EQUIDISTRIBUTION FOR SEQUENCES OF LINE BUNDLES ON NORMAL KÄHLER SPACES

DAN COMAN, XIAONAN MA, AND GEORGE MARINESCU

ABSTRACT. We study the asymptotics of Fubini-Study currents and zeros of holomorphic sections associated to a sequence of singular Hermitian line bundles on a compact normal Kähler complex space.

CONTENTS

1. INTRODUCTION

In this paper we continue the study of equidistribution of Fubini-Study currents and zeros of sequences of holomorphic sections of singular Hermitian holomorphic bundles started in [CM1, CM2, CM3]. We generalize our previous results in two directions. On the one hand, we allow the base space to be singular and work over Kähler spaces. On the

Date: December 4, 2014.
2010 *Mathematics Subject Classification.* Primary 32L10; Secondary 32A60, 32C20, 32U40, 81Q50.
D. Coman is partially supported by the NSF Grant DMS-1300157.
X. Ma is partially supported by Institut Universitaire de France.
G. Marinescu is partially supported by DFG funded projects SFB/TR 12, MA 2469/2-2.
Funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative.

other hand, we consider sequences (L_p, h_p), $p \geq 1$, of singular Hermitian holomorphic line bundles whose Chern curvature satisfy a natural growth condition, instead of sequences of powers (L^p, h^p) of a fixed line bundle (L, h).

Recall that by results of [T, SZ1, DS1] (see also [MM2, Section 5.3]), if (X, ω) is a compact Kähler manifold whose Kähler form is integral, i. e. $[\omega] \in H^2(X, \mathbb{Z})$, and (L, h) is a prequantum line bundle (i. e. the Chern curvature form $c_1(L, h)$ equals ω), then the normalized Fubini-Study currents $\frac{1}{p}\gamma_p$ associated to $H^0(X, L^p)$ converge in the $\mathscr{C}^\infty$ topology to ω and for almost all sequences $\{\sigma_p \in H^0(X, L^p)\}_{p\geq 1}$ the normalized zero-currents $\frac{1}{p}[\sigma_p = 0]$ converge weakly to ω on X. This means that the Kähler form ω can be approximated by various algebraic/analytic objects in the semiclassical limit $p \to \infty$. Equidistribution results [NV, SZ1, SZ2, DS1, DMS] influenced by [FS].

In [CM1, CM2, CM3] we relaxed the smoothness condition on ω and assumed that ω is merely an integral Kähler current. Then there exists a holomorphic line bundle (L, h) endowed with a singular Hermitian metric such that $c_1(L, h)$ equals ω as currents. We established the above approximation results in the sense of currents by working with the spaces of square integrable holomorphic sections $H^0_{(2)}(X, L^p)$. The setting in [CM1, CM2, CM3] allows to deal with several singular Kähler metrics such as singular metrics on big line bundles, metrics with Poincaré growth, Kähler-Einstein metrics singular along a divisor (e. g. good metrics in the sense of Mumford on toroidal compactifications of arithmetic quotiens) or metrics on orbifold line bundles.

We consider in this paper the following setting:

(A) (X, ω) is a compact (reduced) normal Kähler space of pure dimension n, X_{reg} denotes the set of regular points of X, and X_{sing} denotes the set of singular points of X.

(B) (L_p, h_p), $p \geq 1$, is a sequence of holomorphic line bundles on X with singular Hermitian metrics h_p whose curvature currents verify

(1) $$c_1(L_p, h_p) \geq a_p\,\omega \text{ on } X, \text{ where } a_p > 0 \text{ and } \lim_{p\to\infty} a_p = \infty.$$

Let $A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$. If $X_{sing} \neq \emptyset$ we assume moreover that

(2) $$\exists\, T_0 \in \mathscr{T}(X) \text{ such that } c_1(L_p, h_p) \leq A_p T_0\,,\ \forall\, p \geq 1\,.$$

Here we consider currents on the analytic space X in the sense of [D3], and $\mathscr{T}(X)$ denotes the set of positive closed currents of bidegree $(1, 1)$ on X which have a local plurisubharmonic (psh) potential in the neighborhood of each point of X (see Section 2).

We let $H^0_{(2)}(X, L_p)$ be the Bergman space of L^2-holomorphic sections of L_p relative to the metric h_p and the volume form ω^n of X,

(3) $$H^0_{(2)}(X, L_p) = \left\{ S \in H^0(X, L_p) :\ \|S\|_p^2 := \int_{X_{reg}} |S|^2_{h_p}\, \omega^n < \infty \right\},$$

endowed with the obvious inner product. Furthermore, we denote by P_p, resp. γ_p, the Bergman kernel function, resp. the Fubini-Study current, of the space $H^0_{(2)}(X, L_p)$, which are defined as follows. For a (fixed) $p \geq 1$, let $S^p_1, \dots, S^p_{d_p}$ be an orthonormal basis of $H^0_{(2)}(X, L_p)$. If $x \in X$ and e_p is a local holomorphic frame of L_p in a neighborhood U_p of x we write $S^p_j = s^p_j e_p$, where $s^p_j \in \mathcal{O}_X(U_p)$. Then

$$P_p(x) = \sum_{j=1}^{d_p} |S^p_j(x)|^2_{h_p}\ ,\quad \gamma_p|_{U_p} = \frac{1}{2}\, dd^c \log\left(\sum_{j=1}^{d_p} |s^p_j|^2\right)\ , \tag{4}$$

where $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. Note that P_p, γ_p are independent of the choice of basis.

Our main results are the following theorems:

Theorem 1.1. *If (X, ω), (L_p, h_p), $p \geq 1$, verify assumptions (A)-(B), then:*

(i) $\frac{1}{A_p} \log P_p \to 0$ as $p \to \infty$, in $L^1(X, \omega^n)$.

(ii) $\frac{1}{A_p} \left(\gamma_p - c_1(L_p, h_p)\right) \to 0$ as $p \to \infty$, in the weak sense of currents on X.

Moreover, if $\frac{1}{A_p}\, c_1(L_p, h_p) \to T$ for some positive closed current T of bidegree $(1,1)$ on X, then $\frac{1}{A_p}\, \gamma_p \to T$ as $p \to \infty$, in the weak sense of currents on X.

When X is smooth we obtain:

Theorem 1.2. *Let (X, ω) be a compact Kähler manifold of dimension n and (L_p, h_p), $p \geq 1$, be a sequence of singular Hermitian holomorphic line bundles on X which satisfy $c_1(L_p, h_p) \geq a_p\, \omega$, where $a_p > 0$ and $\lim_{p\to\infty} a_p = \infty$. If $A_p = \int_X c_1(L_p, h_p) \wedge \omega^{n-1}$ then $\frac{1}{A_p} \log P_p \to 0$ in $L^1(X, \omega^n)$ and $\frac{1}{A_p} \left(\gamma_p - c_1(L_p, h_p)\right) \to 0$ weakly on X, as $p \to \infty$.*

If $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a fixed singular Hermitian line bundle, we obtain analogues of the equidistributions results from [SZ1, DS1, CM1, CM2, CM3] for compact Kähler spaces. Note that in this case hypothesis (2) is automatically verified as $c_1(L^p, h^p) = p\, c_1(L, h)$, so $A_p = p$ and we can take $T_0 = c_1(L, h)$.

Theorem 1.1 gives semiclassical approximation results for integral Kähler currents. In order to deal with non-integral Kähler currents we consider those Kähler currents which can be approximated by the curvatures of a sequence (L_p, h_p), $p \geq 1$, of singular Hermitian line bundles. Such a sequence can be thought as a "prequantization" of the non-integral Kähler current ω. The asymptotics in Theorems 1.1-1.2 and their consequences are a manifestation of the quantum ergodicity in this context.

An interesting situation is when L_p equals a product of tensor powers of several holomorphic bundles $L_p = F_1^{m_{1,p}} \otimes \ldots \otimes F_k^{m_{k,p}}$, where $\{m_{j,p}\}_p$, $1 \leq j \leq k$, are sequences in $\mathbb{N}$ such that

$$m_{j,p} = r_j\, p + o(p)\,,\ 1 \leq j \leq k, \quad \text{as } p \to \infty\,,$$

where $r_j > 0$, $1 \leq j \leq k$, are given. This means that $(m_{1,p}, \dots, m_{k,p}) \in \mathbb{N}^k$ approximate the semiclassical ray $\mathbb{R}_{>0} \cdot (r_1, \dots, r_k) \in \mathbb{R}^k_{>0}$ with a remainder $o(p)$, as $p \to \infty$. Assume that (F_j, h^{F_j}) are singular Hermitian holomorphic line bundles with $c_1(F_j, h^{F_j}) \geq 0$ for $1 \leq j \leq k$ and one of them is strictly positive, say, $c_1(F_1, h^{F_1}) \geq \varepsilon\omega$, for some $\varepsilon > 0$. Let P_p and γ_p be the Bergman kernel function and Fubini-Study currents associated to

$H^0_{(2)}(X, L_p)$, where $L_p := F_1^{m_{1,p}} \otimes \ldots \otimes F_k^{m_{k,p}}$. Then we have in the weak sense of currents on X,

$$\frac{1}{p}\log P_p \to 0\,, \quad \frac{1}{p}\gamma_p \to T := \sum_{j=1}^{k} r_j\, c_1(F_j, h^{F_j})\,, \quad \text{as } p \to \infty\,,$$

and for almost all sequences $\sigma_p \in \mathcal{S}^p \subset H^0_{(2)}(X, L^p)$, $p \geq 1$, where $\mathcal{S}^p$ is the unit sphere of $H^0_{(2)}(X, L^p)$, the normalized zero-currents $\frac{1}{p}[\sigma_p = 0]$ converge weakly to T on X. For details see Corollary 5.4.

We consider further the situation when the metrics h_p on the bundles L_p are smooth. In [Be, Sect. 2] Berndtsson gave a simple proof for the first order asymptotics of the Bergman kernel function in the case of powers of an ample line bundle. Adapting his methods to our situation we prove the following:

Theorem 1.3. *Let (X, ω) be a compact Kähler manifold of dimension n. Let (L_p, h_p), $p \geq 1$, be a sequence of holomorphic line bundles on X with Hermitian metrics h_p of class $\mathscr{C}^3$ whose curvature forms verify* (1) *and such that*

$$\varepsilon_p := \|h_p\|_3^{1/3} a_p^{-1/2} \to 0 \ \textit{ as } p \to \infty\,. \tag{5}$$

Then there exist a constant $C > 0$ depending only on (X, ω) and $p_0 \in \mathbb{N}$ such that

$$\left| P_p(x)\, \frac{\omega_x^n}{c_1(L_p, h_p)_x^n} - \frac{1}{n!} \right| \leq C\varepsilon_p^{2/3} \tag{6}$$

holds for every $x \in X$ and $p > p_0$.

Here $\|h_p\|_3$ denotes the sup-norm of the third order derivatives of h_p with respect to a reference cover of X as defined in Section 2.5. Theorem 1.3 is a generalization of the first order asymptotic expansion of the Bergman kernel [Be, Ca, DLM1, MM2, MM3, R, T, Z] for $(L_p, h_p) = (L^p, h^p)$, where (L, h) is a positive line bundle with smooth metric h (see Remark 3.3).

The organization of this paper is as follows. In Section 2 we collect the necessary technical facts about complex spaces, plurisubharmonic functions, currents and singular Hermitian metrics on holomorphic line bundles. Section 3 is devoted to the proofs of Theorems 1.1-1.2. In Section 4 we give some applications of these theorems to the equidistribution of zeros of random sequences of holomorphic sections and to the approximation of certain positive closed currents on X by currents of integration along zeros of holomorphic sections.

Let us close the introduction by noting that statistics of zeros of sections and hypersurfaces have been studied also in the context of real manifolds and real vector bundles, see e.g. [GW, NS].

2. Preliminaries

We recall here a few notions and results that will be needed throughout the paper.

2.1. **Plurisubharmonic functions and currents on analytic spaces.** Let X be a complex space. A chart (U, τ, V) on X is a triple consisting of an open set $U \subset X$, a closed complex space $V \subset G \subset \mathbb{C}^N$ in an open set G of $\mathbb{C}^N$ and a biholomorphic map $\tau : U \to V$ (in the category of complex spaces). The map $\tau : U \to G \subset \mathbb{C}^N$ is called a local embedding of the complex space X. We write

$$X = X_{reg} \cup X_{sing} \,,$$

where X_{reg} (resp. X_{sing}) is the set of regular (resp. singular) points of X. Recall that a reduced complex space $(X, \mathscr{O})$ is called normal if for every $x \in X$ the local ring $\mathscr{O}_x$ is integrally closed in its quotient field $\mathscr{M}_x$. Every normal complex space is locally irreducible and locally pure dimensional, cf. [GR2, p. 125], X_{sing} is a closed complex subspace of X with $\operatorname{codim} X_{sing} \geq 2$. Moreover, Riemann's second extension theorem holds on normal complex spaces [GR2, p. 143]. In particular, every holomorphic function on X_{reg} extends uniquely to a holomorphic function on X.

Let X be a complex space. A continuous (resp. smooth) function on X is a function $\varphi : X \to \mathbb{C}$ such that for every $x \in X$ there exists a local embedding $\tau : U \to G \subset \mathbb{C}^N$ with $x \in U$ and a continuous (resp. smooth) function $\widetilde{\varphi} : G \to \mathbb{C}$ such that $\varphi|_U = \widetilde{\varphi} \circ \tau$.

A *(strictly) plurisubharmonic (psh)* function on X is a function $\varphi : X \to [-\infty, \infty)$ such that for every $x \in X$ there exists a local embedding $\tau : U \to G \subset \mathbb{C}^N$ with $x \in U$ and a (strictly) psh function $\widetilde{\varphi} : G \to [-\infty, \infty)$ such that $\varphi|_U = \widetilde{\varphi} \circ \tau$. If $\widetilde{\varphi}$ can be chosen continuous (resp. smooth), then φ is called a continuous (resp. smooth) psh function. The definition is independent of the chart, as is seen from [N, Lemma 4]. It is clear that a continuous psh function is continuous; by a theorem of Richberg [R] the converse also holds true, i.e. a continuous function which is (strictly) psh is also continuous (strictly) psh. The analogue of Riemann's second extension theorem for psh functions holds on normal complex spaces [GR1, Satz 4]. In particular, every psh function on X_{reg} extends uniquely to a psh function on X. We let $PSH(X)$ denote the set of psh functions on X, and refer to [GR1], [N], [FN], [D3] for the properties of psh functions on X. We recall here that psh functions on X are locally integrable with respect to the area measure on X given by any local embedding $\tau : U \to G \subset \mathbb{C}^N$ [D3, Proposition 1.8].

Let X be a complex space of pure dimension n. We consider currents on X as defined in [D3]. The sheaf of smooth (p, q)-forms on X is defined at first locally. Let $\tau : U \to G \subset \mathbb{C}^N$ be a local embedding. We define $\Omega^{p,q}(U)$ to be the image of the morphism $\tau^* : \Omega^{p,q}(G) \to \Omega^{p,q}(U_{reg})$. It can be easily seen that there exists a sheaf $\Omega^{p,q}$ on X whose space of sections on any domain U of local embedding is $\Omega^{p,q}(U)$. Let $\mathcal{D}^{p,q}(X) \subset \Omega^{p,q}(X)$ be the space of forms with compact support, endowed with the inductive limit topology. The dual $\mathcal{D}'_{p,q}(X)$ of $\mathcal{D}_{p,q}(X)$ is the space of currents of bidimension (p, q), or bidegree $(n-p, n-q)$, on X. In particular, if $v \in PSH(X)$ then $dd^c v \in \mathcal{D}'_{n-1,n-1}(X)$ is positive and closed.

Let $\mathscr{T}(X)$ be the space of positive closed currents of bidegree $(1, 1)$ on X which have local psh potentials: $T \in \mathscr{T}(X)$ if every $x \in X$ has a neighborhood U (depending on T) such that there exists a psh function v on U with $T = dd^c v$ on $U \cap X_{reg}$. Most of the currents considered here, like the curvature currents $c_1(L_p, h_p)$ and the Fubini-Study currents γ_p from Theorem 1.1, belong to $\mathscr{T}(X)$. Suppose now that Y is a normal analytic space, $f : Y \longrightarrow X$ is a holomorphic map, and $T \in \mathscr{T}(X)$ is such that if v is a local psh potential of T then $v \circ f$ is not identically $-\infty$ on an open set of Y. Then the pull-back $f^\star T \in \mathscr{T}(Y)$ is a well-defined current whose local psh potentials are $v \circ f$. Some interesting open questions that we will not pursue here are the following: Does every positive closed current of bidegree $(1,1)$ on X belong to $\mathscr{T}(X)$? Is $\mathscr{T}(X)$ closed in the weak* topology on currents? If $T_k, T \in \mathscr{T}(X)$ and $T_k \to T$ weakly on X, does $\{f^\star T_k\}$ converge to $f^\star T$ weakly on Y?

A Kähler form on X is a current $\omega \in \mathscr{T}(X)$ whose local potentials extend to smooth strictly psh functions in local embeddings of X to Euclidean spaces. We call X a Kähler space if X admits a Kähler form (see also [Gr, p. 346], [O], [EGZ, Sec. 5]). A Kähler form is a particular case of a Hermitian form on a complex space. Recall that a Hermitian form on a complex manifold is a smooth positive $(1,1)$-form and can be identified to a Hermitian metric. Now, a Hermitian form on a complex space X is defined as a smooth $(1,1)$-form ω on X such that for every point $x \in X$ there exists a local embedding $\tau : U \to G \subset \mathbb{C}^N$ with $x \in U$ and a Hermitian form $\widetilde{\omega}$ on G with $\omega = \tau^\star\widetilde{\omega}$ on $U \cap X_{reg}$. A Hermitian form on a paracompact complex space space X is constructed as usual by a partition of unity argument. A Hermitian form ω on X clearly induces a Hermitian form in the usual sense (and thus a Hermitian metric) on X_{reg}. Note that ω^n gives locally an area measure on X.

2.2. **Singular Hermitian holomorphic line bundles on analytic spaces.** Let L be a holomorphic line bundle on a normal Kähler space (X,ω). The notion of singular Hermitian metric h on L is defined exactly as in the smooth case (see [D4], [MM2, p. 97])): if e_α is a holomorphic frame of L over an open set $U_\alpha \subset X$ then $|e_\alpha|^2_h = e^{-2\varphi_\alpha}$ where $\varphi_\alpha \in L^1_{loc}(U_\alpha, \omega^n)$. If $g_{\alpha\beta} = e_\beta/e_\alpha \in \mathcal{O}^\star_X(U_\alpha \cap U_\beta)$ are the transition functions of L then $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$. The curvature current $c_1(L,h) \in \mathcal{D}'_{n-1,n-1}(X)$ of h is defined by $c_1(L,h) = dd^c\varphi_\alpha$ on $U_\alpha \cap X_{reg}$. We will denote by h^p the singular Hermitian metric induced by h on $L^p := L^{\otimes p}$. If $c_1(L,h) \geq 0$ then the weight φ_α is psh on $U_\alpha \cap X_{reg}$ and since X is normal it extends to a psh function on U_α [GR1, Satz 4], hence $c_1(L,h) \in \mathscr{T}(X)$.

Let L be a holomorphic line bundle on a compact normal Kähler space (X,ω). Then the space $H^0(X,L)$ of holomorphic sections of L is finite dimensional. This is a special case of the Cartan-Serre finiteness theorem; an elementary proof using the Schwarz lemma can be found in [A, Théorème 1, p.27]. The space $H^0_{(2)}(X,L)$ defined as in (3) is therefore also finite dimensional.

If P_p and γ_p are the Bergman kernel functions, resp. the Fubini-Study currents, of the spaces $H^0_{(2)}(X,L_p)$ from Theorem 1.1, it follows from (4) that $\log P_p \in L^1(X,\omega^n)$ and

$$\gamma_p - c_1(L_p,h_p) = \frac{1}{2}\, dd^c \log P_p\,. \tag{7}$$

Moreover, as in [CM1, CM2], one has for all $x \in X$ that

$$P_p(x) = \max\big\{|S(x)|^2_{h_p} : S \in H^0_{(2)}(X,L_p),\ \|S\|_p = 1\big\}.$$

2.3. **Resolution of singularities.** Bierstone and Milman constructed a resolution of singularities of a compact analytic space X, $\pi : \widetilde{X} \longrightarrow X$, by a finite sequence of blow-ups with smooth center $\sigma_j : X_{j+1} \longrightarrow X_j$, $X_0 = X$, with the property that for any local embedding $X|_U \hookrightarrow \mathbb{C}^N$ this sequence of blow-ups is induced by the embedded desingularization of $X|_U$ [BM, Theorem 13.2]. In [GM, sec. 6] it is shown that the embedded desingularization of $X|_U \hookrightarrow \mathbb{C}^N$ by a finite sequence of blow-ups with smooth center is equivalent to a single blow-up along a coherent sheaf of ideals $\mathcal{I}$ whose support is $X_{sing}|_U$. It follows that every point $x \in X$ has a neighborhood $U \subset X$ for which there exists an ideal $\mathcal{I}_U$ generated by finitely many holomorphic functions on U such that $\pi : \pi^{-1}(U) \longrightarrow U$ is equivalent to the blow-up of $X|_U$ along $\mathcal{I}_U$.

We fix throughout the paper a resolution of singularities $\pi : \widetilde{X} \longrightarrow X$ of our compact normal space X as described above, and we denote by $\Sigma = \pi^{-1}(X_{sing})$ the exceptional divisor. Note that $\pi : \widetilde{X} \setminus \Sigma \longrightarrow X_{reg}$ is a biholomorphism. We will need to consider the singular Hermitian holomorphic line bundles $(\pi^\star L_p, \pi^\star h_p)$ obtained by pulling back (L_p, h_p) to $\widetilde{X}$ by the map π, and their spaces of L^2-holomorphic sections

$$H^0_{(2)}(\widetilde{X}, \pi^\star L_p) = \left\{ \widetilde{S} \in H^0(\widetilde{X}, \pi^\star L_p) : \int_{\widetilde{X}} |\widetilde{S}|^2_{\pi^\star h_p} \, \pi^\star \omega^n < \infty \right\}.$$

Lemma 2.1. *The map $\pi^\star : H^0_{(2)}(X, L_p) \longrightarrow H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ is an isometry and the Bergman kernel function of $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ is $\widetilde{P}_p = P_p \circ \pi$.*

Proof. Let $S^p_1, \ldots, S^p_{d_p}$ be an orthonormal basis of $H^0_{(2)}(X, L_p)$ and $\widetilde{S}^p_j = \pi^\star S^p_j$ be the induced sections of $\pi^\star L^p$. Then $|\widetilde{S}^p_j|_{\pi^\star h_p} = |S^p_j|_{h_p} \circ \pi$ and

$$\int_{\widetilde{X}} |\widetilde{S}^p_j|^2_{\pi^\star h_p} \, \pi^\star \omega^n = \int_{\widetilde{X} \setminus \Sigma} |\widetilde{S}^p_j|^2_{\pi^\star h_p} \, \pi^\star \omega^n = \int_{X_{reg}} |S^p_j|^2_{h_p} \, \omega^n = 1.$$

Suppose now that $\widetilde{S} \in H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ is orthogonal to all $\widetilde{S}^p_j$ and let S be the induced section on X_{reg}. Then, since X is normal, Riemann's second extension theorem [GR2, p. 143] shows that S extends to a section $S \in H^0(X, L_p)$. By the preceding argument, $S \in H^0_{(2)}(X, L_p)$ is orthogonal to all S^p_j, hence $S = 0$ and $\widetilde{S} = 0$. It follows that $\{\widetilde{S}^p_j\}$ is an orthonormal basis of $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ and

$$\widetilde{P}_p = \sum_{j=1}^{d_p} |\widetilde{S}^p_j|^2_{\pi^\star h_p} = \sum_{j=1}^{d_p} |S^p_j|^2_{h_p} \circ \pi = P_p \circ \pi.$$

□

Adapting the proof of Lemma 1 in [Moi] to our situation we obtain the following lemma, whose proof is included for the convenience of the reader.

Lemma 2.2. *Let (X, ω) be a compact (reduced) Hermitian space and $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities as described above, with exceptional divisor Σ. Then there exists a smooth Hermitian metric θ on $F = \mathcal{O}_{\widetilde{X}}(-\Sigma)$ and a constant $C > 0$ such that $\Omega = C\pi^\star\omega + c_1(F, \theta)$ is a Hermitian form on $\widetilde{X}$ and $\Omega \geq \pi^\star\omega$. If ω is Kähler, then Ω is Kähler, too.*

Proof. As in the proof of [Moi, Lemma 1], we can find an open cover $X = \bigcup_{k=1}^{N} U_k$ with the following properties:

(i) There exist a local embedding $\tau_k : U_k \to B_k$ into a ball $B_k \subset \mathbb{C}^{l_k}$ and a smooth Hermitian form ω_k on B_k such that $\omega = \tau_k^* \omega_k$ on $U_k \cap X_{reg}$. Hence we may assume that $U_k \subset B_k$ and ω is the restriction of ω_k to $U_k \cap X_{reg}$. Choose a strictly psh function η_k on B_k such that $\omega_k \geq dd^c \eta_k$ on B_k. If ω is Kähler, we choose η_k such that $\omega_k = dd^c \eta_k$ on B_k, hence $\omega = dd^c \eta_k$ on $U_k \cap X_{reg}$.

(ii) There exist finitely many holomorphic functions $f_{0k}, \ldots, f_{N_k k} \in \mathcal{O}_X(U_k)$ such that over U_k the map $\pi : \pi^{-1}(U_k) \longrightarrow U_k$ and the line bundle $\mathcal{O}_{\widetilde{X}}(\Sigma)|_{\pi^{-1}(U_k)}$ are described as follows. If $\Gamma_k \subset U_k \times \mathbb{P}^{N_k}$ is the graph of the meromorphic map $x \mapsto [f_{0k}(x) : \ldots : f_{N_k k}(x)]$ with canonical projections $\pi_{1k} : \Gamma_k \longrightarrow U_k$ and $\pi_{2k} : \Gamma_k \longrightarrow \mathbb{P}^{N_k}$ then $\pi : \pi^{-1}(U_k) \longrightarrow U_k$ coincides with $\pi_{1k} : \Gamma_k \longrightarrow U_k$ and $\mathcal{O}_{\widetilde{X}}(\Sigma)|_{\pi^{-1}(U_k)}$ can be identified with the line bundle $\pi_{2k}^{\star} \mathcal{O}_{\mathbb{P}^{N_k}}(-1)$ on Γ_k.

Let $[y_{0k} : \ldots : y_{N_k k}]$ denote the homogeneous coordinates on $\mathbb{P}^{N_k}$, $G_{ik} = \{y_{ik} \neq 0\} \subset \mathbb{P}^{N_k}$, $\widetilde{G}_{ik} = \pi_{2k}^{-1}(G_{ik}) \subset \Gamma_k = \pi^{-1}(U_k)$. The transition functions of $\mathcal{O}_{\mathbb{P}^{N_k}}(-1)$ are y_{ik}/y_{jk} and we endow $\mathcal{O}_{\mathbb{P}^{N_k}}(-1)$ with a smooth Hermitian metric with weights ψ'_{ik} on G_{ik} such that $-\psi'_{ik}$ is strictly psh. The corresponding transition functions of $\mathcal{O}_{\widetilde{X}}(\Sigma)|_{\pi^{-1}(U_k)}$ are $\widetilde{g}_{ij,k} = (f_{ik} \circ \pi_{1k})/(f_{jk} \circ \pi_{1k})$ on $\widetilde{G}_{ik} \cap \widetilde{G}_{jk}$, and we denote by ψ_{ik} the weights of the induced Hermitian metric. Then $-\psi_{ik}$ is a smooth psh function on $\widetilde{G}_{ik}$ and $\psi_{ik} = \psi_{jk} + \log |\widetilde{g}_{ij,k}|$ on $\widetilde{G}_{ik} \cap \widetilde{G}_{jk}$.

Consider the open cover $\{\widetilde{G}_{ik} : 1 \leq k \leq N, 0 \leq i \leq N_k\}$ of $\widetilde{X}$. Note that $\pi^{-1}(U_k) = \bigcup_{i=0}^{N_k} \widetilde{G}_{ik}$ and that $\mathcal{O}_{\widetilde{X}}(\Sigma)$ is trivial on $\widetilde{G}_{ik}$. We denote the transition functions of $\mathcal{O}_{\widetilde{X}}(\Sigma)$ by $g_{i_1 k_1, i_2 k_2} \in \mathcal{O}(\widetilde{G}_{i_1 k_1} \cap \widetilde{G}_{i_2 k_2})$. Hence

$$\psi_{i_1 k} = \psi_{i_2 k} + \log |g_{i_1 k, i_2 k}| \,, \quad g_{i_1 k, i_2 k} = \widetilde{g}_{i_1 i_2, k} \,.$$

We now construct a smooth Hermitian metric on $\mathcal{O}_{\widetilde{X}}(\Sigma)$ with weights s_{ik} on $\widetilde{G}_{ik}$. Let $\{\rho_k\}_{1 \leq k \leq N}$ be a smooth partition of unity on X so that $\operatorname{supp} \rho_k \subset U_k$. For a fixed $k_1 \in \{1, \ldots, N\}$ let $\varphi_{k_1}^{ik}$ be a function defined on $\pi^{-1}(U_{k_1}) \cap \widetilde{G}_{ik}$ as follows: if $x \in \widetilde{G}_{i_1 k_1}$ then

$$\varphi_{k_1}^{ik}(x) = \psi_{i_1 k_1}(x) + \log |g_{ik, i_1 k_1}(x)| \,.$$

Note that $\varphi_{k_1}^{ik}$ is well defined and smooth on $\pi^{-1}(U_{k_1}) \cap \widetilde{G}_{ik}$. Indeed, if $x \in \widetilde{G}_{i_1 k_1} \cap \widetilde{G}_{i'_1 k_1} \cap \widetilde{G}_{ik}$ then $g_{i'_1 k_1, i_1 k_1} \cdot g_{i_1 k_1, ik} \cdot g_{ik, i'_1 k_1} = 1$ so

$$\begin{aligned} \psi_{i'_1 k_1}(x) + \log |g_{ik, i'_1 k_1}(x)| &= \psi_{i_1 k_1}(x) + \log |g_{i'_1 k_1, i_1 k_1}(x)| + \log |g_{ik, i'_1 k_1}(x)| \\ &= \psi_{i_1 k_1}(x) + \log |g_{ik, i_1 k_1}(x)| \,. \end{aligned}$$

Next we define

$$s_{ik} = \sum_{k_1=1}^{N} (\rho_{k_1} \circ \pi) \varphi_{k_1}^{ik} \ \text{ on } \widetilde{G}_{ik} \,.$$

We claim that $s_{ik} = s_{i'k'} + \log|g_{ik,i'k'}|$ on $\widetilde{G}_{ik} \cap \widetilde{G}_{i'k'}$, so $\{s_{ik}\}$ defines a smooth Hermitian metric on $\mathcal{O}_{\widetilde{X}}(\Sigma)$. For this we show that if $k_1 \in \{1, \ldots, N\}$ then

$$(\rho_{k_1} \circ \pi)\varphi^{ik}_{k_1} = (\rho_{k_1} \circ \pi)(\varphi^{i'k'}_{k_1} + \log|g_{ik,i'k'}|).$$

Let $x \in \pi^{-1}(U_{k_1})$ and assume $x \in \widetilde{G}_{i_1k_1} \cap \widetilde{G}_{ik} \cap \widetilde{G}_{i'k'}$. Since $g_{i'k',i_1k_1} \cdot g_{i_1k_1,ik} \cdot g_{ik,i'k'} = 1$ we obtain

$$\begin{aligned} \varphi^{ik}_{k_1}(x) &= \psi_{i_1k_1}(x) + \log|g_{ik,i_1k_1}(x)| = \psi_{i_1k_1}(x) + \log|g_{i'k',i_1k_1}(x)| + \log|g_{ik,i'k'}(x)| \\ &= \varphi^{i'k'}_{k_1}(x) + \log|g_{ik,i'k'}(x)|\,, \end{aligned}$$

which proves our claim.

We finally show that the desired metric θ on F is the metric defined by the weights $\{-s_{ik}\}$. By a standard compactness argument it suffices to prove that for every $x \in \widetilde{G}_{ik}$ there exists a constant $C_x > 0$ such that for $C > C_x$ the function $C\eta_k \circ \pi - s_{ik}$ is strictly psh at x. We write $T_x\widetilde{X} = E_x \oplus F_x$, where $E_x = \ker d\pi(x)$. Note that the Levi form

$$\mathcal{L}(\eta_k \circ \pi)(x)(t, t') = \mathcal{L}\eta_k(\pi(x))(d\pi(x)(t), d\pi(x)(t'))\,,$$

so

$$\mathcal{L}(\eta_k \circ \pi)(x)(t, t') = 0\,,\ \ \forall\, t \in T_x\widetilde{X},\, t' \in E_x\,. \tag{8}$$

Moreover, since η_k is strictly psh at $\pi(x) \in B_k$ and $d\pi(x)$ is injective on F_x we deduce that

$$\mathcal{L}(\eta_k \circ \pi)(x)(t, t) > 0\,,\ \ \forall\, t \in F_x \setminus \{0\}\,. \tag{9}$$

The formula of s_{ik} implies that for each $t \in E_x$

$$-\mathcal{L}s_{ik}(x)(t, t) = -\sum_{k_1=1}^{N} \rho_{k_1}(\pi(x))\mathcal{L}\psi_{i_1k_1}(x)(t, t) \geq 0\,,$$

since each function $-\psi_{i_1k_1}$ is psh on $\widetilde{G}_{i_1k_1}$. If $\rho_{k_1}(\pi(x)) > 0$ we may assume that $x \in \widetilde{G}_{i_1k_1} \cap \widetilde{G}_{ik}$ and we will show that $-\mathcal{L}\psi_{i_1k_1}(x)(t, t) > 0$ for all $t \in E_x \setminus \{0\}$. As $\pi^{-1}(U_{k_1}) = \Gamma_{k_1} \subset U_{k_1} \times \mathbb{P}^{N_{k_1}} \subset B_{k_1} \times \mathbb{P}^{N_{k_1}}$, x has a neighborhood $\widetilde{U} \subset \widetilde{X}$ such that $\widetilde{U} \subset B_{k_1} \times G_{i_1k_1}$. Recall that on $\widetilde{U}$, $\psi_{i_1k_1} = \psi'_{i_1k_1} \circ \pi_{2k_1}$ and $\pi = \pi_{1k_1}$. We consider π_{1k_1}, π_{2k_1} as restrictions of the canonical projections $\pi_{1k_1} : B_{k_1} \times G_{i_1k_1} \longrightarrow B_{k_1}$, $\pi_{2k_1} : B_{k_1} \times G_{i_1k_1} \longrightarrow G_{i_1k_1}$. Since $t \in E_x \subset \ker d\pi_{1k_1}(x)$ and $t \neq 0$ it follows that $d\pi_{2k_1}(x)(t) \neq 0$. Therefore

$$-\mathcal{L}\psi_{i_1k_1}(x)(t, t) = -\mathcal{L}\psi'_{i_1k_1}(\pi_{2k_1}(x))(d\pi_{2k_1}(x)(t), d\pi_{2k_1}(x)(t)) > 0\,,$$

as $-\psi'_{i_1k_1}$ is strictly psh on $G_{i_1k_1}$. This yields

$$-\mathcal{L}s_{ik}(x)(t, t) > 0\,,\ \ \forall\, t \in E_x \setminus \{0\}\,. \tag{10}$$

By (8), (9) and (10) we conclude that there exists a constant $C_x > 0$ such that if $C > C_x$ then $\mathcal{L}(C\eta_k \circ \pi - s_{ik})(x)(t, t) > 0$ for all $t \in T_x\widetilde{X} \setminus \{0\}$. This finishes the proof. □

We look now at the nature of the base space X as implied by the hypotheses made on the curvature of the bundles involved in our results. Recall that a compact irreducible complex space X of dimension n is called Moishezon if X possesses n algebraically independent meromorphic functions, i. e. if the transcendence degree of the field of meromorphic functions on X equals the complex dimension of X. Let X' and X be compact irreducible spaces and $h : X' \to X$ be a proper modification. Then h induces an isomorphism of the fields of meromorphic functions on X' and X, respectively, [MM2, Theorem 2.1.18], hence X' is Moishezon if and only if X is Moishezon. Moishezon [Moi] showed that if X is a Moishezon space, then there exists a proper modification $h : X' \to X$, obtained by a finite number of blow-ups with smooth centers, such that X' is a projective algebraic manifold (for a proof see also [MM2, Theorem 2.2.16]).

Lemma 2.2 yields in particular the following:

Proposition 2.3. *If (X,ω) is a compact (reduced) Hermitian space endowed with a singular Hermitian holomorphic line bundle (L,h) such that $c_1(L,h) \geq \varepsilon\omega$ for some constant $\varepsilon > 0$ then X is Moishezon.*

Proof. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities as in Lemma 2.2, with exceptional divisor Σ and with Hermitian form $\Omega = C\pi^\star\omega + c_1(F,\theta)$, where $F = \mathcal{O}_{\widetilde{X}}(-\Sigma)$. Consider the line bundles $E_p = \pi^\star L^p \otimes F$ with singular Hermitian metrics $\theta_p = \pi^\star h^p \otimes \theta$. Then

$$c_1(E_p,\theta_p) = p\pi^\star c_1(L,h) + c_1(F,\theta) \geq p\varepsilon\pi^\star\omega + c_1(F,\theta) \geq \Omega\,,$$

provided that $p\varepsilon \geq C$. Hence $\widetilde{X}$ carries a singular Hermitian holomorphic line bundle with strictly positive curvature in the sense of currents. By [JS] (see also [MM2, Theorem 2.3.8]) it follows that $\widetilde{X}$ is Moishezon and hence X, too. If (X,ω) is Kähler, then $\widetilde{X}$ is already projective. Indeed, $(\widetilde{X},\Omega)$ is Kähler and Moishezon, so by a theorem of Moishezon is projective (see e. g. [MM2, Theorem 2.2.26]). □

The paper [M] (see also Theorems 3.4.10 and 3.4.14 of [MM2]) gives an integral criterion for a complex space with isolated singularities to be Moishezon, generalizing the criterion of Siu and Demailly from the smooth case.

We recall next the following projectivity criterion.

Proposition 2.4 (Grauert)**.** *If (X,ω) is a compact (reduced) Hermitian space endowed with a $\mathscr{C}^2$ Hermitian holomorphic line bundle (L,h) such that $c_1(L,h) \geq \varepsilon\omega$ for some constant $\varepsilon > 0$, then L is ample and X is projective.*

This follows from [Gr, Satz 2, p.343], see also [Gr, Satz 3, p.346]. If X is normal, Grauert shows actually more: if Ω is a continuous Kähler metric whose de Rham cohomology class is integral, then there exists a Hermitian holomorphic line bundle (L,h) with $\mathscr{C}^2$ Hermitian metric and $c_1(L,h) = \Omega$; hence L is ample and X is projective.

2.4. **L^2-estimates for $\overline{\partial}$.** The following version of Demailly's estimates for the $\overline{\partial}$ operator [D2, Théorème 5.1] will be needed in our proofs.

Theorem 2.5 ([D2])**.** *Let Y, $\dim Y = n$, be a complete Kähler manifold and let Ω be a Kähler form on Y (not necessarily complete) such that its Ricci form $\mathrm{Ric}_\Omega \geq -2\pi B\Omega$ on Y, for some constant $B > 0$. Let (L_p,h_p) be singular Hermitian holomorphic line bundles on Y such that $c_1(L_p,h_p) \geq 2a_p\Omega$, where $a_p \to \infty$ as $p \to \infty$, and fix p_0 such that $a_p \geq B$ for all $p > p_0$. If $p > p_0$ and $g \in L^2_{0,1}(Y,L_p,loc)$ verifies $\overline{\partial}g = 0$ and $\int_Y |g|^2_{h_p}\,\Omega^n < \infty$ then there exists $u \in L^2_{0,0}(Y,L_p,loc)$ such that $\overline{\partial}u = g$ and $\int_Y |u|^2_{h_p}\,\Omega^n \leq \frac{1}{a_p}\int_Y |g|^2_{h_p}\,\Omega^n$.*

Proof. We write $L_p = F_p \otimes K_Y$, where $F_p = L_p \otimes K_Y^{-1}$, and K_Y is endowed with the metric h^{K_Y} induced by Ω. If $\theta_p = h_p \otimes h^{K_Y^{-1}}$ is the metric induced on F_p then

$$c_1(F_p, \theta_p) = c_1(L_p, h_p) - c_1(K_Y, h^{K_Y}) = c_1(L_p, h_p) + \frac{1}{2\pi}\,\mathrm{Ric}_\Omega \geq (2a_p - B)\Omega \geq a_p\Omega$$

for $p > p_0$. The theorem follows by using the isometries $L^2_{0,j}(Y, L_p, loc) \cong L^2_{n,j}(Y, F_p, loc)$, $j = 0, 1$, and applying [D2, Théorème 5.1] (see also [CM2, Corollaries 4.5 and 4.6]). □

2.5. **Special weights of Hermitian metrics on reference covers.** Let (X, ω) be a compact Kähler manifold of dimension n. Let (U, z), $z = (z_1, \dots, z_n)$, be local coordinates centered at a point $x \in X$. For $r > 0$ and $y \in U$ we denote by

$$\Delta^n(y, r) = \{z \in U : |z_j - y_j| \leq r,\ j = 1, \dots, n\}$$

the (closed) polydisk of polyradius $(r, \dots, r)$ centered at y. The coordinates (U, z) are called Kähler at $y \in U$ if

$$\omega_z = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j + O(|z - y|^2) \text{ on } U. \tag{11}$$

Definition 2.6. A *reference cover* of X consists of the following data: for $j = 1, \dots, N$, a set of points $x_j \in X$ and

(1) Stein open simply connected coordinate neighborhoods $(U_j, w^{(j)})$ centered at $x_j \equiv 0$,
(2) $R_j > 0$ such that $\Delta^n(x_j, 2R_j) \Subset U_j$ and for every $y \in \Delta^n(x_j, 2R_j)$ there exist coordinates on U_j which are Kähler at y,
(3) $X = \bigcup_{j=1}^N \Delta^n(x_j, R_j)$.

Given the reference cover as above we set $R = \min R_j$.

We can construct a reference cover as follows: For $x \in X$ fix a Stein open simply connected neighborhood U of $x \equiv 0 \in \mathbb{C}^n$ and fix $R > 0$ such that the polydisk $\Delta^n(x, 2R) \Subset U$ and for every $y \in \Delta^n(x, 2R)$ there exist coordinates (U, z) which are Kähler at y. By compactness there exist $x_1, \dots, x_N \in X$ such that the above conditions are fulfilled.

On U_j we consider the differential operators D^α_w, $\alpha \in \mathbb{N}^{2n}$, corresponding to the real coordinates associated to $w = w^{(j)}$. For a function $\varphi \in \mathscr{C}^k(U_j)$ we set

$$\|\varphi\|_k = \|\varphi\|_{k,w} = \sup\big\{|D^\alpha_w \varphi(w)| : w \in \Delta^n(x_j, 2R_j), |\alpha| = k\big\}. \tag{12}$$

Let (L, h) be a Hermitian holomorphic line bundle on X, where the metric h is of class $\mathcal{C}^\ell$. Note that $L|_{U_j}$ is trivial. For $k \leq \ell$ set

$$\begin{aligned} \|h\|_{k,U_j} &= \inf\big\{\|\varphi_j\|_k : \varphi_j \in C^\ell(U_j) \text{ is a weight of } h \text{ on } U_j\big\}, \\ \|h\|_k &= \max\big\{1, \|h\|_{k,U_j} : 1 \leq j \leq N\big\}. \end{aligned} \tag{13}$$

Recall that φ_j is a weight of h on U_j if there exists a holomorphic frame e_j of L on U_j such that $|e_j|_h = e^{-\varphi_j}$.

Lemma 2.7. *There exists a constant $C > 1$ (depending on the reference cover) with the following property: Given any Hermitian line bundle (L, h) on X, any $j \in \{1, \dots, N\}$ and any $x \in \Delta^n(x_j, R_j)$ there exist coordinates $z = (z_1, \dots, z_n)$ on $\Delta^n(x, R)$ which are centered at $x \equiv 0$ and Kähler coordinates for x such that*

(i) $n!\, dm \leq (1 + Cr^2)\omega^n$ and $\omega^n \leq (1 + Cr^2)n!\, dm$ hold on $\Delta^n(x, r)$ for $r < R$ where $dm = dm(z)$ is the Euclidean volume relative to the coordinates z,

(ii) (L, h) has a weight φ on $\Delta^n(x, R)$ with $\varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + \widetilde{\varphi}(z)$, where $\lambda_j \in \mathbb{R}$ and $|\widetilde{\varphi}(z)| \leq C\|h\|_3 |z|^3$ for $z \in \Delta^n(x, R)$.

Proof. By the properties of a reference cover there exist coordinates z on U_j which are Kähler for $x \in \Delta^n(x_j, R_j)$ so $\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z}_j + O(|z - x|^2)$ and (i) holds with a constant C_j uniform for $x \in \Delta^n(x_j, R_j)$. Let e_j be a frame of L on U_j and φ' be a weight of h on U_j with $|e_j|_h = e^{-\varphi'}$ and $\|\varphi'\|_{3,w} \leq 2\|h\|_3$, cf. (12)-(13). By translation we may assume $x = 0$ and write $\varphi'(z) = \operatorname{Re} f(z) + \varphi'_2(z) + \varphi'_3(z)$, where $f(z)$ is a holomorphic polynomial of degree ≤ 2 in z, $\varphi'_2(z) = \sum_{j,k=1}^n \mu_{jk} z_j \bar{z}_k$, and $\operatorname{Re} f(z) + \varphi'_2(z)$ is the Taylor polynomial of order 2 of φ at 0. Note that $\|\varphi'\|_{3,z} \leq C'_j \|\varphi'\|_{3,w} \leq 2C'_j \|h\|_3$, where $\|\varphi'\|_{3,z}$ is the sup norm on $\Delta^n(x, R)$ of the derivatives of order 3 of φ' in the coordinates z and C'_j is a constant uniform for $x \in \Delta^n(x_j, R_j)$. This follows from the fact that z, w are coordinates on $U_j \Supset \Delta^n(x_j, 2R_j)$. We conclude that $|\varphi'_3(z)| \leq 2C'_j \|h\|_3 |z|^3$ for $z \in \Delta^n(x, R)$.

Consider the frame $\widetilde{e}_j = e^f e_j$ of L on U_j. Then $|\widetilde{e}_j|_h = e^{\operatorname{Re} f - \varphi'} = e^{-\varphi}$, so $\varphi(z) := \varphi'_2(z) + \varphi'_3(z)$ is a weight of h on U_j. By a unitary change of coordinates we may assume that $\varphi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + \tilde{\varphi}(z)$. In these new coordinates $\omega^n/n!$ and $\widetilde{\varphi}(z)$ satisfy the desired estimates with a constant C_j uniform for $x \in \Delta^n(x_j, R_j)$. Finally we let $C = \max_{1 \leq j \leq N} C_j$. □

3. Proofs of Theorems 1.1, 1.2 and 1.3

We use here the notations introduced in Section 2 and we start with two lemmas that will be needed in the proof of Theorem 1.1:

Lemma 3.1. *Let D be a divisor in a complex manifold Y and θ be a smooth Hermitian metric on $\mathcal{O}(-D)$ with weight φ over $Y \setminus D$. Then $\lim_{y \to x,\, y \in Y \setminus D} \varphi(y) = -\infty$ for every $x \in D$.*

Proof. Let U_α be a neighborhood of x where D has defining function $f_\alpha \in \mathcal{O}(U_\alpha)$ and θ has weight $\varphi_\alpha \in C^\infty(U_\alpha)$. The transition function of $\mathcal{O}(-D)$ on $U_\alpha \cap (Y \setminus D)$ is $g = 1/f_\alpha$, so $\varphi_\alpha = \varphi + \log|g|$ and $\varphi = \varphi_\alpha + \log|f_\alpha|$. □

Lemma 3.2. *Let (X, ω), (L_p, h_p) verify assumptions (A)-(B), and let $F = \mathcal{O}_{\widetilde{X}}(-\Sigma)$, θ, $\Omega = C\pi^\star\omega + c_1(F, \theta)$ be as in Lemma 2.2. Then there exist numbers $\alpha \in (0, 1)$, $b_p \in \mathbb{N}$, and singular Hermitian metrics $\widetilde{h}_p$ on $\pi^\star L_p|_{\widetilde{X} \setminus \Sigma}$ such that $a_p \geq Cb_p$, $b_p \to \infty$ and $b_p/A_p \to 0$ as $p \to \infty$, $\widetilde{h}_p \geq \alpha^{b_p} \pi^\star h_p$ and $c_1(\pi^\star L_p, \widetilde{h}_p) \geq b_p \Omega$ on $\widetilde{X} \setminus \Sigma$. Moreover, for every open relatively compact subset $\widetilde{U}$ of $\widetilde{X} \setminus \Sigma$ there exists a constant $\beta_{\widetilde{U}} > 1$ such that $\widetilde{h}_p \leq \beta_{\widetilde{U}}^{b_p} \pi^\star h_p$ on $\widetilde{U}$.*

Proof. If h', h'' are singular Hermitian metrics on some holomorphic line bundle G then by $h' \geq ch''$ we mean that $|e|^2_{h'} \geq c|e|^2_{h''}$ for any $e \in G$. Consider the line bundles $E_p =$

$\pi^\star L_p \otimes F^{b_p}$ with metrics $\theta_p = \pi^\star h_p \otimes \theta^{b_p}$, where $b_p \in \mathbb{N}$. Then

$$c_1(E_p, \theta_p) = \pi^\star c_1(L_p, h_p) + b_p c_1(F, \theta) \geq a_p \pi^\star \omega + b_p c_1(F, \theta) \geq b_p \Omega\,,$$

provided that $a_p \geq C b_p$. Since F is trivial on $\widetilde{X} \setminus \Sigma$ we have $\pi^\star L_p|_{\widetilde{X}\setminus\Sigma} \cong E_p|_{\widetilde{X}\setminus\Sigma}$ and we can find a smooth weight φ of θ on $\widetilde{X} \setminus \Sigma$ by setting $|f|^2_\theta = e^{-2\varphi}$, where f is a holomorphic frame of F on $\widetilde{X} \setminus \Sigma$. Let $\widetilde{h}_p$ be the metric of $\pi^\star L_p|_{\widetilde{X}\setminus\Sigma}$ defined by $\widetilde{h}_p = e^{-2b_p\varphi} \pi^\star h_p$. Then $c_1(\pi^\star L_p, \widetilde{h}_p) = c_1(E_p, \theta_p) \geq b_p \Omega$ on $\widetilde{X} \setminus \Sigma$. Since $\varphi \in C^\infty(\widetilde{X} \setminus \Sigma)$ and by Lemma 3.1 $\varphi(y) \to -\infty$ as $y \to \Sigma$, it follows that there exists $\alpha \in (0, 1)$ such that $e^{-2\varphi} \geq \alpha$ on $\widetilde{X} \setminus \Sigma$. Moreover, if $\widetilde{U}$ is an open relatively compact subset of $\widetilde{X} \setminus \Sigma$, there exists $\beta_{\widetilde{U}} > 1$ such that $e^{-2\varphi} \leq \beta_{\widetilde{U}}$ on $\widetilde{U}$. These imply that $\widetilde{h}_p \geq \alpha^{b_p} \pi^\star h_p$ on $\widetilde{X} \setminus \Sigma$ and $\widetilde{h}_p \leq \beta_{\widetilde{U}}^{b_p} \pi^\star h_p$ on $\widetilde{U}$. The lemma follows if we choose $b_p \in \mathbb{N}$ such that $a_p \geq C b_p$, $b_p \to \infty$ and $b_p / A_p \to 0$ as $p \to \infty$.

□

Proof of Theorem 1.1. Note that (ii) follows at once from (i) by using (7). The proof of (i) will be done in two steps.

Step 1. We show here that $\frac{1}{A_p} \log P_p \to 0$ as $p \to \infty$ in $L^1_{loc}(X_{reg}, \omega^n)$. Fix $x \in X_{reg}$, $W \Subset X_{reg}$ a contractible Stein coordinate neighborhood of x, $r_0 > 0$ such that the (closed) ball $V := B(x, 2r_0) \subset W$, and set $U := B(x, r_0)$. Since W is contractible and Stein $L_p|_W$ are trivial so there exist local holomorphic frames $e'_p : W \longrightarrow L_p$, for all p. Let ψ'_p be the corresponding psh weights of h_p on W, $|e'_p|^2_{h_p} = e^{-2\psi'_p}$. The sequence of currents $\{\frac{1}{A_p} c_1(L_p, h_p)\}$ has uniformly bounded mass, so it follows from [DS2, Proposition A.16] (see also [DNS]) that there exist psh functions ψ_p on $\operatorname{int} V$ such that $dd^c \psi_p = c_1(L_p, h_p)$ and the sequence $\{\frac{1}{A_p} \psi_p\}$ is bounded, hence relatively compact [H, Theorem 3.2.12], in $L^1_{loc}(\operatorname{int} V, \omega^n)$. Since $\psi'_p - \psi_p$ is pluriharmonic, we have $\psi'_p - \psi_p = \mathbf{Re}\, f_p$ for some function $f_p \in \mathcal{O}(\operatorname{int} V)$. Considering the local frames $e_p = e^{f_p} e'_p$ of $L_p|_{\operatorname{int} V}$, we obtain

$$|e_p|^2_{h_p} = e^{2\mathbf{Re}\, f_p} |e'_p|^2_{h_p} = e^{-2\psi_p},$$

i.e. ψ_p is the psh weight of h_p relative to the frame e_p.

Let $\{b_p\}$ be as in Lemma 3.2. We prove that there exist constants $C' > 1$ and $p_0 \in \mathbb{N}$ such that

$$-\frac{b_p \log C'}{A_p} \leq \frac{\log P_p(z)}{A_p} \leq \frac{\log(C' r^{-2n})}{A_p} + \frac{2}{A_p} \left(\max_{B(z,r)} \psi_p - \psi_p(z) \right) \tag{14}$$

holds for all $p > p_0$, $0 < r < r_0$, and $z \in U$ with $\psi_p(z) > -\infty$. The upper bound in (14) is proved exactly as the corresponding upper bound from the proof of Theorem 5.1 in [CM1]. For the lower bound, we show that there exist $c \in (0, 1)$ and $p_0 \in \mathbb{N}$ such that if $p > p_0$ and $z \in U$, $\psi_p(z) > -\infty$, then there exists a section $S_{z,p} \in H^0_{(2)}(X, L_p)$ verifying $S_{z,p}(z) \neq 0$ and

$$c^{b_p} \|S_{z,p}\|^2_p \leq |S_{z,p}(z)|^2_{h_p}\,. \tag{15}$$

This implies that

$$\frac{1}{A_p}\,\log P_p(z)=\frac{1}{A_p}\,\max_{\|S\|_p=1}\log|S(z)|^2_{h_p}\geq\frac{b_p\log c}{A_p}\,.$$

To prove (15) we work on $\widetilde{X}\setminus\Sigma$ and recall that $\pi:\widetilde{X}\setminus\Sigma\longrightarrow X_{reg}$ is a biholomorphism. Let $\Omega\geq\pi^\star\omega$ be the Kähler form on $\widetilde{X}$ constructed in Lemma 2.2, and b_p, $\widetilde{h}_p$ be as in Lemma 3.2. Then $b_p\to\infty$ and

$$c_1(\pi^\star L_p|_{\widetilde{X}\setminus\Sigma},\widetilde{h}_p)\geq b_p\Omega\ \text{ on }\ \widetilde{X}\setminus\Sigma.$$

Since Ω is a Kähler form on $\widetilde{X}$ we have $\mathrm{Ric}_\Omega\geq-2\pi B\Omega$ on $\widetilde{X}$, for some constant $B>0$. Moreover, since $\widetilde{X}$ is a compact Kähler manifold, $\widetilde{X}\setminus\Sigma$ has a complete Kähler metric (see [D2], [O]). Repeating the argument in the proof of [CM1, Theorem 5.1] (see also Theorems 4.7 and 4.8 in [CM2]) one applies the Ohsawa-Takegoshi extension theorem [OT] and then solves a suitable $\overline{\partial}$-equation using Theorem 2.5, as in [D5, Proposition 3.1], [D7, Section 9], to show the following: there exist $C''>1$, $p_0\in\mathbb{N}$, such that if $p>p_0$ and $\widetilde{z}\in\pi^{-1}(U)$, $\psi_p\circ\pi(\widetilde{z})>-\infty$, then there exists a section $\widetilde{S}\in H^0(\widetilde{X}\setminus\Sigma,\pi^\star L_p)$ verifying $\widetilde{S}(\widetilde{z})\neq0$ and

$$\int_{\widetilde{X}\setminus\Sigma}|\widetilde{S}|^2_{\widetilde{h}_p}\,\Omega^n\leq C''|\widetilde{S}(\widetilde{z})|^2_{\widetilde{h}_p}\,.$$

It is important to recall here that the weight of $\widetilde{h}_p$ near $\widetilde{z}$ is the sum of $\psi_p\circ\pi$ and a smooth function. By Lemma 3.2 we have for all p that $\widetilde{h}_p\geq\alpha^{b_p}\pi^\star h_p$ on $\widetilde{X}\setminus\Sigma$ and $\widetilde{h}_p\leq\beta^{b_p}\pi^\star h_p$ on $\pi^{-1}(U)$, for some constant $\beta>1$. Since $\Omega\geq\pi^\star\omega$ these imply that

$$\alpha^{b_p}\int_{\widetilde{X}\setminus\Sigma}|\widetilde{S}|^2_{\pi^\star h_p}\,\pi^\star\omega^n\leq C''\beta^{b_p}|\widetilde{S}(\widetilde{z})|^2_{\pi^\star h_p}\,.$$

Fix a constant $c\in(0,1)$ with $C''c^{b_p}\leq(\alpha/\beta)^{b_p}$ for all p and let $S_{z,p}$, where $z=\pi(\widetilde{z})$, be the section of $L_p|_{X_{reg}}$ induced by $\widetilde{S}$. Since X is normal it follows that $S_{z,p}$ extends to a holomorphic section of L_p on X. Moreover, $S_{z,p}(z)\neq0$ and (15) holds for $S_{z,p}$ and c. The proof of (14) is now complete.

To conclude Step 1, it suffices to show that every subsequence of $\{\frac{1}{A_p}\log P_p\}$ has a subsequence convergent to 0 in $L^1(U,\omega^n)$. Without loss of generality, we prove that $\{\frac{1}{A_p}\log P_p\}$ has a subsequence convergent to 0 in $L^1(U,\omega^n)$. Since $\{\frac{1}{A_p}\,\psi_p\}$ is locally uniformly upper bounded in $\operatorname{int}V$ and relatively compact in $L^1_{loc}(\operatorname{int}V,\omega^n)$, there exists a subsequence $\{\psi_{p_j}\}$ so that

$$\frac{1}{A_{p_j}}\,\psi_{p_j}\to\psi=\Big(\limsup\frac{1}{A_{p_j}}\,\psi_{p_j}\Big)^\star$$

in $L^1_{loc}(\operatorname{int}V,\omega^n)$ and a.e. on $\operatorname{int}V$, where $\psi\in PSH(\operatorname{int}V)$. Moreover, by the Hartogs lemma,

$$\limsup\frac{1}{A_{p_j}}\,\max_{B(z,r)}\psi_{p_j}\leq\max_{B(z,r)}\psi\,,$$

for each $z \in U$ and $r < r_0$ (see e.g. [H, Theorem 3.2.13]). Letting $p_j \to \infty$ in (14) we get, since $b_p/A_p \to 0$, that

$$0 \leq \liminf \frac{\log P_{p_j}(z)}{A_{p_j}} \leq \limsup \frac{\log P_{p_j}(z)}{A_{p_j}} \leq 2\left(\max_{B(z,r)} \psi - \psi(z)\right)$$

for a.e. $z \in U$ and every $r < r_0$. Letting $r \searrow 0$ and using the upper semicontinuity of ψ we deduce that $\frac{1}{A_{p_j}} \log P_{p_j} \to 0$ a.e. on U. Since $\{\frac{1}{A_p}\psi_p\}$ is locally uniformly upper bounded in $\operatorname{int} V$ it follows by (14) that there exists a constant $C'' > 0$ such that

$$\left|\frac{1}{A_{p_j}} \log P_{p_j}\right| \leq C'' - \frac{2}{A_{p_j}}\psi_{p_j} \text{ a.e. on } U\,.$$

As $\psi_{p_j}, \psi \in L^1(U,\omega^n)$, $\frac{1}{A_{p_j}}\psi_{p_j} \to \psi$ a.e. on U and in $L^1(U,\omega^n)$, and since $\frac{1}{A_{p_j}} \log P_{p_j} \to 0$ a.e. on U, the generalized Lebesgue dominated convergence theorem implies that $\frac{1}{A_{p_j}} \log P_{p_j} \to 0$ in $L^1(U,\omega^n)$.

Step 2. To complete the proof of (i) we show here that there exists a compact set $K \subset X$ such that $X_{sing} \subset \operatorname{int} K$ and

$$\frac{1}{A_p}\int_K |\log P_p|\,\omega^n \to 0 \text{ as } p \to \infty\,.$$

Let $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ be the Bergman spaces from Lemma 2.1. We note that there exists a constant $M > 0$ such that

$$\int_{\widetilde{X}} c_1(\pi^\star L_p, \pi^\star h_p) \wedge \Omega^{n-1} \leq MA_p\,,\ \ \forall\, p \geq 1\,. \tag{16}$$

Indeed, since $c_1(L_p,h_p),\, T_0 \in \mathscr{T}(X)$ and $c_1(L_p,h_p) \leq A_pT_0$, we have $c_1(\pi^\star L_p, \pi^\star h_p) = \pi^\star c_1(L_p,h_p) \leq A_p\pi^\star T_0$. This yields (16) with $M = \int_{\widetilde{X}} \pi^\star T_0 \wedge \Omega^{n-1}$.

We fix now $y \in \Sigma$ and $\widetilde{W}$ an open neighborhood of y in $\widetilde{X}$ biholomorphic to a ball in $\mathbb{C}^n$. Using (16) and repeating an argument from Step 1, we find holomorphic frames $\widetilde{e}_p$ of $\pi^\star L_p|_{\widetilde{W}}$ for which the corresponding psh weights $\widetilde{\psi}_p$ of $\pi^\star h_p$ are such that $\{\frac{1}{A_p}\widetilde{\psi}_p\}$ is bounded in $L^1_{loc}(\widetilde{W},\Omega^n)$, hence locally uniformly upper bounded on $\widetilde{W}$ and relatively compact in $L^1_{loc}(\widetilde{W},\Omega^n)$. If $\{\widetilde{S}^p_j : 1 \leq j \leq d_p\}$ is an orthonormal basis of $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$ we write $\widetilde{S}^p_j = \widetilde{s}^p_j\widetilde{e}_p$ and let $\widetilde{v}_p = \frac{1}{2}\log(\sum_{j=1}^{d_p}|\widetilde{s}^p_j|^2) \in PSH(\widetilde{W})$. By Lemma 2.1, $P_p \circ \pi$ is the Bergman kernel function of $H^0_{(2)}(\widetilde{X}, \pi^\star L_p)$, hence

$$\frac{1}{A_p}\widetilde{v}_p - \frac{1}{A_p}\widetilde{\psi}_p = \frac{1}{2A_p}\log P_p \circ \pi\,.$$

We claim that $\frac{1}{A_p}\log P_p \circ \pi \to 0$ in $L^1_{loc}(\widetilde{W},\Omega^n)$. As in Step 1, it suffices to produce a subsequence with this property. Since $\{\frac{1}{A_p}\widetilde{\psi}_p\}$ is relatively compact in $L^1_{loc}(\widetilde{W},\Omega^n)$ there is a subsequence $\{\frac{1}{A_{p_j}}\widetilde{\psi}_{p_j}\}$ convergent in $L^1_{loc}(\widetilde{W},\Omega^n)$ to a psh function $\widetilde{\psi}$ on $\widetilde{W}$. By Step 1, $\frac{1}{A_p}\log P_p \to 0$ as $p \to \infty$ in $L^1_{loc}(X_{reg},\omega^n)$, so as $\pi : \widetilde{X} \setminus \Sigma \longrightarrow X_{reg}$ is biholomorphic,

$\frac{1}{A_p}\log P_p\circ\pi\to 0$ in $L^1_{loc}(\widetilde{W}\setminus\Sigma,\Omega^n)$. Thus $\frac{1}{A_{p_j}}\widetilde{v}_{p_j}\to\widetilde{\psi}$ in $L^1_{loc}(\widetilde{W}\setminus\Sigma,\Omega^n)$. By the argument in the proof of Theorem 1.1 (i) from [CM2], we see that $\{\frac{1}{A_{p_j}}\widetilde{v}_{p_j}\}$ is locally uniformly upper bounded in $\widetilde{W}$ and it converges to $\widetilde{\psi}$ in $L^1_{loc}(\widetilde{W},\Omega^n)$. Hence $\frac{1}{A_{p_j}}\log P_{p_j}\circ\pi\to 0$ in $L^1_{loc}(\widetilde{W},\Omega^n)$, which proves our claim.

Since $y\in\Sigma$ was arbitrary and Σ is compact we can find an open set $\widetilde{U}\supset\Sigma$ so that $\frac{1}{A_p}\log P_p\circ\pi\to 0$ in $L^1(\widetilde{U},\Omega^n)$. Then we fix a compact set $K\subset X$ such that $X_{sing}\subset\operatorname{int}K$ and $\pi^{-1}(K)\subset\widetilde{U}$. As $\pi:\widetilde{X}\setminus\Sigma\longrightarrow X_{reg}$ is biholomorphic and $\pi^\star\omega\leq\Omega$ we have

$$\begin{aligned}\frac{1}{A_p}\int_K|\log P_p|\,\omega^n &= \frac{1}{A_p}\int_{K\cap X_{reg}}|\log P_p|\,\omega^n=\frac{1}{A_p}\int_{\pi^{-1}(K)\setminus\Sigma}|\log P_p\circ\pi|\,\pi^\star\omega^n\\ &\leq \frac{1}{A_p}\int_{\pi^{-1}(K)}|\log P_p\circ\pi|\,\Omega^n\to 0\ \text{ as }\ p\to\infty\,.\end{aligned}$$

This concludes Step 2 and finishes the proof of Theorem 1.1. □

Theorem 1.1 holds under the weaker hypotheses obtained by replacing the domination assumption (2) by (16), where $\pi:\widetilde{X}\to X$ is the resolution of singularities fixed in Section 2. The proof goes through without change. Due to the presence of singularities of X, it is not clear whether (16) holds true without the domination condition (2), i. e. if the mass of the pull-back currents $\pi^*c_1(L_p,h_p)$ on $\widetilde{X}$ is dominated by the mass of the currents $c_1(L_p,h_p)$ on X, uniformly in p.

Proof of Theorem 1.2. We repeat the argument in Step 1 from the previous proof, working directly on (X,L_p,h_p) with the Kähler form ω. □

Proof of Theorem 1.3. We use methods from [Be, Sect. 2]. Let us consider a reference cover of X as in Definition 2.6. We fix $x\in X$, so $x\in\Delta^n(x_j,R_j)$ for some j, and we pick coordinates z centered at x as in Lemma 2.7. Let

$$\varphi_p(z)=\varphi_p'(z)+\widetilde{\varphi}_p(z)\,,\ \ \varphi_p'(z)=\sum_{j=1}^n\lambda_j^p|z_j|^2\,,$$

be a weight of h_p on $\Delta^n(x,R)$ so that $\tilde{\varphi}_p$ verifies (ii) in Lemma 2.7 and let e_p be a frame of L_p on U_j with $|e_p|_{h_p}=e^{-\varphi_p}$. Finally, let $r_p\in(0,R/2)$ be an arbitrary number which will be specified later.

We begin by estimating the norm of a section $S\in H^0(X,L_p)$ at x. Writing $S=se_p$, where $s\in\mathcal{O}(\Delta^n(x,R))$, we obtain by the sub-averaging inequality for psh functions

$$|S(x)|^2_{h_p}=|s(0)|^2\leq\frac{\int_{\Delta^n(0,r_p)}|s|^2e^{-2\varphi_p'}\,dm}{\int_{\Delta^n(0,r_p)}e^{-2\varphi_p'}\,dm}\,.$$

If $C>1$ is the constant from Lemma 2.7, we have

$$\begin{aligned}\int_{\Delta^n(0,r_p)}|s|^2e^{-2\varphi'_p}\,dm &\leq (1+Cr_p^2)\exp\big(2\max_{\Delta^n(0,r_p)}\widetilde{\varphi}_p\big)\int_{\Delta^n(0,r_p)}|s|^2e^{-2\varphi_p}\frac{\omega^n}{n!}\\ &\leq (1+Cr_p^2)\exp\big(2C\|h_p\|_3\,r_p^3\big)\frac{1}{n!}\|S\|_p^2\,.\end{aligned}$$

Set

$$E(r):=\int_{|\xi|\leq r}e^{-2|\xi|^2}\,dm(\xi)=\frac{\pi}{2}\left(1-e^{-2r^2}\right),$$

where dm is the Lebesgue measure on $\mathbb{C}$. Since $\lambda_j^p\geq a_p$ we obtain

$$\frac{E(r_p\sqrt{a_p}\,)^n}{\lambda_1^p\ldots\lambda_n^p}\leq\int_{\Delta^n(0,r_p)}e^{-2\varphi'_p}\,dm\leq\int_{\mathbb{C}^n}e^{-2\varphi'_p}\,dm=\frac{(\pi/2)^n}{\lambda_1^p\ldots\lambda_n^p}\,.$$

Combining these estimates it follows that

$$|S(x)|_{h_p}^2\leq\frac{(1+Cr_p^2)\exp\big(2C\|h_p\|_3\,r_p^3\big)}{n!E(r_p\sqrt{a_p})^n}\,\lambda_1^p\ldots\lambda_n^p\,\|S\|_p^2\,. \tag{17}$$

By taking the supremum in (17) over all $S\in H^0(X,L_p)$ with $\|S\|_p=1$ we get

$$\frac{P_p(x)}{\lambda_1^p\ldots\lambda_n^p}\leq\frac{(1+Cr_p^2)\exp\big(2C\|h_p\|_3\,r_p^3\big)}{n!E(r_p\sqrt{a_p})^n}\,,\quad\forall\,r_p\in(0,R/2). \tag{18}$$

For the lower estimate on P_p, let $0\leq\chi\leq1$ be a cut-off function on $\mathbb{C}^n$ with support in $\Delta^n(0,2)$, $\chi\equiv1$ on $\Delta^n(0,1)$, and set $\chi_p(z)=\chi(z/r_p)$. Then $F=\chi_pe_p$ is a section of L_p and $|F(x)|_{h_p}=|e_p(x)|_{h_p}=e^{-\varphi_p(0)}=1$. We have

$$\begin{aligned}\|F\|_p^2&\leq\int_{\Delta^n(0,2r_p)}e^{-2\varphi_p}\omega^n\\ &\leq n!(1+4Cr_p^2)\exp\big(16C\|h_p\|_3\,r_p^3\big)\int_{\Delta^n(0,2r_p)}e^{-2\varphi'_p}\,dm\\ &\leq\left(\frac{\pi}{2}\right)^n n!\,\frac{(1+4Cr_p^2)\exp\big(16C\|h_p\|_3\,r_p^3\big)}{\lambda_1^p\ldots\lambda_n^p}\,.\end{aligned} \tag{19}$$

Set $\alpha=\overline{\partial}F$. Since $\|\overline{\partial}\chi_p\|^2=\|\overline{\partial}\chi\|^2/r_p^2$ we obtain as above

$$\|\alpha\|_p^2=\int_{\Delta^n(0,2r_p)}|\overline{\partial}\chi_p|^2e^{-2\varphi_p}\omega^n\leq\frac{\|\overline{\partial}\chi\|^2}{r_p^2}\left(\frac{\pi}{2}\right)^n n!\,\frac{(1+4Cr_p^2)\exp\big(16C\|h_p\|_3\,r_p^3\big)}{\lambda_1^p\ldots\lambda_n^p}\,.$$

Since $a_p\to\infty$ there exists $p_0\in\mathbb{N}$ such that for $p>p_0$ we can solve the $\overline{\partial}$–equation by Theorem 2.5. We get a smooth section G of L_p with $\overline{\partial}G=\alpha=\overline{\partial}F$ and

$$\|G\|_p^2\leq\frac{1}{a_p}\|\alpha\|_p^2\leq\frac{\|\overline{\partial}\chi\|^2}{a_pr_p^2}\left(\frac{\pi}{2}\right)^n n!\,\frac{(1+4Cr_p^2)\exp\big(16C\|h_p\|_3\,r_p^3\big)}{\lambda_1^p\ldots\lambda_n^p}\,. \tag{20}$$

Since $F = e_p$ is holomorphic on $\Delta^n(0, r_p)$, G is holomorphic on $\Delta^n(0, r_p)$ as $\overline{\partial} G = \overline{\partial} F = 0$ there. So the estimate (17) applies to G on $\Delta^n(0, r_p)$ and gives

$$\begin{aligned} |G(x)|^2_{h_p} &\leq \frac{(1 + Cr_p^2)\exp\big(2C\|h_p\|_3\, r_p^3\big)}{n!E(r_p\sqrt{a_p})^n}\lambda_1^p \ldots \lambda_n^p \|G\|_p^2 \\ &\leq \frac{\|\overline{\partial}\chi\|^2}{a_p r_p^2 E(r_p\sqrt{a_p})^n}\left(\frac{\pi}{2}\right)^n (1 + 4Cr_p^2)^2 \exp\big(18C\|h_p\|_3\, r_p^3\big). \end{aligned}$$

Let $S = F - G \in H^0(X, L_p)$. Then

$$\begin{aligned} |S(x)|^2_{h_p} &\geq (|F(x)|_{h_p} - |G(x)|_{h_p})^2 = (1 - |G(x)|_{h_p})^2 \\ &\geq \left[1 - \left(\frac{\pi}{2}\right)^{n/2} \frac{\|\overline{\partial}\chi\|(1 + 4Cr_p^2)}{r_p\sqrt{a_p}E(r_p\sqrt{a_p})^{n/2}} \exp\big(9C\|h_p\|_3\, r_p^3\big)\right]^2 =: K_1(r_p)\,. \end{aligned}$$

Moreover, by (19) and (20)

$$\|S\|_p^2 \leq (\|F\|_p + \|G\|_p)^2 \leq \left(\frac{\pi}{2}\right)^n n!\, \frac{K_2(r_p)}{\lambda_1^p \ldots \lambda_n^p}\,,$$

where

$$K_2(r_p) = (1 + 4Cr_p^2)\exp\big(16C\|h_p\|_3\, r_p^3\big)\left(1 + \frac{\|\overline{\partial}\chi\|}{r_p\sqrt{a_p}}\right)^2.$$

Therefore

$$P_p(x) \geq \frac{|S(x)|^2_{h_p}}{\|S\|_p^2} \geq \frac{\lambda_1^p \ldots \lambda_n^p}{(\frac{\pi}{2})^n n!}\frac{K_1(r_p)}{K_2(r_p)}\,. \tag{21}$$

Note that at x, $\omega_x = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$, $c_1(L_p, h_p)_x = dd^c\varphi_p(0) = \frac{i}{\pi}\sum_{j=1}^n \lambda_j^p dz_j \wedge d\bar{z}_j$, thus

$$\frac{c_1(L_p, h_p)_x^n}{\omega_x^n} = \left(\frac{2}{\pi}\right)^n \lambda_1^p \ldots \lambda_n^p\,.$$

By (18) and (21) we conclude that

$$\frac{1}{n!}\frac{K_1(r_p)}{K_2(r_p)} \leq P_p(x)\frac{\omega_x^n}{c_1(L_p, h_p)_x^n} \leq \frac{1}{n!}\, K_3(r_p) \tag{22}$$

holds for every $x \in X$, $r_p < R/2$ and $p > p_0$, where

$$K_3(r_p) = \left(\frac{\pi/2}{E(r_p\sqrt{a_p})}\right)^n (1 + Cr_p^2)\exp\big(2C\|h_p\|_3\, r_p^3\big)\,.$$

By (5) we have that $\varepsilon_p = \|h_p\|_3^{1/3} a_p^{-1/2} \to 0$. We set

$$r_p := \varepsilon_p^{1/3}\|h_p\|_3^{-1/3} = \varepsilon_p^{-2/3} a_p^{-1/2}\,, \text{ so } \|h_p\|_3\, r_p^3 = \varepsilon_p\,,\ r_p\sqrt{a_p} = \varepsilon_p^{-2/3}\,.$$

As $\|h_p\|_3 \geq 1$, we have $r_p \leq \varepsilon_p^{1/3}$, thus $r_p \to 0$ as $p \to \infty$. With this choice for r_p we obtain

$$K_3(r_p) \leq \left(\frac{\pi/2}{E(\varepsilon_p^{-2/3})}\right)^n (1 + C\varepsilon_p^{2/3})\exp\big(2C\varepsilon_p\big) \leq 1 + C'\varepsilon_p^{2/3}\,,\quad \frac{K_1(r_p)}{K_2(r_p)} \geq 1 - C'\varepsilon_p^{2/3}\,,$$

where $C' > 0$ is a constant depending only on the reference cover. Therefore (6) follows from (22) and the proof is complete. □

Remark 3.3. Theorem 1.3 shows that

$$\lim_{p\to\infty} P_p(x)\,\frac{\omega_x^n}{c_1(L_p,h_p)_x^n} = \frac{1}{n!} \quad \text{uniformly on } X.$$

This is a generalization of the asymptotic expansion of the Bergman kernel [Ca, DLM1, MM2, MM3, R, T, Z] for $(L_p,h_p) = (L^p,h^p)$, where (L,h) is a positive line bundle with smooth metric h. Indeed, if $(L_p,h_p) = (L^p,h^p)$, we have $a_p = p$ and $\|h_p\|_3 \leq C_h p$, where C_h is a constant depending only on h and the reference cover. Hence

$$\varepsilon_p = \|h_p\|_3^{1/3} a_p^{-1/2} \leq C_h^{1/3} p^{-1/6}\,,$$

so condition (5) is fulfilled. Estimate (6) yields

$$\left|\frac{P_p(x)}{p^n}\,\frac{\omega_x^n}{c_1(L,h)_x^n} - \frac{1}{n!}\right| \leq \frac{C C_h^{2/9}}{p^{1/9}}\,, \tag{23}$$

hence

$$|P_p(x) - b_0(x)p^n| \leq C' p^{n-\frac{1}{9}}\,, \ \text{ where } b_0(x) := \frac{1}{n!}\frac{c_1(L,h)_x^n}{\omega_x^n}\,. \tag{24}$$

By the above mentioned papers there exists $C > 0$ such that $\left|P_p(x) - b_0(x)p^n\right| \leq Cp^{n-1}$ on X, which gives a sharper estimate of $P_p(x)$ than (23)-(24). On the other hand, the method used here can handle the much more general case of sequences of line bundles (L_p,h_p) satisfying the minimal hypotheses of Theorem 1.3. Note also that our normalization for the leading term b_0 of the asymptotics of P_p is different from the one in the papers above by a factor $1/n!$, since we use the volume form ω^n instead of $\omega^n/n!$.

4. ZEROS OF HOLOMORPHIC SECTIONS AND APPROXIMATION RESULTS

We assume as before that (X,ω), $(L_p,h_p) \to X$ satisfy conditions (A) and (B). Consider the unit sphere $\mathcal{S}^p \subset H^0_{(2)}(X,L_p)$, $d_p = \dim H^0_{(2)}(X,L_p)$. We identify the unit sphere $\mathcal{S}^p$ to the unit sphere $\mathbf{S}^{2d_p-1}$ in $\mathbb{C}^{d_p}$ by

$$a = (a_1,\ldots,a_{d_p}) \in \mathbf{S}^{2d_p-1} \longmapsto S_a = \sum_{j=1}^{d_p} a_j S_j^p \in \mathcal{S}^p,$$

and we let λ_p be the probability measure on $\mathcal{S}^p$ induced by the normalized surface measure on $\mathbf{S}^{2d_p-1}$, denoted also by λ_p (i. e. $\lambda_p(\mathbf{S}^{2d_p-1}) = 1$). Consider the probability space $\mathcal{S}_\infty = \prod_{p=1}^\infty \mathcal{S}^p$ endowed with the probability measure $\lambda_\infty = \prod_{p=1}^\infty \lambda_p$. We denote by $[\sigma = 0]$ the current of integration (with multiplicities) over the zero set of a nontrivial section $\sigma \in H^0_{(2)}(X,L_p)$.

Theorem 4.1. *Let* (X,ω)*,* (L_p,h_p)*,* $p\geq 1$*, verify assumptions (A)-(B) and assume that* $\sum_{p=1}^{\infty}\frac{1}{A_p^2}<\infty$*. Then for* λ_∞*-a.e. sequence* $\{\sigma_p\}_{p\geq1}\in\mathcal{S}_\infty$ *we have in the weak sense of currents on* X *that*

$$\lim_{p\to\infty}\frac{1}{A_p}\big([\sigma_p=0]-c_1(L_p,h_p)\big)=0\,.$$

Moreover, if $\lim_{p\to\infty}\frac{1}{A_p}c_1(L_p,h_p)=T$ *for some positive closed current* T *of bidegree* $(1,1)$ *on* X*, then for* λ_∞*-a.e. sequence* $\{\sigma_p\}_{p\geq1}\in\mathcal{S}_\infty$*,*

$$\lim_{p\to\infty}\frac{1}{A_p}[\sigma_p=0]=T\;\text{ weakly on } X.$$

Proof. The arguments in [SZ1, SZ2] (see also [CM2, Section 5.2]; in all these papers $A_p=p$) imply that for λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq1}\in\mathcal{S}_\infty$,

$$\lim_{p\to\infty}\frac{1}{A_p}\big([\sigma_p=0]-\gamma_p\big)=0$$

weakly in the sense of currents on X. Indeed, by working with a countable set of test forms and since

$$\int_X[\sigma_p=0]\wedge\omega^{n-1}=\int_X\gamma_p\wedge\omega^{n-1}=\int_X c_1(L_p,h_p)\wedge\omega^{n-1}=A_p\,,$$

it suffices to show that, for a fixed test form θ, one has

$$\lim_{p\to\infty}\frac{1}{A_p}\big\langle[\sigma_p=0]-\gamma_p,\theta\big\rangle=0\,,\tag{25}$$

for λ_∞-a.e. $\sigma=\{\sigma_p\}_{p\geq1}\in\mathcal{S}_\infty$. Let

$$Y_p:\mathcal{S}_\infty\longrightarrow\mathbb{C}\,,\quad Y_p(\sigma)=\frac{1}{A_p}\big\langle[\sigma_p=0]-\gamma_p,\theta\big\rangle\,.$$

The calculations in [SZ1, Sec. 3.1-3.3] show that

$$\int_{\mathcal{S}_\infty}Y_p\,d\lambda_\infty=0\,,\;\;\int_{\mathcal{S}_\infty}|Y_p|^2\,d\lambda_\infty\leq\frac{AC_\theta}{A_p^2}\,,\;\text{ with } A=\frac{1}{\pi^2}\int_{\mathbb{C}^2}(\log|z_1|)^2e^{-|z_1|^2-|z_2|^2}\,dz\,,$$

where dz is the Lebesgue measure on $\mathbb{C}^2$, and C_θ is a constant depending only on θ. Then (25) follows since

$$\int_{\mathcal{S}_\infty}\left(\sum_{p=1}^{\infty}|Y_p|^2\right)d\lambda_\infty\leq A\,C_\theta\sum_{p=1}^{\infty}\frac{1}{A_p^2}<+\infty\,.$$

We now conclude by Theorem 1.1. □

We show next that equidistribution results hold not only for the zeros of random sequences of holomorphic sections but also for the logarithms of their pointwise norms.

Theorem 4.2. *Let* (X,ω)*,* (L_p,h_p)*,* $p\geq1$*, verify assumptions (A)-(B) and assume that*

$$\liminf_{p\to\infty}\frac{\log d_p}{A_p}=0\,.\tag{26}$$

Then there exists an increasing sequence of natural numbers $\{p_j\}_{j\geq 1}$ such that for λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$ we have

$$\lim_{j\to\infty} \frac{\log |\sigma_{p_j}|_{h_{p_j}}}{A_{p_j}} = 0 \quad \textit{in } L^1(X, \omega^n).$$

Proof. Using (26) we can find a sequence of integers $p_j \nearrow \infty$ such that $\sum_{j=1}^\infty \frac{\log d_{p_j}}{A_{p_j}} < \infty$. We define

$$Y_p : \mathcal{S}_\infty \longrightarrow \mathbb{R}\,,\; Y_p(\sigma) = \frac{1}{A_p} \int_X \log \frac{|\sigma_p|_{h_p}}{\sqrt{P_p}}\, \omega^n \;, \quad \text{where } \sigma = \{\sigma_p\}_{p\geq 1}\,.$$

By Theorem 1.1 we have $\frac{1}{A_p} \log P_p \to 0$ as $p \to \infty$, in $L^1(X, \omega^n)$. Since $\log \frac{|\sigma_p|_{h_p}}{\sqrt{P_p}} \leq 0$ on X for $\sigma = \{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$, it suffices to show that $Y_{p_j}(\sigma) \to 0$ as $j \to \infty$ for λ_∞-a.e. $\sigma \in \mathcal{S}_\infty$. By Tonelli's theorem we have

$$\int_{\mathcal{S}_\infty} Y_p(\sigma)\, d\lambda_\infty = \frac{1}{A_p} \int_{\mathcal{S}^p} \left(\int_X \log \frac{|\sigma_p|_{h_p}}{\sqrt{P_p}}\, \omega^n \right) d\lambda_p = \frac{1}{A_p} \int_X \left(\int_{\mathcal{S}^p} \log \frac{|\sigma_p|_{h_p}}{\sqrt{P_p}}\, d\lambda_p \right) \omega^n\,.$$

For a fixed $x \in X$, we write $S_l^p = s_l^p e_p$ for some holomorphic frame e_p of L_p near x and we set

$$u = (u_1, \ldots, u_{d_p})\;, \quad u_l = \frac{s_l^p}{\sqrt{|s_1^p|^2 + \ldots + |s_{d_p}^p|^2}}\,.$$

Then the integral

$$\int_{\mathcal{S}^p} \log \frac{|\sigma_p(x)|_{h_p}}{\sqrt{P_p(x)}}\, d\lambda_p = \int_{\mathbf{S}^{2d_p-1}} \log |a \cdot u(x)|\, d\lambda_p = -\mathbf{I}(d_p)$$

is independent of x, where $a \cdot u = a_1 u_1 + \ldots + a_{d_p} u_{d_p}$. Here

$$\mathbf{I}(k) := -\frac{1}{\mathrm{area}(\mathbf{S}^{2k-1})} \int_{\mathbf{S}^{2k-1}} \log |z_k|\, d\mathcal{A}\,,$$

$\mathbf{S}^{2k-1}$ is the unit sphere in $\mathbb{C}^k$ with surface measure $d\mathcal{A}$, and $z = (z_1, \ldots, z_k)$. Evaluating this integral by means of spherical coordinates one can show that there exist numbers $a, b > 1$ such that

$$\mathbf{I}(k) \leq a \log k + b\;, \quad \forall\, k \geq 1\,.$$

It follows that

$$\int_{\mathcal{S}_\infty} Y_p(\sigma)\, d\lambda_\infty \geq -\frac{a \log d_p + b}{A_p} \int_X \omega^n\,.$$

The definition of the sequence $\{p_j\}_{j\geq 1}$ shows that

$$\sum_{j=1}^\infty \int_{\mathcal{S}_\infty} Y_{p_j}(\sigma)\, d\lambda_\infty > -\infty\,.$$

Since $Y_p \leq 0$ this implies that $\sum_{j=1}^\infty Y_{p_j}$ converges in $L^1(\mathcal{S}_\infty, \lambda_\infty)$, hence $Y_{p_j}(\sigma) \to 0$ as $j \to \infty$ for λ_∞-a.e. $\sigma \in \mathcal{S}_\infty$. ☐

Let us give two general examples in which condition (26) holds true.

Proposition 4.3. *Let (X,ω), (L_p,h_p), $p\geq 1$, verify assumptions (A)-(B) and assume that X is smooth and that each line bundle L_p has a continuous metric h'_p with the following property: every $x\in X$ has a contractible Stein coordinate neighborhood W_x on which each metric h'_p has a weight ψ'_p such that the family $\{\psi'_p/A_p\}_{p\geq 1}$ is equicontinuous on W_x. Then*

$$\lim_{p\to\infty}\frac{\log\dim H^0(X,L_p)}{A_p}=0\,.$$

Proof. Let $\varepsilon>0$ and let P'_p be the Bergman kernel function of the space $H^0(X,L_p)$ with respect to the metrics h'_p and ω. For $x\in X$ fix $r_x>0$ so that the (closed) ball $B(x,2r_x)\subset W_x$ and let $U_x=B(x,r_x)$. The proof of the upper bound in (14) works for any metric on L_p (see also [CM1, (7)]) and shows that

$$P'_p(z)\leq C_x r^{-2n}\exp\left(2\max_{B(z,r)}\psi'_p-2\psi'_p(z)\right),$$

for any $p\geq 1$, $r<r_x$ and $z\in U_x$, where C_x is a constant depending only on x. The equicontinuity assumption implies that there exists $r_1=r_1(x,\varepsilon)<r_x$ such that $2\max_{B(z,r_1)}\psi'_p-2\psi'_p(z)\leq A_p\varepsilon$ for all $p\geq 1$ and $z\in U_x$, hence $P'_p(z)\leq C_x r_1^{-2n}\exp(A_p\varepsilon)$. A standard compactness argument now shows that there exists a constant $C'=C'(\varepsilon)>0$ such that $P'_p\leq C'\exp(A_p\varepsilon)$ holds on X for all $p\geq 1$. It follows that

$$\dim H^0(X,L_p)=\int_X P'_p\,\omega^n\leq C'\exp(A_p\varepsilon)\int_X\omega^n\ ,\ \ p\geq 1\,,$$

which implies the conclusion of the proposition. □

The second general example is provided by the class of semi-ample line bundles. Recall that a line bundle L on X is called semi-ample if L^k is globally generated for some $k>0$, or, equivalently, the space $H^0(X,L^k)$ has no base locus.

Proposition 4.4. *Let (X,ω), (L_p,h_p), $p\geq 1$, verify assumptions (A)-(B) and assume that X is smooth and that each line bundle L_p is semi-ample. Then there exist an integer $N>0$ and a constant $C>0$ depending only on ω such that*

$$\dim H^0(X,L_p)\leq CA_p^N\ ,\ \ \forall\, p\geq 1\,.$$

Proof. Since L_p is big, X is Moishezon and hence projective since it is Kähler. By the main theorem in [KM], there exists a polynomial $Q(y,z)$ depending only on $\dim X$ such that for any semi-ample line bundle L on X one has that

$$\dim H^0(X,L)\leq Q\left(\int_X c_1(L)^n\ ,\int_X c_1(L)^{n-1}\wedge c_1(X)\right).$$

Lemma 4.5 following this proof implies that there exists a constant $C'>0$ depending only on (X,ω) such that

$$\int_X c_1(L_p)^n\leq C'A_p^n\ ,\ \ \int_X c_1(L_p)^{n-1}\wedge c_1(X)\leq C'A_p^{n-1}\,.$$

The conclusion now follows. □

Lemma 4.5. *Let (X,ω) be a compact Kähler manifold of dimension n and let β be a real valued closed form of type $(1,1)$ on X. Then there exists a constant $C>0$ depending only on ω and β such that for any pseudoeffective class $\alpha\in H^{1,1}(X,\mathbb{R})$ we have*

$$\int_X \alpha^k\wedge\beta^{n-k}\leq C\|\alpha\|^k\ ,\ k=1,\ldots,n\ ,\quad \textit{where } \|\alpha\|=\int_X\alpha\wedge\omega^{n-1}.$$

Proof. If $\|\alpha\|=0$ then $\alpha=0$ since α is pseudoeffective, so we can assume $\|\alpha\|>0$. Let $\theta\in\alpha$ be a smooth form and $T=\theta+dd^c\varphi$ be a positive closed current, where φ is a θ-psh function. The Lelong numbers $\nu(T,x)<C_1\|\alpha\|$ for all $x\in X$, where C_1 is a constant depending only on ω (see e.g. [Bo, Lemma 2.5]). Demailly's regularization theorem [D5] shows that there exists a sequence of smooth functions $\varphi_k\searrow\varphi$ such $\theta+dd^c\varphi_k\geq -C_2\lambda_k\omega$, where λ_k are continuous functions on X, $\lambda_k(x)\searrow\nu(T,x)$ as $k\to\infty$ for every $x\in X$, and C_2 is a constant depending only on ω. We fix k so that $\lambda_k(x)<C_1\|\alpha\|$ for every $x\in X$ and let $R=\theta+dd^c\varphi_k$ and $R'=R+C_3\|\alpha\|\omega$, where $C_3=C_1C_2$, so $R'\geq 0$. Next we set $\beta'=\beta+c\omega$, where $c>0$ is chosen so that $\beta'\geq 0$. Since $R',\beta'\geq 0$ we obtain

$$\begin{aligned}\int_X\alpha^k\wedge\beta^{n-k} &= \int_X R^k\wedge\beta^{n-k}=\int_X(R'-C_3\|\alpha\|\omega)^k\wedge(\beta'-c\omega)^{n-k}\\ &\leq \int_X(R'+C_3\|\alpha\|\omega)^k\wedge(\beta'+c\omega)^{n-k}\leq C_4\|R'+C_3\|\alpha\|\omega\|^k\,,\end{aligned}$$

where C_4 is a constant depending only on the Kähler form $\beta'+c\omega$ (hence on β and ω). The lemma follows since

$$\|R'+C_3\|\alpha\|\omega\|=\int_X(R+2C_3\|\alpha\|\omega)\wedge\omega^{n-1}=\left(1+2C_3\int_X\omega^n\right)\|\alpha\|\,.$$

□

We conclude this section by discussing an application of the above results to the problem of approximation of positive closed currents of bidegree $(1,1)$ on X by currents of integration along analytic hypersurfaces of X. Let $\mathscr{A}(X)$ be the space of positive closed currents $T\in\mathcal{D}'_{n-1,n-1}(X)$ with the property that there exist a sequence of singular Hermitian holomorphic line bundles $\{(F_p,h^{F_p})\}_{p\geq1}$ with $c_1(F_p,h^{F_p})\geq 0$ and a sequence of natural numbers $N_p\to\infty$ such that

$$\lim_{p\to\infty}\frac{1}{N_p}\,c_1(F_p,h^{F_p})=T\,.$$

If $X_{sing}\neq\emptyset$ we require in addition that there exists a current $T_0\in\mathscr{T}(X)$ (depending on T) such that for all $p\geq1$ we have $\frac{1}{N_p}\,c_1(F_p,h^{F_p})\leq T_0$.

When X is smooth the space $\mathscr{A}(X)$ is the closure in $\mathcal{D}'_{n-1,n-1}(X)$ of the convex cone generated by positive closed integral currents. Recall that a real closed current $T\in\mathcal{D}'_{n-1,n-1}(X)$ is called integral if its de Rham cohomology class $[T]$ belongs to $H^{1,1}(X,\mathbb{R})\cap H^2(X,\mathbb{Z})$. A current T is integral if and only if there exists a singular Hermitian holomorphic line bundle (L,h) on X with $c_1(L,h)=T$ (see e.g. [MM2, Lemma 2.3.5]).

Theorem 4.6. *Let (X,ω) be a compact normal Kähler space and (L,h) be a singular Hermitian holomorphic line bundle on X sucht that $c_1(L,h)\geq\varepsilon\omega$ for some $\varepsilon>0$. If $T\in\mathscr{A}(X)$ then there exist a sequence of singular Hermitian holomorphic line bundles $\{(L_p,h_p)\}_{p\geq1}$ with $c_1(L_p,h_p)\geq 0$ and a sequence of natural numbers $N_p\to\infty$ such that for λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq1}\in\mathcal{S}_\infty$,*

$$\lim_{p\to\infty}\frac{1}{N_p}[\sigma_p=0]=T\ \text{ weakly on } X.$$

Here the probability space $(\mathcal{S}_\infty,\lambda_\infty)$ is associated as above to the sequence $\{(L_p,h_p)\}_{p\geq1}$.

Proof. Since $T\in\mathscr{A}(X)$ there exist line bundles $\{(F_p,h^{F_p})\}_{p\geq1}$ with $c_1(F_p,h^{F_p})\geq 0$ and a sequence of natural numbers $N_p\to\infty$, $p\to\infty$, such that $\lim_{p\to\infty}\frac{1}{N_p}c_1(F_p,h^{F_p})=T$. Moreover, if $X_{sing}\neq\emptyset$ there exists a current $T_0\in\mathscr{T}(X)$ such that $c_1(F_p,h^{F_p})\leq N_pT_0$ for all $p\geq1$. We can assume without loss of generality that $N_p\geq p$; otherwise replace (F_p,h^{F_p}) by $(F_p^{m_p},(h^{F_p})^{m_p})$ and N_p by m_pN_p, with a convenient $m_p\in\mathbb{N}$. We fix a sequence $b_p\in\mathbb{N}$ such that $b_p\leq N_p$ for all $p\geq1$ and $b_p\to\infty$, $b_p/N_p\to0$ as $p\to\infty$. Let

$$L_p=F_p\otimes L^{b_p}\,,\quad h_p=h^{F_p}\otimes h^{b_p}\,.$$

The conclusion follows from Theorem 4.1 since $c_1(L_p,h_p)\geq b_p\,c_1(L,h)\geq b_p\,\varepsilon\omega$ and

$$c_1(L_p,h_p)\leq N_pT_0+b_p\,c_1(L,h)\leq N_p\big(T_0+c_1(L,h)\big)\,,$$
$$\frac{c_1(L_p,h_p)}{N_p}\to T\,,\quad \frac{\|c_1(L_p,h_p)\|}{N_p}\to\|T\|\,,\ \text{ as } p\to\infty\,.$$

□

5. APPLICATIONS

5.1. **Powers of a line bundle.** Theorem 1.1 applies for the sequence $(L_p,h_p)=(L^p,h^p)$, where (L,h) is a a singular Hermitian holomorphic line bundle on X with strictly positive curvature current.

Corollary 5.1. *Let (X,ω) be a compact normal Kähler space and (L,h) be a singular Hermitian holomorphic line bundle on X such that $c_1(L,h)\geq\varepsilon\omega$ for some $\varepsilon>0$. Then:*

(i) $\frac{1}{p}\log P_p\to0$ as $p\to\infty$, in $L^1(X,\omega^n)$.

(ii) $\frac{1}{p}\gamma_p\to c_1(L,h)$ as $p\to\infty$, in the weak sense of currents on X.

Indeed, assumptions (A) and (B) are satisfied with $a_p=p\,\varepsilon$ and $T_0=c_1(L,h)/\|c_1(L,h)\|$, where $\|c_1(L,h)\|:=\int_X c_1(L,h)\wedge\omega^{n-1}$.

We consider now the case when the curvature current of the singular metric is not necessarily Kähler.

Corollary 5.2. *Let (L,h) be a singular Hermitian holomorphic line bundle on the compact normal Kähler space (X,ω) such that $c_1(L,h)\geq0$ and assume that L has a singular metric h_0 with $c_1(L,h_0)\geq\varepsilon\omega$ for some $\varepsilon>0$. Let $\{n_p\}_{p\geq1}$ be a sequence of natural numbers such that $n_p\to\infty$ and $n_p/p\to0$ as $p\to\infty$. Let $h_p=h^{p-n_p}\otimes h_0^{n_p}$ and P_p, γ_p, $\mathcal{S}^p$ be the Bergman kernel function, Fubini-Study current, and respectively the unit sphere, associated* *to the spaces* $H^0_{(2)}(X, L_p) = H^0_{(2)}(X, L^p, h_p)$. *Then, as* $p \to \infty$*, we have* $\frac{1}{p} \log P_p \to 0$ *in* $L^1(X, \omega^n)$*,* $\frac{1}{p}\gamma_p \to c_1(L, h)$ *and* $\frac{1}{p}\big[\sigma_p = 0\big] \to c_1(L, h)$ *in the weak sense of currents on* X*, for* λ_∞*-a.e. sequence* $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$.

Proof. Note that

$$c_1(L^p, h_p) = (p - n_p)c_1(L, h) + n_p c_1(L, h_0) \geq \varepsilon n_p \omega \,, \ \frac{A_p}{p} \to \|c_1(L, h)\| = \|c_1(L, h_0)\| > 0 \,.$$

Hence

$$c_1(L^p, h_p) \leq p(c_1(L, h) + c_1(L, h_0)) \leq A_p T_0 \,, \ \text{ with } T_0 = \frac{2}{\|c_1(L, h)\|}\,(c_1(L, h) + c_1(L, h_0)) \,,$$

and Theorems 1.1 and 4.1 apply in this setting and conclude the proof. □

5.2. **Powers of ample line bundles.** We specialize in the sequel the results of the previous corollary to the case when (X, ω) is a projective manifold with a polarization (L, h_0), where L is a positive line bundle on X endowed with a smooth Hermitian metric h_0 such that $c_1(L, h_0) = \omega$. The set of singular Hermitian metrics h on L with $c_1(L, h) \geq 0$ is in one-to-one correspondence to the set $PSH(X, \omega)$ of ω-plurisubharmonic (ω-psh) functions on X, by associating to $\varphi \in PSH(X, \omega)$ the metric $h_\varphi = e^{-2\varphi} h_0$. Note that

$$c_1(L, h_\varphi) = \omega + dd^c\varphi =: \omega_\varphi \,.$$

Corollary 5.3. *Let* (X, ω) *be a compact Kähler manifold and* (L, h_0) *be a positive line bundle on* X *with* $c_1(L, h_0) = \omega$*. Let* $\varphi \in PSH(X, \omega)$ *and* h_p *be the metric on* L^p *constructed as in Corollary 5.2 using the metric* $h = h_\varphi$ *on* L*. Let* γ_p*,* $\mathcal{S}^p$ *be the Fubini-Study current and the unit sphere associated to the space* $H^0_{(2)}(X, L^p, h_p)$*. Then* $\frac{1}{p}\big[\sigma_p = 0\big] \to \omega_\varphi$ *as* $p \to \infty$ *weakly on* X*, for* λ_∞*-a.e. sequence* $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$*. Moreover, if* φ *is continuous then*

$$\frac{1}{p^k}\,\gamma_p^k \to \omega_\varphi^k = (\omega + dd^c\varphi)^k \ \textit{ weakly on } X, \ \textit{ for } k \leq n \,.$$

Proof. The first conclusion follows directly from Corollary 5.2. If φ is continuous and P_p is the Bergman kernel function of the space $H^0_{(2)}(X, L^p, h_p)$ one can proceed as in the proof of [CM1, Theorem 5.3] to show that $\frac{1}{p} \log P_p \to 0$ uniformly on X. This implies the second conclusion of the corollary, as in [CM1, Theorem 5.4]. □

Note that since h_0 is smooth we have that $H^0_{(2)}(X, L^p, h^p) \subset H^0_{(2)}(X, L^p, h_p)$. Moreover, if the metric $h = h_\varphi$ is bounded (i.e. φ is bounded) then equality holds, $H^0_{(2)}(X, L^p, h^p) = H^0_{(2)}(X, L^p, h_p) = H^0(X, L^p)$.

We remark now that, instead of working with random sections of spheres, one can identify $H^0_{(2)}(X, L_p)$ to $\mathbb{C}^{d_p}$ by

$$a = (a_1, \ldots, a_{d_p}) \in \mathbb{C}^{d_p} \longmapsto S_a = \sum_{j=1}^{d_p} a_j S_j^p \in H^0_{(2)}(X, L_p) \,,$$

and one can consider a_j, $1 \leq j \leq d_p$, as independent identically distributed Gaussian random variables on $\mathbb{C}$. Thus the probability space $(\mathcal{S}^p, \lambda_p)$ is replaced by $(H^0_{(2)}(X, L_p), \mu_p)$, where

$$d\mu_p(z) = \pi^{-d_p} e^{-(|z_1|^2+\ldots+|z_{d_p}|^2)} dm(z)\,,$$

and $dm(z)$ is the Lebesgue measure on $\mathbb{C}^{d_p}$. Let $\mu_\infty = \prod_{p=1}^\infty \mu_p$ be the product measure on the space $\mathcal{H} = \prod_{p=1}^\infty H^0_{(2)}(X, L_p)$. Then Theorems 4.1 and 4.2 hold for μ_∞-a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{H}$ with similar proofs. Hence Corollary 5.3 can be seen as a generalization of Theorem 5.2 in [BL] which deals with the special case when $\varphi = \mathcal{V}^\star_{K,q}$ is the weighted ω-psh global extremal function of a compact $K \subset X$. Using different methods and a different inner product on $H^0(X, L^p)$ (note that $\mathcal{V}^\star_{K,q}$ is bounded) it is shown in [BL, Theorem 5.2] that

$$\frac{1}{p}\big[\sigma_p = 0\big] \to \omega + dd^c \mathcal{V}^\star_{K,q} \text{ for } \mu_\infty\text{-a. e. sequences } \{\sigma_p\}_{p\geq 1} \in \prod_{p=1}^\infty H^0(X, L^p).$$

On the other hand [BL, Theorem 5.2] holds for more general probability measures than μ_∞ (see [BL, (2.1) and (2.2)]).

A particularly interesting case is when $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$ and $\omega = \omega_{_{\rm FS}}$ is the Fubini-Study metric in Corollary 5.3. In this case the class $PSH(\mathbb{P}^n, \omega_{_{\rm FS}})$ is in one-to-one correspondence with the Lelong class of psh functions on $\mathbb{C}^n$ of logarithmic growth, and the sections in $H^0(\mathbb{P}^n, \mathcal{O}(p))$ can be identified to polynomials of degree $\leq p$ on $\mathbb{C}^n$ (see e.g. [BL, Sect. 5]). Therefore Corollary 5.3 yields a general equidistribution result for the zeros of a random sequence of polynomials on $\mathbb{C}^n$. For related results see [Ba], [BL] and references therein.

5.3. **Tensor products of powers of several line bundles.** Another typical application of Theorem 1.1 is the following.

Corollary 5.4. *Let (X, ω) be a compact normal Kähler space. Assume that (F_j, h^{F_j}), $1 \leq j \leq k$, are singular Hermitian holomorphic line bundles with $c_1(F_j, h^{F_j}) \geq 0$ and $c_1(F_1, h^{F_1}) \geq \varepsilon\omega$, for some $\varepsilon > 0$. Let $T = \sum_{j=1}^k r_j\, c_1(F_j, h^{F_j})$, where $r_j \geq 0$, and let $\{m_{j,p}\}_p$, $1 \leq j \leq k$, be sequences of natural numbers such that*

$$m_{1,p} \to \infty\,,\ \frac{m_{j,p}}{p} \to r_j\,,\ 1 \leq j \leq k\,,\ \text{ as } p \to \infty\,.$$

Let P_p, γ_p, $\mathcal{S}^p$ be the Bergman kernel function, Fubini-Study current, and respectively the unit sphere, associated to $H^0_{(2)}(X, L_p)$, where

$$L_p = F_1^{m_{1,p}} \otimes \ldots \otimes F_k^{m_{k,p}}\,,\ \ h_p = (h^{F_1})^{m_{1,p}} \otimes \ldots \otimes (h^{F_k})^{m_{k,p}}\,.$$

Then, as $p \to \infty$, we have $\frac{1}{p}\log P_p \to 0$ in $L^1(X, \omega^n)$, $\frac{1}{p}\gamma_p \to T$ and $\frac{1}{p}\big[\sigma_p = 0\big] \to T$ in the weak sense of currents on X, for λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq 1} \in \mathcal{S}_\infty$.

Proof. We may assume that $m_{j,p}/p < r_j + 1$ for all $1 \leq j \leq k$, $p \geq 1$, so

$$\begin{aligned} c_1(L_p, h_p) &= \sum_{j=1}^{k} m_{j,p}\, c_1(F_j, h^{F_j}) \geq \varepsilon m_{1,p}\, \omega\,, \\ c_1(L_p, h_p) &\leq p\, T_0\,, \text{ where } T_0 = \sum_{j=1}^{k} (r_j + 1)\, c_1(F_j, h^{F_j})\,. \end{aligned}$$

Moreover $c_1(L_p, h_p)/p \to T$ as $p \to \infty$. Note that

$$\frac{A_p}{p} = \frac{1}{p} \sum_{j=1}^{k} m_{j,p}\, \|c_1(F_j, h^{F_j})\| \to \sum_{j=1}^{k} r_j \|c_1(F_j, h^{F_j})\|\,.$$

The conclusion follows from Theorems 1.1 and 4.1, since

$$\frac{\gamma_p}{p} = \frac{A_p}{p} \frac{1}{A_p} (\, \gamma_p - c_1(L_p, h_p)) + \frac{1}{p}\, c_1(L_p, h_p)\,.$$

□

We conclude this section by noting that Theorem 4.2 holds in the context of Corollaries 5.1, 5.2, 5.3, 5.4, since the condition (26) holds in all of these situations.

6. Some examples

(1) Satake-Baily-Borel compactifications of arithmetic quotients. Let D be a bounded symmetric domain in $\mathbb{C}^n$ and let Γ be a torsion-free arithmetic lattice. Then $U := D/\Gamma$ is a smooth quasi-projective variety, called an arithmetic variety. Moreover, the Satake-Borel-Baily compactications (Satake [Sa], Borel-Baily [BB]) give in general highly singular compactications $U \subset X$, which are minimal in the sense that, given any normal compactication $U \subset X'$, the identity map on U extends to a holomorphic map $X' \to X$.

Assume that Γ is neat (see [Mu, p. 253]). By [AMRT], U admits a smooth toroidal compactification $\widetilde{X}$. In particular, $\Sigma := \widetilde{X} \setminus U$ is a divisor with normal crossings. The Bergman metric $\omega_D^{\mathcal{B}}$ on D descends to a complete Kähler metric $\omega := \omega_U^{\mathcal{B}}$ on U. Moreover, ω is Kähler-Einstein with $\mathrm{Ric}_\omega = -\omega$. We denote by h^{K_U} the Hermitian metric induced by ω on K_U. Then the metric h^{K_U} defines a singular metric h^L on L such that $c_1(L, h^L)$ is a positive current on $\widetilde{X}$ which extends $\frac{1}{2\pi}\,\omega$. Moreover, $L|_U = \tau^*(F)$, where F is an ample line bundle on X and $\tau : U \to X$ is the inclusion [Mu, Proposition 3.4(b)]. Then the metric h^{K_U} extends to a metric h^F on F over X and ω extends to a current $c_1(F, h^F) = \pi_* c_1(L, h^L)$, where $\pi : \widetilde{X} \to X$. Theorems 1.1-1.2 and their corollaries apply to X, (F, h^F). In particular, zeros of random sequences of cusp forms are equidistributed on X.

(2) Gromov-Hausforff limits of Kähler-Einstein manifolds. In [DoSu, Theorem 1.2] it is proved that the Gromov-Hausdorff limit of any sequence (X_j, ω_j) of Kähler-Einstein Fano manifolds with fixed volume $\int_{X_j} c_1(X_j)^n = V$ is a $\mathbb{Q}$-Fano normal variety X (with log terminal singularities), equipped with a Kähler form ω having continuous local potentials, with $\int_X c_1(X)^n = V$. This metric ω is Kähler-Einstein in the sense that $\mathrm{Ric}_\omega = \omega$ on X_{reg}.

Moreover, ω lies in $c_1(X)$ and K_X^{-1} is ample.

(3) Log Fano pairs. (?) A pair (X, Σ) is the data of a normal compact complex space X and an effective $\mathbb{Q}$-divisor Σ such that $K_X \otimes [\Sigma]$ is $\mathbb{Q}$-Cartier. A log Fano pair is a klt pair (X, D) such that X is projective and $(K_X \otimes [\Sigma])^{-1}$ is ample.

(4) Coverings. Let (X, ω) be a compact Kähler manifold of dimension n and (L_p, h_p), $p \geq 1$, be a sequence of singular Hermitian holomorphic line bundles on X satisfying condition (1). Let $q : \widetilde{X} \to X$ be a (paracompact) Galois covering of X, where q is the canonical projection. Let us denote by $\widetilde{\omega} = q^*\omega$, $\widetilde{L}_p = q^*(L_p)$ and by $\widetilde{h}_p$ the metric on $\widetilde{L}_p$ which is the pull-back of the metric h_p. We let $H^0_{(2)}(\widetilde{X}, \widetilde{L}_p)$ be the Bergman space of L^2-holomorphic sections of $\widetilde{L}_p$ relative to the metric $\widetilde{h}_p$ and the volume form $\widetilde{\omega}^n$ of $\widetilde{X}$, defined as in (3), endowed with the obvious inner product. We define the Bergman kernel function $\widetilde{P}_p$ and Fubini-Study currents $\widetilde{\gamma}_p$ associated to $H^0_{(2)}(\widetilde{X}, \widetilde{L}_p)$ as in (4). In this context, $d_p \in \mathbb{N} \cup \{\infty\}$, and these objects are well defined even for $d_p = \infty$, see [CM1, Lemmas 3.1-3.2].

Note that $\widetilde{\omega}$ is a complete Hermitian metric on $\widetilde{X}$ and $c_1(\widetilde{L}_p, \widetilde{h}_p) = q^* c_1(L_p, h_p)$ so $c_1(\widetilde{L}_p, \widetilde{h}_p) \geq a_p \widetilde{\omega}$. A straightforward adaptation of the proofs given above yield:

$$
\begin{aligned}
&\frac{1}{A_p} \log \widetilde{P}_p \to 0 \text{ in } L^1_{loc}(\widetilde{X}, \widetilde{\omega}^n)\,, \\
&\frac{1}{A_p} (\widetilde{\gamma}_p - c_1(\widetilde{L}_p, \widetilde{h}_p)) \to 0 \text{ weakly on } \widetilde{X}\,.
\end{aligned}
\tag{27}
$$

Assume moreover that the metrics h_p are of class $\mathscr{C}^3$ and condition (5) is fulfilled, that is, $\varepsilon_p := \|h_p\|_3^{1/3} a_p^{-1/2} \to 0$ as $p \to \infty$. Then there exists $p_0 \in \mathbb{N}$ such that for all $p > p_0$

$$
\left| \widetilde{P}_p \frac{\widetilde{\omega}^n}{c_1(\widetilde{L}_p, \widetilde{h}_p)^n} - \frac{1}{n!} \right| \leq C \varepsilon_p^{2/3} \quad \text{on } \widetilde{X},
\tag{28}
$$

where $C > 0$ is a constant depending only on (X, ω). Estimates (6) and (28) show that the asymptotics of the Bergman kernels $\widetilde{P}_p(\widetilde{x})$ and $P_p(x)$, $x = q(\widetilde{x})$, are the same. For $(L_p, h_p) = (L^p, h^p)$ with a smooth metric h satisfying $c_1(L, h) \geq a\omega$ this follows from [MM2, Theorem 6.1.4].

References

[A] A. Andreotti, *Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves*, Bull. Soc. Math. France **91** (1963), 1–38.

[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. S. Tai, *Smooth compactifications of locally symmetric varieties*, 2nd ed. With the collaboration of Peter Scholze. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.

[BB] W. L. Baily and A. Borel, *Compactifcation of arithmetic quotients of bounded symmetric domains*, Ann. Math. **84** (1966), 442–528.

[Ba] T. Bayraktar, *Equidistribution of zeros of random holomorphic sections*, arXiv:1312.0933.

[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. **37** (1976), 1–44.

[BT2] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1–40.

[Be] B. Berndtsson, *Bergman kernels related to Hermitian line bundles over compact complex manifolds*, Explorations in complex and Riemannian geometry, Contemp. Math. **332** 2003, 1–17.

[BM] E. Bierstone and P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), 207–302.

[Bl1] Z. Błocki, *On the definition of the Monge-Ampère operator in* $\mathbb{C}^2$, Math. Ann. **328** (2004), 415–423.

[Bl2] Z. Błocki, *The domain of definition of the complex Monge-Ampère operator*, Amer. J. Math. **128** (2006), 519–530.

[BK] Z. Błocki and S. Kołodziej, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc. **135** (2007), 2089–2093.

[BL] T. Bloom and N. Levenberg, *Random polynomials and pluripotential-theoretic extremal functions*, arXiv:1304.4529.

[Bo] S. Boucksom, *On the volume of a line bundle*, Internat. J. Math. **13** (2002), 1043–1063.

[Ca] D. Catlin, *The Bergman kernel and a theorem of Tian*, in *Analysis and geometry in several complex variables (Katata, 1997)*, 1–23, Trends Math., Birkhäuser, Boston, 1999.

[Ce] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), 159–179.

[CGZ] D. Coman, V. Guedj, and A. Zeriahi, *Domains of definition of Monge-Ampère operators on compact Kähler manifolds*, Math. Z. **259** (2008), 393–418.

[CM1] D. Coman and G. Marinescu, *Equidistribution results for singular metrics on line bundles*, 2011, preprint available at arXiv:1108.5163.

[CM2] D. Coman and G. Marinescu, *Convergence of Fubini-Study currents for orbifold line bundles*, Internat. J. Math. **24** (2013), 1350051, 27 pp.

[CM3] D. Coman and G. Marinescu, *On the approximation of positive closed currents on compact Kähler manifolds*, 2013, preprint available at arXiv:1302.0292.

[DLM1] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), 1–41; announced in C. R. Math. Acad. Sci. Paris **339** (2004), 193–198.

[DLM2] X. Dai, K. Liu, and X. Ma, *A remark on weighted Bergman kernels on orbifolds*, Math. Res. Lett. **19** (2012), 143–148.

[D1] J.-P. Demailly, *Courants positifs extrêmaux et conjecture de Hodge*, Invent. Math. **69** (1982), 347–374.

[D2] J. P. Demailly, *Estimations* L^2 *pour l'opérateur* $\overline{\partial}$ *d'un fibré holomorphe semipositif au–dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. **15** (1982), 457–511.

[D3] J. P. Demailly, *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.) No. 19 (1985), 1–125.

[D4] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, in *Complex algebraic varieties (Bayreuth, 1990)*, Lecture Notes in Math. 1507, Springer, Berlin, 1992, 87–104.

[D5] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), 361–409.

[D6] J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, in *Complex analysis and geometry*, Plenum, New York, 1993, 115–193.

[D7] J. P. Demailly, *A numerical criterion for very ample line bundles*, J. Differential Geom. **37** (1993), 323–374.

[DP] J. P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), 1247–1274.

[DMS] T.-C. Dinh, G. Marinescu and V. Schmidt, *Asymptotic distribution of zeros of holomorphic sections in the non compact setting*, J. Stat. Phys. **148** (2012), 113–136.

[DS1] T.-C. Dinh and N. Sibony, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv. **81** (2006), 221–258.

[DS2] T.-C. Dinh and N. Sibony, *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings*, *Holomorphic dynamical systems*, 165–294, Lecture Notes in Math. **1998**, Springer, Berlin, 2010.

[DNS] T.-C. Dinh, V.-A. Nguyên, and N. Sibony, *Dynamics of horizontal-like maps in higher dimension*, Adv. Math. **219** (2008), 1689–1721.

[DoSu] S. K. Donaldson and S. Sun, *Gromov-Hausdorff limits of Kahler manifolds and lagebraic geometry*, 2012, preprint available at arXiv:1206.2609.

[DuS] J. Duval and N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. **79** (1995), 487–513.

[EGZ] P. Eyssidieux, V. Guedj and A. Zeriahi, *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), 607–639.

[FN] J. E. Fornæss and R. Narasimhan, *The Levi problem on complex spaces with singularities*, Math. Ann. **248** (1980), 47–72.

[FS] J. E. Fornæss and N. Sibony, *Oka's inequality for currents and applications*, Math. Ann. **301** (1995), 399–419.

[GW] D. Gayet and J.-Y. Welschinger, *What is the total Betti number of a random real hypersurface?*, J. Reine Angew. Math. **689** (2014), 137–168.

[GM] C. Grant Melles and P. Milman, *Classical Poincaré metric pulled back off singularities using a Chow-type theorem and desingularization* Ann. Fac. Sci. Toulouse Math. (6) **15** (2006), 689–771.

[Gr] H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[GR1] H. Grauert and R. Remmert, *Plurisubharmonische Funktionen in komplexen Räumen*, Math. Z. **65** (1956), 175–194.

[GR2] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Springer, Berlin, 1984. Grundlehren der Mathematischen Wissenschaften, 265, Springer-Verlag, Berlin, 249 pp., 1984.

[G] V. Guedj, *Approximation of currents on complex manifolds*, Math. Ann. **313** (1999), 437–474.

[GZ1] V. Guedj and A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. **15** (2005), 607–639.

[GZ2] V. Guedj and A. Zeriahi, *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*, J. Func. Anal. **250** (2007), 442–482.

[H] L. Hörmander, *Notions of Convexity*, Reprint of the 1994 edition, Modern Birkhäuser Classics, Basel, Birkhäuser, 2007, viii, 414 pp.

[HsM] C.-Y. Hsiao and G. Marinescu, *The asymptotics for Bergman kernels for lower energy forms and the multiplier ideal Bergman kernel asymptotics*, preprint available at arXiv:1112.5464, to appear in Comm. Anal. Geom.

[JS] S. Ji and B. Shiffman, *Properties of compact complex manifolds carrying closed positive currents*, J. Geom. Anal. **3** (1993), 37–61.

[KM] J. Kollár and T. Matsusaka, *Riemann-Roch type inequalities*, Amer. J. Math. **105** (1983), 229–252.

[MM1] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), 493–498.

[MM2] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Math., vol. 254, Birkhäuser, Basel, 2007, xiii, 422 pp.

[MM3] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, Adv. Math. **217** (2008), 1756–1815.

[M] G. Marinescu, *A criterion for Moishezon spaces with isolated singularities*, Ann. Mat. Pura Appl. (4) **185** (2005), no. 1, 1-16.

[Moi] B. G. Moishezon, *Reduction theorems for compact complex spaces with a sufficiently large field of meromorphic functions. (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **31** 1967, 1385–1414. English translation: *Resolution theorems for compact complex spaces with a sufficiently large field of meromorphic functions*, Math. USSR - Izvestija **1** 1967, no. 6, 1331–1356.

[Mu] D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math. **42** (1977), 239–272.

[N] R. Narasimhan, *The Levi problem for complex spaces II*, Math. Ann. **146** (1962), 195–216.

[NS] L. Nicolaescu and N. Savale, *The Gauss-Bonnet-Chern theorem: a probabilistic perspective*, arXiv:1404.5206.

[NV] S. Nonnenmacher and A. Voros, *Chaotic eigenfunctions in phase space*, J. Stat. Phys. **92** (1998), no. 3-4, 451–518.

[O] T. Ohsawa, *Hodge spectral sequence and symmetry on compact Kähler spaces*, Publ. Res. Inst. Math. Sci. **23** (1987), 613–625.

[OT] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), 197–204.

[R] R. Richberg, *Stetige streng pseudokonvexe Funktionen*, Math. Ann. **175** (1968), 256–286.

[R] W. D. Ruan, *Canonical coordinates and Bergman metrics*, Comm. Anal. Geom. **6** (1998), 589–631.

[Sa] I. Satake, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. Math. **72** (1960), 555–580.

[Sh] B. Shiffman, *Convergence of random zeros on complex manifolds*, Sci. China Ser. A **51** (2008), 707–720.

[SZ1] B. Shiffman and S. Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Comm. Math. Phys. **200** (1999), 661–683.

[SZ2] B. Shiffman and S. Zelditch, *Number variance of random zeros on complex manifolds*, Geom. Funct. Anal. **18** (2008), 1422–1475.

[Si] N. Sibony, *Quelques problèmes de prolongement de courants en analyse complexe*, Duke Math. J. **52** (1985), 157–197.

[Siu] Y. T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974), 53–156.

[T] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), 99–130.

[Z] S. Zelditch, *Szegö kernels and a theorem of Tian*, Internat. Math. Res. Notices **1998**, no. 6, 317–331.

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13244-1150, USA
E-mail address: dcoman@syr.edu

INSTITUT UNIVERSITAIRE DE FRANCE & UNIVERSITÉ PARIS DIDEROT - PARIS 7, UFR DE MATHÉMATIQUES, CASE 7012, 75205 PARIS CEDEX 13, FRANCE
E-mail address: ma@math.jussieu.fr

UNIVERSITÄT ZU KÖLN, MATHEMATISCHES INSTITUT, WEYERTAL 86-90, 50931 KÖLN, DEUTSCHLAND & INSTITUTE OF MATHEMATICS 'SIMION STOILOW', ROMANIAN ACADEMY, BUCHAREST, ROMANIA
E-mail address: gmarines@math.uni-koeln.de