2 e^{-2p\psi_\alpha(z)} \leq C_1 |S_{z,p}(z)|_{h_p}^2, \end{aligned}$$

with a constant $C_1 > 0$ that depends only on x . This concludes the proof of (7).

Recall that $\log P_p \in L^1(X, \Omega^n)$, as it is locally the difference of psh functions. Observe that, by the upper semicontinuity of ψ_α , (7) implies that $\frac{1}{p} \log P_p \rightarrow 0$ as $p \rightarrow \infty$, Ω^n -a.e. on X . Since ψ_α is psh on U_α , it is integrable on $U \Subset U_\alpha$ [37, Corollary 3.2.8]. By dominated convergence, (7) implies that $\frac{1}{p} \log P_p \rightarrow 0$ in $L^1(U, \Omega^n)$, hence in $L^1(X, \Omega^n)$, so

$$\gamma_p - c_1(L, h) = \frac{1}{2p} dd^c \log P_p \rightarrow 0 \text{ weakly on } X.$$

The conclusion about the equidistribution of zeros of random sequences of sections now follows as in [54, Theorem 1.1] (see Section 4 and Theorem 4.3). □

We return to the main setting of the paper, given by assumptions (A)-(C) stated in the introduction, and we take here $f \equiv 1$.

THEOREM 5.3. – *Let $X, \Sigma, (L, h), \Omega$ verify (A)-(B) and assume that X is compact, Ω is a Kähler form, and $c_1(L, h)$ is a strictly positive current on X . Then (3) holds for the Bergman density function P_p defined in (1) for the space $H_{(2)}^0(X \setminus \Sigma, L^p)$.*

Proof. – Let $x \in X \setminus \Sigma, U_\alpha \subset X \setminus \Sigma, \psi_\alpha, V, U$, be as in the proof of Theorem 5.1. Then (7) shows that $\frac{1}{p} \log P_p \rightarrow 0$ as $p \rightarrow \infty$ uniformly on U , thanks to the uniform continuity of ψ_α on V . □

Combining Theorems 1.1, 5.3 and 1.2 we obtain the following equidistribution theorem for big line bundles:

THEOREM 5.4. – *Let (L, h) be a line bundle over the compact Kähler manifold (X, Ω) endowed with a singular Hermitian metric h which is continuous outside a proper analytic subset Σ and so that $\gamma := c_1(L, h)$ is a strictly positive current. If γ_p is the current defined by (2) for the space $H_{(2)}^0(X \setminus \Sigma, L^p)$ then $\frac{1}{p} \gamma_p \rightarrow \gamma$ weakly on X . If $\dim \Sigma \leq n - k$ for some $2 \leq k \leq n$, then the currents γ^k and γ_p^k , for all p sufficiently large, are well defined and $\frac{1}{p^k} \gamma_p^k \rightarrow \gamma^k$ weakly on X . Moreover, the conclusions of Theorems 1.2 and 4.3 hold in this setting.*

Note that in Theorems 5.1 and 5.4 the bundle L is a big line bundle and X is Moishezon, by a theorem of Ji and Shiffman [39] (cf. also [43, Theorem 2.3.28, 2.3.30]). Hence X is in fact a projective manifold, since it is assumed to be Kähler (see e.g., [43, Theorem 2.2.26]).

6. Applications

Let $X, \Sigma, (L, h), f, \Omega$ verify assumptions (A)-(C) stated in the introduction and assume in addition that $\gamma = c_1(L, h)$ is a strictly positive current. To emphasize the metrics that are used, we denote throughout this section the corresponding spaces of L^2 -holomorphic sections by $H_{(2)}^0(X \setminus \Sigma, L^p, h, f\Omega^n)$. We discuss here several important situations in which the Bergman density function P_p defined in (1) satisfies our Hypothesis (3). In Sections 6.1, 6.2 we consider singular Hermitian metrics on big line bundles, and we deduce equidistribution results for L^2 holomorphic sections with respect to the Poincaré metric and for sections of Nadel multiplier sheaves. In Section 6.3 we turn to Zariski-open manifolds with bounded negative Ricci curvature, and we generalize a theorem of Tian [59, Theorem C] in our framework. Natural examples of Kähler-Einstein manifolds of negative Ricci curvature are the arithmetic quotients. We show in Section 6.4 how our results apply for toroidal compactifications of such manifolds. Finally, in Sections 6.5, 6.6 we exhibit some results for 1-convex manifolds.

6.1. Properties of h_ε

For some of the applications, we will have to work with the Poincaré metric Θ on $X \setminus \Sigma$ and with a small perturbation h_ε of the metric h on L . Let us begin by listing certain properties of these special metrics.

We refer to Section 2.3 for the construction of the metrics Θ, h_ε , and we shall use the notations introduced there. In particular, $\Theta^n = f\Omega^n$ with a function f as in (C) (see Section 2.3.1). Note that h_ε is in fact a metric on $L|_{X \setminus Y}$, where $Y \subset \Sigma$ is an analytic subset of dimension $\leq n - 2$ (Section 2.3.2). We recall the following fact:

LEMMA 6.1. – *Let L be a holomorphic line bundle over a complex manifold X and Y be an analytic subvariety of X with $\text{codim } Y \geq 2$. Then any positively curved singular metric h^L on $L|_{X \setminus Y}$ extends to a positively curved singular metric on L . Moreover, if $c_1(L|_{X \setminus Y}, h^L) \geq \delta\Omega$ on $X \setminus Y$, for some $\delta > 0$, then the same estimate holds for the curvature current of the extended metric on X .*

Proof. – If U_α is a neighborhood of some point $y \in Y$ on which L has a holomorphic frame e_α , then $h^L(e_\alpha, e_\alpha) = e^{-2\varphi_\alpha}$ for some psh function φ_α on $U_\alpha \setminus Y$. Since $\text{codim } Y \geq 2$ the function φ_α is locally upper bounded near the points of $U_\alpha \cap Y$, hence it extends to a psh function on U_α . The second conclusion follows since the current $c_1(L, h^L)$ does not charge Y by Federer's support theorem ([26], see also [35, Theorem 1.7]). \square

We denote the extended metric still by h_ε and we let $\omega = c_1(L, h_\varepsilon)$, so ω is a positive closed (1, 1) current on X .

PROPOSITION 6.2. – (i) We have $\omega = \gamma + \pi_*(\theta' + \varepsilon dd^c F)$, where F is defined in (4) and θ' is a smooth real closed $(1, 1)$ form on \tilde{X} .

(ii) Let A be an irreducible component of Σ of dimension $n - 1$. Then the generic Lelong numbers $\nu(\gamma, A) = \nu(\omega, A)$. Moreover, any section in $H_{(2)}^0(X \setminus \Sigma, L^p, h_\varepsilon, f\Omega^n)$ vanishes at least to order $p\nu(\omega, A)$ on A .

Proof. – (i) Recall from Section 2.3.2 that the metric h_ε on $L|_{X \setminus Y}$ was induced via the biholomorphism $\pi : \tilde{X} \setminus E \rightarrow X \setminus Y$ by a metric $h_\varepsilon^{L'}$ on $L' = \pi^*(L|_{X \setminus Y})$ with curvature current $\gamma'_\varepsilon = \pi^*\gamma + \theta' + \varepsilon dd^c F$. The map $\pi : \tilde{X} \rightarrow X$ is proper so $\pi_*\gamma'_\varepsilon$ is a well defined positive closed $(1, 1)$ current on X which satisfies $\pi_*\gamma'_\varepsilon = \omega$ on $X \setminus Y$. As $\dim Y \leq n - 2$, Federer’s support theorem [26] implies that $\pi_*\gamma'_\varepsilon = \omega$ on X . Similarly, we have that $\pi_*\pi^*\gamma = \gamma$ on $X \setminus Y$, and hence on X . The formula for ω in the statement now follows.

(ii) Fix a point $x \in A \setminus Y$. Then $x \in \Sigma_{\text{reg}}^{n-1}$, so we can find a neighborhood $V_x \subset X$ of x and local coordinates z_1, \dots, z_n on V_x so that $\pi : \pi^{-1}(V_x) \rightarrow V_x$ is a biholomorphism, $x = 0$, $\Sigma \cap V_x = A \cap V_x = \{z_1 = 0\}$, and $f \geq c > 0$ on V_x .

By (i) we have

$$\omega = \gamma + (\pi^{-1})^*\theta' + \varepsilon dd^c F \circ \pi^{-1} \text{ on } V_x.$$

We can assume that there exist functions φ, u on V_x so that φ is psh, u is smooth, $dd^c\varphi = \gamma$, $dd^c u = (\pi^{-1})^*\theta'$. Then the function $\varphi_\varepsilon = \varphi + u + \varepsilon F \circ \pi^{-1}$ is psh on V_x and $dd^c\varphi_\varepsilon = \omega$. It follows by the Definition (4) of F that near x we have $F \circ \pi^{-1} = -\log(g - \log|z_1|) + O(1)$, where g is a smooth function. Thus

$$\varphi_\varepsilon = \varphi - \varepsilon \log(g - \log|z_1|) + O(1),$$

which shows that the Lelong numbers $\nu(\varphi_\varepsilon, x) = \nu(\varphi, x)$. Since $x \in A \setminus Y$ was arbitrary this implies that $\nu(\gamma, A) = \nu(\omega, A)$.

Next, let $S \in H_{(2)}^0(X \setminus \Sigma, L^p, h_\varepsilon, f\Omega^n)$ be defined on V_x by $S = se_\alpha^{\otimes p}$, where e_α is a local frame for L , and let $\nu = \nu(\omega, A)$. As $f \geq c$ we have

$$\int_{V_x \setminus A} |s|^2 e^{-2p\varphi_\varepsilon} d\lambda < \infty,$$

where λ is the Lebesgue measure in coordinates. By the results of [58], $dd^c\varphi_\varepsilon = \nu dd^c \log|z_1| + T$, where T is a positive closed current, so $T = dd^c v$ for some psh function v on V_x . It follows that the function $\varphi_\varepsilon - \nu \log|z_1| - v$ is pluriharmonic on V_x . Hence, by shrinking V_x if necessary, we have

$$\varphi_\varepsilon \leq \nu \log|z_1| + O(1), \text{ hence } \int_{V_x \setminus A} |s|^2 |z_1|^{-2p\nu} d\lambda < \infty.$$

This implies that s vanishes at least to order $p\nu$ along A . □

REMARK 6.3. – The proof of Proposition 6.2 shows in fact that the currents ω and γ have the same Lelong numbers at each point of $\Sigma_{\text{reg}}^{n-1}$. However, the Lelong numbers of ω are bigger than those of γ at other points of Σ . For instance, if Σ is a finite set then \tilde{X} is simply the blow-up of X at each of the points of Σ . Then, in local coordinates z near a point $x = 0 \in \Sigma$, we have $\pi_*\theta' = a dd^c \log \|z\|$, for some $a > 0$.

6.2. Singular metrics on big line bundles

Let L be a big line bundle over the compact complex manifold X . Then X is Moishezon and L admits a singular metric h , smooth outside a proper analytic subset Σ of X , and with strictly positive curvature current $\gamma = c_1(L, h)$ (see e.g., [43, Lemma 2.3.6]).

6.2.1. *Special metrics on Moishezon manifolds.* – Let Θ be the Poincaré metric on $X \setminus \Sigma$ and h_ε be the small perturbation of the metric h on L constructed in Section 2.3. It is shown in [42, 44] (see also [43, Chapter 6]) that the Bergman density function P_p of the space $H_{(2)}^0(X \setminus \Sigma, L^p, h_\varepsilon, \Theta^n)$ has a full asymptotic expansion locally uniformly on $X \setminus \Sigma$. This clearly implies (3), so we have the following:

THEOREM 6.4. – *The conclusions of Theorems 1.1, 1.2 and 4.3 hold for the spaces $H_{(2)}^0(X \setminus \Sigma, L^p, h_\varepsilon, \Theta^n)$ and for $\omega = c_1(L, h_\varepsilon)$.*

Note that in this case X is not assumed to be Kähler.

6.2.2. *Multiplier ideal sheaves.* – We recall first the notion of multiplier ideal sheaf. Let $\varphi \in L^1(X, \mathbb{R})$. The *Nadel multiplier ideal sheaf* $\mathcal{I}(\varphi)$ is the ideal subsheaf of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that $|f|^2 e^{-2\varphi}$ is integrable with respect to the Lebesgue measure in local coordinates near x .

If h' is a smooth Hermitian metric on L then $h = h' e^{-2\varphi}$ for some function $\varphi \in L^1(X, \mathbb{R})$. The Nadel multiplier ideal sheaf of h is defined by $\mathcal{I}(h) = \mathcal{I}(\varphi)$; the definition does not depend on the choice of h' . The space of global sections in the sheaf $L \otimes \mathcal{I}(h)$ is given by

$$(8) \quad H^0(X, L \otimes \mathcal{I}(h)) = \left\{ s \in H^0(X, L) : \int_X |s|_h^2 \Omega^n = \int_X |s|_{h'}^2 e^{-2\varphi} \Omega^n < \infty \right\},$$

where Ω is a fixed smooth positive $(1, 1)$ form on X . We have

$$H^0(X, L^p \otimes \mathcal{I}(h_p)) = H_{(2)}^0(X \setminus \Sigma, L^p, h, \Omega^n),$$

where h_p is the metric induced by h on L^p . If $\{S_j^p\}$ is an orthonormal basis of $H^0(X, L^p \otimes \mathcal{I}(h_p))$ we define the Fubini-Study currents γ_p on X as in (2).

THEOREM 6.5. – *Let L be a big line bundle over a compact Kähler manifold X and h be a singular Hermitian metric on L , smooth outside a proper analytic subset Σ of X , and with strictly positive curvature current $\gamma = c_1(L, h)$. The conclusions of Theorems 1.1, 1.2 and 4.3 hold for the spaces $H^0(X, L^p \otimes \mathcal{I}(h_p))$ and for γ .*

Proof. – Conditions (A)-(C) are obviously verified in the present situation. Moreover, (3) follows from Theorem 5.3. It can also be seen as a consequence of the full asymptotic expansion of the Bergman density function proved in [38]. Therefore, Theorem 1.1 implies the desired conclusion. \square

Note that X is in fact a projective manifold, since it is Moishezon and Kähler (see e.g., [43, Theorem 2.2.26]).

6.3. Zariski-open manifolds with bounded negative Ricci curvature

Let (M, J, ω) be a Kähler manifold, let g^{TM} be the Riemannian metric associated to ω by $g^{TM}(Ju, Jv) = g^{TM}(u, v)$ for all $u, v \in T_x M, x \in M$. Let Ric be the Ricci curvature of g^{TM} . The Ricci form Ric_ω is defined as the $(1, 1)$ -form associated to Ric by

$$\text{Ric}_\omega(u, v) = \text{Ric}(Ju, v), \quad \text{for any } u, v \in T_x M, x \in M.$$

The volume form ω^n induces a metric $h^{K_M^*}$ on K_M^* , whose dual metric on K_M is denoted by h^{K_M} . Since the metric g^{TM} is Kähler, we have (see e.g., [43, Prob. 1.7])

$$\text{Ric}_\omega = iR^{K_M^*} = -iR^{K_M}.$$

We denote by $H_{(2)}^0(M, K_M^p)$ the space of L^2 -pluricanonical sections with respect to the metric $h^{K_M^p}$ and the volume form ω^n .

We consider in this section the following setting:

- (I) X is a compact complex manifold of dimension n , Σ is an analytic subvariety of X , $M := X \setminus \Sigma$.
- (II) M admits a complete Kähler metric ω such that $\text{Ric}_\omega \leq -\lambda\omega$, for some constant $\lambda > 0$.

Note that $K_M = K_X|_M$. Moreover, condition (II) implies that the volume form ω^n is integrable over X ; indeed, by Yau’s Schwarz lemma [61, Theorem 3] it follows that $\omega^n \lesssim \Theta^n$, where Θ is the generalized Poincaré metric on M (see e.g., [48, Proposition 1.10]) and Θ^n is integrable over X (see also [43, Lemma 6.2.1]). We have the following:

THEOREM 6.6. – *Let X, Σ, M, ω be as in (I), (II), and assume that $\dim \Sigma \leq n - k, k \geq 2$. Then the following hold:*

- (i) $H_{(2)}^0(M, K_M^p) \subset H^0(X, K_X^p)$.
- (ii) The currents $(-\text{Ric}_\omega)^j, \gamma_p^j, 1 \leq j \leq k$, are well defined on X for p sufficiently large, where γ_p are the Fubini-Study currents defined by (2) for $H_{(2)}^0(M, K_M^p)$.
- (iii) $\frac{1}{p^j} \gamma_p^j \rightarrow (-\frac{1}{2\pi} \text{Ric}_\omega)^j$ weakly on X as $p \rightarrow \infty$, for $1 \leq j \leq k$.

Proof. – We only have to check condition (B). Since $\text{codim } \Sigma \geq 2$, Lemma 6.1 implies that the metric h^{K_M} extends to a positively curved (singular) metric on K_X over X , which we denote by h . Moreover,

$$-\text{Ric}_\omega = iR^{K_M} = 2\pi c_1(K_M, h^{K_M}) = 2\pi c_1(K_X, h)|_M$$

extends to a positive closed current on X .

Condition (3) holds, as shown by Tian [59, Theorem 4.1] (this follows also from the more general result in [43, Theorem 6.1.1]). Therefore, Theorem 1.1 implies the desired conclusion. □

Note that Tian [59, §5] considered the situation when X, Σ, M verify assumptions (I), (II), X is projective and $k = 1$. In that case he shows that the sections of $H_{(2)}^0(M, K_M^p)$ extend meromorphically to X , with poles of order at most $p - 1$ along Σ , and $-\text{Ric}_\omega$ extends to a positive closed current on X [59, Theorem C].

This situation is more difficult, as the metric h^{K_M} does not extend to a positively curved metric on K_X . Nevertheless, we shall now show how this case fits into our framework

from Theorem 1.1. In view of Theorem 6.6 (and its proof), we may assume without loss of generality that

(III) Σ has pure dimension $n - 1$.

For this purpose, consider the line bundle $L := K_X \otimes \mathcal{O}_X(\Sigma)$, where $\mathcal{O}_X(\Sigma)$ is the line bundle associated to the divisor Σ . Let σ be the canonical section of $\mathcal{O}_X(\Sigma)$ (cf. [43, p. 71]) and denote by h_σ the metric induced by σ on $\mathcal{O}_X(\Sigma)$ (cf. [43, Example 2.3.4]). Note also that $c_1(\mathcal{O}_X(\Sigma), h_\sigma) = [\Sigma]$ by [43, (2.3.8)]. Consider the metric naturally defined by h^{K_M} ,

$$(9) \quad h_{M,\sigma} := h^{K_M} \otimes h_\sigma \text{ on } L|_M = K_M \otimes \mathcal{O}_X(\Sigma)|_M \cong K_M.$$

We recall the following simple fact, whose proof is left to the reader.

LEMMA 6.7. – *Let X, Σ, M verify assumptions (I) and (III). Assume that (E, h^E) is a singular Hermitian line bundle on X and $p \geq 1$. Then*

$$\mathcal{J}_\sigma : H^0(M, E|_M) \longrightarrow H^0(M, (E \otimes \mathcal{O}_X(\Sigma)^p)|_M), \quad \mathcal{J}_\sigma(S) = S \otimes \sigma^{\otimes p},$$

is an isomorphism and we have $|S|_{h^E}^2 = |\mathcal{J}_\sigma(S)|_{h^E \otimes h_\sigma^p}$ pointwise on M , where h_σ^p is the metric induced by h_σ on $\mathcal{O}_X(\Sigma)^p$.

LEMMA 6.8. – *Let X, Σ, M, ω verify assumptions (I)-(III). The metric $h_{M,\sigma}$ defined in (9) extends uniquely to a positively curved metric h on L over X . The curvature current $c_1(L, h)$ is independent of the choice of σ and we have $c_1(L, h)|_M = -\frac{1}{2\pi} \text{Ric}_\omega$.*

Proof. – By Lemma 6.1 it suffices to show that the metric $h_{M,\sigma}$ extends near each regular point $x \in \Sigma$. We follow at first the argument of Tian from [59, Lemma 5.1] to estimate the volume of ω as in [59, (5.3)]. Let \mathbb{D} be the unit disc in \mathbb{C} . Then $x \in \Sigma$ has a coordinate neighborhood U_x such that

$$U_x \cong \mathbb{D}^n, \quad x = 0, \quad U_x \cap \Sigma \cong \{z = (z_1, \dots, z_n) : z_1 = 0\}, \quad U_x \cap M \cong \mathbb{D}^* \times \mathbb{D}^{n-1}.$$

Consider the complete hyperbolic metric g_x on $\mathbb{D}^* \times \mathbb{D}^{n-1}$ given by the product of the Poincaré metrics on \mathbb{D}^* and \mathbb{D} . By (II) and Yau’s Schwarz lemma [61, Theorem 3], the volume of ω is dominated on $U_x \cap M$ by a constant multiple of the volume of g_x . On a smaller polydisc $\mathbb{D}_r^* \times \mathbb{D}_r^{n-1}$, $r < 1$, the volume of g_x is $\sim (|z_1| \log |z_1|)^{-2}$. It follows that

$$\det[g_{jk}] \leq C(|z_1| \log |z_1|)^{-2} \text{ on } \mathbb{D}_r^* \times \mathbb{D}_r^{n-1}, \quad \text{where } \omega = i \sum_{j,k=1}^n g_{jk} dz_j \wedge d\bar{z}_k,$$

for some constant $C > 0$.

Note that L is trivial over U_x and the metric $h_{M,\sigma}$ has a weight φ on $U_x \cap M \cong \mathbb{D}^* \times \mathbb{D}^{n-1}$ given by $e^{2\varphi} = |z_1|^2 \det[g_{jk}]$. So $dd^c \varphi = -\frac{1}{2\pi} \text{Ric}_\omega > 0$ and φ is psh on $U_x \cap M$. By the above estimate,

$$e^{2\varphi} = |z_1|^2 \det[g_{jk}] \leq C(\log |z_1|)^{-2} \text{ on } \mathbb{D}_r^* \times \mathbb{D}_r^{n-1},$$

which implies that $\varphi(z) \rightarrow -\infty$ as $z \rightarrow \Sigma$, so φ is upper bounded near x . Hence φ extends to a psh function on U_x , and $h_{M,\sigma}$ extends uniquely to a positively curved metric h on L . Moreover,

$$c_1(L, h)|_M = c_1(K_M, h^{K_M}) + c_1(\mathcal{O}_X(\Sigma)|_M, h_\sigma) = -\frac{1}{2\pi} \text{Ric}_\omega.$$

Since X is compact, any section σ' of $\mathcal{O}_X(\Sigma)$ that vanishes on Σ is a constant multiple of σ , hence the metric $h_{\sigma'}$ is a constant multiple of h_σ . This shows that $c_1(L, h)$ is independent of the choice of σ . \square

THEOREM 6.9. – *Let X, Σ, M, ω verify assumptions (I)-(III). Let h^{K_M} be the metric induced by ω on K_M and $H_{(2)}^0(M, K_M^p)$ be the space of L^2 -pluricanonical sections with respect to the metric $h^{K_M^p}$ and the volume form ω^n . Then we have:*

- (i) *The Fubini-Study currents γ_p of $H_{(2)}^0(M, K_M^p)$ extend naturally as closed currents of order 0 on X defined locally by Formula (2), and $\frac{1}{p} \gamma_p + [\Sigma] \geq 0$ on X .*
- (ii) *$\frac{1}{p} \gamma_p + [\Sigma]$ converge weakly on X to a positive closed current T so that $T|_M = -\frac{1}{2\pi} \text{Ric}_\omega$ and $T = c_1(L, h)$ for a singular Hermitian metric h on $L = K_X \otimes \mathcal{O}_X(\Sigma)$.*

Proof. – By [48, Proposition 1.11] (see also [59, Lemma 5.1]) the sections in $H_{(2)}^0(M, K_M^p)$ extend to meromorphic sections of K_X^p over X , with poles in Σ of order at most $p - 1$. This yields (i).

Let $h_{M,\sigma}$ be the metric defined in (9) on $L|_M$, and h be its extension to L provided in Lemma 6.8, so $c_1(L, h)|_M = -\frac{1}{2\pi} \text{Ric}_\omega$. It follows from Lemma 6.7 and [48, Proposition 1.11] that $\mathcal{I}_\sigma(H_{(2)}^0(M, K_M^p)) = H_{(2)}^0(M, L^p, h, \omega^n) \subset H^0(X, L^p)$. So $X, \Sigma, (L, h)$ and the volume form ω^n verify assumptions (A), (B), (C'), (D) (see Remark 3.6).

Lemmas 6.7 and 6.8 imply that \mathcal{I}_σ maps an orthonormal basis of $H_{(2)}^0(M, K_M^p)$ onto an orthonormal basis of $H_{(2)}^0(M, L^p, h, \omega^n)$ and that the Bergman kernel functions P_p defined by (1) for these spaces are equal. Condition (3) holds, as shown by [59, §4] or [43, Theorem 6.1.1]. By Theorem 1.1 and Remark 3.6 we have $\frac{1}{p} \gamma'_p \rightarrow c_1(L, h)$ weakly on X , where γ'_p are the Fubini-Study currents defined by (2) for $H_{(2)}^0(M, L^p, h, \omega^n)$.

Observe that Lemmas 6.7 and 6.8 imply $\gamma'_p = \gamma_p + p[\Sigma]$ on X . This completes the proof. \square

REMARK 6.10. – *Note that assumptions (I)-(III) are verified if X is a compact projective manifold, Σ is an effective divisor of X , and $L = K_X \otimes \mathcal{O}_X(\Sigma)$ is ample, due to a result by R. Kobayashi [40] about the existence of Kähler-Einstein metrics on $X \setminus \Sigma$. Conversely, let X, Σ, M, ω verify assumptions (I)-(III) as in Theorem 6.9. By the proof of [48, Proposition 1.12] we see that the following properties hold:*

- (a) *There exists p_0 such that $H^0(X, L^{p_0})$ separates the points of M and gives local holomorphic coordinates on M ,*
- (b) *M is biholomorphic to a quasiprojective manifold; in fact the meromorphic Kodaira map $\Phi_{p_0} : X \dashrightarrow \mathbb{P}^N$ defined by $H^0(X, L^{p_0})$ induces a birational morphism to a normal projective variety Y such that $\Phi_{p_0}(M)$ is Zariski open in Y and $\Phi_{p_0} : M \rightarrow \Phi_{p_0}(M)$ is biholomorphic,*
- (c) *L is big and X is Moishezon.*

Note that L is not necessarily ample in the case of toroidal compactifications considered in the Section 6.4.

6.4. Arithmetic quotients

Let D be a bounded symmetric domain in \mathbb{C}^n and let Γ be a neat arithmetic group acting properly discontinuously on D (see [47, p. 253]). Then $U := D/\Gamma$ is a smooth quasi-projective variety, called an arithmetic variety. By [2], U admits a smooth toroidal compactification X . In particular, $\Sigma := X \setminus U$ is a divisor with normal crossings. The Bergman metric ω_D^β on D descends to a complete Kähler metric $\omega := \omega_U^\beta$ on U . Moreover, ω is Kähler-Einstein with $\text{Ric}_\omega = -\omega$ (since the metric ω_D^β has this property). We denote by h^{K_U} the Hermitian metric induced by ω on K_U . We wish to study the spaces $H_{(2)}^0(U, K_U^p)$ of L^2 -pluricanonical sections with respect to the metric $h^{K_U^p}$ and the volume form ω^n .

As in Section 6.3, consider the line bundle $L := K_X \otimes \mathcal{O}_X(\Sigma)$ and the metric $h_{U,\sigma}$ on $L|_U$ defined in (9). By Lemma 6.8 $h_{U,\sigma}$ extends uniquely to a positively curved metric h on L and $c_1(L, h)|_U = \frac{\omega}{2\pi}$. Clearly, Theorem 6.9 holds in this setting:

THEOREM 6.11. – *Let X be a smooth toroidal compactification of an arithmetic quotient $U = D/\Gamma$ and set $\Sigma = X \setminus U$, $L = K_X \otimes \mathcal{O}_X(\Sigma)$. Let ω be the induced Bergman metric on U and let h^{K_U} be the metric induced by ω on K_U . Then we have:*

- (i) *The metric h^{K_U} defines a singular metric h on L such that $c_1(L, h)$ is a positive current on X which extends $\frac{\omega}{2\pi}$.*
- (ii) *$H_{(2)}^0(U, L^p, h, \omega^n) \subset H^0(X, L^p)$ for all $p \geq 1$, so the currents γ_p given by (2) for $H_{(2)}^0(U, L^p, h, \omega^n)$ extend naturally to positive closed currents on X .*
- (iii) *$\frac{1}{p} \gamma_p \rightarrow c_1(L, h)$ and $\frac{1}{p} [\sigma_p = 0] \rightarrow c_1(L, h)$ in the weak sense of currents on X , for λ_∞ -a.e. sequence $\{\sigma_p\}_{p \geq 1} \in \mathcal{I}_\infty$, where $\mathcal{I}_\infty, \lambda_\infty$ are as in Theorem 4.3.*

By Lemma 6.7, $H_{(2)}^0(U, K_U^p) \cong H_{(2)}^0(U, L^p, h, \omega^n)$. Let us describe this space in more detail. By [47, Proposition 3.3, 3.4(b)],

$$H^0(X, L^p) \cong \left\{ \begin{array}{l} \text{modular forms with respect to the } p \text{th power} \\ \text{of the canonical automorphy factor} \end{array} \right\},$$

so $H_{(2)}^0(U, K_U^p) \subset H^0(X, L^p)$ are modular forms. The space

$$H^0(X, L^p \otimes \mathcal{O}_X(\Sigma)^{-1}) = H^0(X, K_X^p \otimes \mathcal{O}_X(\Sigma)^{p-1})$$

of modular forms vanishing on the boundary is called the space of cusp forms.

We will need the following definition from Mumford [47, p. 242]. Let \mathbb{D} be the unit disc in \mathbb{C} . Every $x \in \Sigma$ has a coordinate neighborhood $V_x \cong \mathbb{D}^n$ such that for some $1 \leq l \leq n$,

$$(10) \quad V_x \cong \mathbb{D}^n, \quad x = 0, \quad V_x \cap \Sigma \cong \{z = (z_1, \dots, z_n) : z_1 z_2 \dots z_l = 0\}.$$

DEFINITION 6.12. – *A smooth Hermitian metric h on $L|_U$ is said to be good on X if for all $x \in \Sigma$ and all holomorphic frames e of L in a neighborhood $V_x \cong \mathbb{D}^n$ of x as in (10) we have*

- (i) $|e|_h^2, |e|_h^{-2} \leq C \left| \sum_{k=1}^l \log |z_k| \right|^{2\alpha}$, for some $C > 0, \alpha \geq 1$,
- (ii) *the forms $\eta = \partial \log |e|_h^2$ and $d\eta$ have Poincaré growth on V_x .*

Examples of Hermitian line bundles with good metrics are provided by the following class of line bundles over arithmetic quotients considered by Mumford in [47, p. 256]. Namely, if D is a bounded symmetric domain, then $D \cong K \backslash G$, where G is a semi-simple adjoint group (i.e., isomorphic to its adjoint group) and K a maximal compact subgroup (see e.g., [2, p. 106]). Let E_0 be a G -equivariant holomorphic line bundle over D . Let $U = D/\Gamma$ be an arithmetic quotient and X be a smooth toroidal compactification of U . Then Γ acts on E_0 and $E_U = E_0/\Gamma$ is a holomorphic line bundle on U . Moreover, E_0 carries a G -invariant Hermitian metric h_0 which induces a Hermitian metric h_U on E_U . By [47, Main Theorem 3.1], E_U admits a unique extension to a holomorphic line bundle \overline{E} over X such that the metric h_U on $\overline{E}|_U = E_U$ is good on X .

Consider the G -invariant line bundle $(E_0, h_0) = (K_D, h^{K_D})$ on D , where h^{K_D} is induced by ω_D^β . Note that the Bergman metric ω_D^β is G -invariant and so is h^{K_D} . Then $(E_U, h_U) = (K_U, h^{K_U})$, where h^{K_U} is induced by ω_U^β . By [47, Main Theorem 3.1, Proposition 3.4] the extension \overline{K}_U of K_U satisfies the following condition: for any $x \in \Sigma$ and any open coordinate neighborhood $V_x \cong \mathbb{D}^n$ of x as in (10), a holomorphic frame of $\overline{K}_U|_{V_x}$ is of the form $e = (z_1 z_2 \dots z_l)^{-1} dz_1 \wedge \dots \wedge dz_n$. This shows that $\overline{K}_U \cong K_X \otimes \mathcal{O}_X(\Sigma) =: L$ and the metric $h_{U,\sigma}$ induced by h^{K_U} (see (9)) on $L|_U \cong K_U$ is good on X . Hence we obtain by condition (i) of Definition 6.12 that

$$(11) \quad \omega^n \gtrsim \prod_{j=1}^l |z_j|^{-2} \left(\sum_{k=1}^l \log |z_k| \right)^{-2\alpha} \Omega^n \quad \text{on } V_x \setminus \Sigma,$$

where $\alpha \geq 1$ and Ω is a positive (1,1) form on X , and

$$(12) \quad |e|_{h_{U,\sigma}}^2 \lesssim \left(\sum_{k=1}^l \log |z_k| \right)^{2\alpha} \quad \text{on } V_x \setminus \Sigma.$$

LEMMA 6.13. – *Let $U = D/\Gamma$ be an arithmetic quotient and let X be a smooth toroidal compactification of U . Then $H_{(2)}^0(U, K_U^p) \cong H^0(X, K_X^p \otimes \mathcal{O}_X(\Sigma)^{p-1})$, i.e., the space of L^2 -pluricanonical sections is the space of cusp forms.*

Proof. – By [48, Proposition 1.11] we have $H_{(2)}^0(U, K_U^p) \subset H^0(X, K_X^p \otimes \mathcal{O}_X(\Sigma)^{p-1})$. If $S \in H^0(X, K_X^p \otimes \mathcal{O}_X(\Sigma)^{p-1})$, then $S = fe^{\otimes p}$, in a neighborhood V_x of $x \in \Sigma$ as in (10), where $f \in \mathcal{O}_X(V_x)$ vanishes on Σ and e is a frame of L over V_x . Estimate (12) together with the fact that ω^n is integrable over X [48, Proposition 1.10] imply that

$$\int_{V_x \setminus \Sigma} |S|_{h_{U,\sigma}}^2 \omega^n = \int_{V_x \setminus \Sigma} |f|^2 |e^{\otimes p}|_{h_{U,\sigma}}^2 \omega^n \lesssim \int_{V_x \setminus \Sigma} |f|^2 \left| \sum_{k=1}^l \log |z_k| \right|^{2p\alpha} \omega^n < \infty,$$

thus $S \in H_{(2)}^0(U, L^p, h, \omega^n) \cong H_{(2)}^0(U, K_U^p)$. □

Theorem 6.11 shows that the zero-divisors of random cusp forms $\{\sigma_p\}$ (where σ_p is a p -pluricanonical section) are equidistributed with respect to the extension of the Bergman metric on a smooth toroidal compactification of an arithmetic quotient. The equidistribution on the arithmetic quotient D/Γ itself was shown in [23]. In this framework the equidistribution of zeros is a variant of the Quantum Unique Ergodicity conjecture of Rudnick-Sarnak [52], cf. Rudnick [51], Holowinsky and Soundararajan [36], Marshall [46].

The case of arithmetic quotients of dimension 1 is particularly interesting.

COROLLARY 6.14. – *Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index acting freely and properly discontinuously on the hyperbolic plane \mathbb{H} via linear fractional transformations. Set $U = \mathbb{H}/\Gamma$ and let ω be the induced Poincaré metric on U . Let X be a compact Riemann surface such that $U \subset X$ and $X \setminus U = \Sigma$ is a finite set. Let $L = K_X \otimes \mathcal{O}_X(\Sigma)$. Then the space $S_{2p}(\Gamma)$ of cusp forms of weight $2p$ of Γ is isomorphic to $H_{(2)}^0(U, K_U^p)$ and assertions (i)-(iii) of Theorem 6.11 hold for the Fubini-Study currents γ_p defined by $S_{2p}(\Gamma)$ and for the zero-sets of random sequences of cusp forms.*

We can extend the results of this section for the class of invariant line bundles considered by Mumford [47, p. 256].

THEOREM 6.15. – *Let $D, U, X, (E_0, h_0)$ be as above and assume that $iR^{(E_0, h_0)} \geq \varepsilon\omega_D^\beta$ on D , for some $\varepsilon > 0$. Let (E_U, h_U) be the induced line bundle on U and E be its unique extension to X so that the metric h_U on $E|_U$ is good on X . Then h_U extends to a singular Hermitian metric h on E such that $c_1(E, h)$ is a positive current on X which extends $c_1(E_U, h_U)$, and the conclusions of Theorems 1.1, 1.2 and 4.3 hold for the spaces $H_{(2)}^0(U, E^p, h, \omega^n)$ and for $c_1(E, h)$.*

Proof. – Let $x \in \Sigma$ and V_x be a coordinate neighborhood of x as in (10) on which there exists a holomorphic frame e of E . Then the local weights $\varphi = -\log |e|_{h_U}$ verify

$$-\log \left| \sum_{k=1}^l \log |z_k| \right| - \frac{\log C}{2\alpha} \leq \frac{\varphi}{\alpha} \leq \log \left| \sum_{k=1}^l \log |z_k| \right| + \frac{\log C}{2\alpha} \quad \text{on } V_x \setminus \Sigma.$$

Since the metric h_U is positively curved, the function φ is psh on $U \cap V_x = V_x \setminus \Sigma$. Hence φ is psh on V_x , in view of the previous upper bound and of Lemma 6.16 hereafter. Thus $c_1(E, h) \geq 0$ and condition (B) is fulfilled.

To prove that (C) holds, we write $\omega^n = f\Omega^n$ for some fixed positive (1,1) form Ω on X . Let $x \in \Sigma_{\text{reg}}^{n-1}$ and local coordinates z_1, \dots, z_n be chosen so that $x = 0, \Sigma = \{z_1 = 0\}$. Estimate (11) implies that $f \gtrsim |z_1|^{-2} |\log |z_1||^{-2\alpha}$ near x , where $\alpha \geq 1$. Hence $f \geq c_x > 0$ Ω^n -a.e. in a neighborhood U_x of each $x \in (X \setminus \Sigma) \cup \Sigma_{\text{reg}}^{n-1}$.

Condition (3) holds due to [43, Theorem 6.1.1], which applies since $iR^{(E, h)} \geq \varepsilon\omega_U^\beta$ on U . By Theorem 1.1 we infer the conclusion. \square

LEMMA 6.16. – *Let $V \subset \mathbb{C}^n$ be an open set and Σ be a proper analytic subvariety of V . Suppose that u is a psh function on $V \setminus \Sigma$ which verifies*

$$u(z) \leq C_{z_0} \log |\log \text{dist}(z, \Sigma)|$$

for $z \in V \setminus \Sigma$ near each point $z_0 \in \Sigma_{\text{reg}}^{n-1}$, with a constant $C_{z_0} > 0$. Then u is locally upper bounded near each point of Σ hence it extends to a psh function on V .

Proof. – It suffices to show that u is locally upper bounded near each point $z_0 \in \Sigma_{\text{reg}}^{n-1}$. We may assume that $z_0 = 0, \Sigma = \{z_1 = 0\} \subset V = \mathbb{D}^n$ and that $u(z_1, z') \leq C \log |\log |z_1||$ for $z \in V$ with $0 < |z_1| < e^{-1}$, where $C > 0$ is a constant. The function $u(\cdot, z')$ is subharmonic on $\mathbb{D} \setminus \{0\}$, so $r \rightarrow f(r; z') := \max_{|z_1|=r} u(z_1, z')$ is a convex function of $\log r$

for $r > 0$ [37, Corollary 3.2.22]. If $0 < t < r < s < e^{-1}$ we have by the growth assumption on u and the convexity property of $f(r)$ that

$$\frac{f(r; z') - C \log |\log t|}{\log r - \log t} \leq \frac{f(r; z') - f(t; z')}{\log r - \log t} \leq \frac{f(s; z') - f(r; z')}{\log s - \log r}.$$

Letting $t \searrow 0$ it follows that $f(r; z') \leq f(s; z')$, hence the functions $f(\cdot; z')$ are increasing. This implies that u is upper bounded in a neighborhood of $z_0 = 0$. \square

6.5. 1-convex manifolds

A complex manifold X is called 1-convex if there exists a smooth exhaustion function $\psi : X \rightarrow \mathbb{R}$ which is strictly psh outside a compact set of X . This is equivalent to the following condition (see e.g., [1]): There exist a Stein space Y , a proper holomorphic surjective map $\rho : X \rightarrow Y$ satisfying $\rho_* \mathcal{O}_X = \mathcal{O}_Y$, and a finite set $A \subset Y$ such that the induced map $X \setminus \rho^{-1}(A) \rightarrow Y \setminus A$ is biholomorphic. The Stein space Y is called the Remmert reduction of X and $\Sigma := \rho^{-1}(A)$ the exceptional set of X .

Consider a strictly psh smooth exhaustion function φ_Y of Y , such that $\varphi_Y \geq 0$ and $\{\varphi_Y = 0\} = A$ (see e.g., [13, p. 563]). Then $\varphi = \varphi_Y \circ \rho$ is a smooth psh exhaustion function of X , such that $\varphi \geq 0$, $\{\varphi = 0\} = \Sigma$ and φ is strictly psh on $X \setminus \Sigma$.

We consider in the sequel a holomorphic line bundle (L, h) on X with singular metric h , which is smooth outside the exceptional set Σ and has strictly positive curvature current in a neighborhood U of Σ . By using a modification $\pi : \tilde{X} \rightarrow X$ of X we construct as in Section 2.3 the Poincaré metric Θ on $X \setminus \Sigma$ and also the metric h_ε on $L|_{X \setminus \Sigma}$. We may suppose that Θ is complete on $X \setminus \Sigma$ (the metric $\tilde{\Omega}$ on \tilde{X} may be taken to be complete, by setting $\tilde{\Omega} = \Omega \exp(\eta \circ \varphi \circ \pi)$, where Ω is an arbitrary metric on \tilde{X} and $\eta : \mathbb{R} \rightarrow \mathbb{R}$ is a fast increasing function at infinity).

Let us consider a convex increasing function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ and endow L with the Hermitian metric $h_\varepsilon e^{-\chi(\varphi)}$. Consider the L^2 inner product on the space $\Omega_{0,*}^n(X \setminus \Sigma, L^p)$ of sections with compact support, induced by the metrics $h_\varepsilon e^{-\chi(\varphi)}$ on L and Θ on $X \setminus \Sigma$. Set

$$\begin{aligned} L_{0,*}^2(X \setminus \Sigma, L^p) &:= L_{0,*}^2(X \setminus \Sigma, L^p, h_\varepsilon e^{-\chi(\varphi)}, \Theta^n), \\ H_{(2)}^0(X \setminus \Sigma, L^p) &:= L_{0,0}^2(X \setminus \Sigma, L^p) \cap \ker \bar{\partial}. \end{aligned}$$

We denote by $\bar{\partial}_\chi^*$ and $\square_{p,\chi}$ the adjoint of $\bar{\partial}$ with respect to this L^2 inner product and the corresponding Kodaira Laplace operator.

Let us denote by $\mathcal{T} = [i(\Theta), \partial\Theta]$ the Hermitian torsion of the Poincaré metric Θ . Set $\tilde{L}^p = L^p \otimes K_X^*$. There exists a natural isometry

$$\begin{aligned} \Psi &= \sim : \Lambda^{0,q}(T^*X) \otimes L^p \longrightarrow \Lambda^{n,q}(T^*X) \otimes \tilde{L}^p, \\ \Psi s &= \tilde{s} = (w^1 \wedge \cdots \wedge w^n \wedge s) \otimes (w_1 \wedge \cdots \wedge w_n), \end{aligned}$$

where $\{w_j\}_{j=1}^n$ is a local orthonormal frame of $T^{(1,0)}X$ and $\{w^j\}_{j=1}^n$ is the dual frame. The Bochner-Kodaira-Nakano formula [43, Cor. 1.4.17] shows that for any $s \in \Omega_{0,1}^n(X \setminus \Sigma, L^p)$

we have

$$(13) \quad \frac{3}{2} \left(\|\bar{\partial}s\|^2 + \|\bar{\partial}_\chi^* s\|^2 \right) \geq \left(R^{L^p \otimes K_X^*}(w_j, \bar{w}_k) \bar{w}^k \wedge i_{\bar{w}_j} s, s \right) - \frac{1}{2} \left(\|\mathcal{F}^* \tilde{s}\|^2 + \|\bar{\mathcal{F}} \tilde{s}\|^2 + \|\bar{\mathcal{F}}^* \tilde{s}\|^2 \right).$$

Set $T = \frac{1}{2}(\mathcal{F} \mathcal{F}^* + \bar{\mathcal{F}}^* \bar{\mathcal{F}} + \bar{\mathcal{F}} \bar{\mathcal{F}}^*)$. Define the continuous function

$$(14) \quad \tau : X \setminus \Sigma \rightarrow \mathbb{R}, \quad \tau(x) = \sup \{ \langle T\alpha, \alpha \rangle / \langle \alpha, \alpha \rangle : \alpha \in \Lambda^{n,1} T_x^* X \setminus \{0\} \}.$$

Then for any $x \in X \setminus \Sigma$, $p \in \mathbb{N}$ and $\alpha \in L_x^p \otimes \Lambda^{n,1} T_x^* X$ we have

$$\langle T\alpha, \alpha \rangle \leq \tau(x) \langle \alpha, \alpha \rangle.$$

Hence (13) gives for all $s \in \Omega_0^{0,1}(X \setminus \Sigma, L^p)$

$$(15) \quad \frac{3}{2} \left(\|\bar{\partial}s\|^2 + \|\bar{\partial}_\chi^* s\|^2 \right) \geq \left(R^{L^p \otimes K_X^*}(w_j, \bar{w}_k) \bar{w}^k \wedge i_{\bar{w}_j} s, s \right) - \int_{X \setminus \Sigma} \tau(x) |s|^2.$$

LEMMA 6.17. – *There exist an increasing convex function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ and constants $a, b > 0$, such that:*

$$(16a) \quad c_1(L, h_\varepsilon e^{-\chi(\varphi)}) \geq a\Theta,$$

$$(16b) \quad c_1(L, h_\varepsilon e^{-\chi(\varphi)}) + iR^{K_X^*} - \tau\Theta \geq -b\Theta,$$

on $X \setminus \Sigma$.

Proof. – We have

$$c_1(L, h_\varepsilon e^{-\chi(\varphi)}) = c_1(L, h_\varepsilon) + \frac{1}{2} dd^c \chi(\varphi) = c_1(L, h_\varepsilon) + \frac{i}{2\pi} (\chi'(\varphi) \partial \bar{\partial} \varphi + \chi''(\varphi) \partial \varphi \wedge \bar{\partial} \varphi).$$

Since φ is psh, for any increasing convex function χ this is $\geq c_1(L, h_\varepsilon)$, hence positive on U . Thus (16a) holds on U by the construction of h_ε . Moreover, [43, Lemma 6.2.1] shows that $iR^{K_X^*}$ and the torsion operators of Θ , hence τ , are bounded with respect to Θ on U . Thus (16b) also holds on U , thanks to (16a).

Consider $c > 0$ such that $\Sigma \subset X_c \Subset U$, where $X_c := \{\varphi < c\}$. Note that φ is strictly psh outside X_c . Thus we can choose χ increasing fast enough such that (16a)-(16b) are satisfied on $X \setminus \bar{X}_c$. \square

LEMMA 6.18. – *Let $\chi : \mathbb{R} \rightarrow \mathbb{R}$ be as in Lemma 6.17. Then:*

(i) *There exist constants $a_1, b_1 > 0$ such that for any $p \in \mathbb{N}$ we have*

$$(17) \quad \|\bar{\partial}s\|^2 + \|\bar{\partial}_\chi^* s\|^2 \geq (p a_1 - a_1 - b_1) \|s\|^2, \quad s \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}_\chi^*) \cap L_{0,1}^2(X \setminus \Sigma, L^p).$$

(ii) *The spectrum of $\square_{p,\chi}$ on $L_{0,0}^2(X \setminus \Sigma, L^p)$ satisfies*

$$(18) \quad \text{Spec}(\square_{p,\chi}) \subset \{0\} \cup (p a_1 - a_1 - b_1, +\infty).$$

(iii) *The Bergman density function P_p of $H_{(2)}^0(X \setminus \Sigma, L^p)$ has a full asymptotic expansion on any compact set of $X \setminus \Sigma$.*

Proof. – (i) Since $\Omega_0^{0,1}(X \setminus \Sigma, L^p)$ is dense in $\text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}_\chi^*) \cap L_{0,1}^2(X \setminus \Sigma, L^p)$ (Andreotti-Vesentini density lemma, see [43, Lemma 3.3.1]), it suffices to prove (17) for $s \in \Omega_0^{0,1}(X \setminus \Sigma, L^p)$. But in this case, (17) follows immediately from (15) and (16a)-(16b).

(ii) Once we have (17), the assertion about the spectrum of $\square_{p,\chi}$ on $L_{0,0}^2(X \setminus \Sigma, L^p)$ follows as in the proof of [43, Theorem 6.1.1].

(iii) Since the Kodaira Laplacian $\square_{p,\chi}$ on $L_{0,0}^2(X \setminus \Sigma, L^p)$ has a spectral gap, by the argument in [43, §4.1.2], we can localize the problem, and we obtain the result as in the proof of [43, Theorem 4.1.1] (cf. also [43, Theorem 6.1.1], where a similar situation is treated). \square

THEOREM 6.19. – *Let X be a 1-convex manifold and (L, h) be a holomorphic line bundle on X with singular metric h . Assume that h is smooth outside the exceptional set Σ and that it has strictly positive curvature current in a neighborhood of Σ . Let Θ be a complete Poincaré metric on X , h_ε be constructed in Section 2.3, and let χ be as in Lemma 6.17. The conclusions of Theorem 1.1 hold for the spaces $H_{(2)}^0(X \setminus \Sigma, L^p, h_\varepsilon e^{-\chi(\varphi)}, \Theta^n)$ and for $\omega = c_1(L, h_\varepsilon e^{-\chi(\varphi)})$.*

Proof. – Conditions (A)-(C) are satisfied by construction and condition (3) follows from Lemma 6.18. \square

6.6. Strongly pseudoconvex domains

We give now a variant ‘with boundary’ of the previous result. Let M be a complex manifold and let $X \Subset M$ be a strongly pseudoconvex domain with smooth boundary. We consider a defining function $\varrho \in \mathcal{C}^\infty(M, \mathbb{R})$ of X , i.e., $X = \{x \in M : \varrho(x) < 0\}$ and $d\varrho \neq 0$ on ∂X . Since X is strongly pseudoconvex, the Levi form of ϱ is positive definite on the complex tangent space to ∂X . It is well-known that one can modify the defining function ϱ such that in a neighborhood of ∂X , ϱ is strictly psh and $d\varrho \neq 0$ (see [34, Ch. 9, Sec. A, Proposition 4]). Thus, for $c \geq 0$ small enough, $X_c = \{x \in M : \varrho(x) < c\}$ is strongly pseudoconvex.

Let $\eta_c : (-\infty, c) \rightarrow \mathbb{R}$ be a convex increasing function such that $\eta_c(t) \rightarrow \infty$, as $t \rightarrow c$. Then $\eta_c \circ \varrho$ is an exhaustion function for X_c , which is strictly psh outside a compact set of X_c . Therefore X_c is a 1-convex manifold.

Let Σ be the exceptional set of X_c (it is the same exceptional set as for X) and let $\varphi : X_c \rightarrow \mathbb{R}$ be a smooth psh exhaustion function of X_c , such that $\varphi \geq 0$, $\{\varphi = 0\} = \Sigma$ and φ is strictly psh on $X_c \setminus \Sigma$.

Let (L, h) be a holomorphic line bundle on M with singular metric h which is smooth outside the exceptional set Σ and which has strictly positive curvature current in a neighborhood U of Σ . By using a modification \widetilde{M} of M we construct as in Section 2.3 the Poincaré metric Θ on $M \setminus \Sigma$ and also the metric h_ε on $L|_{M \setminus \Sigma}$.

Let $A > 0$. On the space $\Omega_0^{0,*}(\overline{X} \setminus \Sigma, L^p)$ of sections with compact support in $\overline{X} \setminus \Sigma$ we introduce the L^2 inner product with respect to the metrics Θ and $h_\varepsilon e^{-A\varphi}$ and set

$$\begin{aligned} L_{0,*}^2(X \setminus \Sigma, L^p) &:= L_{0,*}^2(X \setminus \Sigma, L^p, h_\varepsilon e^{-A\varphi}, \Theta^n), \\ H_{(2)}^0(X \setminus \Sigma, L^p) &:= L_{0,0}^2(X \setminus \Sigma, L^p) \cap \ker \bar{\partial}. \end{aligned}$$

We consider the $L^2 \bar{\partial}$ -Neumann problem on $\overline{X} \setminus \Sigma$ and show that the $\bar{\partial}$ -Neumann Laplacian on $L_{0,1}^2(X \setminus \Sigma, L^p)$ has a spectral gap. Here we work with $\bar{\partial}$ -Neumann boundary conditions at the end ∂X of $\overline{X} \setminus \Sigma$ and with a complete metric at the end corresponding to Σ . This kind

of analysis was already used in [45] in connection to the compactification of hyperconcave manifolds.

We denote by $\bar{\partial}^* = \bar{\partial}^{L^p,*}$ the Hilbert space adjoint of the maximal extension of $\bar{\partial}$ on $L^2_{0,1}(X \setminus \Sigma, L^p)$. Denoting by $*$ the Hodge star operator, integration by parts as in [28, Proposition 1.3.1–2] yields

$$\begin{aligned} B^{0,1}(\bar{X} \setminus \Sigma, L^p) &:= \{s \in \Omega^{0,1}_0(\bar{X} \setminus \Sigma, L^p) : *d\varrho \wedge *s = 0 \text{ on } \partial X\} \\ &= \text{Dom}(\bar{\partial}^*) \cap \Omega^{0,1}_0(\bar{X} \setminus \Sigma, L^p). \end{aligned}$$

The space $B^{0,1}(\bar{X} \setminus \Sigma, L^p)$ is dense in $\text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$ with respect to the graph norm $s \mapsto (\|s\|^2 + \|\bar{\partial}s\|^2 + \|\bar{\partial}^*s\|^2)^{1/2}$ (cf. [45, Lemma 2.2]).

Let us consider a defining function ϱ of X such that $|d\varrho| = 1$ on ∂X . We denote by \mathcal{L}_ϱ the Levi form of ϱ (cf. [43, Def. 1.4.20]). The Bochner-Kodaira-Nakano formula with boundary term [43, Cor. 1.4.22] shows that for any $s \in B^{0,1}(\bar{X} \setminus \Sigma, L^p)$ we have

$$\begin{aligned} \frac{3}{2} \left(\|\bar{\partial}s\|^2 + \|\bar{\partial}^*s\|^2 \right) &\geq \left(R^{L^p \otimes K_X^*}(w_j, \bar{w}_k) \bar{w}^k \wedge i_{\bar{w}_j} s, s \right) \\ &\quad + \int_{\partial X} \mathcal{L}_\varrho(s, s) dv_{\partial X} - \frac{1}{2} \left(\|\mathcal{T}^* \tilde{s}\|^2 + \|\bar{\mathcal{T}} \tilde{s}\|^2 + \|\bar{\mathcal{T}}^* \tilde{s}\|^2 \right). \end{aligned}$$

Since X is strongly pseudoconvex the boundary integral is non-negative. Therefore we obtain for all $s \in B^{0,1}(\bar{X} \setminus \Sigma, L^p)$ the estimate

$$\frac{3}{2} \left(\|\bar{\partial}s\|^2 + \|\bar{\partial}^*s\|^2 \right) \geq \left(R^{L^p \otimes K_X^*}(w_j, \bar{w}_k) \bar{w}^k \wedge i_{\bar{w}_j} s, s \right) - \int_{\bar{X} \setminus \Sigma} \tau(x) |s|^2,$$

where τ is defined on $X_c \setminus \Sigma$ as in (14). Making use of the compactness of \bar{X} we obtain:

LEMMA 6.20. – *There exist constants $A_0, a, b > 0$ such that for any $A > A_0$ the $(1, 1)$ current $c_1(L, h_\varepsilon e^{-A\varphi})$ is strictly positive on a neighborhood of \bar{X} and*

$$(19a) \quad c_1(L, h_\varepsilon e^{-A\varphi}) \geq a\Theta,$$

$$(19b) \quad c_1(L, h_\varepsilon e^{-A\varphi}) + iR^{K_X^*} - \tau\Theta \geq -b\Theta,$$

on $\bar{X} \setminus \Sigma$.

Let us now fix $A > 0$ as in Lemma 6.20. Using (15), (19a) and (19b), we deduce immediately the estimate (17) for any $s \in B^{0,1}(\bar{X} \setminus \Sigma, L^p)$ and, by density, for any $s \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \cap L^2_{0,1}(X \setminus \Sigma, L^p)$. This shows that \square_p acting on $L^2_{0,0}(X \setminus \Sigma, L^p)$ has a spectral gap as in (18). Therefore, the Bergman density function P_p of $H^0_{(2)}(X \setminus \Sigma, L^p)$ has a full asymptotic expansion on any compact set of $X \setminus \Sigma$.

The preceding discussion leads to the following:

THEOREM 6.21. – *Let X be a strongly pseudoconvex domain with smooth boundary in a complex manifold M . Let (L, h) be a holomorphic line bundle on M with singular metric h which is smooth outside the exceptional set Σ and which has strictly positive curvature current in a neighborhood U of Σ . The conclusions of Theorem 1.1 hold for the spaces $H^0_{(2)}(X \setminus \Sigma, L^p, h_\varepsilon e^{-A\varphi}, \Theta^n)$ and for $\omega = c_1(L, h_\varepsilon e^{-A\varphi})$, where A is sufficiently large.*

REMARK 6.22. – *In the same vein, we can obtain a variant of Theorem 6.21 for Nadel multiplier ideal sheaves. Assume that $X \Subset M$ is a strongly pseudoconvex domain as above. Let (L, h) be a holomorphic line bundle on M with singular metric h which is smooth outside the exceptional set Σ . Assume for simplicity that the curvature current of h is strictly positive in a whole neighborhood of \overline{X} . The conclusions of Theorem 1.1 hold for the spaces $H^0(X, L^p \otimes \mathcal{I}(h_p))$ (defined as in (8)) and for $\gamma = c_1(L, h)$.*

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