Available online at www.sciencedirect.com
SCIENCE DIRECT
C. R. Acad. Sci. Paris, Ser. I 339 (2004) 493–498
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Differential Geometry

Generalized Bergman kernels on symplectic manifolds

Xiaonan Ma [a], George Marinescu [b,*]

[a] *Centre de mathématiques, UMR 7640 du CNRS, École polytechnique, 91128 Palaiseau cedex, France*
[b] *Humboldt-Universität zu Berlin, Institut für Mathematik, Rudower Chaussee 25, 12489 Berlin, Germany*

Received and accepted 12 July 2004

Available online 12 September 2004

Presented by Jean-Michel Bismut

Abstract

We study the asymptotic of the generalized Bergman kernels of the renormalized Bochner–Laplacian on high tensor powers of a positive line bundle on compact symplectic manifolds. ***To cite this article: X. Ma, G. Marinescu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Noyaux de Bergman généralisés sur les variétés symplectiques. On étudie le développement asymptotique du noyau de Bergman généralisé du Laplacien de Bochner renormalisé associé à une puissance tendant vers l'infini d'un fibré en droites positif sur une variété symplectique compacte. ***Pour citer cet article : X. Ma, G. Marinescu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Dans [4] nous avons étudié le développement asymptotique du noyau de Bergman de l'opérateur de Dirac spin^c associé à une puissance tendant vers $+\infty$ d'un fibré en droites positif sur une variété symplectique, et nous l'avons relié au développement asymptotique du noyau de la chaleur correspondant. Cette approche est inspirée de la théorie de l'indice locale, en particulier de [1, §10, 11]. Dans [4], nous avons aussi étudié le développement asymptotique en dehors de la diagonale [4, Théorème 3.18], qui est nécessaire pour étudier le noyau de Bergman sur un orbifold. On y trouve également une introduction brève au noyau de Bergman sur les variétés projectives.

Cette Note est une continuation de [4]. Nous étudions le développement asymptotique du noyau de Bergman généralisé de l'opérateur de Laplace–Bochner renormalisé associé à une puissance tendant vers l'infini d'un fibré en

* Corresponding author.
E-mail addresses: ma@math.polytechnique.fr (X. Ma), george@mathematik.hu-berlin.de (G. Marinescu).

1631-073X/$ – see front matter © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2004.07.016

droites positif sur une variété symplectique compacte. Dans cette situation, ces opérateurs ont des valeurs propres petites quand la puissance tend vers l'infini (dans [4], la seule valeur propre petite est zéro, d'où nous tirons l'équation-clé [4, (3.89)]). En combinant l'estimation de la norme de Sobolev de [4] et une technique de série formelle, nous démontrons le Théorème 3.1, qui donne le développement du noyau de Bergman généralisé près de la diagonale. Nous obtenons aussi une méthode pour calculer les coefficients de ce développement.

Le détails des démonstrations et des applications de nos résultats sont donnés dans [8].

1. Introduction

In [4] we studied the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on compact symplectic manifolds, and related it to that of the corresponding heat kernel. This approach is inspired by local Index Theory, especially by [1, §10, 11]. In [4], we also focused on the full off-diagonal asymptotic expansion [4, Theorem 3.18] which is needed to study the Bergman kernel on orbifolds. We refer to [4] for a brief introduction to the Bergman kernel on complex projective manifolds.

This Note is a continuation of [4]. We study the asymptotic expansion of the generalized Bergman kernels of the renormalized Bochner–Laplacian on high tensor powers of a positive line bundle on compact symplectic manifolds. In this situation the operators have small eigenvalues when the power $p \to \infty$ (the only small eigenvalue is zero in [4], thus we have the key equation [4, (3.89)]) and we are interested in obtaining Theorem 3.1, the *near* diagonal expansion of the generalized Bergman kernels. This result is enough for most of applications. We will combine the Sobolev norm estimates from [4] and a formal power series trick to obtain Theorem 3.1, and in this way, we have a method to compute the coefficients, which is new also in the case of [4].

The full details and some applications of our results are given in [8].

2. Generalized Bergman kernels

Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle (L,h^L) over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$, where $R^L=(\nabla^L)^2$ is the curvature of (L,∇^L). Let (E,h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and its curvature R^E.

Let g^{TX} be a Riemannian metric on X. Let ∇^{TX} be the Levi-Civita connection on (TX,g^{TX}) with its curvature R^{TX} and its scalar curvature r^X. Let dv_X be the Riemannian volume form of (TX,g^{TX}). The scalar product on $\mathcal{C}^\infty(X,L^p\otimes E)$, the space of smooth sections of $L^p\otimes E$, is given by $\langle s_1,s_2\rangle=\int_X\langle s_1(x),s_2(x)\rangle_{L^p\otimes E}\,dv_X(x)$.

Let $\mathbf{J}:TX\to TX$ be the skew-adjoint linear map which satisfies the relation

$$\omega(u,v)=g^{TX}(\mathbf{J}u,v) \tag{1}$$

for $u,v\in TX$. Let J be an almost complex structure which is (separately) compatible with g^{TX} and ω, especially, $\omega(\cdot,J\cdot)$ defines a metric on TX. Then J commutes also with $\mathbf{J}$. Let $\nabla^X J\in T^*X\otimes\mathrm{End}(TX)$ be the covariant derivative of J induced by ∇^{TX}. Let $\nabla^{L^p\otimes E}$ be the connection on $L^p\otimes E$ induced by ∇^L and ∇^E. Let $\{e_i\}_i$ be an orthonormal frame of (TX,g^{TX}). Let $\Delta^{L^p\otimes E}=-\sum_i[(\nabla^{L^p\otimes E}_{e_i})^2-\nabla^{L^p\otimes E}_{\nabla^{TX}_{e_i}e_i}]$ be the induced Bochner–Laplacian acting on $\mathcal{C}^\infty(X,L^p\otimes E)$. We fix a smooth Hermitian section Φ of $\mathrm{End}(E)$ on X. Set $\tau(x)=-\pi\,\mathrm{Tr}|_{TX}[J\mathbf{J}]$, and

$$\Delta_{p,\Phi}=\Delta^{L^p\otimes E}-p\tau+\Phi. \tag{2}$$

By [7, Cor. 1.2] there exist $\mu_0, C_L>0$ independent of p such that the spectrum of $\Delta_{p,\Phi}$ satisfies

$$\mathrm{Spec}\,\Delta_{p,\Phi}\subset[-C_L,C_L]\cup[2p\mu_0-C_L,+\infty[. \tag{3}$$

Let $P_{0,p}$ be the orthogonal projection from $(\mathcal{C}^\infty(X, L^p \otimes E), \langle\, ,\, \rangle)$ onto the eigenspace of $\Delta_{p,\Phi}$ with the eigenvalues in $[-C_L, C_L]$. We define $P_{q,p}(x, x')$, $q \geqslant 0$ as the smooth kernels of the operators $P_{q,p} = (\Delta_{p,\Phi})^q P_{0,p}$ (we set $(\Delta_{p,\Phi})^0 = 1$) with respect to $dv_X(x')$. They are called the generalized Bergman kernels of the renormalized Bochner–Laplacian $\Delta_{p,\Phi}$. Let $\det \mathbf{J}$ be the determinant function of $\mathbf{J}_x \in \mathrm{End}(T_x X)$.

Theorem 2.1. *There exist smooth coefficients $b_{q,r}(x) \in \mathrm{End}(E)_x$ which are polynomials in R^{TX}, R^E (and R^L, Φ) and their derivatives of order $\leqslant 2(r+q)-1$ (resp. $2(r+q)$) at x, and*

$$b_{0,0} = (\det \mathbf{J})^{1/2}\, \mathrm{Id}_E, \tag{4}$$

such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x \in X$, $p \in \mathbb{N}$,

$$\left| \frac{1}{p^n} P_{q,p}(x, x) - \sum_{r=0}^{k} b_{q,r}(x) p^{-r} \right|_{\mathcal{C}^l} \leqslant C_{k,l}\, p^{-k-1}. \tag{5}$$

Moreover, the expansion is uniform in that for any $k, l \in \mathbb{N}$, there is an integer s such that if all data (g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ) run over a bounded set in the $\mathcal{C}^s$-norm and g^{TX} stays bounded below, the constant $C_{k,l}$ is independent of g^{TX}; and the $\mathcal{C}^l$-norm in (5) *includes also the derivatives on the parameters.*

Theorem 2.2. *If $J = \mathbf{J}$, then for $q \geqslant 1$,*

$$b_{0,1} = \frac{1}{8\pi}\left[r^X + \frac{1}{2}|\nabla^X J|^2 + 2\sqrt{-1} R^E(e_j, Je_j) \right], \tag{6}$$

$$b_{q,0} = \left(\frac{1}{12}|\nabla^X J|^2 + \frac{\sqrt{-1}}{2} R^E(e_j, Je_j) + \Phi \right)^q. \tag{7}$$

Theorem 2.1 for $q = 0$ and (6) generalize the results of [3,6,9,10], to the symplectic case, and we can view (7) as an extension and refinement of the results of [2], [5, §5] about the density of states function of $\Delta_{p,\Phi}$.

3. Idea of the proofs

For $x_0 \in X$, $\varepsilon > 0$, we identify the open ball $B^{T_{x_0}X}(0, \varepsilon)$ in $T_{x_0}X$ with center 0 and radius ε, with a neighborhood of $x_0 \in X$ by means of the exponential map. We also identify the fibers of (L, h^L), (E, h^E) with $(L_{x_0}, h^{L_{x_0}})$, $(E_{x_0}, h^{E_{x_0}})$, respectively, in a neighborhood of x_0, by using the parallel transport with respect to ∇^L, ∇^E along the radial direction.

We apply the strategy from the proof in [4]. First, (3) and the finite propagation speed for hyperbolic equations, allows to localize the problem. In particular, the asymptotics of $P_{q,p}(x_0, x')$ as $p \to \infty$ are localized on a neighborhood of x_0. Thus we can translate our analysis from X to the manifold $\mathbb{R}^{2n} \simeq T_{x_0}X =: X_0$.

We then extend the bundles and connections from a neighborhood of 0 to all of $T_{x_0}X$. In particular, we can extend ∇^L (resp. ∇^E) to a Hermitian connection ∇^{L_0} on $(L_0, h^{L_0}) = (X_0 \times L_{x_0}, h^{L_{x_0}})$ (resp. ∇^{E_0} on $(E_0, h^{E_0}) = (X_0 \times E_{x_0}, h^{E_{x_0}})$) on $T_{x_0}X$ in such a way so that we still have positive curvature R^{L_0}; in addition $R^{L_0} = R^L_{x_0}$ outside a compact set. We also extend the metric g^{TX_0}, the almost complex structure J_0, and the smooth section Φ_0 (resp. the connection ∇^{E_0}) in such a way that they coincide with their values at 0 (resp. the trivial connection) outside a compact set. Moreover, using a fixed unit vector $S_L \in L_{x_0}$ and the above discussion, we construct an isometry $E_0 \otimes L_0^p \simeq E_{x_0}$. Let $\Delta^{X_0}_{p,\Phi_0}$ be the renormalized Bochner–Laplacian on X_0 associated to the above data by a formula analogous to (2). Then (3) still holds for $\Delta^{X_0}_{p,\Phi_0}$ with μ_0 replaced by $\mu_0/2$.

Let $\mathrm{d}v_{TX}$ be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$ and $\kappa(Z)$ be the smooth positive function defined by the equation $\mathrm{d}v_{X_0}(Z) = \kappa(Z)\,\mathrm{d}v_{TX}(Z)$, with $k(0) = 1$. For $s \in \mathcal{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$, $Z \in \mathbb{R}^{2n}$ and $t = 1/\sqrt{p}$, set $\|s\|_0^2 = \int_{\mathbb{R}^{2n}} |s(Z)|^2_{h^{E_{x_0}}}\,\mathrm{d}v_{TX}(Z)$, and consider

$$\mathcal{L}_t = S_t^{-1} t^2 \kappa^{1/2} \Delta^{X_0}_{p,\Phi_0} \kappa^{-1/2} S_t, \quad \text{where } (S_t s)(Z) = s(Z/t). \tag{8}$$

Then $\mathcal{L}_t$ is a family of self-adjoint differential operators with coefficients in $\mathrm{End}(E)_{x_0}$. We denote by $\mathcal{P}_{0,t} : (\mathcal{C}^\infty(X_0, E_{x_0}), \|\ \|_0) \to (\mathcal{C}^\infty(X_0, E_{x_0}), \|\ \|_0)$ the spectral projection of $\mathcal{L}_t$ corresponding to the interval $[-C_{L_0}t^2, C_{L_0}t^2]$. Let $\mathcal{P}_{q,t}(Z, Z') = \mathcal{P}_{q,t,x_0}(Z, Z')$, $(Z, Z' \in X_0, q \geqslant 0)$ be the smooth kernel of $\mathcal{P}_{q,t} = (\mathcal{L}_t)^q\, \mathcal{P}_{0,t}$ with respect to $\mathrm{d}v_{TX}(Z')$. We can view $\mathcal{P}_{q,t,x_0}(Z, Z')$ as a smooth section of $\pi^*\mathrm{End}(E)$ over $TX \times_X TX$, where $\pi : TX \times_X TX \to X$. Here we identify a section $S \in \mathcal{C}^\infty(TX \times_X TX, \pi^*\mathrm{End}(E))$ with the family $(S_x)_{x\in X}$, where $S_x = S|_{\pi^{-1}(x)}$. We denote by $|\quad|_{\mathcal{C}^s(X)}$ a $\mathcal{C}^s$ norm on it for the parameter $x_0 \in X$. Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$. By (3),

$$\mathcal{P}_{q,t} = \frac{1}{2\pi i} \int_\delta \lambda^q (\lambda - \mathcal{L}_t)^{-1}\,\mathrm{d}\lambda. \tag{9}$$

From (3) and (9) we can apply the techniques in [4], which are inspired by [1, §11], to get the following key estimate.

Theorem 3.1. *There exist smooth sections $F_{q,r} \in \mathcal{C}^\infty(TX \times_X TX, \pi^*\mathrm{End}(E))$ such that for $k, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $t \in]0, 1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

$$\sup_{|\alpha|,|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\mathcal{P}_{q,t} - \sum_{r=0}^k F_{q,r} t^r \right)(Z, Z') \right|_{\mathcal{C}^{m'}(X)} \leqslant C t^k. \tag{10}$$

Let $P_{0,q,p}(Z, Z') \in \mathrm{End}(E_{x_0})$ $(Z, Z' \in X_0)$ be the analogue of $P_{q,p}(x, x')$. By (8), for $Z, Z' \in \mathbb{R}^{2n}$,

$$P_{0,q,p}(Z, Z') = t^{-2n-2q}\, \kappa^{-1/2}(Z)\, \mathcal{P}_{q,t}(Z/t, Z'/t)\, \kappa^{-1/2}(Z'). \tag{11}$$

To complete the proof of Theorem 2.1, we finally prove $F_{q,r} = 0$ for $r < 2q$. In fact, (10) and (11) yield

$$b_{q,r}(x_0) = F_{q,2r+2q}(0, 0). \tag{12}$$

4. Evaluation of $F_{q,r}$

The almost complex structure J induces a splitting $T_\mathbb{R}X \otimes_\mathbb{R} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. We choose $\{w_i\}_{i=1}^n$ to be an orthonormal basis of $T^{(1,0)}_{x_0}X$, such that

$$-2\pi\sqrt{-1}\mathbf{J}_{x_0} = \mathrm{diag}(a_1, \dots, a_n) \in \mathrm{End}\big(T^{(1,0)}_{x_0}X\big). \tag{13}$$

We use the orthonormal basis $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \dots, n$ of $T_{x_0}X$ to introduce the normal coordinates as in Section 3. In what follows we will use the complex coordinates $z = (z_1, \dots, z_n)$, thus $Z = z + \bar{z}$, and $w_i = \sqrt{2}\frac{\partial}{\partial z_i}$, $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial \bar{z}_i}$. It is very useful to introduce the creation and annihilation operators b_i, b_i^+,

$$b_i = -2\frac{\partial}{\partial z_i} + \frac{1}{2}a_i\bar{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial \bar{z}_i} + \frac{1}{2}a_i z_i, \qquad b = (b_1, \dots, b_n). \tag{14}$$

Now there are second order differential operators $\mathcal{O}_r$ whose coefficients are polynomials in Z with coefficients being polynomials in R^{TX}, $R^{\det}$, R^E, R^L and their derivatives at x_0, such that

$$\mathcal{L}_t = \mathcal{L}_0 + \sum_{r=1}^{\infty} \mathcal{O}_r t^r, \quad \text{with } \mathcal{L}_0 = \textstyle\sum_i b_i b_i^+. \tag{15}$$

Theorem 4.1. *The spectrum of the restriction of $\mathcal{L}_0$ to $L^2(\mathbb{R}^{2n})$ is given by $\{2\sum_{i=1}^n \alpha_i a_i\colon \alpha_i \in \mathbb{N}\}$ and an orthogonal basis of the eigenspace of $2\sum_{i=1}^n \alpha_i a_i$ is given by*

$$b^\alpha\left(z^\beta \exp\left(-\frac{1}{4}\sum_i a_i|z_i|^2\right)\right), \quad \text{with } \beta \in \mathbb{N}^n. \tag{16}$$

Let $N^\perp$ be the orthogonal space of $N = \operatorname{Ker}\mathcal{L}_0$ in $(L^2(\mathbb{R}^{2n}, E_{x_0}), \|\ \ \|_0)$. Let P^N, $P^{N^\perp}$ be the orthogonal projections from $L^2(\mathbb{R}^{2n}, E_{x_0})$ onto N, $N^\perp$, respectively. Let $P^N(Z, Z')$ be the smooth kernel of the operator P^N with respect to $\mathrm{d}v_{TX}(Z')$. From (16), we get

$$P^N(Z, Z') = \frac{1}{(2\pi)^n}\prod_{i=1}^n a_i \exp\left(-\frac{1}{4}\sum_i a_i\left(|z_i|^2 + |z_i'|^2 - 2z_i\bar{z}_i'\right)\right). \tag{17}$$

Now for $\lambda \in \delta$, we solve for the following formal power series on t, with $g_r(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N)$, $f_r^\perp(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N^\perp)$,

$$(\lambda - \mathcal{L}_t)\sum_{r=0}^{\infty}\left(g_r(\lambda) + f_r^\perp(\lambda)\right)t^r = \operatorname{Id}_{L^2(\mathbb{R}^{2n}, E_{x_0})}. \tag{18}$$

From (9), (18), we claim that

$$F_{q,r} = \frac{1}{2\pi i}\int_\delta \lambda^q g_r(\lambda)\,\mathrm{d}\lambda + \frac{1}{2\pi i}\int_\delta \lambda^q f_r^\perp(\lambda)\,\mathrm{d}\lambda. \tag{19}$$

From Theorem 4.1, (19), the key observation that $P^N\mathcal{O}_1 P^N = 0$, and the residue formula, we can get $F_{q,r}$ by using the operators $\mathcal{L}_0^{-1}$, P^N, $P^{N^\perp}$, $\mathcal{O}_i$ $(i \leqslant r)$. This gives us a method to compute $b_{q,r}$ in view of Theorem 4.1 and (12). Especially, for $q > 0$, $r < 2q$,

$$\begin{aligned}
&F_{0,0} = P^N, \qquad F_{q,r} = 0,\\
&F_{q,2q} = \left(P^N\mathcal{O}_2 P^N - P^N\mathcal{O}_1\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1 P^N\right)^q P^N,\\
&F_{0,2} = \mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1 P^N - \mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_2 P^N + P^N\mathcal{O}_1\mathcal{L}_0^{-1}P^{N^\perp}\mathcal{O}_1\mathcal{L}_0^{-1}P^{N^\perp} - P^N\mathcal{O}_2\mathcal{L}_0^{-1}P^{N^\perp}.
\end{aligned} \tag{20}$$

In fact $\mathcal{L}_0$ and $\mathcal{O}_r$ are formal adjoints with respect to $\|\ \ \|_0$; thus in $F_{0,2}$ we only need to compute the first two terms, as the last two terms are their adjoints. This simplifies the computation in Theorem 2.2.

Acknowledgements

We thank Professors Jean-Michel Bismut, Jean-Michel Bony and Johannes Sjöstrand for useful conversations. It is a pleasure to acknowledge our intellectual debt to Xianzhe Dai and Kefeng Liu.

References

[1] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, Publ. Math. IHES 74 (1991) 1–297.
[2] D. Borthwick, A. Uribe, The spectral density function for the Laplacian on high tensor powers of a line bundle, Ann. Global Anal. Geom. 21 (2002) 269–286.
[3] D. Catlin, The Bergman kernel and a theorem of Tian. Analysis and geometry in several complex variables (Katata, 1997), in: Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1–23.
[4] X. Dai, K. Liu, X. Ma, On the asymptotic expansion of Bergman kernel, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 193–198; The full version: math.DG/0404494.
[5] V. Guillemin, A. Uribe, The Laplace operator on the nth tensor power of a line bundle: eigenvalues which are bounded uniformly in n, Asymptotic Anal. 1 (1988) 105–113.
[6] Z. Lu, On the lower order terms of the asymptotic expansion of Tian–Yau–Zelditch, Am. J. Math. 122 (2000) 235–273.
[7] X. Ma, G. Marinescu, The spin^c Dirac operator on high tensor powers of a line bundle, Math. Z. 240 (2002) 651–664.
[8] X. Ma, G. Marinescu, Generalized Bergman kernels on symplectic manifolds, Preprint.
[9] X. Wang, Thesis, 2002.
[10] S. Zelditch, Szegö kernels and a theorem of Tian, Internat. Math. Res. Notices 6 (1998) 317–331.