Holomorphic Sections of Line Bundles Vanishing along Subvarieties

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ABSTRACT. Let X be a compact normal complex space of dimension n, and L be a holomorphic line bundle on X. Suppose $\Sigma = (\Sigma_1, \dots, \Sigma_\ell)$ is an ℓ-tuple of distinct irreducible proper analytic subsets of X, $\tau = (\tau_1, \dots, \tau_\ell)$ is an ℓ-tuple of positive real numbers, and consider the space $H^0_0(X, L^p)$ of global holomorphic sections of $L^p := L^{\otimes p}$ that vanish to order at least $\tau_j p$ along Σ_j, $1 \le j \le \ell$. We find necessary and sufficient conditions which ensure that $\dim H^0_0(X, L^p) \sim p^n$, analogous to Ji-Shiffman's criterion for big line bundles. We give estimates of the partial Bergman kernel, investigate the convergence of the Fubini-Study currents and their potentials, and the equilibrium distribution of normalized currents of integration along zero divisors of random holomorphic sections in $H^0_0(X, L^p)$ as $p \to \infty$. Regularity results for the equilibrium envelope are also included.

CONTENTS

1. INTRODUCTION

Let (X, L) be a polarized projective manifold of dimension n, let Σ be a complex hypersurface of X, and let τ be a positive real number. The study of holomorphic sections of L^p which vanish to order at least $p\tau$ along Σ received much attention in the past few years. The density function of this space, called partial Bergman kernel, appears in a natural way in several contexts, and especially in Kähler geometry and pluripotential theory, linked to the notion of extremal quasiplurisubharmonic (qpsh) functions with poles along Σ (see, e.g., [Be1, RoS, PS, RWN1, RWN2, CM3, ZZ]). One of the motivations is the notion of slope of the hypersurface Σ in the sense of Ross-Thomas [RT06] and its relation to the existence of a constant scalar curvature Kähler metric in $c_1(L)$.

In this paper, we consider a compact normal complex space X of dimension n, a holomorphic line bundle L over X, and the space $H^0_0(X, L^p)$ of holomorphic sections vanishing to order at least $p\tau_j$ along irreducible proper analytic subsets $\Sigma_j \subset X$, $j = 1, \dots, \ell$. We study algebraic and analytic objects associated with $H^0_0(X, L^p)$, especially the partial Bergman kernels, the Fubini-Study currents, and their potentials.

We first give an analytic characterization for $H^0_0(X, L^p)$ to be big, which means by definition that $\dim H^0_0(X, L^p) \sim p^n$, $p \to \infty$. This criterion, stated in terms of singular Hermitian metrics with positive curvature current in the spirit of the Ji-Shiffman/Bonavero/Takayama criterion for big line bundles, involves a desingularization of X where the Σ_j become divisors.

Next, we prove that under natural hypotheses the Fubini-Study currents associated with $H^0_0(X, L^p)$ and their potentials converge as $p \to \infty$. The limit of the sequence of Fubini-Study potentials is the pushforward φ_{eq} of a certain equilibrium envelope with logarithmic poles defined on a desingularization. The sequence of the Fubini-Study currents converge to the corresponding equilibrium current T_{eq}. These are analogues of Tian's theorem [T], which applies for smooth Hermitian metrics with positive curvature. In the context of singular Hermitian metrics, they were introduced in [CM1, CM2]. The convergence of the Fubini-Study currents/potentials is based on the asymptotics of the logarithm of the partial Bergman kernel (see also [CM1, CM2, CMM, DMM] for results of this type concerning the full Bergman kernel).

Returning to the case of a polarized projective manifold (X, L), Shiffman-Zelditch [SZ] showed how Tian's theorem can be applied to obtain the distribution of the zeros of random holomorphic sections of $H^0(X, L^p)$. Dinh-Sibony [DS] used meromorphic transforms to obtain an estimate on the speed of convergence of zeros to the equilibrium distribution (see also [DMS] for the non-compact setting). Random polynomials (or more generally, holomorphic sections in high tensor powers of a holomorphic line bundle) and the distribution of their zeros represent a classical subject in analysis (see, e.g., [BP, ET, H, Ka]). The result of [SZ] was generalized for singular metrics whose curvature is a Kähler current in [CM1] and for sequences of line bundles over normal complex spaces in [CMM] (see also [CM2, DMM]). We show here that the equilibrium distribution of random zeros of sections from $H^0_0(X, L^p)$ is the equilibrium current $T_{\rm eq}$, and we give an estimate on the convergence speed.

1.1. Background and notation. Let X be a compact normal complex space of dimension n. If L is a holomorphic line bundle on X, we let $L^p := L^{\otimes p}$ and denote by $H^0(X, L^p)$ the space of global holomorphic sections of L^p. Since X is compact, the spaces $H^0(X, L^p)$ are finite dimensional. Given $S \in H^0(X, L^p)$, we denote by $[S = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{S = 0\} \subset X$. If h is a singular Hermitian metric on L we denote by $c_1(L, h)$ its *curvature current*.

Suppose now X is a compact complex manifold. For a closed current T of bidegree $(1, 1)$ on X, let $\{T\}$ denote its class in the Dolbeault cohomology group $H^{1,1}(X)$. If L is a holomorphic line bundle over X we denote by $c_1(L)$ its first Chern class in $H^{1,1}(X, \mathbb{Z})$. We have that $\{c_1(L, h)\} = c_1(L)$, for any singular Hermitian metric h on L. The line bundle L is called *big* if its Kodaira-Iitaka dimension equals the dimension of X (see [MM, Definition 2.2.5]). One has that L is big if and only if $\limsup_{p\to\infty} p^{-n} \dim H^0(X, L^p) > 0$ (see Theorem 2.2.7 in [MM]). By the Ji-Shiffman/Bonavero/Takayama criterion [MM, Theorem 2.3.30], L is big if and only if it admits a strictly positively curved singular Hermitian metric h (see Section 2.1 for definitions).

Throughout the article, we denote by $\lfloor r \rfloor$ the greatest integer $\leq r \in \mathbb{R}$, and we let

$$d^c := (1/(2\pi i))(\partial - \bar\partial),$$

so $dd^c = (i/\pi)\, \partial \bar\partial$.

1.2. Sections vanishing along subvarieties. We consider in this paper the following general setting:

(A) X is a compact, irreducible, normal (reduced) complex space of dimension n, $X_{\rm reg}$ denotes the set of regular points of X, and $X_{\rm sing}$ denotes the set of singular points of X.
(B) L is a holomorphic line bundle on X.
(C) $\Sigma = (\Sigma_1, \dots, \Sigma_\ell)$ is an ℓ-tuple of distinct irreducible proper analytic subsets of X such that $\Sigma_j \not\subset X_{\rm sing}$, for every $j \in \{1, \dots, \ell\}$.
(D) $\tau = (\tau_1, \dots, \tau_\ell)$ is an ℓ-tuple of positive real numbers such that $\tau_j > \tau_k$, for every $j, k \in \{1, \dots, \ell\}$ with $\Sigma_j \subset \Sigma_k$.

For $p \geq 1$ let $H^0_0(X, L^p)$ be the space of sections $S \in H^0(X, L^p)$ that vanish to order at least $\tau_j p$ along Σ_j, for all $1 \leq j \leq \ell$. More precisely, let

$$t_{j,p} = \begin{cases} \tau_j p & \text{if } \tau_j p \in \mathbb{N}, \\ \lfloor \tau_j p \rfloor + 1 & \text{if } \tau_j p \notin \mathbb{N}, \end{cases} \tag{1.1}$$

for $1 \le j \le \ell$, $p \ge 1$. Then,

$$\begin{aligned} (1.2) \qquad H^0_0(X, L^p) &= H^0_0(X, L^p, \Sigma, \tau) \\ &:= \{S \in H^0(X, L^p) : \operatorname{ord}(S, \Sigma_j) \ge t_{j,p},\ 1 \le j \le \ell\}, \end{aligned}$$

where $\operatorname{ord}(S, Z)$ denotes the vanishing order of S along an irreducible analytic subset Z of X, $Z \not\subset X_{\rm sing}$.

Definition 1.1. We say that the triplet (L, Σ, τ) is *big* if

$$\limsup_{p\to\infty} \frac{\dim H^0_0(X, L^p)}{p^n} > 0.$$

The first problem we address in this article is the following.

Problem 1.2. *Characterize the big triplets* (L, Σ, τ).

We first give an answer to Problem 1.2 in the case when X is a complex manifold and Σ_j are irreducible hypersurfaces in X. In particular, we have the following analog of Ji-Shiffman's criterion for big line bundles [JS, Theorem 4.6] (see also [Bon], [MM, Theorem 2.3.30]).

Theorem 1.3. *Let* X, L, Σ, τ *verify assumptions* (A)–(D), *and suppose that* X *is smooth and* $\dim \Sigma_j = n - 1$ *for all* $j = 1, \dots, \ell$. *The following are equivalent:*

(i) (L, Σ, τ) *is big.*

(ii) *There is a singular Hermitian metric* h *on* L *such that* $c_1(L, h) - \sum_{j=1}^{\ell} \tau_j [\Sigma_j]$ *is a Kähler current on* X.

(iii) *There exist* $p_0 \in \mathbb{N}$ *and* $c > 0$ *such that* $\dim H^0_0(X, L^p) \ge cp^n$ *for all* $p \ge p_0$.

Here, $[\Sigma_j]$ denotes the current of integration along Σ_j. Recall that a Kähler current is a positive closed current T of bidegree $(1,1)$ such that $T \ge \varepsilon\omega$ for some number $\varepsilon > 0$ and some Hermitian form ω on X. To find a solution to Problem 1.2 in the general case, we first use Hironaka's theorem on resolution of singularities to prove the following result.

Proposition 1.4. *Let* X *and* Σ *verify assumptions* (A) *and* (C). *Then, there exist a compact complex manifold* $\tilde X$ *of dimension* n *and a surjective holomorphic map* $\pi : \tilde X \to X$, *given as the composition of finitely many blowups with smooth center, with the following properties:*

(i) *There is an analytic subset* Y *of* X *so that* $\dim Y \le n-2$, $Y \subset X_{\rm sing} \cup \bigcup_{j=1}^{\ell} \Sigma_j$, $X_{\rm sing} \subset Y$, $\Sigma_j \subset Y$ *if* $\dim \Sigma_j \le n - 2$, $E = \pi^{-1}(Y)$ *is a divisor in* $\tilde X$ *that has only normal crossings, and* $\pi : \tilde X \setminus E \to X \setminus Y$ *is a biholomorphism.*

(ii) *There exist (connected) smooth complex hypersurfaces* $\tilde\Sigma_1, \dots, \tilde\Sigma_\ell$ *in* $\tilde X$, *which have only normal crossings, such that* $\pi(\tilde\Sigma_j) = \Sigma_j$. *Moreover, if* $\dim \Sigma_j = n - 1$ *then* $\tilde\Sigma_j$ *is the final strict transform of* Σ_j, *and if* $\dim \Sigma_j \le n - 2$ *then* $\tilde\Sigma_j$ *is an irreducible component of* E.

(iii) *If $F \to X$ is a holomorphic line bundle and $S \in H^0(X,F)$, then*

$$\operatorname{ord}(S,\Sigma_j) = \operatorname{ord}(\pi^\star S,\tilde{\Sigma}_j), \quad \text{for all } j = 1,\dots,\ell.$$

Definition 1.5. If $\tilde{X}$, π, $\tilde{\Sigma} := (\tilde{\Sigma}_1,\dots,\tilde{\Sigma}_\ell)$ verify the conclusions of Proposition 1.4, we say that $(\tilde{X},\pi,\tilde{\Sigma})$ is a divisorization of (X,Σ).

Divisorizations are not unique. Note that if X is a manifold and $\Sigma_1,\dots,\Sigma_\ell$ are smooth hypersurfaces with normal crossings, then (X,Id,Σ) is a divisorization of (X,Σ), where Id is the identity map. We now give an answer to Problem 1.2 in the general case.

Theorem 1.6. *Let X,L,Σ,τ verify assumptions (A)–(D). The following are equivalent:*

(i) *(L,Σ,τ) is big.*
(ii) *For every divisorization $(\tilde{X},\pi,\tilde{\Sigma})$ of (X,Σ), there exists a singular Hermitian metric $h^\star$ on $\pi^\star L$ such that $c_1(\pi^\star L,h^\star)-\sum_{j=1}^{\ell}\tau_j[\tilde{\Sigma}_j]$ is a Kähler current on $\tilde{X}$.*
(iii) *There exist a divisorization $(\tilde{X},\pi,\tilde{\Sigma})$ of (X,Σ) and a singular Hermitian metric $h^\star$ on $\pi^\star L$ such that $c_1(\pi^\star L,h^\star)-\sum_{j=1}^{\ell}\tau_j[\tilde{\Sigma}_j]$ is a Kähler current on $\tilde{X}$.*
(iv) *There exist $p_0 \in \mathbb{N}$ and $c > 0$ such that $\dim H^0_0(X,L^p) \geq cp^n$ for all $p \geq p_0$.*

An interesting consequence of Theorem 1.6 is the following. Assume that (L,Σ,τ) is big and all Σ_j have dimension $n-1$. If one fixes proper analytic subsets $\Sigma'_j \subset \Sigma_j$ and considers the subspace $V_p \subset H^0_0(X,L^p)$ of sections that vanish to the higher order $(\tau_j+\delta)p$ along Σ'_j, then it holds as well that $\dim V_p \gtrsim p^n$ for all p large enough, provided that $\delta > 0$ is sufficiently small (see Corollary 3.8 for the precise statement).

Proposition 1.4, Theorem 1.3, and Theorem 1.6 are proved in Section 3.

1.3. Equidistribution of zeros. Let X,L,Σ,τ verify assumptions (A)–(D), and assume in addition there exists a Kähler form ω on X and that h is a singular Hermitian metric on L. We fix a smooth Hermitian metric h_0 on L and write

$$\alpha := c_1(L,h_0), \quad h = h_0 e^{-2\varphi}, \tag{1.3}$$

where $\varphi \in L^1(X,\omega^n)$ is called the *(global) weight of h relative to h_0*. The metric h is called bounded, continuous, respectively Hölder continuous, if φ is a bounded, continuous, respectively Hölder continuous, function on X.

Let $H^0_{(2)}(X,L^p) = H^0_{(2)}(X,L^p,h^p,\omega^n)$ be the Bergman space of L^2-holomorphic sections in L^p relative to the metric $h^p := h^{\otimes p}$ and the volume form ω^n on X, endowed with the inner product

$$(S,S')_p := \int_X \langle S,S'\rangle_{h^p}\frac{\omega^n}{n!},$$

and set $\|S\|_p^2 := (S,S)_p$. Let

$$H^0_{0,(2)}(X,L^p) = H^0_{0,(2)}(X,L^p,\Sigma,\tau,h^p,\omega^n) := H^0_{(2)}(X,L^p) \cap H^0_0(X,L^p)$$

be the Bergman subspace of L^2-holomorphic sections in $H^0_0(X,L^p)$, where the space $H^0_0(X,L^p)$ was defined in (1.2). We assume in the sequel that the metric h is bounded, so

$$H^0_{(2)}(X,L^p) = H^0(X,L^p),$$
$$H^0_{0,(2)}(X,L^p) = H^0_0(X,L^p).$$

For every $p \geq 1$ we consider the projective space

$$\mathbb{X}_p := \mathbb{P}H^0_{0,(2)}(X,L^p),\ d_p := \dim \mathbb{X}_p = \dim H^0_{0,(2)}(X,L^p) - 1,$$

equipped with the Fubini-Study volume $\sigma_p = \omega_{\mathrm{FS}}^{d_p}$, where by ω_{FS} we denote the Fubini-Study Kähler form on a projective space $\mathbb{P}^N$. We also consider the probability space

$$(\mathbb{X}_\infty,\sigma_\infty) := \prod_{p=1}^{\infty}(\mathbb{X}_p,\sigma_p).$$

The second problem we address in this article is the following.

Problem 1.7. *Assume that (L,Σ,τ) is big and the metric h is bounded. Do zeros of sequences from $(\mathbb{X}_\infty,\sigma_\infty)$ equidistribute towards a positive closed current T of bidegree $(1,1)$? That is, for σ_∞-almost every $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$, do we have*

$$\frac{1}{p}[s_p = 0] \to T \quad \text{as } p \to \infty,$$

in the weak sense of currents on X? If yes, express T in terms of h and estimate the speed of convergence.

The bigness of (L,Σ,τ) is a reasonable assumption, in order to ensure that the spaces $H^0_0(X,L^p)$ have sufficiently many sections. Let P_p,γ_p be the Bergman kernel function and Fubini-Study current of $H^0_{0,(2)}(X,L^p)$, defined in (2.1) and (2.3). Then,

$$\frac{1}{p}\gamma_p = c_1(L,h) + \frac{1}{2p}dd^c\log P_p = \alpha + dd^c\varphi_p, \tag{1.4}$$
$$\text{where } \varphi_p = \varphi + \frac{1}{2p}\log P_p.$$

We call the function φ_p the *global Fubini-Study potential* of γ_p. To answer Problem 1.7, we first study the convergence of the Fubini-Study currents. We have the following result.

Theorem 1.8. *Let X, L, Σ, τ verify assumptions* (A)–(D), *and assume* (L, Σ, τ) *is big and there exists a Kähler form ω on X. Let h be a continuous Hermitian metric on L and α, φ_p be defined in* (1.3), *respectively* (1.4). *Then, there exists an α-psh function $\varphi_{\rm eq}$ on X such that, as $p \to \infty$,*

$$\begin{aligned}&\varphi_p \to \varphi_{\rm eq} \quad \text{in } L^1(X, \omega^n),\\ &\frac{1}{p}\gamma_p = \alpha + dd^c\varphi_p \to T_{\rm eq} := \alpha + dd^c\varphi_{\rm eq} \quad \text{weakly on } X.\end{aligned}$$

Moreover, if h is Hölder continuous then there exist a constant $C > 0$ and $p_0 \in \mathbb{N}$ such that

$$\int_X |\varphi_p - \varphi_{\rm eq}|\omega^n \leq C\frac{\log p}{p}, \quad \text{for all } p \geq p_0.$$

Definition 1.9. The current $T_{\rm eq}$ from Theorem 1.8 is called *the equilibrium current associated with* (L, h, Σ, τ).

Theorem 1.8 is proved in Section 5. The function $\varphi_{\rm eq}$ is constructed as follows. Let $(\tilde{X}, \pi, \tilde{\Sigma})$ be a divisorization of (X, Σ) as in Definition 1.5, and let $\tilde{\alpha} = \pi^\star\alpha$, $\tilde{\varphi} = \varphi \circ \pi$. We introduce in Section 4 the equilibrium envelope $\tilde{\varphi}_{\rm eq}$ of $(\tilde{\alpha}, \tilde{\Sigma}, \tau, \tilde{\varphi})$, as the largest $\tilde{\alpha}$-psh function dominated by $\tilde{\varphi}$ on $\tilde{X}$, and with logarithmic poles of order τ_j along $\tilde{\Sigma}_j$, $1 \leq j \leq \ell$ (see (4.5), (5.5)). In Theorem 4.3 we study the regularity of $\tilde{\varphi}_{\rm eq}$ when $\tilde{\varphi}$ is continuous, respectively Hölder continuous, and show that $\tilde{\varphi}_{\rm eq}$ is continuous outside a certain analytic subset of $\tilde{X}$, respectively Hölder continuous, with singularities along that analytic subset (see Definition 4.2). The function $\varphi_{\rm eq}$ is then constructed by pushing down $\tilde{\varphi}_{\rm eq}$ to X.

Theorem 1.8 is a generalization of the following foundational result of Tian [T] (with improvements by [Ca, R, Z1]; see also [MM, Theorem 5.1.4]): if X is a compact Kähler manifold and $(L, h) \to X$ is a positive line bundle (with smooth metric h), then $\varphi_p \to \varphi$ and $(1/p)\gamma_p \to c_1(L, h)$ as $p \to \infty$ in the C^∞-topology. If h is a singular metric whose curvature is a Kähler current, it was shown in [CM1, Theorem 5.1] that $\varphi_p \to \varphi$ in $L^1(X, \omega^n)$ and $(1/p)\gamma_p \to c_1(L, h)$ weakly as $p \to \infty$. On the other hand, Bloom [B1, B2] (cf. also Bloom-Levenberg [BL]) pointed out the role of the extremal plurisubharmonic (psh) functions in equidistribution theory for polynomials and Berman [Be1, Be2] extended this point of view to the context of Kähler manifolds. In [DMM, Theorem 1.3] it is shown that in the case of a polarized projective manifold (X, L) and for a Hölder continuous weight φ, we have $\|\varphi_p - \varphi_{\rm eq}\|_\infty = O(p^{-1}\log p)$ as $p \to \infty$. We also note that the statement of Theorem 1.8 is new even in the case when X is smooth and $\Sigma = \varnothing$ (see Corollary 5.7).

Using Theorem 1.8, we obtain a positive answer to the above equidistribution problem in the case when the metric h is continuous. In this formulation it can be seen as a large deviation principle in this context.

Theorem 1.10. *Let X, L, Σ, τ verify assumptions* (A)–(D)*, let h be a singular Hermitian metric on L, and assume that (L, Σ, τ) is big and there exists a Kähler form ω on X. We have the following:*

(i) *If h is continuous then $(1/p)[s_p = 0] \to T_{\mathrm{eq}}$ as $p \to \infty$, in the weak sense of currents on X, for σ_∞-almost every $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$.*

(ii) *If h is Hölder continuous then there exists a constant $c > 0$ with the following property: for any sequence of positive numbers $\{\lambda_p\}_{p\geq 1}$ such that*

$$\liminf_{p\to\infty} \frac{\lambda_p}{\log p} > (1+n)c,$$

there exist subsets $E_p \subset \mathbb{X}_p$ such that, for all p sufficiently large, the following hold:

(a) *$\sigma_p(E_p) \leq cp^n \exp(-\lambda_p/c)$.*

(b) *If $s_p \in \mathbb{X}_p \setminus E_p$ we have*

$$\left| \left\langle \frac{1}{p}[s_p = 0] - T_{\mathrm{eq}}, \phi \right\rangle \right| \leq \frac{c\lambda_p}{p} \|\phi\|_{C^2},$$

for any $(n-1, n-1)$-form ϕ of class C^2 on X.

In particular, the last estimate holds for σ_∞-almost every $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$ provided that p is large enough.

The proof of Theorem 1.10 is given in Section 6. We refer to [BCM, B1, B2, BL, CM1, CM2, CM3, CMM, CMN2, DMM, DMS, DS, SZ] and to the surveys [BCHM, Z2] for equidistribution results for holomorphic sections in various contexts.

We close the introduction with the following remark concerning the spaces and approach used in this paper. If g is a singular Hermitian metric on L with $c_1(L, g) \geq 0$, it is well known that for p sufficiently large, sections in the Bergman space $H^0_{(2)}(X, L^p, g^p, \omega^n)$ must vanish to high order at the points where the psh weights of g have positive Lelong number. In particular, if the Lelong number equals ν along an analytic hypersurface A, then any section in $H^0_{(2)}(X, L^p, g^p, \omega^n)$ vanishes to order at least $\lfloor \nu p \rfloor$ along A. Note that

$$H^0_{(2)}(X, L^p, g^p, \omega^n) = H^0(X, L^p \otimes \mathcal{I}(g^p)),$$

where $\mathcal{I}(g^p)$ is the multiplier ideal sheaf associated with g^p. One can try to study the dimension growth of our spaces $H^0_0(X, L^p)$ by constructing special metrics g with singularities in Σ such that $H^0_{(2)}(X, L^p, g^p, \omega^n) \subset H^0_0(X, L^p)$. The existence of such metrics is in general unclear, and as seen in Section 4, it requires additional hypotheses even in the simplest case of hypersurfaces. The result of Theorem 1.6 is very general, as it holds for a singular space X which is not assumed to be Kähler. The study of Bergman spaces and their Bergman kernels does require that X be assumed to be Kähler.

2. PRELIMINARIES

We start by recalling a few notions of pluripotential theory on analytic spaces that will be needed throughout the paper. We then recall some basic facts about Bergman kernels and Fubini-Study currents.

2.1. Compact complex manifolds and analytic spaces. Let X be a compact complex manifold, and let ω be a Hermitian form on X. If T is a positive closed current on X we denote by $\nu(T,x)$ the Lelong number of T at $x \in X$ (see, e.g., [D5]). A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is called *quasiplurisubharmonic* (qpsh) if it is locally the sum of a psh function and a smooth one. Let α be a smooth real closed $(1,1)$-form on X. A qpsh function φ is called *α-plurisubharmonic* (α-psh) if $\alpha + dd^c\varphi \geq 0$ in the sense of currents. We denote by $\mathrm{PSH}(X,\alpha)$ the set of all α-psh functions on X. The Lelong number of an α-psh function φ at a point $x \in X$ is defined by $\nu(\varphi, x) := \nu(\alpha + dd^c\varphi, x)$. Note that if $\varphi = u + \chi$ near x, where u is psh and χ is smooth, then $\nu(\varphi,x) = \nu(u,x)$.

Since in general the $\partial\bar\partial$-lemma does not hold on X, we will consider the $\partial\bar\partial$-cohomology and particularly the space $H^{1,1}_{\partial\bar\partial}(X,\mathbb{R})$ (see, e.g., [Bou]). This space is finite dimensional, and if α is a smooth real closed $(1,1)$-form on X we denote its $\partial\bar\partial$-cohomology class by $\{\alpha\}_{\partial\bar\partial}$. Note that if X is a compact Kähler manifold, then by the $\partial\bar\partial$-lemma $H^{1,1}_{\partial\bar\partial}(X,\mathbb{R}) = H^{1,1}(X,\mathbb{R})$, and we write $\{\alpha\}_{\partial\bar\partial} = \{\alpha\}$.

Definition 2.1. A positive closed current T of bidegree $(1,1)$ on X is called a *Kähler current* if $T \geq \varepsilon\omega$ for some number $\varepsilon > 0$. A class $\{\alpha\}_{\partial\bar\partial}$ is called *big* if it contains a Kähler current.

Suppose that $\{\alpha\}_{\partial\bar\partial}$ is big. By Demailly's regularization theorem [D4], one can find a Kähler current $T \in \{\alpha\}_{\partial\bar\partial}$ with *analytic singularities*, that is, of the form $T = \alpha + dd^c\varphi \geq \varepsilon\omega$, where $\varepsilon > 0$ and φ is a qpsh function such that

$$\varphi = c\log\Big(\sum_{j=1}^{N} |g_j|^2\Big) + \chi,$$

locally on X, where $c > 0$, χ is a bounded function, and g_j are holomorphic functions, $1 \leq j \leq N$. Moreover, there is a (global) proper modification $\sigma : \tilde X \to X$, obtained as a sequence of blowups with smooth centers and with blowup locus contained in the analytic subset $\{\varphi = -\infty\}$, such that the functions $\chi \circ \sigma$ are smooth (see, e.g., [DP, Theorem 3.2]).

The *non-ample locus* of $\{\alpha\}_{\partial\bar\partial}$ is defined in [Bou, Definition 3.16] as the set

$$\begin{aligned}\mathrm{NAmp}(\{\alpha\}_{\partial\bar\partial}) &= \bigcap\{E_+(T) : T \in \{\alpha\}_{\partial\bar\partial} \text{ Kähler current}\}\\ &= \bigcap\Big\{E_+(T) : T \in \{\alpha\}_{\partial\bar\partial} \text{ Kähler current}\\ &\qquad\qquad \text{with analytic singularities}\Big\},\end{aligned}$$

where $E_+(T) = \{x \in X : \nu(T, x) > 0\}$, and the second equality follows by Demailly's regularization theorem [D4]. Hence, $\mathrm{NAmp}(\{\alpha\}_{\partial\bar\partial})$ is an analytic subset of X. The *ample locus* of α is $\mathrm{Amp}(\{\alpha\}_{\partial\bar\partial}) := X \setminus \mathrm{NAmp}(\{\alpha\}_{\partial\bar\partial})$. It is shown in [Bou, Theorem 3.17] there exists a Kähler current $T \in \{\alpha\}_{\partial\bar\partial}$ with analytic singularities such that $E_+(T) = \mathrm{NAmp}(\{\alpha\}_{\partial\bar\partial})$.

Let now X be a complex space. A chart (U, ι, V) on X is a triple consisting of an open set $U \subset X$, a closed complex space $V \subset G \subset \mathbb{C}^N$ in an open set G of $\mathbb{C}^N$ and a biholomorphic map $\iota : U \to V$ (in the category of complex spaces). The map $\iota : U \to G \subset \mathbb{C}^N$ is called a local embedding of X. We write $X = X_{\mathrm{reg}} \cup X_{\mathrm{sing}}$, where X_{reg} and X_{sing} are the sets of regular and singular points of X. Recall that a reduced complex space $(X, \mathcal{O})$ is called *normal* if for every $x \in X$ the local ring $\mathcal{O}_x$ is integrally closed in its quotient field $\mathcal{M}_x$ (cf. [GR2, p. 124]). Every normal complex space is locally irreducible and locally pure-dimensional (see [GR2, p. 125]), and X_{sing} is a closed complex subspace of X with $\operatorname{codim} X_{\mathrm{sing}} \geq 2$.

A continuous (respectively, smooth) function on X is a function $\varphi : X \to \mathbb{C}$ such that for every $x \in X$ there exists a local embedding $\iota : U \to G \subset \mathbb{C}^N$ with $x \in U$ and a continuous (respectively, smooth) function $\tilde\varphi : G \to \mathbb{C}$ such that $\varphi|_U = \tilde\varphi \circ \iota$. A (strictly) plurisubharmonic (psh) function on X is a function $\varphi : X \to [-\infty, \infty)$ such that for every $x \in X$ there exists a local embedding $\iota : U \to G \subset \mathbb{C}^N$ with $x \in U$ and a (strictly) psh function $\tilde\varphi : G \to [-\infty, \infty)$ such that $\varphi|_U = \tilde\varphi \circ \iota$. If $\tilde\varphi$ can be chosen continuous (respectively, smooth), then φ is called a continuous (respectively, smooth) psh function. We let $\mathrm{PSH}(X)$ denote the set of all psh functions on X.

Assume now that X has pure dimension n. We consider currents on X as defined in [D2]. If $\mathcal{D}^{p,q}(X)$ is the space of forms with compact support, endowed with the inductive limit topology, then the dual $\mathcal{D}_{p,q}(X)$ of $\mathcal{D}^{p,q}(X)$ is the space of currents of bidimension (p, q), or bidegree $(n-p, n-q)$, on X. If $T \in \mathcal{D}_{n-1,n-1}(X)$ is so that, for every $x \in X$, there is a domain U containing x and $v \in \mathrm{PSH}(U)$ with $T = dd^c v$ on U, then T is positive and closed, and we say that v is a local potential of T. A Hermitian form on X is a smooth $(1,1)$-form ω such that for every point $x \in X$ there exist a local embedding $\iota : U \ni x \to G \subset \mathbb{C}^N$ and a Hermitian form $\tilde\omega$ on G with $\omega = \iota^\star \tilde\omega$ on $U \cap X_{\mathrm{reg}}$. Note that $\omega^n/n!$ gives locally an area measure on X. A Kähler form on X is a current $T \in \mathcal{D}_{n-1,n-1}(X)$ whose local potentials extend to smooth strictly psh functions in local embeddings of X to Euclidean spaces. We call X a Kähler space if X admits a Kähler form (see also [G, p. 346], [O, Section 5]).

The notions of qpsh and α-psh function on X, where α is a smooth real closed $(1,1)$-form on X, are defined exactly as in the case when X is smooth. We denote by $\mathrm{PSH}(X, \alpha)$ the set of all α-psh functions on X. If X is compact, a function $\rho : X \to \mathbb{R}$ is called Hölder continuous if there exists a finite open cover of X by charts (U, ι, V), $V \subset G \subset \mathbb{C}^N$, such that $\rho|_U$ is Hölder continuous with respect to the metric on U induced by the Euclidean distance on $\mathbb{C}^N$.

If (L, h) is a singular Hermitian holomorphic line bundle over X, the *curvature current* $c_1(L, h)$ of h is defined as in the case when X is smooth [D3]. If e_U is a local holomorphic frame of L on some open set $U \subset X$ then $|e_U|_h = e^{-\varphi_U}$, where $\varphi_U \in L^1_{\mathrm{loc}}(U)$ is called the *local weight* of the metric h with respect to e_U, and $c_1(L, h)|_U = dd^c \varphi_U$. We say that h is *positively curved*, respectively *strictly positively curved*, if $c_1(L, h) \geq 0$, respectively $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$ and some Hermitian form ω on X.

2.2. Bergman kernel functions and Fubini-Study currents. Let X be as in (A), ω be a Hermitian form on X, and (L, h) be a singular Hermitian holomorphic line bundle on X. Since X is compact, the space $H^0(X, L)$ is finite dimensional. Let $H^0_{(2)}(X, L) = H^0_{(2)}(X, L, h, \omega^n)$ be the Bergman space of L^2-holomorphic sections of L relative to the metric h and the volume form $\omega^n/n!$ on X, endowed with the inner product

$$(S, S') := \int_X \langle S, S' \rangle_h \frac{\omega^n}{n!}.$$

Set $\|S\|^2 = \|S\|^2_{h,\omega^n} := (S, S)$.

Let V be a subspace of $H^0_{(2)}(X, L)$, $r = \dim V$, and $S_1, \dots, S_r$ be an orthonormal basis of V. The *Bergman kernel function* $P = P_V$ of V is defined by

$$P(x) = \sum_{j=1}^{r} |S_j(x)|_h^2, \quad |S_j(x)|_h^2 := \langle S_j(x), S(x) \rangle_h, \ x \in X. \tag{2.1}$$

Note that this definition is independent of the choice of basis. Let U be an open set in X such that L has a local holomorphic frame e_U on U. Then, $|e_U|_h = e^{-\varphi_U}$, where $\varphi_U \in L^1_{\mathrm{loc}}(U, \omega^n)$, and $S_j = s_j e_U$, where $s_j \in \mathcal{O}_X(U)$. It follows that

$$\log P|_U = \log\Big(\sum_{j=1}^{r} |s_j|^2\Big) - 2\varphi_U, \tag{2.2}$$

which shows that $\log P \in L^1(X, \omega^n)$.

The Kodaira map determined by V is the meromorphic map given by

$$\Phi = \Phi_V : X \dashrightarrow \mathbb{P}(V^\star), \quad \Phi(x) = \{S \in V : S(x) = 0\}, \ x \in X \setminus \mathrm{Bs}(V),$$

where a point in $\mathbb{P}(V^\star)$ is identified with a hyperplane through the origin in V and $\mathrm{Bs}(V) = \{x \in X : S(x) = 0, \ \forall\, S \in V\}$ is the base locus of V. We define the *Fubini-Study current* $\gamma = \gamma_V$ of V by

$$\gamma := \Phi^\star(\omega_{\mathrm{FS}}), \tag{2.3}$$

where ω_{FS} denotes the Fubini-Study form on $\mathbb{P}(V^\star)$. Then, γ is a positive closed current of bidegree $(1,1)$ on X, and if U is as above we have

$$\gamma|_U = \frac{1}{2} dd^c \log\Big(\sum_{j=1}^{r} |s_j|^2\Big).$$

Hence, by (2.2),

$$\gamma = c_1(L,h) + \frac{1}{2} dd^c \log P.$$

Let now X, L, Σ, τ verify assumptions (A)–(D) and $H^0_0(X, L^p)$ be the space defined in (1.2). If h is a bounded metric on L, then

$$H^0_0(X, L^p) \subset H^0_{(2)}(X, L^p, h^p, \omega^n).$$

The Bergman kernel function P_p of $H^0_0(X, L^p)$ is called the *partial Bergman kernel function* of the space of sections that vanish to order τp along Σ. It satisfies the following variational principle:

$$P_p(x) = \max\{|S(x)|^2_{h^p} : S \in H^0_0(X, L^p),\ \|S\|_p = 1\},$$

where $\|\cdot\|_p$ denotes the norm given by the inner product in $H^0_{(2)}(X, L^p, h^p, \omega^n)$.

3. DIMENSION GROWTH OF THE SPACES $H^0_0(X, L^p)$

In this section, we give the proofs of Theorem 1.3, Proposition 1.4, and Theorem 1.6.

3.1. Divisorization. We start by proving the existence of the divisorization of (X, Σ) claimed in Proposition 1.4. We will use the following theorems of Hironaka on resolution of singularities. For the first one we refer the reader to [BM, Theorem 13.2].

Theorem 3.1 (Hironaka). *If X is a compact, reduced complex space then there exists a compact complex manifold $\hat{X}$ and a surjective holomorphic map $\sigma : \hat{X} \to X$ such that $\sigma : \hat{X} \setminus E \to X_{\mathrm{reg}}$ is a biholomorphism, where $E = \sigma^{-1}(X_{\mathrm{sing}})$ is a divisor with only normal crossings. Moreover, if X is irreducible then $\hat{X}$ is connected and* $\dim \hat{X} = \dim X$.

The second one is Hironaka's embedded resolution of singularities theorem (see, e.g., [BM, Theorems 10.7 and 1.6], [MM, Theorem 2.1.13]).

Theorem 3.2 (Hironaka). *Let X be a complex manifold of dimension n, and $A \subset X$ be a compact analytic subset of X. Then, there exist a complex manifold $\tilde{X}$ and a surjective holomorphic map $\sigma : \tilde{X} \to X$, given as the composition of finitely many blowups with smooth center, with the following properties:*

(i) $E = \sigma^{-1}(A_{\rm sing})$ *is a divisor in* $\tilde X$*, and* $\sigma : \tilde X \setminus E \to X \setminus A_{\rm sing}$ *is a biholomorphism.*

(ii) *The strict transform* $A' = \overline{\sigma^{-1}(A_{\rm reg})}$ *is smooth, and* A'*,* E *simultaneously have only normal crossings.*

If $A = A_1 \cup \cdots \cup A_m$ has irreducible components A_j, and A'_j is the strict transform of A_j, it follows from Theorem 3.2 that A'_j are pairwise disjoint connected submanifolds of $\tilde X$. Performing blowups of $\tilde X$ with centers A'_j, for all j with $\dim A_j \le n-2$, one obtains the following version of Theorem 3.2 (see also [CM1, Theorem 2.1]).

Theorem 3.3. *Let* X *be a complex manifold of dimension* n*, and* $A \subset X$ *be a compact analytic subset of* X *with irreducible components* $A_1, \dots, A_m$*. Then, there exist a complex manifold* $\tilde X$ *and a surjective holomorphic map* $\sigma : \tilde X \to X$*, given as the composition of finitely many blowups with smooth center, with the following properties:*

(i) *If* $Y = A_{\rm sing} \cup \bigcup\{A_j : \dim A_j \le n-2\}$ *then* $E = \sigma^{-1}(Y)$ *is the final exceptional divisor,* $\sigma : \tilde X \setminus E \to X \setminus Y$ *is a biholomorphism, and* E *has only normal crossings.*

(ii) *There are (connected) smooth complex hypersurfaces* $\tilde A_1, \dots, \tilde A_m$ *in* $\tilde X$*, which have only normal crossings, such that* $\sigma(\tilde A_j) = A_j$*. Moreover, if* $\dim A_j = n-1$ *then* $\tilde A_j$ *is the final strict transform of* A_j*, and if* $\dim A_j \le n-2$*, then* $\tilde A_j$ *is an irreducible component of* E*.*

(iii) *If* $F \to X$ *is a holomorphic line bundle and* $S \in H^0(X, F)$*, then*

$$\operatorname{ord}(S, A_j) = \operatorname{ord}(\sigma^\star S, \tilde A_j), \quad \text{for all } j = 1, \dots, m.$$

Proof. The existence of $\tilde X$ and σ with properties (i)–(ii) follows directly from Theorem 3.2, as previously described. Property (iii) clearly holds for j with $\dim A_j = n-1$, since $\tilde A_j = \overline{\sigma^{-1}(A_j \setminus Y)}$ is the final strict transform of A_j and $\sigma : \tilde X \setminus E \to X \setminus Y$ is a biholomorphism. If j is such that $\dim A_j \le n-2$ and $A''_j \subset \tilde X_1$ is the strict transform of A_j produced in Theorem 3.2, then $\tilde A_j = \pi^{-1}(A''_j)$, where $\pi : \tilde X \to \tilde X_1$ is the blowup of $\tilde X_1$ with center A''_j. Thus, property (iii) follows easily from the local description of the blowup map π. □

Proof of Proposition 1.4. Let $\hat\sigma : \hat X \to X$ be a desingularization of X as in Theorem 3.1, let $\hat E = \hat\sigma^{-1}(X_{\rm sing})$ be the exceptional divisor, and let $\hat\Sigma_j$ be the strict transform of Σ_j. Hence, $\hat\sigma : \hat X \setminus \hat E \to X_{\rm reg}$ is a biholomorphism. Since $\Sigma_j \not\subset X_{\rm sing}$ we have that $\hat\Sigma_j \neq \emptyset$ and $\dim \hat\Sigma_j = \dim \Sigma_j$. Moreover, $\hat\Sigma_j$ is irreducible since Σ_j is. Note that $\hat\Sigma_j \subset \hat\Sigma_k$ if and only if $\Sigma_j \subset \Sigma_k$.

We next apply Theorem 3.3 repeatedly, starting with $\hat X, \hat\Sigma_1, \dots, \hat\Sigma_\ell$, as follows. Let $\hat\Sigma_{j_1}, \dots, \hat\Sigma_{j_k}$ be the minimal elements of $\{\hat\Sigma_1, \dots, \hat\Sigma_\ell\}$ with respect to inclusion. We apply Theorem 3.3 to $\hat X$ and $A = \hat\Sigma_{j_1} \cup \cdots \cup \hat\Sigma_{j_k}$, to obtain a map

$\sigma_1 : \hat{X}_1 \to \hat{X}$ verifying properties (i)–(iii). Let $\hat{\Sigma}'_j$ be the strict transform of $\hat{\Sigma}_j$ by σ_1, for $j \in \{1, \dots, \ell\} \setminus \{j_1, \dots, j_k\}$, and note that $\hat{\Sigma}'_j \subset \hat{\Sigma}'_k$ if and only if $\hat{\Sigma}_j \subset \hat{\Sigma}_k$. We now apply Theorem 3.3 to $\hat{X}_1$ and the analytic subset given by the union of the minimal elements of

$$\{\hat{\Sigma}'_j : j \in \{1, \dots, \ell\} \setminus \{j_1, \dots, j_k\}\}$$

with respect to inclusion, to obtain a map $\sigma_2 : \hat{X}_2 \to \hat{X}_1$ verifying properties (i)–(iii). Repeating this procedure finitely many times, we resolve all of the sets Σ_j. Finally, we apply Theorem 3.3 one more time in order to make the resulting smooth hypersurfaces $\tilde{\Sigma}_1, \dots, \tilde{\Sigma}_\ell$ and the final exceptional divisor (including the preimage of $\hat{E}$) simultaneously have only normal crossings. Taking the composition of the maps σ_j, we obtain a compact complex manifold $\tilde{X}$ and a surjective holomorphic map $\sigma : \tilde{X} \to \hat{X}$, given as the composition of finitely many blowups with smooth center, with the following properties:

(a) There exists an analytic set $\hat{Y} \subset \hat{E} \cup \bigcup_{j=1}^{\ell} \hat{\Sigma}_j$ such that $\dim \hat{Y} \le n-2$, $E_0 := \sigma^{-1}(\hat{Y})$ is a divisor in $\tilde{X}$ with only normal crossings, and $\sigma : \tilde{X} \setminus E_0 \to \hat{X} \setminus \hat{Y}$ is a biholomorphism.

(b) There exist smooth complex hypersurfaces $\tilde{\Sigma}_j \subset \tilde{X}$, $1 \le j \le \ell$, which have only normal crossings, such that $\sigma(\tilde{\Sigma}_j) = \hat{\Sigma}_j$. Moreover, if $\dim \hat{\Sigma}_j = n-1$ then $\tilde{\Sigma}_j$ is the final strict transform of $\hat{\Sigma}_j$ by σ, and if $\dim \hat{\Sigma}_j \le n-2$, then $\tilde{\Sigma}_j$ is an irreducible component of E_0.

(c) If $\hat{F} \to \hat{X}$ is a holomorphic line bundle and $S \in H^0(\hat{X}, \hat{F})$, then

$$\operatorname{ord}(S, \hat{\Sigma}_j) = \operatorname{ord}(\sigma^\star S, \tilde{\Sigma}_j), \quad \text{for all } j = 1, \dots, \ell.$$

We define $\pi := \hat{\sigma} \circ \sigma : \tilde{X} \to X$, and set $Y := X_{\text{sing}} \cup \hat{\sigma}(\hat{Y})$. Since $\hat{\sigma}(\hat{Y})$ is an analytic subset of X of dimension $\le n-2$, we have $\dim Y \le n-2$.

By (b), if $\dim \hat{\Sigma}_j \le n-2$ then $\tilde{\Sigma}_j \subset E_0$, so $\hat{\Sigma}_j = \sigma(\tilde{\Sigma}_j) \subset \hat{Y}$ and $\Sigma_j = \hat{\sigma}(\hat{\Sigma}_j) \subset \hat{\sigma}(\hat{Y}) \subset Y$. Since $\hat{Y} \subset \hat{E} \cup \bigcup_{j=1}^{\ell} \hat{\Sigma}_j$, we have $\hat{\sigma}(\hat{Y}) \subset X_{\text{sing}} \cup \bigcup_{j=1}^{\ell} \Sigma_j$, so $Y \subset X_{\text{sing}} \cup \bigcup_{j=1}^{\ell} \Sigma_j$. Moreover,

$$\hat{E} \cup \hat{Y} \subset \hat{\sigma}^{-1}(X_{\text{sing}}) \cup \hat{\sigma}^{-1}(\hat{\sigma}(\hat{Y})) = \hat{\sigma}^{-1}(Y),$$
$$\hat{\sigma}^{-1}(\hat{\sigma}(\hat{Y})) \subset \hat{E} \cup \hat{\sigma}^{-1}(\hat{\sigma}(\hat{Y}) \setminus X_{\text{sing}}) = \hat{E} \cup (\hat{Y} \setminus \hat{E}) = \hat{E} \cup \hat{Y}.$$

Thus, $\hat{\sigma}^{-1}(Y) = \hat{E} \cup \hat{Y}$. Let $\hat{E}'$ be the strict transform of $\hat{E}$ by σ. It is easy to see that $\sigma^{-1}(\hat{\sigma}^{-1}(Y)) = \hat{E}' \cup E_0$. Thus, $E := \hat{E}' \cup E_0$ is a divisor in $\tilde{X}$ that has only normal crossings and $E = \pi^{-1}(Y)$. Since $\sigma : \tilde{X} \setminus E_0 \to \hat{X} \setminus \hat{Y}$, $\hat{\sigma} : \hat{X} \setminus \hat{E} \to X_{\text{reg}}$ are biholomorphisms and $X_{\text{sing}} \subset Y$, $\hat{Y} \subset \hat{\sigma}^{-1}(Y)$, we conclude that $\pi : \tilde{X} \setminus E \to X \setminus Y$ is a biholomorphism, so property (i) of Proposition 1.4 is satisfied.

Properties (ii)–(iii) of Proposition 1.4 follow easily from (b) and (c), since $\pi(\tilde{\Sigma}_j) = \hat{\sigma}(\hat{\Sigma}_j) = \Sigma_j$. Further, since $\hat{\Sigma}_j$ is the strict transform of $\Sigma_j \not\subset X_{\text{sing}}$ by $\hat{\sigma}$, we have $\operatorname{ord}(S, \Sigma_j) = \operatorname{ord}(\hat{\sigma}^\star S, \hat{\Sigma}_j)$, for any $S \in H^0(X, F)$ and $j = 1, \ldots, \ell$. $\square$

Proposition 1.4 has the following corollary which will be needed later, for the proof of Theorem 1.6.

Corollary 3.4. *Let X, L, Σ, τ verify assumptions* (A)–(D)*, and let $(\tilde{X}, \pi, \tilde{\Sigma})$ be a divisorization of (X, Σ). Then, $H^0_0(X, L^p, \Sigma, \tau) \cong H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}, \tau)$ for all $p \geq 1$.*

Proof. Fix $p \geq 1$. The map π induces a linear map

$$\pi^\star : H^0(X, L^p) \to H^0(\tilde{X}, \pi^\star L^p), \quad S \to \pi^\star S.$$

We can define a linear map $\pi_\star : H^0(\tilde{X}, \pi^\star L^p) \to H^0(X, L^p)$ as follows: if $\tilde{S} \in H^0(\tilde{X}, \pi^\star L^p)$, set $\pi_\star \tilde{S} = S$, where $S := (\pi^{-1})^\star(\tilde{S}|_{\tilde{X} \setminus E}) \in H^0(X \setminus Y, L^p|_{X \setminus Y})$ extends to a section in $H^0(X, L^p)$ since X is normal and $\dim Y \leq n - 2$, [GR2, p. 143]. Since $\pi : \tilde{X} \setminus E \to X \setminus Y$ is a biholomorphism, it follows that $\pi_\star = (\pi^\star)^{-1}$. Proposition 1.4 (iii) implies $\pi^\star(H^0_0(X, L^p, \Sigma, \tau)) \subset H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}, \tau)$. Moreover, if $\tilde{S} \in H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}, \tau)$ then $\tilde{S} = \pi^\star \pi_\star \tilde{S}$, and so $\operatorname{ord}(\tilde{S}, \tilde{\Sigma}_j) = \operatorname{ord}(\pi_\star \tilde{S}, \Sigma_j)$, so $\pi_\star \tilde{S} \in H^0_0(X, L^p, \Sigma, \tau)$. Thus,

$$\pi^\star(H^0_0(X, L^p, \Sigma, \tau)) = H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}, \tau). \qquad \square$$

Remark 3.5. In hypothesis (D), we make the natural assumption that $\tau_j > \tau_k$, for every $j, k \in \{1, \ldots, \ell\}$ with $\Sigma_j \subset \Sigma_k$. We note that Corollary 3.4 is in fact valid for *every* ℓ-tuple τ of positive real numbers. Suppose that $\Sigma_\ell \subset \Sigma_k$ and τ is such that $\tau_\ell \leq \tau_k$. Set $\Sigma' = (\Sigma_1, \ldots, \Sigma_{\ell-1})$, $\tau' = (\tau_1, \ldots, \tau_{\ell-1})$, and note that if $(\tilde{X}, \pi, \tilde{\Sigma})$ is a divisorization of (X, Σ) then $(\tilde{X}, \pi, \tilde{\Sigma}')$ is a divisorization of (X, Σ'), where $\tilde{\Sigma}' = (\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_{\ell-1})$. Clearly, $H^0_0(X, L^p, \Sigma, \tau) = H^0_0(X, L^p, \Sigma', \tau')$. By Corollary 3.4, we have that $H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}, \tau) = H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}', \tau')$.

3.2. Proofs of Theorems 1.3 and 1.6. Theorem 1.3 will follow from Theorem 3.6 below. Let X be a compact complex manifold of dimension n, $\Sigma_j \subset X$ be irreducible complex hypersurfaces, and let $\tau_j > 0$, where $1 \leq j \leq \ell$. Let L be a holomorphic line bundle over X and consider the following:

$$E_p := L^p \otimes \bigotimes_{j=1}^{\ell} \mathcal{O}_X(-\lfloor \tau_j p \rfloor \Sigma_j), \quad V_p := H^0(X, E_p),$$

$$F_p := L^p \otimes \bigotimes_{j=1}^{\ell} \mathcal{O}_X(-t_{j,p} \Sigma_j), \quad W_p := H^0(X, F_p),$$

where $t_{j,p}$ are defined in (1.1). Note that V_p is isomorphic to the space of sections in $H^0(X, L^p)$ that vanish to order $\lfloor \tau_j p \rfloor$ along Σ_j, $1 \leq j \leq \ell$, while W_p is isomorphic to the space $H^0_0(X, L^p)$ defined in (1.2). Clearly, $\dim W_p \leq \dim V_p$.

Theorem 3.6. *In the above setting, the following are equivalent:*

(i) $\limsup_{p\to\infty} p^{-n}\dim V_p > 0$.
(ii) $\limsup_{p\to\infty} p^{-n}\dim W_p > 0$.
(iii) *There is a singular Hermitian metric* h *on* L *such that* $c_1(L,h)-\sum_{j=1}^{\ell}\tau_j[\Sigma_j]$ *is a Kähler current on* X.
(iv) *There exist* $p_0\in\mathbb{N}$ *and* $c>0$ *such that* $\dim V_p\geq cp^n$ *for all* $p\geq p_0$.
(v) *There exist* $p_0\in\mathbb{N}$ *and* $c>0$ *such that* $\dim W_p\geq cp^n$ *for all* $p\geq p_0$.

Proof. Let ω be a Hermitian form on X.

(i) $\Longrightarrow$ (ii) There exist a constant $c>0$ and a sequence of natural numbers $p_k\nearrow\infty$ such that $\dim V_{p_k}\geq cp_k^n$ for all $k\geq 1$. Let us fix k and assume that $\tau_1 p_k\notin\mathbb{N}$. Consider the short exact sequence

$$0\longrightarrow E_{p_k}\otimes\mathcal{O}_X(-\Sigma_1)\longrightarrow E_{p_k}\longrightarrow E_{p_k}|_{\Sigma_1}\longrightarrow 0,$$

which gives the exact sequence

$$\begin{aligned}0\longrightarrow H^0(X,E_{p_k}\otimes\mathcal{O}_X(-\Sigma_1))&\longrightarrow H^0(X,E_{p_k})\\ \longrightarrow H^0(\Sigma_1,E_{p_k}|_{\Sigma_1})&\longrightarrow\cdots.\end{aligned}$$

It follows that

$$\dim H^0(X,E_{p_k})\leq\dim H^0(X,E_{p_k}\otimes\mathcal{O}_X(-\Sigma_1))+\dim H^0(\Sigma_1,E_{p_k}|_{\Sigma_1}).$$

By Siegel's lemma applied to the analytic subset Σ_1 (see Lemma 3.7 following this proof), there exists a constant $c'>0$ such that

$$\dim H^0(\Sigma_1,E_p|_{\Sigma_1})\leq\dim H^0(\Sigma_1,L^p|_{\Sigma_1})\leq c'p^{n-1},\quad\forall\,p\geq 1.$$

Hence, $\dim H^0(X,E_{p_k}\otimes\mathcal{O}_X(-\Sigma_1))\geq cp_k^n-c'p_k^{n-1}$. If $\tau_2 p_k\notin\mathbb{N}$ we repeat the above argument working with the short exact sequence

$$\begin{aligned}0\longrightarrow E_{p_k}\otimes\mathcal{O}_X(-\Sigma_1)\otimes\mathcal{O}_X(-\Sigma_2)&\longrightarrow E_{p_k}\otimes\mathcal{O}_X(-\Sigma_1)\\ \longrightarrow E_{p_k}\otimes\mathcal{O}_X(-\Sigma_1)|_{\Sigma_2}&\longrightarrow 0,\end{aligned}$$

and so on. This yields (ii).

(ii) $\Longrightarrow$ (iii) We proceed in two steps.

Step 1. Assume that X is projective. We fix a smooth ample divisor $A\subset X$ and consider, as above, the exact sequence

$$\begin{aligned}0\longrightarrow H^0(X,F_p\otimes\mathcal{O}_X(-A))&\longrightarrow W_p=H^0(X,F_p)\\ \longrightarrow H^0(A,F_p|_A)&\longrightarrow\cdots.\end{aligned}$$

Since, by Siegel's lemma [MM, Lemma 2.2.6],

$$\dim H^0(A, F_p\,|_A) \leq \dim H^0(A, L^p\,|_A) = O(p^{n-1}),$$

by (ii) there are $c > 0$ and $p_k \nearrow \infty$ such that $\dim H^0(X, F_{p_k} \otimes \mathcal{O}_X(-A)) \geq c p_k^n$. We fix such a $p = p_k$. Since $H^0(X, F_p \otimes \mathcal{O}_X(-A))$ is nontrivial, there exists an effective divisor $D_p \subset X$ such that $F_p \otimes \mathcal{O}_X(-A) = \mathcal{O}_X(D_p)$. Hence,

$$L^p = \mathcal{O}_X(A) \otimes \mathcal{O}_X(D_p) \otimes \bigotimes_{j=1}^{\ell} \mathcal{O}_X(t_{j,p}\Sigma_j).$$

Let h_A be a smooth positive metric on $\mathcal{O}_X(A)$, and let h_{D_p} (respectively, h_{Σ_j}) be the metric induced on $\mathcal{O}_X(D_p)$ (respectively, on $\mathcal{O}_X(\Sigma_j)$) by the canonical section of $\mathcal{O}_X(D_p)$ (respectively, of $\mathcal{O}_X(\Sigma_j)$), so that

$$c_1(\mathcal{O}_X(D_p), h_{D_p}) = [D_p], \quad c_1(\mathcal{O}_X(\Sigma_j), h_{\Sigma_j}) = [\Sigma_j].$$

We define the metric $h_p := h_A \otimes h_{D_p} \otimes \bigotimes_{j=1}^{\ell} h_{\Sigma_j}^{\otimes t_{j,p}}$ on L^p, and we let $h = h_p^{1/p}$ be the induced metric on L. Then,

$$c_1(L, h) = \frac{1}{p}\Big(\omega_0 + [D_p] + \sum_{j=1}^{\ell} t_{j,p}[\Sigma_j]\Big),$$

where $\omega_0 = c_1(\mathcal{O}_X(A), h_A)$ is a Kähler form on X. Since $t_{j,p} \geq \tau_j p$ we get

$$c_1(L, h) - \sum_{j=1}^{\ell} \tau_j[\Sigma_j] = \frac{1}{p}(\omega_0 + [D_p]) + \sum_{j=1}^{\ell}\left(\frac{t_{j,p}}{p} - \tau_j\right)[\Sigma_j] \geq \frac{1}{p}\omega_0,$$

which proves (iii) in the case when X is projective.

Step 2. In the general case when X is a compact complex manifold, we have by (ii) that $\limsup_{p\to\infty} p^{-n} \dim H^0(X, L^p) > 0$; hence, L is a big line bundle and X is Moishezon (see, e.g., [MM, Theorem 2.2.15]). By a theorem of Moishezon (see, e.g., [MM, Theorem 2.2.16]), there exists a projective manifold $\tilde{X}$ and a surjective holomorphic map $\sigma : \tilde{X} \to X$, given as the composition of finitely many blowups with smooth center, such that $\sigma : \tilde{X} \setminus E \to X \setminus Y$ is a biholomorphism, where $Y \subset X$ is an analytic subset with $\dim Y \leq n-2$, and $E = \sigma^{-1}(Y)$ is the final exceptional divisor. Let Σ_j' be the strict transform of Σ_j by σ, and note that if $S \in H^0(X, L^p)$ then $\sigma^\star S \in H^0(\tilde{X}, \sigma^\star L^p)$ and $\operatorname{ord}(S, \Sigma_j) = \operatorname{ord}(\sigma^\star S, \Sigma_j')$, $1 \leq j \leq \ell$. It follows that

$$W_p \cong W_p' := H^0\Big(\tilde{X}, \sigma^\star L^p \otimes \bigotimes_{j=1}^{\ell} \mathcal{O}_{\tilde{X}}(-t_{j,p}\Sigma_j')\Big), \quad \text{so } \limsup_{p\to\infty} \frac{\dim W_p'}{p^n} > 0.$$

We fix a Hermitian form $\tilde{\omega}$ on $\tilde{X}$. Then, $\sigma^\star\omega + \tilde{\omega}$ is a Hermitian form on $\tilde{X}$, and by Step 1 there exists a singular Hermitian metric $h^\star$ on $\sigma^\star L$ such that

$$c_1(\sigma^\star L, h^\star) - \sum_{j=1}^{\ell} \tau_j[\Sigma'_j] \geq \varepsilon(\sigma^\star\omega + \tilde{\omega}) \geq \varepsilon\sigma^\star\omega, \tag{3.1}$$

for some constant $\varepsilon > 0$. The metric $h = (\sigma^{-1})^\star h^\star$ on $L|_{X\setminus Y}$ extends to a metric on L as follows. If U is a coordinate ball centered at $x \in Y$ and e_U is a frame of $L|_U$, then $\sigma^\star e_U$ is a frame of $\sigma^\star L|_{\sigma^{-1}(U)}$. Let $|\sigma^\star e_U|_{h^\star} = e^{-\varphi^\star}$, where $\varphi^\star \in \mathrm{PSH}(\sigma^{-1}(U))$. The function $\varphi = \varphi^\star \circ \sigma^{-1}$ is psh on $U \setminus Y$, so it extends to a psh function on U since $\dim Y \leq n-2$, and we set $|e_U|_h = e^{-\varphi}$.

We have that $\sigma_\star c_1(\sigma^\star L, h^\star) = c_1(L, h)$ on $X \setminus Y$, and hence on X by the support theorem. Similarly, $\sigma_\star[\Sigma'_j] = [\Sigma_j]$, and $\sigma_\star\sigma^\star\omega = \omega$ since ω is a smooth form. By (3.1) it follows that $c_1(L,h) - \sum_{j=1}^{\ell}\tau_j[\Sigma_j] \geq \varepsilon\omega$, which proves (iii).

(iii) $\Longrightarrow$ (iv) Let h be a singular metric on L so that $R := c_1(L,h) - \sum_{j=1}^{\ell}\tau_j[\Sigma_j] \geq 3\varepsilon\omega$, for some constant $\varepsilon > 0$. We fix a smooth metric h_0 on L, set $\alpha = c_1(L, h_0)$, and write $h = h_0 e^{-2\psi}$, where $\psi \in \mathrm{PSH}(X, \alpha)$ since $c_1(L,h) = \alpha + dd^c\psi \geq 0$. Let g_j be a smooth metric on $\mathcal{O}_X(\Sigma_j)$, s_{Σ_j} be the canonical section of $\mathcal{O}_X(\Sigma_j)$, and set

$$\sigma_j := |s_{\Sigma_j}|_{g_j},\ \beta_j = c_1(\mathcal{O}_X(\Sigma_j), g_j),\ \theta = \alpha - \sum_{j=1}^{\ell}\tau_j\beta_j.$$

By the Lelong-Poincaré formula, $[\Sigma_j] = \beta_j + dd^c\log\sigma_j$. Hence, $R = \theta + dd^c\psi'$, where $\psi' = \psi - \sum_{j=1}^{\ell}\tau_j\log\sigma_j$. The function $\psi' \in L^1(X, \omega^n)$ is defined everywhere on $X \setminus (\bigcup_{j=1}^{\ell}\Sigma_j)$. Since $R \geq 0$ it follows that $\psi' = u$ almost everywhere on X, for some function $u \in \mathrm{PSH}(X, \theta)$, and hence everywhere on $X \setminus (\bigcup_{j=1}^{\ell}\Sigma_j)$ since both ψ', u are qpsh there. Thus, ψ' extends to a θ-psh function on X.

Applying Demailly's regularization theorem [D4] to ψ', it follows that there exists a qpsh function φ with algebraic singularities on X—that is, a function φ as described after Definition 2.1, where $c \in \mathbb{Q}$, such that $T := \theta + dd^c\varphi \geq 2\varepsilon\omega$. Moreover, there is a proper modification $\sigma : \tilde{X} \to X$, obtained as a sequence of blowups with smooth centers and with blowup locus contained in the analytic subset $\{\varphi = -\infty\}$, such that the functions $\chi\circ\sigma$ are smooth. Let $\tilde{\omega}$ be a Hermitian form on $\tilde{X}$ such that $\tilde{\omega} \geq \sigma^\star\omega$.

We take sequences $\{r_{j,k}\}_{k\geq 1} \subset \mathbb{Q}$ such that $r_{j,k} \searrow \tau_j$ as $k \to \infty$, and we consider the qpsh functions

$$\psi_k = \varphi + \sum_{j=1}^{\ell} r_{j,k}\log\sigma_j,$$

with algebraic singularities in $Z := \{\varphi = -\infty\} \cup \bigcup_{j=1}^{\ell} \Sigma_j$. Then,

$$\begin{aligned} T &= \alpha + dd^c\Big(\varphi + \sum_{j=1}^{\ell} r_{j,k}\log\sigma_j\Big) - \sum_{j=1}^{\ell} r_{j,k}(\beta_j + dd^c\log\sigma_j) \\ &\quad + \sum_{j=1}^{\ell}(r_{j,k}-\tau_j)\beta_j \\ &= \alpha + dd^c\psi_k - \sum_{j=1}^{\ell} r_{j,k}[\Sigma_j] + \sum_{j=1}^{\ell}(r_{j,k}-\tau_j)\beta_j. \end{aligned}$$

There exists a constant $C > 0$ such that $\beta_j \leq C\omega$ for $j = 1, \dots, \ell$. We obtain that

$$\begin{aligned} \alpha + dd^c\psi_k &= T + \sum_{j=1}^{\ell} r_{j,k}[\Sigma_j] - \sum_{j=1}^{\ell}(r_{j,k}-\tau_j)\beta_j \\ &\geq 2\varepsilon\omega - C\Big(\sum_{j=1}^{\ell}(r_{j,k}-\tau_j)\Big)\omega \geq \varepsilon\omega, \end{aligned}$$

if k is chosen sufficiently large.

With k fixed as above, we now define the singular metric $h_k = h_0 e^{-2\psi_k}$ on L, which verifies $c_1(L, h_k) = \alpha + dd^c\psi_k \geq \varepsilon\omega$. Let $H^0_{(2)}(X, L^p, h^p_k, \omega^n)$ be the space of L^2 holomorphic sections of L^p with respect to the metric h^p_k on L^p and the volume form ω^n on X. Since $\tilde{\omega} \geq \sigma^\star\omega$, and since $\sigma : \tilde{X} \setminus \sigma^{-1}(Y) \to X \setminus Y$ is biholomorphic, where $Y \subset \{\varphi = -\infty\}$ is an analytic subset with $\dim Y \leq n-2$, it follows that

$$H^0_{(2)}(\tilde{X}, \sigma^\star L^p, \sigma^\star h^p_k, \tilde{\omega}^n) \subseteq H^0_{(2)}(\tilde{X}, \sigma^\star L^p, \sigma^\star h^p_k, \sigma^\star\omega^n) \cong H^0_{(2)}(X, L^p, h^p_k, \omega^n).$$

Note that $c_1(\sigma^\star L, \sigma^\star h_k) \geq \varepsilon\sigma^\star\omega$. Bonavero's singular holomorphic Morse inequalities [Bon] (see also [MM, Theorem 2.3.18]) imply that

$$\begin{aligned} \dim H^0_{(2)}(\tilde{X}, \sigma^\star L^p, \sigma^\star h^p_k, \tilde{\omega}^n) &\geq \frac{p^n}{n!}\int_{\tilde{X}\setminus\sigma^{-1}(Z)} c_1(\sigma^\star L, \sigma^\star h_k)^n + o(p^n) \\ &\geq \frac{\varepsilon^n p^n}{n!}\int_{\tilde{X}} \sigma^\star\omega^n + o(p^n). \end{aligned}$$

Since $\lfloor r_{j,k}p\rfloor \geq \lfloor \tau_j p\rfloor$, it follows from the definition of ψ_k that

$$H^0_{(2)}(X, L^p, h^p_k, \omega^n) \subset V_p,$$

so (iv) holds.

(iv) $\implies$ (v) This now follows by the same argument as the one in the proof of (i) $\implies$ (ii).

(v) $\implies$ (i) This is obvious since $\dim W_p \leq \dim V_p$. □

Lemma 3.7. *Let A be a compact irreducible analytic subset of a complex manifold M, and let $k = \dim A$. If F is a holomorphic line bundle over A then there exists a constant $C > 0$ depending on A, M, F, such that $\dim H^0(A, F^p) \leq Cp^k$ for all $p \geq 1$.*

Proof. By Theorem 3.2, there are a complex manifold $\tilde{M}$ and a surjective holomorphic map $\sigma : \tilde{M} \to M$, given as the composition of finitely many blowups with smooth center, such that $E = \sigma^{-1}(A_{\text{sing}})$ is a divisor in $\tilde{M}$, $\sigma : \tilde{M} \setminus E \to M \setminus A_{\text{sing}}$ is a biholomorphism, and the strict transform $A' = \overline{\sigma^{-1}(A_{\text{reg}})}$ is a connected k-dimensional complex submanifold of $\tilde{M}$. The restriction $\sigma : A' \to A$ is a surjective holomorphic map and the induced map $\sigma^\star : H^0(A, F^p) \to H^0(A', \sigma^\star F^p)$ is injective. Thus, $\dim H^0(A, F^p) \leq \dim H^0(A', \sigma^\star F^p)$, and the lemma follows from Siegel's [MM, Lemma 2.2.6] applied to A' and $\sigma^\star F$. □

Proof of Theorem 1.6. (i) $\implies$ (ii). By Corollary 3.4 we have that

$$H^0_0(X, L^p, \Sigma, \tau) \cong H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}, \tau), \quad \forall\, p \geq 1,$$

and hence $(\pi^\star L, \tilde{\Sigma}, \tau)$ is big and (ii) follows from Theorem 1.3.

(ii) $\implies$ (iii) This is obvious.

(iii) $\implies$ (iv) By Theorem 1.3 with $\tilde{X}, \pi^\star L, \tilde{\Sigma}, \tau$, there exist $p_0 \in \mathbb{N}$ and $c > 0$ such that $\dim H^0_0(\tilde{X}, \pi^\star L^p, \tilde{\Sigma}, \tau) \geq cp^n$ for all $p \geq p_0$. Hence, (iv) follows by using Corollary 3.4.

(iv) $\implies$ (i) This is obvious by Definition 1.1. □

Theorem 1.6 has the following interesting corollary.

Corollary 3.8. *Let X, L, Σ, τ verify assumptions (A)–(D), and suppose that $\dim \Sigma_j = n - 1$ for all $j = 1, \ldots, \ell$. Let $\Sigma'_j \subset \Sigma_j$ be distinct irreducible proper analytic subsets such that $\Sigma'_j \not\subset X_{\text{sing}}$, and let*

$$\begin{aligned} \Sigma' &= (\Sigma_1, \ldots, \Sigma_\ell, \Sigma'_1, \ldots, \Sigma'_\ell), \\ \tau' &= (\tau_1, \ldots, \tau_\ell, \tau_1 + \delta, \ldots, \tau_\ell + \delta), \end{aligned}$$

where $\delta > 0$. If (L, Σ, τ) is big then (L, Σ', τ') is big for $\delta > 0$ sufficiently small.

Proof. Without loss of generality we may assume X is a complex manifold, by first desingularizing X if necessary using Theorem 3.1 and applying Corollary 3.4 to the map σ from Theorem 3.1 and the strict transforms of Σ_j, Σ'_j by σ. Let ω be a Hermitian form on X.

Let $(\tilde{X}, \pi, \tilde{\Sigma}')$ be a divisorization of (X, Σ'), so $\pi : \tilde{X} \setminus E \to X \setminus Y$ is a biholomorphism, where $E = \pi^{-1}(Y)$ is the final exceptional divisor, $\tilde{\Sigma}'_j$ are irreducible components of E, and $\tilde{\Sigma}_j$ is the strict transform of Σ_j by π. Let $\tilde{\omega}$ be a Hermitian form on $\tilde{X}$ such that $\tilde{\omega} \geq \pi^\star\omega$. By Theorem 1.6, there exists a singular Hermitian metric $h^\star$ on $\pi^\star L$ such that

$$T := c_1(\pi^\star L, h^\star) - \sum_{j=1}^{\ell} \tau_j[\tilde{\Sigma}_j] \geq \varepsilon\tilde{\omega} \geq \varepsilon\pi^\star\omega,$$

for some constant $\varepsilon > 0$. As argued in the proof of Theorem 3.6 (Step 2 of the implication (ii) $\Longrightarrow$ (iii)), the metric $h = (\pi^{-1})^\star h^\star$ on $L|_{X\setminus Y}$ extends to a metric on L. Moreover, $\pi_\star c_1(\pi^\star L, h^\star) = c_1(L, h)$, $\pi_\star[\tilde{\Sigma}_j] = [\Sigma_j]$, and $\pi_\star\pi^\star\omega = \omega$. We conclude that

$$S := \pi_\star T = c_1(L, h) - \sum_{j=1}^{\ell} \tau_j[\Sigma_j] \geq \varepsilon\omega.$$

It is well known there exists a smooth Hermitian metric g on the line bundle $\mathcal{O}_{\tilde{X}}(E)$ and a constant $\delta_0 > 0$ such that $\tilde{\omega}_0 := \pi^\star\omega - \delta_0\Theta$ is a Hermitian form on $\tilde{X}$, where $\Theta = c_1(\mathcal{O}_{\tilde{X}}(E), g)$ (see, e.g., [CMM, Lemma 2.2]). If s_E is the canonical section of $\mathcal{O}_{\tilde{X}}(E)$, then by the Lelong-Poincaré formula, $[E] = \Theta + dd^c \log|s_E|_g$.

Let $\delta = \varepsilon\delta_0$. Then, $\pi^\star S - \delta\Theta \geq \varepsilon\pi^\star\omega - \varepsilon\delta_0\Theta = \varepsilon\tilde{\omega}_0$. We introduce the singular Hermitian metric $\tilde{h} = |s_E|_g^{-2\delta}\pi^\star h$ on $\pi^\star L$, so

$$c_1(\pi^\star L, \tilde{h}) = c_1(\pi^\star L, \pi^\star h) + \delta dd^c \log|s_E|_g.$$

Since $\Sigma'_j \subset \Sigma_j$, we have $\pi^\star[\Sigma_j] = [\tilde{\Sigma}_j] + [\tilde{\Sigma}'_j] + R_j$, where R_j are positive closed currents of bidegree $(1,1)$ supported in E. It follows that

$$\begin{aligned}\pi^\star S - \delta\Theta &= c_1(\pi^\star L, \pi^\star h) - \sum_{j=1}^{\ell} \tau_j \pi^\star[\Sigma_j] - \delta([E] - dd^c \log|s_E|_g) \\ &= c_1(\pi^\star L, \tilde{h}) - \sum_{j=1}^{\ell} \tau_j[\tilde{\Sigma}_j] - \sum_{j=1}^{\ell} (\tau_j + \delta)[\tilde{\Sigma}'_j] - R,\end{aligned}$$

where R is a positive closed current of bidegree $(1,1)$ supported in E. Thus,

$$c_1(\pi^\star L, \tilde{h}) - \sum_{j=1}^{\ell} \tau_j[\tilde{\Sigma}_j] - \sum_{j=1}^{\ell} (\tau_j + \delta)[\tilde{\Sigma}'_j] = \pi^\star S - \delta\Theta + R \geq \varepsilon\tilde{\omega}_0,$$

and hence (L, Σ', τ') is big by Theorem 1.6. $\square$

4. ENVELOPES OF QPSH FUNCTIONS WITH POLES ALONG A DIVISOR

In this section, we define the relevant spaces of qpsh functions with poles along a divisor and prove the regularity theorem for their upper envelopes.

Let (X, ω) be a compact Hermitian manifold of dimension n, $\Sigma_j \subset X$ be irreducible complex hypersurfaces, and let $\tau_j > 0$, where $1 \leq j \leq \ell$. We write $\Sigma = (\Sigma_1, \dots, \Sigma_\ell)$, $\tau = (\tau_1, \dots, \tau_\ell)$, and we denote by dist the distance on X induced by ω.

Let α be a smooth closed real $(1,1)$-form on X. We fix a smooth Hermitian metric g_j on $\mathcal{O}_X(\Sigma_j)$, let s_{Σ_j} be the canonical section of $\mathcal{O}_X(\Sigma_j)$, $1 \leq j \leq \ell$, and set

$$\beta_j = c_1(\mathcal{O}_X(\Sigma_j), g_j),\ \theta = \alpha - \sum_{j=1}^{\ell} \tau_j \beta_j,\ \sigma_j := |s_{\Sigma_j}|_{g_j}. \tag{4.1}$$

We let

$$\begin{aligned} \mathcal{L}(X, \alpha, \Sigma, \tau) &= \\ &= \{\psi \in \mathrm{PSH}(X, \alpha) : \nu(\psi, x) \geq \tau_j,\ \forall\, x \in \Sigma_j,\ 1 \leq j \leq \ell\} \end{aligned} \tag{4.2}$$

be the class of α-psh functions with logarithmic poles of order τ_j along Σ_j. Given a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ we consider the following subclasses of qpsh functions and their upper envelopes:

$$\mathcal{A}(X, \alpha, \Sigma, \tau, \varphi) = \{\psi \in \mathcal{L}(X, \alpha, \Sigma, \tau) : \psi \leq \varphi \text{ on } X\}, \tag{4.3}$$

$$\begin{aligned} \mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi) &= \\ &= \Big\{\psi' \in \mathrm{PSH}(X, \theta) : \psi' \leq \varphi - \sum_{j=1}^{\ell} \tau_j \log \sigma_j \text{ on } X \setminus \bigcup_{j=1}^{\ell} \Sigma_j\Big\}, \end{aligned} \tag{4.4}$$

$$\varphi_{\mathrm{eq}} = \varphi_{\mathrm{eq},\Sigma,\tau} = \sup\{\psi : \psi \in \mathcal{A}(X, \alpha, \Sigma, \tau, \varphi)\}, \tag{4.5}$$

$$\varphi_{\mathrm{req}} = \varphi_{\mathrm{req},\Sigma,\tau} = \sup\{\psi' : \psi' \in \mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi)\}. \tag{4.6}$$

The function φ_{eq} defined in (4.5) is the largest α-psh function dominated by φ and with logarithmic poles of order τ_j along Σ_j. We call φ_{eq} the *equilibrium envelope of* $(\alpha, \Sigma, \tau, \varphi)$, and φ_{req} the *reduced equilibrium envelope of* $(\alpha, \Sigma, \tau, \varphi)$. This is motivated by the terminology of *equilibrium metric* used in the case when φ is the weight of a singular metric $h = h_0 e^{-2\varphi}$ on a Hermitian holomorphic line bundle (F, h_0) over X (see below).

Extremal psh functions on domains in Stein manifolds with poles along subvarieties (also known as pluricomplex Green functions) are studied in [LS], [RaS]. In particular, pluricomplex Green functions with finitely many poles were studied by many authors. In the context of metrics on line bundles over compact complex manifolds, the above envelope method is introduced in Section 4.1 of [Be1] for defining equilibrium metrics with poles along a divisor. More generally, equilibrium metrics with prescribed singularities on a line bundle are introduced and studied in [RWN2] (see also [Da, Theorem 3]).

Our first result is concerned with some basic properties of the envelope defined in (4.5) under natural, very general assumptions.

Proposition 4.1. *Let $X, \Sigma, \tau, \alpha, \theta$ be as above, and let $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. Then, the following hold:*

(i) *The mapping* $\mathrm{PSH}(X, \theta) \ni \psi' \mapsto \psi := \psi' + \sum_{j=1}^{\ell} \tau_j \log \sigma_j \in \mathcal{L}(X, \alpha, \Sigma, \tau)$ *is well defined and bijective, with inverse*

$$\mathcal{L}(X, \alpha, \Sigma, \tau) \ni \psi \mapsto \psi' := \psi - \sum_{j=1}^{\ell} \tau_j \log \sigma_j \in \mathrm{PSH}(X, \theta).$$

(ii) *There exists a constant $C > 0$ depending only on $X, \Sigma, \tau, \alpha, \theta$ such that $\sup_X \psi' \leq \sup_X \varphi + C$, for every $\psi' \in \mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi)$.*

(iii) *$\mathcal{A}(X, \alpha, \Sigma, \tau, \varphi) \neq \emptyset$ if and only if $\mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi) \neq \emptyset$. Moreover, in this case we have that $\varphi_{\mathrm{req}} \in \mathcal{A}'(X, \alpha, \Sigma, \tau, \varphi)$, $\varphi_{\mathrm{eq}} \in \mathcal{A}(X, \alpha, \Sigma, \tau, \varphi)$, and*

$$\varphi_{\mathrm{eq}} = \varphi_{\mathrm{req}} + \sum_{j=1}^{\ell} \tau_j \log \sigma_j \quad \text{on } X. \tag{4.7}$$

(iv) *If φ is bounded and there exists a bounded θ-psh function, then φ_{req} is bounded on X.*

(v) *If $\mathrm{PSH}(X, \theta) \neq \emptyset$ and $\varphi_1, \varphi_2 : X \to \mathbb{R}$ are bounded and upper semicontinuous, then*

$$\varphi_{1,\mathrm{req}} - \sup_X |\varphi_1 - \varphi_2| \leq \varphi_{2,\mathrm{req}} \leq \varphi_{1,\mathrm{req}} + \sup_X |\varphi_1 - \varphi_2|$$

holds on X. Moreover, if $\varphi_1 \leq \varphi_2$ then $\varphi_{1,\mathrm{req}} \leq \varphi_{2,\mathrm{req}}$.

Proof. (i) If $\psi' \in \mathrm{PSH}(X, \theta)$ then $\psi = \psi' + \sum_{j=1}^{\ell} \tau_j \log \sigma_j$ is qpsh and

$$\alpha + dd^c \psi = \theta + dd^c \psi' + \sum_{j=1}^{\ell} \tau_j [\Sigma_j] \geq 0,$$

so $\psi \in \mathrm{PSH}(X, \alpha)$. If $x \in \Sigma_j$ then $\nu(\psi, x) \geq \tau_j \nu(\log \sigma_j, x) \geq \tau_j$, and hence $\psi \in \mathcal{L}(X, \alpha, \Sigma, \tau)$.

Conversely, if $\psi \in \mathcal{L}(X, \alpha, \Sigma, \tau)$, let $T = \alpha + dd^c \psi$. As $\nu(T, x) \geq \tau_j$ for all $x \in \Sigma_j$, we have by Siu's decomposition theorem that $T' = T - \sum_{j=1}^{\ell} \tau_j [\Sigma_j]$ is a positive closed current. Also, $T' = \theta + dd^c\psi'$, where $\psi' = \psi - \sum_{j=1}^{\ell}\tau_j\log\sigma_j$. The function $\psi' \in L^1(X,\omega^n)$ is defined everywhere on $X \setminus \bigcup_{j=1}^{\ell}\Sigma_j$. Since $T' \geq 0$ it follows that $\psi' = u$ almost everywhere on X, for some function $u \in \mathrm{PSH}(X,\theta)$, and hence everywhere on $X \setminus \bigcup_{j=1}^{\ell}\Sigma_j$ since both ψ', u are qpsh there. Thus, ψ' extends to a θ-psh function on X.

(ii) There exist points $x_k \in X$, coordinate neighborhoods U_k centered at x_k, and numbers $r_k > 0$, $1 \leq k \leq N$, such that the balls $\bar{\mathbb{B}}(x_k, 2r_k) \subset U_k$ and $X = \bigcup_{k=1}^{N}\mathbb{B}(x_k, r_k)$. Set $r = \min_{1\leq k\leq N} r_k$. Let ρ_k be a smooth function defined in a neighborhood of $\bar{\mathbb{B}}(x_k, 2r_k)$ such that $dd^c\rho_k = \theta$. If $\psi' \in \mathrm{PSH}(X,\theta)$ and $x \in \mathbb{B}(x_k, r_k)$, we have by the subaverage inequality for psh functions that

$$\rho_k(x) + \psi'(x) \leq \frac{n!}{\pi^n r^{2n}}\int_{\mathbb{B}(x,r)}(\rho_k + \psi')\,\mathrm{d}\lambda,$$

where λ is the Lebesgue measure in coordinates. Hence, there exists a constant $C' > 0$ such that for every function $\psi' \in \mathrm{PSH}(X,\theta)$, one has

$$\psi'(x) \leq \frac{n!}{\pi^n r^{2n}}\int_{\mathbb{B}(x,r)}\psi'\,\mathrm{d}\lambda + C', \quad \forall\, x \in \mathbb{B}(x_k, r),\ k = 1,\dots,N.$$

If $\psi' \in \mathcal{A}'(X,\alpha,\Sigma,\tau,\varphi)$ and $x \in \mathbb{B}(x_k, r)$, we have

$$\begin{aligned}\psi'(x) &\leq \frac{n!}{\pi^n r^{2n}}\int_{\mathbb{B}(x,r)}\Big(\varphi - \sum_{j=1}^{\ell}\tau_j\log\sigma_j\Big)\,\mathrm{d}\lambda + C' \\ &\leq \sup_{\mathbb{B}(x,r)}\varphi + \frac{n!}{\pi^n r^{2n}}\int_{\mathbb{B}(x,r)}\Big|\sum_{j=1}^{\ell}\tau_j\log\sigma_j\Big|\,\mathrm{d}\lambda + C' \leq \sup_{\mathbb{B}(x,r)}\varphi + C,\end{aligned}$$

for some constant $C > 0$ depending only on $X, \Sigma, \tau, \alpha, \theta$. Hence, $\sup_X \psi' \leq \sup_X \varphi + C$.

(iii) It follows immediately from (i) that the mapping

$$\mathcal{A}'(X,\alpha,\Sigma,\tau,\varphi) \ni \psi' \longmapsto \psi := \psi' + \sum_{j=1}^{\ell}\tau_j\log\sigma_j \in \mathcal{A}(X,\alpha,\Sigma,\tau,\varphi)$$

is well defined and bijective. By (ii), the family $\mathcal{A}'(X,\alpha,\Sigma,\tau,\varphi)$ of θ-psh functions is uniformly upper bounded, and hence the upper semicontinuous regularization $\varphi_{\mathrm{req}}^{\star}$ of φ_{req} is θ-psh. Since $\varphi_{\mathrm{req}} \leq \varphi - \sum_{j=1}^{\ell}\tau_j\log\sigma_j$ on $X \setminus \bigcup_{j=1}^{\ell}\Sigma_j$ and the latter is upper semicontinuous there, we see that $\varphi_{\mathrm{req}}^{\star} \in \mathcal{A}'(X,\alpha,\Sigma,\tau,\varphi)$, so $\varphi_{\mathrm{req}} = \varphi_{\mathrm{req}}^{\star}$. Moreover, if $\psi \in \mathcal{A}(X,\alpha,\Sigma,\tau,\varphi)$ then $\psi \leq \varphi_{\mathrm{req}} + \sum_{j=1}^{\ell}\tau_j\log\sigma_j$.

It follows that the family $\mathcal{A}(X,\alpha,\Sigma,\tau,\varphi)$ is uniformly upper bounded, the upper semicontinuous regularization $\varphi^\star_{\rm eq}$ of $\varphi_{\rm eq}$ is α-psh, and it verifies $\varphi^\star_{\rm eq}\leq\varphi_{\rm req}+\sum_{j=1}^\ell\tau_j\log\sigma_j$ and $\varphi^\star_{\rm eq}\leq\varphi$ on X, since the functions on the righthand side are upper semicontinuous. Hence, $\varphi^\star_{\rm eq}\in\mathcal{A}(X,\alpha,\Sigma,\tau,\varphi)$, so $\varphi_{\rm eq}=\varphi^\star_{\rm eq}$ and (4.7) is clearly satisfied.

(iv) Since $m:=\inf_X(\varphi-\sum_{j=1}^\ell\tau_j\log\sigma_j)>-\infty$, there exists a bounded θ-psh function ψ' such that $\psi'\leq m$ on X. Thus, $\psi'\leq\varphi_{\rm req}\leq\sup_X\varphi+C$ on X.

(v) As $\mathrm{PSH}(X,\theta)\neq\emptyset$ and φ_j is bounded, it follows that $\mathcal{A}'(X,\alpha,\Sigma,\tau,\varphi_j)\neq\emptyset$, $j=1,2$. Then, (v) follows easily from the definition (4.6) of $\varphi_{\rm req}$. □

The following notion is needed for studying certain regularity properties of the equilibrium envelopes.

Definition 4.2. A function $\phi:X\to[-\infty,\infty)$ is Hölder with singularities along a proper analytic subset $A\subset X$ if there exist constants $c,\varrho>0$ and $0<\nu\leq 1$ such that

$$|\phi(z)-\phi(w)|\leq\frac{c\,\mathrm{dist}(z,w)^\nu}{\min\{\mathrm{dist}(z,A),\mathrm{dist}(w,A)\}^\varrho},\quad\text{for all } z,w\in X\setminus A.$$

We assume now that the class $\{\theta\}_{\partial\bar\partial}$ is big, and we let

$$Z_0:=\mathrm{NAmp}(\{\theta\}_{\partial\bar\partial}).$$

By [Bou, Theorem 3.17] there exists a Kähler current $T_0\in\{\theta\}_{\partial\bar\partial}$ with analytic singularities such that $E_+(T_0)=Z_0$. We let

$$T_0=\theta+dd^c\psi_0\geq\varepsilon_0\omega, \tag{4.8}$$

where $\varepsilon_0>0$ and ψ_0 is a qpsh function with analytic singularities. Thus, $Z_0=\{\psi_0=-\infty\}$. By subtracting a constant we may assume that $\psi_0\leq-1$ on X.

Using the methods developed in [D6] and [BD] (see also [DMN]), we prove next the following regularity result for the functions $\varphi_{\rm req}=\varphi_{{\rm req},\Sigma,\tau}$ defined in (4.6), and $\varphi_{\rm eq}=\varphi_{{\rm eq},\Sigma,\tau}$ defined in (4.5).

If $v=1$ then φ is said to be Lipschitz with singularities along A.

Theorem 4.3. *Let (X,ω) be a compact Hermitian manifold of dimension n, $\Sigma_j\subset X$ be irreducible complex hypersurfaces, and let $\tau_j>0$, where $1\leq j\leq\ell$. Let α be a smooth closed real $(1,1)$-form on X, and θ be as in (4.1). Assume that the class $\{\theta\}_{\partial\bar\partial}$ is big, and let $Z_0:=\mathrm{NAmp}(\{\theta\}_{\partial\bar\partial})$. Then, the following hold:*

(i) *If $\varphi:X\to\mathbb{R}$ is Hölder continuous then $\varphi_{\rm req}$ is Hölder with singularities along Z_0, and $\varphi_{\rm eq}$ is Hölder with singularities along $\Sigma_1\cup\dots\cup\Sigma_\ell\cup Z_0$.*

(ii) *If $\varphi:X\to\mathbb{R}$ is continuous then $\varphi_{\rm req}$ is continuous on $X\setminus Z_0$, and $\varphi_{\rm eq}$ is continuous on $X\setminus(\Sigma_1\cup\dots\cup\Sigma_\ell\cup Z_0)$.*

We will need the following lemma which follows from Lemma 2.8 in [DMN].

Lemma 4.4. *Let $r, \nu \in (0,1)$ and u be a subharmonic function in a neighborhood of the ball $\bar{\mathbb{B}}(0,3r) \subset \mathbb{R}^m$. Suppose there exists a constant $A > 0$ such that $|u(x)| \le A$ for all $x \in \mathbb{B}(0,3r)$, and*

$$\Delta u(\mathbb{B}(x,s)) \le A s^{m-2+\nu},$$

for all $x \in \mathbb{B}(0,2r)$ and $0 < s < r$. Then, there exists a constant $C(m,\nu) > 0$ such that

$$|u(x) - u(y)| \le \frac{C(m,\nu)A}{r^\nu}|x-y|^\nu, \quad \forall\, x,y \in \mathbb{B}(0,r).$$

Proof of Theorem 4.3. (i) Assume φ is a $\mathcal{C}^\nu$ function on X, and let $\psi := \phi_{\mathrm{req}}$. We show that ψ is Hölder with singularities along Z_0 and with Hölder exponent ν. Following [BD], we will regularize ψ using the method introduced by Demailly in [D6].

Consider the exponential map associated with the Chern connection on the tangent bundle TX of X. The *formal holomorphic part* of its Taylor expansion is denoted by

$$\mathrm{exph} : TX \to X \quad \text{with } T_xX \ni \zeta \mapsto \mathrm{exph}_x(\zeta).$$

Let $\chi : \mathbb{R} \to [0,\infty)$ be the smooth function with support in $(-\infty, 1]$ defined by

$$\chi(t) = \begin{cases} \dfrac{\mathrm{const}}{(1-t)^2}\exp\dfrac{1}{t-1} & \text{for } t < 1, \\ 0 & \text{for } t \ge 1, \end{cases}$$

where the constant const is adjusted so that $\int_{|\zeta|\le 1} \chi(|\zeta|^2)\,\mathrm{d\,Leb}(\zeta) = 1$ with respect to the Lebesgue measure $\mathrm{Leb}(\zeta)$ on $\mathbb{C}^n \simeq T_xX$. We fix a constant $\delta_0 > 0$ small enough, and define

$$\Psi(x,t) := \int_{\zeta \in T_xX} \psi(\mathrm{exph}_x(t\zeta))\chi(|\zeta|^2)\,\mathrm{d\,Leb}(\zeta), \quad \text{for } (x,t) \in X \times [0,\delta_0]. \tag{4.9}$$

By [D6], there exists a constant b such that the function $t \mapsto \Psi(x,t) + bt^2$ is increasing for t in $[0,\delta_0]$. Observe also that $\Psi(x,0) = \psi(x)$.

Consider for $c > 0$ and $\delta \in (0,\delta_0]$ the *Kiselman-Legendre transform*

$$\psi_{c,\delta}(x) := \inf_{t\in(0,\delta]} \left(\Psi(x,t) + b(t^2-\delta^2) + b(t-\delta) - c\log\frac{t}{\delta}\right). \tag{4.10}$$

It was shown in [BD, Lemma 1.12] and [KoN, Lemma 4.1] (see also [DiNT, Lemma 3.1]) that $\psi_{c,\delta}$ is qpsh and

$$\theta + dd^c\psi_{c,\delta} \ge -(ac + 2b\delta)\omega, \tag{4.11}$$

where $a > 0$ is a constant (see also [Ki1, Ki2]).

For $t := \delta$ we obtain from (4.10) that $\psi_{c,\delta}(x) \le \Psi(x,\delta)$. From (4.9) we deduce that $\Psi(x,\delta)$ is an average of values of ψ in a ball $\mathbb{B}(x, A\delta)$ in X for some constant A depending only on X and ω. By Proposition 4.1 we have that

$$\psi \le \min\Big\{\varphi - \sum_{j=1}^{\ell} \tau_j \log \sigma_j, \sup_X \varphi + C\Big\} \le \varphi + f \quad \text{on } X,$$

where $f := \min\{-\sum_{j=1}^{\ell} \tau_j \log \sigma_j, \sup_X \varphi - \inf_X \varphi + C\}$. In the sequel, we denote $O(\delta^\nu) := C'\delta^\nu$ with constants $C' > 0$ independent of x and δ. Since $\varphi \in C^\nu$ and f is a Lipschitz function on X, it follows that

$$\begin{aligned}\Psi(x,\delta) &\le \sup_{\mathbb{B}(x,A\delta)} (\varphi + f) \le \varphi(x) + f(x) + O(\delta^\nu)\\ &\le \varphi(x) - \sum_{j=1}^{\ell} \tau_j \log \sigma_j(x) + O(\delta^\nu),\end{aligned}$$

for all $x \in X$ and $0 < \delta \le \delta_0$. Hence, $\psi_{c,\delta} \le \varphi - \sum_{j=1}^{\ell} \tau_j \log \sigma_j + O(\delta^\nu)$. Let ψ_0 be the θ-psh function with analytic singularities in Z_0 defined in (4.8). Since φ is bounded there exists a constant $C_1 > 0$ such that $\psi_0 \le C_1 + \varphi - \sum_{j=1}^{\ell} \tau_j \log \sigma_j$. Thus, $\psi_0 - C_1 \le \psi$, and ψ is locally bounded on $X \setminus Z_0$. Consider the convex combination

$$\xi := \frac{ac + 2b\delta}{\varepsilon_0}\psi_0 + \Big(1 - \frac{ac + 2b\delta}{\varepsilon_0}\Big)\psi_{c,\delta},$$

where we take $c = \delta^\nu$. We deduce from the above upper bounds for $\psi_{c,\delta}$ and ψ_0 that

$$\xi \le \varphi - \sum_{j=1}^{\ell} \tau_j \log \sigma_j + O(\delta^\nu).$$

By (4.8) and (4.11) we have that

$$\theta + dd^c\xi \ge (ac + 2b\delta)\omega - \Big(1 - \frac{ac + 2b\delta}{\varepsilon_0}\Big)(ac + 2b\delta)\omega \ge 0,$$

and hence

$$\psi(x) + O(\delta^\nu) \ge \xi(x) \ge \frac{ac + 2b\delta}{\varepsilon_0}\psi_0(x) + \psi_{c,\delta}(x) - \frac{ac + 2b\delta}{\varepsilon_0}\Psi(x,\delta),$$

for all $x \in X$. Since $c = \delta^\nu$ and by Proposition 4.1 (ii), $\Psi(x,\delta) \le \sup_X \psi \le \sup_X \varphi + C$, it follows that

$$\frac{ac + 2b\delta}{\varepsilon_0}\psi_0 + \psi_{c,\delta} \le \psi + O(\delta^\nu).$$

If $x \in X \setminus Z_0$ then $\Psi(x,0) = \psi(x) > -\infty$, and so the increasing function $t \mapsto \Psi(x,t) + bt^2$ is bounded and the infimum in the definition of $\psi_{c,\delta}(x)$ is reached for some $t = t_{x,\delta} \in (0,\delta]$. Hence,

$$\Psi(x,t_{x,\delta}) + bt_{x,\delta}^2 \le \psi(x) + c\log\frac{t_{x,\delta}}{\delta} - \frac{ac+2b\delta}{\varepsilon_0}\psi_0(x) + O(\delta^\nu). \tag{4.12}$$

Using the fact that $t \mapsto \Psi(x,t) + bt^2$ is increasing, this implies that

$$c\log\frac{t_{x,\delta}}{\delta} - \frac{ac+2b\delta}{\varepsilon_0}\psi_0(x) + O(\delta^\nu) \ge 0.$$

Since $c = \delta^\nu$ and $\psi_0 \le -1$, we infer from the above inequality there exists a constant $C_2 > 0$ such that $e^{C_2\psi_0(x)}\delta \le t_{x,\delta} \le \delta$, for all $x \in X \setminus Z_0$. By (4.12) and using again that $t \mapsto \Psi(x,t) + bt^2$ is increasing, we see that

$$\Psi(x, e^{C_2\psi_0(x)}\delta) - \psi(x) \le |\psi_0(x)|O(\delta^\nu), \quad \forall\, x \in X \setminus Z_0.$$

Since ψ_0 has analytic singularities, it follows by the Lojasiewicz inequality, Theorem 5.2.4 in [KP], that there exists a constant $M > 0$ such that $e^{\psi_0(x)} \gtrsim \operatorname{dist}(x,Z_0)^M$. Letting $t = e^{C_2\psi_0(x)}\delta$ we conclude by the above estimates that there exist constants $\varepsilon_1, N > 0$ such that

$$\Psi(x,t) - \psi(x) \le \frac{O(t^\nu)}{\operatorname{dist}(x,Z_0)^N} \qquad \forall\, x \in X \setminus Z_0,\ 0 < t < \varepsilon_1 \operatorname{dist}(x,Z_0)^N. \tag{4.13}$$

There exists $r_0 > 0$ such that every $x \in X$ has a coordinate neighborhood U_x centered at x with $\bar{\mathbb{B}}(x,3r_0) \subset U_x$ and so that the metric on X coincides at x with the standard metric given by the coordinates. According to [D6, (4.5)] (see also [BD, (1.16)], [DMN, Theorem 2.7]), it follows from a Lelong-Jensen type inequality that

$$\Psi(x,t) - \psi(x) \gtrsim \frac{1}{t^{2n-2}} \int_{\mathbb{B}(x,t/4)} \Delta\psi - O(t^2),$$

where the constants involved are independent of $x \in X$ and $t \in (0,\delta_0)$. Combining this and (4.13), we infer that

$$\int_{\mathbb{B}(x,t)} \Delta\psi \le \frac{O(t^{2n-2+\nu})}{\operatorname{dist}(x,Z_0)^N} \quad \forall\, x \in X \setminus Z_0,\ 0 < t < \min\{r_0, \varepsilon_1 \operatorname{dist}(x,Z_0)^N\}.$$

Lemma 4.4 implies there exist constants $C_3, \varepsilon_2 > 0$ such that if $x \in X \setminus Z_0$ and $\operatorname{dist}(y,x) \le \varepsilon_2 \operatorname{dist}(x,Z_0)^N$, then

$$|\psi(y) - \psi(x)| \le \frac{C_3}{\operatorname{dist}(x,Z_0)^{2N}} \operatorname{dist}(y,x)^\nu.$$

Since $\psi_0 - C_1 \le \psi$, it follows that $|\psi| \le C_4|\log \operatorname{dist}(\cdot, Z_0)| + C_1$ holds on X, for some constant $C_4 > 0$. Assume now that $x, y \in X \setminus Z_0$ and $\operatorname{dist}(y, x) \ge \varepsilon_2 \operatorname{dist}(x, Z_0)^N$. Then,

$$|\psi(y) - \psi(x)| \le (C_4|\log \operatorname{dist}(x, Z_0)| + C_4|\log \operatorname{dist}(y, Z_0)| + 2C_1) \times \frac{\operatorname{dist}(y,x)}{\varepsilon_2 \operatorname{dist}(x, Z_0)^N}.$$

The previous two estimates combined show that $\psi = \varphi_{\mathrm{req}}$ is Hölder with singularities along Z_0 and with Hölder exponent ν. Hence, by (4.7), φ_{eq} is Hölder with singularities along $\Sigma_1 \cup \cdots \cup \Sigma_\ell \cup Z_0$ and with Hölder exponent ν, since $\log \sigma_j$ is Lipschitz with singularities along Σ_j.

(ii) Let $\{\varphi_k\}$ be a sequence of real-valued smooth functions converging uniformly to φ on X. By (i) and Proposition 4.1 (v), $\varphi_{k,\mathrm{req}}$ are continuous and converge uniformly to φ_{req} on $X \setminus Z_0$, and hence φ_{req} is continuous on $X \setminus Z_0$. □

5. Convergence of the Global Fubini-Study Potentials

In this section, we prove the convergence of the Fubini-Study potentials and currents of the space $H^0_0(X, L^p)$ defined in (1.2), towards φ_{eq} and the equilibrium current T_{eq} of (L, h, Σ, τ), respectively (Theorem 1.8), provided X, L, Σ, τ verify assumptions (A)–(D), and assuming in addition there exists a Kähler form ω on X, and that h is a continuous Hermitian metric on L.

Let h_0, φ be as in (1.3). Let P_p, γ_p be the Bergman kernel function and Fubini-Study current of the space $H^0_{0,(2)}(X, L^p)$, and let φ_p be the global Fubini-Study potential of γ_p (see (1.4)). We fix a divisorization $(\tilde X, \pi, \tilde\Sigma)$ of (X, Σ) as in Definition 1.5. Thus, there exists an analytic subset Y of X such that $\dim Y \le n-2$, $X_{\mathrm{sing}} \subset Y$, $E = \pi^{-1}(Y)$ is the final exceptional divisor, and $\pi : \tilde X \setminus E \to X \setminus Y$ is a biholomorphism. We let $\tilde\omega$ be a Kähler form on $\tilde X$ such that $\tilde\omega \ge \pi^\star\omega$ (see, e.g., [CMM, Lemma 2.2]), and denote by dist the distance on $\tilde X$ induced by $\tilde\omega$. Set

$$\tilde L := \pi^\star L,\ \tilde h_0 := \pi^\star h_0,\ \tilde\alpha := \pi^\star\alpha = c_1(\tilde L, \tilde h_0), \quad \tilde\varphi := \varphi \circ \pi,\ \tilde h := \pi^\star h = \tilde h_0 e^{-2\tilde\varphi}. \tag{5.1}$$

We write $\tilde h^p = \tilde h^{\otimes p}$ and $\tilde h_0^p = \tilde h_0^{\otimes p}$. Corollary 3.4 implies that the map

$$S \in H^0_{0,(2)}(X, L^p) \to \pi^\star S \in H^0_{0,(2)}(\tilde X, \tilde L^p) := H^0_{0,(2)}(\tilde X, \tilde L^p, \tilde\Sigma, \tau, \tilde h^p, \pi^\star\omega^n) \tag{5.2}$$

is an isometry. It follows that

$$\tilde P_p = P_p \circ \pi,\ \tilde\gamma_p = \pi^\star\gamma_p \tag{5.3}$$

are the Bergman kernel function, respectively Fubini-Study current, of the space $H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$. Note that

$$\frac{1}{p}\tilde{\gamma}_p = \tilde{\alpha} + dd^c\tilde{\varphi}_p, \quad \text{where } \tilde{\varphi}_p = \tilde{\varphi} + \frac{1}{2p}\log \tilde{P}_p. \tag{5.4}$$

We call the function $\tilde{\varphi}_p$ the *global Fubini-Study potential* of $\tilde{\gamma}_p$.

Let $\tilde{\varphi}_{\rm eq}$ be the equilibrium envelope of $(\tilde{\alpha}, \tilde{\Sigma}, \tau, \tilde{\varphi})$ defined in (4.5),

$$\tilde{\varphi}_{\rm eq} = \tilde{\varphi}_{{\rm eq},\tilde{\Sigma},\tau} = \sup\{\psi : \psi \in \mathcal{L}(\tilde{X}, \tilde{\alpha}, \tilde{\Sigma}, \tau),\ \psi \leq \tilde{\varphi} \text{ on } \tilde{X}\}, \tag{5.5}$$

where $\mathcal{L}(\tilde{X}, \tilde{\alpha}, \tilde{\Sigma}, \tau)$ is defined in (4.2). Let $s_{\tilde{\Sigma}_j}$ be the canonical section of $\mathcal{O}_{\tilde{X}}(\tilde{\Sigma}_j)$, and fix a smooth Hermitian metric g_j on $\mathcal{O}_{\tilde{X}}(\tilde{\Sigma}_j)$ such that

$$\sigma_j := |s_{\tilde{\Sigma}_j}|_{g_j} < 1 \quad \text{on } \tilde{X},\ 1 \leq j \leq \ell.$$

Set

$$\beta_j = c_1(\mathcal{O}_{\tilde{X}}(\tilde{\Sigma}_j), g_j), \quad \tilde{\theta} = \tilde{\alpha} - \sum_{j=1}^{\ell} \tau_j\beta_j. \tag{5.6}$$

Note that $[\tilde{\Sigma}_j] = \beta_j + dd^c \log\sigma_j$, by the Lelong-Poincaré formula.

In the above setting, we have the following theorem which shows that on $\tilde{X}$, the global Fubini-Study potentials $\tilde{\varphi}_p$ converge to the equilibrium envelope $\tilde{\varphi}_{\rm eq}$ of $\tilde{\varphi}$.

Theorem 5.1. *Let X, L, Σ, τ verify assumptions* (A)–(D)*, and assume (L, Σ, τ) is big and there exists a Kähler form ω on X. Let h be a continuous Hermitian metric on L, let φ, $\tilde{\varphi}$, $\tilde{\varphi}_p, \tilde{\varphi}_{\rm eq}, \tilde{\theta}$ be defined in* (1.3), (5.1), (5.4), (5.5) *(respectively* (5.6)*), and set $Z := \tilde{\Sigma}_1 \cup \cdots \cup \tilde{\Sigma}_\ell \cup \mathrm{NAmp}(\{\tilde{\theta}\})$. Then, the following hold:*

(i) *$\tilde{\varphi}_p \to \tilde{\varphi}_{\rm eq}$ in $L^1(\tilde{X}, \tilde{\omega}^n)$ and locally uniformly on $\tilde{X} \setminus Z$ as $p \to \infty$.*

(ii) *If φ is Hölder continuous on X, then there exist a constant $C > 0$ and $p_0 \in \mathbb{N}$ such that, for all $x \in \tilde{X} \setminus Z$ and $p \geq p_0$, we have*

$$|\tilde{\varphi}_p(x) - \tilde{\varphi}_{\rm eq}(x)| \leq \frac{C}{p}(\log p + |\log \mathrm{dist}(x, Z)|).$$

In particular, we have the convergence of the Fubini-Study currents defined in (5.3)*:*

$$\frac{1}{p}\tilde{\gamma}_p = \tilde{\alpha} + dd^c\tilde{\varphi}_p \to \tilde{T}_{\rm eq} := \tilde{\alpha} + dd^c\tilde{\varphi}_{\rm eq}, \quad \text{as } p \to \infty, \text{ weakly on } \tilde{X}.$$

The proof is done by estimating the partial Bergman kernel $\tilde{P}_p$ from (5.3) (see [Be1], [Be2], [CM1], [RWN2] for similar approaches). Let

$$\Omega_{\tilde{\varphi}}(\delta) = \sup\{|\tilde{\varphi}(x) - \tilde{\varphi}(y)| : x, y \in \tilde{X},\ \mathrm{dist}(x, y) < \delta\}$$

be the modulus of continuity of $\tilde{\varphi}$. We first deal with the upper estimate for $\log \tilde{P}_p$.

Proposition 5.2. *In the setting of Theorem 5.1, there exists a constant $C > 0$ such that*

$$\frac{1}{2p} \log \tilde{P}_p(x) \leq \frac{C}{p}(1 - \log\delta) + \delta + \Omega_{\tilde{\varphi}}(\delta),$$

for all $p \geq 1$, $\delta \in (0,1)$, and $x \in \tilde{X}$ with $\mathrm{dist}(x, E) \geq \delta$.

Proof. By compactness, there exist constants $r_0 > 0$, $C_1 > 1$ with the following properties: every $x \in \tilde{X}$ has a contractible Stein coordinate neighborhood U_x centered at x such that the following hold:

(i) The ball $\bar{\mathbb{B}}(x, r_0) \subset U_x$ and the Lebesgue measure in coordinates satisfies $d\lambda \leq C_1 \tilde{\omega}^n / n!$.

(ii) $C_1^{-1}|z - y| \leq \mathrm{dist}(z, y) \leq C_1|z - y|$ holds for $z, y \in \bar{\mathbb{B}}(x, r_0)$.

(iii) $\tilde{L}$ has a local holomorphic frame e_x on U_x such that if $|e_x|_{\tilde{h}_0} = e^{-\psi_x}$ then ψ_x is a Lipschitz function with Lipschitz constant C_1 on U_x.

Moreover, there exists $K > 0$ such that

$$\pi^\star\omega^n(x) \geq C_1^{-1}\,\mathrm{dist}(x, E)^K \tilde{\omega}^n(x), \quad \forall\, x \in \tilde{X}.$$

Indeed, using local embeddings of X into $\mathbb{C}^N$ we have that $\omega \gtrsim i\sum_{j=1}^N dz_j \wedge d\bar{z}_j$, so the above claim follows from the Lojasiewicz inequality.

We let $\delta \in (0,1)$ and fix $x \in \tilde{X} \setminus E$ with $\mathrm{dist}(x, E) \geq \delta$. Let $r < r_0$, $r < (2C_1)^{-1}\mathrm{dist}(x, E)$. If $S \in H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$, $\|S\|_p = 1$, we write $S = se_x^{\otimes p}$, where $s \in \mathcal{O}_{\tilde{X}}(U_x)$. Using the subaverage inequality, we obtain the following:

$$\begin{aligned}|S(x)|^2_{\tilde{h}^p} &= |s(x)|^2 e^{-2p(\psi_x(x) + \tilde{\varphi}(x))} \\ &\leq \frac{n! C_1}{\pi^n r^{2n}} e^{2p(\max_{\mathbb{B}(x,r)}(\psi_x + \tilde{\varphi}) - \psi_x(x) - \tilde{\varphi}(x))} \int_{\mathbb{B}(x,r)} |S|^2_{\tilde{h}^p} \frac{\tilde{\omega}^n}{n!}.\end{aligned}$$

If $y \in \mathbb{B}(x, r)$, then $\mathrm{dist}(y, E) \geq \mathrm{dist}(x, E) - C_1 r \geq \frac{1}{2}\mathrm{dist}(x, E) \geq \delta/2$, and hence

$$\tilde{\omega}^n(y) \leq C_1\,\mathrm{dist}(y, E)^{-K}\pi^\star\omega^n(y) \leq 2^K C_1 \delta^{-K}\pi^\star\omega^n(y).$$

Therefore,

$$|S(x)|^2_{\tilde{h}^p} \leq \frac{C_2}{r^{2n}\delta^K} e^{2p(\max_{\mathbb{B}(x,r)}(\psi_x + \tilde{\varphi}) - \psi_x(x) - \tilde{\varphi}(x))} \int_{\mathbb{B}(x,r)} |S|^2_{\tilde{h}^p} \frac{\pi^\star\omega^n}{n!},$$

where $C_2 = n!\pi^{-n}2^K C_1^2$. Since $S \in H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$ is arbitrary with $\|S\|_p = 1$, we infer that

$$\begin{aligned}\frac{1}{2p}\log \tilde{P}_p(x) &\leq \frac{\log C_2}{2p} - \frac{n}{p}\log r - \frac{K}{2p}\log\delta \\ &\qquad + \max_{\mathbb{B}(x,r)} \psi_x - \psi_x(x) + \max_{\mathbb{B}(x,r)} \tilde{\varphi} - \tilde{\varphi}(x) \\ &\leq \frac{\log C_2}{2p} - \frac{n}{p}\log r - \frac{K}{2p}\log\delta + C_1 r + \Omega_{\tilde{\varphi}}(C_1 r).\end{aligned}$$

Let $M > C_1$ be a constant such that $(2M)^{-1}\operatorname{dist}(x, E) < r_0$ for all $x \in \tilde{X}$. Taking $r = \delta/(2M)$ in the last estimate, we see that the conclusion holds with a constant $C = C(n, C_2, K, M)$. ◻

Remark 5.3. The conclusion of Proposition 5.2 holds in fact for the full Bergman kernel of the space $H^0_{(2)}(\tilde{X}, \tilde{L}^p, \tilde{h}^p, \pi^\star\omega^n)$.

Proposition 5.4. *In the setting of Theorem 5.1, there exists a constant $C > 0$ such that for all $p \geq 1$ and $\delta \in (0,1)$ the following estimate holds on $\tilde{X}$:*

$$\tilde{\varphi}_p \leq \tilde{\varphi}_{\mathrm{eq}} + C\left(\delta + \frac{1}{p} - \frac{\log\delta}{p}\right) + 2\Omega_{\tilde{\varphi}}(C\delta).$$

Proof. If $p \geq 1$ and $0 < \delta < 1$, we have by Proposition 5.2 that

$$\begin{aligned}F_p(\delta) &:= \sup\left\{\frac{1}{2p}\log\tilde{P}_p(x) : x \in \tilde{X},\ \operatorname{dist}(x, E) \geq \delta\right\} \\ &\leq \frac{C}{p}(1 - \log\delta) + \delta + \Omega_{\tilde{\varphi}}(\delta).\end{aligned} \tag{5.7}$$

Let $E_\delta := \{x \in \tilde{X} : \operatorname{dist}(x, E) < \delta\}$, and fix $x_0 \in E$. There exist a coordinate neighborhood U_{x_0} centered at x_0 and a constant $C_1 = C_{1,x_0} > 1$ with the following properties:

(i) The polydisc $\bar{\Delta}^n(0,2) \subset U_{x_0}$ and $C_1^{-1}|z-y| \leq \operatorname{dist}(z,y) \leq C_1|z-y|$ for $z, y \in \bar{\Delta}^n(0,2)$, where $|z| := \max_{1\leq j\leq n}|z_j|$.

(ii) $\tilde{\alpha} = dd^c\rho$ on U_{x_0}, where ρ is a smooth function with Lipschitz constant C_1 on $\bar{\Delta}^n(0,2)$ (with respect to the norm $|z|$ from (i)).

(iii) $E \cap \Delta^n(0,2) = (\bigcup_{j=1}^k \{z_j = 0\}) \cap \Delta^n(0,2)$, for some $1 \leq k \leq n$.

Note that (iii) can be achieved since E is a divisor with only normal crossings. Hence,

$$E_{\delta/C_1} \cap \Delta^n(0,1) \subset \{z \in \Delta^n(0,1) : \min_{1\leq j\leq k}|z_j| < \delta\}.$$

Let $z \in E_{\delta/C_1} \cap \Delta^n(0,1)$ and without loss of generality assume that

$$|z_1| < \delta, \ldots, |z_l| < \delta,\ |z_{l+1}| \geq \delta, \ldots, |z_k| \geq \delta.$$

The function $v := \rho + \tilde{\varphi}_p = \rho + \tilde{\varphi} + (1/(2p)) \log \tilde{P}_p$ is psh on U_{x_0}. By the maximum principle applied on

$$V_l := \{\zeta = (\zeta_1, \dots, \zeta_l, z_{l+1}, \dots, z_n) : |\zeta_j| \leq \delta,\ 1 \leq j \leq l\},$$

it follows that

$$v(z) \leq \max_{\partial_d V_l} v \leq \max_{\partial_d V_l} \rho + \max_{\partial_d V_l} \tilde{\varphi} + F_p\left(\frac{\delta}{C_1}\right),$$

as $|\zeta_1| = \cdots = |\zeta_l| = \delta$ for ζ in the distinguished boundary $\partial_d V_l$, and hence $\operatorname{dist}(\zeta, E) \geq \delta/C_1$. Since

$$\max_{\partial_d V_l} \rho \leq \rho(z) + 2C_1\delta,\ \max_{\partial_d V_l} \tilde{\varphi} \leq \tilde{\varphi}(z) + \Omega_{\tilde{\varphi}}(2C_1\delta),$$

we conclude that

$$\tilde{\varphi}_p(z) \leq \tilde{\varphi}(z) + 2C_1\delta + \Omega_{\tilde{\varphi}}(2C_1\delta) + F_p(\delta/C_1), \quad \forall\, z \in E_{\delta/C_1} \cap \Delta^n(0,1).$$

Using a finite cover of E with neighborhoods $\Delta^n(0,1) \subset U_{x_0}$, $x_0 \in E$, we infer by above that there exists a constant $C' > 1$ such that for all $p \geq 1$ and $0 < \delta < 1$, we have

$$\tilde{\varphi}_p(x) \leq \tilde{\varphi}(x) + C'\delta + \Omega_{\tilde{\varphi}}(C'\delta) + F_p\left(\frac{\delta}{C'}\right) \qquad \text{for all } x \in E_{\delta/C'}. \tag{5.8}$$

Note that

$$\tilde{\varphi}_p(x) = \tilde{\varphi}(x) + \frac{1}{2p} \log \tilde{P}_p \leq \tilde{\varphi}(x) + F_p\left(\frac{\delta}{C'}\right) \quad \text{for } x \in \tilde{X} \setminus E_{\delta/C'},$$

so the estimate (5.8) holds for all $x \in \tilde{X}$, $p \geq 1$, and $0 < \delta < 1$. Since $\tilde{P}_p$ is the Bergman kernel function of the space $H^0_0(\tilde{X}, \tilde{L}^p, \tilde{\Sigma}, \tau)$, it follows that $\tilde{\varphi}_p$ has Lelong number $\geq t_{j,p}/p \geq \tau_j$ along $\tilde{\Sigma}_j$, $1 \leq j \leq \ell$. Therefore, by (5.5) and (5.7),

$$\begin{aligned}
\tilde{\varphi}_p &\leq \tilde{\varphi}_{\mathrm{eq}} + C'\delta + \Omega_{\tilde{\varphi}}(C'\delta) + F_p\left(\frac{\delta}{C'}\right) \\
&\leq \tilde{\varphi}_{\mathrm{eq}} + (C'+1)\delta + 2\Omega_{\tilde{\varphi}}(C'\delta) + \frac{C}{p}\left(1 - \log \frac{\delta}{C'}\right),
\end{aligned}$$

which concludes the proof. □

We next deal with the lower bound for $\log \tilde{P}_p$ and $\tilde{\varphi}_p$. The following form of the L^2-estimates of Hörmander/Andreotti-Vesentini for $\bar{\partial}$ is due to Demailly [D1, Théorème 5.1] (see also [CM1, Theorem 5.2], [CMM, Theorem 2.5]).

Theorem 5.5. *Let M be a complete Kähler manifold of dimension n, Ω be a (not necessarily complete) Kähler form on M, χ be a* qpsh *function on M, and (F,h) be a singular Hermitian holomorphic line bundle on M. Assume there exist constants $A,B>0$ such that*

$$\operatorname{Ric}\Omega \geq -2\pi B\Omega, \quad dd^c\chi \geq -A\Omega,\ c_1(F,h) \geq (1+B+A/2)\Omega.$$

If $g \in L^2_{0,1}(M,F,\mathrm{loc})$ satisfies $\bar\partial g = 0$ and $\int_M |g|^2_h e^{-\chi}\Omega^n < +\infty$, then there exists $u \in L^2_{0,0}(M,F,\mathrm{loc})$ with $\bar\partial u = g$ and $\int_M |u|^2_h e^{-\chi}\Omega^n \leq \int_M |g|^2_h e^{-\chi}\Omega^n$.

Since (L,Σ,τ) is big and $\tilde X$ is Kähler, it follows from Theorem 1.6 that the class $\{\tilde\theta\} = \{\tilde\theta\}_{\partial\bar\partial}$ is big, where $\tilde\theta$ is defined in (5.6). By [D4] and Theorem 3.17 in [Bou], there exists a $\tilde\theta$-psh function η with analytic singularities on $\tilde X$ such that

$$\{\eta=-\infty\} = \operatorname{NAmp}(\{\tilde\theta\}),\ \eta \leq -1,\ \text{and } \tilde\theta + dd^c\eta \geq \varepsilon_0\tilde\omega \geq \varepsilon_0\pi^\star\omega$$

hold on $\tilde X$, for some constant $\varepsilon_0 > 0$.

Proposition 5.6. *In the setting of Theorem 5.1, there exist a constant $C>0$ and $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, the following estimate holds on $\tilde X \setminus Z$:*

$$\tilde\varphi_p \geq \tilde\varphi_{\mathrm{eq}} + \frac{C}{p}\eta + \frac{1}{p}\sum_{j=1}^{\ell}\log\sigma_j.$$

Proof. We consider the Bergman space $H^0_{(2)}(\tilde X,\tilde L^p,H_p,\tilde\omega^n)$ of L^2-integrable sections of $\tilde L^p$ with respect to the volume form $\tilde\omega^n$ on $\tilde X$ and the metric $H_p := \tilde h_0^p e^{-2\psi_p}$ on $\tilde L^p$, where

$$\psi_p = (p-p_0)\tilde\varphi_{\mathrm{eq}} + p_0\eta + \sum_{j=1}^{\ell}(p_0\tau_j+1)\log\sigma_j,$$

and $p_0 \in \mathbb{N}$ will be specified later. Since $\sigma_j < 1$, $\eta < 0$, and $\tilde\varphi_{\mathrm{eq}} \leq \tilde\varphi$, we have for $p \geq p_0$ that $\psi_p \leq (p-p_0)\tilde\varphi$. Moreover,

$$\begin{aligned} c_1(\tilde L^p,H_p) &\\ &= p\tilde\alpha + (p-p_0)dd^c\tilde\varphi_{\mathrm{eq}} + p_0dd^c\eta + \sum_{j=1}^{\ell}(p_0\tau_j+1)([\tilde\Sigma_j]-\beta_j) \\ &= (p-p_0)(\tilde\alpha + dd^c\tilde\varphi_{\mathrm{eq}}) + p_0(\tilde\theta+dd^c\eta) + \sum_{j=1}^{\ell}(p_0\tau_j+1)[\tilde\Sigma_j] - \sum_{j=1}^{\ell}\beta_j \\ &\geq (p_0\varepsilon_0 - C_1)\tilde\omega, \end{aligned}$$

where $C_1 > 0$ is a constant such that $\sum_{j=1}^{\ell} \beta_j \leq C_1 \tilde{\omega}$. If p_0 is chosen large enough (i.e., $p_0 \varepsilon_0 - C_1 \geq 1 + B + A/2$, where A, B are as in Theorem 5.5) and $p \geq p_0$, we use Theorem 5.5 for $\tilde{L}^p$, with suitable weights χ as in the proof of [CM1, Theorem 5.1], to show there exists a constant $C_2 > 0$ such that for all $p \geq p_0$ and $x \in \tilde{X} \setminus Z$, there exists $S_x \in H^0_{(2)}(\tilde{X}, \tilde{L}^p, H_p, \tilde{\omega}^n)$ with $S_x(x) \neq 0$ and

$$\|S_x\|^2_{H_p,\tilde{\omega}^n} \leq C_2 |S_x(x)|^2_{H_p}.$$

Note that $H_p = \tilde{h}^p e^{2F_p}$, where $F_p = p\tilde{\varphi} - \psi_p \geq p_0 \tilde{\varphi}$. Let $a := \min_{\tilde{X}} \tilde{\varphi}$. Then, $F_p \geq a p_0$, and since $\tilde{\omega} \geq \pi^\star \omega$, we obtain

$$\|S_x\|^2_{H_p,\tilde{\omega}^n} = \int_{\tilde{X}} |S_x|^2_{\tilde{h}^p} e^{2F_p} \tilde{\omega}^n \geq e^{2ap_0} \|S_x\|^2_{\tilde{h}^p,\pi^\star \omega^n}.$$

By (4.7), we have that $\tilde{\varphi}_{\rm eq} = \tilde{\varphi}_{\rm req} + \sum_{j=1}^{\ell} \tau_j \log \sigma_j$ on $\tilde{X}$, where $\tilde{\varphi}_{\rm req} = \tilde{\varphi}_{{\rm req},\tilde{\Sigma},\tau}$ is the reduced equilibrium envelope of $(\tilde{\alpha}, \tilde{\Sigma}, \tau, \tilde{\varphi})$ defined in (4.6). Hence,

$$\psi_p = (p - p_0)\tilde{\varphi}_{\rm req} + p_0 \eta + \sum_{j=1}^{\ell} (p\tau_j + 1) \log \sigma_j$$

is a qpsh function with Lelong numbers $\geq p\tau_j + 1$ along $\tilde{\Sigma}_j$. Since

$$\|S_x\|^2_{H_p,\tilde{\omega}^n} < +\infty,$$

this shows that $\operatorname{ord}(S_x, \tilde{\Sigma}_j) \geq \lfloor \tau_j p \rfloor + 1 \geq t_{j,p}$, so $S_x \in H^0_0(\tilde{X}, \tilde{L}^p, \tilde{\Sigma}, \tau)$. Moreover, since

$$e^{2ap_0} \|S_x\|^2_{\tilde{h}^p,\pi^\star \omega^n} \leq C_2 |S_x(x)|^2_{\tilde{h}^p} e^{2F_p(x)},$$

we infer that

$$\tilde{P}_p(x) \geq C_2^{-1} e^{2ap_0 - 2F_p(x)}.$$

Note that on $\tilde{X}$ we have, for some constant $C_3 > 0$,

$$\begin{aligned} F_p = p\tilde{\varphi} - \psi_p &= p(\tilde{\varphi} - \tilde{\varphi}_{\rm eq}) + p_0 \tilde{\varphi}_{\rm req} - p_0 \eta - \sum_{j=1}^{\ell} \log \sigma_j \\ &\leq p(\tilde{\varphi} - \tilde{\varphi}_{\rm eq}) + C_3 - p_0 \eta - \sum_{j=1}^{\ell} \log \sigma_j. \end{aligned}$$

It follows there exists a constant $C_4 > 0$ such that for all $p \geq p_0$,

$$\frac{1}{2p} \log \tilde{P}_p \geq \tilde{\varphi}_{\rm eq} - \tilde{\varphi} - \frac{C_4}{p} + \frac{p_0}{p} \eta + \frac{1}{p} \sum_{j=1}^{\ell} \log \sigma_j \tag{5.9}$$

holds on $\tilde{X} \setminus Z$. By using (5.4) and since $\eta \le -1$, (5.9) implies that

$$\tilde{\varphi}_p \ge \tilde{\varphi}_{\mathrm{eq}} + \frac{C_4 + p_0}{p}\eta + \frac{1}{p}\sum_{j=1}^{\ell} \log \sigma_j$$

holds on $\tilde{X} \setminus Z$ for all $p \ge p_0$. ◻

Proof of Theorem 5.1. Since η, $\log \sigma_j$ are qpsh functions on $\tilde{X}$ with analytic singularities along $\mathrm{NAmp}(\{\tilde{\theta}\})$ (respectively, $\tilde{\Sigma}_j$), there exist constants $N_j, M_j > 0$, $0 \le j \le \ell$, such that

$$\begin{aligned} \eta(x) &\ge -N_0 |\log \mathrm{dist}(x, \mathrm{NAmp}(\{\tilde{\theta}\}))| - M_0, \\ \log \sigma_j(x) &\ge -N_j |\log \mathrm{dist}(x, \tilde{\Sigma}_j)| - M_j, \end{aligned}$$

for all $x \in \tilde{X}$. Together with Proposition 5.6, these imply that there exist a constant $C_1 > 0$ and $p_0 \in \mathbb{N}$ such that if $p \ge p_0$ then

$$\tilde{\varphi}_p \ge \tilde{\varphi}_{\mathrm{eq}} - \frac{C_1}{p}(|\log \mathrm{dist}(x, Z)| + 1) \tag{5.10}$$

holds on $\tilde{X}$.

Since φ is continuous, we have that $\tilde{\varphi}$ is continuous. Let $\varepsilon > 0$ and fix $\delta = \delta(\varepsilon)$ such that $C\delta + 2\Omega_{\tilde{\varphi}}(C\delta) < \varepsilon/2$, where C is the constant from Proposition 5.4. There exists p_ε such that $(C/p)(1 - \log \delta) < \varepsilon/2$ for $p \ge p_\varepsilon$. Hence, by Proposition 5.4,

$$\tilde{\varphi}_p \le \tilde{\varphi}_{\mathrm{eq}} + C\left(\delta + \frac{1}{p} - \frac{\log \delta}{p}\right) + 2\Omega_{\tilde{\varphi}}(C\delta) \le \tilde{\varphi}_{\mathrm{eq}} + \varepsilon \tag{5.11}$$

holds on $\tilde{X}$ for $p \ge p_\varepsilon$. Note that $\log \mathrm{dist}(\cdot, Z) \in L^1(\tilde{X}, \tilde{\omega}^n)$ (see, e.g., Lemma 5.2 in [CMN1] and its proof). Assertion (i) of Theorem 5.1 now follows from (5.10) and (5.11).

Assume next that the function φ is Hölder continuous on X. Then, $\tilde{\varphi}$ is Hölder continuous on $\tilde{X}$, so $\Omega_{\tilde{\varphi}}(\delta) \le C_2 \delta^\nu$ for some constant $C_2 > 0$, where ν is the Hölder exponent of $\tilde{\varphi}$. Taking $\delta = p^{-1/\nu}$ in Proposition 5.4, we see there exists a constant $C_3 > 0$ such that

$$\tilde{\varphi}_p \le \tilde{\varphi}_{\mathrm{eq}} + C\left(p^{-1/\nu} + p^{-1} + \frac{\log p}{\nu p}\right) + 2C_2 \frac{C^\nu}{p} \le \tilde{\varphi}_{\mathrm{eq}} + C_3 \frac{\log p}{p} \tag{5.12}$$

holds on $\tilde{X}$ for $p \ge 2$.

Assertion (ii) follows immediately from (5.10) and (5.12). ◻

Using Theorem 5.1 we can prove the convergence of the Fubini-Study currents and their global potentials on X.

Proof of Theorem 1.8. Let $(\tilde{X}, \pi, \tilde{\Sigma})$ be a divisorization of (X, Σ) as in Definition 1.5. Then, $\pi : \tilde{X} \setminus E \to X \setminus Y$ is a biholomorphism, where $Y \supset X_{\rm sing}$ is an analytic subset of X with $\dim Y \leq n-2$ and $E = \pi^{-1}(Y)$. By Theorem 5.1, $\tilde{\varphi}_p = \varphi_p \circ \pi \to \tilde{\varphi}_{\rm eq}$ in $L^1(\tilde{X}, \tilde{\omega}^n)$, where $\tilde{\omega}$ is a Kähler form on $\tilde{X}$ such that $\tilde{\omega} \geq \pi^\star \omega$. Recall that the functions $\tilde{\varphi}_p, \tilde{\varphi}_{\rm eq}$ are $\tilde{\alpha}$-psh on $\tilde{X}$, where $\tilde{\alpha} = \pi^\star \alpha$. We define $\varphi_{\rm eq} := \tilde{\varphi}_{\rm eq} \circ \pi^{-1}$ on $X \setminus Y \subset X_{\rm reg}$. Then,

$$
\begin{aligned}
(5.13) \qquad \int_{X\setminus Y} |\varphi_p - \varphi_{\rm eq}| \omega^n &= \int_{\tilde{X}\setminus E} |\tilde{\varphi}_p - \tilde{\varphi}_{\rm eq}| \pi^\star \omega^n \\
&\leq \int_{\tilde{X}} |\tilde{\varphi}_p - \tilde{\varphi}_{\rm eq}| \tilde{\omega}^n \to 0 \quad \text{as } p \to \infty.
\end{aligned}
$$

Since $\pi^\star(\alpha + dd^c \varphi_{\rm eq}) = \tilde{\alpha} + dd^c \tilde{\varphi}_{\rm eq} \geq 0$, it follows that $\varphi_{\rm eq}$ is α-psh on $X \setminus Y$.

It remains to show that $\varphi_{\rm eq}$ extends to an α-psh function on X. Let $x_0 \in Y$ and U_{x_0} be a neighborhood of x_0 in X on which L has a local holomorphic frame e_{x_0}, and let $|e_{x_0}|_{h_0} = e^{-\rho}$, where ρ is a smooth function on U_{x_0}. Then, $dd^c\rho = \alpha$. Since $\rho \circ \pi + \tilde{\varphi}_{\rm eq}$ is psh on $\pi^{-1}(U_{x_0}) \setminus E$, we infer that $v := \rho + \varphi_{\rm eq}$ is psh on $U_{x_0} \setminus Y$. Hence, v extends to a psh function on U_{x_0} since X is normal and $\dim Y \leq n-2$ [GR1, Satz 4]. Therefore, $\varphi_{\rm eq}$ extends to an α-psh function on X.

The second assertion of Theorem 1.8 follows at once from (5.13) and Theorem 5.1 (ii), since the function $\log \operatorname{dist}(\cdot, Z) \in L^1(\tilde{X}, \tilde{\omega}^n)$. ◻

We record here an immediate consequence of Theorems 1.8 and 1.10 for the case when $\Sigma = \varnothing$.

Corollary 5.7. *Let (X, ω) be an irreducible compact normal Kähler space of dimension n, and let L be a big line bundle on X endowed with a continuous Hermitian metric h. We denote by P_p the Bergman kernel function of $H^0(X, L^p)$ relative to h^p and $\omega^n/n!$, and by γ_p the corresponding Fubini-Study current* (1.4). *Let h_0 be a smooth metric on L, and denote by φ_p the global Fubini-Study potential relative to h_0* (1.4). *Then, the following assertions hold:*

(a) *There is an α-psh function $\varphi_{\rm eq}$ on X such that as $p \to \infty$ we have $\varphi_p \to \varphi_{\rm eq}$ in $L^1(X, \omega^n)$, $(1/p)\gamma_p \to T_{\rm eq} := \alpha + dd^c \varphi_{\rm eq}$, and $(1/p)[s_p = 0] \to T_{\rm eq}$, in the weak sense of currents on X, for σ_∞-almost every $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$.*

(b) *If, in addition, h is Hölder continuous, then there exist a constant $C > 0$ and $p_0 \in \mathbb{N}$ such that $\|\varphi_p - \varphi_{\rm eq}\|_{L^1(X,\omega^n)} \leq C\,(\log p)/p$ for all $p \geq p_0$, and the large deviation principle of Theorem 1.10* (ii) *holds.*

Note that if X is smooth, then $\varphi_{\rm eq} = \sup\{\psi \in \mathrm{PSH}(X, \alpha) : \psi \leq \varphi \text{ on } X\}$ is the usual upper envelope.

6. PROOF OF THE EQUIDISTRIBUTION THEOREM 1.10

Let h, h_0, φ be as in (1.3). Let P_p, γ_p be the Bergman kernel function and Fubini-Study current of the space $H^0_{0,(2)}(X, L^p)$, and let φ_p be the global Fubini-Study potential of γ_p (see (1.4)). We start by proving that zero divisors of random sections distribute like the Fubini-Study currents.

Theorem 6.1. *Let X, L, Σ, τ verify assumptions (A)–(D), let h be a bounded singular Hermitian metric on L, and assume (L, Σ, τ) is big and there exists a Kähler form ω on X. Then, there exists a constant $c > 0$ with the following property. For any sequence of positive numbers $\{\lambda_p\}_{p\geq 1}$ such that*

$$\liminf_{p\to\infty} \frac{\lambda_p}{\log p} > (1+n)c,$$

there exist subsets $E_p \subset \mathbb{X}_p$ such that the following hold:

(a) *$\sigma_p(E_p) \leq cp^n \exp(-\lambda_p/c)$ holds for all p sufficiently large.*

(b) *If $s_p \in \mathbb{X}_p \setminus E_p$ we have*

$$\left| \frac{1}{p}\langle [s_p = 0] - \gamma_p, \phi\rangle \right| \leq \frac{c\lambda_p}{p} \|\phi\|_{C^2},$$

for any $(n-1, n-1)$-form ϕ of class C^2 on X.

In particular, the last estimate holds for σ_∞-almost every $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$ provided that p is large enough.

Proof. We apply the Dinh-Sibony equidistribution theorem for meromorphic transforms [DS, Theorem 4.1], as in the proof of [CMN1, Theorem 4.2]. Our present situation is easier as we only deal with currents of bidegree $(1,1)$. We fix a divisorization $(\tilde{X}, \pi, \tilde{\Sigma})$ of (X, Σ) as in Definition 1.5, and let $\tilde{\omega}$ be a Kähler form on $\tilde{X}$. Let $\tilde{L}, \tilde{h}$ be as in (5.1), and $\tilde{P}_p, \tilde{\gamma}_p$ be the Bergman kernel function and Fubini-Study current of the space $H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$ (see (5.3)). By (5.2), $H^0_{0,(2)}(\tilde{X}, \tilde{L}^p), H^0_{0,(2)}(X, L^p)$ are isometric. We proceed in two steps.

Step 1. We prove here that Theorem 6.1 holds for the spaces $H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$. Set

$$\tilde{\mathbb{X}}_p := \mathbb{P}H^0_{0,(2)}(\tilde{X}, \tilde{L}^p),\ \sigma_p = \omega_{\mathrm{FS}}^{d_p},\ (\tilde{\mathbb{X}}_\infty, \sigma_\infty) := \prod_{p=1}^{\infty} (\tilde{\mathbb{X}}_p, \sigma_p),$$

$$\text{where } d_p = \dim \tilde{\mathbb{X}}_p = \dim \mathbb{X}_p.$$

We proceed as in [CMN1, Section 4] and consider the Kodaira maps as meromorphic transforms of codimension $n-1$, $\Phi_p : \tilde{X} \dashrightarrow \mathbb{P}H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$, with graph

$$\Gamma_p = \{(x, \tilde{s}) \in \tilde{X} \times \mathbb{P}H^0_{0,(2)}(\tilde{X}, \tilde{L}^p) : \tilde{s}(x) = 0\}.$$

If $\delta_{\tilde{s}_p}$ is the Dirac mass at $\tilde{s}_p \in \mathbb{P}H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$, then $\Phi_p^\star(\delta_{\tilde{s}_p})$ is well defined for generic $\tilde{s}_p$ and $\Phi_p^\star(\delta_{\tilde{s}_p}) = [\tilde{s}_p = 0]$. Moreover, by [CMN1, Lemma 4.5] (see also [SZ]) we have

$$\langle \Phi_p^\star(\sigma_p), \phi \rangle = \int_{\tilde{\mathbb{X}}_p} \langle [\tilde{s}_p = 0], \phi \rangle \, \mathrm{d}\sigma_p(\tilde{s}_p) = \langle \tilde{\gamma}_p, \phi \rangle,$$

where ϕ is a smooth $(n-1, n-1)$-form on $\tilde{X}$. The intermediate degrees of Φ_p are

$$d(\Phi_p) := \int_{\tilde{X}} \Phi_p^\star(\sigma_p) \wedge \tilde{\omega}^{n-1} = p \int_{\tilde{X}} c_1(\tilde{L}, \tilde{h}) \wedge \tilde{\omega}^{n-1},$$
$$\delta(\Phi_p) := \int_{\tilde{X}} \Phi_p^\star(\omega_{\mathrm{FS}}^{d_p - 1}) \wedge \tilde{\omega}^n = \int_{\tilde{X}} \tilde{\omega}^n.$$

For $\varepsilon > 0$ let

$$\tilde{E}_p(\varepsilon) := \bigcup_{\|\phi\|_{C^2} \le 1} \{ \tilde{s} \in \tilde{\mathbb{X}}_p : \big| \langle [\tilde{s} = 0] - \tilde{\gamma}_p, \phi \rangle \big| \ge d(\Phi_p)\varepsilon \}.$$

By [DS, Lemma 4.2 (d)], we infer, using the estimates of [CMN1, Lemma 4.6] for a projective space, that there exist constants $C_1, a_1, M_1 > 0$ such that

$$\sigma_p(\tilde{E}_p(\varepsilon)) \le C_1 d_p e^{-a_1 \varepsilon p + M_1 \log d_p}, \quad \forall\, \varepsilon > 0, \ p \ge 1.$$

By Siegel's lemma $d_p = O(p^n)$, so $\sigma_p(\tilde{E}_p(\varepsilon)) \le C_2 p^n e^{-a_1 \varepsilon p + C_2 \log p}$ for some constant $C_2 > 0$.

We let $\varepsilon_p := \lambda_p / p$ and $\tilde{E}_p := \tilde{E}_p(\varepsilon_p)$. If $\liminf_{p \to \infty}(\lambda_p / \log p) > 2C_2/a_1$, it follows that $\sigma_p(\tilde{E}_p) \le C_2 p^n e^{-a_1 \lambda_p / 2}$ for all p sufficiently large. Set

$$c = \max\Big(\frac{2C_2}{a_1(1+n)}, \frac{2}{a_1}, C_2, \int_{\tilde{X}} c_1(\tilde{L}, \tilde{h}) \wedge \tilde{\omega}^{n-1} \Big).$$

If $\liminf_{p \to \infty}(\lambda_p / \log p) > (1+n)c$ then $\sigma_p(\tilde{E}_p) \le c p^n e^{-\lambda_p / c}$ for all p large enough. Moreover,

$$\left| \frac{1}{p} \langle [\tilde{s}_p = 0] - \tilde{\gamma}_p, \phi \rangle \right| \le \frac{d(\Phi_p)}{p} \frac{\lambda_p}{p} \|\phi\|_{C^2} \le \frac{c \lambda_p}{p} \|\phi\|_{C^2}$$

holds for any $\tilde{s}_p \in \tilde{\mathbb{X}}_p \setminus \tilde{E}_p$ and any $(n-1, n-1)$-form ϕ of class C^2 on $\tilde{X}$.

Step 2. We complete now the proof of the theorem when X is singular. Let $c > 0$ be the constant constructed in Step 1, let $\{\lambda_p\}_{p\geq 1}$ be a sequence of positive numbers such that $\liminf_{p\to\infty}(\lambda_p/\log p) > (1+n)c$, and let $\tilde{E}_p \in \tilde{\mathbb{X}}_p$ be the corresponding sets constructed in Step 1. We now recall by (5.2) that the map $S \in H^0_{0,(2)}(X, L^p) \to \pi^\star S \in H^0_{0,(2)}(\tilde{X}, \tilde{L}^p)$ is an isometry, and let $E_p = F_p(\tilde{E}_p)$, where $F_p : \tilde{\mathbb{X}}_p \to \mathbb{X}_p$ is the isomorphism induced by this isometry. Then, we have $\sigma_p(E_p) \leq cp^n e^{-\lambda_p/c}$ for all p sufficiently large. Note that $\pi_\star \tilde{\gamma}_p = \gamma_p$ and, if $s \in \mathbb{X}_p$, then $\pi_\star[\tilde{s} = 0] = [s = 0]$, where $s = F_p(\tilde{s})$. Hence, if $s_p \in \mathbb{X}_p \setminus E_p$ and ϕ is any $(n-1, n-1)$-form of class C^2 on X, we have

$$\begin{aligned}\left|\frac{1}{p}\langle [s_p = 0] - \gamma_p, \phi\rangle\right| &= \left|\frac{1}{p}\langle [\tilde{s}_p = 0] - \tilde{\gamma}_p, \pi^\star\phi\rangle\right| \\ &\leq \frac{c\lambda_p}{p}\|\pi^\star\phi\|_{C^2} \leq \frac{cc_1\lambda_p}{p}\|\phi\|_{C^2},\end{aligned}$$

for some constant $c_1 > 0$. The last assertion of Theorem 6.1 follows as in [CMN1, Theorem 4.2]. ◻

Proof of Theorem 1.10. If h is continuous, then by Theorem 1.8,

$$\frac{1}{p}\gamma_p \to T_{\mathrm{eq}} \quad \text{weakly on } X, \text{ as } p \to \infty.$$

Moreover, by Theorem 6.1, $(1/p)([s_p = 0] - \gamma_p) \to 0$ weakly on X, for σ_∞-almost every $\{s_p\}_{p\geq 1} \in \mathbb{X}_\infty$. This proves assertion (i).

Assume now h is Hölder continuous. There exists a constant $C' > 0$ such that

$$-C'\|\phi\|_{C^2}\omega^n \leq dd^c\phi \leq C'\|\phi\|_{C^2}\omega^n,$$

for every real valued $(n-1, n-1)$-form ϕ of class C^2 on X. Hence, the total variation of $dd^c\phi$ satisfies $|dd^c\phi| \leq C'\|\phi\|_{C^2}\omega^n$ (see, e.g., [BCM]). Using Theorem 1.8, we infer that

$$\begin{aligned}\left|\left\langle \frac{1}{p}\gamma_p - T_{\mathrm{eq}}, \phi\right\rangle\right| &= \left|\int_X (\varphi_p - \varphi_{\mathrm{eq}})dd^c\phi\right| \leq C'\|\phi\|_{C^2}\int_X |\varphi_p - \varphi_{\mathrm{eq}}|\omega^n \\ &\leq CC'\|\phi\|_{C^2}\frac{\log p}{p},\end{aligned}$$

for all $p \geq p_0$ and ϕ as above. Assertion (ii) follows by combining this and Theorem 6.1. ◻

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References

[BCM] T. Bayraktar, D. Coman, and G. Marinescu, *Universality results for zeros of random holomorphic sections*, Trans. Amer. Math. Soc. **373** (2020), no. 6, 3765–3791. https://dx.doi.org/10.1090/tran/7807. MR4105509.

[BCHM] T. Bayraktar, D. Coman, H. Herrmann, and G. Marinescu, *A survey on zeros of random holomorphic sections*, Dolomites Res. Notes Approx. **11** (2018), no. Special Issue Norm Levenberg, 1–19. MR3895931.

[Be1] R. J. Berman, *Bergman kernels and equilibrium measures for ample line bundles* (2007), preprint. https://dx.doi.org/10.48550/arXiv.0704.1640.

[Be2] ———, *Bergman kernels and equilibrium measures for line bundles over projective manifolds*, Amer. J. Math. **131** (2009), no. 5, 1485–1524. https://dx.doi.org/10.1353/ajm.0.0077. MR2559862.

[BD] R. J. Berman and J.-P. Demailly, *Regularity of plurisubharmonic upper envelopes in big cohomology classes*, Perspectives in Analysis, Geometry, and Topology, Progr. Math., vol. 296, Birkhäuser/Springer, New York, 2012, pp. 39–66. https://dx.doi.org/10.1007/978-0-8176-8277-4_3. MR2884031.

[BM] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. https://dx.doi.org/10.1007/s002220050141. MR1440306.

[BP] A. Bloch and G. Pólya, *On the roots of certain algebraic equations*, Proc. London Math. Soc. (2) **33** (1931), no. 2, 102–114. https://dx.doi.org/10.1112/plms/s2-33.1.102. MR1576817.

[B1] T. Bloom, *Random polynomials and Green functions*, Int. Math. Res. Not. **28** (2005), 1689–1708. https://dx.doi.org/10.1155/IMRN.2005.1689. MR2172337.

[B2] ———, *Weighted polynomials and weighted pluripotential theory*, Trans. Amer. Math. Soc. **361** (2009), no. 4, 2163–2179. https://dx.doi.org/10.1090/S0002-9947-08-04607-2. MR2465832.

[BL] T. Bloom and N. Levenberg, *Random polynomials and pluripotential-theoretic extremal functions*, Potential Anal. **42** (2015), no. 2, 311–334. https://dx.doi.org/10.1007/s11118-014-9435-4. MR3306686.

[Bon] L. Bonavero, *Inégalités de morse holomorphes singulières*, J. Geom. Anal. **8** (1998), no. 3, 409–425, Announced in C. R. Acad. Sci. Paris Sér. I Math. **317** (1993), no. 12, 1163–1166) (French, with English summary). https://dx.doi.org/10.1007/BF02921793. MR1707735.

[Bou] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76 (English, with English and French summaries). https://dx.doi.org/10.1016/j.ansens.2003.04.002. MR2050205.

[Ca] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and Geometry in Several Complex Variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1–23. http://dx.doi.org/10.1007/978-1-4612-2166-1_1. MR1699887.

[CM1] D. Coman and G. Marinescu, *Equidistribution results for singular metrics on line bundles*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 3, 497–536 (English, with English and French summaries). https://dx.doi.org/10.24033/asens.2250. MR3377051.

[CM2] ______, *Convergence of Fubini-study currents for orbifold line bundles*, Internat. J. Math. **24** (2013), no. 7, 1350051, 27 pp. https://dx.doi.org/10.1142/S0129167X13500511. MR3084732.

[CM3] ______, *On the first order asymptotics of partial Bergman kernels*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 5, 1193–1210 (English, with English and French summaries). https://dx.doi.org/10.5802/afst.1564. MR3746626.

[CMM] D. COMAN, X. MA, AND G. MARINESCU, *Equidistribution for sequences of line bundles on normal Kähler spaces*, Geom. Topol. **21** (2017), no. 2, 923–962. https://dx.doi.org/10.2140/gt.2017.21.923. MR3626594.

[CMN1] D. COMAN, G. MARINESCU, AND V.-A. NGUYÊN, *Hölder singular metrics on big line bundles and equidistribution*, Int. Math. Res. Not. IMRN **16** (2016), 5048–5075. https://dx.doi.org/10.1093/imrn/rnv303. MR3556432.

[CMN2] ______, *Approximation and equidistribution results for pseudo-effective line bundles*, J. Math. Pures Appl. (9) **115** (2018), 218–236 (English, with English and French summaries). https://dx.doi.org/10.1016/j.matpur.2017.09.010. MR3808345.

[Da] T. DARVAS, *Weak geodesic rays in the space of Kähler potentials and the class $\mathcal{E}(X,\omega)$*, J. Inst. Math. Jussieu **16** (2017), no. 4, 837–858. https://dx.doi.org/10.1017/S1474748015000316. MR3680345.

[D1] J.-P. DEMAILLY, *Estimations L^2 pour l'opérateur $\bar\partial$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 457–511 (French). http://dx.doi.org/10.24033/asens.1434. MR690650.

[D2] ______, *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.) **19** (1985), 128 pp. (French, with English summary). http://dx.doi.org/10.24033/msmf.320. MR813252.

[D3] ______, *Singular Hermitian metrics on positive line bundles*, Complex Algebraic Varieties (Bayreuth, 1990), Lecture Notes in Math., vol. 1507, Springer, Berlin, 1992, pp. 87–104. https://dx.doi.org/10.1007/BFb0094512. MR1178721.

[D4] ______, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), no. 3, 361–409. MR1158622.

[D5] ______, *Monge-Ampère operators, Lelong numbers and intersection theory*, Complex Analysis and Geometry, Univ. Ser. Math., Plenum, New York, 1993, pp. 115–193. http://dx.doi.org/10.1007/978-1-4757-9771-8_4. MR1211880.

[D6] ______, *Regularization of closed positive currents of type $(1,1)$ by the flow of a Chern connection*, Contributions to Complex Analysis and Analytic Geometry, Aspects Math., E26, Friedr. Vieweg, Braunschweig, 1994, pp. 105–126. http://dx.doi.org/10.1007/978-3-663-14196-9_4. MR1319346.

[DP] J. P. DEMAILLY AND M. PAUN, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), no. 3, 1247–1274. https://dx.doi.org/10.4007/annals.2004.159.1247. MR2113021.

[DiNT] E. DI NEZZA AND S. TRAPANI, *The regularity of envelopes*, Ann. Sci. Éc. Norm. Supér (4), to appear.

[DMM] T.-C. DINH, X. MA, AND G. MARINESCU, *Equidistribution and convergence speed for zeros of holomorphic sections of singular Hermitian line bundles*, J. Funct. Anal. **271** (2016), no. 11, 3082–3110. https://dx.doi.org/10.1016/j.jfa.2016.09.007. MR3554701.

[DMN] T.-C. DINH, X. MA, AND V.-A. NGUYÊN, *Equidistribution speed for Fekete points associated with an ample line bundle*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 3, 545–578 (English, with English and French summaries). https://dx.doi.org/10.24033/asens.2327. MR3665550.

[DMS] T.-C. DINH, G. MARINESCU, AND V. SCHMIDT, *Asymptotic distribution of zeros of holomorphic sections in the non-compact setting*, J. Stat. Phys. **148** (2012), no. 1, 113–136. https://dx.doi.org/10.1007/s10955-012-0526-6.

[DS] T.-C. DINH AND N. SIBONY, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv. **81** (2006), no. 1, 221–258 (French, with English summary). https://dx.doi.org/10.4171/CMH/50. MR2208805.

[ET] P. ERDŐS AND P. TURÁN, *On the distribution of roots of polynomials*, Ann. of Math. (2) **51** (1950), 105–119. https://dx.doi.org/10.2307/1969500. MR33372.

[G] H. GRAUERT, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368 (German). https://dx.doi.org/10.1007/BF01441136. MR137127.

[GR1] H. GRAUERT AND R. REMMERT, *Plurisubharmonische Funktionen in komplexen Räumen*, Math. Z. **65** (1956), 175–194 (German). https://dx.doi.org/10.1007/BF01473877. MR81960.

[GR2] ———, *Coherent Analytic Sheaves*, Die Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 265, Springer-Verlag, Berlin, 1984. https://dx.doi.org/10.1007/978-3-642-69582-7. MR755331.

[H] J. M. HAMMERSLEY, *The zeros of a random polynomial*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, University of California Press, Berkeley-Los Angeles, Calif., 1956, pp. 89–111. http://dx.doi.org/10.1525/9780520350670-010. MR0084888.

[JS] S. JI AND B. SHIFFMAN, *Properties of compact complex manifolds carrying closed positive currents*, J. Geom. Anal. **3** (1993), no. 1, 37–61. http://dx.doi.org/10.1007/BF02921329. MR1197016.

[Ka] M. KAC, *On the average number of real roots of a random algebraic equation, II*, Proc. London Math. Soc. (2) **50** (1949), no. 6, 390–408. https://dx.doi.org/10.1112/plms/s2-50.6.401. MR1575245.

[Ki1] C. O. KISELMAN, *The partial Legendre transformation for plurisubharmonic functions*, Invent. Math. **49** (1978), no. 2, 137–148. https://dx.doi.org/10.1007/BF01403083. MR511187.

[Ki2] ———, *Attenuating the singularities of plurisubharmonic functions*, Ann. Polon. Math. **60** (1994), no. 2, 173–197 (English, with English and Esperanto summaries). https://dx.doi.org/10.4064/ap-60-2-173-197. MR1301603.

[KoN] S. KOŁODZIEJ AND N. C. NGUYEN, *Stability and regularity of solutions of the Monge-Ampère equation on Hermitian manifolds*, Adv. Math. **346** (2019), 264–304. https://dx.doi.org/10.1016/j.aim.2019.02.004. MR3910489.

[KP] S. G. KRANTZ AND H. R. PARKS, *A Primer of Real Analytic Functions*, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2002. https://dx.doi.org/10.1007/978-0-8176-8134-0. MR1916029.

[LS] F. LÁRUSSON AND R. SIGURDSSON, *Plurisubharmonic extremal functions, Lelong numbers and coherent ideal sheaves*, Indiana Univ. Math. J. **48** (1999), no. 4, 1513–1534. https://dx.doi.org/10.1512/iumj.1999.48.1767. MR1757081.

[MM] X. MA AND G. MARINESCU, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007. https://dx.doi.org/10.1007/978-3-7643-8115-8. MR2339952.

[O] T. OHSAWA, *Hodge spectral sequence and symmetry on compact Kähler spaces*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 4, 613–625. https://dx.doi.org/10.2977/prims/1195176250. MR918517.

[PS] F. T. POKORNY AND M. SINGER, *Toric partial density functions and stability of toric varieties*, Math. Ann. **358** (2014), no. 3–4, 879–923. https://dx.doi.org/10.1007/s00208-013-0978-2. MR3175144.

[RaS] A. RASHKOVSKII AND R. SIGURDSSON, *Green functions with singularities along complex spaces*, Internat. J. Math. **16** (2005), no. 4, 333–355. https://dx.doi.org/10.1142/S0129167X05002904. MR2133260.

[RoS] J. ROSS AND M. SINGER, *Asymptotics of partial density functions for divisors*, J. Geom. Anal. **27** (2017), no. 3, 1803–1854. https://dx.doi.org/10.1007/s12220-016-9741-8. MR3667411.

[RT06] J. ROSS AND R. THOMAS, *An obstruction to the existence of constant scalar curvature Kähler metrics*, J. Differential Geom. **72** (2006), no. 3, 429–466. http://dx.doi.org/10.4310/jdg/1143593746. MR2219940.

[RWN1] J. ROSS AND D. W. NYSTRÖM, *Analytic test configurations and geodesic rays*, J. Symplectic Geom. **12** (2014), no. 1, 125–169. https://dx.doi.org/10.4310/JSG.2014.v12.n1.a5. MR3194078.

[RWN2] ———, *Envelopes of positive metrics with prescribed singularities*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 3, 687–728 (English, with English and French summaries). https://dx.doi.org/10.5802/afst.1549. MR3669969.

[R] W.-D. RUAN, *Canonical coordinates and Bergmann [Bergman] metrics*, Comm. Anal. Geom. **6** (1998), no. 3, 589–631. https://dx.doi.org/10.4310/CAG.1998.v6.n3.a5. MR1638878.

[SZ] B. SHIFFMAN AND S. ZELDITCH, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Comm. Math. Phys. **200** (1999), no. 3, 661–683. https://dx.doi.org/10.1007/s002200050544. MR1675133.

[T] G. TIAN, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130. http://dx.doi.org/10.4310/jdg/1214445039. MR1064867.

[Z1] S. ZELDITCH, *Szegő kernels and a theorem of Tian*, Internat. Math. Res. Notices **6** (1998), 317–331. https://dx.doi.org/10.1155/S107379289800021X. MR1616718.

[Z2] ———, *Quantum ergodic sequences and equilibrium measures*, Constr. Approx. **47** (2018), no. 1, 89–118. https://dx.doi.org/10.1007/s00365-017-9397-z. MR3742811.

[ZZ] S. ZELDITCH AND P. ZHOU, *Interface asymptotics of partial Bergman kernels on S^1-symmetric Kähler manifolds*, J. Symplectic Geom. **17** (2019), no. 3, 793–856. https://dx.doi.org/10.4310/JSG.2019.v17.n3.a6. MR4022214.

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