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Approximation and equidistribution results for pseudo-effective line bundles ☆

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A B S T R A C T

We study the distribution of the common zero sets of m-tuples of holomorphic sections of powers of m singular Hermitian pseudo-effective line bundles on a compact Kähler manifold. As an application, we obtain sufficient conditions which ensure that the wedge product of the curvature currents of these line bundles can be approximated by analytic cycles.

R É S U M É

Nous étudions la distribution de l'ensemble des zéros communs de m-tuplets de sections holomorphes de puissances de m fibrés en droites hermitiens singuliers pseudo-effectifs sur une variété kählérienne compacte. Comme application, nous obtenons des conditions suffisantes pour que le produit extérieur des courants de courbure de ces fibrés puisse être approché par des cycles analytiques.

1. Introduction

A fundamental problem in pluripotential theory is to characterize those positive closed currents on a projective complex manifold which can be approximated by effective cycles with real coefficients. The case of bidegree $(1, 1)$ was originally introduced by Lelong [26] and intensively studied since then, see e.g. [13, 15,20,25]. Approximation results for positive currents have applications to analytic geometry (analyticity

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of level sets of Lelong numbers [32]), complex algebraic geometry (invariance of plurigenera [33,29]) and number theory (algebraic values of meromorphic maps [4,27,34]).

The problem of approximation by analytic cycles is widely open for higher bidegree currents and is linked to the Hodge conjecture [11]. It turns out that there are counter-examples [1] to the strong Hodge conjecture formulated in [11]. It is thus interesting to identify large classes of currents which can be approximated by analytic/rational cycles. The purpose of this paper is to study the case of currents obtained as the wedge product of curvature currents of singular Hermitian line bundles, by assuming that the analytic sets where the metrics are singular (unbounded or discontinuous) are in general position. Note that the study of singular metrics of holomorphic line bundles is central to several questions of complex algebraic geometry [4,12–15, 28].

The main ideas are an equidistribution property of the common zeros of random holomorphic sections of powers of line bundles and the compactness of the space of currents approachable by cycles. The equidistribution result enters the program of studying equidistribution in the setting of singular Hermitian metrics [5–9,16,17,30,31]. Our main tools are Bertini type results from [5,9] and Theorem 4.4, a version of an abstract equidistribution theorem due to Dinh–Sibony [18], which is a Large Deviation Principle in this setting. Another ingredient is an estimate on the Bergman kernel function (Theorem 3.1), which is of independent interest.

In the remaining of the introduction we give a short discussion about the background of this work and then state our approximation and equidistribution results.

1.1. Background

Let (X, ω) be a compact Kähler manifold of dimension n, dist be the distance on X induced by ω, and K_X be the *canonical line bundle* of X. If (L, h) is a holomorphic line bundle on X endowed with a singular Hermitian metric h, we denote by $c_1(L, h)$ its *curvature current,* cf. [12], [28, Section 2.3.1]. Recall that if e_L is a holomorphic frame of L on some open set $U \subset X$ then $|e_L|_h^2 = e^{-2\phi}$, where $\phi \in L^1_{loc}(U)$ is called the *local weight* of the metric h with respect to e_L, and $c_1(L, h)|_U = dd^c\phi$. Here $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. We say that h is *positively curved* (resp. *strictly positively curved*) if $c_1(L, h) \geq 0$ (resp. $c_1(L, h) \geq \varepsilon\omega$ for some $\varepsilon > 0$) in the sense of currents. This is equivalent to saying that the local weights ϕ are plurisubharmonic (psh for short) (resp. strictly plurisubharmonic). We say that (L, h) is *pseudo-effective* if the metric h is positively curved. For $p \in \mathbb{N}$ and L a holomorphic line bundle on X, let $L^p := L^{\otimes p}$. Given a holomorphic section $s \in H^0(X, L^p)$, we denote by $[s = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{s = 0\} \subset X$.

Recall that a holomorphic line bundle L is called *big* if its Kodaira–Iitaka dimension equals the dimension of X (see [28, Definition 2.2.5]). By the Shiffman–Ji–Bonavero–Takayama criterion [28, Lemma 2.3.6], L is big if and only if it admits a strictly positively curved singular Hermitian metric h. In this case we also say that (L, h) is big.

Recall from [9, Definition 1.1] the following concept. We say that the analytic subsets $A_1, \ldots, A_m \subset X$ are *in general position* if $\operatorname{codim}(A_{i_1} \cap \ldots \cap A_{i_k}) \geq k$ for every $1 \leq k \leq m$ and $1 \leq i_1 < \ldots < i_k \leq m$.

Let L_k, $1 \leq k \leq m \leq n$, be m holomorphic line bundles on (X, ω). For each $p \in \mathbb{N}^*$, we define $\mathscr{A}^p(L_1, \ldots, L_m)$ to be the space of all positive closed currents R of bidegree (m, m) on X of the form

$$R = \frac{1}{p^m}[s_{p1} = 0] \wedge \ldots \wedge [s_{pm} = 0], \;\; s_{pj} \in H^0(X, L_j^p), \tag{1}$$

where s_{pj} are such that the hypersurfaces $\{s_{p1} = 0\}, \ldots, \{s_{pm} = 0\}$ are in general position. This condition ensures that $[s_{p1} = 0] \wedge \ldots \wedge [s_{pm} = 0]$ is a well-defined positive closed current of bidegree (m, m) which is equal to the current of integration with multiplicities along the analytic set $\{s_{p1} = 0\} \cap \ldots \cap \{s_{pm} = 0\}$ (see e.g. [14, Corollary 2.11, Proposition 2.12] and [22, Theorem 3.5]).

When $L_1 = \cdots = L_m = L$, $\mathscr{A}^p(L, \ldots, L)$ is related with the space $\mathcal{A}_m(L^p)$ introduced by the first and second authors in [7] as

$$\mathcal{A}_m(L^p) = \left\{ R = \frac{1}{N} \sum_{l=1}^{N} R_l : \quad R_l \in \mathscr{A}^p(L, \ldots, L), \ N \in \mathbb{N}^* \right\}.$$

In particular, $\mathscr{A}^p(L, \ldots, L) \subset \mathcal{A}_m(L^p)$.

For each $p \in \mathbb{N}^*$, we define $\mathscr{A}^p_K(L_1, \ldots, L_m)$ to be the space of all positive closed currents R of bidegree (m, m) on X of the form

$$R = \frac{1}{p^m}[s_{p1} = 0] \wedge \ldots \wedge [s_{pm} = 0], \ \ s_{pj} \in H^0(X, L_j^p \otimes K_X), \tag{2}$$

where s_{pj} are such that the hypersurfaces $\{s_{p1} = 0\}, \ldots, \{s_{pm} = 0\}$ are in general position.

1.2. Approximation results

Here is our first approximation result using the sequence of spaces $\mathscr{A}^p(L_1, \ldots, L_m)$, $p \geq 1$.

Theorem 1.1. *Let (X, ω) be a compact Kähler manifold of dimension n and $1 \leq m \leq n$ be an integer. For $1 \leq k \leq m$ let L_k be a holomorphic line bundle on X endowed with two singular Hermitian metrics g_k and h_k such that:*

(i) *g_k and h_k are locally bounded outside a proper analytic subset $\Sigma_k \subset X$;*
(ii) *$c_1(L_k, g_k) \geq \varepsilon\omega$ on X for some $\varepsilon > 0$ and $c_1(L_k, h_k) \geq 0$ on X;*
(iii) *$\Sigma_1, \ldots, \Sigma_m$ are in general position.*

Then there exists a sequence of currents $R_j \in \mathscr{A}^{p_j}(L_1, \ldots, L_m)$, where $p_j \nearrow \infty$, such that R_j converges weakly on X to $c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m)$ as $j \to \infty$.

Working with sections of adjoint line bundles, i.e. using the sequence of spaces $\mathscr{A}^p_K(L_1, \ldots, L_m)$, $p \geq 1$, we obtain a more general approximation result than Theorem 1.1, in the sense that the metrics g_k are assumed to verify a weaker positivity condition. The next theorem only deals with two line bundles. However, it requires a very weak assumption on the sets where the metrics may not be continuous.

Theorem 1.2. *Let (X, ω) be a compact Kähler manifold of dimension $n \geq 2$, and for $1 \leq k \leq 2$ let L_k be a holomorphic line bundle on X endowed with two singular Hermitian metrics g_k and h_k such that:*

(i) *g_k and h_k are continuous outside a proper analytic subset $\Sigma_k \subset X$;*
(ii) *$c_1(L_k, h_k) \geq 0$ on X, $c_1(L_k, g_k) \geq 0$ on X, and for every $x \in X \setminus \Sigma_k$ there is a constant $c_x > 0$ so that $c_1(L_k, g_k) \geq c_x\omega$ in a neighborhood of x;*
(iii) *Σ_1 and Σ_2 are in general position.*

Then there exists a sequence of currents $R_j \in \mathscr{A}^{p_j}_K(L_1, L_2)$, where $p_j \nearrow \infty$, such that R_j converges weakly on X to $c_1(L_1, h_1) \wedge c_1(L_2, h_2)$ as $j \to \infty$.

The last approximation result deals with several line bundles. However, it requires a strong assumption on the set where the metrics may not be continuous.

Theorem 1.3. *Let (X, ω) be a compact Kähler manifold of dimension n and $1 \leq m \leq n$ be an integer. For $1 \leq k \leq m$ let L_k be a holomorphic line bundle on X endowed with two singular Hermitian metrics g_k and h_k such that:*

(i) *g_k and h_k are continuous outside a proper analytic subset $\Sigma \subset X$;*
(ii) *$c_1(L_k, h_k) \geq 0$ on X, $c_1(L_k, g_k) \geq 0$ on X, and for every $x \in X \setminus \Sigma$ there is a constant $c_x > 0$ so that $c_1(L_k, g_k) \geq c_x \omega$ in a neighborhood of x;*
(iii) *$\operatorname{codim}(\Sigma) \geq m$.*

Then there exists a sequence of currents $R_j \in \mathscr{A}_K^{p_j}(L_1, \ldots, L_m)$, where $p_j \nearrow \infty$, such that R_j converges weakly on X to $c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m)$ as $j \to \infty$.

1.3. Equidistribution results

In order to investigate the equidistribution problem, we need to introduce some more notation and terminology. Let (L_k, h_k), $1 \leq k \leq m \leq n$, be m singular Hermitian holomorphic line bundles on (X, ω). Let $H^0_{(2)}(X, L_k^p)$ (resp. $H^0_{(2)}(X, L_k^p \otimes K_X)$) be the Bergman space of L^2-holomorphic sections of L_k^p (resp. of $L_k^p \otimes K_X$) relative to the metric $h_{k,p} := h_k^{\otimes p}$ induced by h_k, and the metric h^{K_X} on K_X induced by the volume form ω^n on X. These spaces are endowed with the respective inner product

$$
\begin{aligned}
(S, S')_{k,p} &:= \int_X \langle S, S' \rangle_{h_{k,p}} \, \omega^n \,, \quad S, S' \in H^0_{(2)}(X, L_k^p), \\
(S, S')^K_{k,p} &:= \int_X \langle S, S' \rangle_{h_{k,p} \otimes h^{K_X}} \, \omega^n \,, \quad S, S' \in H^0_{(2)}(X, L_k^p \otimes K_X).
\end{aligned}
\tag{3}
$$

For every $p \geq 1$ and $1 \leq k \leq m$, let $\sigma_{k,p}$ be the Fubini–Study volume form on the projective space $\mathbb{P}H^0_{(2)}(X, L_k^p)$ (resp. $\mathbb{P}H^0_{(2)}(X, L_k^p \otimes K_X)$) which is the projectivization of the finite-dimensional complex vector spaces $H^0_{(2)}(X, L_k^p)$ (resp. $H^0_{(2)}(X, L_k^p \otimes K_X)$) endowed with the above inner product $(S, S')_{k,p}$ (resp. $(S, S')^K_{k,p}$). Clearly, the measure $\sigma_{k,p}$ depends not only on L_k and p, but also on h_k.

For every $p \geq 1$ we consider the multi-projective spaces

$$
\begin{aligned}
\mathbb{X}_p &:= \mathbb{P}H^0_{(2)}(X, L_1^p) \times \ldots \times \mathbb{P}H^0_{(2)}(X, L_m^p), \\
\mathbb{X}_{K,p} &:= \mathbb{P}H^0_{(2)}(X, L_1^p \otimes K_X) \times \ldots \times \mathbb{P}H^0_{(2)}(X, L_m^p \otimes K_X)
\end{aligned}
\tag{4}
$$

equipped with the probability measure σ_p which is the product of the Fubini–Study volumes on the components. If $S \in H^0(X, L_k^p)$ (resp. $S \in H^0(X, L_k^p \otimes K_X)$), we denote by $[S = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{S = 0\}$ of X. Set

$$
[\mathbf{s}_p = 0] := [s_{p1} = 0] \wedge \ldots \wedge [s_{pm} = 0] \,, \text{ for } \mathbf{s}_p = (s_{p1}, \ldots, s_{pm}) \in \mathbb{X}_p \text{ or } \in \mathbb{X}_{K,p},
$$

whenever the hypersurfaces $\{s_{p1} = 0\}, \ldots, \{s_{pm} = 0\}$ of X are in general position. We also consider the probability spaces

$$
\begin{aligned}
(\Omega, \sigma_\infty) &:= \prod_{p=1}^{\infty} (\mathbb{X}_p, \sigma_p), \\
(\Omega_K, \sigma_\infty) &:= \prod_{p=1}^{\infty} (\mathbb{X}_{K,p}, \sigma_p) \,.
\end{aligned}
\tag{5}
$$

For the sake of clarity we may write $\Omega(h_1,\dots,h_m)$, $\sigma_\infty(h_1,\dots,h_m)$ (resp. $\Omega_K(h_1,\dots,h_m)$, $\sigma_\infty(h_1,\dots,h_m)$) in order to make precise the dependence of (Ω,σ_∞) (resp. (Ω_K,σ_∞)) on the metrics $h_1,\dots,h_m$. Assume that for σ_∞-a.e. $\{\mathbf{s}_p\}_{p\geq 1}\in\Omega$ (resp. $\in\Omega_K$), the hypersurfaces $\{s_{p1}=0\},\dots,\{s_{pm}=0\}$ of X are in general position for all p sufficiently large.

Definition 1.4. We say that (Ω,σ_∞) (resp. (Ω_K,σ_∞)) (or simply Ω (resp. Ω_K) if σ_∞ is clear from the context) *equidistributes toward a positive closed* (m,m) *current* T defined on X if for σ_∞-a.e. $\{\mathbf{s}_p\}_{p\geq 1}\in\Omega$ (resp. $\in\Omega_K$), we have in the weak sense of currents on X,

$$\frac{1}{p^m}\,[\mathbf{s}_p=0]\to T \ \text{ as } p\to\infty\,.$$

Our first equidistribution theorem only deals with two line bundles. However, it requires a very weak assumption on the sets where the metrics may not be continuous.

Theorem 1.5. *Let (X,ω) be a compact Kähler manifold of dimension $n\geq 2$ and (L_k,h_k), $1\leq k\leq 2$, be two singular Hermitian holomorphic line bundles on X such that:*

(i) *h_k is continuous outside a proper analytic subset $\Sigma_k\subset X$;*
(ii) *$c_1(L_k,h_k)\geq 0$ on X, and for every $x\in X\setminus\Sigma_k$ there is a constant $c_x>0$ so that $c_1(L_k,h_k)\geq c_x\omega$ in a neighborhood of x;*
(iii) *Σ_1 and Σ_2 are in general position.*

Then Ω_K equidistributes toward the current $c_1(L_1,h_1)\wedge c_1(L_2,h_2)$.

The last equidistribution result deals with several line bundles. However, it requires a strong assumption on the set where the metrics may not be continuous.

Theorem 1.6. *Let (X,ω) be a compact Kähler manifold of dimension n and (L_k,h_k), $1\leq k\leq m\leq n$, be m singular Hermitian holomorphic line bundles on X such that:*

(i) *h_k is continuous outside a proper analytic subset $\Sigma\subset X$;*
(ii) *$c_1(L_k,h_k)\geq 0$ on X, and for every $x\in X\setminus\Sigma$ there is a constant $c_x>0$ so that $c_1(L_k,h_k)\geq c_x\omega$ in a neighborhood of x;*
(iii) *$\operatorname{codim}(\Sigma)\geq m$.*

Then Ω_K equidistributes toward the current $c_1(L_1,h_1)\wedge\ldots\wedge c_1(L_m,h_m)$.

This paper is organized as follows. In Section 2 we first present a method which allows us to deduce approximation results from equidistribution theorems. Using this method as well as our previous work [9], in the remainder of the section we give the proof of Theorem 1.1 and we show that equidistribution theorems (Theorems 1.5–1.6) imply approximations theorems (Theorems 1.2–1.3).

The next two sections develop the necessary tools. Section 3 studies the dimension growth of section spaces and Bergman kernel functions. Section 4 establishes the convergence towards intersection of Fubini–Study currents. Here we apply equidistribution results due to Dinh–Sibony for meromorphic transforms [18].

Having these two tools at hand and using the intersection theory of positive closed currents, our first equidistribution theorem (for two line bundles), Theorem 1.5, will be proved in Section 5.

Finally, Section 6 concludes the article with the proof of our second equidistribution theorem, Theorem 1.6.

2. Equidistribution implies approximation

Let (X, ω) be a compact Kähler manifold of dimension n and $0 \leq m \leq n$ be an integer. In [19] Dinh and Sibony have introduced the following natural metric on the space of positive closed currents of bidegree (m, m) on X. If R and S are such currents, define

$$\operatorname{dist}(R, S) := \sup_{\|\Phi\|_{\mathscr{C}^1} \leq 1} |\langle R - S, \Phi \rangle|,$$

where Φ is a smooth $(n-m, n-m)$-form on X and we use the sum of $\mathscr{C}^1$-norms of its coefficients for a fixed atlas on X.

Lemma 2.1. (i) *On the convex set of positive closed currents of bidegree (m, m) and of mass ≤ 1 on X, the topology induced by the above distance coincides with the weak topology.*

(ii) *Assume that T, $(T_j)_{j=1}^{\infty}$, $(T_{jp})_{j,p=1}^{\infty}$ are positive closed currents of bidegree (m, m) on X such that $T_j \to T$ as $j \to \infty$ and that for each $j \in \mathbb{N}^*$, $T_{jp} \to T_j$ as $p \to \infty$. Then there is a subsequence $(p_j)_{j=1}^{\infty} \subset \mathbb{N}^* \nearrow \infty$ such that $T_{jp_j} \to T$ as $j \to \infty$.*

Proof. Assertion (i) has been proved in [19, Proposition 2.1.4].

By passing to a subsequence if necessary, we may assume without loss of generality that the masses of currents $T, (T_j)_{j=1}^{\infty}, (T_{jp})_{j,p=1}^{\infty}$ are all bounded from above by a common finite constant. Therefore, applying assertion (i), we obtain that

$$\lim_{j \to \infty} \operatorname{dist}(T_j, T) = 0 \quad \text{and} \quad \lim_{p \to \infty} \operatorname{dist}(T_{jp}, T_j) = 0 \quad \text{for each } j \in \mathbb{N}^*.$$

The second limit shows that for every $j \in \mathbb{N}^*$, there is $p_j > p_{j-1}$ such that $\operatorname{dist}(T_{jp_j}, T_j) \leq 1/j$. This, combined with the first limit, implies that $\lim_{j \to \infty} \operatorname{dist}(T_{jp_j}, T) = 0$, proving assertion (ii) in view of assertion (i). □

The following result shows that equidistribution implies approximation:

Proposition 2.2. *Let (X, ω) be a compact Kähler manifold of dimension n and $1 \leq m \leq n$ be an integer. For $1 \leq k \leq m$ let L_k be a holomorphic line bundle on X endowed with two singular Hermitian metrics g_k and h_k such that*

- (i) *g_k and h_k are locally bounded outside a proper analytic subset $\Sigma_k \subset X$;*
- (ii) *$c_1(L_k, g_k) \geq 0$ and $c_1(L_k, h_k) \geq 0$ on X;*
- (iii) *$\Sigma_1, \ldots, \Sigma_m$ are in general position;*
- (iv) *there is a sequence $\epsilon_j \searrow 0$ such that $\Omega(h_1^{1-\epsilon_j} g_1^{\epsilon_j}, \ldots, h_m^{1-\epsilon_j} g_m^{\epsilon_j})$ (resp. $\Omega_K(h_1^{1-\epsilon_j} g_1^{\epsilon_j}, \ldots, h_m^{1-\epsilon_j} g_m^{\epsilon_j})$) equidistributes towards the current $c_1(L_1, h_1^{1-\epsilon_j} g_1^{\epsilon_j}) \wedge \ldots \wedge c_1(L_m, h_m^{1-\epsilon_j} g_m^{\epsilon_j})$ for all $j \in \mathbb{N}^*$.*

Then there exists a sequence of currents $R_j \in \mathscr{A}^{p_j}(L_1, \ldots, L_m)$ (resp. $R_j \in \mathscr{A}_K^{p_j}(L_1, \ldots, L_m)$), where $p_j \nearrow \infty$, such that R_j converges weakly on X to $c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m)$ as $j \to \infty$.

Proof. We only give the proof when the space Ω is considered in (iv). The case of Ω_K is identical to this one.

For each $1 \leq k \leq m$ and $j \in \mathbb{N}^*$, observe that the metric $h_k^{1-\epsilon_j} g_k^{\epsilon_j}$ is locally bounded outside Σ_k by (i), and that

$$c_1(L_k, h_k^{1-\epsilon_j} g_k^{\epsilon_j}) = (1 - \epsilon_j) c_1(L_k, h_k) + \epsilon_j \, c_1(L_k, g_k) \geq 0$$

by (ii). Consequently, using (iii) and applying [14, Corollary 2.11, Proposition 2.12] or [22, Theorem 3.5], we get that

$$\begin{aligned} T_j &:= c_1(L_1, h_1^{1-\epsilon_j} g_1^{\epsilon_j}) \wedge \ldots \wedge c_1(L_m, h_m^{1-\epsilon_j} g_m^{\epsilon_j}) \\ &= (1-\epsilon_j)^m c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m) + \sum_J \epsilon_j^{m-|J|}(1-\epsilon_j)^{|J|} \bigwedge_{k\in J} c_1(L_k, h_k) \wedge \bigwedge_{k\in J'} c_1(L_k, g_k), \end{aligned}$$

where the last sum is taken over all subsets $J \subsetneq \{1, \ldots, m\}$ with cardinal $|J|$ and $J' := \{1, \ldots, m\} \setminus J$. Since for such a subset J, the current $\bigwedge_{k\in J} c_1(L_k, h_k) \wedge \bigwedge_{k\in J'} c_1(L_k, g_k)$ is a well-defined positive closed current and $\epsilon_j^{m-|J|} \searrow 0$ as $j \nearrow \infty$, it follows that

$$c_1(L_1, h_1^{1-\epsilon_j} g_1^{\epsilon_j}) \wedge \ldots \wedge c_1(L_m, h_m^{1-\epsilon_j} g_m^{\epsilon_j}) \to c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m) =: T \quad \text{as } j \to \infty.$$

In other words, $T_j \to T$ as $j \to \infty$.

On the other hand, by (iv) for each $j \in \mathbb{N}^*$ and each $p \in \mathbb{N}^*$ there is a current $T_{jp} \in \mathscr{A}^p(L_1, \ldots, L_m)$ such that $T_{jp} \to T_j$ as $p \to \infty$. Applying Lemma 2.1 (ii) to the above family of currents T, $(T_j)_{j=1}^\infty$ and $(T_{jp})_{j,p=1}^\infty$, we can find a subsequence $(p_j)_{j=1}^\infty \subset \mathbb{N}^* \nearrow \infty$ such that

$$T_{jp_j} \to c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m) \quad \text{as } j \to \infty.$$

Since $R_j := T_{jp_j} \in \mathscr{A}^{p_j}(L_1, \ldots, L_m)$ and $p_j \nearrow \infty$ as $j \nearrow \infty$, the proof is complete. □

To illustrate the usefulness of Proposition 2.2, we give in the remainder of the section the proof of Theorems 1.1–1.3, by taking Theorems 1.5 and 1.6 for granted. The following equidistribution result is needed.

Theorem 2.3. *Let (X, ω) be a compact Kähler manifold of dimension n and (L_k, h_k), $1 \le k \le m \le n$, be m singular Hermitian holomorphic line bundles on X such that:*

(i) *h_k is locally bounded outside a proper analytic subset $\Sigma_k \subset X$;*
(ii) *$c_1(L_k, h_k) \ge \varepsilon\omega$ on X for some $\varepsilon > 0$;*
(iii) *$\Sigma_1, \ldots, \Sigma_m$ are in general position.*

Then $\Omega(h_1, \ldots, h_m)$ equidistributes toward the current $c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m)$.

Proof. When (i) is replaced by the stronger condition that h_k is continuous outside Σ_k, the theorem was proved in our previous work [9, Theorem 1.2]. A careful verification shows that our proof still works assuming the weaker condition (i). Indeed, let $P_{k,p}$ be the Bergman kernel function of $(L_k^p, h_{k,p})$ (see [9, eq. (4)]). Then the fact that $\frac{1}{p} \log P_{k,p} \to 0$ in $L^1(X, \omega^n)$ as $p \to \infty$, as well as the estimate [9, eq. (14)] hold under the more general assumption (i) (see also [5, Theorem 5.1]). Moreover, [9, Proposition 4.7] holds with the same proof for metrics that are locally bounded outside an analytic subset of X. □

Proof of Theorem 1.1. Fix a sequence $\epsilon_j \searrow 0$ as $j \nearrow \infty$. By (ii) we get that

$$c_1(L_k, h_k^{1-\epsilon_j} g_k^{\epsilon_j}) = (1-\epsilon_j) c_1(L_k, h_k) + \epsilon_j\, c_1(L_k, g_k) \ge \epsilon_j \varepsilon\, \omega, \qquad 1 \le k \le m.$$

Therefore, applying Theorem 2.3 to the singular Hermitian holomorphic big line bundles $(L_k, h_k^{1-\epsilon_j} g_k^{\epsilon_j})$, $1 \le k \le m$, we infer that $\Omega(h_1^{1-\epsilon_j} g_1^{\epsilon_j}, \ldots, h_m^{1-\epsilon_j} g_m^{\epsilon_j})$ equidistributes toward the current $c_1(L_1, h_1^{1-\epsilon_j} g_1^{\epsilon_j}) \wedge \ldots \wedge c_1(L_m, h_m^{1-\epsilon_j} g_m^{\epsilon_j})$ for all $j \in \mathbb{N}^*$. Putting this together with (i) and (iii), we are in the position to apply Proposition 2.2, and hence the proof is complete. □

Proofs of Theorem 1.2 and Theorem 1.3. Theorem 1.2 (resp. Theorem 1.3) follows from Theorem 1.5 (resp. Theorem 1.6). We will only give here the proof that Theorem 1.5 implies Theorem 1.2. The other implication can be proved similarly. Fix a sequence $\epsilon_j \searrow 0$ as $j \nearrow \infty$. By (ii) we get that

$$c_1(L_k, h_k^{1-\epsilon_j} g_k^{\epsilon_j}) = (1-\epsilon_j)c_1(L_k, h_k) + \epsilon_j\, c_1(L_k, g_k) \geq \epsilon_j\, c_1(L_k, g_k), \qquad 1 \leq k \leq m.$$

Therefore, applying Theorem 1.5 to the singular Hermitian holomorphic line bundles $(L_k, h_k^{1-\epsilon_j} g_k^{\epsilon_j})$, $1 \leq k \leq 2$, we infer that $\Omega_K(h_1^{1-\epsilon_j} g_1^{\epsilon_j}, h_2^{1-\epsilon_j} g_2^{\epsilon_j})$ equidistributes toward the current $c_1(L_1, h_1^{1-\epsilon_j} g_1^{\epsilon_j}) \wedge c_1(L_2, h_2^{1-\epsilon_j} g_2^{\epsilon_j})$ for all $j \in \mathbb{N}^*$. Putting this together with (i) and (iii), we are in the position to apply Proposition 2.2, and hence the proof of Theorem 1.2 is complete. □

3. Dimension growth of section spaces and Bergman kernel functions

In this section we prove a theorem about the dimension growth of section spaces and the asymptotic behavior of the Bergman kernel function of adjoint line bundles.

Let (L, h) be a singular Hermitian holomorphic line bundle over a compact Kähler manifold (X, ω) of dimension n. Consider the space $H^0_{(2)}(X, L^p \otimes K_X)$ of L^2-holomorphic sections of L^p relative to the metric $h_p := h^{\otimes p}$ induced by h, h^{K_X} on K_X and the volume form ω^n on X, endowed with the natural inner product (see (3)). Since $H^0_{(2)}(X, L^p \otimes K_X)$ is finite dimensional, let

$$d_p := \dim H^0_{(2)}(X, L^p \otimes K_X) - 1, \tag{6}$$

and when $d_p \geq 0$ let $\{S_j^p\}_{j=0}^{d_p}$ be an orthonormal basis. We denote by P_p the Bergman kernel function defined by

$$P_p(x) = \sum_{j=0}^{d_p} |S_j^p(x)|^2_{h_p \otimes h^{K_X}}, \quad |S_j^p(x)|^2_{h^p \otimes h^{K_X}} := \langle S_j^p(x), S_j^p(x)\rangle_{h_p \otimes h^{K_X}}, \ x \in X. \tag{7}$$

Note that this definition is independent of the choice of basis.

Theorem 3.1. *Let (X, ω) be a compact Kähler manifold of dimension n, (L, h) be a singular Hermitian holomorphic line bundle on X, and $\Sigma \subset X$ be a proper analytic subset such that:*

(i) *h is continuous outside Σ;*
(ii) *$c_1(L, h) \geq 0$ on X, and for every $x \in X \setminus \Sigma$ there is a constant $c_x > 0$ so that $c_1(L, h) \geq c_x \omega$ in a neighborhood of x.*

For every $p \geq 1$, let d_p be given by (6) and P_p be the Bergman kernel function defined by (7) for the space $H^0_{(2)}(X, L^p \otimes K_X)$. Then

1) $\lim_{p\to\infty} \frac{1}{p} \log P_p(x) = 0$ *locally uniformly on $X \setminus \Sigma$.*
2) *There is a constant $c > 1$ such that $c^{-1} \leq d_p/p^n \leq c$ for all $p \geq 1$.*

In order to prove our theorems we need the following variant of the existence theorem for $\overline{\partial}$ in the case of singular Hermitian metrics. The smooth case goes back to Andreotti–Vesentini and Hörmander, while the singular case was first observed by Bombieri and Skoda and proved in the present general form by Demailly [10, Theorem 5.1].

Theorem 3.2 *(L^2-estimates for $\overline{\partial}$). Let (M,ω) be a Kähler manifold of dimension n which admits a complete Kähler metric. Let (L,h) be a singular Hermitian holomorphic line bundle and let $\lambda : M \to [0,\infty)$ be a continuous function such that $c_1(L,h) \geq \lambda\omega$. Then for any form $g \in L^2_{n,1}(M,L,\mathrm{loc})$ satisfying*

$$\overline{\partial} g = 0, \qquad \int_M \lambda^{-1} |g|^2_{\omega,h} \omega^n < \infty\,,$$

there exists $u \in L^2_{n,0}(M,L,\mathrm{loc})$ with $\overline{\partial} u = g$ and

$$\int_M |u|^2_{\omega,h} \omega^n \leq \int_M \lambda^{-1} |g|^2_{\omega,h} \omega^n\,.$$

Proof. See [6, Corollary 4.2]. □

Proof of Theorem 3.1. Following the arguments of [13,5,6] we will first establish the following upper and lower estimates (8)–(9) for $\frac{1}{p}\log P_p$.

To state the upper estimate (8), let $x \in X$ and let $U_\alpha \subset X$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_α of L and e'_α of K_X. Let ψ_α be a psh weight of h and ρ_α be a smooth weight of h^{K_X} on U_α. Fix $r_0 > 0$ so that the ball $V := B(x, 2r_0) \subset\subset U_\alpha$ and let $U := B(x, r_0)$. Then (8) says that there exist constants $C > 0$, $p_0 \in \mathbb{N}$, so that

$$\frac{\log P_p(z)}{p} \leq \frac{\log(Cr^{-2n})}{p} + 2\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right) \tag{8}$$

holds for all $p > p_0$, $0 < r < r_0$, and $z \in U$ with $\psi_\alpha(z) > -\infty$.

The lower estimate (9) says that for every $x \in X \setminus \Sigma$, there exist a constant $C = C_x$, $p_0 \in \mathbb{N}$ large enough and an open neighborhood U of x such that

$$-\frac{\log C}{p} \leq \frac{1}{p} \log P_p(z) \tag{9}$$

holds for all $p > p_0$ and $z \in U$.

For the upper estimate (8), let $S \in H^0_{(2)}(X, L^p \otimes K_X)$ with $\|S\| = 1$ and write $S = s e_\alpha^{\otimes p} \otimes e'_\alpha$. Repeating an argument of Demailly we obtain, for $0 < r < r_0$,

$$\begin{aligned}
|S(z)|^2_{h_p \otimes h^{K_X}} &= |s(z)|^2 e^{-2p\psi_\alpha(z) - 2\rho_\alpha(z)} \leq e^{-2p\psi_\alpha(z) - 2\rho_\alpha(z)} \frac{C_1}{r^{2n}} \int_{B(z,r)} |s|^2\, \omega^n \\
&\leq \frac{C_1}{r^{2n}} \exp\left(2p\left(\max_{B(z,r)} (\psi_\alpha + \frac{\rho_\alpha}{p}) - \psi_\alpha(z) - \frac{\rho_\alpha(z)}{p}\right)\right) \int_{B(z,r)} |s|^2 e^{-2p\psi_\alpha - 2\rho_\alpha}\, \omega^n \\
&\leq \frac{C_2}{r^{2n}} \exp\left(2p\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right)\right),
\end{aligned}$$

where C_1, C_2 are constants that depend only on x, and we use here the fact that ρ_α is smooth. Hence

$$\frac{1}{p}\log P_p(z) = \frac{1}{p} \max_{\|S\|=1} \log |S(z)|^2_{h_p \otimes h^{K_X}} \leq \frac{\log(C_2 r^{-2n})}{p} + 2\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right).$$

Note that this estimate holds for all p and it does not require the strict positivity of the current $c_1(L,h)$, nor the hypotheses that X is compact or ω is a Kähler form. Covering X by a finite number of such open set U, the last estimate implies (8).

For the lower estimate (9), let $x \in X \setminus \Sigma$ and $U_\alpha \Subset X \setminus \Sigma$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_α of L and e'_α of K_X. Let ψ_α be a psh weight of h and ρ_α be a smooth weight of h^{K_X} on U_α. Fix $r_0 > 0$ so that the ball $V := B(x, 2r_0) \subset\subset U_\alpha$ and let $U := B(x, r_0)$. Next, we proceed as in [15, Section 9] and [6, Theorem 4.2] to show that there exist a constant $C = C_x > 0$ and $p_0 \in \mathbb{N}$ large enough such that for all $p > p_0$ and all $z \in U$ (note that $\psi_\alpha > -\infty$ on U) there is a section $S_{z,p} \in H^0_{(2)}(X, L^p \otimes K_X)$ with $S_{z,p}(z) \neq 0$ and

$$\|S_{z,p}\|^2 \leq C|S_{z,p}(z)|^2_{h_p \otimes h^{K_X}}\,.$$

Observe that this implies that

$$\frac{1}{p}\log P_p(z) = \frac{1}{p}\max_{\|S\|=1}\log|S(z)|^2_{h_p\otimes h^{K_X}} \geq -\frac{\log C}{p}\,.$$

Note that, by the continuity of ψ_α, putting (8) and (9) together implies that $\frac{1}{p}\log P_p \to 0$ as $p \to \infty$ locally uniformly on $V \setminus \Sigma$. This proves Part 1).

Now we turn to the proof of Part 2). Let $x \in X \setminus \Sigma$ and $U_\alpha \Subset X \setminus \Sigma$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_α of L and e'_α of K_X. Let ψ_α be a psh weight of h and ρ_α be a smooth weight of h^{K_X} on U_α. Fix $r_0 > 0$ so that the ball $V := B(x, 2r_0) \subset\subset U_\alpha$ and let $U := B(x, r_0)$. Let $\theta \in \mathscr{C}^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \theta \leq 1$, $\theta(t) = 1$ for $|t| \leq \frac{1}{2}$, $\theta(t) = 0$ for $|t| \geq 1$. For $z \in U$, define the quasi-psh function φ_z on X by

$$\varphi_z(y) = \begin{cases} 2\theta\big(\frac{|y-z|}{r_0}\big)\log\big(\frac{|y-z|}{r_0}\big)\,, & \text{for } y \in U_\alpha\,, \\ 0, \quad \text{for } y \in X \setminus B(z, r_0)\,. \end{cases} \tag{10}$$

Note the function φ_z is psh, hence $dd^c\varphi_z \geq 0$, on $\{y : \ |y-z| \leq r_0/2\}$. Since $V \Subset U_\alpha$, it follows that there exists a constant $c' > 0$ such that for all $z \in U$ we have $dd^c\varphi_z \geq -c'\omega$ on X and $dd^c\varphi_z = 0$ outside $\overline{V}$. By assumption (ii), there is a constant $c > 0$ such that $c_1(L,h) \geq c\omega$ on a neighborhood of $\overline{V}$. Therefore, there exist $0 < a, b < 1$ and $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$ and all $z \in U$

$$\begin{aligned} &c_1(L^p, h_p e^{-bp\varphi_z}) \geq 0 && \text{on } X, \\ &c_1(L^p, h_p e^{-bp\varphi_z}) \geq ap\omega && \text{on a neighborhood of } \overline{V}. \end{aligned}$$

Let $\lambda : \ X \to [0,\infty)$ be a continuous function such that $\lambda = ap$ on $\overline{V}$ and

$$c_1(L^p, h_p e^{-bp\varphi_z}) \geq \lambda\omega \qquad \text{on} \qquad X.$$

By identifying V to an open ball in $\mathbb{C}^n$, we may write $y = (y_1, \ldots, y_n)$ for $y \in V$. Fix $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ with $\sum_{j=1}^n \beta_j \leq [bp] - n$. Let $v_{z,p,\beta} \in \mathscr{O}(V)$ be given by

$$v_{z,p,\beta}(y) := (y_1 - z_1)^{\beta_1}\cdots(y_n - z_n)^{\beta_n} \qquad \text{for} \qquad y \in V. \tag{11}$$

Consider the form

$$g_{z,p,\beta} \in L^2_{n,1}(X, L^p),\ g_{z,p,\beta} := \overline{\partial}\big(v_{z,p,\beta}\theta\big(\tfrac{|y-z|}{r_0}\big)e_\alpha^{\otimes p} \otimes e'_\alpha\big).$$

As $g_{z,p,\beta}=0$ outside V, we get that

$$\int_X \frac{1}{\lambda}|g_{z,p,\beta}|^2_{h_p\otimes h^{K_X}}e^{-bp\varphi_z}\omega^n=\int_V \frac{1}{\lambda}|g_{z,p,\beta}|^2_{h_p\otimes h^{K_X}}e^{-bp\varphi_z}\omega^n=\frac{1}{ap}\int_V |g_{z,p,\beta}|^2_{h_p\otimes h^{K_X}}e^{-bp\varphi_z}\omega^n\,.$$

Note that the integral at the right is finite since ψ_α is bounded on V, $g_{z,p,\beta}=0$ on $B(z,r_0/2)$, and φ_z is bounded on $V\setminus B(z,r_0/2)$, so

$$\begin{aligned}\int_V |g_{z,p,\beta}|^2_{h_p\otimes h^{K_X}}e^{-bp\varphi_z}\omega^n&=\int_{V\setminus B(z,r_0/2)}|v_{z,p,\beta}|^2|\overline{\partial}\theta(\tfrac{|y-z|}{r_0})|^2e^{-2p\psi_\alpha}e^{-bp\varphi_z}\omega^n\\&\le C''_p\int_V|v_{z,p,\beta}|^2e^{-2p\psi_\alpha}\omega^n<\infty,\end{aligned}$$

where $C''_p>0$ is a constant that depends only on x and p.

The hypotheses of Theorem 3.2 are satisfied for the complete Kähler manifold (X,ω), the semipositive line bundle $(L^p,h_pe^{-bp\varphi_z})$ and the form $g_{z,p,\beta}$, for all $p\ge p_0$ and $z\in U$. So by Theorem 3.2, there exists $u_{z,p,\beta}\in L^2_{n,0}(X,L^p)$ such that $\overline{\partial}u_{z,p,\beta}=g_{z,p,\beta}$ and

$$\int_X |u_{z,p,\beta}|^2_{h_p\otimes h^{K_X}}e^{-bp\varphi_z}\omega^n\le\int_X\frac{1}{\lambda}|g_{z,p,\beta}|^2_{h_p\otimes h^{K_X}}e^{-bp\varphi_z}\omega^n\le\frac{C''_p}{ap}\int_V|v_{z,p,\beta}|^2e^{-2p\psi_\alpha}\omega^n. \tag{12}$$

Define

$$S_{z,p,\beta}:=v_{z,p,\beta}\theta\big(\tfrac{|y-z|}{r_0}\big)e_\alpha^{\otimes p}\otimes e'_\alpha-u_{z,p,\beta}.$$

Then $\overline{\partial}S_{z,p,\beta}=0$ and $S_{z,p,\beta}\in H^0_{(2)}(X,L^p\otimes K_X)$. Moreover, by (10) we get that

$$S_{z,p,\beta}(y)=v_{z,p,\beta}(y)e_\alpha^{\otimes p}\otimes e'_\alpha-u_{z,p,\beta}(y)\quad\text{for}\quad y\in B(z,r_0/2). \tag{13}$$

Therefore, we deduce from this and (11) that $\overline{\partial}u_{z,p,\beta}=\overline{\partial}S_{z,p,\beta}=0$. Thus $u_{z,p,\beta}$ is a $(n,0)$-holomorphic form near z.

Let $\mathscr{J}$ be the sheaf of holomorphic functions on X vanishing at z and let $\mathbf{m}\subset\mathscr{O}_{X,z}$ the maximal ideal of the ring of germs of holomorphic functions at z. For $k,p\in\mathbb{N}$ we have a canonical residue map $L^p\otimes K_X\to L^p\otimes K_X\otimes(\mathscr{O}_X/\mathscr{J}^{k+1})$ which induces in cohomology a map which associates to each global L^2-holomorphic section of $L^p\otimes K_X$ its *k-jet* at z:

$$J^k_p:\ H^0_{(2)}(X,L^p\otimes K_X)\to H^0\big(X,L^p\otimes K_X\otimes(\mathscr{O}_X/\mathscr{J}^{k+1})\big)=(L_z)^{\otimes p}\otimes(K_X)_z\otimes(\mathscr{O}_{X,z}/\mathbf{m}^{k+1}).$$

The right hand side is called *the space of k-jets of L^2-holomorphic sections of $L^p\otimes K_X$ at z.*

Near z, $e^{-bp\varphi_z(y)}=r_0^{2bp}|y-z|^{-2bp}$. It is well-known (see [28, p. 103]) that for $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{N}^n$ and $z=(z_1,\dots,z_n)\in\mathbb{C}^n$, the integral

$$\int_{|y_1-z_1|<1,\dots,|y_n-z_n|<1}|y_1-z_1|^{2\gamma_1}\cdots|y_n-z_n|^{2\gamma_n}|y-z|^{-2bp}\cdot i^ndy_1\wedge d\bar{y}_1\wedge\cdots\wedge dy_n\wedge d\bar{y}_n$$

is finite if and only if $\sum_{j=1}^n\gamma_j\ge[bp]-n+1$. Putting this together with (13), (11) and (12) and the fact that $u_{z,p,\beta}$ is an $(n,0)$-holomorphic form near z, we see that the $([bp]-n)$-jet of $S_{z,p,\beta}$ coincides with $v_{z,p,\beta}$.

Summarizing what has been done so far, we have shown that the map $J_p^{[bp]-n}$ is surjective. Hence, there is a constant $c > 1$ such that for all p sufficiently large

$$d_p = \dim H^0_{(2)}(X, L^p \otimes K_X) - 1 \geq \dim(\mathscr{O}_X / \mathscr{J}^{[bp]-n+1}) - 1 = \binom{[bp]}{[bp]-n} - 1 \geq c^{-1} p^n.$$

On the other hand, arguing as in the proof of Siegel's lemma [28, Lemma 2.2.6], there is a constant $c > 1$ such that $d_p \leq cp^n$ for all $p \geq 1$. This completes the proof. □

4. Convergence towards intersection of Fubini–Study currents

In this section we show that the intersection of the Fubini–Study currents associated with line bundles as in Theorem 1.5 is well-defined. Moreover, we show that the sequence of wedge products of normalized Fubini–Study currents converges weakly to the wedge product of the curvature currents of (L_k, h_k). We then prove that almost all zero-divisors of sections of large powers of these bundles are in general position.

Let V be a complex vector space of dimension $d + 1$. If V is endowed with a Hermitian metric, then we denote by ω_{FS} the induced Fubini–Study form on the projective space $\mathbb{P}(V)$ (see [28, pp. 65, 212]) normalized so that ω^d_{FS} is a probability measure. We also use the same notations for $\mathbb{P}(V^*)$.

We return to the setting of Theorem 1.5. In fact, for the results of this section it suffices to assume that the metrics involved are only locally bounded. Namely, (L_k, h_k), $1 \leq k \leq m \leq n$, are singular Hermitian holomorphic line bundles on the compact Kähler manifold (X, ω) of dimension n, such that

(i) h_k is *locally bounded* outside a proper analytic subset $\Sigma_k \subset X$;
(ii) $c_1(L_k, h_k) \geq 0$ on X, and for every $x \in X \setminus \Sigma_k$ there is a constant $c_x > 0$ so that $c_1(L_k, h_k) \geq c_x \omega$ in a neighborhood of x;
(iii) $\Sigma_1, \dots, \Sigma_m$ are in general position.

Consider the space $H^0_{(2)}(X, L^p_k \otimes K_X)$ of L^2-holomorphic sections of $L^p_k \otimes K_X$ endowed with the inner product (3). Let

$$d_{k,p} := \dim H^0_{(2)}(X, L^p_k \otimes K_X) - 1.$$

By Part 2) of Theorem 3.1, there is a constant $c > 1$ such that

$$c^{-1} p^n \leq d_{k,p} \leq cp^n. \tag{14}$$

The *Kodaira map* associated with $(L^p_k \otimes K_X, h_{k,p} \otimes h^{K_X})$ is defined by

$$\Phi_{k,p} : \; X \dashrightarrow \mathbb{G}(d_{k,p}, H^0_{(2)}(X, L^p_k \otimes K_X)), \;\; \Phi_{k,p}(x) := \Big\{ s \in H^0_{(2)}(X, L^p_k \otimes K_X) : \; s(x) = 0 \Big\}, \tag{15}$$

where $\mathbb{G}(d_{k,p}, H^0_{(2)}(X, L^p_k \otimes K_X))$ denotes the Grassmannian of hyperplanes in $H^0_{(2)}(X, L^p_k \otimes K_X)$ (see [28, p. 82]). Let us identify $\mathbb{G}(d_{k,p}, H^0_{(2)}(X, L^p_k \otimes K_X))$ with $\mathbb{P}(H^0_{(2)}(X, L^p_k \otimes K_X)^*)$ by sending a hyperplane to an equivalence class of non-zero complex linear functionals on $H^0_{(2)}(X, L^p_k \otimes K_X)$ having the hyperplane as their common kernel. By composing $\Phi_{k,p}$ with this identification, we obtain a meromorphic map

$$\Phi_{k,p} : \; X \dashrightarrow \mathbb{P}(H^0_{(2)}(X, L^p_k \otimes K_X)^*). \tag{16}$$

To get an analytic description of $\Phi_{k,p}$, let

$$S_j^{k,p} \in H^0_{(2)}(X, L_k^p \otimes K_X),\ j = 0, \dots, d_{k,p}\,, \tag{17}$$

be an orthonormal basis and denote by $P_{k,p}$ the Bergman kernel function of the space $H^0_{(2)}(X, L_k^p \otimes K_X)$ defined as in (7). This basis gives identifications $H^0_{(2)}(X, L_k^p \otimes K_X) \simeq \mathbb{C}^{d_{k,p}+1}$ and $\mathbb{P}(H^0_{(2)}(X, L_k^p \otimes K_X)^*) \simeq \mathbb{P}^{d_{k,p}}$. Let U be a contractible Stein open set in X, let e_k, e' be local holomorphic frames on U for L_k, respectively K_X, and write $S_j^{k,p} = s_j^{k,p} e_k^{\otimes p} \otimes e'$, where $s_j^{k,p}$ is a holomorphic function on U. By composing $\Phi_{k,p}$ given in (16) with the last identification, we obtain a meromorphic map $\Phi_{k,p} : X \dashrightarrow \mathbb{P}^{d_{k,p}}$ which has the following local expression

$$\Phi_{k,p}(x) = [s_0^{k,p}(x) : \dots : s_{d_{k,p}}^{k,p}(x)] \ \text{ for } x \in U. \tag{18}$$

It is called *the Kodaira map defined by the basis* $\{S_j^{k,p}\}_{j=0}^{d_{k,p}}$.

Next, we define the *Fubini–Study currents* $\gamma_{k,p}$ of $H^0_{(2)}(X, L_k^p \otimes K_X)$ by

$$\gamma_{k,p}|_U = \frac{1}{2}\, dd^c \log \sum_{j=0}^{d_{k,p}} |s_j^{k,p}|^2, \tag{19}$$

where the open set U and the holomorphic functions $s_j^{k,p}$ are as above. Note that $\gamma_{k,p}$ is a positive closed current of bidegree $(1,1)$ on X, and is independent of the choice of basis.

Actually, the Fubini–Study currents are pullbacks of the Fubini–Study forms by Kodaira maps, which justifies their name. If ω_{FS} is the Fubini–Study form on $\mathbb{P}^{d_{k,p}}$ then by (18) and (19),

$$\gamma_{k,p} = \Phi_{k,p}^*(\omega_{\mathrm{FS}}), \quad 1 \le k \le m. \tag{20}$$

Using (7) we introduce the psh function

$$u_{k,p} := \frac{1}{2p} \log \sum_{j=0}^{d_{k,p}} |s_j^{k,p}|^2 = u_k + \frac{\rho}{p} + \frac{1}{2p} \log P_{k,p} \quad \text{on } U\,, \tag{21}$$

where u_k (resp. ρ) is the weight of the metric h_k (resp. h^{K_X}) on U corresponding to e_k (resp. e'), i.e. $|e_k|_{h_k} = e^{-u_k}$, $|e'|_{h^{K_X}} = e^{-\rho}$. Clearly, by (19) and (21), $dd^c u_{k,p} = \frac{1}{p}\, \gamma_{k,p}$. Note that $\rho/p \to 0$ uniformly as $p \to \infty$ because the metric h^{K_X} is smooth. Moreover, note that by (21), $\log P_{k,p} \in L^1(X, \omega^n)$ and

$$\frac{1}{p}\, \gamma_{k,p} = c_1(L_k, h_k) + \frac{1}{p} c_1(K_X, h^{K_X}) + \frac{1}{2p}\, dd^c \log P_{k,p} \tag{22}$$

as currents on X. For $p \ge 1$ consider the following analytic subsets of X:

$$\Sigma_{k,p} := \left\{ x \in X :\ S_j^{k,p}(x) = 0,\ 0 \le j \le d_{k,p} \right\},\ 1 \le k \le m\,.$$

Hence $\Sigma_{k,p}$ is the base locus of $H^0_{(2)}(X, L_k^p \otimes K_X)$, and $\Sigma_{k,p} \cap U = \{u_{k,p} = -\infty\}$. Note also that $\Sigma_k \cap U \supset \{u_k = -\infty\}$.

Proposition 4.1. *In the above hypotheses we have the following:*

(i) *For all p sufficiently large and every $J \subset \{1, \dots, m\}$ the analytic sets $\Sigma_{k,p}$, $k \in J$, Σ_ℓ, $\ell \in J' := \{1, \dots, m\} \setminus J$, are in general position.*

(ii) *If p is sufficiently large then the currents*

$$\bigwedge_{k\in J}\gamma_{k,p}\ \wedge\ \bigwedge_{\ell\in J'}c_1(L_\ell,h_\ell)$$

are well defined on X, for every $J\subset\{1,\dots,m\}$.

Proof. (i) We show that for p large enough, $\operatorname{codim}(\Sigma_{J,J',p})\geq m$, where

$$\Sigma_{J,J',p}:=\bigcap_{k\in J}\Sigma_{k,p}\cap\bigcap_{\ell\in J'}\Sigma_\ell.$$

The remaining assertions of (i) are proved in a similar way. Assume for a contradiction that there exists a sequence $p_r\to\infty$ such that Σ_{J,J',p_r} has an irreducible component Y_r of dimension $n-m+s$ for some $s\geq 1$. Note that the estimate (9) from the proof of Part 1) of Theorem 3.1 holds in the case that the metric h is locally bounded away from Σ. It implies that for every compact $K\subset X\setminus\Sigma_k$ there exist $c_{k,K}>0$ and $p_{k,K}\in\mathbb{N}$ such that $P_{k,p}\geq c_{k,K}$ holds on K for $p\geq p_{k,K}$, where $1\leq k\leq m$. Using (21) we infer that, given any ϵ-neighborhood $V_{k,\epsilon}$ of Σ_k, $\Sigma_{k,p_r}\subset V_{k,\epsilon}$ for all r sufficiently large. Hence $Y_r\to\bigcap_{k\in J}\Sigma_k\cap\bigcap_{\ell\in J'}\Sigma_\ell=\Sigma_1\cap\dots\cap\Sigma_m$ as $r\to\infty$. Let $R_r=[Y_r]/|Y_r|$, where $[Y_r]$ denotes the current of integration on Y_r and $|Y_r|=\int_{Y_r}\omega^{n-m+s}$. Since R_r have unit mass, we may assume by passing to a subsequence that R_r converges weakly to a positive closed current R of bidimension $(n-m+s,n-m+s)$ and unit mass. But R is supported in $\Sigma_1\cap\dots\cap\Sigma_m$ which has dimension $\leq n-m$, so $R=0$ by the support theorem ([21], see also [23, Theorem 1.7]), a contradiction.

(ii) Using (i) and [14, Corollary 2.11], assertion (ii) follows. □

The following version of Bertini's theorem is proved in [9, Proposition 3.2].

Proposition 4.2. *Let $L_k\longrightarrow X$, $1\leq k\leq m\leq n$, be holomorphic line bundles over a compact complex manifold X of dimension n. Assume that:*

(i) *V_k is a vector subspace of $H^0(X,L_k)$ with basis $S_{k,0},\dots,S_{k,d_k}$, base locus $\operatorname{Bs}V_k:=\{S_{k,0}=\dots=S_{k,d_k}=0\}\subset X$, such that $d_k\geq 1$ and the analytic sets $\operatorname{Bs}V_1,\dots,\operatorname{Bs}V_m$ are in general position.*
(ii) *$Z(t_k):=\{x\in X:\sum_{j=0}^{d_k}t_{k,j}S_{k,j}(x)=0\}$, where $t_k=[t_{k,0}:\ldots:t_{k,d_k}]\in\mathbb{P}^{d_k}$.*
(iii) *$\nu=\mu_1\times\ldots\times\mu_m$ is the product measure on $\mathbb{P}^{d_1}\times\ldots\times\mathbb{P}^{d_m}$, where μ_k is the Fubini–Study volume on $\mathbb{P}^{d_k}$.*

Then the analytic sets $Z(t_1),\dots,Z(t_m)$ are in general position for ν-a.e. $(t_1,\dots,t_m)\in\mathbb{P}^{d_1}\times\ldots\times\mathbb{P}^{d_m}$.

We keep the hypotheses (i), (ii), (iii) at the beginning of the section. If $\{S_j^{k,p}\}_{j=0}^{d_{k,p}}$ is an orthonormal basis of $H^0_{(2)}(X,L_k^p\otimes K_X)$, we define the analytic hypersurface $Z(t_k)\subset X$, for $t_k=[t_{k,0}:\ldots:t_{k,d_{k,p}}]\in\mathbb{P}^{d_{k,p}}$, as in Proposition 4.2 (ii). Let $\mu_{k,p}$ be the Fubini–Study volume on $\mathbb{P}^{d_{k,p}}$, $1\leq k\leq m$, $p\geq 1$, and let $\mu_p=\mu_{1,p}\times\ldots\times\mu_{m,p}$ be the product measure on $\mathbb{P}^{d_{1,p}}\times\ldots\times\mathbb{P}^{d_{m,p}}$. Applying Proposition 4.2 we obtain:

Proposition 4.3. *If p is sufficiently large then for μ_p-a.e. $(t_1,\dots,t_m)\in\mathbb{P}^{d_{1,p}}\times\ldots\times\mathbb{P}^{d_{m,p}}$ the analytic subsets $Z(t_1),\dots,Z(t_m)\subset X$ are in general position, and $Z(t_{i_1})\cap\ldots\cap Z(t_{i_k})$ has pure dimension $n-k$ for each $1\leq k\leq m$, $1\leq i_1<\ldots<i_k\leq m$.*

Proof. Let $V_{k,p}:=H^0_{(2)}(X,L_k^p\otimes K_X)$, so $\operatorname{Bs}V_{k,p}=\Sigma_{k,p}$. By Proposition 4.1 (i), $\Sigma_{1,p},\dots,\Sigma_{m,p}$ are in general position for all p sufficiently large. We fix such p and denote by $[Z(t_k)]$ the current of integration along the analytic hypersurface $Z(t_k)$; it has the same cohomology class as $pc_1(L_k,h_k)+c_1(K_X,h^{K_X})$. Proposition 4.2 shows that the analytic subsets $Z(t_1),\dots,Z(t_m)$ are in general position for μ_p-a.e. $(t_1,\dots,t_m)\in\mathbb{P}^{d_{1,p}}\times\dots\times\mathbb{P}^{d_{m,p}}$. Hence if $1\le k\le m$, $1\le i_1<\dots<i_k\le m$, the current $[Z(t_{i_1})]\wedge\dots\wedge[Z(t_{i_k})]$ is well defined by [14, Corollary 2.11] and it is supported in $Z(t_{i_1})\cap\dots\cap Z(t_{i_k})$. By the Lelong–Poincaré formula [28, Theorem 2.3.3] and hypothesis (ii) it follows that

$$\frac{1}{p^k}\int_X[Z(t_{i_1})]\wedge\dots\wedge[Z(t_{i_k})]\wedge\omega^{n-k}=\int_X c_1(L_{i_1},h_{i_1})\wedge\dots\wedge c_1(L_{i_k},h_{i_k})\wedge\omega^{n-k}+O(p^{-1})>0.$$

So $Z(t_{i_1})\cap\dots\cap Z(t_{i_k})\neq\emptyset$, hence it has pure dimension $n-k$. □

The main result of this section is the following theorem, which is a Large Deviation Principle in our setting.

Theorem 4.4. *We keep the hypotheses (i), (ii), (iii) at the beginning of the section and use the notation introduced in (4)–(5). Then there exist a constant $\xi>0$ depending only on m and a constant $c=c(X,L_1,h_1,\dots,L_m,h_m)>0$ with the following property: For any sequence of positive numbers $\{\lambda_p\}_{p\ge1}$ with*

$$\liminf_{p\to\infty}\frac{\lambda_p}{\log p}>(1+\xi n)c,$$

there are subsets $E_p\subset\mathbb{X}_{K,p}$ such that

(a) $\sigma_p(E_p)\le cp^{\xi n}\exp(-\lambda_p/c)$ for all p large enough;
(b) if $\mathbf{s}_p\in\mathbb{X}_{K,p}\setminus E_p$ we have that the estimate

$$\left|\frac{1}{p^m}\big\langle[\mathbf{s}_p=0]-\gamma_{1,p}\wedge\dots\wedge\gamma_{m,p}\,,\phi\big\rangle\right|\le c\,\frac{\lambda_p}{p}\,\|\phi\|_{\mathscr{C}^2}$$

holds for every $(n-m,n-m)$-form ϕ of class $\mathscr{C}^2$.

In particular, for σ_∞-a.e. $\mathbf{s}\in\Omega_K$ the estimate from (b) holds for all p sufficiently large.

Proof. We repeat the proof of [9, Theorem 4.2] by making the necessary changes. In fact, we apply Dinh–Sibony's equidistribution results for meromorphic transforms [18] and Propositions 4.1 and 4.3. Here the main point is that the dimension estimate (14) plays the role of [9, Proposition 4.7]. □

5. Equidistribution for sections of two adjoint line bundles

The main purpose of this section is to prove Theorem 1.5. Let $\gamma_{k,p}$, $k=1,2$, be the Fubini–Study currents of the spaces $H^0_{(2)}(X,L_k^p\otimes K_X)$ as defined in (19).

Theorem 5.1. *In the setting of Theorem 1.5 we have*

$$\frac{1}{p^2}\,\gamma_{1,p}\wedge\gamma_{2,p}\to c_1(L_1,h_1)\wedge c_1(L_2,h_2)\ \ \textit{as}\ \ p\to\infty\,,$$

in the weak sense of currents on X.

Taking for granted the above result, we arrive at the

Proof of Theorem 1.5. This follows directly from Theorem 4.4 and Theorem 5.1. □

The remainder of the section is devoted to the proof of Theorem 5.1. Let us start with the following lemma.

Lemma 5.2. *Let U be an open set in $\mathbb{C}^n$, A, B be proper analytic subvarieties of U with* $\operatorname{codim} A \cap B \geq 2$, *and u, v, u_p, v_p, $p \geq 1$, be psh functions on U such that:*

(i) *u is continuous on $U \setminus A$ and $u_p \to u$ as $p \to \infty$ locally uniformly on $U \setminus A$.*
(ii) *v is continuous on $U \setminus B$ and $v_p \to v$ as $p \to \infty$ locally uniformly on $U \setminus B$.*
(iii) *The currents $dd^c u_p \wedge dd^c v_p = dd^c(u_p dd^c v_p) = dd^c(v_p dd^c u_p)$ are well defined.*

Then $dd^c u_p \wedge dd^c v_p \to dd^c u \wedge dd^c v$ in the weak sense of currents on $U \setminus (A \cap B)$. Moreover, if $n = 2$ then $dd^c u_p \wedge dd^c v_p \to dd^c u \wedge dd^c v$ as measures on U.

Proof. We recall that the current $dd^c \rho \wedge T := dd^c(\rho T)$ is well defined, where ρ is a psh function and T a positive closed current on U, if ρ is locally integrable on U with respect to the trace measure of T. The current $dd^c u \wedge dd^c v$ is well defined on U since $\operatorname{codim} A \cap B \geq 2$ and u, v are locally bounded on $U \setminus A$, resp. on $U \setminus B$ [14, Corollary 2.11] (see also [22]). Since $u_p \to u$ locally uniformly on $U \setminus A$ and u is continuous there, we have by [5, Theorem 3.4] that $u_p \to u$ in $L^1_{loc}(U)$ hence $dd^c u_p \to dd^c u$ weakly on U. Similarly, $dd^c v_p \to dd^c v$ weakly on U. Using again the uniform convergence of u_p on $U \setminus A$ and the continuity of u there, it follows that $u_p dd^c v_p \to u\, dd^c v$, hence $dd^c u_p \wedge dd^c v_p \to dd^c u \wedge dd^c v$, weakly on $U \setminus A$ (see e.g. [2,3], [14, Corollary 1.6]). Similarly one has that $v_p dd^c u_p \to v\, dd^c u$, hence $dd^c u_p \wedge dd^c v_p \to dd^c u \wedge dd^c v$, weakly on $U \setminus B$. Thus $dd^c u_p \wedge dd^c v_p \to dd^c u \wedge dd^c v$ weakly on $U \setminus (A \cap B)$.

We prove now that $u_p dd^c v_p \to u\, dd^c v$ weakly on $U \setminus B$ as well. Indeed, note that by [24, Theorem 4.1.8] we have $u_p \to u$, $v_p \to v$ in $L^p_{loc}(U)$ for any $1 \leq p < \infty$, and in the Sobolev space $W^{1,p}_{loc}(U)$ for any $1 \leq p < 2$. If χ is a test form supported in $U \setminus B$ then

$$\int u_p dd^c v_p \wedge \chi = \int v_p dd^c(u_p \chi)$$
$$= \int v_p dd^c u_p \wedge \chi + \int v_p(du_p \wedge d^c \chi - d^c u_p \wedge d\chi + u_p dd^c \chi)\,.$$

Now our claim follows since $v_p dd^c u_p \to v\, dd^c u$ weakly on $U \setminus B$ and since $v_p du_p \to v\, du$, $v_p d^c u_p \to v\, d^c u$, $v_p u_p \to vu$ in $L^1_{loc}(U)$. Therefore we have in fact that $u_p dd^c v_p \to u\, dd^c v$ weakly on $U \setminus (A \cap B)$.

We consider finally the case $n = 2$, so $A \cap B$ consists of isolated points. Let $x \in A \cap B$ and $\chi \geq 0$ be a smooth function with compact support in U so that $\chi = 1$ near x and $\operatorname{supp} \chi \cap (A \cap B) = \{x\}$. Since $u_p dd^c v_p \to u\, dd^c v$ weakly on $U \setminus (A \cap B) \supset \operatorname{supp} dd^c \chi$ we obtain

$$\int \chi\, dd^c u_p \wedge dd^c v_p = \int u_p dd^c v_p \wedge dd^c \chi \to \int u\, dd^c v \wedge dd^c \chi = \int \chi\, dd^c u \wedge dd^c v\,.$$

Hence the sequence of positive measures $dd^c u_p \wedge dd^c v_p$ has locally bounded mass and any weak limit point μ satisfies $\mu(\{x\}) = dd^c u \wedge dd^c v(\{x\})$ for $x \in A \cap B$. It follows that $dd^c u_p \wedge dd^c v_p \to dd^c u \wedge dd^c v$ as measures on U. □

Proof of Theorem 5.1. Recall that the currents $\gamma_{1,p} \wedge \gamma_{2,p}$ and $c_1(L_1, h_1) \wedge c_1(L_2, h_2)$ are well defined by Proposition 4.1. Formula (22) implies that

$$\frac{1}{p^2}\int_X \gamma_{1,p}\wedge\gamma_{2,p}\wedge\omega^{n-2}=\int_X c_1(L_1,h_1)\wedge c_1(L_2,h_2)\wedge\omega^{n-2}+O(\frac{1}{p}).$$

Hence it suffices to show that if T is a limit point of the sequence $\left\{\frac{1}{p^2}\gamma_{1,p}\wedge\gamma_{2,p}\right\}$ then $T=c_1(L_1,h_1)\wedge c_1(L_2,h_2)$. For simplicity, we may assume that $\frac{1}{p^2}\gamma_{1,p}\wedge\gamma_{2,p}\to T$ as $p\to\infty$. Since T and $c_1(L_1,h_1)\wedge c_1(L_2,h_2)$ have the same mass, it is enough to prove that $T\geq c_1(L_1,h_1)\wedge c_1(L_2,h_2)$.

We fix $x\in X$ and let U be a neighborhood of x such that there exist holomorphic frames e_1 of L_1, e_2 of L_2, and e' of K_X, over U. Using the notation from Section 3, we let u_1,u_2,ρ be the weights of h_1,h_2,h^{K_X} on U corresponding to these frames, and let $u_{k,p}$ be the psh functions defined in (21). Then $\frac{1}{p}\gamma_{k,p}=dd^cu_{k,p}$ and $c_1(L_k,h_k)=dd^cu_k$ on U. Note that u_k is continuous on $U\setminus\Sigma(h_k)$. By (21) and by Part 1) of Theorem 3.1 and by the smoothness of h^{K_X}, we have

$$u_{k,p}-u_k=\frac{1}{2p}\log P_{k,p}+\frac{\rho}{p}\to 0,$$

locally uniformly on $U\setminus\Sigma(h_k)$. It follows by Lemma 5.2 that $T=c_1(L_1,h_1)\wedge c_1(L_2,h_2)$ on $U\setminus\Gamma$, and hence on $X\setminus\Gamma$, where $\Gamma:=\Sigma(h_1)\cap\Sigma(h_2)$.

Next we write $\Gamma=Y\cup(\cup_{j\geq1}Y_j)$, where Y_j are the irreducible components of dimension $n-2$ and $\dim Y\leq n-3$. Then by Federer's support theorem ([21], see also [23, Theorem 1.7]), $T=c_1(L_1,h_1)\wedge c_1(L_2,h_2)$ on $D=X\setminus\cup_{j\geq1}Y_j$, since Y is an analytic subset of D of dimension $\leq n-3$. Siu's decomposition formula ([32], see also [14, Theorem 6.19]) implies that

$$T=R+\sum_{j\geq1}c_j[Y_j]\,,\ c_1(L_1,h_1)\wedge c_1(L_2,h_2)=R+\sum_{j\geq1}d_j[Y_j]\,,\tag{23}$$

where $[Y_j]$ denotes the current of integration on Y_j, $c_j,d_j\geq0$, and R is a positive closed current of bidegree $(2,2)$ on X which does not charge any Y_j. To conclude the proof of Theorem 5.1 we show that $c_j\geq d_j$ for each j, by using slicing as in the proof of [5, Theorem 3.4].

Without loss of generality, let $j=1$ and $x\in Y_1$ be a regular point of Γ with a neighborhood U as above. By a change of coordinates $z=(z',z'')$ near x we may assume that $x=0\in\overline{\Delta}^n\subset U$ and $\Gamma\cap\Delta^n=Y_1\cap\Delta^n=\{z'=0\}$, where Δ is the unit disk in $\mathbb{C}$, $z'=(z_1,z_2)$, $z''=(z_3,\dots,z_n)$. Let $\chi_1(z')\geq0$ (resp. $\chi_2(z'')\geq0$) be a smooth function with compact support in Δ^2 (resp. in Δ^{n-2}) so that $\chi_1=1$ near $0\in\mathbb{C}^2$ (resp. $\chi_2=1$ near $0\in\mathbb{C}^{n-2}$), and let $\beta=i/2\sum_{j=3}^n dz_j\wedge d\overline{z}_j$ be the standard Kähler form in $\mathbb{C}^{n-2}$. We set

$$u^{z''}_{k,p}(z')=u_{k,p}(z',z'')\,,\ \ u^{z''}_k(z')=u_k(z',z'')\,.$$

Let $\Sigma_{k,p}$ denote the base locus of $H^0_{(2)}(X,L_k^p\otimes K_X)$ and set $\Sigma_p=\Sigma_{1,p}\cap\Sigma_{2,p}$. Then $\Sigma_{k,p}\cap U=\{u_{k,p}=-\infty\}$. Since $u_{k,p}\to u_k$ locally uniformly on $U\setminus\Sigma(h_k)$ and u_k is continuous there, it follows that $\Sigma_p\cap\Delta^n\subset\{(z',z'')\in\Delta^n:|z'|<1/2\}$ for all p sufficiently large. Thus for each $z''\in\Delta^{n-2}$ the analytic set $\{z'\in\Delta^2:(z',z'')\in\Sigma_p\cap\Delta^n\}$ is compact, hence finite, so the measures $dd^cu^{z''}_{1,p}\wedge dd^cu^{z''}_{2,p}$ are well defined [14, Corollary 2.11]. Moreover,

$$\mu^{z''}_p:=dd^cu^{z''}_{1,p}\wedge dd^cu^{z''}_{2,p}\to\mu^{z''}:=dd^cu^{z''}_1\wedge dd^cu^{z''}_2$$

weakly as measures on Δ^2 by Lemma 5.2. One has the slicing formula (see e.g. [18, formula (2.1)])

$$\int_{\Delta^n}\chi_1(z')\chi_2(z'')dd^cu_{1,p}\wedge dd^cu_{2,p}\wedge\beta^{n-2}=\int_{\Delta^{n-2}}\left(\int_{\Delta^2}\chi_1(z')\,d\mu^{z''}_p(z')\right)\chi_2(z'')\beta^{n-2},$$

and similarly for $dd^c u_1 \wedge dd^c u_2$. Since $dd^c u_{1,p} \wedge dd^c u_{2,p} \to T$ it follows from Fatou's lemma that

$$\int\limits_{\Delta^n} \chi_1(z')\chi_2(z'')T \wedge \beta^{n-2} \geq \int\limits_{\Delta^{n-2}} \lim_{p\to\infty} \left(\int\limits_{\Delta^2} \chi_1(z')\, d\mu_p^{z''}(z') \right) \chi_2(z'')\beta^{n-2}$$
$$= \int\limits_{\Delta^{n-2}} \left(\int\limits_{\Delta^2} \chi_1(z')\, d\mu^{z''}(z') \right) \chi_2(z'')\beta^{n-2}$$
$$= \int\limits_{\Delta^n} \chi_1(z')\chi_2(z'')dd^c u_1 \wedge dd^c u_2 \wedge \beta^{n-2}.$$

This implies that $c_1 \geq d_1$, since by (23), $T = R + c_1[z' = 0]$ and $dd^c u_1 \wedge dd^c u_2 = R + d_1[z' = 0]$ on Δ^n. □

6. Equidistribution for sections of several adjoint line bundles

We prove here Theorem 1.6. We will need the following local property of the complex Monge–Ampère operator:

Proposition 6.1. *Let U be an open set in $\mathbb{C}^n$, Σ be a proper analytic subset of U, and $u_1, \dots, u_m$ be psh functions on U which are continuous on $U \setminus \Sigma$. Assume that $\dim \Sigma \leq n - m$ and that $u_{k,p}$, where $1 \leq k \leq m$ and $p \geq 1$, are psh functions on U so that $u_{k,p} \to u_k$ locally uniformly on $U \setminus \Sigma$. Then the currents $dd^c u_{1,p} \wedge \ldots \wedge dd^c u_{m,p}$ are well defined on U for p sufficiently enough, and $dd^c u_{1,p} \wedge \ldots \wedge dd^c u_{m,p} \to dd^c u_1 \wedge \ldots \wedge dd^c u_m$ weakly as $p \to \infty$ in the sense of currents on U.*

Proof. It follows along the same lines as those given in the proof of [5, Theorem 3.4]. □

Proof of Theorem 1.6. Let $U \subset X$ be a contractible Stein open set, $u_{k,p}$, u_k be the psh functions defined in (21), so $dd^c u_k = c_1(L_k, h_k)$ and $dd^c u_{k,p} = \frac{1}{p}\gamma_{k,p}$ on U. By Part 1) of Theorem 3.1 we have that $\frac{1}{p}\log P_{k,p} \to 0$ locally uniformly on $U \setminus \Sigma$, hence by (21), $u_{k,p} \to u_k$ locally uniformly on $U \setminus \Sigma$ as $p \to \infty$, for each $1 \leq k \leq m$. Therefore, Proposition 6.1 implies that $dd^c u_{1,p} \wedge \ldots \wedge dd^c u_{m,p} \to dd^c u_1 \wedge \ldots \wedge dd^c u_m$ weakly on U as $p \to \infty$. Thus, we have shown that

$$\frac{1}{p^m}\gamma_{1,p} \wedge \ldots \wedge \gamma_{m,p} \to c_1(L_1, h_1) \wedge \ldots \wedge c_1(L_m, h_m)$$

as $p \to \infty$, in the weak sense of currents on X. This, combined with Theorem 4.4, implies Theorem 1.6. □

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