L^2 holomorphic sections of bundles over weakly pseudoconvex coverings

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Abstract. We show the existence of L^2 holomorphic sections of invariant line bundles over normal coverings of pseudoconvex domains using a L^2 generalization of Demailly's Weyl type formula.

Keywords: pseudoconvex domain, von Neumann dimension, semi–classical estimate, Demailly's asymptotic formula

Abbreviations: psh – plurisubharmonic

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1. Introduction

A well–known theorem of Grauert [11] (his solution of the Levi problem) asserts that a strongly pseudoconvex domain is holomorphically convex. In particular the dimension of the vector space of holomorphic functions has infinite dimension. One can also ask about the existence of L^2 holomorphic functions (with respect to a hermitian metric in the neighbourhood of the closure of the domain). Gromov, Henkin and Shubin [13] compute actually the von Neumann dimension of the space of L^2 holomorphic functions on coverings of strongly pseudoconvex domains. The von Neumann dimension turns out to be infinite. This type of existence results were introduced by Atiyah [1].

Our goal is to extend this result to situations where weaker pseudoconvexity conditions are required. We consider two classes of (weak) pseudoconvex manifolds. A smooth domain M in a complex manifold X is called *pseudoconvex* if the Levi form of the defining function is positive semi–definite when restricted to the holomorphic tangent bundle of the boundary. (The domain is called *strongly pseudoconvex* if we replace 'semi–definite' by 'definite'.) As shown by Grauert [12], there exists pseudoconvex domains which possess only constant holomorphic functions, but are domains of meromorphy. Thus, it is natural to study the existence of sections in holomorphic line bundles over pseudoconvex domains.

Let E be a semipositive line bundle defined in a neighbourhood of $\overline{M}$ and which is positive near the boundary. We note that the examples constructed by Grauert possess

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such line bundles. Then we show that the dimension of the space of holomorphic L^2 sections with values in the k-th tensor powers E^k, for large k, grows faster than a polynomial in k^n with leading coefficient the integral over M of the curvature $(\frac{\imath}{2\pi}\mathbf{c}(E))^n$, where $n = \dim M$. If E is positive everywhere this is the volume of the domain in the metric given by the curvature of the bundle. Moreover, the same estimate holds if we replace the domain with one of its Galois coverings and the usual dimension with the von Neumann dimension. This yields the generalization of the theorem of Gromov–Henkin–Shubin alluded to before.

The above mentioned results are consequences of Theorem 3.8. We also find there an upper bound for the von Neumann dimension of the L^2 reduced cohomology groups of type $(0,q)$, $q \geqslant 1$, with values in E^k, with k very large.

We consider further the following class of manifolds. A manifold X is called *weakly* 1–*complete* if admits a smooth psh exhaustion function φ (see [20]). The sublevel sets $X_c = \{\varphi < c\}$ for regular values c of φ are pseudoconvex domains. For such domains let us consider a semipositive line bundle which is positive near the boundary. Then the dimension of the space of holomorphic L^2 sections with values in the high tensor powers E^k grows faster than *any* polynomial in k^n, $n = \dim M$. The existence of an exhaustion function permits us to endow the manifold a complete metric of infinite volume and apply the previous estimate for sublevel sets larger than M_c. Moreover, the estimate holds also for the von Neumann dimension of the space of L^2 sections over a normal covering.

Let us note that the estimates for the dimension of the L^2 holomorphic sections for pseudoconvex coverings are new, even in the case of relatively compact pseudoconvex domains. Due to the weak positivity conditions (weak pseudoconvexity and semipositivity), we cannot solve directly the $\overline{\partial}$–equation in order to get holomorphic sections. We argue indirectly, by using the asymptotic Morse inequalities introduced by Siu [22] and Demailly [7] for compact manifolds (see also Berndtsson [3]; for the non–compact case, Nadel–Tsuji [18], Bouche [4], and [17]).

In [26] we gave a generalization of the asymptotic Morse inequalities for coverings of complete hermitian manifolds. Here we work with incomplete metrics (and therefore with the laplacian with $\overline{\partial}$–Neumann boundary conditions), since we cannot apply [26] to obtain the main result of the present paper (see the remarks at the end of Section 3). However, one of the principal ingredients of the proof is taken from [26]. It is the generalization to the covering case of Demailly's Weyl type formula for the counting function of the Dirichlet laplacian Δ''_k acting on high tensor powers E^k.

The plan of the paper is as follows. In Section 2 we remind the necessary background and results from [26]. In Section 3 we use the Bochner–Kodaira formula in the form given by Andreotti–Vesentini [2] and Griffiths [10] to reduce the study of the laplacian with $\overline{\partial}$–Neumann boundary conditions on ∂M to the study of the Dirichlet laplacian on a smaller and invariant domain. In Section 4 we apply our main result (Theorem 3.8) to the case of weakly 1–complete manifolds and strongly pseudoconvex domains.

2. Estimates of the spectrum distribution function

One of the main ingredients in Demailly's proof of the asymptotic Morse inequalities is a Weyl type formula for the counting function of the laplacian Δ_k'' acting on E^k. On the other hand, Shubin gave in [23] a proof of the L^2 Morse inequalities for covering manifolds in the spirit of Witten's paper [27]. Our proof uses the technique of [23] to generalize Demailly's formula for the Dirichlet laplacian on a covering manifold.

Let M be a complex analytic manifold of complex dimension n on which a discrete group Γ acts freely and properly discontinuously. Let $\widetilde{M} = M/\Gamma$ and let $\pi : M \longrightarrow \widetilde{M}$ be the canonical projection. We assume M is paracompact so that Γ will be countable. Suppose we are given a holomorphic vector bundle $\widetilde{G}$ on $\widetilde{M}$ and take its pull-back $G = \pi^*\widetilde{G}$, which is a Γ–invariant bundle on M. We also fix Γ–invariant hermitian metrics on M and on F.

We consider a relatively compact open set $\widetilde{\Omega} \Subset \widetilde{M}$ with smooth boundary and its preimage $\Omega = \pi^{-1}\widetilde{\Omega}$; Γ acts on $\widetilde{\Omega}$ and $\Omega/\Gamma = \widetilde{\Omega}$. In general we will decorate by tildes the objects living on the quotient. Let U be a fundamental domain of the action of Γ on Ω. This means that (see e.g. [1]): a) Ω is covered by the translations of $\overline{U}$, b) different translations of U have empty intersection and c) $\overline{U} \setminus U$ has zero measure (since $\partial\widetilde{\Omega}$ is smooth). Since $\widetilde{\Omega}$ is relatively compact U has the same property.

Let us define the space of square integrable sections $L^2(\Omega, G)$ with respect to a Γ–invariant metric on M (and its volume form) and a Γ–invariant metric on G. Then $L^2(U, G)$ is constructed with respect to the same metrics. There is a unitary action of Γ on $L^2(\Omega, G)$. In fact it is easy to see that $L^2(\Omega, G) \cong L^2\Gamma \otimes L^2(U, G) \cong L^2\Gamma \otimes L^2(\widetilde{\Omega}, G)$. We have a unitary action of Γ on $L^2\Gamma$ by left translations: $\gamma \longmapsto l_\gamma$ where $l_\gamma f(x) = f(\gamma^{-1}x)$ for $x \in \Gamma$, $f \in L^2\Gamma$. It induces an action on $L^2(\Omega, G)$ by $\gamma \longmapsto L_\gamma = l_\gamma \otimes \mathrm{Id}$. Finally we denote by $C_c(\cdot,\cdot)$ the various spaces of smooth compactly supported sections.

Let us consider a formally self-adjoint, strongly elliptic, positive differential operator $\widetilde{P}$ on $\widetilde{M}$ acting on sections of $\widetilde{G}$. Denote by P the Γ–invariant differential operator which is its pull-back to M. From P we construct the following operators: the Friedrichs extension in $L^2(\Omega, G)$ of $P|_{C_c(\Omega,G)}$ and the Friedrichs extension in $L^2(U, G)$ of $P|_{C_c(U,G)}$. From now on we denote these extensions by P and P_0. They are closed self-adjoint positive operators and coincide with the Dirichlet laplacians on Ω and U.

It is known that P is also Γ–invariant i.e. it commutes with all L_γ. This amounts to saying that $\boldsymbol{E}_\lambda(P)$ commutes with L_γ, $\gamma \in \Gamma$, where $(\boldsymbol{E}_\lambda(P))_\lambda$ is the spectral family of P. On the other hand the Rellich lemma tells that P_0 has compact resolvent and hence discrete spectrum.

Definition 2.1. For any Hilbert space $\mathcal{H}$ we call the Hilbert space $L^2\Gamma \otimes \mathcal{H}$ a free Hilbert Γ–module. Γ–invariant closed spaces of free Hilbert Γ–modules are called Γ–modules.

Since $\boldsymbol{E}_\lambda$ is Γ–invariant its image $\mathrm{Ran}\boldsymbol{E}_\lambda$ is a Γ–module of the free Hilbert Γ–module $L^2\Gamma \otimes L^2(U, G) \cong L^2(\Omega, G)$. The action of Γ is defined as above by $\gamma \longmapsto L_\gamma = l_\gamma \otimes \mathrm{Id}$.

For Γ–modules one can associate a positive, possibly infinite real number, called von Neumann or Γ–dimension, denoted $\dim_\Gamma$. For notions involving the Γ–dimension and linear algebra for Γ–modules we refer the reader to [1], [23] and [16, pp. 75–80] (in the latter proofs from scratch are given).

Let us denote by $\mathcal{A}_\Gamma$ the von Neumann algebra which consists of bounded linear operators in $L^2\Gamma\otimes\mathcal{H}$ which commute to the action of Γ. To describe $\mathcal{A}_\Gamma$ let us consider the von Neumann $\mathcal{R}_\Gamma$ algebra of bounded operators on $L^2\Gamma$ which commute with all L_γ. It is generated by right translations. If we consider the orthonormal basis $(\delta_\gamma)_\gamma$ in $L^2\Gamma$ where δ_γ is the Dirac delta function at γ, then the matrix of any operator $A\in\mathcal{R}_\Gamma$ has the property that all its diagonal elements are equal. Therefore we define a natural trace on $\mathcal{R}_\Gamma$ as the diagonal element, that is, $\operatorname{tr}_\Gamma A=(A\delta_e,\delta_e)$ where e is the identity element. $\mathcal{A}_\Gamma$ is the tensor product of $\mathcal{R}_\Gamma$ and the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. If Tr is the usual trace on $\mathcal{B}(\mathcal{H})$ then we have a trace on $\mathcal{A}_\Gamma$ by $\operatorname{Tr}_\Gamma=\operatorname{tr}_\Gamma\otimes\operatorname{Tr}$.

Definition 2.2. For any Γ–module, the projection $P_L\in\mathcal{A}_\Gamma$ and we define $\dim_\Gamma L=\operatorname{Tr}_\Gamma P_L$.

We also introduce the notion of Γ–morphisms. If L_1, L_2 are two Γ–modules then a bounded linear operator $T:L_1\longrightarrow L_2$ is called a Γ–morphism if it commutes with the action of Γ. As for the usual dimension the following statements are true (see [16, pp. 75–80]). If T is injective, $\dim_\Gamma L_1\leqslant\dim_\Gamma L_2$, and if T has dense image, $\dim_\Gamma L_1\geqslant\dim_\Gamma L_2$. We denote by $\operatorname{rank}_\Gamma T=\dim_\Gamma[\operatorname{Ran}(T)]$ where $[V]$ stands for the closure of a vector space V.

We denote in the sequel $N_\Gamma(\lambda,P)=\dim_\Gamma\operatorname{Ran}\boldsymbol{E}_\lambda(P)$. Similary we consider the spectral distribution (counting) function $N(\lambda,P_0)=\dim\operatorname{Ran}\boldsymbol{E}_\lambda(P_0)$ where $\boldsymbol{E}_\lambda(P_0)$ is the spectral family of P_0; it equals the number of eigenvalues $\leqslant\lambda$.

To compare $N_\Gamma(\lambda,P)$ and $N(\lambda,P_0)$ we use essentially the analysis of Shubin [23]. However there exist a difference in our method. Namely, we take from the beginning the model operator P_0 to be the operator P itself with Dirichlet boundary conditions on U, whereas Shubin considers a direct sum of tangent operators to P. So we do not have to truncate from the outset the eigenfunctions of the model P_0. (See also [23, Remark 1.3].) The next result [23, Lemma 2.4] is fundamental.

Proposition 2.3 (Variational principle). *Let P be a Γ–invariant self-adjoint positive operator on a free Γ–module $L^2\Gamma\otimes\mathcal{H}$ where $\mathcal{H}$ is Hilbert space. Then*

$$N_\Gamma(\lambda,P)=\sup\Big\{\dim_\Gamma L\mid L\subset\operatorname{Dom}(Q),\quad Q(f,f)\le\lambda\|f\|^2,\forall f\in L\Big\}.\tag{2.1}$$

where $L\subset\operatorname{Dom}(Q)$ runs over Γ–modules and Q is the quadratic form of P.

With the requisite maximum–minimum result in place we now begin to examine the relation between the two distribution functions $N_\Gamma(\lambda,P)$ and $N(\lambda,P_0)$.

Proposition 2.4 (Estimate from below). *The counting functions of P and P_0 satisfy the inequality*

$$N_\Gamma(\lambda,P)\geqslant N(\lambda,P_0)\,,\quad\lambda\in\mathbb{R}\tag{2.2}$$

Proof. See [26, Proposition 1.1]. □

In order to get an estimate from above we have to enlarge a little bit the fundamental domain U and compare the counting function of P to the counting function of the Friedrichs extension of P restricted to compactly supported forms in the enlarged domain. For $h>0$, the enlarged domain is $U_h=\{x\in\Omega \mid d(x,U)<h\}$ where d is the distance on M associated to the Riemann metric on M. We consider the operator P with domain $C_c(U_h,G)$ and take its Friedrichs extension, denoted $P_0^{(h)}$.

Proposition 2.5 (Estimate from above). *There is a constant $C>0$ such that*

$$N_\Gamma(\lambda,P)\leqslant N\left(\lambda+\tfrac{C}{h^2},P_0^{(h)}\right)\quad \lambda\in\mathbb{R},\quad h>0 \tag{2.3}$$

Proof. See [26, Proposition 1.4]. □

The estimates from below and above for $N_\Gamma(\lambda,P)$ enable us to study as a by–product the behaviour for $\lambda\longrightarrow\infty$ to obtain the Weyl asymptotics for periodic operators (due to M. Shubin, see [21] and the references therein).

Corollary 2.6. *If P is a periodic, positive, second order elliptic operator as above then*

$$\begin{aligned}\lim_{\lambda\to\infty}\lambda^{-n/2}N_\Gamma(\lambda,P) &= \lim_{\lambda\to\infty}\lambda^{-n/2}N(\lambda,P_0)\\ &= (2\pi)^{-n}\int_U\int_{T_x^*M}N(1,\sigma_0(P)(x,\xi))d\xi\,dx.\end{aligned}$$

$\sigma_0(P)(x,\xi)\in\mathrm{Herm}(F,F)$ *is the principal symbol of P and $N(1,\sigma_0(P)(x,\xi))$ is the counting function for the eigenvalues of this hermitian matrix.*

Proof. First let us remark that the last equality is the classical Weyl type formula as established by Carleman, Gårding and others, see [21, p. 72]. It is obvious that $\underline{\lim}\,\lambda^{-n/2}N(\lambda,P_0)\leqslant\underline{\lim}\,\lambda^{-n/2}N_\Gamma(\lambda,P)$ by the estimate from below. On the other hand the estimate from above gives for $\lambda\longrightarrow\infty$

$$\begin{aligned}\overline{\lim}\,\lambda^{-n/2}N_\Gamma(\lambda,P)&\leqslant\overline{\lim}\,\left(1+\tfrac{C}{\lambda h^2}\right)^{n/2}\left(\lambda+\tfrac{C}{h^2}\right)^{-n/2}N\left(\lambda+\tfrac{C}{h^2},P_0^{(h)}\right)\\ &\leqslant\overline{\lim}\,\mu^{-n/2}N(\mu,P_0^{(h)})=(2\pi)^{-n}\int_{U_h}\int_{T_x^*M}N(1,\sigma_0(P)(x,\xi))d\xi\,dx\end{aligned}$$

for a fixed small h. We make $h\longrightarrow 0$ and obtain the desired formula. □

We are going to apply the above results to the semi-classical asymptotics as $k\longrightarrow\infty$ of the spectral distribution function of the laplacian $\frac{1}{k}\Delta_k''$ on M. Let $\widetilde{G}$ be a hermitian holomorphic bundle on $\widetilde{M}$ and $G=p^*\widetilde{G}$ its pull-back. We define $C_c^{(0,q)}(\cdot,\cdot)$ to be the space of smooth compactly supported $(0,q)$ forms. We denote by $L^{0,q}(M,G)$ the space of $(0,q)$–forms which are L^2 on M with respect to smooth Γ–invariant metrics on X and G. Let $\overline{\partial}:C_c^{0,q}(M,G)\longrightarrow C_c^{0,q+1}(M,G)$ be the Cauchy–Riemann operator and $\vartheta:C_c^{0,q+1}(M,G)\longrightarrow C_c^{0,q}(M,G)$ the formal adjoint of $\overline{\partial}$ with respect to the given hermitian metrics on M and G. Then $\Delta''=\overline{\partial}\vartheta+\vartheta\overline{\partial}$ is a formally self-adjoint, strongly elliptic, positive and Γ–invariant differential operator.

We take E and F two Γ–invariant holomorphic bundles. Let us form the Laplace–Beltrami operator Δ_k'' on $(0,q)$ forms with values in $E^k\otimes F$. Thus we will consider the

Γ–invariant hermitian bundle $G=\Lambda^{(0,q)}T^*M\otimes E^k\otimes F$ and apply the previous results for $P=\frac{1}{k}\Delta''_k\restriction_\Omega$ where the index Ω emphasises that the Friedrichs extension gives the operator of the Dirichlet problem on Ω. Now we have to make a good choice of the parameter h. We take $h=k^{-1/4}$. By (2.2) and (2.3), we obtain the following semi–classical estimate for laplacian.

Proposition 2.7. *There exists a constant* $C>0$ *such that for* $\lambda\in\mathbb{R}$ *and* $k>0$ *we have*

$$N\left(\lambda,\tfrac{1}{k}\Delta''_k\restriction_U\right)\leqslant N_\Gamma\left(\lambda,\tfrac{1}{k}\Delta''_k\restriction_\Omega\right)\leqslant N\left(\lambda+\tfrac{C}{\sqrt{k}},\tfrac{1}{k}\Delta''_k\restriction_{U_{k^{-1/4}}}\right). \tag{2.4}$$

Demailly has determined the distribution of spectrum for the Dirichlet problem for $\frac{1}{k}\Delta''_k$ in [7, Theorem 3.14]. For this purpose he introduces ([7, (1.5)]) the function $\nu_E: M\times\mathbb{R}\longrightarrow\mathbb{R}$ depending on the curvature of E and then considers the function $\bar{\nu}_E(x,\lambda)=\lim_{\varepsilon\searrow 0}\nu_E(x,\lambda+\varepsilon)$. The function $\bar{\nu}_E(x,\lambda)$ is right continuous in λ and bounded above on compacts of M. Denote by $\alpha_1(x),\dots,\alpha_n(x)$ the eigenvalues of of the curvature form $i\mathbf{c}(E)(x)$ with respect to the metric on M. We also denote for a multiindex $J\subset\{1,\dots,n\}$, $\alpha_J=\sum_{j\in J}\alpha_j$ and $C(J)=\{1,\dots,n\}\setminus J$. For $V\Subset M$ we introduce

$$I^q(V,\mu)=\sum_{|J|=q}\int_V\bar{\nu}_E(2\mu+\alpha_{C(J)}-\alpha_J)\,d\sigma.$$

Proposition 2.8 (Demailly). *Assume that* ∂V *has measure zero and that the laplacian acts on* $(0,q)$ *forms. Then*

$$\overline{\lim_{k\longrightarrow\infty}}\,k^{-n}N(\lambda,\tfrac{1}{k}\Delta''_k\restriction_V)\leqslant I^q(V,\lambda).$$

Moreover there exists an at most countable set $\mathcal{D}\subset\mathbb{R}$ *such that for* $\lambda\in\mathbb{R}\setminus\mathcal{D}$ *the limit of the left–hand side expression exists and we have equality.*

We return now to the case of a covering manifold and apply Demailly's formula in (2.4). Let us fix $\varepsilon>0$. For sufficiently large k we have $U_{k^{-1/4}}\subset U_\varepsilon$ so the fact that the counting function is increasing and the variational principle (2.1) yield

$$N\left(\lambda+\tfrac{C}{\sqrt{k}},\tfrac{1}{k}\Delta''\restriction_{U_{k^{-1/4}}}\right)\leqslant N\left(\lambda+\varepsilon,\tfrac{1}{k}\Delta''\restriction_{U_{k^{-1/4}}}\right)\leqslant N\left(\lambda+\varepsilon,\tfrac{1}{k}\Delta''\restriction_{U_\varepsilon}\right).$$

Hence by (2.4) and Proposition 2.8 (∂U_ε is negligible for small ε),

$$\limsup_k k^{-n}N_\Gamma(\lambda,\tfrac{1}{k}\Delta''_k\restriction_\Omega)\leqslant I^q(U_\varepsilon,\lambda+\varepsilon).$$

The use of dominated convergence to make $\varepsilon\longrightarrow 0$ in the last integral yield the following asymptotic formula for the laplacian on a covering manifold.

Theorem 2.9. *The spectral distribution function of* $\frac{1}{k}\Delta''_k\restriction_\Omega$ *on* $L^{0,q}(\Omega,E^k\otimes F)$ *with Dirichlet boundary conditions satisfies*

$$\overline{\lim_{k\longrightarrow\infty}}\,k^{-n}N_\Gamma\left(\lambda,\tfrac{1}{k}\Delta''_k\restriction_\Omega\right)\leqslant I^q(U,\lambda). \tag{2.5}$$

Moreover, there exists an at most countable set $\mathcal{D}\subset\mathbb{R}$ *such that for* $\lambda\in\mathbb{R}\setminus\mathcal{D}$ *the limit exists and we have equality in* (2.11).

We need to know the bahaviour of $I^q(U,\lambda)$ for $\lambda \longrightarrow 0$ which is given in [7]. First it is clear that $\lim_{\lambda\longrightarrow 0} I^q(U,\lambda) = I^q(U,0) = \sum_{|J|=q} \int_U \bar{\nu}_E(\alpha_{C(J)} - \alpha_J)\, d\sigma$. By definition ([7, (1.5)]),

$$\bar{\nu}_E(\alpha_{C(J)} - \alpha_J) = \tfrac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)} |\alpha_1 \cdots \alpha_s| \sum_{p\in\mathbb{N}^s} \Big\{\alpha_{C(J)} - \alpha_J - \sum_{j=1}^{s}(2p_j+1)|\alpha_j|\Big\}_+^{n-s},$$

where the eigenvalues satisfy $|\alpha_1| \geqslant |\alpha_2| \ldots \geqslant |\alpha_s| > 0 = |\alpha_{s+1}| = \ldots = |\alpha_n|$ with the notation $\lambda_+^{n/2}$ ($[\lambda]_+^0 = 0$ for $\lambda < 0$, $[\lambda]_+^0 = 1$ for $\lambda \geqslant 0$). It is clear that for $p \in \mathbb{N}^s$, the corresponding summand in the above expression vanishes unless $s = n$, $p_1 = \cdots = p_n = 0$, $\alpha_j < 0$ for $j \in J$ and $\alpha_j > 0$ for $j \in C(J)$. In particular, if $\bar{\nu}_E(\alpha_{C(J)} - \alpha_J) \neq 0$, $\imath\mathbf{c}(E)$ is non–degenerate and has exactly q negative eigenvalues.

Let $U(q)$ be the set of points x of U such that $\imath\mathbf{c}(E)(x)$ is non–degenerate and has exactly q negative eigenvalues. For $x \in U(q)$ and $|J| = q$ it follows $\bar{\nu}_E(\alpha_{C(J)} - \alpha_J) = (2\pi)^{-n}|\alpha_1 \cdots \alpha_n| = (-1)^q(2\pi)^{-n}(\imath\mathbf{c}(E))^n$ for $J = J(x)$ where $J(x) = \{j : \alpha_j(x) < 0\}$ and $\bar{\nu}_E(\alpha_{C(J)} - \alpha_J) = 0$ for $J \neq J(x)$. Therefore

$$\lim_{\lambda\longrightarrow 0} I^q(U,\lambda) = \tfrac{1}{n!}\int_{U(q)} (-1)^q \Big(\tfrac{\imath}{2\pi}\mathbf{c}(E)\Big)^n. \tag{2.6}$$

For the formulation of the main result we set $U(\leqslant 1) := U(0) \cup U(1)$.

3. Pseudoconvex manifolds

Let X be a complex manifold acted upon freely by a discrete group of automorphisms Γ and $G \longrightarrow X$ be a Γ–invariant holomorphic vector bundle. We fix on X and G Γ–invariant hermitian metrics.

Let $M \subset X$ be a smooth open set such that:

– M is pseudoconvex,

– $\overline{M}$ is Γ–invariant,

– $\overline{M}/\Gamma$ is compact.

Thus $\overline{M}/\Gamma$ is a pseudoconvex relatively compact domain in the manifold X/Γ. We introduce the functional spaces $C^{0,q}(M,G)$ of smooth $(0,q)$–forms on M and $C_c^{0,q}(\overline{M},G)$ the subspace of forms smooth up to the boundary and with compact support in $\overline{M}$. We remind that $L^{0,q}(M,G)$ is the L^2 space of $(0,q)$–forms, with respect to the fixed Γ–invariant metrics on M and G.

The object of our study is the reduced L^2 cohomology for $\overline{\partial}$. To define it, we consider the weak maximal extension $\overline{\partial} : L^{0,q}(M,G) \longrightarrow L^{0,q+1}(M,G)$, which is a closed densely defined operator with domain $\mathrm{Dom}(\overline{\partial})$ (consisting of elements u such that $\overline{\partial}u$ calculated in distributional sense is in L^2). The reduced L^2 Dolbeault cohomology is defined to be

$$H^{0,q}_{(2)}(M,G) := \mathrm{Ker}\,\overline{\partial} \cap L^{0,q}(M,G) \Big/ \left[\mathrm{Ran}\,\overline{\partial} \cap L^{0,q}(M,G)\right],$$

where $[V]$ denotes the closure of the space V. We need to take the closure in order to make sure that $H^{0,q}_{(2)}(M,G)$ is a Hilbert space. The operator $\overline{\partial}$ is obviously Γ–invariant so that $H^{0,q}_{(2)}(M,G)$ is a Γ–module.

We link the L^2 cohomology to the theory of elliptic operators via the Hodge decomposition for a certain Laplace–Beltrami operator. Let $\overline{\partial}^*$ be the Hilbert space adjoint of $\overline{\partial}$. We introduce the operator Δ'' defined by

$$\begin{aligned} \mathrm{Dom}(\Delta'') &:= \big\{u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) : \overline{\partial}u \in \mathrm{Dom}(\overline{\partial}^*)\,,\ \overline{\partial}^* u \in \mathrm{Dom}(\overline{\partial})\big\}\,, \\ \Delta'' u &= \overline{\partial}\,\overline{\partial}^* u + \overline{\partial}^*\overline{\partial} u \ \text{ for } u \in \mathrm{Dom}(\Delta'')\,. \end{aligned}$$

By general arguments of functional analysis we know that Δ'' is selfadjoint ([9, Proposition 1.3.8]). The advantage of using Δ'' is that it leads to the Hodge decomposition, since $\mathrm{Ker}\,\Delta'' = \mathrm{Ker}\,\overline{\partial} \cap \mathrm{Ker}\,\overline{\partial}^*$. It is clear that $\overline{\partial}^*$ and Δ'' are Γ–invariant and we have the following lemma (for a proof see e.g. [13, Proposition 1.4]).

Lemma 3.1. *The following weak Hodge decomposition holds:*

$$\begin{aligned} L^{0,\bullet}(M,G) &= \mathrm{Ker}\,\Delta'' \oplus \big[\mathrm{Ran}\,\overline{\partial}\big] \oplus \big[\mathrm{Ran}\,\overline{\partial}^*\big]\,, \\ \mathrm{Ker}\,\overline{\partial} &= \mathrm{Ker}\,\Delta'' \oplus \big[\mathrm{Ran}\,\overline{\partial}\big]\,. \end{aligned}$$

In particular we have an isomorphism of Γ*–modules:*

$$H^{0,q}_{(2)}(M,G) \cong \mathrm{Ker}\,\Delta'' \cap L^{0,q}(M,G)\,. \tag{3.1}$$

We will work with the quadratic form canonically associated to Δ'', rather than directly with Δ''. Let us recall that the closed quadratic form $q = q(A)$ associated to a positive selfadjoint operator A is defined by $\mathrm{Dom}(q) = \mathrm{Dom}(A^{1/2})$, $q(u,v) = (A^{1/2}u, A^{1/2}v)$ for $u, v \in \mathrm{Dom}(A^{1/2})$.

Lemma 3.2. *The quadratic form associated to* Δ'' *is the form* Q *given by* $\mathrm{Dom}(Q) := \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$ *and*

$$Q(u,v) = (\overline{\partial}u, \overline{\partial}v) + (\overline{\partial}^*u, \overline{\partial}^*v)\,, \quad u, v \in \mathrm{Dom}(Q)$$

Proof. First remark that Q is a closed form. It is well–known (see e.g. [6, pp. 81–83]) that any closed, positive form is associated to a unique selfadjoint, positive operator F. The domain of F consists of elements $u \in \mathrm{Dom}(Q)$ such that there exists w with

$$Q(u,v) = (w,v) \quad \text{for any } u \in \mathrm{Dom}(Q)\,. \tag{3.2}$$

Moreover, for such u, $Fu = w$. For $u \in \mathrm{Dom}(\Delta'')$ it is clear that $u \in \mathrm{Dom}(Q)$ and u satisfies (3.2) with $w = \Delta'' u$. Therefore $\Delta'' \subset F$ and, since both operators are selfadjoint, $F = \Delta''$. □

We describe next a core form for Q, i.e. a dense subspace of $\mathrm{Dom}(Q)$ in the norm $\|u\|_Q = \big(\|u\|^2 + Q(u,u)\big)^{1/2}$. Let $M = \{x \in X : r(x) < 0\}$ where r is a smooth function on X which has non–vanishing gradient on ∂M. We denote

$$B^{0,q}(M,G) = \{u \in C^{0,q}_c(\overline{M},G) : \partial r \wedge * u = 0 \text{ on } \partial M\}\,.$$

Integration by parts ([9, Propositions 1.3.1–2]) shows that

$$B^{0,q}(M,G) = C_c^{0,q}(\overline{M},G) \cap \mathrm{Dom}(\overline{\partial}^*)\,, \quad \overline{\partial}^* = \vartheta \text{ on } B^{0,q}(M,G)\,. \tag{3.3}$$

Note that, by (3.3), the operator $\overline{\partial}\vartheta + \vartheta\overline{\partial}$ on $\{u \in B^{0,q}(M,G) : \overline{\partial}u \in B^{0,q+1}(M,G)\}$ is positive and Δ'' is one of its selfadjoint extensions.

Lemma 3.3 (Approximation). *$B^{0,q}(M,G)$ is a core form for Q.*

Proof. In the case the element from $\mathrm{Dom}(Q) = \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$ to be approximated in $\|\cdot\|_Q$ has compact support, the lemma is well known, see [15, Proposition 1.2.4]. Thus the only thing to do is to approximate elements from $\mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$ with elements with compact support. We can do this thanks to the Γ–action, which guarantees that the Γ–invariant metric on M is in some sense 'complete'. For the details, see [13, Lemma 1.1]. ☐

Next let us use a form of the Bochner–Kodaira formula introduced by Andreotti–Vesentini [2] and Griffiths [10]. Let us assume that there exists a Γ–invariant hermitian metric on X, Kähler near ∂M, written in local coordinates z^α as a smooth positive definite matrix $(g_{\alpha\beta})$. We also fix a Γ–invariant hermitian metric on G, written locally in a holomorphic frame f of G, as a smooth positive function h.

There exists a canonical connection on G, compatible with the complex structure and the metric. The curvature of this connection is denoted $\mathbf{c}(G)$. It is a $(1,1)$-form on X, $\mathbf{c}(G) = \sum \theta_{\alpha\beta}\, dz^\alpha \wedge d\overline{z}^\beta$, where $\theta_{\alpha\beta} = -\partial_{z^\alpha}\partial_{\overline{z}^\beta} \log h$. Let θ^μ_β be the curvature tensor with the first index raised. Let

$$u = \tfrac{1}{q!} \sum u_{\lambda_1 \dots \lambda_q}\, d\overline{z}^{\lambda_1} \wedge \cdots \wedge d\overline{z}^{\lambda_q} \otimes f$$

be a G–valued $(0,q)$–form on X. We define the $(0,q)$–form

$$\mathbf{c}(G)u = \tfrac{1}{q!} \sum \theta^\mu_{\lambda_1} u_{\mu\lambda_2 \dots \lambda_q}\, d\overline{z}^{\lambda_1} \wedge \cdots \wedge d\overline{z}^{\lambda_q} \otimes f\,.$$

We also introduce the Ricci curvature $\mathrm{Ric} = -\mathbf{c}(K_X)$, where K_X is the canonical bundle of X. For a G–valued $(0,q)$–form u, we set $\mathrm{Ric}\, u = -\mathbf{c}(K_X)u$.

We define next the Levi operator (see [10, p.418]). First let us remark that we can choose a Γ–invariant defining function r for M such that $|\partial r| = 1$ in a neighbourhood of ∂M, with respect to the hermitian metric on X. Let us pick, near a boundary point of M, an orthonormal frame $\omega^1, \dots, \omega^n$ for the bundle of $(1,0)$–forms, such that $\omega^n = \partial r$. We have

$$\partial\overline{\partial} r = -\overline{\partial}\omega^n = \sum l_{\alpha\beta}\, \omega^\alpha \wedge \overline{\omega}^\beta\,, \quad (1 \leqslant \alpha, \beta \leqslant n)\,.$$

Definition 3.4. The Levi form of ∂M is the restriction of $\partial\overline{\partial} r$ to the holomorphic tangent bundle of ∂M; it is given in the dual frame of $\omega^1, \dots, \omega^{n-1}$ by the matrix $(l_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant n-1}$. M is pseudoconvex if the Levi form is everywhere positive semi–definite.

For a $(0,q)$–form written locally

$$u = \tfrac{1}{q!} \sum u_{\alpha_1 \dots \beta_q}\, \overline{\omega}^{\alpha_1} \wedge \cdots \wedge \overline{\omega}^{\alpha_q} \otimes f$$

where f is an orthonormal frame in G, we set

$$\mathcal{L}(u,u)=\frac{1}{(q-1)!}\sum l_{\alpha\beta}\,u_{\alpha\beta_1\ldots\beta_{q-1}}\,\overline{u}_{\beta\beta_1\ldots\beta_{q-1}}\quad(1\leqslant\alpha,\beta\leqslant n)\,.$$

The form $u\in B^{0,q}(M,G)$ if and only if $u_{\alpha_1\ldots\alpha_q}=0$ for $n\in\{\alpha_1,\ldots,\alpha_q\}$. Therefore for $u\in B^{0,q}(M,G)$ the summation restricts over $1\leqslant\alpha,\beta\leqslant n-1$ and

$$\mathcal{L}(u,u)=\frac{1}{(q-1)!}\sum l_{\alpha\beta}\,u_{\alpha\beta_1\ldots\beta_{q-1}}\,\overline{u}_{\beta\beta_1\ldots\beta_{q-1}}\;\geqslant 0\,. \tag{3.4}$$

Finally, let $\overline{\nabla}$ denote the covariant derivative in the $(0,1)$–direction.

Lemma 3.5. *Assume that the Γ–invariant metric on X is Kähler in a Γ–invariant neighbourhood U of ∂M. Then for any $u\in B^{0,q}(M,G)$ with support in U we have*

$$Q(u,u)=\big\|\overline{\nabla}u\big\|^2+\big(\mathbf{c}(G)u,u\big)+\big(\mathrm{Ric}\,u,u\big)+\int_{\partial M}\mathcal{L}(u,u)\,dS \tag{3.5}$$

Proof. This formula was given by Griffiths [10, p. 429, (7.14)]. □

Next we specialize to the case $G=E^k$ for a Γ–invariant hermitian holomorphic line bundle E. Let Δ''_k the laplacian acting on forms with values in E^k. We shall normalize the quadratic form of the laplacian:

$$Q_k(u,u)=\tfrac{1}{k}\left(\|\overline{\partial}u\|^2+\|\overline{\partial}^*u\|^2\right)\,,$$

for $u\in\mathrm{Dom}(Q_k):=\mathrm{Dom}(\overline{\partial})\cap\mathrm{Dom}(\overline{\partial}^*)\cap L^{p,q}(M,E^k)$. The bundle E is endowed with a Γ–invariant hermitian metric whose curvature is positive near ∂M. From now on we fix a Γ–invariant hermitian metric on X which coincides with $\mathbf{c}(E)$ near ∂M. Since any two Γ–invariant metric on M are equivalent the L^2 spaces over M do not change.

Lemma 3.6. *Let E be a Γ–invariant holomorphic line bundle on X endowed with a Γ–invariant hermitian metric which is positive in a neighbourhood U of ∂M. Let $V\subset U$ be another Γ–invariant neighbourhood of ∂M and $0\leqslant\rho\leqslant 1$ be a smooth Γ–invariant function which equals 1 on V and vanishes outside U. Then for sufficiently large k, any $u\in\mathrm{Dom}(Q_k)\cap L^{0,q}(M,E^k)$, $q\geqslant 1$, satisfies the apriori estimate:*

$$\|u\|^2\leqslant 12\,Q_k(u,u)+4\int_{\Omega}\big|(1-\rho)u\big|^2, \tag{3.6}$$

where $\Omega:=M\smallsetminus\overline{V}$.

Proof. Let $u\in B^{0,q}(M,E^k)$, $q\geqslant 1$, with support in U. By (3.5) and (3.4) and $\mathbf{c}(E^k)=k\mathbf{c}(E)$,

$$Q_k(u,u)\geqslant\big(\mathbf{c}(E)u,u\big)+\tfrac{1}{k}\big(\mathrm{Ric}\,u,u\big)\,.$$

Since the metric coincides with $\mathbf{c}(E)$ near ∂M we have $\mathbf{c}(E)u=u$. Therefore there exists a constant C_0 (depending only of the Ricci curvature of the invariant metric on $\overline{M}$) such that

$$Q_k(u,u)\geqslant\tfrac{1}{2}\|u\|^2,\quad k\geqslant C_0\,. \tag{3.7}$$

We use now the elementary estimate:

$$Q_k(\rho u,\rho u) \leqslant \tfrac{3}{2}Q_k(u,u) + \tfrac{6}{k}\sup|d\rho|^2\,\|u\|^2\,. \tag{3.8}$$

Obviously $C_1 = 6\sup|d\rho|^2 < \infty$. If $\Omega = M \smallsetminus \overline{V}$ estimates (3.7) and (3.8) yield

$$\|u\|^2 \leqslant 12\,Q_k(u,u) + 4\int_\Omega \big|(1-\rho)u\big|^2, \quad k \geqslant \max\{C_0, 8C_1\}\,,$$

for any $u \in B^{0,q}(M,E^k)$. By the Approximation Lemma 3.3 the estimate holds also for $u \in \operatorname{Dom} Q_k$. This proves our contention. □

From relation (3.6) we infer next that the spectral spaces corresponding to the lower part of the spectrum of $\frac{1}{k}\Delta''_k$ on $(0,q)$–forms, $q \geqslant 1$, can be injected into the spectral spaces of the Γ–invariant operator $\frac{1}{k}\Delta''_k\restriction_\Omega$, which correspond to the Dirichlet problem on Ω for $\frac{1}{k}\Delta''_k$. We may apply thus the results from section 2. This idea appears in Witten's proof of the Morse inequalities (see [27] [14]) and in [4] in the context of q–convex manifolds in the sense of Andreotti–Grauert.

Let us introduce the following spectral spaces: $\mathcal{E}^q_k(\lambda) = \operatorname{Ran}\boldsymbol{E}_\lambda(\frac{1}{k}\Delta''_k)$ is the spectral space of $\frac{1}{k}\Delta''_k$ on $L^{0,q}(M,E^k)$, $\mathcal{E}^q_{k,\Omega}(\mu) = \operatorname{Ran}\boldsymbol{E}_\mu(\frac{1}{k}\Delta''_k\restriction_\Omega)$, the spectral spaces of $\frac{1}{k}\Delta''_k\restriction_\Omega$. As before $\boldsymbol{E}_\lambda$ stands for the spectral projection. We denote $N^q_\Gamma\left(\lambda, \frac{1}{k}\Delta''_k\right) = \dim_\Gamma \mathcal{E}^q_k(\lambda)$ the von Neumann dimension of the spectral spaces of $\frac{1}{k}\Delta''_k$.

Lemma 3.7. *Assume we are in the hypotheses of Lemma 3.6. Then for sufficiently large k, for $q \geqslant 1$ and any $\lambda < 1/24$ the following map is an injective Γ–morphism :*

$$\mathcal{E}^q_k(\lambda) \longrightarrow \mathcal{E}^q_{k,\Omega}\left(12\lambda + \tfrac{8C_1}{k}\right), \quad u \longmapsto \boldsymbol{E}_{12\lambda+\frac{8C_1}{k}}\left(\tfrac{1}{k}\Delta''_k\restriction_\Omega\right)(1-\rho)u\,. \tag{3.9}$$

In particular,

$$N^q_\Gamma\left(\lambda, \tfrac{1}{k}\Delta''_k\right) \leqslant N^q_\Gamma\left(12\lambda + \tfrac{8C_1}{k}, \tfrac{1}{k}\Delta''_k\restriction_\Omega\right), \quad q \geqslant 1\,,\ \lambda < 1/24\,, \tag{3.10}$$

The meaning of the map (3.9) is that we cut–off the form u and then we project it on the spectral space of the Dirichlet laplacian. We note that (3.10) shows that the spectral spaces $\mathcal{E}^q_k(\lambda)$, for $q \geqslant 1$, $\lambda < 1/24$, are of finite Γ–dimension.

Proof. To prove the claim let us remark that the map (3.9) is the restriction of an operator on $L^{0,q}(M,E^k)$ of the same form; this is continuous and Γ–invariant being a composition of a multiplication with a bounded Γ–invariant function and a Γ–invariant projection. To prove the injectivity we choose $u \in \mathcal{E}^q_k(\lambda)$, $\lambda < 1/24$ to the effect that $Q_k(u,u) \leqslant \lambda\|u\|^2 \leqslant \frac{1}{24}\|u\|^2$. Plugging this relation in (3.6) we get

$$\|u\|^2 \leqslant 8\int_\Omega \big|(1-\rho)u\big|^2, \quad u \in \mathcal{E}^q_k(\lambda)\,, \quad \lambda < 1/24\,. \tag{3.11}$$

Let us denote by $Q_{k,\Omega}$ the quadratic form of $\frac{1}{k}\Delta''_k\restriction_\Omega$. Then by (3.8) and (3.11),

$$Q_{k,\Omega}\big((1-\rho)u,(1-\rho)u\big) \leqslant \tfrac{3}{2}\,Q_k(u,u) + \tfrac{C_1}{k}\|u\|^2 \leqslant \left(12\lambda + \tfrac{8C_1}{k}\right)\int_\Omega \big|(1-\rho)u\big|^2$$

which shows that $\boldsymbol{E}_{12\lambda+\frac{8C_1}{k}}\left(\frac{1}{k}\Delta''_k\restriction_\Omega\right)(1-\rho)u=0$ imply $(1-\rho)u=0$ so that $u=0$ by (3.11). This achieves the proof. □

We now state our main result. We find a lower bound for the Γ–dimension of the group $H^0_{(2)}(M,E^k)$ of L^2 holomorphic sections in E^k and give an upper bound for the growth of the Γ–dimension of the higher L^2 cohomology groups. The L^2 condition is understood with respect to Γ–invariant metrics on the base X and along the fibers of the bundle.

Theorem 3.8. *Let $\overline{M}$ be a smooth pseudoconvex domain which admits a free holomorphic action of a discrete group Γ such that the action extends in a complex neighbourhood X of $\overline{M}$. Assume that $\overline{M}/\Gamma$ is compact and there exists a hermitian holomorphic Γ–invariant line bundle E on X which is positive near ∂M. Then for $k\longrightarrow\infty$,*

$$\dim_\Gamma H^0_{(2)}(M,E^k)\geqslant\frac{k^n}{n!}\int_{(M/\Gamma)(\leqslant 1)}\left(\tfrac{\imath}{2\pi}\mathbf{c}(E)\right)^n+o(k^n)\,,\qquad(3.12)$$

where $n=\dim M$.

The higher L^2 reduced cohomology groups $(q\geqslant 1)$ have finite Γ–dimension and satisfy the inequalities for $k\longrightarrow\infty$:

$$\dim_\Gamma H^{0,q}_{(2)}(M,E^k)\leqslant\frac{k^n}{n!}\int_{(M/\Gamma)(q)}(-1)^q\left(\tfrac{\imath}{2\pi}\mathbf{c}(E)\right)^n+o(k^n)\,,\qquad(3.13)$$

Proof. Since $\operatorname{Dom}Q_{k,\Omega}$ can be embedded in $\operatorname{Dom}Q_k$, an easy consequence of the variational principle (2.1) is that $N^0_\Gamma(\lambda,\frac{1}{k}\Delta''_k)\geqslant N^0_\Gamma(\lambda,\frac{1}{k}\Delta''_k\restriction_\Omega)$. By Theorem 2.9 for $q=0$

$$\varliminf_k k^{-n}N^0_\Gamma\left(\lambda,\tfrac{1}{k}\Delta''_k\right)\geqslant I^0(U,\lambda)\,,\quad\lambda<1/24\,,\quad\lambda\in\mathbb{R}\smallsetminus\mathcal{D}\qquad(3.14)$$

(Note that Ω can be chosen smooth.)

We find now an upper bound for $N^1_\Gamma\left(\lambda,\frac{1}{k}\Delta''_k\right)$. Fix an arbitrary $\delta>0$. For $k>8C_1/\delta$ we have

$$N^1(\lambda,\tfrac{1}{k}\Delta''_k)\leqslant N^1_\Gamma(12\lambda+\tfrac{8C_1}{k},\tfrac{1}{k}\Delta''_k\restriction_\Omega)\leqslant N^1_\Gamma(12\lambda+\delta,\tfrac{1}{k}\Delta''_k\restriction_\Omega)\,,$$

hence by (2.5), $\varlimsup_k k^{-n}N^1_\Gamma(\lambda,\frac{1}{k}\Delta''_k)\leqslant I^1(U,12\lambda+\delta)$. We can let $\delta\longrightarrow 0$ so that

$$\varlimsup_k k^{-n}N^1_\Gamma\left(\lambda,\frac{1}{k}\Delta''_k\right)\leqslant I^1(U,12\lambda)\,,\quad\lambda<1/24\,,\quad\lambda\in\mathbb{R}\smallsetminus\mathcal{D}\,.\qquad(3.15)$$

We consider next $\overline\partial\colon L^{0,0}(M,E^k)\longrightarrow L^{0,1}(M,E^k)$. Since Δ''_k commutes with $\overline\partial$ it follows that the spectral projections of Δ''_k commute with $\overline\partial$ too, showing thus $\overline\partial\mathcal{E}^0_k(\lambda)\subset\mathcal{E}^1_k(\lambda)$. Therefore we have the Γ–morphism $\overline\partial\colon\mathcal{E}^0_k(\lambda)\longrightarrow\mathcal{E}^1_k(\lambda)$, where $\overline\partial_\lambda$ denotes the restriction of $\overline\partial$ (by the definition of $\mathcal{E}^0_k(\lambda)$, $\overline\partial_\lambda$ is bounded by $k\lambda$). Since for any Γ–morphism A we have $\dim_\Gamma[\operatorname{Ran}(A)]=\dim_\Gamma\operatorname{Ker}(\underline{A})^\perp$, we see that $\dim_\Gamma\operatorname{Ker}\overline\partial_\lambda+\dim_\Gamma\left[\operatorname{Ran}(\overline\partial_\lambda)\right]=\dim_\Gamma\mathcal{E}^0_k(\lambda)$. Moreover $\dim_\Gamma\left[\operatorname{Ran}(\overline\partial_\lambda)\right]\leqslant\dim_\Gamma\mathcal{E}^1_k(\lambda)$ and they are finite. Therefore by (3.14) and (3.15),

$$\dim_\Gamma H^0_{(2)}(M,E^k)\geqslant\dim_\Gamma\operatorname{Ker}\overline\partial_\lambda\geqslant k^n\left[I^0(U,\lambda)-I^1(U,12\lambda)\right]+o(k^n)\,,$$

for $\lambda < 1/24$ and $\lambda \in \mathbb{R} \smallsetminus \mathcal{D}$. We can now let λ go to zero through these values. The limits $I^0(U,0)$ and $I^1(U,0)$ are calculated in [7] (see (2.6)) and if we identify the fundamental domain U with Ω/Γ we get

$$\dim_\Gamma H^0_{(2)}(M,E^k) \geqslant \frac{k^n}{n!}\int_{(\Omega/\Gamma)(\leqslant 1)} \left(\tfrac{\imath}{2\pi}\mathbf{c}(E)\right)^n + o(k^n)\,.$$

Now we can let Ω exhaust M and the desired formula (3.12) follows.

To prove (3.13) we remark that for $\lambda \geqslant 0$,

$$\dim_\Gamma \operatorname{Ker}\left(\tfrac{1}{k}\Delta''_k\right) \cap L^{0,q}(M,E^k) \leqslant N^q_\Gamma\left(\lambda, \tfrac{1}{k}\Delta''_k\right)\,,$$

and as in formula (3.15),

$$\varlimsup_k k^{-n} \dim_\Gamma \operatorname{Ker}\left(\tfrac{1}{k}\Delta''_k\right) \cap L^{0,q}(M,E^k) \leqslant I^q(U,12\lambda)\,,\quad \lambda < 1/24\,,\quad \lambda \in \mathbb{R}\smallsetminus\mathcal{D}\,.$$

We invoke now the Hodge isomorphism (3.1)

$$\operatorname{Ker}\left(\tfrac{1}{k}\Delta''_k\right) \cap L^{0,q}(M,E^k) \cong H^{0,q}_{(2)}(M,E^k)\,,$$

and let $\lambda \longrightarrow 0$ through values $\lambda \in \mathbb{R}\smallsetminus\mathcal{D}$. By (2.6) we get the inequality (3.13) for the domain of integration Ω/Γ so we can let $\Omega \longrightarrow M$. The proof is thus complete. □

Corollary 3.9. *In the conditions of Theorem 3.8 assume that*

$$\int_{(M/\Gamma)(\leqslant 1)} \left(\tfrac{\imath}{2\pi}\mathbf{c}(E)\right)^n > 0\,. \tag{3.16}$$

Then for large k the space $H^0_{(2)}(M,E^k)$ is non–trivial. More precisely,

$$\dim_\Gamma H^0_{(2)}(M,E^k) \gtrsim k^n\,.$$

Remarks.

(i) Condition (3.16) may be seen as a very weak positivity condition. It means that the integral of the curvature over its positivity points exceeds the integral of the curvature over the points where the curvature is non–degenerate and has exactly one negative eigenvalue. So the curvature may have negative eigenvalues. The inequality (3.16) is certainly satisfied if the curvature is everywhere semipositive, to the effect that $(M/\Gamma)(1) = \varnothing$. Therefore the integral in (3.12) extends over all M/Γ. If E is everywhere positive this integal expresses the volume of M/Γ in the metric given by $\mathbf{c}(E)$.

(ii) Assume that E is positive everywhere on $\overline{M}$. Then (3.6) becomes

$$\|u\|^2 \leqslant 12\,Q_k(u,u)\,,\quad k \gg 1\,,\quad q \geqslant 1\,. \tag{3.17}$$

Thus $\operatorname{Ker}\Delta''_k = 0$ and by Hodge isomorphism we obtain the asymptotic vanishing result

$$H^{0,q}_{(2)}(M,E^k) = 0\,,\quad k \gg 1\,,\quad q \geqslant 1\,.$$

For $\Gamma = \{\mathrm{Id}\}$, this is implicit in Takegoshi [25]. Moreover, (3.17) shows that $\operatorname{Ran}(\overline{\partial}) \cap L^{0,q}(M,E^k)$ is closed for very large k and $q \geqslant 1$. Therefore, for any $f \in L^{0,q}(M,E^k)$,

$q \geqslant 1$, $k \gg 1$, with $\overline{\partial} f = 0$, there exists $u \in L^{0,q-1}(M,E^k)$ with $\overline{\partial} u = f$. By solving the $\overline{\partial}$–equation with singular logarithmic weights we can show that the L^2 holomorphic sections of E^k, $k \gg 1$, separate points of M and give local coordinates on M (see [13, Theorem 0.7] for the case when M/Γ is Stein).

(iii) The inequalities (3.12), (3.13) remain true if $\partial M = \varnothing$. This implies that M/Γ is a compact manifold and we have obtained estimates for the growth of the von Neumann dimension of the L^2 cohomology analogous to the Morse inequalities of Demailly [7]. Note that since $\partial M = \varnothing$ we need no positivity condition for E near the boundary. Corollary 3.9 holds true in this situation too. It can be seen as a generalization of a theorem of Napier [19] which shows that if E is positive then there are sufficiently many sections in E^k (more precisely, M is E^k–convex).

(iv) Let us explain briefly why it is necessary to work directly with incomplete metrics. Let M be a relatively compact pseudoconvex domain. If there exists a Kähler metric near ∂M, there exists a complete metric on M which is Kähler near ∂M. Indeed, if we denote with d a function which, near ∂M, equals the distance to ∂M, it is known [8, Principal Lemma], that the complex hessian $\partial\overline{\partial}\chi$ of $\chi = -\log(d)$ is uniformly bounded from below near ∂M. By adding $\varepsilon\partial\overline{\partial}\chi$, ($\varepsilon \ll 1$), to a Kähler metric we obtain the desired complete metric.

In order to apply the method from [26] we need to have an analogue of estimate (3.6). But, on one hand, the boundedness from below of the eigenvalues of $\mathbf{c}(E)$ with respect to the complete metric just introduced can not be verified. On the other hand, if we want to get around this difficulty by introducing the weight $\exp(-\chi) = d$ along the fibers of E, we obtain, in degree 0, an L^2 cohomology strictly larger as the one considered in this paper. Therefore, the results in [26] do not imply Theorem 3.8.

4. Weakly 1–complete manifolds

In this section we apply the main Theorem 3.8 to the case of weakly 1–complete manifolds and strongly pseudoconvex domains. We consider coverings of pseudoconvex domains which appear as sublevel sets of a weakly 1–complete manifold $\widetilde{X}$. We denote by $\varphi : \widetilde{X} \longrightarrow \mathbb{R}$ the smooth psh exhaustion function and $\widetilde{X}_c = \{\varphi < c\}$ for $c \in \mathbb{R}$. Let $n = \dim X$.

Theorem 4.1. *Let X be a complex manifold which admits a free holomorphic action of a discrete group Γ such that $X/\Gamma = \widetilde{X}$ is a weakly 1–complete non–compact manifold. Let $X_c = \pi^{-1}(\widetilde{X}_c)$ where π is the covering map. Assume that E is a Γ–invariant hermitian holomorphic line bundle in a neighbourhood of $\overline{X}_c$ and is positive near ∂X_c. Then,*

$$\lim_{k\to\infty} k^{-n} \dim_\Gamma H^0_{(2)}(X_c, E^k) = \infty$$

where the L^2 condition is considred with respect to any Γ-invariant metrics on $\overline{X}_c$ and $E\restriction_{\overline{X}_c}$.

A similar result for the trivial covering and E an everywhere positive line bundle has been obtained by S. Takayama [24].

Proof. Consider $d > c$ such that E is defined on $X_d = \pi^{-1}(\widetilde{X}_d)$. We construct a metric of infinite volume on $\widetilde{X}_d$ in the following way. Let $\widetilde{h}$ be a metric on $\widetilde{E} = E/\Gamma$ which has positive curvature on $\widetilde{X}_d \smallsetminus \widetilde{X}_{c-\varepsilon}$ where $\varepsilon > 0$ is small enough ($\widetilde{h}$ is the push down of the given metric on E). Let $\widetilde{\omega}$ be a metric on $\widetilde{X}_d$ which coincides with $\imath\mathbf{c}(\widetilde{E})$ on $\widetilde{X}_d \smallsetminus \widetilde{X}_{c-\varepsilon}$. Let $\chi : (-\infty, d) \longrightarrow \mathbb{R}$ be a convex increasing function such that $\lim_{t\to c}\chi(t) = \infty$ (e.g. $\chi(t) = (c-t)^{-2}$). Then $\widetilde{\omega}_0 = \widetilde{\omega} + \imath\partial\overline{\partial}\chi(\varphi)$ is a metric of infinite volume if χ increases very fast. We consider the new metric $\widetilde{h}_0 = \widetilde{h}\exp(-\chi(\varphi))$ with curvature $\imath\mathbf{c}(\widetilde{E},\widetilde{h}_0) = \imath\mathbf{c}(\widetilde{E}) + \imath\partial\overline{\partial}\chi(\varphi) = \widetilde{\omega}_0$. We take the pull–backs of the metrics $\widetilde{\omega}_0$ and $\widetilde{h}_0$ on X_d and denote them with ω_0 and h_0. We consider a pseudoconvex domain $X_e = \pi^{-1}\widetilde{X}_e$ (e regular value of φ, $c < e < d$) and apply Theorem 3.8 for X_e and the metric h_0. In particular (3.12) becomes

$$\varliminf_k k^{-n}\dim_\Gamma H^0_{(2)}(X_e, E^k) \geqslant \frac{1}{n!}\int_{(\widetilde{X}_e)(\leqslant 1)}\left(\tfrac{\imath}{2\pi}\mathbf{c}(\widetilde{E},\widetilde{h}_0)\right)^n.$$

The restriction morphism

$$\Phi_k : H^0_{(2)}(X_e, E^k) \longrightarrow H^0_{(2)}(X_c, E^k)$$

is clearly an injective Γ–morphism. Moreover the L^2 condition in the latter space may be taken with respect to any Γ–invariant metrics in the neighbourhood of $\overline{X}_c$. Hence

$$\varliminf_k k^{-n}\dim_\Gamma H^0_{(2)}(X_c, E^k) \geqslant \frac{1}{n!}\int_{(\widetilde{X}_e)(\leqslant 1)}\left(\tfrac{\imath}{2\pi}\mathbf{c}(\widetilde{E},\widetilde{h}_0)\right)^n. \tag{4.1}$$

We split the integral in (4.1) in two parts:

$$\int_{(\widetilde{X}_e)(\leqslant 1)}\left(\tfrac{\imath}{2\pi}\mathbf{c}(\widetilde{E},\widetilde{h}_0)\right)^n = \int_{(\widetilde{X}_c)(\leqslant 1)}\left(\tfrac{\imath}{2\pi}\mathbf{c}(\widetilde{E},\widetilde{h}_0)\right)^n + \int_{\widetilde{X}_e\setminus\widetilde{X}_c}\left(\tfrac{\widetilde{\omega}_0}{2\pi}\right)^n.$$

The first integral is finite and may be negative. The second represents the volume of $\widetilde{X}_e \smallsetminus \widetilde{X}_c$ in the metric $\widetilde{\omega}_0$. Since $\widetilde{\omega}_0$ has infinite volume on $\widetilde{X}_d$ the second term goes to infinity as $e \longrightarrow d$. Therefore by letting $e \longrightarrow d$ in (4.1) we obtain the desired result. □

Remarks.

(i) We can actually prove more. Namely, for each k there exists a sequence of Γ–invariant spaces $L_k \subset H^0_{(2)}(X_c, E^k) \cap C^{0,0}(\overline{X}_c, E^k)$ such that

$$k^{-n}\dim_\Gamma[L_k] \longrightarrow \infty.$$

This follows from a variant of the main Theorem 3.8 for complete hermitian manifolds proved in [26]. Indeed, the metric ω_0 is complete on X_d and $\mathbf{c}(E, h_0) = \omega_0$ at least outside X_c. Then, by [26], for the L^2 cohomology with respect to ω_0 and h_0 we have $\lim_k k^{-n}\dim_\Gamma H^0_{(2)}(X_d, E^k) = \infty$ since the volume of $\widetilde{X}_d$ is infinite. The range L_k of the injective Γ–morphism $H^0_{(2)}(X_d, E^k) \longrightarrow H^0_{(2)}(X_c, E^k)$ is contained in $C^{0,0}(\overline{X}_c, E^k)$ and $[L_k]$ has the same von Neumann dimension as $H^0_{(2)}(X_d, E^k)$.

(ii) Using (3.13) we obtain for $q \geqslant 1$ and $k \gg 1$:

$$\dim_\Gamma H^{0,q}_{(2)}(X_c, E^k) \leqslant \frac{k^n}{n!}\int_{(X_c/\Gamma)(q)}(-1)^q\left(\tfrac{\imath}{2\pi}\mathbf{c}(E)\right)^n + o(k^n). \tag{4.2}$$

Let us examine (4.2) for the case $\Gamma = \{\mathrm{Id}\}$. Then by the representation theorem of Takegoshi ([25], Theorem 6.2) $H^{0,q}_{(2)}(X_c, E^k) \cong H^q(X_c, \mathcal{O}(E^k))$ for $q \geqslant 1$ and $k \gg 1$. We obtain thus that the growth of the dimension of the analytic sheaf cohomology $\dim H^q(X_c, \mathcal{O}(E^k))$ is at most polynomial in k^n. This was proved in [17] using the $\overline{\partial}$ operator theory without boundary conditions as an answer to a question of Ohsawa [20].

We show next how we can recover some results from Gromov–Henkin–Shubin [13]. As before we consider a complex manifold X with a free action of a discrete group Γ and a Γ–invariant domain $\overline{M}$.

Proposition 4.2 ([13]). *Assume that $\overline{M}$ is a strongly pseudoconvex domain such that $\overline{M}/\Gamma$ is compact. Then* $\dim_\Gamma \left[H^0_{(2)}(M) \cap C^{0,0}(\overline{M})\right] = \infty$. *Moreover* $\dim_\Gamma H^{0,q}_{(2)}(M) < \infty$ *for $q \geqslant 1$.*

Proof. Since $\widetilde{M} = M/\Gamma$ is a compact strongly pseudoconvex domain in X/Γ we may take a defining function $\widetilde{\varphi}$ for $\widetilde{M}$ such that $\widetilde{M} = \{\widetilde{\varphi} < 0\}$ and $\partial\overline{\partial}\widetilde{\varphi}$ is positive definite near $\partial\widetilde{M}$. This follows immediately from the definition by composition of a defining function as in definition with an increasing convex function (e.g. $t \longmapsto \exp(At) - 1$ for large $A > 0$). Consider $\varepsilon > 0$ sufficiently small such that $\widetilde{M}_\varepsilon = \{\widetilde{\varphi} < \varepsilon\}$ is also strongly pseudoconvex. Let $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $\chi(t) = 0$ for $t \leqslant 0$ and $\chi'(t) > 0$, $\chi''(t) > 0$ for $t > 0$. Then by replacing $\widetilde{\varphi}$ with $\chi(\widetilde{\varphi})$ we can assume that $\partial\overline{\partial}\widetilde{\varphi} \geqslant 0$ everywhere and $\partial\overline{\partial}\widetilde{\varphi} > 0$ near $\partial\widetilde{M}_\varepsilon$. Let φ be the pull–back of $\widetilde{\varphi}$ to X and let M_ε the pre–image of $\widetilde{M}_\varepsilon$ by the covering map. M_ε is a pseudoconvex domain and the trivial bundle $E = X \times \mathbb{C}$ endowed with the metric $\exp(-\varphi)$ is semipositive and positive near ∂M_ε. By Theorem 3.8,

$$\dim_\Gamma H^0_{(2)}(M_\varepsilon, E^k) \geqslant \frac{k^n}{n!}\int_{\widetilde{M}_\varepsilon} \left(\tfrac{i}{2\pi}\partial\overline{\partial}\widetilde{\varphi}\right)^n + o(k^n)\,, \tag{4.3}$$

for $k \longrightarrow \infty$. Since

$$H^0_{(2)}(M_\varepsilon, E^k) = \left\{ f : M_\varepsilon \longrightarrow \mathbb{C} \,:\, \int_{M_\varepsilon} |f|^2 \exp(-k\varphi) < \infty \right\}$$

and $\varphi < \varepsilon$ on M_ε we see that $H^0_{(2)}(M_\varepsilon, E^k) \subset H^0_{(2)}(M_\varepsilon)$ for any k. By (4.3),

$$\dim_\Gamma H^0_{(2)}(M_\varepsilon) \geqslant C k^n + o(k^n)$$

for some $C > 0$. By letting $k \longrightarrow \infty$ we obtain $\dim_\Gamma H^0_{(2)}(M_\varepsilon) = \infty$. We take L to be the range of the injective Γ–morphism of restriction $H^0_{(2)}(M_\varepsilon) \longrightarrow H^0_{(2)}(M)$. Then $L \subset C^{0,0}(\overline{M})$ and $\dim_\Gamma [L] = \infty$ as claimed.

To prove the finiteness of the Γ–dimension for $q \geqslant 1$ we use the finiteness statement in Theorem 3.8. By this result we know that $\dim_\Gamma H^{0,q}_{(2)}(M, E^k) < \infty$ for sufficiently large k. We fix such a k. The sections with $L^{0,\bullet}(M, E^k)$ are forms with coefficients in the weighted L^2 space $L^2(M, \exp(-k\varphi))$. Since $|\varphi|$ is bounded on M we see that

$L^{0,\bullet}(M,E^k) = L^{0,q}(M)$ as sets and the two norms are equivalent. Hence $H^{0,q}_{(2)}(M,E^k) \cong H^{0,q}_{(2)}(M)$. □

Proposition 4.2 recovers a part of [13, Theorems 0.1–2], where it is moreover proved that any point of ∂M is a peak point for $H^0_{(2)}(M) \cap C^{0,0}(\overline{M})$. In [13, Theorem 0.3], it is proved that for any integer $N > 0$ there exists a Γ–invariant subspace $L \subset H^0_{(2)}(M) \cap C^{0,0}(\overline{M})$ such that $\dim_\Gamma[L] = N$. See also Remark (i) above.

Proposition 4.3 ([13]). *Let M be a pseudoconvex domain with holomorphic action of a discrete group Γ on a complex neighbourhood X of $\overline{M}$ such that $\overline{M}$ is invariant and $\overline{M}/\Gamma$ is compact. Suppose that in a Γ–invariant neighbourhood of ∂M there exists a Γ–invariant smooth strictly psh function φ. Then $\dim_\Gamma H^0_{(2)}(M) = \infty$ and $\dim_\Gamma H^{0,q}_{(2)}(M) < \infty$ for $q \geqslant 1$.*

Proof. Suppose that $M = \{\rho < 0\}$, for a smooth invariant function ρ. Choose $\varepsilon > 0$ such that φ is strictly psh on $\{-\varepsilon < \varphi < \varepsilon\}$. Let $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $\chi(t) = 0$ for $t \leqslant -\varepsilon$ and $\chi'(t) > 0$, $\chi''(t) > 0$ for $t > -\varepsilon$. Then $\chi(\varphi)$ is a Γ–invariant psh function which is strictly psh near ∂M. Therefore the trivial bundle $E = X \times \mathbb{C}$ endowed with the metric $\exp(-\chi(\varphi))$ is semipositive and positive near ∂M. The conclusion of the Proposition follow now from Theorem 3.8 as in the preceding proof. □

Proposition 4.3 recovers a part of [13, Theorems 0.5–6], where φ is allowed to be non–smooth and it is moreover proved that any point of strong pseudoconvexity in ∂M is a peak point for $H^0_{(2)}(M) \cap C^{0,0}(\overline{M})$.

Remark. The results in Sections 3 and 4 are still valid if we replace E^k with $E^k \otimes F$ where F is a Γ–invariant hermitian holomorphic vector bundle defined on the neighbourhood of $\overline{M}$. Therefore we can replace in these sections the L^2–cohomology for $(0,q)$–forms with the L^2 cohomology for (p,q)–forms.

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