D. Coman et al. (2016) "Hölder Singular Metrics on Big Line Bundles and Equidistribution,"
International Mathematics Research Notices, Vol. 2016, No. 16, pp. 5048–5075
Advance Access Publication October 15, 2015
doi:10.1093/imrn/rnv303

Hölder Singular Metrics on Big Line Bundles and Equidistribution

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We show that normalized currents of integration along the common zeros of random m-tuples of sections of powers of m singular Hermitian big line bundles on a compact Kähler manifold distribute asymptotically to the wedge product of the curvature currents of the metrics. If the Hermitian metrics are Hölder with singularities we also estimate the speed of convergence.

1 Introduction

Random polynomials or more generally holomorphic sections and the distribution of their zeros represent a classical subject in analysis [5, 25, 28, 31], and they have been more recently used to model quantum chaotic eigenfunctions [8, 37].

This area witnessed intense activity recently [6, 7, 9, 22, 23, 39–41], and especially results about equidistribution of holomorphic sections in singular Hermitian holomorphic bundles were obtained [11–14, 20] with emphasis on the speed

Received June 4, 2015; Revised September 9, 2015; Accepted September 18, 2015
Communicated by Prof. Thomas Bloom

of convergence. The equidistribution is linked to the Quantum Unique Ergodicity conjecture of Rudnick-Sarnak [38], cf. [29, 35].

The equidistribution of common zeros of several sections is particularly interesting. Their study is difficult in the singular context and equidistribution with the estimate of convergence speed was established in [20] for Hölder continuous metrics.

In this paper, we obtain the equidistribution of common zeros of sections of m singular Hermitian line bundles under the hypothesis that the metrics are continuous outside analytic sets intersecting generically. We will moreover introduce the notion of Hölder metric with singularities and establish the equidistribution with convergence speed of common zeros.

Let (X, ω) be a compact Kähler manifold of dimension n and dist be the distance on X induced by ω. If (L, h) is a singular Hermitian holomorphic line bundle on X we denote by $c_1(L, h)$ its curvature current. Recall that if e_L is a holomorphic frame of L on some open set $U \subset X$, then $|e_L|^2_h = e^{-2\phi}$, where $\phi \in L^1_{loc}(U)$ is called the *local weight* of the metric h with respect to e_L, and $c_1(L, h)|_U = dd^c\phi$. Here $d = \partial + \bar{\partial}$, $d^c = \frac{1}{2\pi \mathrm{i}}(\partial - \bar{\partial})$. We say that h is *positively curved* if $c_1(L, h) \geq 0$ in the sense of currents. This is equivalent to saying that the local weights ϕ are plurisubharmonic (psh).

Recall that a holomorphic line bundle L is called *big* if its Kodaira–Iitaka dimension equals the dimension of X (see [33, Definition 2.2.5]). By the Shiffman–Ji–Bonavero–Takayama criterion [33, Lemma 2.3.6], L is big if and only if it admits a singular metric h with $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$.

Let (L_k, h_k), $1 \leq k \leq m \leq n$, be m singular Hermitian holomorphic line bundles on (X, ω). Let $H^0_{(2)}(X, L^p_k)$ be the Bergman space of L^2-holomorphic sections of $L^p_k := L^{\otimes p}_k$ relative to the metric $h_{k,p} := h^{\otimes p}_k$ induced by h_k and the volume form ω^n on X, endowed with the inner product

$$\left(S, S'\right)_{k,p} := \int_X \langle S, S' \rangle_{h_{k,p}} \omega^n, \quad S, S' \in H^0_{(2)}\left(X, L^p_k\right). \tag{1}$$

Set $\|S\|^2_{k,p} = (S, S)_{k,p}$, $d_{k,p} = \dim H^0_{(2)}(X, L^p_k) - 1$. For every $p \geq 1$ we consider the multi-projective space

$$\mathbb{X}_p := \mathbb{P}H^0_{(2)}\left(X, L^p_1\right) \times \cdots \times \mathbb{P}H^0_{(2)}\left(X, L^p_m\right) \tag{2}$$

equipped with the probability measure σ_p which is the product of the Fubini-Study volumes on the components. If $S \in H^0(X, L^p_k)$ we denote by $[S = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{S = 0\}$ of X. Set

$$[\mathbf{s}_p = 0] := [s_{p1} = 0] \wedge \cdots \wedge [s_{pm} = 0], \quad \text{for } \mathbf{s}_p = \left(s_{p1}, \ldots, s_{pm}\right) \in \mathbb{X}_p,$$

whenever this is well defined (cf. Section 3). We also consider the probability space

$$(\Omega, \sigma_\infty) := \prod_{p=1}^{\infty} \left(\mathbb{X}_p, \sigma_p\right).$$

Let us recall the following definition.

Definition 1.1. We say that the analytic subsets $A_1, \ldots, A_m$, $m \le n$, of a compact complex manifold X of dimension n are in general position if $\operatorname{codim} A_{i_1} \cap \cdots \cap A_{i_k} \ge k$ for every $1 \le k \le m$ and $1 \le i_1 < \cdots < i_k \le m$. □

Here is our first main result.

Theorem 1.2. Let (X, ω) be a compact Kähler manifold of dimension n and (L_k, h_k), $1 \le k \le m \le n$, be m singular Hermitian holomorphic line bundles on X such that h_k is continuous outside a proper analytic subset $\Sigma(h_k) \subset X$, $c_1(L_k, h_k) \ge \varepsilon\omega$ on X for some $\varepsilon > 0$, and $\Sigma(h_1), \ldots, \Sigma(h_m)$ are in general position. Then for σ_∞-a.e. $\{\mathbf{s}_p\}_{p\ge1} \in \Omega$, we have in the weak sense of currents on X,

$$\frac{1}{p^m}[\mathbf{s}_p = 0] \to c_1\left(L_1, h_1\right) \wedge \cdots \wedge c_1\left(L_m, h_m\right) \quad \text{as } p \to \infty.$$ □

In order to prove this theorem we show in Theorem 4.2 that the currents $\frac{1}{p^m}[\mathbf{s}_p = 0]$ distribute as $p \to \infty$ like the wedge product of the normalized Fubini-Study currents of the spaces $H^0_{(2)}(X, L_k^p)$ defined in (10). Then in Proposition 3.1 we prove that the latter sequence of currents converges to $c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_m, h_m)$.

Our second main result gives an estimate of the speed of convergence in Theorem 1.2 in the case when the metrics are Hölder with singularities.

Definition 1.3. We say that a function $\phi : U \to [-\infty, \infty)$ defined on an open subset $U \subset X$ is Hölder with singularities along a proper analytic subset $\Sigma \subset X$ if there exist constants $c, \varrho > 0$ and $0 < \nu \le 1$ such that

$$|\phi(z) - \phi(w)| \le \frac{c \operatorname{dist}(z, w)^\nu}{\min\{\operatorname{dist}(z, \Sigma), \operatorname{dist}(w, \Sigma)\}^\varrho} \tag{3}$$

holds for all $z, w \in U \setminus \Sigma$. A singular metric h on L is called Hölder with singularities along a proper analytic subset $\Sigma \subset X$ if all its local weights are Hölder functions with singularities along Σ. □

Hölder singular Hermitian metrics appear frequently in complex geometry and pluri-potential theory. Let us first observe that metrics with analytic singularities

[33, Definition 2.3.9], which are very important for the regularization of currents and for transcendental methods in algebraic geometry [3, 13, 16, 17, 19], are Hölder metrics with singularities. The class of Hölder metrics with singularities is invariant under pull-back and push-forward by meromorphic maps. In particular, this class is invariant under birational maps, for example, blow-up and blow-down. They occur also as certain quasiplurisubharmonic upper envelopes (e.g., Hermitian metrics with minimal singularities on a big line bundle, equilibrium metrics, see [2, 20, 21], especially [3, Theorem 1.4]).

Theorem 1.4. In the setting of Theorem 1.2 assume in addition that h_k is Hölder with singularities along $\Sigma(h_k)$. Then there exist a constant $\xi > 0$ depending only on m and a constant $c = c(X, L_1, h_1, \dots, L_m, h_m) > 0$ with the following property: for any sequence of positive numbers $\{\lambda_p\}_{p\geq 1}$ such that

$$\liminf_{p\to\infty} \frac{\lambda_p}{\log p} > (1+\xi n)\, c,$$

there are subsets $E_p \subset \mathbb{X}_p$ such that for p large enough,

(a) $\sigma_p(E_p) \leq c p^{\xi n} \exp(-\lambda_p / c)$,
(b) if $\mathbf{s}_p \in \mathbb{X}_p \setminus E_p$ we have

$$\left| \left\langle \frac{1}{p^m}[\mathbf{s}_p = 0] - \bigwedge_{k=1}^{m} c_1\left(L_k, h_k\right), \phi \right\rangle \right| \leq \frac{c\lambda_p}{p} \|\phi\|_{\mathscr{C}^2},$$

for any form ϕ of class $\mathscr{C}^2$.

In particular, the last estimate holds for σ_∞-a.e. sequence $\{\mathbf{s}_p\}_{p\geq 1} \in \Omega$ provided that p is large enough. □

Let P_p be the Bergman kernel function of the space $H^0_{(2)}(X, L^p)$ defined in (4) below. The proof of Theorem 1.4 uses the estimate for P_p obtained in Theorem 2.1 in the case when the metric h on L is Hölder with singularities.

One can readily specialize Theorem 1.4 to study the asymptotics with speed of common zeros of random m-tuples of sections of a (single) big line bundle endowed with a Hölder Hermitian metric with singularities along a proper analytic subset $\Sigma \subset X$ of codimension $\geq m$. Let (L, h) be a singular Hermitian holomorphic line bundle on (X, ω) and $H^0_{(2)}(X, L^p)$ be the corresponding spaces of L^2-holomorphic sections. Consider the multi-projective space

$$\mathbb{X}'_p := \left(\mathbb{P}H^0_{(2)}\left(X, L^p\right)\right)^m$$

endowed with the product probability measure σ'_p induced by the Fubini-Study volume on $\mathbb{P}H^0_{(2)}(X, L^p)$, and let

$$(\Omega', \sigma'_\infty) := \prod_{p=1}^{\infty} (\mathbb{X}'_p, \sigma'_p) .$$

If $\mathbf{s}_p = (s_{p1}, \ldots, s_{pm}) \in \mathbb{X}'_p$ we set $[\mathbf{s}_p = 0] := [s_{p1} = 0] \wedge \cdots \wedge [s_{pm} = 0]$, provided this current is well-defined. Applying Theorem 1.4 with $(L_k, h_k) = (L, h)$, $1 \le k \le m$, and for the sequence $\lambda_p = (2 + \xi n) c \log p$, we obtain:

Theorem 1.5. Let (X, ω) be a compact Kähler manifold of dimension n and (L, h) be a singular Hermitian holomorphic line bundle on X such that h is Hölder with singularities along a proper analytic subset $\Sigma \subset X$ of codimension $\ge m$, and $c_1(L, h) \ge \varepsilon\omega$ for some $\varepsilon > 0$. Then there exist a constant $C > 0$ depending only on (X, ω, L, h), and subsets $E_p \subset \mathbb{X}'_p$, such that for p large enough,

(a) $\sigma'_p(E_p) \le C p^{-2}$,

(b) if $\mathbf{s}_p \in \mathbb{X}'_p \setminus E_p$ we have

$$\left| \left\langle \frac{1}{p^m} [\mathbf{s}_p = 0] - c_1 (L, h)^m , \phi \right\rangle \right| \le C \frac{\log p}{p} \|\phi\|_{\mathscr{C}^2},$$

for any form ϕ of class $\mathscr{C}^2$.

In particular, the last estimate holds for σ'_∞-a.e. sequence $\{\mathbf{s}_p\}_{p \ge 1} \in \Omega'$ provided that p is large enough. □

This paper is organized as follows. In Section 2, we prove a pointwise estimate for the Bergman kernel function in the case of Hölder metrics with singularities. Section 3 is devoted to the study of the intersection of Fubini-Study currents and to a version of the Bertini theorem. In Section 4, we consider the Kodaira map as a meromorphic transform and estimate the speed of convergence of the intersection of zero-divisors of m bundles. We use this to prove Theorem 1.2. Finally, in Section 5 we prove Theorem 1.4.

2 Asymptotic Behavior of Bergman Kernel Functions

In this section, we prove a theorem about the asymptotic behavior of the Bergman kernel function in the case when the metric is Hölder with singularities.

Let (L, h) be a holomorphic line bundle over a compact Kähler manifold (X, ω) of dimension n, where h is a singular Hermitian metric on L. Consider the space $H^0_{(2)}(X, L^p)$ of L^2-holomorphic sections of L^p relative to the metric $h^p := h^{\otimes p}$ induced by h and the volume form ω^n on X, endowed with the natural inner product (see (1)). Since $H^0_{(2)}(X, L^p)$ is finite dimensional, let $\{S^p_j\}_{j=0}^{d_p}$ be an orthonormal basis and denote by P_p the Bergman kernel function defined by

$$P_p(x) = \sum_{j=0}^{d_p} \left|S^p_j(x)\right|^2_{h^p}, \quad \left|S^p_j(x)\right|^2_{h^p} := \left\langle S^p_j(x), S^p_j(x)\right\rangle_{h^p}, \; x \in X. \tag{4}$$

Note that this definition is independent of the choice of basis.

Theorem 2.1. Let (X, ω) be a compact Kähler manifold of dimension n and (L, h) be a singular Hermitian holomorphic line bundle on X such that $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Assume that h is Hölder with singularities along a proper analytic subset Σ of X and with parameters ν, ϱ as in (3). If P_p is the Bergman kernel function defined by (4) for the space $H^0_{(2)}(X, L^p)$, then there exist a constant $c > 1$ and $p_0 \in \mathbb{N}$ which depend only on (X, ω, L, h) such that for all $z \in X \setminus \Sigma$ and all $p \geq p_0$

$$\frac{1}{c} \leq P_p(z) \leq \frac{cp^{2n/\nu}}{\operatorname{dist}(z, \Sigma)^{2n\varrho/\nu}}. \tag{5}$$

□

Recall that by Theorem 5.3 in [12] we have $\lim_{p\to\infty} \frac{1}{p}\log P_p(z) = 0$ locally uniformly on $X \setminus \Sigma$ for any metric h which is only continuous outside of Σ. Theorem 2.1 refines [12, Theorem 5.3] in this context, and it is interesting to compare it with the asymptotic expansion of the Bergman kernel function in the case of smooth metrics [4, 10, 30, 33, 34, 42, 45].

Proof. The proof follows from [12, Section 5], which is based on techniques of Demailly [15, Proposition 3.1; 20, Section 9]. Let $x \in X$ and $U_\alpha \subset X$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_α of L. Let ψ_α be a psh weight of h on U_α. Fix $r_0 > 0$ so that the (closed) ball $V := B(x, 2r_0) \Subset U_\alpha$ and let $U := B(x, r_0)$. By [12, (7)] there exist constants $c_1 > 0$, $p_0 \in \mathbb{N}$ so that

$$-\frac{\log c_1}{p} \leq \frac{1}{p}\log P_p(z) \leq \frac{\log\left(c_1 r^{-2n}\right)}{p} + 2\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right)$$

holds for all $p > p_0$, $0 < r < r_0$ and $z \in U$ with $\psi_\alpha(z) > -\infty$.

For $z \in U \setminus \Sigma$ and $r < \min\{\operatorname{dist}(z, \Sigma), r_0\}$ we have since ψ_α is Hölder that

$$\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z) \leq \frac{cr^\nu}{(\operatorname{dist}(z, \Sigma) - r)^\varrho},$$

where $c>0$ depends only on x. Taking $r=\operatorname{dist}(z,\Sigma)^{\varrho/\nu}p^{-1/\nu}<\operatorname{dist}(z,\Sigma)/2$ (for p_0 large enough), we obtain

$$-\log c_1 \le \log P_p(z) \le \log c_1 - 2n\log r + 2^{\varrho+1}cpr^{\nu}\operatorname{dist}(z,\Sigma)^{-\varrho}$$
$$= c_0 + 2n\log\left(\operatorname{dist}(z,\Sigma)^{-\varrho/\nu}\,p^{1/\nu}\right).$$

This holds for all $z\in U\setminus\Sigma$ and $p>p_0$, with constants r_0, p_0, c_0, c_1 depending only on x. A standard compactness argument now finishes the proof. ∎

3 Intersection of Fubini-Study Currents and Bertini Type Theorem

In this section, we show that the intersection of the Fubini-Study currents associated with line bundles as in Theorem 1.2 is well defined. Moreover, we show that the sequence of wedge products of normalized Fubini-Study currents converges weakly to the wedge product of the curvature currents of (L_k, h_k). We then prove that almost all zero-divisors of sections of large powers of these bundles are in general position in the sense of Definition 1.1.

Let V be a vector space of complex dimension $d+1$. If V is endowed with a Hermitian metric, then we denote by $\omega_{\rm FS}$ the induced Fubini-Study form on the projective space $\mathbb{P}(V)$ (see [33, p. 65, 212]) normalized so that $\omega_{\rm FS}^d$ is a probability measure. We also use the same notations for $\mathbb{P}(V^*)$.

We keep the hypotheses and notation of Theorem 1.2. Namely, (L_k, h_k), $1\le k\le m\le n$, are singular Hermitian holomorphic line bundles on the compact Kähler manifold (X,ω) of dimension n, such that h_k is continuous outside a proper analytic subset $\Sigma(h_k)\subset X$, $c_1(L_k,h_k)\ge\varepsilon\omega$ for some $\varepsilon>0$, and $\Sigma(h_1),\dots,\Sigma(h_m)$ are in general position in the sense of Definition 1.1.

Consider the space $H^0_{(2)}(X,L_k^p)$ of L^2-holomorphic sections of L_k^p endowed with the inner product (1). Since $c_1(L_k,h_k)\ge\varepsilon\omega$, it is well known that $H^0_{(2)}(X,L_k^p)$ is nontrivial for p sufficiently large, see for example, Proposition 4.7. Let

$$d_{k,p} := \dim H^0_{(2)}\left(X, L_k^p\right) - 1.$$

The *Kodaira map* associated with $(L_k^p, h_{k,p})$ is defined by

$$\Phi_{k,p}: X \dashrightarrow \mathbb{G}\left(d_{k,p}, H^0_{(2)}\left(X,L_k^p\right)\right),\quad \Phi_{k,p}(x) := \left\{s\in H^0_{(2)}\left(X,L_k^p\right): s(x)=0\right\}, \tag{6}$$

where $\mathbb{G}\left(d_{k,p}, H^0_{(2)}\left(X,L_k^p\right)\right)$ denotes the Grassmannian of hyperplanes in $H^0_{(2)}(X,L_k^p)$ (see [33, p. 82]). Let us identify $\mathbb{G}\left(d_{k,p}, H^0_{(2)}\left(X,L_k^p\right)\right)$ with $\mathbb{P}\left(H^0_{(2)}\left(X,L_k^p\right)^*\right)$ by sending a hyperplane to an equivalence class of non-zero complex linear functionals on $H^0_{(2)}\left(X, L_k^p\right)$ having the hyperplane as their common kernel. By composing $\Phi_{k,p}$ with this identification, we obtain a meromorphic map

$$\Phi_{k,p}\colon X \dashrightarrow \mathbb{P}\left(H^0_{(2)}\left(X, L_k^p\right)^*\right). \tag{7}$$

To get an analytic description of $\Phi_{k,p}$, let

$$S_j^{k,p} \in H^0_{(2)}\left(X, L_k^p\right), \quad j = 0, \ldots, d_{k,p}, \tag{8}$$

be an orthonormal basis and denote by $P_{k,p}$ the Bergman kernel function of the space $H^0_{(2)}\left(X, L_k^p\right)$ defined as in (4). This basis gives identifications $H^0_{(2)}\left(X, L_k^p\right) \simeq \mathbb{C}^{d_{k,p}+1}$ and $\mathbb{P}\left(H^0_{(2)}\left(X, L_k^p\right)^*\right) \simeq \mathbb{P}^{d_{k,p}}$. Let U be a contractible Stein open set in X, let e_k be a local holomorphic frame for L_k on U, and write $S_j^{k,p} = s_j^{k,p} e_k^{\otimes p}$, where $s_j^{k,p}$ is a holomorphic function on U. By composing $\Phi_{k,p}$ given in (7) with the last identification, we obtain a meromorphic map $\Phi_{k,p}\colon X \dashrightarrow \mathbb{P}^{d_{k,p}}$ which has the following local expression

$$\Phi_{k,p}(x) = \left[s_0^{k,p}(x) : \ldots : s_{d_{k,p}}^{k,p}(x)\right] \quad \text{for } x \in U. \tag{9}$$

It is called *the Kodaira map defined by the basis* $\left\{S_j^{k,p}\right\}_{j=0}^{d_{k,p}}$.

Next, we define the *Fubini-Study currents* $\gamma_{k,p}$ of $H^0_{(2)}\left(X, L_k^p\right)$ by

$$\gamma_{k,p}|_U = \frac{1}{2}\, dd^c \log \sum_{j=0}^{d_{k,p}} |s_j^{k,p}|^2, \tag{10}$$

where the open set U and the holomorphic functions $s_j^{k,p}$ are as above. Note that $\gamma_{k,p}$ is a positive closed current of bidegree $(1,1)$ on X, and is independent of the choice of basis. Actually, the Fubini-Study currents are pullbacks of the Fubini-Study forms by Kodaira maps, which justifies their name.

Let ω_{FS} be the Fubini-Study form on $\mathbb{P}^{d_{k,p}}$. By (9) and (10), the currents $\gamma_{k,p}$ can be described as pullbacks

$$\gamma_{k,p} = \Phi_{k,p}^*(\omega_{\mathrm{FS}}), \quad 1 \le k \le m. \tag{11}$$

We introduce the psh function

$$u_{k,p} := \frac{1}{2p} \log \sum_{j=0}^{d_{k,p}} \left|s_j^{k,p}\right|^2 = u_k + \frac{1}{2p} \log P_{k,p} \quad \text{on } U, \tag{12}$$

where u_k is the weight of the metric h_k on U corresponding to e_k, so $|e_k|_{h_k} = e^{-u_k}$. Clearly, by (10) and (12), $dd^c u_{k,p} = \frac{1}{p}\gamma_{k,p}$. Moreover, note that by (12), $\log P_{k,p} \in L^1(X, \omega^n)$ and

$$\frac{1}{p}\gamma_{k,p} = c_1(L_k, h_k) + \frac{1}{2p} dd^c \log P_{k,p} \tag{13}$$

as currents on X. By [12, Theorems 5.1 and 5.3] (see also [12, (7)]) there exist $c > 0$, $p_0 \in \mathbb{N}$, such that if $p \geq p_0$, $1 \leq k \leq m$ and $z \in X \setminus \Sigma(h_k)$, then $P_{k,p}(z) \geq c$. By (12) it follows that

$$u_{k,p}(z) \geq u_k(z) + \frac{\log c}{2p}, \quad z \in U, \ p \geq p_0, \ 1 \leq k \leq m. \tag{14}$$

For $p \geq 1$ consider the following analytic subsets of X:

$$\Sigma_{k,p} := \left\{ x \in X : S_j^{k,p}(x) = 0, \quad 0 \leq j \leq d_{k,p} \right\}, \ 1 \leq k \leq m.$$

Hence $\Sigma_{k,p}$ is the base locus of $H^0_{(2)}(X, L_k^p)$, and $\Sigma_{k,p} \cap U = \{u_{k,p} = -\infty\}$. Note also that $\Sigma(h_k) \cap U \supset \{u_k = -\infty\}$ and by (14) we have $\Sigma_{k,p} \subset \Sigma(h_k)$ for $p \geq p_0$.

Proposition 3.1. In the hypotheses of Theorem 1.2 we have the following:

(i) For all p sufficiently large and every $J \subset \{1, \dots, m\}$ the analytic sets $\Sigma_{k,p}$, $k \in J$, $\Sigma(h_\ell)$, $\ell \in J' := \{1, \dots, m\} \setminus J$, are in general position.

(ii) If p is sufficiently large then the currents

$$\bigwedge_{k \in J} \gamma_{k,p} \wedge \bigwedge_{\ell \in J'} c_1(L_\ell, h_\ell)$$

are well defined on X, for every $J \subset \{1, \dots, m\}$.

(iii) $\frac{1}{p^m}\gamma_{1,p} \wedge \dots \wedge \gamma_{m,p} \to c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m)$ as $p \to \infty$, in the weak sense of currents on X. □

Proof. As noted above we have by (14) that $\Sigma_{k,p} \subset \Sigma(h_k)$ for all p sufficiently large. Since $\Sigma(h_1), \dots, \Sigma(h_m)$ are in general position this implies (i). Then (ii) follows by [18, Corollary 2.11].

(iii) Let $U \subset X$ be a contractible Stein open set as above, $u_{k,p}$, u_k be the psh functions defined in (12), so $dd^c u_k = c_1(L_k, h_k)$ and $dd^c u_{k,p} = \frac{1}{p}\gamma_{k,p}$ on U. By [12, Theorem 5.1], we have that $\frac{1}{p}\log P_{k,p} \to 0$ in $L^1(X, \omega^n)$, hence by (12), $u_{k,p} \to u_k$ in $L^1_{loc}(U)$, as $p \to \infty$, for each $1 \leq k \leq m$. Recall that by (14), $u_{k,p} \geq u_k - \frac{C}{p}$ holds on U for all p sufficiently large and some constant $C > 0$. Then [26, Theorem 3.5] implies that $dd^c u_{1,p} \wedge \dots \wedge dd^c u_{m,p} \to dd^c u_1 \wedge \dots \wedge dd^c u_m$ weakly on U as $p \to \infty$. ■

We will need the following version of Bertini's theorem. The corresponding statement for the case of a single line bundle is proved in [12, Proposition 4.1].

Proposition 3.2. Let $L_k \longrightarrow X$, $1 \le k \le m \le n$, be holomorphic line bundles over a compact complex manifold X of dimension n. Assume that

(i) V_k is a vector subspace of $H^0(X, L_k)$ with basis $S_{k,0}, \dots, S_{k,d_k}$, base locus $\mathrm{Bs}\, V_k := \{S_{k,0} = \cdots = S_{k,d_k} = 0\} \subset X$, such that $d_k \ge 1$ and the analytic sets $\mathrm{Bs}\, V_1, \dots, \mathrm{Bs}\, V_m$ are in general position in the sense of Definition 1.1.

(ii) $Z(t_k) := \left\{x \in X : \sum_{j=0}^{d_k} t_{k,j} S_{k,j}(x) = 0\right\}$, where $t_k = [t_{k,0} : \dots : t_{k,d_k}] \in \mathbb{P}^{d_k}$.

(iii) $\nu = \mu_1 \times \cdots \times \mu_m$ is the product measure on $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_m}$, where μ_k is the Fubini-Study volume on $\mathbb{P}^{d_k}$.

Then the analytic sets $Z(t_1), \dots, Z(t_m)$ are in general position for ν-a.e. $(t_1, \dots, t_m) \in \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_m}$. □

Proof. If $1 \le l_1 < \cdots < l_k \le m$ let $\nu_{l_1 \dots l_k} = \mu_{l_1} \times \cdots \times \mu_{l_k}$ be the product measure on $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_k}$. For $1 \le k \le m$ consider the sets

$$U_k = \left\{ \left(t_{l_1}, \dots, t_{l_k}\right) \in \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_k} : \dim Z\left(t_{l_1}\right) \cap \cdots \cap Z\left(t_{l_k}\right) \cap A_j \le n - k - j \right\},$$

where $1 \le l_1 < \cdots < l_k \le m$, $j = 0$ and $A_0 = \emptyset$, or $1 \le j \le m - k$ and $A_j = \mathrm{Bs}\, V_{i_1} \cap \cdots \cap \mathrm{Bs}\, V_{i_j}$ for some $i_1 < \cdots < i_j$ in $\{1, \dots, m\} \setminus \{l_1, \dots, l_k\}$.

The proposition follows if we prove by induction on k that

$$\nu_{l_1 \dots l_k}(U_k) = 1$$

for every set U_k with $1 \le l_1 < \cdots < l_k \le m$, $0 \le j \le m - k$ and A_j as above. Clearly, it suffices to consider the case $\{l_1, \dots, l_k\} = \{1, \dots, k\}$. To simplify notation we set $\nu_k := \nu_{1 \dots k}$.

Let $k = 1$. If $j = 0$, $A_0 = \emptyset$, so $U_1 = \left\{t_1 \in \mathbb{P}^{d_1} : \dim Z(t_1) \le n - 1\right\} = \mathbb{P}^{d_1}$. Assume next that $1 \le j \le m - 1$ and write $A_j = \bigcup_{l=1}^{N} D_l \cup B$, where D_l are the irreducible components of A_j of dimension $n - j$ and $\dim B \le n - j - 1$. We have that $\left\{t_1 \in \mathbb{P}^{d_1} : D_l \subset Z(t_1)\right\}$ is a proper linear subspace of $\mathbb{P}^{d_1}$. Indeed, otherwise $D_l \subset \mathrm{Bs}\, V_1$, so $\dim A_j \cap \mathrm{Bs}\, V_1 = n - j$, which contradicts the hypothesis that $\mathrm{Bs}\, V_1, \dots, \mathrm{Bs}\, V_m$ are in general position. If $t_1 \in \mathbb{P}^{d_1} \setminus U_1$, then $\dim Z(t_1) \cap A_j \ge n - j$. Since $Z(t_1) \cap A_j$ is an analytic subset of A_j, it follows that $D_l \subset Z(t_1) \cap A_j$ for some l, hence $\mathbb{P}^{d_1} \setminus U_1 = \bigcup_{l=1}^{N} \left\{t_1 \in \mathbb{P}^{d_1} : D_l \subset Z(t_1)\right\}$. Therefore, $\mu_1(\mathbb{P}^{d_1} \setminus U_1) = 0$.

We assume now that $\nu_k(U_k)=1$ for any set U_k as above. Let

$$U_{k+1}=\left\{(t_1,\dots,t_{k+1})\in\mathbb{P}^{d_1}\times\cdots\times\mathbb{P}^{d_{k+1}}:\dim Z(t_1)\cap\cdots\cap Z(t_{k+1})\cap A_j\le n-k-1-j\right\},$$

where $0\le j\le m-k-1$, $A_0=\emptyset$, or $A_j=\operatorname{Bs}V_{i_1}\cap\cdots\cap\operatorname{Bs}V_{i_j}$ with $k+2\le i_1<\cdots<i_j\le m$. Consider the set $U=U'\cap U''$, where

$$U'=\left\{(t_1,\dots,t_k)\in\mathbb{P}^{d_1}\times\cdots\times\mathbb{P}^{d_k}:\dim Z(t_1)\cap\cdots\cap Z(t_k)\cap A_j\le n-k-j\right\},$$

$$U''=\left\{(t_1,\dots,t_k)\in\mathbb{P}^{d_1}\times\cdots\times\mathbb{P}^{d_k}:\dim Z(t_1)\cap\cdots\cap Z(t_k)\cap\operatorname{Bs}V_{k+1}\cap A_j\le n-k-j-1\right\}.$$

By the induction hypothesis we have $\nu_k(U')=\nu_k(U'')=1$, so $\nu_k(U)=1$. To prove that $\nu_{k+1}(U_{k+1})=1$ it suffices to show that

$$\nu_{k+1}(W)=0,\quad\text{where}\quad W:=\left(U\times\mathbb{P}^{d_{k+1}}\right)\setminus U_{k+1}.$$

To this end we fix $t:=(t_1,\dots,t_k)\in U$, we let

$$Z(t):=Z(t_1)\cap\cdots\cap Z(t_k),\ W(t):=\left\{t_{k+1}\in\mathbb{P}^{d_{k+1}}:\dim Z(t)\cap A_j\cap Z(t_{k+1})\ge n-k-j\right\},$$

and prove that $\mu_{k+1}(W(t))=0$.

Since $t\in U\subset U'$ we can write $Z(t)\cap A_j=\bigcup_{l=1}^N D_l\cup B$, where D_l are the irreducible components of $Z(t)\cap A_j$ of dimension $n-k-j$ and $\dim B\le n-k-j-1$. If $t_{k+1}\in W(t)$, then $Z(t)\cap A_j\cap Z(t_{k+1})$ is an analytic subset of $Z(t)\cap A_j$ of dimension $n-k-j$, so $D_l\subset Z(t)\cap A_j\cap Z(t_{k+1})$ for some l. Thus

$$W(t)=\bigcup_{l=1}^N F_l(t),\quad\text{where } F_l(t):=\left\{t_{k+1}\in\mathbb{P}^{d_{k+1}}:D_l\subset Z(t_{k+1})\right\}.$$

If $D_l\subset\operatorname{Bs}V_{k+1}$, then $\dim Z(t)\cap A_j\cap\operatorname{Bs}V_{k+1}=n-k-j$, which contradicts the fact that $t\in U''$. Hence the sections in V_{k+1} cannot all vanish on D_l, so we may assume that $S_{k+1,d_{k+1}}\not\equiv 0$ on D_l. We have $F_l(t)\subset\{t_{k+1,0}=0\}\cup H_l(t)$ where

$$H_l(t):=\left\{\left[1:t_{k+1,1}:\dots:t_{k+1,d_{k+1}}\right]\in\mathbb{P}^{d_{k+1}}:D_l\subset Z\left(\left[1:t_{k+1,1}:\dots:t_{k+1,d_{k+1}}\right]\right)\right\}.$$

For each $(t_{k+1,1}:\dots:t_{k+1,d_{k+1}-1})\in\mathbb{C}^{d_{k+1}-1}$ there exists at most one $\zeta\in\mathbb{C}$ with $[1:t_{k+1,1}:\dots:t_{k+1,d_{k+1}-1}:\zeta]\in H_l(t)$. Indeed, if $\zeta\ne\zeta'$ have this property then

$$S_{k+1,0}+t_{k+1,1}S_{k+1,1}+\cdots+t_{k+1,d_{k+1}-1}S_{k+1,d_{k+1}-1}+aS_{k+1,d_{k+1}}\equiv 0\quad\text{on } D_l,$$

for $a=\zeta, \zeta'$, hence $S_{k+1,d_{k+1}} \equiv 0$ on D_l, a contradiction. It follows that $\mu_{k+1}(H_l(t))=0$, so $\mu_{k+1}(F_l(t))=0$. Hence $\mu_{k+1}(W(t))=0$ and the proof is complete. ■

We return now to the setting of Theorem 1.2. If $\left\{S_j^{k,p}\right\}_{j=0}^{d_{k,p}}$ is an orthonormal basis of $H^0_{(2)}\left(X, L_k^p\right)$, we define the analytic hypersurface $Z(t_k) \subset X$, for $t_k=\left[t_{k,0}:\ldots:t_{k,d_{k,p}}\right] \in \mathbb{P}^{d_{k,p}}$, as in Proposition 3.2 (*ii*). Let $\mu_{k,p}$ be the Fubini-Study volume on $\mathbb{P}^{d_{k,p}}$, $1 \le k \le m$, $p \ge 1$, and let $\mu_p=\mu_{1,p}\times\cdots\times\mu_{m,p}$ be the product measure on $\mathbb{P}^{d_{1,p}}\times\cdots\times\mathbb{P}^{d_{m,p}}$. Applying Proposition 3.2 we obtain:

Proposition 3.3. In the above setting, if p is sufficiently large, then for μ_p-a.e. $(t_1,\ldots,t_m) \in \mathbb{P}^{d_{1,p}}\times\cdots\times\mathbb{P}^{d_{m,p}}$ the analytic subsets $Z(t_1),\ldots,Z(t_m) \subset X$ are in general position, and $Z(t_{i_1})\cap\cdots\cap Z(t_{i_k})$ has pure dimension $n-k$ for each $1\le k\le m$, $1\le i_1<\cdots< i_k\le m$. □

Proof. Let $V_{k,p} := H^0_{(2)}\left(X, L_k^p\right)$, so $\operatorname{Bs} V_{k,p}=\Sigma_{k,p}$. By Proposition 3.1, $\Sigma_{1,p},\ldots,\Sigma_{m,p}$ are in general position for all p sufficiently large. We fix such p and denote by $[Z(t_k)]$ the current of integration along the analytic hypersurface $Z(t_k)$; it has the same cohomology class as $pc_1(L_k, h_k)$. Proposition 3.2 shows that the analytic subsets $Z(t_1),\ldots,Z(t_m)$ are in general position for μ_p-a.e. $(t_1,\ldots,t_m)\in\mathbb{P}^{d_{1,p}}\times\cdots\times\mathbb{P}^{d_{m,p}}$. Hence if $1\le k\le m$, $1\le i_1<\cdots<i_k\le m$, the current $[Z(t_{i_1})]\wedge\cdots\wedge[Z(t_{i_k})]$ is well defined by [18, Corollary 2.11] and it is supported in $Z(t_{i_1})\cap\cdots\cap Z(t_{i_k})$. Since $c_1(L_k,h_k)\ge\varepsilon\omega$, it follows that

$$\int_X [Z\left(t_{i_1}\right)]\wedge\cdots\wedge[Z\left(t_{i_k}\right)]\wedge\omega^{n-k}=p^k\int_X c_1\left(L_{i_1},h_{i_1}\right)\wedge\cdots\wedge c_1\left(L_{i_k},h_{i_k}\right)\wedge\omega^{n-k}\ge p^k\varepsilon^k\int_X\omega^n.$$

So $Z(t_{i_1})\cap\cdots\cap Z(t_{i_k})\neq\emptyset$, hence it has pure dimension $n-k$. ■

4 Convergence Speed Towards Intersection of Fubini-Study Currents

In this section, we rely on techniques introduced by Dinh–Sibony [23], based on the notion of meromorphic transform, in order to estimate the speed of equidistribution of the common zeros of m-tuples of sections of the considered big line bundles towards the intersection of the Fubini-Study currents. We then prove Theorem 1.2.

4.1 Dinh–Sibony equidistribution theorem

A meromorphic transform $F : X \dashrightarrow Y$ between two compact Kähler manifolds (X,ω) of dimension n and (Y,ω_Y) of dimension m is the data of an analytic subset $\Gamma\subset X\times Y$ (called *the graph of* F) of pure dimension $m+k$ such that the projections $\pi_1 : X\times Y\to X$

and $\pi_2: X\times Y\to Y$ restricted to each irreducible component of Γ are surjective. We set formally $F=\pi_2\circ(\pi_1|_\Gamma)^{-1}$. For $y\in Y$ generic (that is, outside a proper analytic subset), the dimension of the fiber $F^{-1}(y):=\pi_1\left(\pi_2^{-1}|_\Gamma(y)\right)$ is equal to k. This is called *the codimension* of F. We consider two of the *intermediate degrees* for F (see [23, Section 3.1]):

$$d(F):=\int_X F^*\left(\omega_Y^m\right)\wedge\omega^k \quad \text{and} \quad \delta(F):=\int_X F^*\left(\omega_Y^{m-1}\right)\wedge\omega^{k+1}.$$

By [23, Proposition 2.2], there exists $r:=r(Y,\omega_Y)$ such that for every positive closed current T of bidegree $(1,1)$ on Y with $\|T\|=1$ there is a smooth $(1,1)$-form α which depends uniquely on the class $\{T\}$ and a quasi-plurisubharmonic (qpsh) function φ such that $-r\omega_Y\le\alpha\le r\omega_Y$ and $dd^c\varphi-T=\alpha$. If Y is the projective space $\mathbb{P}^\ell$ equipped with the Fubiny-Study form $\omega_{\rm FS}$, then we have $r\left(\mathbb{P}^\ell,\omega_{\rm FS}\right)=1$. Consider the class

$$\mathcal{Q}(Y,\omega_Y):=\left\{\varphi \quad \text{qpsh on } Y,\ dd^c\varphi\ge -r(Y,\omega_Y)\,\omega_Y\right\}.$$

A positive measure μ on Y is called a BP measure if all qpsh functions on Y are integrable with respect to μ. When $\dim Y=1$, it is well known that μ is BP if and only if it admits locally a bounded potential. The terminology BP comes from this fact (see [23]).

If μ is a BP measure on Y and $t\in\mathbb{R}$, we let

$$R(Y,\omega_Y,\mu):=\sup\left\{\max_Y\varphi:\ \varphi\in\mathcal{Q}(Y,\omega_Y),\ \int_Y\varphi\,{\rm d}\mu=0\right\},$$

$$\Delta(Y,\omega_Y,\mu,t):=\sup\left\{\mu(\varphi<-t):\ \varphi\in\mathcal{Q}(Y,\omega_Y),\ \int_Y\varphi\,{\rm d}\mu=0\right\}.$$

These constants are related to the Alexander–Dinh–Sibony capacity [1, 23, 27].

Let Φ_p be a sequence of meromorphic transforms from a compact Kähler manifold (X,ω) into compact Kähler manifolds $(\mathbb{X}_p,\omega_p)$ of the same codimension k, where $\mathbb{X}_p$ is defined in (2). Let ν_p be a BP probability measure on $\mathbb{X}_p$ and $\nu_\infty=\prod_{p\ge1}\nu_p$ be the product measure on $\Omega:=\prod_{p\ge1}\mathbb{X}_p$. For every $p>0$ and $\varepsilon>0$ let

$$E_p(\varepsilon):=\bigcup_{\|\phi\|_{\mathscr{C}^2}\le1}\left\{x_p\in\mathbb{X}_p:\ \left|\left\langle\Phi_p^*\left(\delta_{x_p}\right)-\Phi_p^*\left(\nu_p\right),\phi\right\rangle\right|\ge d\left(\Phi_p\right)\varepsilon\right\},$$

where δ_{x_p} is the Dirac mass at x_p. Note that $\Phi_p^*(\delta_{x_p})$ and $\Phi_p^*(\nu_p)$ are positive closed currents of bidimension (k,k) on X, and the former is well defined for the generic point $x_p\in\mathbb{X}_p$ (see [23, Section 3.1]). Now we are in position to state the part which deals with the quantified speed of convergence in the Dinh–Sibony equidistribution theorem [23, Theorem 4.1].

Theorem 4.1 ([23, Lemma 4.2 (d)]). In the above setting the following estimate holds:

$$\nu_p\left(E_p(\varepsilon)\right) \le \Delta\left(\mathbb{X}_p, \omega_p, \nu_p, \eta_{\varepsilon,p}\right),$$

where $\eta_{\varepsilon,p} := \varepsilon\delta(\Phi_p)^{-1}d(\Phi_p) - 3R\left(\mathbb{X}_p, \omega_p, \nu_p\right)$. □

4.2 Equidistribution of pullbacks of Dirac masses by Kodaira maps

Let (X,ω) be a compact Kähler manifold of dimension n and (L_k, h_k), $1 \le k \le m \le n$, be singular Hermitian holomorphic line bundles on X such that h_k is continuous outside a proper analytic subset $\Sigma(h_k) \subset X$, $c_1(L_k, h_k) \ge \varepsilon\omega$ on X for some $\varepsilon > 0$, and $\Sigma(h_1), \dots, \Sigma(h_m)$ are in general position. Recall from Section 1 that

$$\mathbb{X}_p := \mathbb{P}H^0_{(2)}\left(X, L_1^p\right) \times \cdots \times \mathbb{P}H^0_{(2)}\left(X, L_m^p\right), \quad (\Omega, \sigma_\infty) := \prod_{p=1}^{\infty}\left(\mathbb{X}_p, \sigma_p\right),$$

where the probability measure σ_p is the product of the Fubini-Study volume on each factor. From now on let $p \in \mathbb{N}$ be large enough. Fix an orthonormal basis $\{S_j^{k,p}\}_{j=0}^{d_{k,p}}$ as in (8) and let $\Phi_{k,p} : X \dashrightarrow \mathbb{P}^{d_{k,p}}$ be the Kodaira map defined by this basis (see (9)). By (11) we have that $\Phi_{k,p}^*\omega_{\mathrm{FS}} = \gamma_{k,p}$, where $\gamma_{k,p}$ is the Fubini-Study current of the space $H^0_{(2)}(X, L_k^p)$ as defined in (10).

We consider now the Kodaira maps as meromorphic transforms from X to $\mathbb{P}H^0_{(2)}(X, L_k^p)$ which we denote still by $\Phi_{k,p} : X \dashrightarrow \mathbb{P}H^0_{(2)}(X, L_k^p)$. Precisely, this is the meromorphic transform with graph

$$\Gamma_{k,p} = \left\{(x, s) \in X \times \mathbb{P}H^0_{(2)}\left(X, L_k^p\right) : s(x) = 0\right\}, \quad 1 \le k \le m.$$

Indeed, since $\dim H^0_{(2)}(X, L_k^p) \ge 2$ (see e.g., Proposition 4.7), there exists, for every $x \in X$, a section $s \in H^0_{(2)}(X, L_k^p)$ with $s(x) = 0$, so the projection $\Gamma_{k,p} \longrightarrow X$ is surjective. Moreover, since L_k^p is non-trivial, every global holomorphic section of L_k^p must vanish at some $x \in X$, hence the projection $\Gamma_{k,p} \longrightarrow \mathbb{P}H^0_{(2)}(X, L_k^p)$ is surjective. Note that

$$\Phi_{k,p}(x) = \left\{s \in \mathbb{P}H^0_{(2)}\left(X, L_k^p\right) : s(x) = 0\right\}, \quad \Phi_{k,p}^{-1}(s) = \{x \in X : s(x) = 0\}.$$

Let Φ_p be the product transform of $\Phi_{1,p}, \dots, \Phi_{m,p}$ (see [23, Section 3.3]). It is the meromorphic transform with graph

$$\Gamma_p = \left\{\left(x, s_{p1}, \dots, s_{pm}\right) \in X \times \mathbb{X}_p : s_{p1}(x) = \cdots = s_{pm}(x) = 0\right\}. \tag{15}$$

By above, the projection $\Pi_1 : \Gamma_p \longrightarrow X$ is surjective. The second projection $\Pi_2 : \Gamma_p \longrightarrow \mathbb{X}_p$ is proper, hence by Remmert's theorem $\Pi_2(\Gamma_p)$ is an analytic subvariety of $\mathbb{X}_p$. Proposition 3.3 implies that $\Pi_2(\Gamma_p)$ has full measure in $\mathbb{X}_p$, so Π_2 is surjective and Φ_p is a meromorphic transform of codimension $n-m$, with fibers

$$\Phi_p^{-1}\left(\mathbf{s}_p\right) = \left\{x \in X : s_{p1}(x) = \cdots = s_{pm}(x) = 0\right\}, \quad \text{where } \mathbf{s}_p = \left(s_{p1}, \ldots, s_{pm}\right) \in \mathbb{X}_p.$$

Considering the product transform of any $\Phi_{i_1,p}, \ldots, \Phi_{i_k,p}$, $1 \le i_1 < \cdots < i_k \le m$, and arguing as above it follows that, for $\mathbf{s}_p = (s_{p1}, \ldots, s_{pm}) \in \mathbb{X}_p$ generic, the analytic sets $\{s_{p1} = 0\}, \ldots, \{s_{pm} = 0\}$ are in general position. Hence by [18, Corollary 2.11] the following current of bidegree (m, m) is well defined on X:

$$\Phi_p^*\left(\delta_{\mathbf{s}_p}\right) = [\mathbf{s}_p = 0] = [s_{p1} = 0] \wedge \cdots \wedge [s_{pm} = 0] = \Phi_{1,p}^*\left(\delta_{s_{p1}}\right) \wedge \cdots \wedge \Phi_{m,p}^*\left(\delta_{s_{pm}}\right).$$

The main result of this section is the following theorem.

Theorem 4.2. Under the hypotheses of Theorem 1.2 there exist a constant $\xi > 0$ depending only on m and a constant $c = c(X, L_1, h_1, \ldots, L_m, h_m) > 0$ with the following property: for any sequence of positive numbers $\{\lambda_p\}_{p\ge 1}$ with

$$\liminf_{p\to\infty} \frac{\lambda_p}{\log p} > (1 + \xi n)\, c,$$

there are subsets $E_p \subset \mathbb{X}_p$ such that

(a) $\sigma_p(E_p) \le c p^{\xi n} \exp(-\lambda_p / c)$ for all p large enough;
(b) if $\mathbf{s}_p \in \mathbb{X}_p \setminus E_p$ we have that the estimate

$$\left| \frac{1}{p^m} \langle [\mathbf{s}_p = 0] - \gamma_{1,p} \wedge \cdots \wedge \gamma_{m,p}, \phi \rangle \right| \le c \frac{\lambda_p}{p} \|\phi\|_{\mathscr{C}^2}$$

holds for every $(n-m, n-m)$ form ϕ of class $\mathscr{C}^2$.

In particular, for σ_∞-a.e. $\mathbf{s} \in \Omega$ the estimate from (b) holds for all p sufficiently large. □

Prior to the proof we need to establish some preparatory results. Let

$$d_{0,p} = d_{1,p} + \cdots + d_{m,p}$$

be the dimension of $\mathbb{X}_p$ and π_k be the canonical projection of $\mathbb{X}_p$ on to its kth factor. Let

$$\omega_p := c_p \left(\pi_1^* \omega_{\mathrm{FS}} + \cdots + \pi_m^* \omega_{\mathrm{FS}}\right), \quad \text{so } \sigma_p = \omega_p^{d_{0,p}}.$$

Here ω_{FS} denotes, as usual, the Fubini-Study form on each factor $\mathbb{P}H^0_{(2)}(X, L_k^p)$, and the constant c_p is chosen so that σ_p is a probability measure on $\mathbb{X}_p$, thus

$$(c_p)^{-d_{0,p}} = \frac{d_{0,p}!}{d_{1,p}! \dots d_{m,p}!}. \tag{16}$$

Lemma 4.3. There is a constant $c_0 > 0$ such that $c_p \geq c_0$ for all $p \geq 1$. □

Proof. Fix $p \geq 1$ large enough. For each $1 \leq k \leq m$, let $l_k := d_{k,p}$. Using Stirling's formula $\ell! \approx (\ell/e)^\ell \sqrt{2\pi\ell}$ it suffices to show that there is a constant $c > 0$ such that for all $l_1, \dots, l_m \geq 1$,

$$\log(l_1 + \dots + l_m) - \left(\frac{l_1 \log l_1}{l_1 + \dots + l_m} + \dots + \frac{l_m \log l_m}{l_1 + \dots + l_m} \right) \leq c.$$

Since the function $t \mapsto t \log t$, $t > 0$, is convex, we infer that

$$\frac{1}{m} (l_1 \log l_1 + \dots + l_m \log l_m) \geq \frac{l_1 + \dots + l_m}{m} \log \frac{l_1 + \dots + l_m}{m}.$$

This implies the required estimate with $c := \log m$. ■

Following Section 4.1, we consider two intermediate degrees for the Kodaira maps Φ_p:

$$d_p = d(\Phi_p) := \int_X \Phi_p^* \left(\omega_p^{d_{0,p}} \right) \wedge \omega^{n-m} \quad \text{and} \quad \delta_p = \delta(\Phi_p) := \int_X \Phi_p^* \left(\omega_p^{d_{0,p}-1} \right) \wedge \omega^{n-m+1}.$$

The next result gives the asymptotic behavior of d_p and δ_p as $p \to \infty$.

Lemma 4.4. We have $d_p = p^m \| c_1(L_1, h_1) \wedge \dots \wedge c_1(L_m, h_m) \|$ and

$$\delta_p = \frac{p^{m-1}}{c_p} \sum_{k=1}^m \frac{d_{k,p}}{d_{0,p}} \left\| \bigwedge_{l=1, l \neq k}^m c_1(L_l, h_l) \right\| \leq C p^{m-1},$$

where $C > 0$ is a constant depending on (L_k, h_k), $1 \leq k \leq m$. □

Proof. We use a cohomological argument. For the first identity we replace $\omega_p^{d_{0,p}}$ by a Dirac mass $\delta_{\mathbf{s}}$, where $\mathbf{s} := (s_1, \dots, s_m) \in \mathbb{X}_p$ is such that $\{s_1 = 0\}, \dots, \{s_m = 0\}$ are in general position, so the current $\Phi_p^*(\delta_{\mathbf{s}}) = [s_1 = 0] \wedge \dots \wedge [s_m = 0]$ is well defined (see Proposition 3.3). By the Poincaré-Lelong formula [33, Theorem 2.3.3],

$$[s_k = 0] = p c_1(L_k, h_k) + dd^c \log |s_k|_{h_{k,p}}, \quad 1 \leq k \leq m.$$

Since the current $c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_m, h_m)$ is well defined (see Proposition 3.1) it follows that

$$\int_X \Phi_p^*(\delta_{\mathbf{s}}) \wedge \omega^{n-m} = p^m \int_X \theta_1 \wedge \cdots \wedge \theta_m \wedge \omega^{n-m} = p^m \int_X c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_m, h_m) \wedge \omega^{n-m},$$

where θ_k is a smooth closed $(1,1)$ form in the cohomology class of $c_1(L_k, h_k)$. Thus

$$d_p = \int_X \Phi_p^*\left(\omega_p^{d_{0,p}}\right) \wedge \omega^{n-m} = \int_X \Phi_p^*(\delta_{\mathbf{s}}) \wedge \omega^{n-m} = p^m \| c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_m, h_m) \|.$$

For the second identity, a straightforward computation shows that

$$\omega_p^{d_{0,p}-1} = \sum_{k=1}^m \frac{c_p^{d_{0,p}-1}(d_{0,p}-1)!}{d_{1,p}! \ldots (d_{k,p}-1)! \ldots d_{m,p}!} \quad \pi_1^* \omega_{\mathrm{FS}}^{d_{1,p}} \wedge \cdots \wedge \pi_k^* \omega_{\mathrm{FS}}^{d_{k,p}-1} \wedge \cdots \wedge \pi_m^* \omega_{\mathrm{FS}}^{d_{m,p}}.$$

Using (16) and replacing $\omega_{\mathrm{FS}}^{d_{k,p}}$ (resp. $\omega_{\mathrm{FS}}^{d_{k,p}-1}$) by a generic point (resp. a generic complex line) in $\mathbb{P}H^0_{(2)}(X, L_k^p)$, we may replace $\omega_p^{d_{0,p}-1}$ by a current of the form

$$T := \sum_{k=1}^m \frac{d_{k,p}}{c_p d_{0,p}} [\{s_1\} \times \cdots \times \mathcal{D}_k \times \cdots \times \{s_m\}].$$

Here, $\mathcal{D}_k$ is a generic complex line in $\mathbb{P}H^0_{(2)}(X, L_k^p)$ and $(s_1, \ldots, s_m)$ is a generic point in $\mathbb{X}_p$. The genericity of $\mathcal{D}_k$ implies that $\Phi_{k,p}^*(\mathcal{D}_k) = X$, so

$$\Phi_p^*([\{s_1\} \times \cdots \times \mathcal{D}_k \times \cdots \times \{s_m\}]) = \bigwedge_{l=1, l\neq k}^m [s_l = 0].$$

The Poincaré–Lelong formula yields

$$\left\| \Phi_p^*([\{s_1\} \times \cdots \times \mathcal{D}_k \times \cdots \times \{s_m\}]) \right\| = p^{m-1} \left\| \bigwedge_{l=1, l\neq k}^m c_1(L_l, h_l) \right\|.$$

Since $\delta_p = \|\Phi_p^*(T)\|$, the second identity follows. Using Lemma 4.3, this yields the upper bound on δ_p. ■

Lemma 4.5. For all p sufficiently large we have $\Phi_p^*(\sigma_p) = \gamma_{1,p} \wedge \cdots \wedge \gamma_{m,p}$. □

Proof. Let us write $\mathbb{X}_p = X_{1,p} \times \cdots \times X_{m,p}$ and $\sigma_p = \sigma_{1,p} \times \cdots \times \sigma_{m,p}$, where $X_{k,p} = \mathbb{P}H^0_{(2)}(X, L_k^p)$ and $\sigma_{k,p}$ is the Fubini–Study volume on $X_{k,p}$. Recall that the meromorphic transform Φ_p has graph Γ_p defined in (15), and $\Pi_1 : \Gamma_p \longrightarrow X$, $\Pi_2 : \Gamma_p \longrightarrow \mathbb{X}_p$, denote the canonical projections. By the definition of $\Phi_p^*(\sigma_p)$ (see [23, Section 3.1]) we have

$$\langle \Phi_p^*(\sigma_p), \phi\rangle = \int_{\Gamma_p} \Pi_1^*(\phi)\wedge \Pi_2^*(\sigma_p) = \int_{\mathbb{X}_p} \Pi_{2*}\Pi_1^*(\phi)\wedge\sigma_p = \int_{\mathbb{X}_p} \langle [\mathbf{s}_p = 0], \phi\rangle\, d\sigma_p(\mathbf{s}_p),$$

where ϕ is a smooth $(n-m, n-m)$ form on X. Thanks to Propositions 3.1 and 3.2, we can apply [12, Proposition 4.2] as in the proof of [12, Theorem 1.2] to show that

$$\begin{aligned}\langle \Phi_p^*(\sigma_p), \phi\rangle &= \int_{X_{m,p}} \dots \int_{X_{1,p}} \langle [s_{p1}=0]\wedge\dots\wedge[s_{pm}=0], \phi\rangle\, d\sigma_{1,p}(s_{p1})\dots d\sigma_{m,p}(s_{pm}) \\ &= \int_{X_{m,p}} \dots \int_{X_{2,p}} \langle \gamma_{1,p}\wedge[s_{p2}=0]\wedge\dots\wedge[s_{pm}=0], \phi\rangle\, d\sigma_{2,p}(s_{p2})\dots d\sigma_{m,p}(s_{pm}) \\ &= \dots = \langle \gamma_{1,p}\wedge\dots\wedge\gamma_{m,p}, \phi\rangle.\end{aligned}$$

This concludes the proof of the lemma. ■

Lemma 4.6. There exist absolute constants $C_1, \alpha > 0$, and constants $C_2, \alpha', \xi > 0$ depending only on $m \geq 1$, such that for all $\ell, \ell_1, \dots, \ell_m \geq 1$, and $t \geq 0$,

$$R\left(\mathbb{P}^\ell, \omega_{\mathrm{FS}}, \omega_{\mathrm{FS}}^\ell\right) \leq \frac{1}{2}\,(1+\log\ell),$$

$$\Delta\left(\mathbb{P}^\ell, \omega_{\mathrm{FS}}, \omega_{\mathrm{FS}}^\ell, t\right) \leq C_1 \ell\, e^{-\alpha t},$$

$$r\left(\mathbb{P}^{\ell_1}\times\dots\times\mathbb{P}^{\ell_m}, \omega_{\mathrm{MP}}\right) \leq r(\ell_1,\dots,\ell_m) := \max_{1\leq k\leq m}\frac{d}{\ell_k},$$

$$R\left(\mathbb{P}^{\ell_1}\times\dots\times\mathbb{P}^{\ell_m}, \omega_{\mathrm{MP}}, \omega_{\mathrm{MP}}^d\right) \leq C_2 r(\ell_1,\dots,\ell_m)\,(1+\log d),$$

$$\Delta\left(\mathbb{P}^{\ell_1}\times\dots\times\mathbb{P}^{\ell_m}, \omega_{\mathrm{MP}}, \omega_{\mathrm{MP}}^d, t\right) \leq C_2 d^\xi e^{-\alpha' t/r(\ell_1,\dots,\ell_m)},$$

where

$$d = \ell_1+\dots+\ell_m, \quad \omega_{\mathrm{MP}} := c\left(\pi_1^*(\omega_{\mathrm{FS}})+\dots+\pi_m^*(\omega_{\mathrm{FS}})\right), \quad c^{-d} = \frac{d!}{\ell_1!\dots\ell_m!},$$

so ω_{MP}^d is a probability measure on $\mathbb{P}^{\ell_1}\times\dots\times\mathbb{P}^{\ell_m}$. □

Proof. The first two inequalities are proved in Proposition A.3 and Corollary A.5 from [23]. If T is a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^{\ell_1}\times\dots\times\mathbb{P}^{\ell_m}$ with $\|T\| = 1$, then T is in the cohomology class of $\alpha = a_1\pi_1^*(\omega_{\mathrm{FS}})+\dots+a_m\pi_m^*(\omega_{\mathrm{FS}})$, for some

$a_k \geq 0$. Hence

$$0 \leq \alpha \leq \left(\max_{1 \leq k \leq m} \frac{a_k}{c} \right) \omega_{\mathrm{MP}}.$$

Now

$$1 = \|T\| = \int_{\mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_m}} \alpha \wedge \omega_{\mathrm{MP}}^{d-1} = \sum_{k=1}^{m} \frac{a_k \ell_k}{cd},$$

so $a_k/c \leq d/\ell_k$. Thus $r(\mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_m}, \omega_{\mathrm{MP}}) \leq \max_{1 \leq k \leq m} \frac{d}{\ell_k}$. The last two inequalities follow from these estimates by applying [23, Propositions A.8 and A.9]. ∎

We will also need the following lower estimate for the dimension $d_{k,p}$.

Proposition 4.7. Let (X, ω) be a compact Kähler manifold of dimension n. Let $(L, h) \to X$ be a singular Hermitian holomorphic line bundle such that $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$ and h is continuous outside a proper analytic subset of X. Then there exists $C > 0$ and $p_0 \in \mathbb{N}$ such that

$$\dim H^0_{(2)}\left(X, L^p\right) \geq C p^n \quad \forall\, p \geq p_0.$$ □

Proof. Let $\Sigma \subset X$ be a proper analytic set such that h is continuous on $X \setminus \Sigma$. We fix $x_0 \in X \setminus \Sigma$ and $r > 0$ such that $B(x_0, 2r) \cap \Sigma = \emptyset$. Let $0 \leq \chi \leq 1$ be a smooth cut-off function that equals 1 on $\bar{B}(x_0, r)$ and is supported in $B(x_0, 2r)$. We consider the function $\psi : X \to [-\infty, \infty)$, $\psi(x) = \eta\chi(x) \log|x - x_0|$, where $\eta > 0$.

Consider the metric $h_0 = h\exp(-\psi)$ on L. We choose η sufficiently small such that

$$c_1\left(L, h_0\right) \geq \frac{\varepsilon}{2}\omega \quad \text{on } X.$$

Let us denote by $\mathcal{I}(h^p)$ the multiplier ideal sheaf associated with h^p. Note that $H^0_{(2)}(X, L^p) = H^0(X, L^p \otimes \mathcal{I}(h^p))$. The Nadel vanishing theorem [19, 36] shows that there exists $p_0 \in \mathbb{N}$ such that

$$H^1\left(X, L^p \otimes \mathcal{I}\left(h_0^p\right)\right) = 0, \quad p \geq p_0. \tag{17}$$

Note that $\mathcal{I}(h_0^p) = \mathcal{I}(h^p) \otimes \mathcal{I}(p\psi)$. Consider the exact sequence

$$0 \to L^p \otimes \mathcal{I}\left(h^p\right) \otimes \mathcal{I}\left(p\psi\right) \to L^p \otimes \mathcal{I}\left(h^p\right) \to L^p \otimes \mathcal{I}\left(h^p\right) \otimes \mathcal{O}_X/\mathcal{I}\left(p\psi\right) \to 0. \tag{18}$$

Thanks to (17) applied to the long exact cohomology sequence associated with (18) we have

$$H^0\left(X, L^p \otimes \mathcal{I}\left(h^p\right)\right) \to H^0\left(X, L^p \otimes \mathcal{I}\left(h^p\right) \otimes \mathcal{O}_X/\mathcal{I}\left(p\psi\right)\right) \to 0, \quad p \geq p_0. \tag{19}$$

Now, for $x \neq x_0$, $\mathcal{I}(p\psi)_x = \mathcal{O}_{X,x}$ hence $\mathcal{O}_{X,x}/\mathcal{I}(p\psi)_x = 0$. Moreover, $\mathcal{I}(h^p)_{x_0} = \mathcal{O}_{X,x_0}$ since h is continuous at x_0. Hence

$$H^0\left(X, L^p \otimes \mathcal{I}\left(h^p\right) \otimes \mathcal{O}_X/\mathcal{I}\left(p\psi\right)\right) = L^p_{x_0} \otimes \mathcal{I}\left(h^p\right)_{x_0} \otimes \mathcal{O}_{X,x_0}/\mathcal{I}\left(p\psi\right)_{x_0}$$
$$= L^p_{x_0} \otimes \mathcal{O}_{X,x_0}/\mathcal{I}\left(p\psi\right)_{x_0}, \qquad (20)$$

so

$$H^0\left(X, L^p \otimes \mathcal{I}\left(h^p\right)\right) \to L^p_{x_0} \otimes \mathcal{O}_{X,x_0}/\mathcal{I}\left(p\psi\right)_{x_0} \to 0, \quad p \geq p_0. \qquad (21)$$

Denote by $\mathcal{M}_{X,x_0}$ the maximal ideal of $\mathcal{O}_{X,x_0}$ (i.e., germs of holomorphic functions vanishing at x_0). We have $\mathcal{I}(p\psi)_{x_0} \subset \mathcal{M}^{[p\eta]-n+1}_{X,x_0}$ and $\dim \mathcal{O}_{X,x_0}/\mathcal{M}^{k+1}_{X,x_0} = \binom{k+n}{k}$, which together with (21) implies the conclusion. ■

Proof of Theorem 4.2. We will apply Theorem 4.1 to the meromorphic transforms Φ_p from X to the multi-projective space $(\mathbb{X}_p, \omega_p)$ defined above, and the BP measures $\nu_p := \sigma_p$ on $\mathbb{X}_p$. For $t \in \mathbb{R}$ and $\varepsilon > 0$ let

$$R_p := R\left(\mathbb{X}_p, \omega_p, \sigma_p\right), \quad \Delta_p(t) := \Delta\left(\mathbb{X}_p, \omega_p, \sigma_p, t\right),$$
$$E_p(\varepsilon) := \bigcup_{\|\phi\|_{\mathscr{C}^2} \leq 1} \{\mathbf{s} \in \mathbb{X}_p : |\langle[\mathbf{s} = 0] - \gamma_{1,p} \wedge \cdots \wedge \gamma_{m,p}, \phi\rangle| \geq d_p\varepsilon\}. \qquad (22)$$

It follows from Siegel's lemma [33, Lemma 2.2.6] and Proposition 4.7 that there exist $C_3 > 0$ depending only on $(X, L_k, h_k)_{1\leq k\leq m}$ and $p_0 \in \mathbb{N}$ such that

$$p^n/C_3 \leq d_{k,p} \leq C_3 p^n, \ \ p \geq p_0, \ \ 1 \leq k \leq m.$$

By the last two inequalities in Lemma 4.6 we obtain for $p \geq p_0$ and $t \geq 0$,

$$R_p \leq mC_2C_3^2\left(1 + \log\left(mC_3p^n\right)\right) \leq C_4 \log p,$$
$$\Delta_p(t) \leq C_2\left(mC_3p^n\right)^{\xi} \exp\left(\frac{-\alpha' t}{mC_3^2}\right) \leq C_4 p^{\xi n} e^{-t/C_4}, \qquad (23)$$

where C_4 is a constant depending only on $(X, L_k, h_k)_{1\leq k\leq m}$. Now set

$$\varepsilon_p := \lambda_p/p, \quad \eta_p := \varepsilon_p d_p/\delta_p - 3R_p.$$

Lemma 4.4 implies that $d_p \approx p^m$, $\delta_p \lesssim p^{m-1}$, so

$$\eta_p \geq C_5\lambda_p - 3C_4 \log p, \quad p \geq p_0,$$

where C_5 is a constant depending only on $(X, L_k, h_k)_{1\leq k\leq m}$. Note that for all p sufficiently large,

$$\eta_p > \frac{C_5}{2}\lambda_p, \text{ provided that } \liminf_{p\to\infty} \frac{\lambda_p}{\log p} > 6C_4/C_5.$$

If $E_p = E_p(\varepsilon_p)$ then it follows from Theorem 4.1 and Lemma 4.5 that for all p sufficiently large

$$\sigma_p\left(E_p\right) \leq \Delta_p\left(\eta_p\right) \leq C_4 p^{\xi n} \exp\left(\frac{-C_5\lambda_p}{2C_4}\right),$$

where for the last estimate we used (23). Let

$$c = \max\left(\frac{6C_4}{C_5\,(1+\xi n)}, \frac{2C_4}{C_5}, C_4, \|c_1\left(L_1, h_1\right) \wedge \cdots \wedge c_1\left(L_m, h_m\right)\|\right).$$

If $\liminf_{p\to\infty}(\lambda_p/\log p) > (1+\xi n)c$, then for all p sufficiently large

$$\sigma_p\left(E_p\right) \leq C_4 p^{\xi n} \exp\left(\frac{-C_5\lambda_p}{2C_4}\right) \leq cp^{\xi n} \exp\left(\frac{-\lambda_p}{c}\right).$$

On the other hand, we have by the definition of E_p that if $\mathbf{s}_p \in \mathbb{X}_p \setminus E_p$ and ϕ is an $(n-m, n-m)$ form of class $\mathscr{C}^2$, then

$$\left|\frac{1}{p^m}\left\langle [\mathbf{s}_p = 0] - \gamma_{1,p} \wedge \cdots \wedge \gamma_{m,p}, \phi \right\rangle\right| \leq \frac{d_p}{p^m}\frac{\lambda_p}{p}\|\phi\|_{\mathscr{C}^2} \leq c\frac{\lambda_p}{p}\|\phi\|_{\mathscr{C}^2}.$$

In the last inequality we used the fact that $d_p \leq c\,p^m$ by Lemma 4.4.

For the last conclusion of Theorem 4.2 we proceed as in [20, p. 9]. The assumption on $\lambda_p/\log p$ and (a) imply that

$$\sum_{p=1}^{\infty} \sigma_p\left(E_p\right) \leq c' \sum_{p=1}^{\infty} \frac{1}{p^\eta} < \infty$$

for some $c' > 0$ and $\eta > 1$. Hence the set

$$E := \{\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, \ldots) \in \Omega : \mathbf{s}_p \in E_p \text{ for infinitely many } p\}$$

satisfies $\sigma_\infty(E) = 0$. Indeed, for every $N \geq 1$, E is contained in the set

$$\{\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, \ldots) \in \Omega : \mathbf{s}_p \in E_p \text{ for at least one } p \geq N\},$$

whose σ_∞-measure is at most

$$\sum_{p=N}^{\infty} \sigma_p\left(E_p\right) \leq c' \sum_{p=N}^{\infty} \frac{1}{p^\eta} \to 0 \quad \text{as } N \to \infty.$$

The proof of the theorem is thereby completed. ■

Proof of Theorem 1.2. Theorem 1.2 follows directly from Theorem 4.2 and Proposition 3.1(iii). ■

5 Equidistribution with Convergence Speed for Hölder Singular Metrics

In this section, we prove Theorem 1.4. We close with more examples of Hölder metrics with singularities. Theorem 1.4 follows at once from Theorem 4.2 and the next result.

Theorem 5.1. In the setting of Theorem 1.4, there exists a constant $c > 0$ depending only on $(X, L_1, h_1, \dots, L_m, h_m)$ such that for all p sufficiently large the estimate

$$\left|\left\langle \frac{1}{p^m} \bigwedge_{k=1}^{m} \gamma_{k,p} - \bigwedge_{k=1}^{m} c_1(L_k, h_k), \phi \right\rangle\right| \leq \frac{c \log p}{p} \|\phi\|_{\mathscr{C}^2}$$

holds for every $(n-m, n-m)$ form ϕ of class $\mathscr{C}^2$. □

Proof. We may assume that ϕ is real. There exists a constant $c' > 0$ such that for every real $(n-m, n-m)$ form ϕ of class $\mathscr{C}^2$,

$$-c' \|\phi\|_{\mathscr{C}^2}\, \omega^{n-m+1} \leq dd^c\phi \leq c' \|\phi\|_{\mathscr{C}^2}\, \omega^{n-m+1}. \tag{24}$$

Using Proposition 3.1 and (13) we can write

$$\left\langle \frac{1}{p^m} \bigwedge_{k=1}^{m} \gamma_{k,p} - \bigwedge_{k=1}^{m} c_1(L_k, h_k), \phi \right\rangle = \sum_{k=1}^{m} I_k,$$

where

$$\begin{aligned} I_k &= \left\langle c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_{k-1}, h_{k-1}) \wedge \left(\frac{\gamma_{k,p}}{p} - c_1(L_k, h_k)\right) \wedge \frac{\gamma_{k+1,p}}{p} \wedge \cdots \wedge \frac{\gamma_{m,p}}{p}, \phi \right\rangle \\ &= \left\langle c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_{k-1}, h_{k-1}) \wedge \frac{dd^c \log P_{k,p}}{2p} \wedge \frac{\gamma_{k+1,p}}{p} \wedge \cdots \wedge \frac{\gamma_{m,p}}{p}, \phi \right\rangle \\ &= \int_X \frac{\log P_{k,p}}{2p} c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_{k-1}, h_{k-1}) \wedge \frac{\gamma_{k+1,p}}{p} \wedge \cdots \wedge \frac{\gamma_{m,p}}{p} \wedge dd^c\phi. \end{aligned}$$

By (24) the total variation of the measure in the last integral is dominated by the positive measure $c' \|\phi\|_{\mathscr{C}^2}\, \mu$, where

$$\mu := c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_{k-1}, h_{k-1}) \wedge \frac{\gamma_{k+1,p}}{p} \wedge \cdots \wedge \frac{\gamma_{m,p}}{p} \wedge \omega^{n-m+1},$$

hence

$$|I_k| \leq c' \|\phi\|_{\mathscr{C}^2} \int_X \frac{|\log P_{k,p}|}{2p}\, d\mu.$$

Theorem 2.1 implies that there exist a constant $c''>0$ and $p_0\in\mathbb{N}$ such that for all $z\in X\setminus\Sigma(h_k)$ and all $p\geq p_0$ one has

$$-c''\leq\log P_{k,p}\leq c''\log p+c''|\log\operatorname{dist}(z,\Sigma(h_k))|,\quad 1\leq k\leq m.$$

We obtain that

$$|I_k|\leq\frac{c'c''\|\phi\|_{\mathscr{C}^2}}{2p}\int_X(\log p+|\log\operatorname{dist}(z,\Sigma(h_k))|)\,d\mu\quad\forall p\geq p_0,\,1\leq k\leq m.$$

Applying Lemma 5.2 below to the right-hand side yields that

$$|I_k|\leq\frac{c\log p}{mp}\|\phi\|_{\mathscr{C}^2}\quad\forall p\geq p_0,\,1\leq k\leq m,$$

with some constant $c=c(X,L_1,h_1,\dots,L_m,h_m)>0$. The proof is thereby completed. ■

The following crucial estimate was used in the proof of Theorem 5.1.

Lemma 5.2. In the setting of Theorem 5.1 there exists a constant $C>0$ such that for every $1\leq k\leq m$,

$$\int_X|\log\operatorname{dist}(z,\Sigma(h_k))|\,c_1(L_1,h_1)\wedge\dots\wedge c_1(L_{k-1},h_{k-1})\wedge\frac{\gamma_{k+1,p}}{p}\wedge\dots\wedge\frac{\gamma_{m,p}}{p}\wedge\omega^{n-m+1}<C.$$

□

Proof. Let $U\subset X$ be a contractible Stein open set as in Section 3. For $1\leq j\leq m$ and $p\geq 1$, let $u_{j,p}$, u_j be the psh functions defined in (12), so $dd^cu_j=c_1(L_j,h_j)$ and $dd^cu_{j,p}=\frac{1}{p}\gamma_{j,p}$ on U.

By shrinking U if necessary, we may construct a negative psh function v_k on U such that $v_k\leq\log\operatorname{dist}(z,\Sigma(h_k))<0$ and v_k is smooth outside $\Sigma(h_k)$. Indeed, let $f_1,\dots,f_N$ be holomorphic functions defined on a neighborhood of $\bar{U}$ such that

$$\Sigma(h_k)\cap U=\{z\in U:\ f_1(z)=\dots=f_N(z)=0\}.$$

We see easily that the function $v_k(z):=\log\left(|f_1(z)|^2+\dots+|f_N(z)|^2\right)-c'$, $z\in U$, with a suitable constant $c'>0$, does the job. Indeed, we may assume that the function $h(z)=|f_1(z)|^2+\dots+|f_N(z)|^2$ is Lipschitz on U, so there exist constants $c_1,c_2>0$ such that in local coordinates z on U, we have $|h(z)|\leq c_1|z-w|$ for all $z\in U$, $w\in\Sigma(h_k)$, hence $|h(z)|\leq c_2\operatorname{dist}(z,\Sigma(h_k))$, so $\log|h(z)|\leq\log\operatorname{dist}(z,\Sigma(h_k))+\log c_2$.

By [12, Theorem 5.1] we have that $\frac{1}{p}\log P_{j,p}\to 0$ in $L^1(X,\omega^n)$, hence by (12), $u_{j,p}\to u_j$ in $L^1_{loc}(U)$, as $p\to\infty$, for each $k+1\leq j\leq m$. Recall that by (14), $u_{j,p}\geq u_j-\frac{c''}{p}$ holds on

U for all p sufficiently large and some constant $c'' > 0$. Using the assumption that $\Sigma(h_j)$ are in general position, [26, Theorem 3.5, Corollary 3.6] implies that

$$v_k dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c u_{k+1,p} \wedge \cdots \wedge dd^c u_{m,p}$$
$$\to v_k dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c u_{k+1} \wedge \cdots \wedge dd^c u_m$$

weakly on U as $p \to \infty$, and that the right-hand side has locally bounded mass on U. Since X is compact, we may cover X by a finite number of open sets U as above. Writing ω^{n-m+1} as a finite sum of smooth positive forms such that each form is supported in at least one open set U, the lemma follows from the last limit. ■

Let us close the paper with more examples of Hölder metrics with singularities:

(1) Consider a projective manifold X and a smooth divisor $\Sigma \subset X$. By [32, 44], if $L = K_X \otimes \mathscr{O}_X(\Sigma)$ is ample, there exists a complete Kähler–Einstein metric ω on $M := X \setminus \Sigma$ with $\mathrm{Ric}_\omega = -\omega$. This metric has Poincaré type singularities, described as follows. We denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$. Each $x \in \Sigma$ has a coordinate neighborhood U_x such that

$$U_x \cong \mathbb{D}^n,\ x = 0, \quad U_x \cap \Sigma \cong \{z = (z_1, \ldots, z_n) : z_1 = 0\}, \quad U_x \cap M \cong \mathbb{D}^\star \times \mathbb{D}^{n-1}.$$

Then $\omega = \frac{i}{2} \sum_{j,k=1}^n g_{jk} dz_j \wedge d\bar{z}_k$ is quasi-isometric to the Poincaré-type metric

$$\omega_P = \frac{i}{2} \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2 \left(\log |z_1|^2\right)^2} + \frac{i}{2} \sum_{j=2}^n dz_j \wedge d\bar{z}_j.$$

Let σ be the canonical section of $\mathscr{O}_X(\Sigma)$ (cf. [33, p. 71]) and denote by h_σ the metric induced by σ on $\mathscr{O}_X(\Sigma)$ (cf. [33, Example 2.3.4]). Note also that $c_1(\mathscr{O}_X(\Sigma), h_\sigma) = [\Sigma]$ by [33, (2.3.8)]. Consider the metric

$$h_{M,\sigma} := h^{K_M} \otimes h_\sigma \quad \text{on } L\,|_M = K_M \otimes \mathscr{O}_X(\Sigma)\,|_M \cong K_M. \tag{25}$$

Note that L is trivial over U_x and the metric $h_{M,\sigma}$ has a weight φ on $U_x \cap M \cong \mathbb{D}^\star \times \mathbb{D}^{n-1}$ given by $e^{2\varphi} = |z_1|^2 \det[g_{jk}]$. So $dd^c\varphi = -\frac{1}{2\pi}\mathrm{Ric}_\omega > 0$ and φ is psh on $U_x \cap M$. We see as in [12, Lemma 6.8] that φ extends to a psh function on U_x, and $h_{M,\sigma}$ extends uniquely to a positively curved metric h^L on L. By construction, h^L is a Hölder metric with singularities.

(2) Let us specialize the previous example to the case of Riemann surfaces. Let X be a compact Riemann surface of genus g and let $\Sigma = \{p_1, \ldots, p_d\} \subset X$. It follows from

the Uniformization Theorem that the following conditions are equivalent:

(i) $U = X \setminus \Sigma$ admits a complete Kähler–Einstein metric ω with $\mathrm{Ric}_\omega = -\omega$,

(ii) $2g - 2 + d > 0$,

(iii) $L = K_X \otimes \mathscr{O}_X(\Sigma)$ is ample,

(iv) the universal cover of U is the upper-half plane $\mathbb{H}$.

If one of these equivalent conditions is satisfied, the Kähler–Einstein metric ω is induced by the Poincaré metric on $\mathbb{H}$. In local coordinate z centered at $p \in \Sigma$ we have $\omega = \frac{i}{2} g dz \wedge d\bar{z}$, where g satisfies $c|z|^{-2}(\log|z|^2)^{-2} \leq g(z) \leq c^{-1}|z|^{-2}(\log|z|^2)^{-2}$, for some $c > 0$, in a punctured neighborhood of p. Note that ω extends to a closed strictly positive $(1,1)$-current, that is, a positive measure of finite mass, on X. By [12, Lemma 6.8] there exists a singular metric h^L on L such that $c_1(L, h^L) = \frac{1}{2\pi}\omega$ on X. The weight of h^L near a point $p \in \Sigma$ has the form $\varphi(z) = \frac{1}{2}\log(|z|^2 g(z))$, which is Hölder with singularities.

(3) Let X be a complex manifold, (L, h_0^L) a holomorphic line bundle on X with smooth Hermitian metric such that $c_1(L, h_0^L)$ is a Kähler metric. Let Σ be a compact divisor with normal crossings. Let $\Sigma_1, \dots, \Sigma_N$ be the irreducible components of Σ, so Σ_j is a smooth hypersurface in X. Let σ_j be holomorphic sections of the associated holomorphic line bundle $\mathscr{O}_X(\Sigma_j)$ vanishing to first order on Σ_j and let $|\cdot|_j$ be a smooth Hermitian metric on $\mathscr{O}_X(\Sigma_j)$ so that $|\sigma_j|_j < 1$ and $|\sigma_j|_j = 1/e$ outside a relatively compact open set containing Σ. Set

$$\Theta_\delta = \Omega + \delta dd^c F, \quad \text{where } \delta > 0, \quad F = -\frac{1}{2}\sum_{j=1}^{N} \log\left(-\log|\sigma_j|_j\right).$$

For δ sufficiently small Θ_δ defines the generalized Poincaré metric [33, Lemma 6.2.1], [12, Section 2.3]. For $\varepsilon > 0$,

$$h_\varepsilon^L = h_0^L \prod_{j=1}^{N} \left(-\log|\sigma_j|_j\right)^\varepsilon$$

is a singular Hermitian metric on L which is Hölder with singularities. The curvature $c_1\left(L, h_\varepsilon^L\right)$ is a strictly positive current on X, provided that ε is sufficiently small (cf. [33, Lemma 6.2.1]). When X is compact the curvature current of h_ε dominates a small multiple of Θ_δ on $X \setminus \Sigma$.

(4) Let X be a Fano manifold. Fix a Hermitian metric h_0 on K_X^{-1} such that $\omega := c_1(K_X^{-1}, h_0)$ is a Kähler metric. We denote by $PSH(X,\omega)$ the set of ω-plurisubharmonic functions on X. Let Σ be a smooth divisor in the linear system defined by $K_X^{-\ell}$, so there exists a section $s \in H^0(X, K_X^{-\ell})$ with $\Sigma = \mathrm{Div}(s)$.

Fix a smooth metric h on the bundle $\mathscr{O}_X(\Sigma)$ and let $\beta \in (0,1]$. A conic Kähler–Einstein metric $\hat{\omega}$ on X with cone angle β along Σ, cf. [24, 43], is a current $\hat{\omega} = \omega_\varphi = \omega + dd^c\varphi \in c_1(X)$, where $\varphi = \psi + |s|_h^{2\beta} \in PSH(X,\omega)$ and $\psi \in \mathscr{C}^\infty(X) \cap PSH(X,\omega)$. In a neighborhood of a point of Σ where Σ is given by $z_1 = 0$ the metric $\hat{\omega}$ is equivalent to the cone metric $\frac{i}{2}\left(|z_1|^{2\beta-2} dz_1 \wedge d\bar{z}_1 + \sum_{j=2}^n dz_j \wedge d\bar{z}_j\right)$.

The metric $\hat{\omega}$ defines a singular metric $h_{\hat{\omega}}$ on K_X^{-1} which is Hölder with singularities. Its curvature current is $\mathrm{Ric}_{\hat{\omega}} := c_1\left(K_X^{-1}, h_{\hat{\omega}}\right) = (1-\ell(1-\beta))\hat{\omega} + (1-\beta)[\Sigma]$, where $[\Sigma]$ is the current of integration on Σ.

Funding

D.C. is partially supported by the NSF Grant DMS-1300157. G.M. is partially supported by DFG funded projects SFB/TR 12, MA 2469/2-2. V.-A.N. is partially supported by Humboldt Foundation, Max-Planck Institute for Mathematics and Vietnam Institute for Advanced Study in Mathematics (VIASM). Funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative.

Acknowledgement

We thank Vincent Guedj for useful discussions regarding this paper. We are grateful to the referee for a nice idea to improve an earlier version of Theorem 5.1.

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