The Laplace Operator on High Tensor Powers of Line Bundles

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CHAPTER 1

Introduction

Many important results in algebraic and complex geometry are derived by combining a vanishing with an index theorem. The vanishing theorems we will encounter are in turn obtained via harmonic theory and the Bochner technique. The key remark is that the spectrum of the Laplace operator acting on $(0,q)$-forms, $q \geqslant 1$, with values in the tensor powers of a positive line bundle shifts to the right linearly in the tensor power.

An important generalization which we will emphasize are the asymptotic Morse inequalities of Demailly. They give asymptotic bounds on the Morse sums of the Betti numbers of $\overline{\partial}$ on high tensor powers of a holomorphic hermitian line bundle in terms of certain integrals of the curvature form. The asymptotic Morse inequalities provide a useful tool in complex geometry. They are again based on the asymptotic spectral behaviour of the Laplace operator.

The applications of vanishing theorems and Morse inequalities are numerous. Let us mention here only the Kodaira embedding theorem, the classical Lefschetz hyperplane theorem for projective manifolds, Donaldson's version for symplectic ones, the computation of the asymptotics of the Ray-Singer analytic torsion by Bismut and Vasserot, as well as the solution of the Grauert-Riemenschneider conjecture by Siu and Demailly or the compactification of complete Kähler manifolds of negative Ricci curvature by Nadel and Tsuji.

The holomorphic Morse inequalities are global statements which can be deduced from local informations such as the behaviour of the heat or Bergman kernels. In this refined form they can be used for the study of the existence of Kähler-Einstein metrics in relation to Mumford-Chow stability, convergence of the induced Fubini-Study metric, distribution of zeroes of random and quantum polynomials or sections, Berezin-Toeplitz quantization and sampling problems.

Our goal is to study the interplay between the spectral properties of the Laplacian on high tensor powers of line bundles and the analytic and geometric properties of the underlying manifolds.

Some words are in order about the conception of these notes. We wished to follow a thought in some of its proteic transformations, from Witten's proof of the Morse inequalities to the asymptotic expansion of the Bergman kernel. So we started from the basic results and included many natural applications, alongside with our original contributions. The result is sometimes baroque in form, but could serve as reference for me and hopefully (if well rewritten!) also for others. We would be delighted if these notes could as well achieve the aim of being a successful Habilitation Thesis.

The rest of the introduction consists of two parallel sections. Section 1.1 illustrates the richness of the subject and presents well-known results, announcing the themes of the present work. In Section 1.2 we highlight our contribution to these questions and in the same time describe the contents of each chapter.

1.1. Motivation and examples

Kodaira vanishing theorem. We begin with the famous Kodaira vanishing theorem [**88**]. Let X be a compact complex manifold and $L \longrightarrow X$ a positive line bundle. Then

$$H^p(X,\mathscr{O}(L^k)) = 0, \quad p \geqslant 1, \quad \text{for large } k. \tag{1.1}$$

By the Hirzebruch-Riemann-Roch formula (which for a general complex manifold is a consequence of the index theorem of Atiyah-Singer [**17**])

$$\sum_{p=1}^{n} (-1)^p \dim H^p(X,\mathscr{O}(L^k)) = \frac{k^n}{n!}\operatorname{vol}(X) + P_{n-1}(k)\,, \tag{1.2}$$

where $n = \dim X$, $P_{n-1}(k) \in \mathbb{Q}[k]$ is a polynomial of degree $n-1$, and $\operatorname{vol}(X)$ is the volume of X in the metric given by the curvature $\frac{\sqrt{-1}}{2\pi}R^L$. By (1.1) all higher cohomology groups in (1.2) vanish, so

$$\dim H^0(X,\mathscr{O}(L^k)) = \frac{k^n}{n!}\operatorname{vol}(X) + o(k^n), \quad k \longrightarrow \infty. \tag{1.3}$$

An important particular case had been considered by Poincaré. If X is a compact Riemann surface of genus $g \geqslant 2$, the universal covering of X is the unit disc $D \subset \mathbb{C}$. The Poincaré metric on D, $\omega_P = 4\big(1-|z|^2\big)^{-2} dz\wedge d\bar{z}$, is invariant under automorphisms of D, and descends to a metric on X, denoted with the same symbol. Moreover ω_P is a Kähler–Einstein, with Ricci curvature

$$R^{\det} = -\omega_P\,. \tag{1.4}$$

Let K_X be the canonical bundle of X (generated by holomorphic 1–forms). Equation (1.4) implies $R^{K_X} = \omega_P$, so K_X is positive. By Kodaira's theory there are a lot of sections in K_X^k. They correspond in fact to automorphic forms of degree k.

The analytic method of proving (1.1) consists in applying the Dolbeault isomorphism to identify the sheaf cohomology $H^p(X,\mathscr{O}(L^k))$ to the Dolbeault cohomology $H^{0,p}(X,L^k)$ and then, via Hodge theory, to the space of harmonic space $\mathscr{H}^{0,p}(X,L^k)$. Let us denote by $\overline{\partial}^E$ the $\overline{\partial}$-operator acting on a holomorphic vector bundle E and by ϑ^E its formal adjoint. The Kodaira-Laplacian is then $\square^E = \big(\overline{\partial}^E + \vartheta^E\big)^2$. The Bochner technique delivers, due to the positivity of the curvature R^L, that $\big(\square^{L^k}u,u\big) \geqslant Ck\|u\|^2$ for u a $(0,p)$-form with values in L^k, $p \geqslant 1$ and $k \gg 1$. We see that the spectrum is contained in the set $[Ck,+\infty)$. This is very similar to the spectral gap developed by the Witten Laplacian Δ_t (see (1.7)), for t going to $+\infty$. The spectral gap will be important also for more refined questions such as the asymptotic expansion of the Bergman kernel.

L^2 **estimates for** $\overline{\partial}$**.** There are several wide–ranging and deep generalizations of the Kodaira vanishing theorem. One of them is the L^2 method for the $\overline{\partial}$–equation. We briefly state some of the results obtained by this method. Although we shall not use it directly, it stays always in the background of our work, since it tackles the problem of finding holomorphic sections.

The L^2 estimates for $\overline{\partial}$ on complete manifolds and were introduced by Andreotti–Vesentini [**15**], Hörmander [**80**] and applied to a variety of problems by Nakano [**97**], Skoda [**118**], Demailly [**51**], Ohsawa [**99**]. The solution of the $\overline{\partial}$–Neumann problem by Kohn, Morrey, Hörmander can be seen as an extension of the Kodaira technique for manifolds with boundary.

Bombieri [**34**] and Skoda [**118**] introduced a new technique to deal with singular hermitian metrics. This was generalized by Nadel [**95**] and Demailly [**54**] which introduced the so called Nadel multiplier sheaf. If (L,h^L) is a line bundle with singular metric h^L let $\mathscr{I}(h^L)$ be the ideal sheaf of holomorphic functions square integrable with respect to the local weights of h^L. The Nadel–Demailly vanishing theorem asserts that $H^q(X,L\otimes K_X\otimes \mathscr{I}(h^L))=0$ for $q\geqslant 1$, if $\sqrt{-1}R^L\geqslant \varepsilon\omega$ in the sense of currents. By using the liberty of choosing local weights one can produce holomorphic sections with given jets at a finite set of points. The Nadel–Demailly vanishing theorem implies the Kawamata–Viehweg vanishing theorem, one of the cornerstones of modern algebraic geometry.

Returning to smooth complete metrics, the usual Bochner–Kodaira technique is not sufficient to explain all vanishing theorems. For example, the fact that $H^{1,1}_{(2)}(B)=0$ where B is the unit disc in $\mathbb{C}$ endowed with the Bergman metric. Donnelly and Fefferman [**63**] first found a method to get around this difficulty. This leads to important discoveries such as the Oshawa-Takegoshi-Manivel extension theorem [**102, 24, 92, 56**] (which in turn has many applications e.g. to the invariance of plurigenera of varieties of general type [**117**] or Fujita conjecture [**116**]) or the solution of the Cheeger–Goreski–McPherson conjecture for isolated singularities by Ohsawa [**100**].

The vanishing theorem of Donnelly–Fefferman asserts that $H^{p,q}_{(2)}(D)=0$ for $p+q\neq n$ if D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ endowed with the Bergman metric. The proof depends on the fact that the Bergman metric has a global potential whose gradient is bounded (this follows from the asymptotic expansion of the Bergman kernel of Fefferman and Boutet de Monvel–Sjöstrand). Gromov [**74**] generalized the Donnelly–Fefferman condition and proved the vanishing theorem for complete Kähler manifolds (X,ω) with $\omega=d\eta$ for some form η with bounded norm. This yields a solution of the Hopf conjecture in the Kähler case. Modified Bochner-Kodaira method which lead to a simplified proof of the Donnelly-Fefferman vanishing theorem were found by Berndtsson [**23, 24**] and Siu [**116**]. We will apply related ideas to the compactification of hyperconcave manifolds in Chapter 6.

As references for the L^2 method for $\overline{\partial}$ let us mention Hörmander [**80, 81, 82**] Vesentini [**122**], Demailly [**57**], Ohsawa [**101**].

Kodaira embedding and generalizations. The Kodaira vanishing theorem implies the Kodaira embedding theorem, to the effect that for large k, the sections of L^k give an embedding of X in the projective space. It is worthwhile to mention that to get the result, we need only use and refine the vanishing of the first cohomology group $H^1\big(X,\mathscr{O}(L^k)\big)$. Let us define the Kodaira map

$$\Phi_k : X\smallsetminus \mathrm{Bl}_k \longrightarrow \mathbb{P}H^0(X,L^k)^*\,,\quad \Phi_k(x)=\{s\in H^0(X,L^k): s(x)=0\}\,, \tag{1.5}$$

which associates to each point x outside the base locus Bl_k (by definition, the set of points where all sections of $H^0(X,L^k)$ vanish) the hyperplane of sections vanishing at x. The Kodaira's embedding theorem gives an intrinsic characterization of projective manifolds:

1.1. KODAIRA EMBEDDING THEOREM. *If L is positive the Kodaira map Φ_k is defined everywhere and is an embedding for k sufficiently large. Therefore, X projective if and only if X possesses a positive line bundle.*

The generalization of the Kodaira embedding theorem was proposed by Grauert and Riemenschneider in connection to the characterization of Moishezon varieties. Moishezon varieties are simply compact complex spaces such that the transcendence degree of the meromorphic function field equals their complex dimension. They are so important in algebraic geometry because most of the natural modifications of algebraic varieties can be performed in the category of Moishezon varieties but sometimes not in the category of algebraic varieties.

Projective varieties are Moishezon. Moreover, a fundamental result of Moishezon [**93**] asserts that a Moishezon manifold X can be transformed into a projective manifold $\widehat{X}$ by a finite number of blow ups along smooth centers. By taking the push-forward of the positive line bundle on $\widehat{X}$ we obtain in general a sheaf on X, which is free outside a proper analytic set and has a smooth metric of positive curvature. Such sheaves are called quasi-positive. The question is, whether this property characterizes Moishezon manifolds. Since the Moishezon property is bimeromorphically invariant, we can blow up X in order to obtain a manifold X' possessing a line bundle with semi-positive curvature everywhere and positive outside a proper analytic set. If we show that X' is Moishezon, it follows that X is Moishezon too.

1.2. GRAUERT-RIEMENSCHNEIDER CRITERION. *If X possesses a smooth hermitian line bundle which is semi–positive everywhere and positive on an open dense set, X is Moishezon. Therefore, X is Moishezon if and only if X carries a quasi-positive sheaf.*

The criterion was known as the Grauert–Riemenschneider conjecture until it was solved by Siu [**113, 114**], who used an *asymptotic* vanishing theorem. He showed namely, that

$$\dim H^p\left(X,\mathscr{O}(L^k)\right)=o\big(k^n\big)\quad \text{for } p\geqslant 1 \text{ and } k\longrightarrow\infty\,. \tag{1.6}$$

By Hirzebruch-Riemann-Roch (1.2), $\dim H^0\left(X,\mathscr{O}(L^k)\right)=O\big(k^n\big)$, which implies that L is big and X is Moishezon. Recall that L is called big if its Kodaira–Iitaka dimension $\kappa(L)=\dim X$. By definition $\kappa(L):=\max\{\operatorname{rank}\Phi_k : k\geqslant 1\}$, where Φ_k are the Kodaira maps (1.5). If L is big, by taking quotients of sections of L^k for k large, we obtain enough meromorphic functions on X.

In this situation zero might be in the spectrum of $\square^{L^k}$ on $(0,1)$–forms and is certainly in the spectrum of $\square^{L^k}$ on sections. Siu [**113**, p. 433] raised however the following conjecture and proved that it implies the Grauert-Riemenschneider conjecture. Let X be a compact complex manifold and L a hermitian holomorphic line bundle over X whose curvature form is positive semidefinite everywhere and positive definite at some point. Then $\inf_k\lambda_1(X,L^k)>0$, where $\lambda_1(X,L^k)$ is the smallest positive eigenvalue of the Laplacian $\square^{L^k}$ on L^2 sections of L^k.

There are also generalizations of the Kodaira emebedding Theorem to the case of singular vareties, cf. Schumacher–Tsuji [**108**]. We will also be concerned with algebraicity criteria for singular spaces in Chapter 4.

Demailly's Holomorphic Morse Inequalities. Siu's argument in the proof of the Grauert-Riemenschneider conjecture used all the higher p-th cohomology groups, $p\geqslant 1$. We see, however, that in the proof of Kodaira embedding theorem only the vanishing of the first cohomology group $H^1\big(X,\mathscr{O}(L^k)\big)$ matters. This can also be adapted for a new proof of Grauert-Riemenschneider conjecture and this discovery was triggered by developments in other areas.

In 1982, E. Witten [**124**] gave a new analytic proof of the Morse inequalities, by analyzing the spectrum of the Schrödinger operator

$$\Delta_t=\Delta+t^2|df|^2+tV, \tag{1.7}$$

where $t>0$ is a real parameter, f is a Morse function and V is a 0-order operator. For $t\longrightarrow\infty$, the spectrum of Δ_t approaches the spectrum of a sum of harmonic oscillators attached to the critical points of f.

Using the same philosophy, Demailly [**52**] succeeded in proving asymptotic Morse inequalities in the holomorphic setting. Heat equation proofs were subsequently given by Bismut [**27**], Demailly [**53**] and Bouche [**41**]. Of particular importance are the strong Morse inequalities, which involve partial sums of the Euler-Poincaré characteristic:

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(X,\mathscr{O}(L^k))\leqslant\frac{k^n}{n!}\int_{X(\leqslant q)}(-1)^q\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n) \tag{1.8}$$

as $k\longrightarrow\infty$. Here $X(\leqslant q)$ is the set of points where $\sqrt{-1}R^L$ is non-degenerate and has at most q negative eigenvalues. For $q=n$ we have equality, so we obtain an asymptotic Hirzebruch-Riemann-Roch formula, weaker than (1.2). For $q=1$ however, we get the very precious

$$\dim H^0(X,\mathscr{O}(L^k))\geqslant\frac{k^n}{n!}\int_{X(\leqslant 1)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n)\,,\quad k\longrightarrow\infty\,. \tag{1.9}$$

We obtain therefore:

1.3. DEMAILLY CRITERION. *If L satisfies*

$$\int_{X(\leqslant 1)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n>0\,, \tag{1.10}$$

relation (1.3) *is satisfied and X is Moishezon.*

This also solves the Grauert-Riemenschneider conjecture, for the integral (1.10) is certainly positive if L is semi-positive and positive at one point.

This proof of the Grauert-Riemenschneider criterion deals with smooth hermitian line bundles. However, smooth hermitian metrics with semi-positive curvature do not *characterize* Moishezon manifolds, since there exists examples of Moishezon manifolds which do not possess a line bundle satisfying (1.10) for a *smooth* metric [**89**]. Nevertheless, returning to Moishezon's theorem, it can easily be seen, that the push-forward of the curvature of a positive line bundle on $\widehat{X}$ (the projective blow-up of X) forms an integral Kähler current on X. It implies the existence of a holomorphic line bundle on X, possessing a *singular* hermitian metric with positive curvature (in the sense of currents). So arises the following.

1.4. SHIFFMAN-JI-BONAVERO CRITERION. *X is Moishezon if and only if X possesses a singular hermitian line bundle with positive curvature.*

This was proved independently by Ji and Shiffman [**86**] and Bonavero [**35**] who proposed a proof using Morse inequalities (1.8) for singular hermitian metrics. The new element in Bonavero's paper is the introduction of the Nadel multiplier sheaf in the Morse inequalities. Namely, if $T\in c_1(L)$ is a closed $(1,1)$-current with algebraic singularities, (1.9) becomes

$$\dim H^0(X,\mathscr{O}(L^k)\otimes\mathscr{I}(kT))\geqslant\frac{k^n}{n!}\int_{X(T,\leqslant 1)}T^n_{ac}+o(k^n)\,,\quad k\longrightarrow\infty\,. \tag{1.9$'$}$$

where T_{ac} is the absolute continuous part of T, $\mathscr{I}(kT)$ is the Nadel multiplier sheaf of kT, and the index set $X(T,\leqslant 1)$ is the set of points where T_{ac} is non-degenerate and has at most one negative eigenvalue. It is an interesting problem to prove the singular Morse inequalities for transcendental currents T, that is, without the hypothesis that T has algebraic singularities.

Takayama [**120**] presented a proof using a variant of (1.8) for non-compact manifolds and the generalized Poincaré metric.

The Shiffman–Ji–Bonavero criterion has many applications. An important one is the projectivity criterion for hyperkähler manifolds given by Huybrechts [**83, 84**]. Let X be a compact hyperkähler manifold. Then X is projective if and only if there exists a line bundle L on X such that $q_X(c_1(L)) > 0$, where q_X is the Beauville–Bogomolov quadratic form on the second cohomology of an connected symplectic manifold. Another notable application of the strong Morse inequalities is to the proof of the effective Matsusaka theorem by Siu [**115, 55**].

Recently, the Morse inequalities were used by Boucksom [**42**] to calculate the volume of a pseudoeffective line bundle over a compact Kähler manifold X and to prove a Fujita theorem for big classes. Using the singular Morse inequalities of Bonavero, Boucksom shows that for a pseudoeffective line bundle L and $T \in c_1(L)$ we have

$$\operatorname{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n} \dim H^0(X, \mathscr{O}(L^k)) \geqslant \int_X T^n_{ac}\,, \quad k \longrightarrow \infty\,.$$

Therefore, one has a Grauert-Riemanschneider-type criterion: if L is pseudoeffective and its Chern class $c_1(L)$ contains a current T with $\int_X T^n_{ac} > 0$, $\operatorname{vol}(L) > 0$ and L is big. It is shown then in [**42**, Th.4.7] (using techniques from the proof of the Nakai-Moishezon criterion of Demailly-Paun [**58**]) that the criterion extends to non-necesarily rational pseudoeffevctive classes.

Non–compact generalizations. The importance of inequality (1.9) lies in the fact that it provides us with a substitute for the Riemann-Roch formula, which is particularly suitable to generalization to non-compact manifolds. The reason is that the usual Riemann-Roch formula may break down, for example if the higher cohomology groups are infinite dimensional.

The first use of (1.9) for non-compact manifolds are due to Nadel and Tsuji [**96**]. They prove that, if (X,ω) is a complete Kähler manifold of dimension n with Ricci curvature $R^{\det} \leqslant -\omega$,

$$\dim H^0(X, \mathscr{O}(K_X^k)) \geqslant \frac{k^n}{n!} \operatorname{vol}(X) + o(k^n)\,, \quad k \longrightarrow \infty\,, \tag{1.11}$$

where K_X is the canonical bundle of X. The proof is based on the fact that $\square^{L^k}$ on $(0,1)$-forms has no spectrum in an interval $(0, Ck)$. As a consequence, they obtain a generalization with a new analytic proof of the compactification of arithmetic quotients:

1.5. THEOREM (Nadel–Tsuji [**96**]). *Let (X,ω) be a complete Kähler manifold of dimension n of negative Ricci curvature. Assume that X is uniformized by a Stein manifold and that X is very strongly $(n-2)$-pseudoconcave. Then, X is biholomorphic to a quasi-projective variety.*

Napier and Ramachandran applied (1.11) to generalize a theorem of Burns which states that a quotient of the unit ball in $\mathbb{C}^n$ $(n \geqslant 3)$ by a discrete group of automorphisms which has a strongly pseudoconvex boundary component has only finitely many ends. They proved the following result.

1.6. THEOREM (Napier–Ramachandran [**98**]). *If a complete Hermitian manifold* (X,ω) *of complex dimension* $n \geqslant 3$ *has a strongly pseudoconvex end and its Ricci curvature satisfies* $R^{\det} < -C\omega$ *for some positive constant* C*, then, away from the strongly pseudoconvex end, the manifold has finite volume.*

Atiyah [**16**] initiated the index theory on covering manifolds. He shows that, if X is a compact manifold and $\widetilde{X}$ is a covering with $X = \widetilde{X}/\Gamma$ and P is an elliptic operator on X whose lifting to $\widetilde{X}$ is $\widetilde{P}$, then the von Neumann index of $\widetilde{P}$ equals the index of P. Moreover, the vanishing theorems from the compact case carry over to vanishing theorems for the L^2–cohomology with respect to invariant metrics on the covering (Kollár, Demailly–Campana, Eyssidieux, Braverman).

A nice example of how these ideas combine is the non-vanishing theorem of Kollár, which asserts that, if a projective manifold X has generically large fundamental group and is of general type, $\dim H^0(X,\mathscr{O}(K_X^k)) \geqslant 1$ for $k \geqslant 2$ and $\dim H^0(X,\mathscr{O}(K_X^k)) \geqslant 2$ for $k \geqslant 4$. The idea is that the von Neumann dimension $\dim_\Gamma H^0_{(2)}(\widetilde{X},K^k_{\widetilde{X}})$ equals $\dim H^0(X,\mathscr{O}(K_X^k))$, where $\widetilde{X}$ is a Galois covering of X with Galois group Γ [**90**, 15.5]. The study of L^2–sections of bundles over coverings proves fruitful also for deriving Lefschetz–type theorems à la Nori [**98**].

1.7. THEOREM (Napier–Ramachandran). *If* X *and* Y *are connected smooth projective varieties of positive dimension and if* $f : Y \longrightarrow X$ *is a holomorphic immersion with ample normal bundle, the image of* $\pi_1(Y)$ *in* $\pi_1(X)$ *is of finite index.*

The proof is done by looking at the covering $\widetilde{X} \longrightarrow X$ with Galois group associated to the image of $\pi_1(Y)$ in $\pi_1(X)$. One then constructs holomorphic L^2–sections of an appropriate line bundle using the L^2–method of Skoda–Bombieri–Hörmander and finds a bound for the degree of the covering $\widetilde{X}$.

Analysis of the Bergman Kernel. In 1907 Poincaré proved that the ball and ellipsoid in $\mathbb{C}^2$ are not biholomorphically equivalent. The problem was raised to classify domains under biholomorphic maps. The start is the theorem of Fefferman which asserts that a biholomorphic map $\Phi : D_1 \longrightarrow D_2$ between smoothly bounded strictly pseudoconvex domains extends to a diffeomorphism $\Phi : \overline{D}_1 \longrightarrow \overline{D}_2$. The main tool in Fefferman's proof is the asymptotic expansion of the Bergman kernel $P(z,z)$ as z approaches ∂D. Recall that the Bergman kernel of a domain D in $\mathbb{C}^n$ is the smooth kernel of the projection on the space of L^2 (with respect to the Lebesgue metric) holomorphic functions on D. Boutet de Monvel and Sjöstrand [**44**] related the analysis of $P(z,z)$ to the analysis of the Szegö projection defined on $L^2(\partial D)$ with values in the space of boundary values of holomorphic functions in D, or equivalently, functions on ∂D annihilated by the tangential Cauchy–Riemann operator $\overline{\partial}_b$.

A tradition says that Stefan Bergman discovered the Bergman kernel as a freshman student at a German university. Due to his poor German he misunderstood an exercise (Übungsaufgabe) about the unit interval in $\mathbb{R}$ and worked it out for the unit disc in $\mathbb{C}$. A similar "error" leads to a fertile point of view in the study of polarized projective manifolds. Namely, consider a compact Kähler manifold (X,ω) and (L,h^L) a holomorphic hermitian line bundle such that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$. The Bergman projection is the orthogonal projection

$$P_k : \boldsymbol{L}^2(X,L^k) \longrightarrow H^0(X,L^k)\,. \tag{1.12}$$

from the space of L^2 sections to the space of holomorphic sections of L^k. The Bergman kernel $P_k(x,x')$ is now the smooth kernel the Bergman projection

$$(P_kS)(z) = \int_X P_k(z,z')S(z')\,dv_X(z'), \quad S \in \boldsymbol{L}^2(X,L^k). \tag{1.13}$$

We denote by $B_k(z) = P_k(z,z)$ the restriction on the diagonal. The Bergman kernel is linked to the Kodaira embedding in the following way. Let $\Phi_k : X \longrightarrow \mathbb{P}H^0(X,L^k)^*$ be the Kodaira map (1.5) and let ω_{FS} be the Fubini-Study form on $\mathbb{P}H^0(X,L^k)^*$. Then

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \omega = \tfrac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log B_k(z) \tag{1.14}$$

which shows that $\partial\overline{\partial}\log B_k(x)$ measure how far is Φ_k from being an isometry. For abelian manifolds, Kempf [**87**] and then Ji [**85**] showed that the induced Fubini-Study $\frac{1}{k}\Phi_k^*\omega_{FS}$ metric converges to ω. Tian [**121**] extended the result to general projective manifolds and showed the convergence is in the $\mathscr{C}^2$ topology, so that Φ_k is asymptotically an isometry. Bouche [**41**] gave later a heat kernel proof. The main motivation was the general philosophy of Yau that the stability of embeddings in the sense of Chow–Mumford is connected to the existence of Kähler–Einstein metrics. This was further substantiated in a groundbreaking paper of by Donaldson [**61**] who used a refinement of the results of Tian–Bouche. This is the Tian-Yau-Zelditch asymptotic expansion [**125, 49**].

1.8. THEOREM (Zelditch, Catlin). *Let $L \longrightarrow X$ be a positive line bundle over a compact manifold X and let B_k be the Bergmann kernel associated to L^k by* (1.13). *There exist smooth functions b_j on X such that*

$$B_k(x) = k^n(b_0 + b_1k^{-1} + b_2k^{-2} + \cdots). \tag{1.15a}$$

More precisely, for any $R, l \geqslant 0$ there exists a constant $C_{R,l}$ such that

$$\|B_k - k^n \sum_{j<R} b_j k^{-j}\|_{\mathscr{C}^l} \leqslant C_{R,l}k^{n-R}. \tag{1.15b}$$

The proof of (1.15a) is based on some beautiful observations which permit the application of the asymptotic expansion of Boutet–Sjöstrand [**44**]. Let us define the Grauert tube $T = \{v \in L^* : |v|_{h^{L^*}} = 1\}$ (or generalized Hardy space) to be the associated circle bundle of L^*. Then T is the boundary of the strongly pseudoconvex domain $\{v \in L^* : |v|_{h^{L^*}} < 1\}$ as a consequence of the positivity of R^L. Then sections of L^k may be identified to equivariant functions on T, and the family $\overline{\partial}^{L^k}$ can be identified with the tangential Cauchy–Riemann operator $\overline{\partial}_b$ on T. We have actually

$$H^0(X,L^k) \cong \left\{ f : T \to \mathbb{C} : \overline{\partial}_b f = 0,\ f(e^{\sqrt{-1}\vartheta}w) = e^{\sqrt{-1}k\vartheta} f(w),\ \text{for } w \in T,\ \vartheta \in \mathbb{R} \right\} \tag{1.16}$$

Now, the Bergman kernels B_k are the Fourier coefficients of the Szegö kernel on T and the result of Boutet–Sjöstrand applies. The identification (1.16) was already used by Grauert [**72**] to generalize the Kodaira embedding theorem to singular varieties.

An essential ingredient in Donaldson's result is the calculation of the first coefficients of the expansion (1.15a).

1.9. THEOREM ([**91**]). *Assume that X is endowed with the metric $\omega = \frac{\sqrt{-1}}{2\pi}R^L$. The functions b_j are polynomials in the curvature of ω and its covariant derivatives (with control over the number of differentiations required). Moreover $b_0 = 1$, $b_1 = \frac{1}{8\pi}r^X$ where r^X is the scalar curvature of (X,ω).*

Assume now that the bundle L is not necessarily positive. Although it seems that the asymptotic expansion (1.15a) of B_k does not exists in general, Berndtsson [**25**] and Berman [**21**] determined the expansion up to order $o(k^n)$ and reproved in this way the Morse inequalities of Demailly.

Random and quantum chaotic sections, supersymmetric vacua. There is a growing interest of physicists for the distribution of zeroes of eigenfunctions of quantum maps [**33, 77**] and random polynomials or sections of positive line bundles are a good model for them. In a series of papers Bleher, Shiffman and Zelditch [**30, 32, 29, 31**] studied how the zeroes are disributed and correlated. In the simplest case let us consider sections over $X=\mathbb{P}^n$ of powers L^k of the hyperplane line bundle $L=\mathscr{O}(1)$. As usual we identify sections of L^k with homogeneous polynomials of degree k over $\mathbb{C}^{n+1}$ and introduce an $SU(n+1)$–invariant Gaussian probability measure on the space of such polynomials. It is then proven in [**109**] that if a sequence $\{p_k\}$ is chosen independently and randomly from the spaces of homogeneous polynomials of degree k and L^2 norm one, the zero sets $\{p_k=0\}$ almost surely become uniformly distributed with respect to the Fubini–Study volume form. The expansion of the Bergman kernel is again the main technical tool. These result have interesting extensions in complex dynamics, see Dinh-Sibony [**59**].

In a very recent development Douglas, Shiffman and Zelditch [**64, 65**] studied the statistics of vacua in string/M theory. Vacua are critical points $\nabla s(z)=0$ of a holomorphic section s of a line bundle L, where ∇ is the Chern connection. Physically, they model extremal black holes in addition to vacua of string/M theory [**46, 69**].

Symplectic geometry. In an important work [**60**] (see also [**112**]), Donaldson found a method of producing symplectic submanifolds of symplectic manifolds by extending to the almost–complex case results linked to the existence of holomorphic sections of positive line bundles. Let (X,ω) be a compact Kähler manifold and assume that $\omega=\frac{\sqrt{-1}}{2\pi}R^L$, where (L,h^L) is a holomorphic line bundle. Then the Poincaré dual of the cohomology class $k[\omega]$ is represented by a divisor for large k. The divisor is given as the zero set of a generic holomorphic section of L^k.

Now, if (X,ω) is a general symplectic manifold, the bundle L^k may have no holomorphic sections, so Donaldson proves the symplectic version of the above result by introducing the notion of “asymptotically holomorphic sections” and eventually proves the symplectic Lefschetz hyperplane section theorem. He constructs sections $s_k\in H^0(X,L^k)$ such that $|\overline{\partial}s_k|\leqslant\frac{1}{k}|\partial s|$ on $\{s=0\}$ and the method is reminiscent of the peak section construction of Tian (although the L^2 estimates for $\overline{\partial}$ are not available).

The work of Donaldson, Auroux [**18**] and others to find symplectic analogues of the objects in complex geometry motivated the introduction of a microlocal point of view by Borthwick–Uribe [**39**] and Shiffman–Zelditch [**110**]. They define “almost holomorphic sections” by a method of Boutet de Monvel–Guillemin [**43**]. Remember the identification (1.16) of the holomorphic sections with equivariant functions on the Grauert tube which are solutions of the system $\overline{\partial}_b f=0$. In the non–integrable case Boutet–Guillemin [**43**] define an analogue of the $\overline{\partial}_b$–operator, which is a first order pseudo–differential operator D_b on Y, with the same microlocal properties of $\overline{\partial}_b$. The “almost holomorphic sections” are obtained as sections of L^k corresponding to equivariant functions on Y annihilated by D_b. Although D_b is not canonically defined, Shiffman–Zelditch show that they have typically the same properties as the “asymptotically holomorphic sections” of Donaldson. The proof is based on the near–diagonal asymptotic expansion of the Szegö kernel on the circle bundle Y, which generalizes Theorem 1.8. The asymptotic expansion implies also a symplectic version of the Tian's theorem of convergence of the Fubini–Study metrics and a "nearly holomorphic" and "nearly isometric" embedding theorem [**39**], [**110**].

We will explain below a new geometric approach based on another substitute for holomorphic sections proposed by Guillemin–Uribe [**76**].

1.2. Contents and results

The previous examples illustrate how the spectrum of the Laplace operator on high tensor powers of certain line bundles gives information about analytical and geometrical properties of the manifold.

Our goal is to continue and complement this study. We shall consider

- ◦ singular varieties,
- ◦ manifolds satisfying complex convexity conditions,
- ◦ covering manifolds and
- ◦ symplectic manifolds.

As a warm up, we include in ***Chapter 2*** Witten's analytic proof of the usual Morse inequalities. Witten's approach is to deform the de Rham complex in a manner depending on a Morse function, so that the low-energy eigenvectors of the corresponding Laplacians (1.7), called Witten Laplacians, become concentrated near the critical points. The Witten Laplacians written in Morse coordinates in the neighbourhood of critical points are given essentially by harmonic oscillators. Comparing their spectra with the help of the minimax principle finishes the proof.

The first rigorous account of the analytic proof of the Morse inequalities appeared in the paper by Helffer-Sjöstrand [**78**], based on their results on Schrödinger operators and in a paper by Bismut [**26**] where a proof by heat equation methods was presented (se also [**48**]).

Chapter 3 is devoted to the proof of Demailly's generalization of Weyl's formula for the asymptotic behaviour of the Kodaira–Laplacian $\square^{L^k}$ acting on high tensor powers of a hermitian holomorphic line bundle (L,h^L) on a complex manifold. Later, Bismut and Bouche proved local Morse inequalities, in the sense of index theory. Namely, the global Morse inequalities are derived by integrating the local ones. This is the approach we follow here.

In ***Chapter 4*** we prove the global holomorphic Morse inequalities. We start by proving the Morse inequalities for the Dolbeault L^2–cohomology spaces for a manifold satisfying the fundamental estimate (Poincaré inequality) at infinity. It is straightforward to obtain from here the holomorphic Morse inequalities of Demailly on compact manifolds. After introducing the necessary apparatus of complex geometry (Siegel's lemma, independence of meromorphic functions and Moishezon spaces) we prove the Grauert-Riemenschneider criterion.

Using the abstract formulation of the Morse inequalities we can find a lower bound for the growth of holomorphic section space for uniformly positive line bundles (Theorem 4.30). From this we deduce the Ji–Shiffman–Bonavero criterion, by working on a Zariski open set endowed with the generalized Poincaré metric. Using the same approach we can obtain a sharper result in the case of isolated singularities, mamely a tale quale extension of the Grauert-Riemenschneider criterion:

1.10. THEOREM ([**3**]). *Let X be a compact complex space of dimension $n \geqslant 2$ and with isolated singularities. Suppose that we have one the following conditions.*

(i) *There exists a holomorphic hermitian line bundle L on X_{reg} which is semi-positive in a deleted neighbourhood of X_{sing} and satisfies condition* (1.10) *on X_{reg}.*

(ii) *Assume that L is defined over all X, the hermitian metric may be singular at X_{sing} but the cuvature current R^L is dominated by the euclidian metric near X_{sing} and moreover condition* (1.10) *is fulfilled on X_{reg}.*

Then X is Moishezon.

The result is linked to the interesting question of extending to singular varieties the harmonic theory and its consequences. For example, Brüning and Lesch [**47**] proved that the L^2-Kähler package holds on conformally conic Kähler manifolds and Pardon and Stern [**104**] extended the Kodaira vanishing theorem to singular manifolds. In the case of compact analytic surfaces in $\mathbb{R}^n$ D. Grieser [**73**] has given an L^2 Gauss-Bonnet theorem for the regular part, endowed with a riemannian metric induced from the ambient space. For an account of a differential geometric approach to stratified spaces endowed with a smooth structure we refer to the monograph of M. Pflaum [**105**].

We end the chapter with a study of a class of manifolds satisfying pseudoconvexity conditions in the sense of Andreotti-Grauert, namely q-convex and weakly 1-complete manifolds. Manifolds satisfying complex convexity conditions are very important in complex geometry and analysis. The definition of pseudoconvexity or pseudoconcavity postulates the existence of an exhaustion function whose complex hessian has certain positive or negative eigenvalues so that, morally, the situation is similar to the usual Morse theory. Technically, we use the representation of the Dolbeault cohomology by harmonic forms satisfying $\overline{\partial}$–Neumann conditions and reduce to the problem studied before of the distribution of small eigenvalues of the Bochner-Laplacian.

In ***Chapter 5*** we deal also with Morse inequalities on coverings in the framework of Atiyah. Here the usual dimension is replaced by the von Neumann dimension. Our main technical device comes from Shubin's generalization [**111**] of the usual Morse inequalities, the so–called Novikov–Shubin inequalities. Actually, we generalize the Weyl type formula of Demailly by describing the asymptotic behaviour of the spectrum of a Γ–invariant laplacian associated to high powers of a Γ–invariant line bundle. As a consequence we have the following.

1.11. THEOREM ([**5**]). *Let (X,ω) be an n–dimensional complete hermitian manifold and let (L,h^L) be a holomorphic hermitian line bundle. Let $K \Subset M$ and a constant $C_0 > 0$ such that $\sqrt{-1}R^L \geqslant C_0\,\omega$ on $X \smallsetminus K$. Let $\pi_\Gamma : \widetilde{X} \to X$ be a Galois covering of Galois group Γ, $\widetilde{L} = \pi_\Gamma^*(L)$ and let U be any open subset with smooth boundary such that $K \Subset U \Subset X$. Then, for $k \longrightarrow \infty$,*

$$\dim_\Gamma H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k) \geqslant \frac{k^n}{n!}\int_{U(\leqslant 1,h^L)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n + o(k^n)\,, \tag{1.17}$$

where $H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)$ is the space of $(n,0)$–forms with values in $\widetilde{L}^k$ which are L^2 with respect to any metric on $\widetilde{X}$ and the metric $\pi_\Gamma^*(h^L)$ on $\widetilde{L}$.

Our result pertains also to the work of Gromov, Henkin and Shubin [**75**] in which the authors compute the von Neumann dimension of the space of L^2 holomorphic functions on coverings of strictly pseudoconvex domains. Their work was generalized for weakly pseudoconvex manifolds in [**5**] by using a variant for incomplete metrics of Theorem 1.11. The case of the trivial covering was treated in [**1, 120**].

A nice application of theorem 1.11 are weak Lefschetz theorems à la Nori by extending the method of Napier–Ramachandran.

1.12. THEOREM ([**6**]). *Let (X,ω) be an n–dimensional complete hermitian manifold and let (L,h^L) be a holomorphic hermitian line bundle such that $\sqrt{-1}R^L > C\omega$, $C>0$, outside a compact set. Let $Y \longrightarrow X$ be a holomorphic immersion with ample normal bundle and assume that the image G of $\pi_1(Y)$ in $\pi_1(X)$ is a normal subgroup. Then G has finite index in $\pi_1(X)$.*

As a consequence we derive weak Lefschetz theorems for Zariski open sets in Moishezon manifolds (cf. Corollary 5.19).

Chapter 6 discusses the problem of compactifying complex manifolds. Satake [**107**], Baily [**19**] and Baily–Borel [**20**] endowed the quotients of bounded symmetric domains with a complex structure making them into Zariski–open sets of a projective algebraic variety, called the Satake–Baily–Borel compactification. Their methods were algebraic. We will be interested in the complex analytic and differential–geometric methods of compactifying arithmetic quotients. Andreotti and Grauert [**13**] verified that certain arithmetic quotients of the Siegel upper half plane are pseudoconcave and it was proved later that all irreducible arithmetic quotients of dimension $\geqslant 2$ are pseudoconcave (Spilker [**119**], Borel [**37**]). This is important since one hopes to apply a Kodaira type theorem for pseudoconcave manifolds (such as Andreotti–Tomassini theorem [**14**]) and embed them as open sets of projective manifolds and prove eventually quasi–projectivity.

In the line of thought of Andreotti–Grauert we introduce a type of pseudoconcavity which models arithmetic quotients of rank one and more generally spaces with isolated singularities. A manifold X is called a hyperconcave end if there exists $\varphi : X \longrightarrow \mathbb{R}$, proper, smooth which is strictly plurisubharmonic on a set of the form $\{\varphi < a\}$, $a \in \mathbb{R}$. The following four results have been proved in [**7**].

1.13. THEOREM. *Any hyperconcave end X can be compactified, i.e., there exist a complex manifold $\widehat{X}$ such that X is (biholomorphic to) an open set in $\widehat{X}$ and $\widehat{X} \smallsetminus X \cup \{\varphi \leqslant d\}$ is compact for any $d < a$. More specifically, if φ is strictly plurisubharmonic on the whole X, $\widehat{X}$ can be chosen a Stein space with at worst isolated singularities.*

One of the main points of the proof is to produce non-constant holomorphic on X via the finiteness of the L^2-cohomology in bidegree $(0,1)$ with respect to a complete Kähler metric satisfying the Donnelly-Fefferman condition. One application is the embedding of sasakian manifolds in the euclidian space (cf. Theorem 6.37 and [**8**]). Theorem 1.13 was independently used by Ornea-Verbitsky [**103**] to prove the embeddability into a sphere.

From the point of view of application it is desirable to find natural conditions for X to be a Zariski open set in a compact manifold.

1.14. THEOREM. *Let X be a hyperconcave end and let $\widehat{X}$ be a smooth completion of X. Assume that X can be covered by Zariski-open sets which are uniformized by Stein manifolds. Then $\widehat{X} \smallsetminus X$ is the union of a finite set D' and an exceptional analytic set which can be blown down to a finite set D. Each connected component of X_c, for sufficiently small c, can be analytically compactified by one point from $D' \cup D$. If X itself has a Stein cover, $D' = \varnothing$ and D consists of the singular set of the Remmert reduction of $\widehat{X}$.*

As a consequence we have the following characterization which answers [**94**, Problem 1] for the case $q=0$.

1.15. THEOREM. *Let X be a connected manifold of dimension $n\geqslant 2$. The following conditions are necessary and sufficient for X to be a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many points.*

(i) *X is hyper 1-concave.*
(ii) *X admits a positive line bundle E.*
(iii) *X can be covered by Zariski-open sets which can be uniformized by Stein manifolds.*

Our motivation was to give a complex analytic proof of Siu–Yau's theorem, which generalizes the compactification of arithmetic quotients of rank one. The analysis of the Busemann function shows that such manifolds are hyper 1–concave, so we can deduce the following slightly sharper form from Theorem 1.15.

1.16. COROLLARY (Siu-Yau). *Let X be a complete Kähler manifold of finite volume and bounded negative sectional curvature. If $\dim X\geqslant 2$, X is biholomorphic to a quasiprojective manifold which can be compactified by adding finitely many points to a Moishezon space.*

We present further a version "with boundary" of the Siu–Yau theorem.

1.17. THEOREM ([**8**]). *Let X be a connected complex manifold with compact strongly pseudoconvex boundary and of complex dimension $n\geqslant 2$. Assume that X is endowed with a complete Kähler metric with pinched negative curvature.*
(i) *The following assertions are equivalent*

(1) *∂X is embeddable in some $\mathbb{C}^N$*
(2) *X has finite volume away from a neighbourhood of ∂X*

(ii) *Assume that one of the equivalent assertions in* (i) *holds true. Then X can be compactified to a strongly pseudoconvex domain in a projective variety by adding an exceptional analytic set, that is, there exists a compact strongly pseudoconvex domain D in a smooth projective variety and an embedding $h:X\longrightarrow D$ which is a biholomorphism between $\mathrm{Int}X$ and $h(\mathrm{Int}X)$, $h(\partial X)=\partial D$, and $D\smallsetminus h(X)$ is an exceptional analytic set which can be blown down to a finite set of singular points.*

The implication $(1)\Rightarrow(2)$ uses the holomorphic Morse inequalities as in Theorem 1.5 and is contained in Theorem 1.6. For the reverse implication we show that with the exception of the end corresponding to the strongly pseudoconvex boundary, all the ends are hyperconcave.

Nadel and Tsuji [**96**] generalized the compactification of arithmetic quotients of any rank, by showing that certain pseudoconcave manifolds are quasiprojective. We will discuss the proof in Section 6.5. Let us note that in dimension two their condition coincides with hyperconcavity. Theorem 1.15 yields, in dimension two, a stronger version of their theorem together with a completely complex-analytic proof of the compactification of arithmetic quotients, cf. Remark 6.48.

We consider then the problem of finding a Moishezon compactification for general q–concave manifolds. Even the following innocent looking particular case is very interesting. Consider a compact manifold Y and let $X=Y\smallsetminus B$ the complement of a ball in some coordinate patch. Let $L \longrightarrow Y$ be a line bundle which is positive on X. Is then X (and with it Y) Moishezon? What is the growth of $\dim H^0(X,L^k)$ as $k \to \infty$?

The study of such questions was initiated in [**2, 6**] where some particular cases linked to isolated singularities were settled. It was also conjectured that the Grauert–Riemenschneider–Siu criterion holds in the case of 1–concave manifolds. We could verify the conjecture if, roughly speaking, the volume of the manifold is more important than the volume of the boundary. This follows from the following estimate.

1.18. THEOREM ([**4**]). *Let $D \Subset X$ be a smooth domain in a complex manifold X such that the Levi form of ∂D possesses at least 2 negative eigenvalues. Let (L,h^L) be a holomorphic line bundle on X which is assumed to be positive on a neighbourhood of $\overline{D}$. Then*

$$\liminf_{k\longrightarrow\infty} k^{-n}\dim H^0(D,\mathscr{O}(L^k)) \geqslant \int_D \Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n - C(\varphi,L)\int_{\partial D}\frac{dS_L}{|d\varphi|_L} \tag{1.18}$$

The constant $C(\varphi,L)$ depends explicitely on the curvature R^L and on the Levi form $\sqrt{-1}\partial\overline{\partial}\varphi$.

The notations dS_L and $|d\varphi|_L$ mean the boundary volume form and the norm of $d\varphi$ induced by the metric $\sqrt{-1}R^L$.

The theorem was obtained first in collaboration with Prof. G. Henkin by using Siu's original method and the some result of Henkin-Leiterer on the solution of the $\overline{\partial}$-equation near the boundary points using integral representations [**79**]. R. Berman [**22**] proved related results by identifying more precisely the boundary integral. We apply Theorem 1.18 to the deformation theory of concave manifolds. This is a very lively research area, see Epstein–Henkin [**66, 67, 68**] and the references therein. As a consequence we obtain the following stability result.

1.19. COROLLARY. *Let X be a projective manifold with positive canonical bundle $x_0 \in X$. There exists an $r_0 > 0$ such that for any $r < r_0$ and any sufficiently small perturbation of the complex structure of $X \smallsetminus B(x_0,r)$, the new manifold compactifies to a Moishezon manifold.*

In ***Chapter 7*** we take up the study of the Bergman kernel on symplectic manifolds. The Bergman kernel for complex projective manifolds is the smooth kernel of the orthogonal projection from the space of smooth sections of a positive line bundle L on the space of holomorphic sections of L, or, equivalently, on the kernel of the Kodaira-Laplacian $\square^L = \overline{\partial}^L\overline{\partial}^{L*} + \overline{\partial}^{L*}\overline{\partial}^L$ on L. It is studied in Tian [**121**], Ruan [**106**], Zelditch [**125**], Catlin [**49**], Bleher-Shiffman-Zelditch [**30**], Z. Lu [**91**] in various generalities, establishing the asymptotic expansion for high powers of L. Moreover, the coefficients in the asymptotic expansion encode geometric information about the underlying complex projective manifolds.

Since on a symplectic manifold there are in general no holomorphic sections, we have to look for a replacement of the $\overline{\partial}$ operator. One option is the spin^c Dirac operator. In [**50**], Dai, Liu and Ma studied the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on a compact symplectic manifold was studied, in relation to that of the corresponding heat kernel. As a by product, a new proof of the above results is obtained. The approach is inspired by Local Index Theory, especially by Bismut-Lebeau [**28**, §11].

We wish to propose another natural generalization of the operator $\square^L$ in symplectic geometry, which was initiated by Guillemin and Uribe [**76**]. In this very interesting short paper, they introduce a renormalized Bochner–Laplacian (cf. (1.19)) which is exactly $2\square^L$ in the Kähler case. The asymptotic of the spectrum of the renormalized Bochner–Laplacian on L^k when $k\to\infty$ is studied in various generalities in [**38, 45, 76**] by applying the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [**43**], and in [**9**] as a direct application of Lichnerowicz formula.

Of course, there exists also a replacement of the $\overline{\partial}$–operator and of the notion of holomorphic section based on a construction of Boutet de Monvel–Guillemin [**43**] of a first order pseudodifferential operator D_b which mimic the $\overline{\partial}_b$ operator on the circle bundle associated to L. However, D_b is neither canonically defined nor unique. This point of view was adopted in a series of papers [**39, 110, 32**]. Bleher, Shiffman and Zelditch used it in order to study the probabilistic behaviour of sequences of 'almost–holomorphic' sections of L^k.

Here we will study the asymptotic expansion of the generalized Bergman kernel of the renormalized Bochner-Laplacian, namely the smooth kernel of the projection on its bound states as $k\to\infty$. The advantage of this approach is that the renormalized Bochner-Laplacian has geometric meaning and is canonically defined. Moreover, it does not require the passage to the associated circle bundle as we can work directly on the base manifold. Let's explain our results in detail.

Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$, where $R^L=(\nabla^L)^2$ is the curvature of (L,∇^L). Let (E,h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and curvature R^E. Introduce a Riemannian metric g^{TX} on X with Levi-Civita connection ∇^{TX}, curvature R^{TX} and scalar curvature r^X. If dv_X denotes the Riemannian volume form of (TX,g^{TX}), the scalar product on $\mathscr{C}^\infty(X,L^k\otimes E)$, the space of smooth sections of $L^k\otimes E$, is given by $(s_1,s_2)=\int_X\langle s_1(x),s_2(x)\rangle_{L^p\otimes E}\,dv_X(x)$.

We introduce now the Bochner-Laplacian, defined by $\Delta^{L^k\otimes E}=(\nabla^{L^k\otimes E})^*\nabla^{L^k\otimes E}$, where $\nabla^{L^k\otimes E}$ is the induced connection from ∇^L and ∇^E. Let $\mathbf{J}:TX\longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation $\omega(u,v)=g^{TX}(\mathbf{J}u,v)$, for $u,v\in TX$. There exists an almost complex structure J which is (separately) compatible with g^{TX} and ω, especially, $\omega(\cdot,J\cdot)$ defines a metric on TX. Moreover J commutes also with $\mathbf{J}$. We fix a smooth hermitian section Φ of $\mathrm{End}(E)$ on X Set $\tau(x)=-\pi\operatorname{Tr}_{|TX}[J\mathbf{J}]$, and let

$$\Delta_{k,\Phi}=\Delta^{L^k\otimes E}-k\tau+\Phi \tag{1.19}$$

be the renormalized Bochner-Laplace operator. In Corollary 7.2 we will prove:

1.20. THEOREM ([**9**, Cor. 1.2]). *There exist μ_0, $C_L>0$ independent of p such that the spectrum of $\Delta_{k,\Phi}$ satisfies* $\operatorname{Spec}\Delta_{k,\Phi}\subset[-C_L,C_L]\cup[2p\mu_0-C_L,+\infty[$. *For k large enough, the number d_k of eigenvalues on the interval $[-C_L,C_L]$ satisfies $d_k=\langle\operatorname{ch}(L^k\otimes E)\operatorname{Td}(TX),[X]\rangle$. In particular $d_k\sim k^n(\operatorname{rank}E)\operatorname{vol}_\omega(X)$.*

This means that $\Delta_{k,\Phi}$ has d_k bound states whose energies are bounded uniformly independent of k and the rest of the spectrum drifts to the right at linear rate, as $k\to\infty$. Therefore, we can use the space $\mathscr{H}_k$ as a replacement of the space of holomorphic functions for the symplectic manifolds. Let $P_{0,k}$ be the orthogonal projection from $(\mathscr{C}^\infty(X,L^k\otimes E),(\ ,\ ))$ onto the eigenspace of $\Delta_{k,\Phi}$ with the eigenvalues in $[-C_L,C_L]$. We define $P_{q,k}(x,x')$, $q\geqslant 0$ as the smooth kernels of the operators $P_{q,k}=(\Delta_{k,\Phi})^qP_{0,k}$ (we set $(\Delta_{k,\Phi})^0=1$) with respect to $dv_X(x')$. They are called the generalized Bergman kernels of the renormalized Bochner-Laplacian $\Delta_{k,\Phi}$. We denote by $B_{q,k}(x) = P_{q,k}(x,x)$. Let $\det \mathbf{J}$ be the determinant function of $\mathbf{J}_x \in \mathrm{End}(T_xX)$.

1.21. THEOREM ([**10, 11**]). *There exist smooth coefficients $b_{q,r}(x) \in \mathrm{End}(E)_x$ which are polynomials in R^{TX}, R^E (and R^L, Φ) and their derivatives of order $\leqslant 2(r+q)-1$ (resp. $2(r+q)$) at x, and*

$$b_{0,0} = (\det \mathbf{J})^{1/2}\, \mathrm{Id}_E, \tag{1.20}$$

such that for any $p,l \in \mathbb{N}$, there exists $C_{p,l} > 0$ such that for any $x \in X$, $k \in \mathbb{N}$,

$$\Big|\frac{1}{k^n}B_{q,k}(x) - \sum_{r=0}^{p} b_{q,r}(x)k^{-r}\Big|_{\mathscr{C}^l} \leqslant C_{p,l}\, k^{-p-1}. \tag{1.21}$$

Moreover, the expansion is uniform in the following sense: for any fixed $k,l \in \mathbb{N}$, assume that the derivatives of g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ with order $\leqslant 2n+2k+2q+l+2$ run over a set bounded in the $\mathscr{C}^l$–norm taken with respect to the parameter $x \in X$ and, moreover, g^{TX} runs over a set bounded below. Then the constant $C_{k,l}$ is independent of g^{TX}; and the $\mathscr{C}^l$-norm in (1.21) *includes also the derivatives on the parameters.*

By derivatives with respect to the parameters we mean directional derivatives in the spaces of all appropriate g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ (on which $B_{q,p}$ and $b_{q,r}$ implicitly depend).

We calculate further the coefficients $b_{0,1}$ and $b_{q,0}$, $q \geqslant 1$ as follows[1].

1.22. THEOREM. *If $J = \mathbf{J}$, then for $q \geqslant 1$,*

$$b_{0,1} = \frac{1}{8\pi}\Big[r^X + \frac{1}{4}|\nabla^X J|^2 + 2\sqrt{-1}R^E(e_j, Je_j)\Big], \tag{1.22}$$

$$b_{q,0} = \Big(\frac{1}{24}|\nabla^X J|^2 + \frac{\sqrt{-1}}{2}R^E(e_j, Je_j) + \Phi\Big)^q. \tag{1.23}$$

The formulas are compatible with the Atiyah-Singer formula. Theorem 1.21 for $q = 0$ and (1.22) generalize the results of [**49, 125, 91**] and [**123**] to the symplectic case.

The term $r^X + \frac{1}{4}|\nabla^X J|^2$ in (1.22) is called the Hermitian scalar curvature in the literature [**71**, Chap. 10] and is a natural substitute for the Riemannian scalar curvature in the almost-Kähler case. It was used by Donaldson [**62**] to define the moment map on the space of compatible almost-complex structures.

We can view (1.23) as an extension and refinement of the results of [**40**], [**76**, §5] about the density of states function of $\Delta_{k,\Phi}$.

We apply the previous theorems to obtain a symplectic version of the convergence of the induced Fubini-Study metric, cf. Theorem 7.28. This generalizes the Theorems of Tian [**121**] and Bouche [**41**] and also gives a symplectic version of the Kodaira embedding theorem. Our method extends also to non-compact manifolds in Section 7.3.5. For example, by integrating the expansion of the Bergman kernel we can also derive Morse inequalities and reprove some of the results obtained in the previous chapters. We see therefore that the analysis of the Bergman metric yields a unified treatment of the convergence of the induced Fubini–Study metric, the holomorphic Morse inequalities and the characterization of Moishezon spaces.

[1] Here $|\nabla^X J|^2 = \sum_{ij} |(\nabla^X_{e_i} J)e_j|^2$ which is two times the corresponding $|\nabla^X J|^2$ from [**10**].

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CHAPTER 2

Witten's proof of the Morse Inequalities

In this chapter we present, as a warm up, a short analytic proof of the Morse inequalities, following Witten [**14**]. The ideas emerging here are ubiquitvous in our paper. Morse theory was developped by Martson Morse in the 30's and received a new impetus in topology in the 50's through the work of Bott, Milnor and Smale. The standard references for classical Morse theory are the books of Milnor [**10, 11**]. Witten introduced a new approach to Morse theory in 1982 (for a beautiful historical account, see [**3**]). The new approach consists in viewing the manifold together with a Morse function as a phisycal system of harmonic oscillators attached to the critical points. Along the way, Witten interprets from this point of wiew the Thom–Smale complex [**9**], a developement which inspired the introduction of the Floer homology [**6**]. The first rigorous account of Witten's ideas appeared in Helffer–Sjöstrand [**8**] where the authors make use of the theory of Schrödinger operators (see also [**5**]). Later, Bismut [**1**] gave a heat equation proof. Bismut and Zhang used the Witten deformation to give a proof of the equality of Redemeister and Ray–Singer torsions. In his book [**15**], Zhang presents a short proof based on [**2**]. Our proof differs from his in the use of the min-max principle in Section 2.2.2.

2.1. Witten deformation

2.1.1. Morse functions. Let X be an n–dimensional compact manifold. Let $f\in\mathscr{C}^\infty(X)$ be a smooth function on X. A point $p\in X$ is called a *critical point* of f if $df(p)=0$. The set of critical points of f is denoted by $\operatorname{Crit}(f)$. At a critical point p we can define symmetric bilinear form

$$d^2f(p):T_pX\times T_pX\longrightarrow T_pX$$

called the Hessian of f at p. Indeed, let $v,w\in T_pX$ and set $\tilde v,\tilde w$ arbitrary smooth extensions of v,w to smooth vector fields on X. Since

$$\tilde v(\tilde w f)(p)-\tilde w(\tilde v f)(p)=df([\tilde v,\tilde w])(p)=0$$

it follows $v(\tilde w f)(p)=\tilde v(\tilde w f)(p)=\tilde w(\tilde v f)(p)=w(\tilde v f)(p)$. The map

$$d^2f(p)(v,w):=v(\tilde w f)(p)=w(\tilde v f)(p)$$

depends only on v,w and it is obviously bilinear symmetric. In local coordinates $(x^1,\dots,x^n)$, $d^2f(p)$ is represented by the matrix $\left(\frac{\partial^2 f}{\partial x^j\partial x^k}(p)\right)$.

A critical point is called *nondegenerate* if the Hessian $d^2f(p)$ is non–singular, that is, the nullity vanishes. This condition means that, as submanifolds of TX, X (the zero cross–section) and $df(X)$ are transversal at p. From here follows that p is isolated in the set $\operatorname{Crit}(f)=X\cap df(X)$.

The function $f\in\mathscr{C}^\infty(X)$ is called a *Morse function* if all its critical points are nondegenerate. For a Morse function the set $\operatorname{Crit}(f)$ is therefore discrete, and since X is compact, $\operatorname{Crit}(f)$ is finite. It is well known that the Morse functions on X are dense in the space of all smooth functions on X.

From now on we fix a Morse function f on X. The index of the bilinear form $d^2 f(p)$, $p \in \mathrm{Crit}(p)$, is called the index of f at x, denoted $\mathrm{ind}_f(x)$. We denote $\mathrm{Crit}(f;l)$ the set of critical points with index l. The following lemma describes the local behavior of a Morse function near a critical point. For a proof we refer to [**10**, p. 6].

2.1. MORSE LEMMA. *For any critical point p of a Morse function f exists a coordinate system $(U_p, x^1, \dots, x^n)$ such that*

$$f(x^1,\dots,x^n) = f(p) - \frac{1}{2}(x^1)^2 - \dots - \frac{1}{2}(x^l)^2 + \frac{1}{2}(x^{l+1})^2 + \dots + \frac{1}{2}(x^n)^2$$

where l is the index of p.

Since we have a finite number of critical points we may assume that U_p, $p \in \mathrm{Crit}(f)$, are pairwise disjoint. Let m_l be the cardinal of $\mathrm{Crit}(f;l)$.

2.1.2. Betti numbers and Morse Inequalities. The Betti numbers of X are defined by $b_l = \dim H^l_{sing}(X,\mathbb{R})$, the dimension of the l–th singular cohomology of X. The Betti numbers can be calculated with the help of the de Rham complex $\big(\Omega^\bullet(X), d\big)$:

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\ d\ } \Omega^1(X) \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^n(X) \longrightarrow 0$$

The cohomology of this complex is called the de Rham cohomology of X, denoted $H^\bullet_{dR}(X,\mathbb{R})$. By the de Rham theorem [**13**] there exist a canonical isomorphism $H^\bullet_{sing}(X,\mathbb{R}) \cong H^\bullet_{dR}(X,\mathbb{R})$. Moreover, by a Mayer–Vietoris argument [**4**] , the de Rham cohomology is finite dimensional. Thus $b_l = \dim H^l_{dR}(X,\mathbb{R})$ is finite for $l = 0,1,\dots,n$.

Our purpose is to give an analytic proof of the following result known as the Morse inequalities.

2.2. THEOREM. *Let X be a compact differentiable manifold and let $f : X \longrightarrow \mathbb{R}$ be a Morse function. Then the following relations hold for all $l = 0,1,\dots,n$:*

(i) *Weak Morse inequalities: $b_l \leqslant m_l$,*

(ii) *Strong Morse inequalities:*

$$\sum_{j=0}^{l} (-1)^{l-j} b_j \leqslant \sum_{j=0}^{l} (-1)^{l-j} m_j,$$

with equality for $l = n$, that is,

(iii) $\chi(X) = \sum_{j=0}^{n} (-1)^j m_j$.

We refer to [**10**] for a topological proof of this result.

2.1.3. Witten's complex. Given the Morse function f, Witten deformed the exterior derivative by conjugating with e^{tf}, $t \geqslant 0$:

$$d_t = e^{-tf}\, d\, e^{tf} \tag{2.1}$$

Since $d^2 = 0$, $d_t^2 = 0$ and we obtain the deformed de Rahm complex $\big(\Omega^\bullet(X), d_t\big)$

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\ d_t\ } \Omega^1(X) \xrightarrow{\ d_t\ } \cdots \xrightarrow{\ d_t\ } \Omega^n(X) \longrightarrow 0 \tag{2.2}$$

Set $H^\bullet_{t,dR}(X,\mathbb{R})$ for the cohomology of this complex. It is immediate that all the deformed complexes are isomorphic and hence have the same cohomology.

2.3. THEOREM. *The multiplication with $e^{-tf} : \big(\Omega^\bullet(X), d\big) \longrightarrow \big(\Omega^\bullet(X), d_t\big)$ induces an isomorphism in cohomology $H^\bullet_{dR}(X) \cong H^\bullet_{t,dR}(X)$. The inverse isomorphism is induced by $e^{tf} : \big(\Omega^\bullet(X), d_t\big) \longrightarrow \big(\Omega^\bullet(X), d\big)$.*

2.1.4. Some operators in riemannian geometry. Let g^{TX} be a riemannian metric on TX. Later we shall make a particular choice of g^{TX}, but for the moment we keep g^{TX} arbitrary.

g^{TX} induces riemannian metrics on T^*X and $\Lambda^\bullet T^*X$ in a canonical way. The scalar product of two forms $\alpha,\beta \in \Lambda^\bullet T_xX$ is denoted $\langle\alpha,\beta\rangle$. In order to define a global scalar product we introduce the canonical measure $v_X : \mathscr{C}_0(X) \longrightarrow \mathbb{R}$ so that for a chart $(U,x^1,\dots,x^n)$

$$\int \varphi\, dv_X = \int_{\mathbb{R}^n} \varphi(x^1,\dots,x^n)\sqrt{\det(g_{ij})}\, dx^1\dots dx^n$$

for all $\varphi \in \mathscr{C}^\infty(X)$, $\operatorname{supp}\varphi \subset U$. If X is orientable, v_X is induced by the volume form of the metric. We have then a global scalar product

$$(\alpha,\beta) = \int_X \langle\alpha,\beta\rangle\, dv_X\,, \quad \alpha,\beta \in \Omega^\bullet(X). \tag{2.3}$$

Let us now introduce some useful operators in riemannian geometry.

The metric g^{TX} induces bundle isomorphism $\flat : TX \longrightarrow T^*X$ (flat) and its inverse $\sharp : T^*X \longrightarrow TX$ (sharp). Given a covector $\xi \in T^*_xX$ we associate the exterior product with ξ, $\xi\wedge : \Lambda^\bullet T^*_xX \longrightarrow \Lambda^\bullet T^*_xX$. Given a vector $e \in T_xX$ we associate the interior product with e, $i_e : \Lambda^\bullet T^*_xX \longrightarrow \Lambda^\bullet T^*_xX$. The adjoint of $\xi\wedge$ with respect to the induced scalar product on $\Lambda^\bullet T_xX$ is given by $i_{\xi^\sharp}$, that is, $(\xi\wedge)^* = i_{\xi^\sharp}$, or explicitely

$$\langle\xi\wedge\alpha,\beta\rangle = \langle\alpha, i_{\xi^\sharp(\beta)}\rangle, \quad \alpha \in \Lambda^l T^*_xX, \beta \in \Lambda^{l+1}T^*_xX.$$

We denote by $\nabla^{\Lambda^\bullet T^*X}$ the Levi–Civita connection on X and the induced connection on $\Lambda^\bullet T^*X$. Let $(e_1,\dots,e_n)$ be a local frame of TX and $(e^1,\dots,e^n)$ the dual frame. The exterior differential may be expressed as

$$d = \sum_{j=1}^n e^j \wedge \nabla^{\Lambda^\bullet T^*X}_{e_j} \tag{2.4}$$

The formal adjoint of d is a differential operator of degree -1 which satisfies

$$(d\alpha,\beta) = (\alpha,\delta\beta), \quad \alpha \in \Omega^l(X), \beta \in \Omega^{l+1}(X). \tag{2.5}$$

It is given explicitely by

$$\delta = -\sum_{j=1}^n (e^j\wedge)^* \nabla^{\Lambda^\bullet T^*X}_{e_j} = -\sum_{j=1}^n i_{(e^j)^\sharp}\nabla^{\Lambda^\bullet T^*X}_{e_j}. \tag{2.6}$$

Assume for a moment that the frame $(e_1,\dots,e_n)$ is orthonormal. Then $(e^j)^\sharp = e_j$ and we have the formula $\delta = -\sum_{j=1}^n i_{e_j}\nabla_{e_j}$. Witten notes that the operators $a^j = i_{e_j}$ and $a^{j*} = e^j$ are called in the physics literature 'fermion annihilation and creation operators'.

The Laplace–Beltrami operator is a second-order, degree 0 differential operator

$$\Delta : \Omega^\bullet(X) \longrightarrow \Omega^\bullet(X), \quad \Delta = d\delta + \delta d\,. \tag{2.7}$$

2.1.5. Witten's laplacian. We now define the Witten Laplacian. It is easily seen that the formal adjoint of d_t satisfies $\delta_t := d_t^* = e^{tf}\delta e^{-tf}$. The Witten Laplace operator is then

$$\Delta_t := d_t\delta_t + \delta_t d_t\,. \tag{2.8}$$

Since for $\alpha,\beta \in \Omega^l(X)$, $\quad (\Delta_t\alpha,\beta) = (d_t\alpha,d_t\beta) + (\delta_t\alpha,\delta_t\beta)$ we see that $(\Delta_t\alpha,\beta) = (\alpha,\Delta_t\beta)$, i. e. Δ_t is symmetric. Moreover, $(\Delta_t\alpha,\alpha) \geqslant 0$, $\quad \alpha \in \Omega^l(X)$, that is, Δ_t is positive.

2.4. PROPOSITION. *Δ_t is an elliptic operator. Actually, it has the same symbol as Δ.*

PROOF. It is clear that

$$d_t = d + t\, df\wedge, \tag{2.9}$$

so d_t is the sum of d and a 0-order differential operator. Thus d_t and d have the same symbol. For a function $\varphi \in \mathscr{C}^\infty(X)$, we define the gradient by $\operatorname{grad}\varphi := (d\varphi)^\sharp$.

It is then easily checked, that

$$\delta(\varphi\alpha) = \varphi\delta\alpha - i_{\operatorname{grad}\varphi}(\alpha), \quad \alpha \in \Omega^\bullet(X), \varphi \in \mathscr{C}^\infty(X). \tag{2.10}$$

This yields

$$\delta_t = \delta + t\, i_{\operatorname{grad} f}, \tag{2.11}$$

so δ_t is the sum of δ and a 0-order differential operator and δ_t has the same symbol as δ. By the formal calculation rules of symbols, Δ_t has the same symbol as Δ and is elliptic. □

2.1.6. Hodge Theory for the deformed complex. We can apply now the standard elliptic theory to Δ_t. For a vector bundle $E \longrightarrow X$, endowed with a riemannian metric, we can introduce a global scalar product as in (2.3). Let $\boldsymbol{L}^2(X,E)$ be the completion of the space of smooth sections $\Omega(X,E)$ under this scalar product. In the case of $E = \Lambda^l T^*X$ we denote $\boldsymbol{L}^2(X,\Lambda^l T^*X)$ by $\boldsymbol{L}^l_2(X)$. Since Δ_t is elliptic, by Theorems A.8 and A.33, it is an essentialy selfadjoint operator with compact resolvent, which we still denote by Δ_t. The spectrum of the selfadjoint extension consists of a sequence $0 \leqslant \lambda_0(t) \leqslant \lambda_1(t) \leqslant \ldots \longrightarrow +\infty$ of eigenvalues of finite multiplicity. The eigenspace $\ker(\Delta_t - \mu\operatorname{Id})$ has smooth elements and we denote by $\mathscr{E}^l(\lambda,\Delta_t) = \oplus_{\mu\leqslant\lambda}\ker(\Delta_t - \mu\operatorname{Id})$ the sum of eigenspaces corresponding to the eigenvalues less or equal to λ. We have

$$N^l(\lambda,\Delta_t) := \dim \mathscr{E}^l_t(\lambda,\Delta_t) = \#\{j : \lambda_j \leqslant \lambda\},$$

the number of eigenvalues less or equal than λ. $N^l(\lambda,\Delta_t)$ is called the spectral counting function.

We introduce a subcomplex of the Witten deformed complex. First observe that $d_t\Delta_t = d_t\delta_t d_t = \Delta_t d_t$ so

$$d_t \ker(\Delta_t - \mu\operatorname{Id}) \subset \ker(\Delta_t - \mu\operatorname{Id}) \tag{2.12}$$

Let $E(\lambda,\Delta_t) : \boldsymbol{L}^\bullet_2(X) \longrightarrow \mathscr{E}^\bullet(\lambda,\Delta_t)$ be the orthogonal projections. By (2.12),

$$d_t\mathscr{E}^l(\lambda,\Delta_t) \subset \mathscr{E}^{l+1}(\lambda,\Delta_t)$$

$$P_t(\lambda)d_t = d_t P_t(\lambda)$$

2.5. PROPOSITION. *$\big(\mathscr{E}^\bullet(\lambda,\Delta_t), d_t\big)$ is a subcomplex of $\big(\mathscr{E}^\bullet, d_t\big)$ having the same cohomology.*

PROOF. By the definition (A.45) of the Green operator G_t of Δ_t, and by (A.46)

$$\operatorname{Id} - P_t(\lambda) = \big(\Delta_t G + P_t(0)\big)\big(\operatorname{Id} - P_t(\lambda)\big) = d_t\big[\delta_t G\big(\operatorname{Id} - P_t(\lambda)\big)\big] + \big[\delta_t G_t\big(\operatorname{Id} - P_t(\lambda)\big)\big]d_t$$

so $\delta_t G_t\big(\operatorname{Id} - P_t(\lambda)\big)$ is a homotopy operator between Id and $P_t(\lambda)$. □

By Theorem 2.3 we get

$$H^\bullet\big(\mathscr{E}^\bullet(\lambda,\Delta_t), d_t\big) \cong H^\bullet_{dR,t}(X) \cong H^\bullet_{dR}(X)$$

2.1.7. A Bochner–type formula for Δ_t. We deduce a formula for Δ_t which shows the role of the Morse function. Δ_t can be written as a Schrödinger operator with potential $t^2|df|^2$, which outside $\mathrm{Crit}(f)$ is very large, for big t. This will permit to localize the problem to a neighbourhood of $\mathrm{Crit}(f)$. By formulas (2.9), (2.11) we get for $\alpha \in \mathscr{E}(X)$,

$$\Delta_t\alpha = (d_t\delta_t + d_t\delta_t)\alpha = \Delta\alpha + t^2\big(df\wedge i_{df}\alpha + i_{df}(df\wedge\alpha)\big) + t\big(d(i_{df}\alpha) + df\wedge\delta\alpha + \delta(df\wedge\alpha) + i_{df}(d\alpha)\big)$$

The coefficient of t^2 is easily calculated by the well known formula

$$\xi\wedge i_\xi\alpha + i_\xi(\xi\wedge\alpha) = |\xi|^2\alpha,\ \alpha\in\Lambda^\bullet T^*X.$$

To calculate the coefficient of t, we denote

$$A(\alpha) = d(i_{df}\alpha) + df\wedge\delta\alpha + \delta(df\wedge\alpha) + i_{df}(d\alpha),\ \alpha\in\Lambda^\bullet T^*X.$$

Upon using (2.10) and the fact that d and i_ξ are derivations on $\mathscr{E}^\bullet(X)$, we see that $A(\varphi\alpha) = \varphi A(\alpha)$, for $\alpha\in\mathscr{E}^\bullet(X)$, $\varphi\in\mathscr{C}^\infty(X)$. Thus A is defined by a bundle map, denoted still $A:\Lambda^\bullet T^*X \longrightarrow \Lambda^\bullet T^*X$. To compute A it is sufficient to calculate $A(dx^1\wedge\ldots\wedge dx^p)$ where $(x^1,\ldots,x^n)$ is an arbitrary coordinate system.

We fix a point $x_0\in X$ and consider normal geodesic coordinates around x_0. Then

$$g_{ij}(x_0) = \delta_{ij},\quad dg^{ij}(x_0) = 0,$$

$$\Gamma^l_{jk}(x_0) = 0, \tag{2.13}$$

$$\nabla_{\frac{\partial}{\partial x^j}}(dx^1\wedge\ldots\wedge dx^p)(x_0) = 0, \tag{2.14}$$

$$\delta(dx^1\wedge\ldots\wedge dx^p)(x_0) = 0 \tag{2.15}$$

(see (2.6) and (2.14)). We get then

$$A(dx^1\wedge\ldots\wedge dx^p) = \sum_{k,l}\frac{\partial^2 f}{\partial x^l\partial x^k}(dx^l\wedge i_{dx^l} - i_{dx^k}dx^p\wedge)(dx^1\wedge\ldots\wedge dx^p) + \\ + \sum_k\frac{\partial f}{\partial x^k}d\big(i_{dx^k}(dx^1\wedge\ldots\wedge dx^p)\big) + \sum_k\frac{\partial f}{\partial x^k}\delta(dx^k\wedge\ldots\wedge x^p).$$

At x_0, the second term vanishes since x_0 is a critical point and the third term vanish by (2.15). Moreover, the components of the Hessian form $\mathrm{Hess}_f = \nabla^{T^*X}df \in \mathscr{E}(T^*X\otimes T^*X)$ satisfy

$$\begin{aligned}\mathrm{Hess}_f\Big(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x_k}\Big) &:= \frac{\partial^2 f}{\partial x^j\partial x^k} - \frac{\partial f}{\partial x^l}\Gamma^l_{jk}, &&\text{by definition}\\ &= \frac{\partial^2 f}{\partial x^j\partial x^k}, &&\text{by (2.13).}\end{aligned}$$

We infer

$$A = \sum_{k,l}\mathrm{Hess}_f\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)[dx^l\wedge, i_{dx^k}]\quad\text{at}\quad x_0.$$

Since this formula is coordinate–invariant, it holds in all coordinate systems. All in all, we have proved the following Bochner formula:

$$\Delta_t = \Delta + t^2|df|^2 + t\sum_{l,k}\mathrm{Hess}_f\big(\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^k}\big)[dx^l\wedge, i_{dx^k}] \tag{2.16}$$

The formula shows that Δ_t a Schrödinger operator whose dominant part of the potential is $t^2|df|^2$. When $t\longrightarrow+\infty$ the potential is huge on the set $\{df\neq 0\}$. This is why the L^2–norm of the eigenforms will concentrate asymptotically in a neighbourhood of $\mathrm{Crit}\,(f)$.

We study in the sequel what happens to Δ_t in the neighbourhood of $\mathrm{Crit}\,(f)$.

2.1.8. The model operator. Let $p\in\mathrm{Crit}\,(f;r)$ i.e. of index r. Let $(U_p,x^1\dots x^n)$ be a neighbourhood of p as in section 2.2. Therefore $|df|^2=|x|^2$ on U_p. If $\alpha\in\mathscr{C}_0^\infty(U_p)$, $|J|=q$ (2.16) delivers

$$\Delta_t(\alpha dx^J)=\sum_{i=1}^n\left[-\big(\frac{\partial}{\partial x^j}\big)^2\alpha+t|x^j|^2\alpha\right](dx^J)+\alpha\sum_{j=1}^n\varepsilon_j\left[dx^j\wedge,i_{dx^j}\right](dx^J)$$

where $\varepsilon_j=-1$ for $j\leqslant r$, $\varepsilon_j=+1$ for $j\geqslant r+1$. We consider then the following model operator on $\boldsymbol{L}_2^\bullet(\mathbb{R}^n)$:

$$\Delta'_{t,r}=\sum_{j=1}^n H_j+\sum_{j=1}^n\varepsilon_j K_j$$

where $H_j=-\big(\frac{\partial}{\partial x_j}\big)^2+t^2|x^j|^2$ acts componentwise, and $K_j=\left[dx^j\wedge,i_{dx^j}\right]$ is a bundle morphism $\Lambda^\bullet T^*\mathbb{R}^n\longrightarrow\Lambda^\bullet T^*\mathbb{R}^n$.

We want to compute the spectrum of $\Delta'_{t,l}$. By [**12**, Vol. 1,p. 142], [**7**, p. 12] we know that the spectrum of the harmonic oscillator $-\big(\frac{\partial}{\partial y}\big)^2+y^2$ on $\boldsymbol{L}^2(\mathbb{R})$ consist of the eigenvalues with multiplicity one $\{2N+1: N=0,1,2,\dots\}$ with corresponding eigenfunctions

$$\Phi_N(y)=(2^N\cdot N!)^{-1/2}(-1)^N\pi^{-1/4}e^{y^2/2}\big(\tfrac{\partial}{\partial y}\big)^N e^{-y^2}. \tag{2.17}$$

We infer that the spectrum of $-\big(\frac{\partial}{\partial x^k}\big)^2+t|x^k|^2$ on $\boldsymbol{L}_2^q(\mathbb{R}^n)$ consists of the eigenvalues $\{t(2N+1): N=0,1,2,\dots\}$ with multiplicity $\binom{n}{q}$. The corresponding eigenforms are $\Phi_N(\sqrt{t}y)dy^J$, $|J|=q$.

On the other hand $K_j(dx^J)=\varepsilon_j^J dx^J$, where $\varepsilon_j^J=1$ if $j\in J$, $\quad\varepsilon_j^J=-1$ if $j\notin J$. Since the operators H_k an H_j commute, we find that $\boldsymbol{L}_2^q(\mathbb{R}^n)$ has the following ONB of eigenforms of $\Delta'_{t,r}$:

$$\left\{\Phi_{N_1}(\sqrt{t}x^1)\cdot\ldots\cdot\Phi_{N_n}(\sqrt{t}x^n)dx^J:\quad N_1,\dots,N_n\in\mathbb{N}\cup\{0\},\,|J|=q\right\}.$$

The corresponding eigenvalues are

$$\left\{t\sum_{j=1}^n(2N_j+1+\varepsilon_j\varepsilon_j^J):\quad N_1,\dots,N_n\in\mathbb{N}\cup\{0\},|J|=q\right\} \tag{2.18}$$

2.6. THEOREM. *The spectrum of $\Delta'_{t,r}$ on $\boldsymbol{L}_2^q(\mathbb{R}^n)$ is* (2.18). *Moreover*

$$\mathrm{Ker}\left(\Delta'_{t,r}|_{L_2^q(\mathbb{R}^n)}\right)=\begin{cases}0 & ,\ if\ r\neq q\\ \mathbb{R}\,e^{-\frac{t|x|^2}{2}}dx^1\wedge\ldots\wedge dx^l & ,\ r=q\end{cases} \tag{2.19}$$

All other eigenvalues are $O(t)$, $t\longrightarrow\infty$.

PROOF. An eigenvalue $t\sum_{j=1}^{N}(2N_j+1+\varepsilon_j\varepsilon_j^J)$ vanishes if and only if all parenthesis vanish (since they are all positive). This is the case if and only if $N_j=0$ for $\varepsilon_j\varepsilon_j^J=-1$ and all $j=1,\dots,n$. This means precisely $J=\{1,\dots,l\}$, after the definitions of ε_j and ε_j^J. The corresponding eigenvalue is

$$\Phi_0(\sqrt{t}x^1)\cdots\Phi_0(\sqrt{t}x^n)dx^1\wedge\ldots\wedge dx^l=e^{-\frac{t|x|^2}{2}}dx^1\wedge\ldots\wedge x^l.$$

□

2.2. Proof of the Morse Inequalities

2.2.1. Start of the proof. We need the following easy result of linear algebra.

2.7. LEMMA. *Let*

$$0\longrightarrow V^0\xrightarrow{d^0}V^1\xrightarrow{d^1}\cdots\longrightarrow V^n\longrightarrow 0$$

be a complex of finite–dimensional vector spaces, with $\dim V^l=m_l$, $\dim H^l(V^\bullet)=b_l$. *Then we have the following inequalities for* $0\leqslant l\leqslant n$

(i) *Weak Morse inequalities:* $b_l\leqslant m_l$,

(ii) *Strong Morse inequalities:*

$$\sum_{j=0}^{l}(-1)^{l-j}b_j\leqslant\sum_{j=0}^{l}(-1)^{l-j}m_j$$

(iii) *Equality of the Euler–Poincaré characteristics:*

$$\sum_{j=0}^{n}(-1)^jb_j=\sum_{j=0}^{n}(-1)^jm_j.$$

PROOF. Set $z_l=\dim\operatorname{Ker}d^l$, $r_l=\dim\operatorname{Im}d^{l-1}$. Then $m_l=z_l+r_l$ and $b_l=z_l-r_{l-1}$. Thus

$$\sum_{j=0}^{l}(-1)^{l-j}m_j=r_l+\sum_{j=0}^{l}(-1)^{q-j}b_j\,.$$

Since $r_{-1}=r_n=0$ and $r_j\geqslant 0$ for all j we obtain the inequalities. □

The Morse inequalities are a consequence of the following result of Witten, to be proven in the section 2.2.2.

We fix from now on a metric g^{TX} such that we have $g^{TX}=d(x^1)^2+\ldots+d(x^n)^2$, in the neighbourhood U_p with coordinates $(x^1,\dots,x^n)$ given by the Morse Lemma. Let p run in $\operatorname{Crit}(f)$ and let U_p be a neighbourhood of p as in 2.1.1.

2.8. THEOREM. *For* $\lambda>0$ *sufficiently large there exists* $t(\lambda)$ *such that for all* $t\geqslant t(\lambda)$,

$$N^l(\lambda,\Delta_t)=m_l.$$

PROOF OF THE MORSE INEQUALITIES. By applying the algebraic Lemma to the complex $\left(\mathscr{E}^\bullet(\lambda,\Delta_t),d_t\right)$ together with Theorem 2.8 we get Theorem 2.2. □

2.9. REMARK. The teorem (2.8) holds true without the hypothesis on the form of the form of the metric and of the function. In the non–euclidian case the estimate of the quadratic forms Q_t in 2.10 involves supplementary terms which are however $O(t)$ for $t\longrightarrow\infty$. The theorem in the present form suffices for the proof of the Morse inequalities.

2.2.2. The spectral gap of Δ_t. One of the key remarks of Witten [**14**, p. 666] is that the eigenforms of the modified laplacian Δ_t concentrate near the critical points of f as $t \longrightarrow \infty$. We shall use in the proof of Theorem 2.10 below, especially in §2.2.4.

In this section we prove Theorem 2.8. For this purpose we study the spectrum of Δ_t comparing it with the spectrum of Δ'_t by means of the min–max principle. The result is as follows:

2.10. THEOREM. *There exist constants $C_1, C_2 > 0$ such that*

$$\operatorname{Spec}(\Delta_t) \subset \left[0, e^{-C_1 t}\right] \cup \left[C_2 t, +\infty\right), \quad t \gg 1.$$

$\Delta_t|_{L_2^l(X)}$ *has exactly m_l eigenvalues (counted with multiplicity) in $[0, e^{-C_1 t}]$, that is, $N^l(e^{-C_1 t}, \Delta_t) = m_l$.*

It is obvious that Theorem 2.10 implies theorem 2.8. We prove Theorem 2.10 in two steps.

Step 1: We show that there at least m_l eigenvalues in $[0, e^{-C_1 t}]$.

Step 2: We show that the (m_l+1)–th eigenvalue is an $O(t), t \to \infty$. The proof shows that $\operatorname{Spec}(\Delta_t|_{L_2^l})$ approaches for large t the spectrum of the sum over all critical points of index l of model operators $\Delta'_{t,l}$.

Each critical point contributes with one ground state, in total m_l, the other states correspond to eigenvalues wich are $O(t), t \to \infty$.

2.2.3. Proof of Step 1. Let $\eta \in \mathscr{C}_0^\infty(\mathbb{R}), \operatorname{supp}\eta = [-2,2], \eta = 1$ on $[-1,1]$. Set $\eta_\varepsilon(t) = \eta(\varepsilon t)$, for $\varepsilon > 0$. We fix $l \in \{0,1,\ldots,n\}$. For the points of $\operatorname{Crit}(f;l)$ we consider pairwise disjoint coordinate neighbourhoods $(U_p, x^1, \ldots, x^n)$, $p \in \operatorname{Crit}(f;l)$, on which the metric g^{TX} is euclidian and f is a quadratic form (see section 2.2). We fix $\varepsilon > 0$ sufficiently small and we consider $\psi_{p,\varepsilon} : U_p \longrightarrow \mathbb{R}$, $\psi_{p,\varepsilon}(x) = \eta_\varepsilon(x^1)\cdot\ldots\cdot\eta_\varepsilon(x^n)$ where $(x^1,\ldots,x^n)$ are the coordinates of U_p.

Then $\operatorname{supp}\psi_{p,\varepsilon} = \{x \in U_p : |x^i| \leqslant 2\varepsilon, i = 1,\ldots,n\}$. We set

$$\omega_{p,t} = \begin{cases} \dfrac{1}{\sqrt{a_t^n}} e^{-t|x|^2}\psi_{p,\varepsilon}(x)dx^1\wedge\ldots\wedge dx^l & \text{on } U_p \\ 0 & \text{on } X \smallsetminus \operatorname{supp}\psi_{p,\varepsilon} \end{cases}$$

where $a_t = \int_{\mathbb{R}} e^{-ty^2}\eta_\varepsilon^2(y)dy$.

The forms $\omega_{p,t}$ are obtained by transplanting to X the ground states (2.19) of $\Delta'_{t,l}|_{L_2^l(\mathbb{R}^n)}$, by identifying U_p to an open set in $\mathbb{R}^n$. By definition $\|\omega_{p,t}\| = 1$. Since $\operatorname{supp}\omega_{p,t} \subset U_p$, the forms $\omega_{p,t}$ are linearly independent when p runs in $\operatorname{Crit}(f;l)$. We set

$$F_t^l = \oplus\{\mathbb{R}\omega_{p,t} : p \in \operatorname{Crit}(f;l)\}.$$

2.11. LEMMA. *There exists $C_1 > 0$ such that for large t,*

$$(\Delta_t\omega_{p,t}, \omega_{p,t}) \leqslant e^{-C_1 t} \quad , p \in \operatorname{Crit}(f;l).$$

PROOF. We compute

$$(\Delta_t\omega_{p,t}, \omega_{p,t}) = (\Delta'_t\omega_{p,t}, \omega_{p,t}) = \frac{n}{a_t}\int_{\mathbb{R}}[-\eta''_\varepsilon(y)\eta_\varepsilon(y) + 2ty\eta'_\varepsilon(y)\eta_\varepsilon(y)]e^{-ty^2}dy$$

Since the support of the function in brackets is $[\varepsilon, 2\varepsilon]$ we can estimate the integral by $Ce^{-t\varepsilon^2}$. Moreover $a_t \geqslant \frac{1}{2}(\frac{\pi}{t})^{1/2}$ for large t. It follows that

$$(\Delta_t\omega_{p,t}, \omega_{p,t}) \leqslant e^{-t\varepsilon^2/2} \quad , t \gg 1.$$

Let us denote by $\lambda_1(t) \leqslant \lambda_2(t) \leqslant \dots$ the spectrum of Δ_t on $L^l_2(X)$.

We define the quadratic form associated to Δ_t:

$$Q_t(u) = \|d_t u\|^2 + \|\delta_t u\|^2, \quad u \in W^l_1(X)\,.$$

By the min–max principle A.30 we obtain

$$\lambda_{m_l}(t) \leqslant \max_{\substack{u\in F\\ \|u\|=1}} Q_t(u) \leqslant e^{-C_1 t}$$

since $\dim F^l_t = m_l$. The proof of Step 1 is finished. □

2.2.4. Proof of Step 2. By the min–max principle it is sufficient to prove that there exists $C_2 > 0$ such that

$$Q_t(u) \geqslant C_2 t \|u\|^2 \quad , u \in W^l_1(X),\, u \perp F^l_t. \tag{2.20}$$

It follows from (2.20) that $\lambda_{m_l+1}(t) \geqslant C_2 t$ for large t, so this proves Step 2.

To prove (2.20) we use a localisation procedure. We construct first a cover $\mathscr{U}$ of X as follows. Let $U_0 = X \smallsetminus \cup\{\overline{U}_p : p \in \operatorname{Crit}(f)\}$ and set $\mathscr{U} = \{U_0\} \cup \{U_p : p \in \operatorname{Crit}(f)\}$. We consider a partition of unity $\{\varphi_U : U \in \mathscr{U}\}$ subordinated to $\mathscr{U}$, with

$$\sum \varphi_U^2 = 1, \quad \varphi_{U_p} = 1 \text{ on } \operatorname{supp} \omega_{p,t}\,. \tag{2.21}$$

Remember that an explicit formula for Q_t is

$$Q_t(u) = \int_X \big(|du|^2 + |\delta u|^2 + t^2 |df|^2 |u|^2 + t\langle Au, u\rangle\big) dv_X \tag{2.22}$$

where $A \in \operatorname{End}(\Lambda^l T^* X)$ is a symmetric operator determined in (2.16).

By (2.21) we obtain $\sum \varphi_U d\varphi_U = 0$ and

$$\sum |d(\varphi_U u)|^2 = |du|^2 + \sum |d\varphi_U \wedge u|^2$$
$$\sum |\delta(\varphi_U u)|^2 = |\delta u|^2 + \sum |i_{d\varphi_U} u|^2$$

Thus for $u \in W^l_1(X)$

$$\sum Q_t(\varphi_U u) = \sum \int_X \big(|d\varphi_U \wedge u|^2 + |i_{d\varphi_U} \alpha|^2\big) dV + Q_t(u)$$

so there exist a constant $C > 0$ with

$$Q_t(u) \geqslant \sum Q_t(\varphi_U u) - C\|u\|^2, \quad u \in W^l_1(X). \tag{2.23}$$

We examine Q_t for each $U \in \mathscr{U}$. For $U \in \mathscr{U}$ we consider the Sobolev space $W^l_{1,0}(U)$ which is the closure of $\{u \in \Omega^l(X) : \operatorname{supp} u \subset U\}$ in $W^l_1(X)$.

2.12. LEMMA. *There exist $C > 0$ such that for $t \gg 1$*

$$Q_t(u) \geqslant Ct\|u\|^2, \quad u \in W^l_{1,0}(U_0).$$

PROOF. This follows immediately from (2.22) since $|df| \geqslant c > 0$ on U_0 and A is bounded from below on X. □

2.13. LEMMA. *For any $p \in \operatorname{Crit}(f) \smallsetminus \operatorname{Crit}(f; l)$ there exists $C > 0$ such that*

$$Q_t(u) \geqslant Ct\|u\|^2, \quad u \in W^l_{1,0}(U_p).$$

PROOF. Let r be the index of p. Then $\Delta_t = \Delta_{t,r}$ on U_p. Since $r \neq l$ and $\Delta_{t,r}$ acts on l–forms, Theorem 2.1.8 entails the result. □

2.14. LEMMA. *For each $p \in \mathrm{Crit}(f;l)$ there exists $C>0$ such that for $t \gg 1$,*

$$Q_t(u) \geqslant Ct\|u\|^2, \quad u \in W^l_{1,0}(U_p), u \perp F^l_t.$$

PROOF. Let us identify U_p with an open set of $\mathbb{R}^n$ endowed with the euclidian metric. We denote by $\widetilde{u}$ the form on $\mathbb{R}^n$ which extend u with 0 outside U_p. Then $\|u\| = \|\widetilde{u}\|$. On U_p we have $\Delta_t = \Delta'_{t,l}$. If we denote by Q'_t the quadratic form of $\Delta'_{t,l}$ we have $Q'_t(\widetilde{u}) = Q_t(u)$. Relation $u \perp F^l_t$ means $u \perp \mathbb{R}\,\omega_{p,t}$. From this follows that $\widetilde{u}$ is asymptotically orthogonal to $\mathrm{Ker}\,\Delta'_{t,l} = \mathbb{R}\,e^{-t|x|^2/2}dx^1 \wedge \ldots \wedge dx^l$. Indeed

$$\begin{aligned}|(\widetilde{u}, e^{-t|x|^2/2}dx^1 \wedge \ldots \wedge dx^l)| &= |(u, e^{-t|x|^2/2}dx^1 \wedge \ldots \wedge dx^l)| \\ &= |(u,(1-\psi_{\varepsilon,p})e^{-t|x|^2/2}dx^1 \wedge \ldots \wedge dx^l)| \leqslant C\,e^{-t\varepsilon^2/4}\|u\| = C\,e^{-t\varepsilon^2/4}\|\widetilde{u}\| \quad (2.24)\end{aligned}$$

by the Cauchy–Schwarz inequality and the fact that $1-\psi_{\varepsilon,p}$ vanishes on $[-\varepsilon,\varepsilon]^n$. Let use the orthogonal decomposition $\widetilde{u} = \widetilde{u}_1 + \widetilde{u}_2$, $u_1 \in \mathrm{Ker}\,\Delta'_{t,l}$, $u_2 \perp \mathrm{Ker}\,\Delta'_{t,l}$. We have actually $u_1 = (u, e^{-t|x|^2/2}\,dx^1 \wedge \ldots \wedge dx^l)e^{-t|x|^2/2}\,dx^1 \wedge \ldots \wedge dx^n$. From (2.24) we learn that

$$\|\widetilde{u}_1\| \leqslant Ce^{-t\varepsilon^2/4}\|\widetilde{u}\|$$

From Theorem 2.1.8 we have moreover

$$(\Delta'_{t,l}\widetilde{u}_2, \widetilde{u}_2) \geqslant Ct\|\widetilde{u}_2\|^2$$

Therefore

$$Q_t(u) = Q'_t(\widetilde{u}) = (\Delta'_{t,l}\widetilde{u}, \widetilde{u}) = (\Delta'_{t,l}\widetilde{u}_2, \widetilde{u}_2) \geqslant C\,t(1-C\,e^{-t\varepsilon^2/4})\|\widetilde{u}\|^2 \geqslant C\,t\|\widetilde{u}\|^2 = C\,t\|u\|^2$$

for large t. □

It is now easy to prove (2.20). Let $u \in W^l_1(X)$, $u \perp F^l_t$. Then $(\eta_U u, \omega_{p,t}) = (u, \omega_{p,t}) = 0$, by (2.21). We can apply thus lemmata 2.12, 2.13 and 2.14 for $\eta_U u$, for all $U \in \mathscr{U}$. Together with (2.23) this implies (2.20).

This achieves the proof of Step 2 and of Theorem 2.10.

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CHAPTER 3

Local asymptotic Morse inequalities

At the foundation of many of the results of our paper lies a Weyl type formula of Demailly for the spectrum of the Kodaira-Laplacian $\frac{1}{k}\square^{F_k}$ acting on twisted tensor powers $F_k = L^k \otimes F$ of a line bundle L. The analysis of Demailly is based on the the fact that in the Bochner-Kodaira formula for $\frac{1}{k}\square^{F_k}$ on $L^k \otimes F$ the metric of L formally plays the role of the Morse function in the expresion of Witten's laplacian (1.7). J.-P. Demailly [**3**] used H. Weyl's original localization procedure, which consists in dividing the manifold in small pieces on which the spectrum of Dirichlet problem looks more and more as the spectrum of a harmonic oscillator. The Weyl-type asymptotics holds actually for a Schrödinger operator with magnetic field and is valid in the riemannian case.

Later, J.-M. Bismut [**1**] gave a heat equation proof based on his approach to the index theorem. The inequalities obtained are *pointwise* in the sense of index theory, namely, they yield through integration the global inequalities. We shall present here a related method due to T. Bouche [**4, 2**]. One can also apply a similar method to that employed in Chapter 7.

3.1. Asymptotic of the heat kernel

This section is organized as follows. In 3.1.1 we introduce the Bochner Laplacians and an associated Schrödinger operator $\Delta_{k,\tau}$ together with its heat kernels and the heat kernels acting on L^k and we state the asymptotics of the heat kernel as $k \longrightarrow \infty$.The proof starts in 3.1.2 by proving that the problem can be localized on balls of radius r_k, where $kr_k^2 \longrightarrow +\infty$, $kr_k^3 \longrightarrow 0$. In 3.6 we compute the heat kernel of the tangent $\Delta^0_{k,\tau}$ operator to $\Delta_{k,\tau}$ at a point with the help of Mehler's formula. Finally, we show in 3.7 that the heat kernel of Schrödinger operator is an infinite sum depending only on the tangent operator $\Delta^0_{k,\tau}$ on a ball of radius r_k. We can use then the explicit computation of 3.6 and the localization of 3.1.2 to achieve the proof. As a corollary we obtain in 3.10 the asymptotic distribution of the eigenvalues of the laplacians. We refine the result on the asymptotic in 3.1.6. This estimate will be used to determine the asymptotic of the heat kernel of the Kodaira Laplacian in the next section.

3.1.1. Statement of the result. Now, we introduce our notations: (X, g^{TX}) is a compact riemannian manifold of dimension m, with associated volume element dv_X. The manifold X may have a non-empty boundary ∂X. We denote $\overline{X} = X \cup \partial X$. Let (L, h^L),(F, h^F) be two hermitian complex vector bundles endowed with hermitian connections ∇^L and ∇^F respectively. We assume $\operatorname{rank} L = 1$ and $\operatorname{rank} F = r$. Let us fix an isometric local trivialisation. The connection ∇^L has then the form $\nabla^L = d + \Gamma^L \wedge$ where Γ^L is a 1–form. Since ∇^L is hermitian, $\Gamma^L = -\sqrt{-1}A^L$ with a real 1–form A^L. Let $R^L = (\nabla^L)^2$ be the curvature of (L, h^L). In the local trivialisation, $R^L = d\Gamma^L$. The curvature depends only on the connection and not on the trivialisation. We set $R^L = -\sqrt{-1}\omega$, where $\omega = \sqrt{-1}R^L$ is a 2-form on X, called the *magnetic field* of the connection. Fix a point $x_0 \in X$. There exists a coordinate system $(U,x^1,\dots,x^m)$ such that $dx^1,\dots,dx^m$ is an orthonormal frame of $T^*_{x_0}X$ and such that

$$\omega(x_0)=\sum_{j=1}^{s(x_0)}\omega_j(x_0)\,dx^j\wedge dx^{j+s} \tag{3.1}$$

where $2s=2s(x_0)\leqslant m$ is the rank of the skew-symmetric 2-form ω at x_0 and $\omega_1(x_0)\geqslant\ldots\geqslant\omega_s(x_0)>0=\omega_{s+1}(x_0)=\ldots=\omega_n(x_0)$ are its eigenvalues with respect to the metric $g^{T_{x_0}X}$. Along the fibers of $\Lambda^\bullet T^*X\otimes L^k\otimes F$, we consider the pointwise scalar product $\langle\cdot,\cdot\rangle$ induced by g^{TX}, h^L and h^F. The L^2–scalar product on $\Omega^\bullet(X,L^k\otimes F)$, the space of smooth sections of $\Lambda^\bullet T^*X\otimes L^k\otimes F$, is given by

$$(s_1,s_2)=\int_X\langle s_1(x),s_2(x)\rangle\,dv_X(x)\,. \tag{3.2}$$

We denote the corresponding norm with $\|\cdot\|$.

We will be primarily concerned with the spectral distribution of a Schrödinger operator constructed as follows. Let us consider a hermitian section $\tau\in\mathscr{E}(X,\operatorname{End}(F))$, identified to $\operatorname{Id}_{L^k}\otimes\tau\in\operatorname{End}(L^k\otimes F)$.

3.1. DEFINITION. Let $\nabla^{L^k\otimes F}$ be the connection on $L^k\otimes F$ induced by ∇^L and ∇^F. The *Bochner Laplacian* is the operator

$$\Delta^{L^k\otimes F}=\big(\nabla^{L^k\otimes F}\big)^*\nabla^{L^k\otimes F}+\nabla^{L^k\otimes F}\big(\nabla^{L^k\otimes F}\big)^* \tag{3.3}$$

on $\Omega^\bullet(X,L^k\otimes F)$. In the sequel we also use the notation $F_k=L^k\otimes F$. The Schrödinger operator associated to τ is

$$\Delta_{k,\tau}=\frac{1}{k}\Delta^{F_k}-\tau. \tag{3.4}$$

The heat operator is given by

$$P_k=\frac{\partial}{\partial t}-\Delta_{k,\tau}. \tag{3.5}$$

The selfadjoint extension of $\Delta_{k,\tau}$ which we consider is the operator $\Delta_{k,\tau}$ with Dirichlet boundary conditions (see A.2.3). In general, if U is an open set with smooth boundary we denote the operator $\Delta_{k,\tau}$ with Dirichlet boundary conditions by $\Delta_{k,\tau,U}$. But in case $U=X$ we omit the subscript and write simply $\Delta_{k,\tau}$ for this extension.

The quadratic form associated to $\Delta_{k,\tau}$ is

$$Q_{k,\tau}(u)=\int_X\big(\tfrac{1}{k}|\nabla^{L^k\otimes F}u|^2-\langle\tau u,u\rangle\big)dv_X\,,\quad u\in W_{1,0}(X,L^k\otimes F). \tag{3.6}$$

As $\Delta_{k,\tau}$ is elliptic and X is compact, $\Delta_{k,\tau}$ has a discrete spectrum which can be recovered from the heat kernel $e_k(t,x,y)$ which is the smooth kernel of the operator $\exp(-t\Delta_{k,\tau})$. If we want to indicate the open set where $\Delta_{k,\tau}$ acts, we denote the heat kernel of $\exp(-t\Delta_{k,\tau,U})$ by $e_{k,U}(t,x,y)$.

The heat kernel enjoys the following expansion: for $j=0,1,\cdots$, let λ_j^k be the eigenvalues of $\Delta_{k,\tau}$ (counted with multiplicities), and $(S_j^k)_j$ be an orthonormal L_2 basis of eigenforms associated to the eigenvalues λ_j^k, then

$$e_k(t,x,y)=\sum_{j\geqslant 0}\exp\big(-t\lambda_j^k\big)S_j^k(x)\otimes S_j^k(x)^*, \tag{3.7}$$

and is characterized by the following properties:

$$e_k \in \mathscr{C}^\infty\big((0,\infty)\times\overline{X}\times\overline{X}, \operatorname{End}\big(\Lambda^{0,q}T^*X\otimes L^k\otimes F\big)\big)\,, \tag{3.8a}$$

$$P_k e_k = 0 \text{ on } X, \text{ where } \Delta_{k,\tau} \text{ acts on the first variable}\,, \tag{3.8b}$$

$$e_k(t,x,y) \longrightarrow \delta_y \text{ (Dirac } \delta\text{-function at point } y) \text{ if } t \longrightarrow 0\,, \tag{3.8c}$$

$$e_k(t,x,y) = e_k^*(t,x,y)\,. \tag{3.8d}$$

$$e_k(t,\partial X,y) = \{0\}\,, \tag{3.8e}$$

Our aim is to prove the following result from [**2**]. Let $\omega(x)$ be the magnetic field at a point $x\in X$ written as in (3.1). We denote

$$e_\infty(t,x) = \frac{\exp(t\,\tau(x))}{(4\pi)^{m/2}\,t^{m/2-s(x)}}\prod_{j=1}^{s(x)}\frac{\omega_j(x)}{\sinh\omega_j(x)t} \in \operatorname{End}(E_x)\,. \tag{3.9}$$

3.2. THEOREM (Bouche). *There exists $\varepsilon\in(0,1)$, not depending on k, such that the heat kernel $k^{-m/2}e_k(t,x,x)$ converges to $e_\infty(t,x)$, as $k\longrightarrow+\infty$, uniformly with respect to $x\in X$ and $t\in[t_0,t_1]\subset(0,+\infty)$.*

The rest of the section is devoted to the proof of Theorem 3.2.

3.1.2. Localization. Let us fix a point $x_0\in X$ and local coordinates around this point. Let B_r be the ball of center x_0 and radius r. Set

$$\overline{\tau}_r = \sup_{x\in B_r}\|\tau(x)\|$$

We prove now that the problem is local by comparing the heat kernel $e_k(t,x_0,x_0)$ to the heat kernel $e_{k,B_r}(t,x_0,x_0)$ over the ball B_r.

3.3. PROPOSITION (Localization). *There exist positive constants C_1 and ε_1 such that for any $t\in(0,\min(k\varepsilon_1,kr^2/2m))$ we have*

$$|e_k(t,x_0,x_0)-e_{k,B_r}(t,x_0,x_0)| \leqslant C_1\frac{k^{\frac{m}{2}}}{t^{\frac{m}{2}}}\exp\left(-\frac{kr^2}{4t}+2t\overline{\tau}_r\right)\,. \tag{3.10}$$

As preparation we need the following.

3.4. LEMMA. *Let $d(\cdot,\cdot)$ the geodesic distance associated to the metric g^{TX}. Then for all $t\in(0,k\varepsilon_1]$ and all $x,y\in X$ we have*

$$|e_k(t,x,y)| \leqslant C_1\frac{k^{\frac{m}{2}}}{t^{\frac{m}{2}}}\exp\left(-k\frac{d(x,y)^2}{4t}+t\overline{\tau}_r\right)\,. \tag{3.11}$$

PROOF. Let us denote by $e_g(w,x,y)$ the heat kernel associated to Δ_g and $\hat{e}_k(w,x,y)$ the heat kernel associated to $k\Delta_{k,\tau} = \Delta^{F_k} - k\tau$. Kato's inequality for Δ^{F_k} reads

$$\langle\Delta^{F_k}u,u\rangle \geqslant |u|\Delta_g|u|\,,\quad \text{for } u\in\Omega(B_r,L^k\otimes F)\,, \tag{3.12}$$

and hence

$$\langle k\Delta_{k,\tau}u,u\rangle \geqslant |u|\,((\Delta_g-k\overline{\tau}_r)|u|) \tag{3.13}$$

By [**5**, Theorem 3.1], (3.13) entails

$$|\hat{e}_k(w,x,y)| \leqslant \operatorname{rank}F\,\exp(k\overline{\tau}_r)\,e_g(w,x,y)\,. \tag{3.14}$$

Since $e_g(w,x,y)$ admits the asymptotic expansion in a neighbourhood of $w=0$,

$$e_g(w,x,y) \sim u_0(4\pi w)^{-\frac{m}{2}}\exp\left(-\frac{d^2(x,y)}{4r}\right) \tag{3.15}$$

we infer the existence of constants $u_1, \varepsilon > 0$ such that

$$e_g(w,x,y) \leqslant u_1(4\pi w)^{-\frac{m}{2}} \exp\left(-\frac{d^2(x,y)}{4r}\right), \quad \text{for } w \leqslant \varepsilon\,. \tag{3.16}$$

The statement of the lemma is now a consequence of (3.14) and (3.16) by setting $w = t/k$. □

PROOF OF PROPOSITION 3.3. Let φ be a smooth function supported in B_r. We set

$$\psi(t,x) = \int_{B_r} (e_{k,B_r} - e_k)\varphi dv_X$$

which, by (3.8b), (3.8c) has the properties

$$P_k\psi = \frac{\partial\psi}{\partial t} + \Delta_{k,\tau}\psi = 0 \quad \text{on } (0,+\infty)\times B_r\,, \tag{3.17a}$$

$$\psi \longrightarrow 0 \quad \text{as } t \longrightarrow 0\,. \tag{3.17b}$$

By Kato's inequality, $\exp(-t\overline{\tau}_r)|\psi|$ satisfies the maximum principle for the heat operator P_k. Indeed,

$$\left\langle \tfrac{\partial}{\partial t}\psi, \psi \right\rangle = \tfrac{1}{2}\tfrac{\partial}{\partial t}|\psi|^2 = |\psi|\tfrac{\partial}{\partial t}|\psi| \tag{3.18}$$

and (3.12) entail

$$\langle P_k\psi, \psi\rangle \geqslant |\psi|\big(\tfrac{\partial}{\partial t} + \tfrac{1}{k}\Delta_g - \overline{\tau}_r\big)|\psi| \tag{3.19}$$

hence

$$\left(t\tfrac{\partial}{\partial t} + \tfrac{1}{k}\Delta_g - \overline{\tau}_r\right)|\psi| \leqslant 0 \tag{3.20}$$

that is

$$\big(\tfrac{\partial}{\partial t} + \tfrac{1}{k}\Delta_g\big)\exp(-t\overline{\tau}_r)|\psi| \leqslant 0\,. \tag{3.21}$$

Thus the function $\exp(-t\overline{\tau}_r)|\psi|$ takes its maximum on $\{0\}\times B_r \cup [0,t]\times\partial B_r$. Therefore

$$\begin{aligned} |\psi(t,x_0)| &\leqslant \exp(t\overline{\tau}_r)\sup\{|\psi| : (r,x)\in[0,t]\times\partial B_r\} \\ &\leqslant C_1\frac{k^{\frac{m}{2}}}{t^{\frac{m}{2}}}\exp\left(-\frac{k}{t}d^2(\sup\varphi,\partial B_r) + 2t\overline{\tau}_r\right)\|\varphi\|_1 \end{aligned} \tag{3.22}$$

if $t \leqslant 2kd^2(\operatorname{supp}\varphi,\partial B_r)/2$ where $\|\varphi\|_1$ is the Sobolev norm of φ. We let now $\varphi \longrightarrow \delta_{x_0}$ in (3.22) and we obtain (3.10). □

3.5. REMARK. We first remark that we may choose C_1 and ε_1 uniform in $x \in X$. In order to have uniformity with respect to t of (3.10) we need to apply the estimate on balls of radius r_k such that

$$\lim_{k\longrightarrow\infty} kr_k^2 = +\infty.$$

If we choose $r_k = k^{-\frac{5}{12}}$ then $kr_k^2 = k^{\frac{1}{6}}$ so (3.10) holds for $t \in (0, k^{\frac{1}{6}})$ and $k \geqslant k_0$, where k_0 is uniform in $x \in X$.

3.1.3. The heat kernel of the tangent operator. We consider now the operator $\Delta^0_{k,\tau}$ on $\mathbb{R}^m$ which is the tangent operator to $\Delta_{k,\tau}$ at x_0. We endow $\mathbb{R}^m$ with the metric $g_0 = \sum dx^j \wedge dx^j$ and the trivial line bundle $L_0 = \mathbb{R}^m \times \mathbb{C}$ with the connection

$$\nabla^{L_0} = d - \sqrt{-1}\sum_1^s \omega_j x^j\, dx^{j+s}\,, \quad A^{L_0} = \sum_{j=1}^s \omega_j x^j dx^{j+s} \tag{3.23}$$

having curvature $\omega = \sum_{j=1}^s \omega_j dx^j \wedge dx^{j+s}$. We take ∇^E flat and τ constant, $\langle \tau u, u\rangle = \sum_{1\leqslant\lambda\leqslant r} \tau_\lambda |u_\lambda|^2$ for $u = \sum_{\lambda=1}^r u_\lambda \otimes e_\lambda$.

3.6. PROPOSITION. *With this choices the heat kernel of $\Delta^0_{k,\tau}$ has the form:*

$$e(t_0,x,x) = k^{m/2}\frac{\exp(t\tau)}{(4\pi)^{m/2}t^{m/2-s}}\prod_{j=1}^s \frac{\omega_j}{\sinh \omega_j t}. \tag{3.24}$$

PROOF. By (3.6) and (3.23) the quadratic form of $\Delta^0_{k,\tau}$ is given by:

$$Q_k(u) = \int_{\mathbb{R}^m} \frac{1}{k}\Big[\sum_{\substack{i\leqslant j\leqslant s\\ 1\leqslant\lambda\leqslant r}} \Big(\Big|\frac{\partial u_\lambda}{\partial x^j}\Big|^2 + \Big|\frac{\partial u_\lambda}{\partial x^{j+s}} - ik\omega_j x^j u_\lambda\Big|^2\Big) + \sum_{\substack{j>2s\\ 1\leqslant\lambda\leqslant r}} \Big|\frac{du_\lambda}{dx^j}\Big|^2\Big] - \sum_{1\leqslant\lambda\leqslant r} V_\lambda |u_\lambda|^2.$$

In this situation, Q_k is a direct sum of quadratic form acting on each component u_λ and the computation of $\exp(-t\Delta_{k,\tau})$ is reduced to the following simple cases (3.25) and (3.26). The first case is:

$$Q(f) = \int_{\mathbb{R}} \Big|\frac{df}{dx}\Big|^2, \quad \text{associated to the operator} - d^2 f/dx^2 \tag{3.25}$$

Then, it is well known that the heat kernel is given by

$$e(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}.$$

The second case is:

$$Q(f) = \int_{\mathbb{R}^2} \Big|\frac{df}{dx^1}\Big|^2 + \Big|\frac{df}{dx^2} - iax^1 f\Big|^2. \tag{3.26}$$

A partial Fourier transform in the x^2 variable gives

$$Q(f) = \int_{\mathbb{R}^2} \Big|\frac{d\widehat{f}}{dx^1}(x^1,\xi^2)\Big|^2 + a^2\Big(x^1 - \frac{\xi^2}{a}\Big)^2 |\widehat{f}(x^1,\xi^2)|^2$$

and the change variables $(x^1)' = x^1 - x^2/a$, $(x^2)' = \xi^2$ leads to the so called “harmonic oscillator” energy functional

$$q(u) = \int_{\mathbb{R}} \Big|\frac{du}{dx}\Big|^2 + a^2x^2|u|^2, \quad \text{associated to} \quad \square = -\frac{d^2}{dx^2} + a^2x^2.$$

The heat kernel of this operator is given by Mehler’s formula:

$$e(t,x,y) = \sqrt{\frac{a}{2\pi \sinh 2at}} \exp\Big(-\frac{a}{2}(\coth 2at)(x-y)^2 - a(\tanh at)xy\Big).$$

To see this we can use the form of the eigenvalues of the harmonic oscillator $\square$ as given in (2.17):

$$\big(2^p p!\sqrt{\pi/a}\big)^{-1/2}\Phi_p(\sqrt{a}x),\ p = 0,1,2,\ldots,$$

with associated eigenvalues $(2p+1)a$, where (Φ_p) is the sequence of functions associated to Hermite polynomials:

$$\Phi_p(x) = e^{x^2/2}\frac{d^p}{dx^p}(e^{-x^2}).$$

Therefore we have

$$e(t,x,y) = \sqrt{\frac{a}{\pi}}e^{a(x^2+y^2)/2}\sum_{p=0}^{+\infty}\frac{e^{-(2p+1)}}{2^p p! a^p}\frac{d^p}{dx^p}(e^{-ax^2})\frac{d^p}{dy^p}(e^{-ay^2})$$

and the summation $\sum(x,y)$ can be computed from its Fourier transform

$$\widehat{\sum}(\xi,\eta) = e^{-at}\exp\big(-\frac{1}{2a}e^{-2at}\xi\eta\big)\cdot\frac{1}{\sqrt{2a}}e^{-(\xi^2+\eta^2)/4a}.$$

The heat kernel operator of Q from (3.26) is thus given by

$$(e^{-t\Box}f)^\wedge(x^1,\xi_2) = \int_{\mathbb{R}} e\Big(t,x^1-\frac{\xi_2}{a},y^1-\frac{\xi_2}{a}\Big)\widehat{f}(y^1,\xi_2)dy^1.$$

which by inverse Fourier transform reads:

$$\begin{aligned} e(t;x^1,x^2;y^1,y^2) = \frac{a}{4\pi\sinh at}\exp\Big(-\frac{a}{4}(\coth at)\big((x^1-y^1)^2+(x^2-y^2)^2\big)\Big) \\ \times\exp\big(\frac{i}{2}a(x^1+y^1)(x^2-y^2)\big). \end{aligned}$$

The heat kernel associated to a sum of (pairwise commuting) operators $\Box_1,\ldots,\Box_m$ acting on disjoint sets of variables is the product of all heat kernels $\exp(-t\Box_j)$. Let $e^{\lambda}(t,x,y)$ be the heat kernel of Q_k acting on a single component u_λ. The factor in the heat kernel corresponding to the pair of variables (x^j,x^{j+s}), $1\leqslant j\leqslant s$ is obtained when substituting $k\omega_j$ to a and t/k to t. Thus

$$\begin{aligned} e_k^0(t,x,y) = \prod_{j=1}^{s}\frac{k\omega_j}{4\pi\sinh\omega_j t}\exp\Big(\frac{-k\omega_j}{4}\coth\omega_j t((x^j-y^j)^2+(x^{j+s}-y^{j+s})^2) \\ +\frac{\sqrt{-1}}{2}k\omega_j(x^j+y^j)(x^{j+s}-y^{j+s})\Big) \\ \times\exp(t\tau)\Big(\frac{k}{4\pi t}\Big)^{\frac{m-2s}{2}}\exp\Big(-k\sum_{j>2s}(x^j-y^j)^2/4t\Big) \quad (3.27) \end{aligned}$$

Restricting to the diagonal we obtain (3.24). □

3.1.4. Heat asymptotic of the localized operator. Using a chart we identify now a neighbourhood of x_0 with $\mathbb{R}^m$. We define a new operator $\widetilde{\Delta}_{k,\tau}$ on $\mathbb{R}^m$ which coincides with $\Delta_{k,\tau}$ on a small ball. Let $B_k = B_{r_k}$ with $r_k = k^{-5/12}$. We construct the operator $\widetilde{\Delta}_{k,\tau}$ wich coincides with $\Delta_{k,\tau}$ on B_k and with $\Delta^0_{k,\tau}$ on $\mathbb{R}^m\setminus 2B_k$. We achieve this in the following manner. Let φ a smooth function with support in $2B_k$ which equals 1 on B_k. We consider on $\mathbb{R}^m$ the metric $\widetilde{g} = \varphi g^{TX}+(1-\varphi)g^0$, on $L_0 = \mathbb{R}^m\times\mathbb{C}$ the connection $\widetilde{\nabla}^L = \varphi\nabla^L+(1-\varphi)\nabla^{L_0}$ and on F the flat connection. We define $\widetilde{\Delta}^{L^k\otimes F}$ as the laplacian associated to this connection. We set $\widetilde{\tau} = \varphi\tau+(1-\varphi)\tau(x_0)$ and we get as in (3.4) the operator $\widetilde{\Delta}_{k,\tau}$.

If we denote by $A=\sqrt{-1}\Gamma^L$ we know that

$$\Delta^{F_k}u = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_l}\big(g^{jl}\sqrt{\det g}\frac{\partial u}{\partial x_j}\big) + \sqrt{-1}k\frac{\partial u}{\partial x_j}\overline{A}_l g^{jl} + \sqrt{-1}k\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_l}(A_j g^{jl}\sqrt{\det g}\,u) + k^2|A|^2$$

This entails the estimate

$$\widetilde{\Delta}_{k,\tau} - \Delta^0_{k,\tau} = \begin{cases} O\Big(\frac{|x-x_0|}{k}\nabla^2 + (\frac{1}{k}+|x-x_0|^2)\nabla + |x-x_0| + k|x-x_0|^3\Big) \\ 0 \quad \text{outside } 2B_k \end{cases} \tag{3.28}$$

where ∇ is the trivial connection on $\mathbb{R}^m$ and A is chosen such that $|A(x)-A(x_0)| \leqslant C_2|x-x_0|^2$. There is possible since $|\omega(x)-\omega(x_0)| = O(|x-x_0|)$ and by the homotopy formula on a ball, used to construct the potential A. Let $\widetilde{e}_k(t,x,y)$ be the heat kernel of $\widetilde{\Delta}_{k,\tau}$.

Our purpose is to express the heat kernel $\widetilde{e}_k(t,x,y)$ as an infinite sum depending only on $\Delta^0_{k,\tau}$ on a ball of radius $r_k = k^{-\frac{5}{12}}$. Using Lemma 3.3 and properties $(3.6a-d)$ we obtain

$$\begin{aligned}
\widetilde{e}_k(t,x,y) - e^0_k(t,x,y) &= \int_{\mathbb{R}^m} e^0_k(0,z,y)\widetilde{e}_k(t,x,z)dz - \int_{\mathbb{R}^m}\widetilde{e}_k(0,x,z)e^0_k(t,z,y)dz \\
&= \int_0^t dw\frac{\partial}{\partial r}\int_{\mathbb{R}^m}\widetilde{e}_k(w,x,z)e^0_k(t-w,z,y)dz \\
&= -\int_0^t dw\int_{\mathbb{R}^m}\{(\widetilde{\Delta}_{k,\tau})^*\widetilde{e}_k(w,x,z)\,e^0_k(t-w,z,y) \\
&\quad - \widetilde{e}_k(t,x,z)\Delta^{\#,0}_k e^0_k(t-w,z,y)\}dz \\
&= -\int_0^t dw\int_{\mathbb{R}^m}\widetilde{e}_k(w,x,z)(\widetilde{\Delta}_{k,\tau} - \Delta^{\#,0}_k)_z\, e^0_k(t-w,z,y)dz \\
&= -\int_0^t dw\int_{2B_k}\widetilde{e}_k(w,x,z)(\widetilde{\Delta}_{k,\tau} - \Delta^{\#,0}_k)_z\, e^0_k(t-w,z,y)dz
\end{aligned}$$

Set $f_k(t,x,y) = (\widetilde{\Delta}_{k,\tau} - \Delta^{\#,0}_k)e^0_k(t,x,y)$ and denote the previous equality by $\widetilde{e}_k - e^0_k = \widetilde{e}_k\sharp f_k$. We obtain the formal Levi sum

$$\widetilde{e}_k = e^0_k + e^0_k\sharp f_k + \cdots + e^0_k\sharp f_k\sharp\cdots\sharp f_k + \cdots = \sum_{p\geqslant 0} e^0_k\sharp f_k^{\sharp p} \tag{3.29}$$

3.7. LEMMA. *The sum* (3.29) *converges to* $\widetilde{e}_k$.

PROOF. In the sequel C stands for possibly different constants. By (3.27) we have

$$\begin{aligned}
|\nabla e^0_k(t,z,y)| &\leqslant C\Big(\frac{k}{t}|z-y|\Big) + k(|z|+|y|)\,|e^0_k(t,z,y)| \\
|\nabla^2 e^0_k(t,z,y)| &\leqslant C\Big(\frac{k}{t} + k + \frac{k^2}{t^2}|y-z|^2 + k^2(|y|+|z|)^2 \\
&\qquad + \frac{k^2}{t}|y-z|(|y|+|z|)\Big)\,|e^0_k(t,z,y)|
\end{aligned}$$

hence by (3.28)

$$|\,f_k(t,z,y)| \leqslant C\Big(\frac{k|z-y|^3}{t^2} + \frac{|z-y|+k|z-y|^2}{t} + kr_k^3 + r_k^2\Big)\,|e^0_k(t,z,y)|\,.$$

Since $r_k = k^{-5/12}$ and the function $(0,+\infty) \longmapsto x^p \exp(-\alpha x)$ is bounded for all $p \geqslant 0$, $\alpha > 0$ we find $a_0 > 0$ such that for k sufficiently large

$$|f_k(t,z,y)| \leqslant C\frac{k^{\frac{m}{2}-\frac{1}{4}}}{\sqrt{t}} \exp(t\overline{\tau}_{r_k})\, t^{-\frac{m}{2}} \exp\Big(\frac{-a_0 k|x-y|^2}{t}\Big) \tag{3.30}$$

hence

$$\begin{aligned} |e_k^0 \sharp f_k(t,x,y)| &\leqslant C k^{\frac{m}{2}-\frac{1}{4}} \exp(t\overline{\tau}_{r_k})\cdot \\ &\cdot \int_0^t dw \int_{\mathbb{R}^m} w^{-\frac{m+1}{2}} (t-w)^{-\frac{m}{2}} \exp\left(-a_0 k\Big(\frac{|z-y|^2}{w} + \frac{|z-x|^2}{t-w}\Big)\right) dz \end{aligned} \tag{3.31}$$

Since

$$\begin{aligned} \int_{\mathbb{R}_m} w^{-\frac{m+1}{2}} (t-w)^{-\frac{m}{2}} \exp\Big(-a_0\, k\Big(\frac{|z-y|^2}{w} + \frac{|z-x|^2}{t-w}\Big)\Big) dz = \\ \Big(\frac{2\pi}{a_0}\Big)^{\frac{m}{2}} \frac{(kt)^{-\frac{m}{2}}}{\sqrt{w}} \exp\Big(-a_0\, k\frac{|x-y|^2}{t}\Big) \end{aligned} \tag{3.32}$$

Therefore

$$\begin{cases} |e_k^0 \sharp f_k(t,x,y)| \leqslant C^2 k^{\frac{m}{2}-\frac{1}{4}} \exp(t\overline{\tau}_{r_k})\, \frac{1}{2} t^{-\frac{m+1}{2}} \exp\Big(\frac{-a_0 k}{t}|x-y|^2\Big) \\ \cdots \\ |e_k^0 \sharp \cdots \sharp f_k(t,x,y)| \leqslant C^p k^{\frac{m}{2}-\frac{p}{4}} \exp(t\overline{\tau}_{r_k}) \frac{1}{p!} t^{-\frac{m-1}{2}+p} \exp\Big(\frac{-a_0 k}{t}|x-y|^2\Big) \end{cases} \tag{3.33}$$

By (3.33) the series (3.29) and its derivatives converges on any compact of $(0,\infty)\times\mathbb{R}^m$. Therefore for any section $S \in \Omega\big((0,\infty)\times\mathbb{R}^m, \mathrm{End}(E)\big)$ we get

$$\tfrac{\partial}{\partial t}(e_k^0 \sharp S) = S \quad \text{and} \quad f_k \sharp S = (\widetilde{\Delta}_{k,\tau} - \Delta^0_{k,\tau})(e_k^0 \sharp S)$$

Hence

$$\begin{aligned} \tfrac{\partial}{\partial t}(e_k^0 + e_k^0 \sharp f + \cdots) &= -\Delta^0_{k,\tau} e_k^0 - (\widetilde{\Delta}_{k,\tau} - \Delta^0_{k,\tau}) e_k^0 - (\widetilde{\Delta}_{k,\tau} - \Delta^0_{k,\tau}) e_k^0 \sharp f_k + \cdots \\ &= -\widetilde{\Delta}_{k,\tau}(e_k^0 + e_k^0 \sharp f_k + \cdots) \end{aligned}$$

By $e_k^0(0,x,y) = \delta_x(y)$ and (3.33),

$$|\widetilde{e}_k(t,x,y) - e_k^0(t,x,y)| \leqslant k^{\frac{m}{2}} t^{-\frac{m-1}{2}} \exp(t\overline{\tau}_{r_k})\big(\exp(Ctk^{-1/4}) - 1\big) \exp\big(-\frac{a_0 k}{t}|x-y|^2\big), \tag{3.34}$$

where the second term converges to 0 in $L^1(0,\infty)$ for $t \longrightarrow 0$.

Since the heat kernel is unique this achieves the proof of Lemma 3.7. □

PROOF OF THEOREM 3.2. Proposition 3.3 and Lemma 3.7 imply Theorem 3.2 due to estimate (3.34). □

3.1.5. Asymptotic distribution of eigenvalues. Theorem 3.2 permits to determine the asymptotic behaviour of the eigenvalue distribution of the operator $\Delta_{k,\tau} = \frac{1}{k}\Delta^{F_k} - \tau$ for $k \longrightarrow \infty$.

3.8. PROPOSITION. *For $j = 0, 1, \ldots$, let λ_j^k be the eigenvalues of $\Delta_{k,\tau}$ counted with multiplicities. The following relation holds for all $t > 0$:*

$$\lim_{k\to\infty} k^{-m/2} \sum_{j=0}^{\infty} \exp(-t\lambda_j^k) = \int_X \mathrm{Tr}_F\, e_\infty(t,x)\, dv_X\,. \tag{3.35}$$

PROOF. The definition of the heat kernel (3.7) implies

$$\operatorname{Tr}_F e_k(t,x,x) = \sum_{j=0}^{\infty} \exp(-t\,\lambda_j^k)|S_j^k|^2\,,$$

and by integration

$$\int_X \operatorname{Tr}_F e_k(t,x,x) = \sum_{j=0}^{\infty} \exp(-t\,\lambda_j^k)\,.$$

The uniform convergence for $x \in X$ and t fixed of $k^{-m/2}\operatorname{Tr}_F e_k(t,x,x)$ to $\operatorname{Tr}_F e_\infty(t,x)$ as $k \longrightarrow \infty$ implies then the statement. □

We define the spectrum counting function of $\Delta_{k,\tau}$ by

$$N(\lambda,\Delta_{k,\tau}) = \dim\operatorname{Im} E_\lambda(\Delta_{k,\tau}) = \operatorname{card}\left\{ j \,:\, \lambda_j^k \leqslant \lambda \right\}\,, \tag{3.37}$$

and by

$$\mu_k = \frac{1}{k^{m/2}}\frac{d}{d\lambda}N(\lambda,\Delta_{k,\tau}) = \frac{1}{k^{m/2}}\sum_{j=0}^{\infty}\delta(\lambda_j^k)\,, \tag{3.38}$$

the spectral density measure, where $\delta(\lambda_j^k)$ is the Dirac measure at λ_j^k, so that, μ_k is a sum of Dirac measures on $\mathbb{R}$ supported on $\operatorname{Spec}\Delta_{k,\tau}$.

3.9. PROPOSITION. *The sequence of measures μ_k converges weakly to a measure μ whose Laplace transform is $\int_X \operatorname{Tr}_F e_\infty(t,x)\,dv_X$.*

Indeed, (3.35) can be written $\lim_{k\to\infty}\int_{\mathbb{R}}\exp(-t\lambda)d\mu_k = \int_X e_\infty(t,x)\,dv_X$.

Let us intoduce the function $\nu_{\omega(x)}(\lambda)$ on $X \times \mathbb{R}$ given by

$$\nu_\omega(\lambda) = \frac{2^{s-m}\pi^{-m/2}}{\Gamma(\frac{m}{2}-s+1)}\;\omega_1\cdots\omega_s \sum_{(p_1,\dots,p_s)\in\mathbb{N}^s}\left[\lambda - \sum(2p_j+1)\omega_j\right]_+^{\frac{m}{2}-s}\,, \tag{3.39}$$

with the convention $[\lambda]_+^0 = 0$ for $\lambda \leqslant 0$, $[\lambda]_+^0 = 1$ for $\lambda > 0$. It is an increasing function, left-continuous in λ and lower semicontinuous on X. We also consider the function

$$\overline{\nu}_\omega(\lambda) = \lim_{\varepsilon\to 0+}\nu_\omega(\lambda+\varepsilon)\,. \tag{3.40}$$

which is increasing and right-continuous in λ and upper semicontinuous on X. It has the form

$$\overline{\nu}_\omega(\lambda) = \frac{2^{s-m}\pi^{-m/2}}{\Gamma(\frac{m}{2}-s+1)}\;\omega_1\cdots\omega_s \sum_{(p_1,\dots,p_s)\in\mathbb{N}^s}\left\{\lambda - \sum(2p_j+1)\omega_j\right\}_+^{\frac{m}{2}-s}\,, \tag{3.41}$$

with the convention $\{\lambda\}_+^0 = 0$ for $\lambda < 0$, $\{\lambda\}_+^0 = 1$ for $\lambda \geqslant 0$.

Since $\nu_\omega(\lambda) \leqslant \overline{\nu}_\omega(\lambda) \leqslant \lambda_+^{m/2}$, the functions $\nu_\omega(\lambda+\tau_l(x))$ and $\overline{\nu}_\omega(\lambda+\tau_l(x))$ are bounded on compact sets of the form $\mathbb{R}\times X$.

Let $r := \operatorname{rank} F$. By taking the inverse Laplace transformation of $\int_X e_\infty(t,x)\,dv_X$ we obtain:

3.10. COROLLARY. *We have the following estimates:*

$$\liminf_{k\to\infty} k^{-m/2} N(\lambda,\Delta_{k,\tau}) \geqslant \int_X \sum_{l=1}^r \nu_{\omega(x)}(\lambda+\tau_l(x))\, dv_X\,, \tag{3.42a}$$

$$\limsup_{k\to\infty} k^{-m/2} N(\lambda,\Delta_{k,\tau}) \leqslant \int_X \sum_{l=1}^r \overline{\nu}_{\omega(x)}(\lambda+\tau_l(x))\, dv_X\,, \tag{3.42b}$$

There exists an at most countable set $\mathscr{D}\subset\mathbb{R}$ such that for $\lambda\in\mathbb{R}\smallsetminus\mathscr{D}$

$$\lim_{k\to\infty} k^{-m/2} N(\lambda,\Delta_{k,\tau}) = \mu\big((-\infty,\lambda]\big) = \int_X \sum_{l=1}^r \nu_{\omega(x)}(\lambda+\tau_l(x))\, dv_X\,, \tag{3.43}$$

Indeed, the Lebesgue dominated convergence theorem shows that the function

$$g:\mathbb{R}\to\mathbb{R},\quad g(\lambda)=\int_X \sum_{l=1}^r \nu_{\omega(x)}(\lambda+\tau_l(x))\, dv_X$$

is left-continuous and increasing. Moreover

$$\lim_{\varepsilon\to 0+} g(\lambda+\varepsilon) = \int_X \sum_{l=1}^r \overline{\nu}_{\omega(x)}(\lambda+\tau_l(x))\, dv_X\,.$$

The set $\mathscr{D}$ is the set of discontinuities of g.

3.1.6. A more refined estimate. For further use we need a more precise version of Theorem 3.2:

3.11. THEOREM (Bouche). *There exists $\varepsilon\in(0,1)$ such that on the set*

$$\Big\{\tau(x)-\sum_{j=1}^s \omega_j(x)\,\mathrm{Id}_F\leqslant 0\Big\}$$

we have

$$\Big|e_k(t,x,x)-k^{m/2}e_\infty(t,x)\Big|_{\mathscr{C}^0}\leqslant C\Big(1+\frac{1}{t^{m/2}}\Big)k^{-1/5} \tag{3.44}$$

uniformly with respect to $t\in[t_0,k^{\varepsilon}]$ for any $t_0>0$.

PROOF. We want to obtain a better order of convergence for $\widetilde{e}_k$ as in (3.33). The proof of (3.30) shows in fact that

$$\begin{aligned}|f_k(t,z,y)|\leqslant C\frac{k^{\frac{m}{2}-\frac14}}{\sqrt{t}}\exp(t\overline{\tau}_{r_k})\prod_{j=1}^{2s}\frac{\sqrt{\omega_j}\exp(-a_0k\omega_j\coth\omega_j t|y^j-z^j|^2)}{\sqrt{\sinh\omega_j t}}\\ \times\prod_{j>2s}\frac{\exp(-\frac{a_0k}{t}|y^j-z^j|^2)}{\sqrt{t}}\,.\end{aligned} \tag{3.45}$$

We make in the sequel the convention $\omega_j=\omega_{j-s}$ for $j\in\{s+1,\dots,2s\}$. We obtain

$$\begin{aligned}&|e_k^0\sharp f_k(t,x_0,x_0)|\leqslant k^{m-\frac14}\exp(t\overline{\tau}_{r_k})\cdot\\ &\cdot\int_o^t dw\int_{\mathbb{R}^m}\prod_{j=1}^{2s}\frac{\omega_j\exp(-a_0k\omega_j(\coth\omega_j w+\coth\omega_j(t-w))|z^j-x_0^j|^2)}{\sqrt{\sinh\omega_j w\sinh\omega_j(t-w)}}\\ &\times\frac{1}{\sqrt{w}}\times\prod_{j>2s}\frac{\exp(a_0k(\frac1w+\frac{1}{t-w})|z^j-x_0^j|^2)}{\sqrt{w(t-w)}}\end{aligned} \tag{3.46}$$

For $j > 2s$ we already know that

$$\int_{\mathbb{R}} \frac{\exp\big(a_0 k(\frac{(t-w)w}{t})|z^j - x_0^j|^2\big)}{\sqrt{w(t-w)}} dz^j = \sqrt{\frac{2\pi}{k a_0 t}} \tag{3.47}$$

and for $j \in \{1, \dots, 2s\}$ we have

$$\begin{aligned}\int_{\mathbb{R}} \frac{\omega_j \exp\big(-a_0 k \omega_j (\coth \omega_j w + \coth \omega_j (t-w))|z^j - x_0^j|^2\big)}{\sqrt{\sinh \omega_j w \sinh \omega_j (t-w)}} dz^j = \sqrt{\frac{2\pi\omega_j}{a_0 k}} \\ \times \frac{1}{\sqrt{\sinh \omega_j t}}\end{aligned} \tag{3.48}$$

By (3.46),

$$\begin{cases} |e_k^0 \sharp f_k(t,x_0,x_0)| \leqslant C k^{\frac{m}{2}-\frac{1}{4}} \exp(t\overline{\tau}_{r_k}) \prod_{j=1}^s \dfrac{\omega_j}{\sinh \omega_j t} \cdot \dfrac{t^{s-\frac{m}{2}+1}}{2} \\ \dots \\ |e_k^0 \sharp \cdots \sharp f_k(t,x_0,x_0)| \leqslant C^p k^{\frac{m}{2}-\frac{p}{4}} \exp(t\overline{\tau}_{r_k}) \prod_{j=1}^s \dfrac{\omega_j}{\sinh \omega_j t} \cdot \dfrac{t^{s-\frac{m}{2}+p}}{(p+1)!} \end{cases} \tag{3.49}$$

We obtain finally the following estimate

$$\begin{aligned} |\widetilde{e}_k - e_k^0| &\leqslant (e^{Ct/k^{1/4}} - 1) k^{\frac{m}{2}} \exp(t\overline{\tau}_{r_k}) t^{s-\frac{m}{2}} \prod_{j=1}^s \frac{\omega_j}{\sinh \omega_j t} \\ &\leqslant C k^{\frac{m}{2}-\frac{1}{4}} t^{s-\frac{m}{2}+1} \exp(t\overline{\tau}_{r_k}) \prod_{j=1}^s \frac{\omega_j}{\sinh \omega_j t} \quad \text{if} t \leqslant k^{1/4} \\ &\leqslant C k^{-\frac{1}{4}} t \exp(C t r_k) |e_k^0(t,x_0,x_0)| \end{aligned} \tag{3.50}$$

Since $r_k = k^{-5/12}$ and since $\varepsilon = 1/6$ is the critical value for the validity of Proposition 3.3, Theorem 3.11 is a consequence of (3.50) with $\varepsilon < 1/6$, since $k^{-m/2} \operatorname{Tr}_F e_k(t,x_0,x_0)$ is bounded as function of t and k as long as $\tau(x_0) - \sum_{j=1}^s \omega_j(x_0) \operatorname{Id}_F \leqslant 0$. □

3.2. Pointwise Morse inequalities

Let us consider the compact manifold X with boundary and the vector bundles L and F as in 3.1.1. Now we assume moreover that X is a connected complex manifold of complex dimension n and real dimension $m = 2n$. Assume also that g^{TX} is compatible with the complex structure J. We denote by η the Kähler form associated to the hermitian metric induced by g^{TX}. Let L, F be holomorphic vector bundles on X, and let ∇^L, ∇^F be the canonical connections, compatible with the complex structure and with the hermitian metrics. For the curvature forms we use the notation $R^L = (\nabla^L)^2$, so that with the previous notation $\omega = \sqrt{-1}R^L)$ is a 2-form on X, called the *magnetic field* of the connection.

3.2.1. Statement of the result. Let us denote $F_k := L^k \otimes F$. Our goal is to study the Laplace-Beltrami operator

$$\square^{F_k} = \overline{\partial}^{F_k} \vartheta^{F_k} + \vartheta^{F_k} \overline{\partial}^{F_k} \quad \text{on } L^k \otimes F\,. \tag{3.51}$$

We introduce a scalar product on $\Omega^{0,\bullet}(X, L^k \otimes F)$ as in (3.2):

$$(s_1, s_2) \doteq \int_X \langle s_1(x), s_2(x) \rangle_{h^k} dv_X(x)\,, \quad dv_X = \eta^n/n!\,. \tag{3.52}$$

$\square^{F_k}$ acting on $\Omega^{0,q}(X,L^k\otimes F)$ is elliptic and has discrete spectrum. We denote for $j=0,1,\dots$ by $\lambda_j^{k,q}$, the eigenvalues and by $S_j^{k,q}$ an orthonormal basis of $\boldsymbol{L}_2^{0,q}(X,L^k\otimes F)$ such that $\square^{F_k}S_j^{k,q}=\lambda_j^{k,q}S_j^{k,q}$.

The operator $\exp\big(-\frac{2t}{k}\square^{F_k}\big)$ has a Schwartz kernel whose restriction to the diagonal has the expression:

$$e_k^{(0,q)}(t,x,x)=\sum_{j\geqslant 0}\exp\big(-\tfrac{2t}{k}\lambda_j^{k,q}\big)S_j^{k,q}(x)\otimes S_j^{k,q}(x)^*\in\operatorname{End}(F_x\otimes\Lambda^{0,q}T_x^*X),\tag{3.53}$$

In order to determine the asymptotics of $e_k^{(0,q)}(t,x,x)$ as $k\to\infty$ we intoduce the eigenvalues $\alpha_1(x),\dots,\alpha_n(x)$ of $\sqrt{-1}R^L$ with respect to η at x. For a multiindex $J\subset\{1,2,\dots,n\}$ we set

$$\alpha_J\doteq\sum_{j\in J}\alpha_j\,,\quad \complement J\doteq\{1,2,\dots,n\}\smallsetminus J\,.\tag{3.54}$$

In accordance to (3.9) let us denote by

$$e_\infty^{(0,q)}(t,x)=\frac{\sum_{|J|=q}\exp t(\alpha_{\complement J}-\alpha_J)}{(4\pi)^n t^{n-s}}\prod_{j=1}^{s}\frac{|\alpha_j(x)|}{\sinh|\alpha_j(x)|t}\operatorname{Id}_F\,.\tag{3.55}$$

3.12. THEOREM. *Let* $e_k^{(0,q)}(t,x,x)$ *be the heat kernel associated to* $\exp(-\frac{2t}{k}\square^{F_k})$ *in bidigree* $(0,q)$. *Then there exists* $\varepsilon\in(0,1)$ *such that the following asymptotic holds*:

$$\big|\operatorname{Tr}_{\Lambda^{0,q}T^*X}e_k^{(0,q)}(t,x,x)-k^n e_\infty^{(0,q)}(t,x)\big|_{\mathscr{C}^0}=o(k^n)\,,\quad k\longrightarrow\infty\tag{3.56}$$

uniformly with respect to $t\in[t_0,k^\varepsilon]$ *for any* $t_0>0$.

The next two sections are devoted to the proof of Theorem 3.12.

3.2.2. $\square^{F_k}$ as a Schrödinger operator. In order to apply Theorem 3.11 we have to express the laplacian $\frac{2}{k}\square^{F_k}$ as a Schrödinger operator $\Delta_{k,\tau}=\frac{1}{k}\Delta_k-\tau$ as defined in (3.4).

For this purpose we shall consider a holomorphic hermitian vector bundle G over a complex manifold X and derive the relation between the Kodaira-Laplace operator $\square^G$ acting on sections of G and the Bochner-Laplace operator on $\Lambda^{0,q}T^*X\otimes G$.

Consider the holomorphic vector bundle $\widetilde{G}=G\otimes\Lambda^nTX$. We denote by $\sim:\Lambda^{0,q}T^*X\otimes G\longrightarrow\Lambda^{n,q}T^*X\otimes\widetilde{G}$, $u\mapsto\widetilde{u}$ the natural isometry.

3.13. PROPOSITION. *Let G be a holomorphic hermitian vector bundle over the hermitian manifold* (X,ω) *with torsion operator T. There exists a bundle morphism*

$$V:\Lambda^{0,q}T^*X\otimes G\longrightarrow\Lambda^1T^*X\otimes\Lambda^{0,q}T^*X\otimes G\,,\tag{3.57}$$

depending only on the metric ω such that for $u\in\Omega_0^{0,q}(G)$,

$$\begin{aligned}2(\square^G u,u)&=\|\nabla^{\Lambda^{0,q}T^*X\otimes G}u+Vu\|^2+\big([\sqrt{-1}R^G,\Lambda]u,u\big)\\&\quad+\big([\sqrt{-1}R^{\widetilde{G}},\Lambda]\widetilde{u},\widetilde{u}\big)+\big(Su,u\big)+\big(S\widetilde{u},\widetilde{u}\big)\,,\end{aligned}\tag{3.58}$$

where S is the operator given by (B.22).

PROOF. Let us assume first that ω is Kähler. Let $u\in\Omega_0^{p,q}(G)$ be a smooth (p,q)–form with compact support and values in G. The Bochner–Kodaira–Nakano formula (B.20) yields by integration by parts

$$(\square^G u,u)=\|(\nabla^G)'u\|^2+\|(\nabla^G)'^*u\|^2+\big([\sqrt{-1}R^G,\Lambda]u,u\big).\tag{3.59}$$

Let now $u \in \Omega_0^{0,q}(G)$. Since $(\nabla^G)'^* u = 0$ (the type of $(\nabla^G)'^*$ is $(-1,0)$) (3.59) gives

$$(\square^G u, u) = \|(\nabla^G)' u\|^2 + \big([\sqrt{-1}R^G, \Lambda]u, u\big). \tag{3.60}$$

The natural isometry $\sim : G \longrightarrow \widetilde{G}$, $u \mapsto \widetilde{u}$ extends to isometries $\sim : \Lambda^{0,q}T^*X \otimes G \longrightarrow \Lambda^{n,q}T^*X \otimes \widetilde{G}$. We have the following commutative diagrams:

$$\begin{array}{ccc} \Omega^{0,q}(G) & \xrightarrow{(\nabla^G)''} & \Omega^{0,q+1}(G) \\ \sim\big\downarrow & & \big\downarrow\sim \\ \Omega^{n,q}(\widetilde{G}) & \xrightarrow{(\nabla^{\widetilde{G}})''} & \Omega^{n,q+1}(\widetilde{G}) \end{array} \tag{3.61a}$$

and

$$\begin{array}{ccc} \Omega^{0,q+1}(G) & \xrightarrow{(\nabla^G)''^*} & \Omega^{0,q}(G) \\ \sim\big\downarrow & & \big\downarrow\sim \\ \Omega^{n,q+1}(\widetilde{G}) & \xrightarrow{(\nabla^{\widetilde{G}})''^*} & \Omega^{n,q}(\widetilde{G}) \end{array} \tag{3.61b}$$

which are consequences of the fact that $\Lambda^n TX$ is a holomorphic bundle.

Combining this two diagrams of (3.61a) and (3.61b) we obtain

$$\begin{array}{ccc} \Omega^{0,q}(G) & \xrightarrow{\square^G} & \Omega^{0,q}(G) \\ \sim\big\downarrow & & \big\downarrow\sim \\ \Omega^{n,q}(\widetilde{G}) & \xrightarrow{\square^{\widetilde{G}}} & \Omega^{n,q}(\widetilde{G}) \end{array} \tag{3.61c}$$

By applying (3.61c) and (3.60) for $\widetilde{u}$ we obtain for $u \in \Omega_0^{0,q}(G)$:

$$\begin{aligned} (\square^G u, u) &= (\square^{\widetilde{G}} \widetilde{u}, \widetilde{u}) \\ &= \|(\nabla^{\widetilde{G}})'^* \widetilde{u}\|^2 + \big([\sqrt{-1}R^{\widetilde{G}}, \Lambda]\widetilde{u}, \widetilde{u}\big) \end{aligned} \tag{3.62}$$

Let us recall the construction of $\nabla^{\Lambda^{0,q}T^*X \otimes G}$. The bundle $\Lambda^{q,0}T^*X$ is holomorphic and has a Chern connection which induces on the conjugate bundle $\Lambda^{0,q}T^*X$ a connexion $\nabla^{\Lambda^{0,q}T^*X}$ whose $(1,0)$ component coincides with ∂.

The connection $\nabla^{\Lambda^{0,q}T^*X \otimes G}$ is obtained from $\nabla^{\Lambda^{0,q}T^*X}$ and the Chern connection ∇^G on G. Since $(\nabla^{\Lambda^{0,q}T^*X})' = \partial$ we obtain

$$\big(\nabla^{\Lambda^{0,q}T^*X \otimes G}\big)' = \big(\nabla^G\big)' : \Omega^{0,0}(\Lambda^{0,q}T^*X \otimes G) \longrightarrow \Omega^{0,1}(\Lambda^{0,q}T^*X \otimes G) \tag{3.63}$$

Let us define the map

$$\Psi_0 : \Lambda^{0,1}T^*X \otimes G \xrightarrow{\ \sim\ } \Lambda^{n,1}T^*X \otimes \widetilde{G} \xrightarrow{\ *\ } \Lambda^{n-1,0}T^*X \otimes \widetilde{G}$$

and

$$\Psi = \mathrm{Id}_{\Lambda^{0,q}T^*X} \otimes (-\sqrt{-1})^{-n^2} \Psi_0 : \Omega^{0,1}(\Lambda^{0,q}T^*X \otimes G) \longrightarrow \Omega^{n-1,q}(\widetilde{G})$$

We have then the commutation relation

$$\begin{array}{ccc} \Omega^{0,0}(\Lambda^{0,q}T^*X\otimes G) & \xrightarrow{\left(\nabla^{\Lambda^{0,q}T^*X\otimes G}\right)''} & \Omega^{0,q}(\Lambda^{0,1}T^*X\otimes G) \\ \sim\big\downarrow & & \big\downarrow\Psi \\ \Omega^{n,q}(\widetilde{G}) & \xrightarrow{\left(\nabla^{\widetilde{G}}\right)'^*} & \Omega^{n-1,q}(\widetilde{G}) \end{array} \tag{3.64}$$

By (3.63) and (3.64) we have

$$|\big(\nabla^G\big)'u|^2 = |\big(\nabla^{\Lambda^{0,q}T^*X\otimes G}\big)'u|^2$$
$$|\big(\nabla^{\widetilde{G}}\big)'^*u|^2 = |\big(\nabla^{\Lambda^{0,q}T^*X\otimes G}\big)''u|^2$$

By (3.59) and (3.62) we obtain

$$2(R^G u,u) = \|\nabla^{\Lambda^{0,q}T^*X\otimes G}u\|^2 + \big([\sqrt{-1}R^G,\Lambda]u,u\big) + \big([\sqrt{-1}R^{\widetilde{G}},\Lambda]\widetilde{u},\widetilde{u}\big)$$

which proves (3.58) in the Kähler case.

If ω is not Kähler, we apply the Bochner-Kodaira-Nakano formula as given in (B.21). We replace thus in (3.60) $(\nabla^G)'u$ by $(\nabla^G)'u+Tu$ and add the term (Su,u). Accordingly we replace $(\nabla^G)'^*\widetilde{u}$ by $(\nabla^G)'^*\widetilde{u}+T^*\widetilde{u}$ and add $(S\widetilde{u},\widetilde{u})$ to (3.62). We define

$$V' : \Lambda^{0,q}T^*X\otimes G \longrightarrow \Lambda^{1,0}T^*X\otimes\Lambda^{0,q}T^*X\otimes G\,, \tag{3.65a}$$
$$V'' : \Lambda^{0,q}T^*X\otimes G \longrightarrow \Lambda^{0,1}T^*X\otimes\Lambda^{0,q}T^*X\otimes G\,. \tag{3.65b}$$

where $V'=T=[\Lambda,\partial\omega]$ and V'' is obtained by composing with $\sim$ and Ψ the morphism

$$T^* = [(\partial\omega)^*,\omega\wedge] : \Lambda^{n,q}T^*X\otimes\widetilde{G} \longrightarrow \Lambda^{n-1,q}T^*X\otimes\widetilde{G}$$

By (3.63) and (3.64) we have

$$\big|\big((\nabla^G)'+T\big)u\big|^2 = \big|\big(\nabla^{\Lambda^{0,q}T^*X\otimes G}\big)'u+V'u\big|^2$$
$$\big|\big((\nabla^{\widetilde{G}})'^*+T^*\big)u\big|^2 = \big|\big(\nabla^{\Lambda^{0,q}T^*X\otimes G}\big)''u+V''u\big|^2$$

and we finally define $V=V'\oplus V''$. Using these relations as in the Kähler case we get (3.58). □

3.2.3. Proof of Theorem 3.12. The proof consists in applying Theorem 3.11 together with Proposition 3.13 for $G=L^k\otimes F$. Let us examine the term

$$\begin{aligned} &\big([\sqrt{-1}R^{L^k\otimes F},\Lambda]u,u\big) + \big([\sqrt{-1}R^{\widetilde{(L^k\otimes F)}},\Lambda]\widetilde{u},\widetilde{u}\big) \\ &\quad = k\big([\sqrt{-1}R^L,\Lambda]u,u\big) + k\big([\sqrt{-1}R^L,\Lambda]\widetilde{u},\widetilde{u}\big) \\ &\qquad + \big([\sqrt{-1}R^F,\Lambda]u,u\big) + \big([\sqrt{-1}R^{\widetilde{F}},\Lambda]\widetilde{u},\widetilde{u}\big). \end{aligned} \tag{3.66}$$

We examine the term corresponding to $\sqrt{-1}R^L$. Let x be a fixed point of X. We can find complex coordinates $(z_1,\ldots,z_n)$ centered at x such that both η and $\sqrt{-1}R^L$ are diagonal at x and $\eta(x)$ is euclidian:

$$\begin{aligned} \eta &= \tfrac{\sqrt{-1}}{2}\sum dz^j\wedge d\overline{z}_j\,, \\ \sqrt{-1}R^L &= \tfrac{\sqrt{-1}}{2}\sum \alpha_j(x)\,dz^j\wedge d\overline{z}_j\,, \end{aligned} \tag{3.67}$$

where $\alpha_1(x),\dots,\alpha_n(x)$ are the eigenvalues of $\sqrt{-1}R^L$ with respect to η at x. Let (f_λ) an orthonormal frame of $L^k_x\otimes F_x$. For a (p,q)–form with values in $L^k\otimes F$, we write at x

$$u=\sum_{|I|=p,|J|=q,\lambda} u_{I,J,\lambda}\, f_\lambda\otimes dz_I\wedge d\bar z_J\,,\quad |u|^2=\sum_{|I|=p,|J|=q,\lambda}|u_{I,J,\lambda}|^2 \tag{3.68}$$

Let us define the hermitian endomorphism $\tau\in\operatorname{End}(\Lambda^{0,q}T^*X)$ by

$$\tau\Big(\sum_{|J|=q}u_J d\bar z_J\Big)=\sum_{|J|=q}(\alpha_{\complement J}-\alpha_J)u_J d\bar z_J\quad \text{at}\, x.$$

We shall also denote with τ the endomorphism $\tau\otimes\operatorname{Id}_{L^k\otimes F}$ which satisfies

$$\tau\Big(\sum_{|J|=q,\lambda}u_J f_\lambda\otimes d\bar z_J\Big)=\sum_{|J|=q}(\alpha_{\complement J}-\alpha_J)u_J f_\lambda\otimes d\bar z_J\quad \text{at}\, x. \tag{3.69}$$

By [**3**, (3.9)] we have

$$\big\langle[\sqrt{-1}R^L,\Lambda]u,u\big\rangle=\sum_{|I|=p,|J|=q,\lambda}(\alpha_I-\alpha_{\complement J})|u_{I,J,\lambda}|^2\,, \tag{3.70}$$

where the multiindex notation of (3.54) was used. Let $u\in\Omega^{0,q}(X,L^k\otimes F)$ written as in (3.68) ($I=\varnothing$). After (3.70)

$$\begin{aligned}\langle[\sqrt{-1}R^L,\Lambda]u,u\rangle&=-\sum\alpha_{\complement J}|u_{J,\lambda}|^2\\ \langle[\sqrt{-1}R^L,\Lambda]\widetilde u,\widetilde u\rangle&=\sum\alpha_J|u_{J,\lambda}|^2\end{aligned}$$

Therefore

$$\langle\tau u,u\rangle=-\langle[\sqrt{-1}R^L,\Lambda]u,u\rangle-\langle[\sqrt{-1}R^L,\Lambda]\widetilde u,\widetilde u\rangle=\sum_{|J|=q,\lambda}(\alpha_{\complement J}-\alpha_J)|u_{J,\lambda}|^2. \tag{3.71}$$

Let $\Xi\in\operatorname{End}(\Lambda^{0,q}T^*X\otimes F)$ be the hermitian endomorphism given by

$$\begin{aligned}\langle\Xi u,u\rangle=\langle[\sqrt{-1}R^F,\Lambda]u,u\rangle+\langle[\sqrt{-1}R^{\widetilde F},\Lambda]\widetilde u,\widetilde u\rangle\\ +\langle Su,u\rangle+\langle S\widetilde u,\widetilde u\rangle.\end{aligned} \tag{3.72}$$

Let Q_k be the quadratic form

$$\begin{aligned}Q_k(u)=\int_X\Big(\frac1k|\nabla^{L^k\otimes\Lambda^{0,q}T^*X\otimes F}u+Vu|^2-\langle\tau u,u\rangle+\frac1k\langle\Xi u,u\rangle\Big)dv_X\,,\\ u\in W_{1,0}(X,L^k\otimes\Lambda^{0,q}T^*X\otimes F).\end{aligned} \tag{3.73}$$

By (3.58) we have

$$Q_k(u)=\Big(\tfrac2k\Box^{F_k}u,u\Big),\quad u\in W_{1,0}(X,\Lambda^{0,q}T^*X\otimes L^k\otimes F). \tag{3.74}$$

Let us compare Q_k with $Q^\#_k$ defined as in (3.6), where τ is given by (3.69). Since V and Ξ are bounded operators acting only on the component $\Lambda^{0,q}T^*X\otimes F$, we infer that there exists a constant $C>0$ such that for all $\varepsilon>0$, there exists $k(\varepsilon)$ with

$$\begin{aligned}(1-\varepsilon)Q^\#_k(u)-C\varepsilon\|u\|^2\leqslant Q_k(u)\leqslant(1+\varepsilon)Q^\#_k(u)+C\varepsilon\|u\|^2,\quad u\in W_{1,0}(X,L^k\otimes\Lambda^{0,q}T^*X\otimes F)\\ \text{for all}\quad k\geqslant k(\varepsilon).\end{aligned} \tag{3.75}$$

The heat kernel of the quadratic form Q_k is denoted by $e_{Q_k}(t,x,y)$ and the heat kernel of $Q^\#_k$ is denoted by $e_k(t,x,y)$. From (3.75) we deduce that

$$e_{Q_k}(t,x,x)\sim e_k(t,x,x)\quad \text{as}\, k\longrightarrow\infty. \tag{3.76}$$

But by (3.74), Q_k is the quadratic form associated to $\frac{2}{k}\square^{F_k}$. We obtain therefore from (3.76)

$$\mathrm{Tr}_{\Lambda^{0,q}T^*X}\, e_k^{(0,q)}(t,x,x) = \mathrm{Tr}_{\Lambda^{0,q}T^*X}\, e_{Q_k}(t,x,x) \simeq \mathrm{Tr}_{\Lambda^{0,q}T^*X}\, e_k(t,x,x)\,, \quad k \longrightarrow \infty\,.$$

We apply now Theorem 3.11 to compute the asymptotic of $e_k(t,x,x)$. In Theorem 3.11 we replace F with $F\otimes\Lambda^{0,q}T^*X$ and take $\omega = \sqrt{-1}R^L$ and τ given by (3.69). By (3.67) we have

$$\omega = \sqrt{-1}R^L = \sum_{j=1}^{n} \alpha_j\, dx^j \wedge dy^j\,, \text{if } z^j = x^j + \sqrt{-1}y^j. \tag{3.77}$$

Let $2s = 2s(x) \leqslant 2n$ be the rank of the skew–symmetric 2–form $\omega(x)$. We order the eigenvalues (by a linear change of coordinates) such that

$$|\alpha_1(x)| \geqslant |\alpha_2(x)| \geqslant \ldots \geqslant |\alpha_s(x)| > 0 = \alpha_{s+1}(x) = \ldots = \alpha_n(x). \tag{3.78}$$

By a new linear change of coordinates we can put ω in the form (3.1):

$$\omega(x) = \sum_{j=1}^{s(x)} \omega_j(x)\, dx^j \wedge dx^{j+s}\,, \tag{3.79a}$$

$$\omega_j(x) = |\alpha_j(x)|\,,\ j = 1,2,\ldots,n\,. \tag{3.79b}$$

By (3.69) we have $\tau(x) - \sum_{j=1}^{s}\omega_j(x)\,\mathrm{Id}_F \leqslant 0$ for all $x\in X$. By (3.44) we obtain

$$k^{-n}e_k(t,x,x) - \frac{\exp(t\tau(x))}{(4\pi)^n t^{n-s}} \prod_{j=1}^{s} \frac{|\alpha_j(x)|}{\sinh|\alpha_j(x)|t} = o(k^n)\,, \quad k \longrightarrow \infty$$

uniformly on $(t,x) \in [t_0,k^{\varepsilon}]\times X$, $t_0 > 0$. By taking the trace $\mathrm{Tr}_{\Lambda^{0,q}T^*X}$ of the left–hand side and using

$$\mathrm{Tr}_{\Lambda^{0,q}T^*X}\exp(t\tau(x)) = \sum_{|J|=q} (\alpha_{\complement J}(x) - \alpha_J(x)) \quad \text{by (3.69)}$$

we get (3.56). The proof of Theorem 3.12 is achieved.

3.2.4. Asymptotic distribution of eigenvalues. Let X be a complex compact manifold with boundary. In Corollary 3.10 we determined the asymptotic behaviour of the eigenvalue distribution of the operator $\Delta_{k,\tau} = \frac{1}{k}\Delta^{F_k} - \tau$ for $k \longrightarrow \infty$. We specialize here the general results to the holomorphic case.

In the sequel $\frac{1}{k}\square^{F_k} = \frac{1}{k}\square^{F_k}_X$ is the Kodaira–Laplacian operator with Dirichlet boundary conditions. In Chapter 4, we'll aplly to good effect the result for for a relatively compact domain U and the Dirichlet Laplacian $\frac{1}{k}\square^{F_k}_U$.

For $j = 0,1,\ldots$, let $\lambda_j^{k,q}$ be the eigenvalues (counted with multiplicities) of $\frac{1}{k}\square^{F_k}$ acting on $(0,q)$-forms with values in $L^k\otimes F$.

3.14. PROPOSITION. *The following relation holds for all $t > 0$:*

$$\lim_{k\to\infty} k^{-n} \sum_{j=0}^{\infty} \exp(-2t\,\lambda_j^{k,q}) = \int_X \mathrm{Tr}_F\, e_\infty^{(0,q)}(t,x)\, dv_X\,. \tag{3.80}$$

where $e_\infty^{(0,q)}(t,x)$ is given by (3.55)*. The sequence of measures*

$$\mu_k^q = \frac{1}{k^{n/2}}\frac{d}{d\lambda}N(\lambda,\tfrac{2}{k}\square^{F_k}) = \frac{1}{k^n}\sum_{j=0}^{\infty}\delta(2\lambda_j^{k,q})\,, \tag{3.81}$$

converges weakly to a measure μ^q whose Laplace transform is $\int_X \mathrm{Tr}_F\, e_\infty^{(0,q)}(t,x)\, dv_X$.

PROOF. The first statement follows immediately from Theorem 3.12 by taking Tr_F in (3.56) and the second is the analog of Proposition 3.9. □

According to (3.1), (3.77), (3.78), and the definition (3.39) of the function $v_{\omega(x)}(\lambda)$ on $X\times\mathbb{R}$ transfoms to

$$v_{R^L(x)}(\lambda)=\frac{2^{s-2n}\pi^n}{\Gamma(n-s+1)}\;\;|\alpha_1(x)\cdots\alpha_s(x)|\sum_{(p_1,\dots,p_s)\in\mathbb{N}^s}\big[\lambda-\textstyle\sum(2p_j+1)|\alpha_j(x)|\big]_+^{n-s}\,, \tag{3.82}$$

with the convention $[\lambda]_+^0=0$ for $\lambda\leqslant 0$ and $[\lambda]_+^0=1$ for $\lambda>0$. We define $\overline{v}_{R^L}$ as in (3.41), by replacing in (3.82) the symbol $[\lambda]_+^0$ with $\{\lambda\}_+^0$, where $\{\lambda\}_+^0=0$ for $\lambda<0$, $\{\lambda\}_+^0=1$ for $\lambda\geqslant 0$.
Set

$$N^q(\lambda,\tfrac{1}{k}\Box^{F_k})=\operatorname{card}\big\{\,j\,:\,\lambda_j^{k,q}\leqslant\lambda\big\}=\dim\operatorname{Im}F_\lambda\big(\tfrac{1}{k}\Box^{F_k}\restriction_{L_2^{0,q}(X,L^k\otimes F)}\big) \tag{3.83}$$

3.15. THEOREM. *We have the following estimates:*

$$\liminf_{k\to\infty}k^{-n/2}N(\lambda,\tfrac{1}{k}\Box^{F_k})\geqslant(\operatorname{rank}F)\int_X\sum_{|J|=q}v_{R^L(x)}(2\lambda+\alpha_{\complement J}(x)-\alpha_J(x))\,dv_X\,. \tag{3.84a}$$

$$\limsup_{k\to\infty}k^{-n/2}N(\lambda,\tfrac{1}{k}\Box^{F_k})\leqslant(\operatorname{rank}F)\int_X\sum_{|J|=q}\overline{v}_{R^L(x)}(2\lambda+\alpha_{\complement J}(x)-\alpha_J(x))\,dv_X\,. \tag{3.84b}$$

There exists at most countable sets $\mathscr{D}^q\subset\mathbb{R}$ such that for $\lambda\in\mathbb{R}\smallsetminus\mathscr{D}^q$,

$$\lim_{k\to\infty}k^{-n}N^q(\lambda,\tfrac{1}{k}\Box^{F_k})=I^q(X,\lambda)\,, \tag{3.85}$$

$$I^q(X,\lambda)=(\operatorname{rank}F)\int_X\sum_{|J|=q}\overline{v}_{R^L(x)}(2\lambda+\alpha_{\complement J}(x)-\alpha_J(x))\,dv_X\,. \tag{3.86}$$

Moreover,

$$\lim_{\lambda\longrightarrow 0}I^q(X,\lambda)=I^q(X,0)=\frac{1}{n!}(\operatorname{rank}F)\int_{X(q)}(-1)^q\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n. \tag{3.87}$$

PROOF. The proof is parallel to Corollary 3.10. Formulas (3.84a), (3.84b) and (3.86) follow from Proposition 3.14 and are obtained by taking the inverse Laplace transform of $\int_X\operatorname{Tr}_F e_\infty^{(0,q)}(t,x)\,dv_X$. We use here $N(\lambda,\frac{1}{k}\Box^{F_k})=N(\frac{\lambda}{2},\frac{2}{k}\Box^{F_k})$ and the form (3.69) of the endomorphism τ.

We compute the bahaviour of $I^q(X,\lambda)$ for $\lambda\longrightarrow 0$. First it is clear that

$$\lim_{\lambda\longrightarrow+0}I^q(X,\lambda)=I^q(X,0)=(\operatorname{rank}F)\int_X\sum_{|J|=q}v_{R^L}(\alpha_{\complement J}-\alpha_J)\,dv_X\,.$$

Since $\alpha_{\complement J}-\alpha_J-\sum(2p_j+1)|\alpha_j|\leqslant 0$ for all $p\in\mathbb{N}^s$, it is clear that for a given $p\in\mathbb{N}^s$,

$$\big\{\alpha_{\complement J}-\alpha_J-\textstyle\sum(2p_j+1)|\alpha_j|\big\}_+^{n-s}$$

vanishes unless $s=n$ (i.e. $\sqrt{-1}R^L(x)$ is non–degenerate) and $\alpha_{\complement J}-\alpha_J-\sum(2p_j+1)|\alpha_j|=0$. The last equality holds if and only if $p_1=\cdots=p_n=0$, $\alpha_j<0$ for $j\in J$ and $\alpha_j>0$ for $j\in\complement J$. In particular, if $v_{R^L}(\alpha_{\complement J}-\alpha_J)\neq 0$, $\sqrt{-1}R^L$ is non–degenerate and has exactly q negative eigenvalues.

Let $X(q)$ be the set of points x of X such that $\sqrt{-1}R^L(x)$ is non–degenerate and has exactly q negative eigenvalues. For $x \in X(q)$, set $J(x) = \{j : \alpha_j(x) < 0\}$. For $|J| = q$ it follows

$$\overline{\nu}_{R^L}(\alpha_{\complement J} - \alpha_J) = (2\pi)^{-n}|\alpha_1 \cdots \alpha_n| = (-1)^q(2\pi)^{-n}\alpha_1 \cdots \alpha_n\,, \tag{3.88}$$

for $J = J(x)$ and

$$\overline{\nu}_{R^L}(\alpha_{\complement J} - \alpha_J) = 0 \quad \text{for } J \neq J(x). \tag{3.89}$$

By (3.88), (3.89) and the relation

$$\alpha_1 \cdots \alpha_n \, dv_X = (\sqrt{-1}R^L)^n$$

we get (3.87). □

To shorten the notation we set

$$\overline{\nu}^q_{R^L}(\lambda, x) = \sum_{|J|=q} \overline{\nu}_{R^L(x)}(2\lambda + \alpha_{\complement J}(x) - \alpha_J(x)) \tag{3.90}$$

so that

$$I^q(X,\lambda) = (\operatorname{rank} F) \int_X \overline{\nu}^q_{R^L}(\lambda, x)\, dv_X\,. \tag{3.91}$$

3.16. REMARK. We can determine in the same way the asymptotic behavior of the spectrum of $\frac{1}{k}\square^{F_k}$ acting on (p,q)–forms with values in $L^k \otimes F$. For this purpose we identify $\Lambda^{p,q}T^*X \otimes L^k \otimes F$ to $\Lambda^{0,q}T^*X \otimes L^k \otimes (\Lambda^{0,p}T^*X \otimes F)$, that is, we replace F by $\Lambda^{0,p}T^*X \otimes F$. Therefore Theorem 3.15 remains true by replacing $\operatorname{rank} F$ with $\binom{n}{p} \operatorname{rank} F$. Set

$$N^{p,q}(\lambda, \tfrac{1}{k}\square^{F_k}) = \dim \operatorname{Im} F_\lambda(\tfrac{1}{k}\square^{F_k} \restriction_{L^{p,q}_2(X, L^k \otimes F)}) \tag{3.92}$$

Then there exists at most countable sets $\mathscr{D}^{p,q} \subset \mathbb{R}$ such that for $\lambda \in \mathbb{R} \smallsetminus \mathscr{D}^q$,

$$\lim_{k\to\infty} k^{-n} N^{p,q}(\lambda, \tfrac{1}{k}\square^{F_k}) = I^{p,q}(X,\lambda)\,, \tag{3.93}$$

$$I^{p,q}(X,\lambda) = \binom{n}{p} I^q(X,\lambda)\,. \tag{3.94}$$

Moreover,

$$\lim_{\lambda \longrightarrow 0} I^{p,q}(X,\lambda) = I^{p,q}(X,0) = \frac{1}{n!}\binom{n}{p}(\operatorname{rank} F) \int_{X(q)} (-1)^q \Big(\frac{\sqrt{-1}}{2\pi} R^L\Big)^n. \tag{3.95}$$

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CHAPTER 4

Global asymptotic Morse inequalities

In this chapter we pass from local Morse inequalities to global ones, that is, we find bounds for the cohomology of complex manifolds.

In Section 4.1 we prove Morse inequalities for the L^2–cohomology in a quite general contex, namely, when the fundamental estimate (4.1) holds. Almost all global Morse inequalities treated here can be reduced to this situation (with the exception of the covering manifolds in Chapter 5; but there we can analyse a fundamental domain).

The main idea, going back to Witten and used by Nadel–Tsuji and Bouche, is to show that the spectral spaces of the Laplacian, corresponding to small eigenvalues, inject in the spectral spaces of the Laplacian with Dirichlet boundary conditions on a smooth relatively domain containing the spectrum for the Dirichlet–Laplacian were calculated in Theorem 3.15.

The first particular case we treat is of course the case of compact manifolds. We obtain then Demailly's original Morse inequalities and as a corollary the solution of the Grauert–Riemenschneider conjecture in Section 4.2. In this section we collected background material about vanishing theorems, the Kodaira embedding and Moishezon manifolds.

In Section 4.3 we specialize the abstract Morse inequalities to a geometric situation and we assume that we have a line bundle which is uniformly positive on a complete hermitian manifold. An immediate consequence is the Nadel–Tsuji cohomology estimate on complete Kähler manifolds with negative Ricci curvature.

The Morse inequalities for uniformly positive line bundles are applied in Sections 4.3.2 and 4.4 to extend Demailly's criterion for compact complex spaces with isolated singularities and prove Theorem 1.10 from the Introduction.

We turn next to Zariski open sets in compact complex spaces possesing a singular hermitian line bundle in the sense of currents. Following Takayama we work on the regular part of the space and of the curvature current and introduce the generalized Poincaré metric and change the hermitian metric on the bundle, which implies the fundamental estimate. We obtain in this way a proof of the Shiffman–Ji–Bonavero criterion.

Finally, Section 4.6 treats the q–convex and weakly 1–complete manifolds. We will revisit some of these topics in Chapter 7 and treat then from the point of view of Bergman kernels.

4.1. Abstract Morse inequalities for the L^2–cohomology

We shall examine a general situation which permits to prove asymptotic Morse inequalities for the L^2–Dolbeault cohomology groups. Let (X,ω) be a hermitian manifold, (L,h^L) and (F,h^F) holomorphic hermitian vector bundles of $\operatorname{rank} L = 1$, $\operatorname{rank} F = r$. We postulate a general estimate for the quadratic form of the $\overline{\partial}$–laplacian $\square^{F_k}$ acting on the bundle $F_k := L^k \otimes F$, which implies estimates from above of the spectral function (4.2). Using the estimate from below of lemma 4.4 we prove in Theorems 4.6 and 4.7 the abstract Morse inequalities.

4.1.1. The Fundamental Estimate. We consider the weak maximal extension $\overline{\partial}^{F_k}$: $\boldsymbol{L}_2^{p,q}(X,L^k\otimes F)\longrightarrow \boldsymbol{L}_2^{p,q+1}(X,L^k\otimes F)$, which is a closed densely defined operator with domain $\operatorname{Dom}(\overline{\partial}^{F_k})$ (consisting of elements u such that $\overline{\partial}^{F_k}u$ calculated in distributional sense is in L^2). We denote by $\overline{\partial}^{F_k *}$ the Hilbert-space adjoint of $\overline{\partial}^{F_k}$.

4.1. FUNDAMENTAL ESTIMATE. We say that the fundamental estimate holds in bidigree (p,q) for forms with values in $L^k\otimes F$ if there exists a compact $K\subset X$ and $C>0$ such that for sufficiently large k we have

$$\begin{aligned}&\|u\|^2\leqslant \frac{C}{k}\big(\|\overline{\partial}^{F_k}u\|^2+\|\overline{\partial}^{F_k *}u\|^2\big)+C\int_K |u|^2\,dv_X\,,\\ &u\in \operatorname{Dom}(\overline{\partial}^{F_k})\cap\operatorname{Dom}(\overline{\partial}^{F_k *})\cap \boldsymbol{L}_2^{p,q}(X,L^k\otimes F).\end{aligned}\tag{4.1}$$

K is called the exceptional compact of the estimate.

The estimate (4.1) is of course a variant with parameters of the fundamental estimate (A.33). It affords to compare the spectral spaces of the laplacian on X and the spectral spaces of the Dirichlet laplacian $\square^{F_k}_U$ on a relatively compact domain U containing K.

First, we introduce the functional spaces and operators. We consider the self-adjoint extension of the $\overline{\partial}$-laplacian given by the Gaffney extension (cf. (A.24)):

$$\begin{aligned}\operatorname{Dom}(\square^{F_k}):=\Big\{u\in \operatorname{Dom}(\overline{\partial}^{F_k})\cap\operatorname{Dom}(\overline{\partial}^{F_k *})\,:\,\overline{\partial}^{F_k}u\in\operatorname{Dom}(D^*)\,,\,\overline{\partial}^{F_k *}u\in\operatorname{Dom}(\overline{\partial}^{F_k})\Big\}\,,\\ \square^{F_k}u=\overline{\partial}^{F_k}\overline{\partial}^{F_k *}u+\overline{\partial}^{F_k *}\overline{\partial}^{F_k}u\ \text{ for } u\in\operatorname{Dom}(\square^{F_k})\,.\end{aligned}\tag{4.2}$$

The use of the Gaffney extension permits to treat at the same time the case of a complete manifold and the case of a manifold with boundary. In the first case the $\overline{\partial}$-laplacian is essentially self-adjoint by Corollary A.14 and the Gaffney extension coincides with its unique self-adjoint extension. If the manifold has non-empty boundary, the Gaffney extension coincides with the $\overline{\partial}$-Neumann laplacian by Proposition A.22.

We normalize the operator (4.2) by $\frac{1}{k}\square^{F_k}$. According to Proposition A.18 the quadratic form associated to $\frac{1}{k}\square^{F_k}$ is

$$Q_k(u,u)=\frac{1}{k}\left(\|\overline{\partial}^{F_k}u\|^2+\|\overline{\partial}^{F_k *}u\|^2\right),\quad \operatorname{Dom}Q_k=\operatorname{Dom}(\overline{\partial}^{F_k})\cap\operatorname{Dom}(\overline{\partial}^{F_k *})\,.\tag{4.3}$$

Let $\{E_\lambda(\frac{1}{k}\square^{F_k})\}_\lambda$ be the spectral resolution of $\frac{1}{k}\square^{F_k}$ and by $\mathscr{E}(\lambda,\frac{1}{k}\square^{F_k})=\operatorname{Im}E_\lambda(\frac{1}{k}\square^{F_k})$ the corresponding spectral spaces (compare definition A.28). All these objects decompose in a direct sum according to the decomposition of forms after bidegree.

Let us fix an open, relatively compact neighbourhood U of K with smooth boundary. We consider the laplacian with Dirichlet boundary conditions on U (see Example A.16) associated to $\overline{\partial}^{F_k}\vartheta^{F_k}+\vartheta^{F_k}db^{F_k}$, denoted $\frac{1}{k}\square^{F_k}_U$. Let $\{E_\lambda(\frac{1}{k}\square^{F_k}_U)\}_\lambda$ be its spectral resolution and set:

$$\begin{aligned}\mathscr{E}(\lambda,\tfrac{1}{k}\square^{F_k}_U))&=\operatorname{Im}E_\lambda(\tfrac{1}{k}\square^{F_k}_U)\\ N^{\bullet,\bullet}(\lambda,\tfrac{1}{k}\square^{F_k})&=\dim\mathscr{E}^{\bullet,\bullet}(\lambda,\tfrac{1}{k}\square^{F_k})\\ N^{\bullet,\bullet}(\lambda,\tfrac{1}{k}\square^{F_k}_U)&=\dim\mathscr{E}^{\bullet,\bullet}(\lambda,\tfrac{1}{k}\square^{F_k}_U)\end{aligned}\tag{4.4}$$

One of the tools fo the proof of the Morse inequalities is to estimate $N^{\bullet,\bullet}(\lambda,\frac{1}{k}\square^{F_k})$ from above and from below. We do this by a localization procedure as in Section 2.2.2 thanks to a remark of Witten (see [**47**, p. 666]): the L^2 norm of the eigenforms of $\frac{1}{k}\square^{F_k}$

on (p,q)–forms concentrates asymptotically for $k \longrightarrow \infty$ on the critical set $X(q)$. In the original setting of classical Morse theory the rôle of the curvature is played by the hessian of a Morse function f and the eigenforms of the modified laplacian $\Delta_t = (d_t + d_t^*)^2$ where $d_t = e^{-tf} d e^{tf}$ and $t > 0$, concentrate near the critical points of f as $t \longrightarrow \infty$. This idea was introduced in the complex geometry setting by Demailly [**12**], Nadel-Tsuji [**31**] and Bouche [**8**].

We show now that if the fundamental estimate holds, the essential spectrum of $\frac{1}{k}\square^{F_k}$ does not contain the open interval $[0, \frac{1}{C_0}]$. Moreover, we can compare the counting function on this interval with the counting function of the same operator considered with Dirichlet boundary conditions on U. For the following compare [**31**, Proposition 2.1] and [**8**, Théorème 2.1].

Consider a smooth function ρ on X, $0 \leqslant \rho \leqslant 1$ such that $\rho = 1$ on K and $\rho = 0$ on $X \smallsetminus \overline{U}$. Set $C_1 = \sup |d\rho|^2$

4.2. THEOREM. *Assume the fundamental estimate* (4.1) *holds in bidegree* (p,q). *Then*

(1) $\frac{1}{k}\square^{F_k}$ *on* $L_2^{p,q}(X, L^k \otimes F)$*, has discrete spectrum in* $[0, 1/C_0]$ *for large k.*
(2) *There exists a constant* C_2 *depending only on* C_0 *and* C_1 *such that for* $\lambda < 1/(2C_0)$ *the maps*

$$\begin{aligned} &\mathscr{E}^{p,q}(\lambda, \tfrac{1}{k}\square^{F_k}) \longrightarrow \mathscr{E}^{p,q}(3C_0\lambda + C_2 k^{-1}, \tfrac{1}{k}\square_U^{F_k}) \\ &u \longrightarrow E_{3C_0\lambda + C_2 k^{-1}}(\tfrac{1}{k}\square_U^{F_k})(\rho u) \end{aligned} \tag{4.5}$$

are injective for k sufficiently large. In particular

$$N^{p,q}(\lambda, \tfrac{1}{k}\square^{F_k}) \leqslant N^{p,q}(3C_0\lambda + C_2 k^{-1}, \tfrac{1}{k}\square_U^{F_k}), \quad \lambda < 1/(2C_0)\,, k \gg 1 \tag{4.6}$$

PROOF. (1) By the decomposition principle A.32, $\frac{1}{k}\square^{F_k}$ has the same essential spectrum as the Dirichlet laplacian $\frac{1}{k}\square^{F_k}_{X\smallsetminus U}$, where U is a compact manifold with boundary containing K. Let $Q_{k,M\smallsetminus U}$ be the associated Dirichlet form.
The fundamental estimate (4.1) shows then that $Q_{k,X\smallsetminus U}(u) \geqslant \frac{1}{C_0}\|u\|^2$, $u \in \operatorname{Dom}(Q_{k,M\smallsetminus U})$, since $\operatorname{Dom}(Q_{k,M\smallsetminus U}) \subset \operatorname{Dom}(Q_k)$. It follows that $\frac{1}{k}\square_U^{F_k}$ has no essential spectrum in $[0, \frac{1}{C_0}]$ and $\frac{1}{k}\square^{F_k}$ has the same property.

(2) We need the following elementary estimate.

4.3. LEMMA. *Let* $0 \leqslant \rho \leqslant 1$ *be a smooth function with bounded gradient. Then*

$$\|\overline{\partial}^{F_k}(\rho u)\|^2 + \|\overline{\partial}^{F_k *}(\rho u)\|^2 \leqslant \frac{3}{2}(\|\overline{\partial}^{F_k} u\|^2 + \|\overline{\partial}^{F_k *} u\|^2) + 6 \sup |d\rho|^2 \cdot \|u\|^2 \tag{4.7}$$

for all $u \in \operatorname{Dom}(\overline{\partial}^{F_k}) \cap \operatorname{Dom}(\overline{\partial}^{F_k *})$.

PROOF. By Leibniz formula

$$\|\overline{\partial}^{F_k}(\rho u)\|^2 + \|\overline{\partial}^{F_k *}(\rho u)\|^2 = \|\rho\overline{\partial}^{F_k} u + \overline{\partial}\rho \wedge u\|^2 + \|\rho\overline{\partial}^{F_k *} u + i(\partial\rho)u\|^2$$

Using the inequality $(x+y)^2 \leqslant \frac{3}{2}x^2 + 3y^2$ together with the triangle inequality we obtain (4.7). ☐

Let $u \in \mathscr{E}^{p,q}(\lambda, \frac{1}{k}\square^{F_k})$, $\lambda < 1/(2C_0)$. Then $\|\overline{\partial}^{F_k} u\|^2 + \|\overline{\partial}^{F_k *} u\|^2 \leqslant \lambda k \|u\|^2$. Plugging this in the relation (4.1) we get

$$\|u\|^2 \leqslant 2C_0 \int_K |u|^2 dv_X \tag{4.8}$$

Moreover by (4.7) we have

$$\begin{aligned} Q_{k,U}(\rho u,\rho u) &\leqslant \frac{3}{2}Q_k(u)+\frac{6C_1}{k}\|u\|^2 \\ &\leqslant \Big(\frac{3}{2}\lambda+\frac{6C_1}{k}\Big)\|u\|^2 \\ &\leqslant \Big(\frac{3}{2}\lambda+\frac{6C_1}{k}\Big)\cdot 2C_0\int_K |u|^2\,dv_X \\ &\leqslant \Big(3C_0\lambda+\frac{12C_0C_1}{k}\Big)\int_U |u|^2\,dv_X\,. \end{aligned} \tag{4.9}$$

We set $C_2=12C_0C_1$. The inequality (4.9) shows that $\rho u\in\mathscr{E}^{p,q}(3C_0\lambda+C_2k^{-1},\frac{1}{k}\square^{F_k}_U$ and if $E_{3C_0\lambda+C_2k^{-1}}(\frac{1}{k}\square^{F_k}_U)(\rho u)=0$ implies $\rho u=0$. Since $\rho=1$ on K, it follows $u=0$ on K and by (4.8) we infer $u=0$. Thus (4.5) is injective. □

As for a lower bound of the counting function $N^{p,q}(\lambda,\frac{1}{k}\square^{F_k})$ we have a general result which does not depend on the fundamental estimate.

4.4. LEMMA. *The following estimate from below holds:*

$$N^{p,q}(\lambda,\tfrac{1}{k}\square^{F_k})\geqslant N^{p,q}(\lambda,\tfrac{1}{k}\square^{F_k}_U)\,. \tag{4.10}$$

PROOF. This is an immediate consequence of the Glazman's lemma A.29. Let P be a self-adjoint positive operator on a Hilbert space $\mathscr{H}$. Then the spectrum distribution function $N(\lambda,P):=\dim\operatorname{Im}E_\lambda(P)$ satisfies:

$$N(\lambda,P)=\sup\Big\{\dim V\mid V\text{ closed }\subset\operatorname{Dom}(Q),\,Q(f,f)\leqslant\lambda\|f\|^2,\forall f\in V\Big\} \tag{4.11}$$

where Q is the quadratic form of P. The Lemma follows by the variational principle and the simple remark that $\operatorname{Dom}(Q_k)\supset\operatorname{Dom}(Q_{k,U})$. Indeed, let us denote by $\lambda_0\leqslant\lambda_1\leqslant\dots$ the spectrum of $\frac{1}{k}\square^{F_k}_U$ acting on (p,q)–forms. Let $\{e_i\}_i$ be an orthonormal basis which consists of eigenforms corresponding to the eigenvalues $\{\lambda_i\}_i$; if we let $\widetilde{e}_i=0$ on $X\smallsetminus U$ and $\widetilde{e}_i=e_i$ on U, $\widetilde{e}_i\in\operatorname{Dom}(Q_k)$ and $Q_k(\widetilde{e}_i,\widetilde{e}_j)=\delta_{i,j}\lambda_i$. Let Φ^0_λ be the subspace spanned by $\{e_i:\lambda_i\leqslant\lambda\}$ in $\boldsymbol{L}^{p,q}_2(U,L^k\otimes F)$ and Φ_λ the closed subspace spanned by $\{\widetilde{e}_i:\lambda_i\leqslant\lambda\}$ in $\boldsymbol{L}^{p,q}_2(X,L^k\otimes F)$. Then $\dim\Phi_\lambda=\dim\Phi^0_\lambda=N(\lambda,\frac{1}{k}\square^{F_k}_U)$. If f is a linear combination of $\{\widetilde{e}_i:\lambda_i\leqslant\lambda\}$, $Q_k(f,f)\leqslant\lambda\|f\|^2$ and, as $\operatorname{Dom}(Q_k)$ is complete in the graph norm, we obtain $\Phi_\lambda\subset\operatorname{Dom}(Q_k)$ and $Q_k(f,f)\leqslant\lambda\|f\|^2$, $f\in\Phi_\lambda$. The variational principle implies now the Lemma. □

4.1.2. Estimate of the cohomology in bidigree $(0,0)$ and $(n,0)$. In the sequel we study the L^2 cohomology in bidegree $(p,0)$ where $p=0$ and $p=n=\dim X$. In the next subsection we will study the general Morse inequalities, but given their importance we prefer to treat this case separately. Our assumption is that the fundamental estimate holds in bidigree $(p,1)$. We set

$$H^{p,0}_{(2)}(X,L^k\otimes F)=\{u\in\boldsymbol{L}^{p,0}_2(X,L^k\otimes F):\overline{\partial}^{F_k}u=0\}\,, \tag{4.12}$$

is the space of $(p,0)$–forms with values in $L^k\otimes F$ which are L^2 with respect to ω, h^L and h^E.

We start with a lemma which gives a lower bound of $\dim H^{p,0}_{(2)}(M,L^k\otimes F)$.

4.5. LEMMA. *Asssume that the fundamental estimate holds in bidigree* $(p,1)$. *For* $\lambda < 1/(2C_0)$ *and sufficiently large k we have*

$$\dim H^{p,0}_{(2)}(X, L^k \otimes F) \geqslant N^{p,0}(\lambda, \tfrac{1}{k}\Box^{F_k}) - N^{p,1}(\lambda, \tfrac{1}{k}\Box^{F_k}). \tag{4.13}$$

PROOF. Since $\Box^{F_k}$ commutes with $\overline{\partial}^{F_k}$ it follows that the spectral projections of $\frac{1}{k}\Box^{F_k}$ commute with $\overline{\partial}^{F_k}$ too, showing thus $\overline{\partial}^{F_k}\mathscr{E}^{p,0}(\lambda, \frac{1}{k}\Box^{F_k}) \subset \mathscr{E}^{p,1}(\lambda, \frac{1}{k}\Box^{F_k})$ and therefore we have the bounded operator $\overline{\partial}^{F_k}_\lambda : \mathscr{E}^{p,0}(\lambda, \frac{1}{k}\Box^{F_k}) \longrightarrow \mathscr{E}^{p,1}(\lambda, \frac{1}{k}\Box^{F_k})$ where $\overline{\partial}^{F_k}_\lambda$ denotes the restriction of $\overline{\partial}$ (by the definition of $\mathscr{E}^{p,0}(\lambda, \frac{1}{k}\Box^{F_k})$, $\overline{\partial}^{F_k}_\lambda$ is bounded by $\sqrt{k\lambda}$). Thus $N^{p,0}(\lambda, \frac{1}{k}\Box^{F_k}) = \dim\ker\overline{\partial}^{F_k}_\lambda + \dim\operatorname{Im}\overline{\partial}^{F_k}_\lambda$. By Theorem 4.2, $N^{p,1}(\lambda, \frac{1}{k}\Box^{F_k})$ is finite dimensional. Obviously $\dim\operatorname{Im}\overline{\partial}^{F_k}_\lambda \leqslant N^1(\lambda, \frac{1}{k}\Box^{F_k})$ and $\operatorname{Ker}\overline{\partial}^{F_k}_\lambda = H^{p,0}_{(2)}(M, L^k \otimes F)$ whereby the desired inequality. $\Box$

Note that both sides of (4.13) may be infinite. This happens if $\dim H^{p,0}(X, L^k \otimes F) = \infty$.

4.6. THEOREM. *Let* (X, ω) *be an n–dimensional complete hermitian manifold such that the fundamental estimate holds in bidigree* $(p,1)$. *Let U be any open set with smooth boundary,* $K \Subset U \Subset X$. *Then, for* $k \longrightarrow \infty$,

$$\dim H^{p,0}_{(2)}(X, L^k \otimes F) \geqslant \frac{k^n}{n!}\int_{U(\leqslant 1, h^L)} \left(\frac{\sqrt{-1}}{2\pi}R^L\right)^n + o(k^n), \tag{4.14}$$

where $U(\leqslant 1, h)$ *is the subset of U where* $\sqrt{-1}R^L)$ *is non–degenerate and has at most one negative eigenvalue.*

PROOF. Let us consider $\lambda < 1/(2C_0)$ and $\delta > 0$. For $k > C_2/\delta$ we have

$$N^{p,1}(3C_0\lambda + C_2k^{-1}, \tfrac{1}{k}\Box^{F_k}_U) \leqslant N^{p,1}(3C_0\lambda + \delta, \tfrac{1}{k}\Box^{F_k}_U) \tag{4.15}$$

The asymptotic of the left–hand side is computed in Theorem 3.15. By (4.6), (4.15) and (3.84b)

$$\begin{aligned} \limsup_{k\longrightarrow\infty} k^{-n}N^{p,1}(\lambda, \tfrac{1}{k}\Box^{F_k}) &\leqslant \limsup_{k\longrightarrow\infty} k^{-n}N^{p,1}(3C_0\lambda + \delta, \tfrac{1}{k}\Box^{F_k}_U) \\ &\leqslant (\operatorname{rank} F)\int_U \overline{\nu}^1_\omega(3C_0\lambda + \delta)\, dv_X \end{aligned} \tag{4.16}$$

Since $\overline{\nu}^1_\omega$ is right–continuous in λ and bounded on U, we can use the Lebesgue dominated convergence theorem to let $\delta \longrightarrow 0$ Hence

$$\limsup_{k\longrightarrow\infty} k^{-n}N^1(\lambda, \tfrac{1}{k}\Box^{F_k}) \leqslant (\operatorname{rank} F)\int_U \overline{\nu}^1_\omega(3C_0\lambda)\, dv_X = I^1(U, 3C_0\lambda) \tag{4.17}$$

On the other hand, by (4.10) and (3.85) we obtain for $\lambda \notin \mathscr{D}^0$

$$\liminf_{k\longrightarrow\infty} k^{-n}N^{p,0}(\lambda, \tfrac{1}{k}\Box^{F_k}) \geqslant \liminf_{k\longrightarrow\infty} k^{-n}N^{p,0}(\lambda, \tfrac{1}{k}\Box^{F_k}_U) = I^0(U, \lambda) \tag{4.18}$$

The estimates (4.16), (4.18) and (4.13) imply

$$\liminf_{k\longrightarrow\infty} k^{-n}\dim H^{p,0}_{(2)}(X, L^k \otimes F) \geqslant I^0(U, \lambda) - I^1(U, 3C_0\lambda).$$

for $\lambda \notin \mathscr{D}^0$. By passing to the limit $\lambda \to 0$ through values $\lambda \notin \mathscr{D}^0$ we obtain (4.14) invoking (3.87). $\Box$

4.1.3. Estimates for the cohomology in arbitrary degree. We study now the general Morse inequalities for cohomology groups in arbitrary bidegree.

4.7. THEOREM. *Let X be an n-dimensional hermitian manifold and (L,h^L), (F,h^F) be holomorphic hermitian bundles over X of* rank 1 *and r, respectively.*

(i) *Assume that there exists an integer $0\leqslant m\leqslant n$ such that the fundamental estimate holds for (p,q)–forms with $q\geqslant m$. Let U be a relatively compact open set with smooth boundary such that $K\Subset U$. As $k\longrightarrow\infty$, the following strong Morse inequalities hold for every $q\geqslant m$:*

$$\sum_{j=q}^{n}(-1)^{j-q}\dim H^{p,j}_{(2)}(X,L^k\otimes F)\leqslant r\binom{n}{p}\frac{k^n}{n!}\int_{U(\geqslant q)}(-1)^q\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n) \tag{4.19}$$

In particular, we get the weak Morse inequalities:

$$\dim H^{q}_{(2)}(X,L^k\otimes F)\leqslant r\binom{n}{p}\frac{k^n}{n!}\int_{U(q)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n). \tag{4.20}$$

(ii) *Assume that there exists an integer $0\leqslant m\leqslant n$ such that the fundamental estimate holds for (p,q)–forms with $q\leqslant m$. Let U be a relatively compact open set with smooth boundary such that $K\Subset U$. As $k\longrightarrow\infty$, the following strong Morse inequalities hold for every $q\leqslant m$:*

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^{p,j}_{(2)}(X,L^k\otimes F)\leqslant r\binom{n}{p}\frac{k^n}{n!}\int_{U(\leqslant q)}(-1)^q\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n) \tag{4.21}$$

In particular, we get the weak Morse inequalities:

$$\dim H^{q}_{(2)}(X,L^k\otimes F)\leqslant r\binom{n}{p}\frac{k^n}{n!}\int_{U(q)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n). \tag{4.22}$$

For the proof we use the same steps as for compact manifolds.

Hodge theory for the L^2–Dolbeault complex. Let $\big(\mathrm{Dom}(\overline{\partial}^{F_k})\cap \boldsymbol{L}^{p,\bullet}_2(X,L^k\otimes F),\overline{\partial}^{F_k}\big)$ be the $\overline{\partial}$–complex of densely defined closed operators, shortly $(\boldsymbol{L}^{p,\bullet}_2(X,L^k\otimes F),\overline{\partial}^{F_k})$. Let

$$\mathscr{E}^{p,\bullet}(\lambda,\tfrac{1}{k}\square^{F_k})=\mathrm{Im}E_\lambda\big(\tfrac{1}{k}\square^{F_k}\big)\cap\boldsymbol{L}^2(X,L^k\otimes F)\,.$$

Since $\overline{\partial}^{F_k}$ commutes to $\frac{1}{k}\square^{F_k}$, it commutes to the spectral projections i.e.

$$\overline{\partial}^{F_k}E_\lambda\big(\tfrac{1}{k}\square^{F_k}\big)=E_\lambda\big(\tfrac{1}{k}\square^{F_k}\big)\overline{\partial}^{F_k}.$$

We obtain therefore a subcomplex

$$\Big(\mathscr{E}^{p,\bullet}\big(\lambda,\tfrac{1}{k}\square^{F_k}\big),\overline{\partial}^{F_k}\Big)\longhookrightarrow\Big(\mathrm{Dom}(\overline{\partial}^{F_k})\cap\boldsymbol{L}^{p,\bullet}_2(X,L^k\otimes F),\overline{\partial}^{F_k}\Big) \tag{4.23}$$

The cohomology of this complex is denoted by $H^\bullet\Big(\mathscr{E}^{p,\bullet}\big(\lambda,\tfrac{1}{k}\square^{F_k}\big),\overline{\partial}^{F_k}\Big)$.

4.8. PROPOSITION. *Let $\lambda\geqslant 0$. If the fundamental estimate holds for $q\geqslant m$ (resp. $q\leqslant m$),*

$$H^q\Big(\mathscr{E}^{p,\bullet}\big(\lambda,\tfrac{1}{k}\square^{F_k}\big)\Big)\cong\mathscr{H}^{p,q}(X,L^k\otimes F)\cong H^{p,q}_{(2)}(X,L^k\otimes F)$$

for $q\geqslant m$ (resp. $q\leqslant m$).

PROOF. By Theorem A.26 the strong Hodge decomposition holds in bidigrees (p,q), $q \geqslant m$. Let us restrict the Hodge decomposition to the complex (4.23):

$$\mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big) = \mathscr{H}^{p,q}(X, L^k \otimes F) \oplus \big(\operatorname{Im}(\overline{\partial}^{F_k}) \cap \mathscr{E}^{p,q}(\lambda, \tfrac{1}{k}\square^{F_k})\big) \\ \oplus \big(\operatorname{Im}(\overline{\partial}^{F_k *}) \cap \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)\big).$$

Due to the commutation of $\overline{\partial}^{F_k}$ to the spectral projections we see that

$$\operatorname{Im}(\overline{\partial}^{F_k}) \cap \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big) = \operatorname{Im}(\overline{\partial}^{F_k} \restriction \mathscr{E}^{p,q}(\lambda, \tfrac{1}{k}\square^{F_k}))\,.$$

On the other hand $\overline{\partial}^{F_k *}$ commutes to the spectral projections too, so we easily obtain $\operatorname{Im}(\overline{\partial}^{F_k *}) \cap \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big) = \operatorname{Im}(\overline{\partial}^{F_k *} \restriction \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big))$. Therefore

$$\mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big) = \mathscr{H}^{p,q}(X, L^k \otimes F) \oplus \operatorname{Im}\big(\overline{\partial}^{F_k} \restriction \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)\big) \\ \oplus \operatorname{Im}\big(\overline{\partial}^{F_k *} \restriction \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)\big)\,,$$

and

$$\operatorname{Ker}(\overline{\partial}^{F_k} \restriction \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)) = \mathscr{H}^{p,q}(X, L^k \otimes F) \oplus \operatorname{Im}\big(\overline{\partial}^{F_k} \restriction \mathscr{E}^{p,q}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)\big)$$

It follows that $H^q\Big(\mathscr{E}^{p,\bullet}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)\Big) \cong \mathscr{H}^{p,q}(X, L^k \otimes F)$. By Theorem A.26, we have also

$$\mathscr{H}^{p,q}(X, L^k \otimes F) \cong H^{p,q}_{(2)}(X, L^k \otimes F).$$

which finishes the proof. □

Algebraic Morse Inequalities. We also need a variant of the algebraic lemma 2.7.

4.9. LEMMA. *Let*

$$0 \longrightarrow V^0 \xrightarrow{\;d^0\;} V^1 \xrightarrow{\;d^1\;} \dots \xrightarrow{\;d^{n-1}\;} V^n \longrightarrow 0$$

be a complex of vector spaces. Set $\dim D^q = c^q$ *and* $h^q = \dim H^q(V^0)$.

(*i*) *If* $c^q < \infty$ *for* $q \geqslant m$, *we have*

$$\sum_{j=q}^{n} (-1)^{j-q} h^j \leqslant \sum_{j=q}^{n} (-1)^{j-q} c^j\,, \textit{for}\, q \geqslant m.$$

(*ii*) *If* $c^q < \infty$ *for* $q \leqslant m$, *we have*

$$\sum_{j=0}^{q} (-1)^{q-j} h^j \leqslant \sum_{j=0}^{q} (-1)^{q-j} c^j\,, \textit{for}\, q \leqslant m.$$

PROOF. Set $z^j = \dim \operatorname{Ker} d^j$, $r^j = \dim \operatorname{Im} d^j$. Then $c^j = z^j + r^j$, $h^j = z^j - r^{j-1}$ and $\sum_{j=q}^{n} (-1)^{j-q} h^j = -r^{q-1} + \sum_{j=q}^{n} (-1)^{j-q} c^j$ and $\sum_{j=0}^{q} (-1)^{q-j} h^j = r^q + \sum_{j=0}^{q} (-1)^{q-j} c^j$. Now the proof follows in the same way as the proof of Lemma 2.7. □

PROOF OF THE THEOREM 4.7. We prove only part (i) since (ii) is similar. Lemma 4.9 applied to the complex (4.23) delivers

$$\sum_{j=q}^{n} (-1)^{j-q} \dim H^j\Big(\mathscr{E}^{p,\bullet}\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)\Big) \leqslant \sum_{j=q}^{n} (-1)^{j-q} N^j\big(\lambda, \tfrac{1}{k}\square^{F_k}\big)$$

Proposition 4.8, Theorem4.2 and (4.10) show then

$$\sum_{j=q}^{n}(-1)^{j-q}\dim H^{j}_{(2)}(X,L^{k}\otimes F)\leqslant \sum_{j=q}^{n}(-1)^{j-q}N^{j}(\lambda(j-q),\tfrac{1}{k}\square^{F_k}_{U})$$

where $\lambda(l)=\lambda$ for l even, $\lambda(l)=3C_0\lambda+C_2k^{-1}$ for all l odd. By proceeding as in the proof of Theorem 4.6 to obtain

$$\sum_{j=q}^{n}(-1)^{j-q}\dim H^{j}_{(2)}(X,L^{k}\otimes F)\leqslant k^{n}\sum_{j=q}^{n}(-1)^{j-q}I^{j}(U,0)+o(k^{n})$$

where $I^{j}(U,0)$ is given in (3.87). □

Let us note that if X is compact, the hypothesis of Theorem 4.7 (ii) are trivially satisfied for $m=n$, so that the following original holomorphic Morse inequalities is a special case of Theorem 4.7. Indeed, the L^2–cohomology is in this case isomorphic to the usual Dolbeault cohomology by (A.40).

4.10. THEOREM (Demailly). *Let X be a compact manifold and L, F holomorphic line bundles,* $\operatorname{rank}L=1$, $\operatorname{rank}F=r$. *As $k\to\infty$, the following strong Morse inequalities hold for every $q=0,1,\dots,n$:*

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^{j}(X,L^{k}\otimes F)\leqslant r\frac{k^{n}}{n!}\int_{X(\leqslant q)}(-1)^{q}\left(\tfrac{\sqrt{-1}}{2\pi}R^{L}\right)^{n}+o(k^{n}). \tag{4.24}$$

with equality for $q=n$ (asymptotic Riemann-Roch formula).

In particular, we get the weak Morse inequalities

$$\dim H^{q}(X,L^{k}\otimes F)\leqslant \frac{k^{n}}{n!}\int_{X(q)}\left(\tfrac{\sqrt{-1}}{2\pi}R^{L}\right)^{n}+o(k^{n}). \tag{4.25}$$

4.11. REMARK. In the compact case we can give a direct proof using the asymptotic of the heat kernel, following [**5, 13**]. Set $h^q_k=\dim H^q(X,L^k\otimes F)$. Then for every $t>0$

$$h^{q}_{k}-h^{q-1}_{k}+\dots+(-1)^{q}h^{0}_{k}\leqslant\sum_{l=0}^{q}(-1)^{q-l}\sum_{j=1}^{+\infty}e^{-t\lambda^{k,l}_{j}}.$$

where $\lambda^{k,l}_j$, $j=0,1,\dots$ is the spectrum of $\square^{F_k}$ acting on $\Omega^{0,l}(X)$. The left hand side is the contribution of the 0 eigenvalues in the right hand side. All we have to check is that the contribution of the other eigenvalues is $\geqslant 0$. The contribution of the eigenvalues such that $\lambda^{k,l}_j=\lambda>0$ is

$$e^{-t\lambda}\sum_{l=0}^{q}(-1)^{q-l}\dim\mathscr{E}^{0,l}(\lambda,\tfrac{1}{k}\square^{F_k}). \tag{4.26}$$

As $\mathscr{E}^{0,\bullet}(\lambda,\frac{1}{k}\square^{F_k})$ has trivial cohomology if $\lambda>0$, one easily sees that the sum (4.26) is equal to the dimension of $\overline{\partial}\mathscr{E}^{0,q}(\lambda,\frac{1}{k}\square^{F_k})\subset\mathscr{E}^{0,q+1}(\lambda,\frac{1}{k}\square^{F_k})$, hence $\geqslant 0$. By Theorem 3.12 we have

$$h^{q}_{k}-h^{q-1}_{k}+\dots+(-1)^{q}h^{0}_{k}\leqslant rk^{n}\sum_{l=0}^{q}(-1)^{q-l}\sum_{|J|=l}\int_{X}\frac{\prod_{j\leqslant s}|\alpha_j|\cdot e^{t(\alpha_{\complement J}-\alpha_J-\sum|\alpha_j|)}}{2^{2n-s}\pi^{n}t^{n-s}\prod_{j\leqslant s}(1-e^{-2t|\alpha_j|})}+o(k^{n}).$$

as $k\to\infty$, uniformly with respect to $t\in[t_0,k^{\varepsilon}]$, for any $t_0>0$. We let $t=k^{\varepsilon}$ tend to $+\infty$. It is clear that $\alpha_{\complement J}-\alpha_J-\sum|\alpha_j|$ is always $\leqslant 0$, thus the integrand tends to 0 at every point where $s<n$. When $s=n$, we have $\alpha_{\complement J}(x)-\alpha_J(x)-\sum|\alpha_j(x)|=0$ if and only if $\alpha_j(x)>0$ for every $j\in\complement J$ and $\alpha_j(x)<0$ for every $j\in J$. This implies $x\in X(l,E)$; in this case there is only one multi–index J satisfying the above conditions and the limit is $(2\pi)^{-n}|\alpha_1\cdots\alpha_n|$. By the monotone convergence theorem, our sum of integrals converges to

$$\sum_{l=0}^{q}(-1)^{q-l}\int_{X(l,E)}(2\pi)^{-n}|\alpha_1\cdots\alpha_n|\,dv_X=\frac{1}{n!}\int_{X(\leqslant q,L)}(-1)^q\big(\tfrac{\sqrt{-1}}{2\pi}R^L\big)^n.$$

4.12. COROLLARY. *In the same situation*

$$\dim H^0(X,L^k\otimes F)\geqslant\frac{k^n}{n!}\int_{X(\leqslant 1)}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n). \tag{4.27}$$

It follows that if (L,h^L) satisfies

$$\int_{X(\leqslant 1)}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n>0\,, \tag{4.28}$$

there exists $C>0$ and k_0 such that we have

$$\dim H^0(X,L^k\otimes F)\geqslant Ck^n\,,\quad k>k_0\,. \tag{4.29}$$

4.2. The Grauert–Riemenschneider Criterion

In this section we will prove the Kodaira embedding theorem in order to explain the context and the terminology of the Grauert–Riemenschneider criterion. Then we show the connection between existence of meromorphic functions and the growth of the dimension of the space of holomorphic sections of a line bundle.

For the basic definitions of positivity and complex projective spaces we refer to Appendix B.4.

4.2.1. Some vanishing theorems. We start by recalling three important vanishing theorems.

4.13. KODAIRA VANISHING THEOREM. *Let X be a compact manifold and let (L,h^L) be a holomorphic hermitian line bundle of positive curvature. Then*

(i) (coarse vanishing) *For any (E,h^E) holomorphic hermitian vector bundle (E,h^E) over X, $H^p(X,\mathscr{O}(L^k\otimes E))=0$ for $p\geqslant 1$ and k sufficiently large.*

(ii) (precise vanishing) $H^p(X,\mathscr{O}(L\otimes K_X))=0$ *for* $p\geqslant 1$.

PROOF. (i) By applying the Bochner–Kodaira–Nakano formula as in (3.62) (or (B.25) with void boundary) we have

$$(\square^{L^k\otimes E}u,u)\geqslant k([\sqrt{-1}R^L,\Lambda]\widetilde{u},\widetilde{u})+([\sqrt{-1}(R^{\det}+R^E),\Lambda]\widetilde{u},\widetilde{u})$$

for any $u\in\Omega^{0,p}(X,L^k\otimes E)$, $p\geqslant 1$, where

$$\widetilde{\ }:\Lambda^{0,p}T^*X\otimes L^k\otimes E\longrightarrow\Lambda^{n,p}T^*X\otimes(L^k\otimes E\otimes\Lambda^nTX)$$

is the natural isometry. We have to pass to the (n,p)-form $\widetilde{u}$ since the curvature term in the Bochner–Kodaira–Nakano formula (3.70) does not permit to exploit directly the positivity of the curvature for $(0,p)$-forms. We choose now the metric $\omega=\sqrt{-1}R^L$ and then, from (3.70),

$$([\sqrt{-1}R^L,\Lambda]\widetilde{u},\widetilde{u})\geqslant kp\|\widetilde{u}\|^2=kp\|u\|^2\,. \tag{4.30}$$

Therefore $(\square^{L^k\otimes E}u,u)\geqslant(kp-C)\|u\|^2$, for some $C>0$ depending on the Ricci curvature of ω and R^E on X. As a consequence any harmonic form $u\in\mathscr{H}^{0,p}(X,L^k)$ vanishes for $k>C/p$. By Hodge theory we obtain $H^p(X,\mathscr{O}(L^k\otimes E))=0$ for $p\geqslant 1$ and k large enough.

(ii) is straightforward since we apply the Bochner-Kodaira-Nakano formula directly for (n,p)-forms and we obtain (4.30). □

We quote now Grauert's generalization of Kodaira's theorem. It is remarkable not only for the fact that it treats the singular case, but also for the beautiful method. It reduces a problem about the cohomology of the base manifold to a problem about the disc bundle of L^*, the so called Grauert tube. For all definitions we refer to Appendix B.

4.14. GRAUERT VANISHING THEOREM. *Let X be a compact complex space and let L be a Grauert–positive line bundle. Then for any coherent analytic sheaf $\mathscr{F}$ on X, $H^q(X, \mathscr{O}(L^k)\otimes\mathscr{F}) = 0$ for $q \geqslant 1$ and k sufficiently large.*

PROOF. The proof uses Grauert's solution of the Levi problem for the Grauert tube $T \subset L^*$ (see (B.28)), more precisely the finiteness property $\dim H^q(T, \widetilde{\mathscr{F}}) < \infty$, $q \geqslant 1$, for all coherent analytic sheaves $\widetilde{\mathscr{F}}$ on T.

Let $\mathscr{F}$ be a coherent analytic sheaf on X and denote $\pi : T \longrightarrow X$ the natural projection. We have then an injective map

$$\bigoplus_{k\geqslant 0} H^q(X, \mathscr{F}\otimes\mathscr{O}(L^k)) \longrightarrow H^q(T, \pi^*\mathscr{F}) \tag{4.31}$$

It s is a section of L^k, we obtain a function on T by setting $\sigma(v) = v^{\otimes k}(s(x))$, where $v \in L^*$, $\pi(v) = x$.

The restriction of σ to the fiber L^*_x is a homogeneous function of order k on L_x. This construction carried at cohomology level gives rise to (4.31). It follows that $H^q(X, \mathscr{F}\otimes\mathscr{O}(L^k)) = 0$ for $q \geqslant 1$ and k large. □

We recall now the standard L^2 existence theorem of Hörmander–Andreotti–Vesentini.

4.15. THEOREM (Hörmander-Andreotti-Vesentini). *Let (X, ω) be a complete Kähler manifold of dimension n and let (L, h^L) be a positive line bundle. Let $\gamma_1 \leqslant \ldots \leqslant \gamma_n$ be the eigenvalues of $\sqrt{-1}R^L$ with respect to ω. Then for any form $f \in L^{n,q}_2(X, L)$ satisfying $\overline{\partial}^L f = 0$ and $\int_X (\gamma_1 + \ldots + \gamma_q)^{-1}|f|^2\, dv_X < \infty$ there exists $u \in L^{n,q-1}_2(X, L)$ such that $\overline{\partial}^L u = f$ and*

$$\int_X |u|^2\, dv_X \leqslant \int_X (\gamma_1 + \ldots + \gamma_q)^{-1}|f|^2\, dv_X\,.$$

In the seminal works of Bombieri [**6**] and Skoda [**42**] it has been observed that the above theorem still applies if h^L is singular. We recall first the terminology abourt currents and singular hermitian metrics [**14, 17**]. Let X be a complex manifold. we denote by $\Omega^{p,q}(X)$ the space of (p,q)–forms, endowed with the $\mathscr{C}^\infty$–topology. A $(1,1)$–current on X is a continous linear functional $T : \Omega^{n-1,n-1}(X) \longrightarrow \mathbb{C}$. The current T is called:

- *closed*, if $T(d\alpha) = 0$, for any smooth form α with $d\alpha \in \Omega^{n-1,n-1}(X)$.
- *real*, if $\overline{T} = T$, i.e., $\overline{T(\varphi)} = T(\overline{\varphi})$, for any $\alpha \in \Omega^{n-1,n-1}(X)$.
- *positive*, if $\left(\sqrt{-1}\right)^{(n-1)^2} T(\beta\wedge\overline{\beta}) \geqslant 0$, for any $\beta \in \Omega^{n-1,0}(X)$.
- *strictly positive*, if there exists a hermitian metric on X whose associated $(1,1)$–form ω has the property that $T - \omega$ is positive.
- *Kähler*, if it is closed and strictly positive.
- *integral*, if it is closed and its cohomology class $\{T\} \in H^2(X, \mathbb{R})$ lies in the image of $H^2(X, \mathbb{Z})$, under the natural map.

Let X be compact complex manifold and let L be a holomorphic line bundle over X. A *singular hermitian metric h^L* on L is a choice of a sesquilinear, hermitian–symmetric form h^L_x on each fiber L_x, such that, in any trivialization $\vartheta : L\upharpoonright_U \longrightarrow U\times\mathbb{C}$, we have $h^L_x(v,v) = |\vartheta(v)|^2 \exp(-\varphi_\vartheta(x))$, $v \in L_x$, $x \in U$, with $\varphi_\vartheta \in L^1(U, \mathrm{loc})$. If $\varphi_\vartheta \in \mathscr{C}^\infty(U)$, we obtain the usual definition of a hermitian metric. Another way to define a singular hermitian metric h^L is to give a smooth hermitian metric h^L_D and a function $\varphi \in L^1_{\rm loc}(X,\mathbb{R})$ and then set $h^L = h^L_0 e^{-\varphi}$.

As in the smooth case, the local $(1,1)$–currents $\sqrt{-1}\partial\overline{\partial}\varphi_\vartheta$ patch together to give a global $(1,1)$– current $\sqrt{-1}R^{(L,h^L)}$, called the curvature current of (L,h^L). $\sqrt{-1}R^{(L,h^L)}$ is obviously a closed and integral, representing the Chern class of L. If $h^L = h^L_0 e^{-\varphi}$ the curvature current is $R^{(L,h^L)} = R^{(L,h^L_0)} + \partial\overline{\partial}\varphi$ (and it does not depend on the choice of h^L and φ such that $h^L = h^L_0 e^{-\varphi}$).

4.16. THEOREM (Hörmander-Bombieri-Skoda). *Let (X,ω) be a complete Kähler manifold,* $\dim X = n$, *and let (L,h^L) be a singular hermitian line bundle such that $\sqrt{-1}R^L \geqslant \varepsilon\omega$ in the sense of currents, for some constant $\varepsilon > 0$. Then for any form $f \in L^{n,q}_2(X,L)$ satisfying $\overline{\partial}^L f = 0$ there exists $u \in L^{n,q}_2(X,L)$ such that $\overline{\partial}^2 f = 0$ and*

$$\int_X |u|^2_{h^L} \leqslant \frac{1}{q\varepsilon}\int_X |f|^2_{h^L}\, dv_X\,.$$

The *multiplier ideal sheaf* $\mathscr{I}(h^L)$ of a singular metric $h^L = h^L_0 e^{-\varphi}$ is defined by

$$\mathscr{I}(h^L)(U) = \{f \in \mathscr{O}_X(U) : |f|^2 e^{-\varphi} \in L^1(U;\mathrm{loc})\}$$

A basic result of Nadel [**30**] says that $\mathscr{I}(h^L)$ is a coherent analytic sheaf if the curvature current $\sqrt{-1}R^L$ is positive.

4.17. NADEL VANISHING THEOREM ([**30**], [**16**]). *Let (X,ω) be a compact Kähler manifold and let (L,h^L) be a singular hermitian line bundle such that $\sqrt{-1}R^L \geqslant \varepsilon\omega$ in the sense of currents, for some constant $\varepsilon > 0$.*

Then $H^q(X,\mathscr{O}(L\otimes K_X)\otimes\mathscr{I}(h^L)) = 0$ for $q \geqslant 1$.

4.2.2. The Kodaira embedding theorem. Let us introduce some piece of terminology. Let X be a complex manifold and L be a holomorphic line bundle. We consider the graded ring

$$\mathscr{A}(X,L) = \oplus_{k\geqslant 0} H^0(X,\mathscr{O}(L^k)) \tag{4.32}$$

of a holomorphic sections of the tensor powers of L (here L^0 is the trivial line bundle).

(a) We say that $\mathscr{A}(X,L)$ separates two points $x \neq y$ in X, there exists $k(x,y) = k > 0$ and sections $s,t \in H^0(X,\mathscr{O}(L^k))$ such that $s(x) = 0$, $s(y) \neq 0$ and $t(y) = 0$, $t(x) \neq 0$. $\mathscr{A}(X,L)$ separates points on a set W if it separates all pairs (x,y) with $x \neq y$. This means that the meromorphic function s/t takes different values at x and y.

(b) We say that $\mathscr{A}(X,L)$ gives local coordinates at a point $x \in X$, if there exists $k = k(x) > 0$ and sections $s_0,\dots,s_n \in H^0(X,\mathscr{O}(L^k))$ such that $s_0(x) \neq 0$ and $d(\frac{s_1}{s_0})\wedge\dots\wedge d(\frac{s_n}{s_0}) \neq 0$ at x. In other words, the meromorphic functions $\frac{s_1}{s_0},\dots,\frac{s_n}{s_0}$ are holomorphic at x and provide local coordinates.

Let us interpret these notions with the help of the Kodaira map (1.5). Let $\{s_1,\dots,s_{d_k}\}$ be a basis of $H^0(X,\mathscr{O}(L^k))$ which induces an identification

$$H^0(X,\mathscr{O}(L^k)) \cong H^0(X,\mathscr{O}(L^k))^* \cong \mathbb{C}^{d_k} \quad \text{and} \quad \mathbb{P}H^0(X,\mathscr{O}(L^k))^* \cong \mathbb{CP}^{d_k-1}\,.$$

The base locus Bl_k of $H^0(X,\mathscr{O}(L^k))$ is the set of points of X where all sections of $H^0(X,\mathscr{O}(L^k))$ vanish. We define $\widetilde{\Phi}_k$ by means of the following commutative diagramm.

$$\begin{array}{ccc} X\smallsetminus \mathrm{Bl}_k & \xrightarrow{\Phi_k} & \mathbb{P}H^0(X,\mathscr{O}(L^k))^* \\ \Big\downarrow \mathrm{Id} & & \Big\downarrow \cong \\ X\smallsetminus \mathrm{Bl}_k & \xrightarrow{\widetilde{\Phi}_k} & \mathbb{C}\mathbb{P}^{d_k-1} \end{array}$$

Let us choose a local holomorphic frame e_L of L in the neighbourhood of x and set $s_i = f_i e_L^{\otimes k}$, for some (local) holomorphic functions f_i. We check then that

$$\widetilde{\Phi}_k(x) = [f_1(x);\ldots;f_{d_k}(x)]$$

and this does not depend on the choice of e_L. We can also write $\widetilde{\Phi}_k(x) = [s_1(x);\ldots;s_{d_k}(x)]$ keeping in mind that the quotient of two sections is a meromorphic function.

Now, (a) is equivalent to the existence of $k = k(x,y) > 0$ such that $\Phi_k(x) \neq \Phi_k(y)$ and (b) to the existence of $k = k(x) > 0$ such that Φ_k is an immersion at x i.e. $\operatorname{rank}\Phi_k(x) = \dim X = n$. (Note that a set of sections satisfying (b) are linearly independent and can be completed to a basis of $H^0(X,\mathscr{O}(L^k))$.) We say that a bundle L is very ample if $H^0(X,\mathscr{O}(L))$ is base point free and the Kodaira map Φ_1 is an embedding. The bundle L is called ample if there exists k_0 such that L^k is very ample for $k \geqslant k_0$.

We want to find a sufficient conditions for $\mathscr{A}(X,L)$ to separate points and to give local coordinates. We say that $\mathscr{A}(X,L)$ spans m-jets on a finite set $\{x_1,\ldots,x_N\}$ if there exists k such that the map

$$H^0(X,\mathscr{O}(L^k)) \longrightarrow \oplus L^k_{x_j}\otimes \mathscr{O}_{X,x_j}/\mathscr{M}^{m+1}_{x_j} \tag{4.33}$$

which sends a section to its m-jet at $\{x_1,\ldots,x_N\}$ is onto. Of course, if $\mathscr{A}(X,L)$ spans m-jets ($m \geqslant 1$) on finite sets of X, $\mathscr{A}(X,L)$ separates points and gives local coordinates on X. On the other hand, a sufficient condition for (4.33) to hold is the vanishing of the cohomology group

$$H^1(X,\mathscr{O}(L^k)\otimes \mathscr{I}^{m+1}_{x_1}\otimes\cdots\otimes \mathscr{I}^{m+1}_{x_N}), \tag{4.34}$$

where $\mathscr{I}^{m+1}_{x_i}$ is the ideal sheaf of holomorphic functions vanishing up to order $m+1$ at x_i.

4.18. KODAIRA EMBEDDING THEOREM. *Let X be a compact complex manifold and L be a holomorphic line bundle. Then L is ample if and only if L admits a hermitian metric of positive curvature.*

PROOF. The approach of Kodaira is to use the harmonic theory in order to obtain the vanishing of the sheaf cohomology group (4.34). For this purpose we blow-up the sheaves and transform them into free sheaves of sections in a holomorphic vector bundle. Let $x \neq y \in X$. We blow up the points x and y and denote by $\pi : \widetilde{X} \longrightarrow X$ the blow up and D the exceptional divisor. Let $\pi^* : H^0(X,\mathscr{O}(L^k)) \to H^0(D,\mathscr{O}(\pi^*L^k))$ be the pull-back map, which is surjective by the Hartogs extension theorem. We notice that π^*L^k is trivial on the exeptional divisor and $\pi^* : H^0(D,\mathscr{O}_D(\pi^*L^k)) \xrightarrow{\sim} L^k_x\otimes L^k_y$. We have the commutative diagramm

$$\begin{array}{ccc} H^0(X,\mathscr{O}(L^k)) & \xrightarrow{r^k_{x,y}} & L^k_x\oplus L^k_y \\ \Big\downarrow \pi^* & & \Big\downarrow \pi^* \\ H^0(\widetilde{X},\mathscr{O}(\pi^*L^k)) & \xrightarrow{\widetilde{r}^k_{x,y}} & H^0(D,\mathscr{O}_D(\pi^*L^k)) \end{array}$$

and the exact sheaf sequence

$$0 \longrightarrow \mathscr{O}(\pi^* L^k \otimes [D]^{-1}) \longrightarrow \mathscr{O}(\pi^* L^k) \longrightarrow \mathscr{O}_D(\pi^* L^k) \longrightarrow 0$$

whose associated exact cohomology sequence shows that the map $\tilde{r}^k_{x,y}$ (and thus $r^k_{x,y}$) is surjective if $H^1(\widetilde{X}, \mathscr{O}(\pi^* L^k \otimes [D]^{-1}))$ vanishes. From Theorem 4.13 we know that this is the case for $k > k(x,y)$. Here $k(x,y)$ depends on the curvature of the exceptional divisor $[D]$ and the Ricci curvature of $\widetilde{X}$. Note that $K_{\widetilde{X}} = \pi^* K_X \otimes [D]^{n-1}$. We can see therefore that we can choose $k(x',y') = k(x,y)$ for (x',y') in the neighbourhood of (x,y). Taking into account the compactness of X we can choose k_0 such that $r^k_{x,y}$ is surjective for all $x \neq y$ if $k > k_0$. It follows that Φ_k is well defined on X and injective for $k > k_0$. In a similar manner we show that there exists k_1 such that the restriction map $H^0(X, \mathscr{O}(L^k)) \to L^k_x \otimes \mathscr{O}_{X,x}/\mathscr{M}^2_x$ is surjective for all $k > k_1$ and all $x \in X$. Therefore Φ_k is regular on X for $k > k_1$. This implies that L^k is very ample for all $k > \max\{k_0, k_1\}$.

Using the Nadel vanishing theorem we obtain directly the vanishing of the sheaf cohomology (4.34). Namely consider the singular metric $h^L e^{-\varphi}$, where φ is smooth on $X \smallsetminus \{x,y\}$ (resp. on $X \smallsetminus \{x\}$) and equals $n \log|z-x|^2$ and $n \log|z-y|^2$ (resp. $(n+1)\log|z-x|^2$) in a neighbourhood of x and y (resp. x). Then $\mathscr{I}(h^L e^{-\varphi}) = \mathscr{I}_{x,y}$ (resp. $\mathscr{I}^2_x$). □

4.19. REMARK. By using the curvature definition of the positivity we were able to show that the bounds $k(x,y)$ and $k(x)$ are uniform in some neighbourhood of (x,y) and x. So we obtain that *all* the maps Φ_k are embeddings for $k \geqslant k_0$. If we use the Grauert approach we obtain the following

(i) the ring $\mathscr{A}(X,L)$ of a positive line bundle gives local coordinates and separates points.

From this we infer that

(ii) Φ_k is an embedding for some k. We have thus:

4.20. GRAUERT EMBEDDING THEOREM ([**19**, Satz 2]). *Let X be a compact complex space and $L \to X$ a Grauert positive line bundle. Then X is projective algebraic.*

Let us also note that the results of this section can be easily generalized by twisting the powers of L with an arbitrary holomorphic vector bundle E over X. We then consider the graded vector space $\oplus_{k \geqslant 0} H^0(X, L^k \otimes E)$. If L is positive,

$$H^0(X, \mathscr{O}(L^k \otimes E)) \longrightarrow \oplus L^k_{x_j} \otimes E \otimes \mathscr{O}_{X,x_j}/\mathscr{M}^{m+1}_{x_j} \tag{4.35}$$

is surjective for large k. We denote $d_k = \dim H^0(X, L^k \otimes E)$. Then the Kodaira map $\Phi_k : X \longrightarrow G(d_k, d_k - \operatorname{rank}(E))$ which associates to each point x the $d_k - \operatorname{rank}(E)$ dimensional of sections from $H^0(X, L^k \otimes E)$ which vanish at x, is well defined and an embedding for k sufficiently large.

4.2.3. Algebraic dependence and Moishezon spaces. Let X be a complex space and $f_1, \dots, f_k$ meromorphic functions on X. We say that these functions are algebraically dependent, if there exists a non–trivial polynomial $P \in \mathbb{C}[z_1, \dots, z_k]$ such that $P(f_1, \dots, f_k) = 0$ wherever it is defined.

Let us denote by $a(X)$ the transcendence degree of $\mathscr{M}(X)$ over $\mathbb{C}$ and call it the algebraic dimension of X. We have the following fundamental result of Siegel:

4.21. SIEGEL-THIMM-REMMERT THEOREM. *Let X be a compact complex space. Then the field of meromorphic functions $\mathscr{M}(X)$ is an algebraic field of transcendence degree $a(X) \leqslant \dim_{\mathbb{C}} X$.*

This theorem has a long and rich history, detailed in [**39**] and [**34**]. In the case of complex tori the result was communicated in 1860 by Riemman to Hermite and it was stated by Weierstrass in 1869. There were many attempts to generalize the result to the case of several variables which failed due to the incomplete understanding of the indeterminacies of meromorphic functions. The first complete proofs were given by Thimm [**46**] and Siegel [**38**] in some special cases. Thimm follows Weierstrass original idea and gives a proof in the case of a complex space X with $\dim X$ independent meromorphic functions. Siegel uses an ansatz of Poincaré and considers quotients of holomorphy domains in $\mathbb{C}^n$. In the case of manifolds he gives an elementary proof based on the Schwarz lemma. The general theorem, without any hypotesis on the number of independent meromorphic functions was stated by Chow and proved by Remmert [**34**]. Finally, Andreotti extends Siegel's method to pseudoconcave complex spaces [**1**]. A generalization for CR manifolds has recently been given by C. Denson–Hill and M. Nacinovich [**22**]. It is course hard to assign names to this theorem whitout exceeding four or five authors. In [**18**] it is called theorem of Weierstrass–Siegel–Thimm.

PROOF OF SIEGEL–THIMM–REMMERT THEOREM. For the proof we refer to the papers cited above. It is based on the fact that on a compact complex space analytic dependence and algebraic dependece of meromorphic functions coincide. We want here only to justify that $a(X) \leqslant n$, where $n = \dim X$. Since X is bimeromorphic to a smooth manifold by the desingularization theorem of Hironaka, we are entitled to assume X smooth.

The proof exploits the relations between the existence of independent meromorphic functions and the growth of the spaces of holomorphic sections in holomorphic line bundles.

Let L be a holomorphic line bundle over a compact complex manifold X of dimension n. For the proof of Siegel's theorem we need only to show that there exists $C > 0$ such that $\dim H^0(X,L^k) \leqslant Ck^n$, for all $k \geqslant 0$. For other applications we shall prove a sharper statement. Let us denote by ρ_k the maximal rank of Φ_k on X. If $\dim H^0(X,L^k) = 0$ we set $\rho_k = -\infty$. The following lemma is crucial.

4.22. LEMMA (Siegel's lemma). *Let X be a compact complex manifold and $L \to X$ be a holomorphic line bundle. Then there exists $C > 0$ such that*

$$\dim H^0(X,L^k) \leqslant Ck^{\rho_k},\, k \geqslant 0.$$

Let $f_1,\dots,f_m \in \mathscr{M}(X)$ algebraically independent. We denote by D the divisor of poles of all f_j and by $[D]$ the associated line bundle [**20**, p. 134]. By a basic correspondence in algebraic geometry [**20**, p. 136], the space $H^0(X,[D]^k)$ is identified to the space of meromorphic functions f satisfying $\operatorname{div}(f) + [D]^k \geqslant 0$.

If $P \in \mathbb{C}[z_1,\dots,z_m]$, $\deg P \leqslant k$, the meromorphic function $P(f_1,\dots,f_m)$ has the former property. By hypothesis, the linear map which associates to a polynomial P of degree $\leqslant k$ the function $P(f_1,\dots,f_m)$ is injective. Therefore $\dim H^0(X,[D]^k) \geqslant \binom{m+k}{m}$ for all k. For $k \to \infty$ this dimension grows like k^m. But by Siegel's lemma $\dim H^0(X,[D]^k) \leqslant Ck^n$ for all k. Therefore $m \leqslant n$. We have thus showed that the transcendence degree of $\mathscr{M}(X)$ is less than or equal to $n = \dim_{\mathbb{C}} X$. □

PROOF OF LEMMA 4.22. For a point $x \in X$ we denote by $P(a,r)$ the polydisc $\{y \in U : |y_i| < r\}$ where $(U,y_1,\dots,y_n)$ is a coordinate system centered at x.

The set of points where Φ_k has rank less than ρ_k is a proper analytic set of X, so $\{x \in X : \operatorname{rank}_x \Phi_k = \rho_k\}$ is dense in X.

Let $a_1,\dots,a_m \in X$ such that

$$X \subset \cup_{i=1}^m P(a_i, r_i e^{-1})$$

and $r_1,\dots,r_m \in \mathbb{R}_+$ and Φ_k has rank ρ_k at each a_j.

Since Φ_k is a subimmersion at a_j there exists a submanifold M_j in the neighbourhood of a_j which is transversal in a_j to the fibre $\Phi_k^{-1}(\Phi_k(a_j))$ and $\dim M_j = \rho_k$. Assume that the line bundle L is given by the transition functions

$$c_{ij} : \overline{P}(a_i, r_i e^{-1}) \cap \overline{P}(a_j, r_j e^{-1}) \longrightarrow \mathbb{C}^*$$

Set

$$\|L\| = \sup\{|c_{ij}(x)| : x \in \overline{P}(a_i, r_i e^{-1}) \cap \overline{P}(a_j, r_j e^{-1}) \,\text{for all}\, i, j\} = e^{\mu}.$$

Since $c_{ij} = c_{ji}^{-1}$, $\mu > 0$. Consider a section $s \in \Gamma(X, \mathscr{O}(L^k))$ which vanishes up to order $h = k([\mu]+1)$ at each a_j along M_j ($[\mu]$ is the integral part of μ). But s vanishes on the fibre which passes form a_j, hence s vanishes up to order h at a_j on X. Assume that s is given on $P(a_i, r_i)$ by $s_i : P(a_i, r_i) \longrightarrow \mathbb{C}$. Set $\|s\| = \sup\{|s_j(x)| : x \in \overline{P}(a_i, r_i e^{-1}) \,\text{for all}\, i\}$.

There exists $q \in \{1, 2, \dots, m\}$ such that for some $w \in S(P(a_q, r_q))$, $|s_q(w)| = \|s\|$. We can find $j \neq q$ such that $w \in P(a_j, r_j)$. Hence $s_q(w) = c_{qj}(w) s_j(w)$ so that

$$\|s\| = |s_q(w)| = |c_{qj}(w) s_j(w)| \leqslant \|L^k\| \, |s_j(w)|.$$

By applying the Schwarz inequality to s_j in $P(a_j, r_j)$ we get $|s_j(w)| \leqslant \|s\| \, |w|^h r_j^{-h}$ where $|w| = \sup |p_j(w) - p_j(a_j)| \leqslant r_j e^{-1}$. Consequently, $\|s\| \leqslant \|s\| \, \|L^k\| e^{-h}$. If s is not identically zero this leads to a contradiction, by our choice of h. Consider the map

$$H^0(X, \mathscr{O}(L^k)) \longrightarrow \prod_{1 \leqslant j \leqslant m} \mathscr{O}_{M_j, a_j} / \mathscr{M}^h_{M_j, a_j}$$

where $\mathscr{M}^h_{M_j, a_j}$ is the maximal ideal of the ring $\mathscr{O}_{M_j, a_j}$, which sends every section in his Taylor developpment of order h at a_j along M_j. By the preceding argument this map is injective. Since the dimension of the target space satisfies the desired estimate we are done. □

4.2.4. Moishezon spaces. Let X be an irreducible compact complex space of dimension n. X is called *Moishezon space* if it possesses n independent meromorphic functions, i.e. if $a(x) = n$. Assume that X is projective. Then X can be realized as finite cover of $\mathbb{P}^n$ and we can pull-back n algebraically independent meromorphic functions on $\mathbb{P}^n$. Thus, every reduced compact projective space is Moishezon. Let X' be a projective space and $\varphi : X' \longrightarrow X$ be a proper modification. We know then that X is Moishezon.

The spaces with the property $a(X) = \dim X$ were named by Artin after B. G. Moishezon (also transliterated Moĭšezon) who proved in [**28**] the following fundamental result.

4.23. THEOREM (Moishezon). *Let X be an irreducible compact complex space. Then there exist a proper modification $\pi : X' \longrightarrow X$, obtained by a finite number of blowings–up, such that X' is a projective algebraic variety.*

Actually, Moishezon proves more. Let us introduce four classes of manifolds. The first, $A^{(1)}$, is the class of complex manifolds obtained from algebraic varieties by a sequence of elementary contractions (that is to say, transformations which are the inverses of monoidal transformations with non-singular centers). The second class, $A^{(2)}$, consists of those manifolds satisfying the Chow lemma: namely, X is in $A^{(2)}$ if and only if there is a regular modification $f : X' \to X$, where X' is a projective algebraic variety. $A^{(3)}$ is the class of compact complex manifolds connected with algebraic varieties by bimeromorphic maps. $A^{(4)}$ is the class of n-dimensional compact complex manifolds with n algebraically independent meromorphic functions (called today Moishezon manifolds). It turns out in the course of his three part work [**27, 29, 29**] that $A^{(1)} = A^{(2)} = A^{(3)} = A^{(4)}$.

From the theorem we infer easily that a a smooth surface is Moishezon if and only if it is projective. Indeed, we can blow up only points in dimension two and the blow up of a manifold at a point is projective if and anly if the manifold is. It has been previously proved by Kodaira and Chow that any two dimensional Moishezon manifold is projective algebraic.

Moishezon gives moreover two criteria for an n-dimensional manifold possessing n algebraically independent meromorphic functions to be projective algebraic. One of them [**27**] is given in terms of Kähler metrics. If X is an irreducible compact complex n-dimensional manifold with n algebraically independent meromorphic functions, then it is projective algebraic if and only if it has a Kähler metric. The case $n = 2$ was previously settled by Chow and Kodaira [**10**].

Let us remark that not all Moishezon spaces are projective. It seems that the first example appeared as folklore in Russia during the 50's and was named in Princeton the "algebraic Sputnik". Grauert constructed an example of a two-dimensional normal Moishezon space in [**19**, §4]. As we saw, smooth Moishezon surfaces are always projective. So we should look for a smooth example starting with dimension three. This was realized first by Hironaka and we refer to the books of Hartshorne [**21**, Example 3.4.2] and Shafarevich [**36**, Ch. 8, §3].

We wish to give a simple characterization of Moishezon manifolds in terms of order of growth of spaces of sections of line bundles.

Let us define the Kodaira–Iitaka dimension of a line bundle $L \longrightarrow X$ as

$$\kappa(L) = \max\{\rho_k = \operatorname{rank}\Phi_k : k > 0\}$$

The bundle L is said to be *big* if $\kappa(L) = n = \dim X$. It is clear that L is big if and only if $\mathscr{A}(X, L)$ gives local coordinates at a point.

4.24. PROPOSITION. *X is Moishezon if and only if it carries a big line bundle.*

PROOF. If X is Moishezon there exist $n = \dim X$ algebraically independent meromorphic functions. We can find a line bundle L such that these functions have the form $s_1/s_0, \dots, s_n/s_0$ where $s_0, \dots, s_n \in H^0(X, \mathscr{O}(L))$. (see [**1**]). Since the algebraic independence implies the analytic independence it follows that $d(s_1/s_0) \wedge \dots \wedge d(s_n/s_0) \neq 0$ on the set where the left–hand side is defined. By completing $\{s_0, \dots, s_n\}$ to a basis of $H^0(X, \mathscr{O}(L))$, we see that the Kodaira map $\Phi_1 : X \smallsetminus \mathrm{Bl}_1 \longrightarrow \mathbb{P}H^0(H, \mathscr{O}(L))^*$ has maximal rank i.e. $\rho_1 = n$ and hence $\kappa(L) = n$.

Conversely, if L is big, there exists $k > 0$ such that $\rho_k = n$. Then the image $\Phi_k(L)$ is an algebraic variety of dimension n. By pulling back n independent rational functions on $\Phi_k(X)$ to X via Φ_k we obtain n independent meromorphic functions. $\square$

4.2.5. Proof of the Grauert–Riemenschneider Criterion. Let X be a compact complex manifold and $L \to X$ be a hermitian holomorphic line bundle (L, h^L) which satisfies Demailly's condition

$$\int_{X(\leqslant 1)} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n > 0\,, \tag{4.36}$$

By Corollary 4.12 (especially (4.29)) and Siegel's Lemma 4.22 there exists $C_1, C_2 > 0$ and k_0 such that

$$C_1 k^{\rho_k} \geqslant \dim H^0(X, L^k) \geqslant C_2 k^n\,, \quad \text{for } k > k_0\,. \tag{4.37}$$

Therefore $\rho_k = n$ for $k > k_0$ so $\kappa(L) = n$ and L is big. By Proposition 4.24 we conclude that X is Moishezon. This proves the Demailly's Criterion 1.3 which of course implies the Grauert-Riemenschneider Criterion 1.2, since a semipositive line bundle which is positive at one point abviously satisfies (4.36).

Let us close by saying that we cannot prove the Grauert-Riemenschneider Criterion by using the L^2 method for the $\overline{\partial}$ operator. This is due to the non-Kähler character of the manifold X. There were a lot of attempts to prove the theorem until the paper of Siu [**40**] (where some of the history is presented). That's why the Morse inequaltities can be seen as a quantitative version of the standard L^2 estimates for $\overline{\partial}$ of Hörmander-Bombieri-Skoda 4.16.

4.2.6. Generalization to pseudococave manifolds.

4.25. DEFINITION. We will call a connected manifold X *Andreotti pseudoconcave* if there exists a smooth non–empty $D \Subset X$ such that the Levi form of D restricted to the analytic tangent plane $T^{1,0}(\partial D)$ has at least one negative eigenvalue at each point of ∂D.

The notion of Andreotti–pseudoconcavity is more general, see [**1**], [**2**], but for our purposes this definition is sufficient. Immediate examples are q–concave manifolds, $q \leqslant n-1$. Indeed, let X be a q–concave manifold as in definition B.29. The definition function of $X_c = \{\varphi > c\} \Subset X$ is $c - \varphi$ and for c sufficiently close to a, the Levi form of $c - \varphi$ has at least $n - q + 1$ negative eigenvalues in a neighbourhood of ∂X_c. Thus, the restriction of the Levi form on the analytic tangent space $T^{1,0}(\partial X_c)$ has at least $n - q \geqslant 1$ negative eigenvalues.

4.26. LEMMA. *For each point $x \in \overline{D}$ we can choose holomorphic coordinates (U, y) and a coordinate polydisc $P(x, r) \subset U$ centred at x such that the Silov boundary $S(P(x, r)) = \{y \in U : |y_i| = r\} \subset D$.*

PROOF. Let ρ be the defining function of D near $x = 0$. We know that $\mathscr{L}_\rho(0)$ restricted to $T_0^{1,0}(\partial D)$ has one negative eigenvalue.

After a suitable change of coordinates we can assume that

$$\rho(z) = 2\operatorname{Re} z_1 + \sum_{j=1}^{n} \alpha_j z_1 \overline{z}_j - |z_2|^2 + \sum_{j=3}^{n} \beta_j |z_j|^2 + O(|z|^3)$$

Geometrically this means that the tangent space at 0 to ∂D is $\{\operatorname{Re} z_1 = 0\}$ and the Levi form is negative definite on the plane $\{z_1 = 0, z_3 = \ldots = z_n = 0\}$. For r sufficiently small we have

$$\{z_1 = 0, \frac{r}{2} < |z_2| \leqslant r, z_3 = \ldots = z_n = 0\} \subset D\,.$$

The lemma follows now easily by a continuity argument. □

4.27. THEOREM. *If X be an Andreotti–pseudoconcave manifold, Siegel's Lemma 4.22 holds on X. The field of meromorphic functions $\mathscr{M}(X)$ is an algebraic field of transcendence degree $a(X) \leqslant \dim_{\mathbb{C}} X$.*

PROOF. The proof of Siegel's lemma in the pseudoconcave case follows the proof in the compact case. First we choose a set $D \Subset X$ as in Definition 4.25 and coordinate polydiscs $P(a_i,r_i)$, $i=1,\dots,m$ such that $\overline{D} \subset \cup_{i=1}^m P(a_i,r_ie^{-1})$, L is trivial on $P(a_i,r_i)$ and the Silov boundary $S(P(a_i,r_i)) \subset D$, for $i=1,\dots,m$. Then the proof goes through as in the compact case by observing that when choosing w with $|s_q(w)| = \|s\|$, we can assume that $w \in S(P(a_q,r_q))$. This is true since the maximum of the modules of a holomorphic function on the closure of a polydisc is attained on its Silov boundary. Thus we may take $w \in D$.

The proof of the second statement is completely analogous to that of Theorem 4.21. □

4.28. REMARK. We observe that Siegel's lemma 4.22 holds for the adjoint bundles, that is $\dim H^0(X,L^k \otimes K_X) \leqslant Ck^{\rho_k}$, where $\rho_k = \operatorname{rank}\Phi_k$.

Theorem 4.27 allows to extend the notion of Moishezon manifold for the case of Andreotti-pseudoconcave manifolds. Thus, an Andreotti-pseudoconcave manifold is called Moishezon if $a(X) = \dim_{\mathbb{C}} X$.

4.29. COROLLARY. *Let X be an Andreotti–pseudoconcave manifold and L be a holomorphic line bundle over X. If $\operatorname{rank}\Phi_k = \dim X$, where Φ_k is the Kodaira map asociated to $H^0(X,L^k)$ or $H^0(X,L^k \otimes K_X)$, X is Moishezon.*

4.3. Uniformly positive line bundles

In this section we apply the results from the previous one to the study of the L^2 cohomology of complex manifolds satisfying certain curvature conditions. If X is a complete Kähler manifold and L a positive line bundle on X the L^2 estimates of Andreotti–Vesentini–Hörmander allow to find a lot of sections of $L^k \otimes K_X$. We prove now a "compact perturbation" of this result. In this case the underlying complete metric is no more assumed to be Kähler, but we assume instead the existence of a uniformly positive line bundle outside a compact set. As application we prove the Nadel–Tsuji theorem in Corollary 4.31 and the Morse inequalities for hyperconcave manifolds.

4.3.1. A general cohomology estimate.

4.30. THEOREM. *Let (X,ω) be an n–dimensional complete hermitian manifold and let (L,h^L) be a holomorphic hermitian line bundle. Let $K \Subset X$ and a constant $C_0 > 0$ such that $\sqrt{-1}R^L \geqslant C_0\,\omega$ on $X \smallsetminus K$.*

(i) Then, for $k \longrightarrow \infty$,

$$\dim H^{n,0}_{(2)}(X,L^k) \geqslant \frac{k^n}{n!}\int_{X(\leqslant 1,h^L)} \Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)\Big)^n + o(k^n)\,, \tag{4.38}$$

where $H^{n,0}_{(2)}(X,L^k)$ is the space of $(n,0)$–forms with values in L^k which are L_2 with respect to any metric on L and the metric h^L on L.

(ii) Assume moreover that the torsion of ω is bounded and the Ricci curvature $R^{\det}$ is bounded from below with respect to ω. Then, for $k \longrightarrow \infty$

$$\dim H^{0,0}_{(2)}(X,L^k) \geqslant \frac{k^n}{n!}\int_{X(\leqslant 1)} \big(\frac{\sqrt{-1}}{2\pi}R^L\big)^n + o(k^n) \tag{4.39}$$

where $H^{0,0}_{(2)}(X,L^k)$ is the space of holomorphic sections in L^k which are L^2 with respect to the metrics ω on X and h^L on L.

As usual, $U(\leqslant 1, h^L)$ is the subset of U where $\sqrt{-1}R^L)$ is non–degenerate and has at most one negative eigenvalue. We can state the theorem without reference to the auxiliary metric ω, by saying that (L,h^L) is positive outside a compact set and the curvature $\sqrt{-1}R^L$ defines a complete metric on X (by extending it to a metric over X).

PROOF. (i) Let us endow X with a metric ω_0 such that $\omega_L = \sqrt{-1}R^L$ outside K, which is complete, for $\omega_L \geqslant C_0\omega$ on $X \smallsetminus K$. The Bochner–Kodaira–Nakano formula (B.20) gives

$$(\square^{L^k}u,u) \geqslant \big([\sqrt{-1}R^{L^k},\Lambda]u,u\big)\,, \quad u \in \Omega^{n,1}_0(X\smallsetminus K, L^k)$$

since ω_0 is Kähler outside K. By (3.70) we know that

$$\big\langle [\sqrt{-1}R^{L^k},\Lambda]u,u\big\rangle \geqslant k\alpha_1(x)|u|^2$$

where $\alpha_1 \leqslant \ldots \leqslant \alpha_n$ are the eigenvalues of $\sqrt{-1}R^L$ with respect to ω_0. In our case $\alpha_1 = \ldots = \alpha_n = 1$ outside K. Hence

$$\|u\|^2 \leqslant \frac{1}{k}(\|\overline{\partial}^{L^k}u\|^2 + \|\overline{\partial}^{L^k*}u\|^2)\,, \quad u \in \Omega^{n,1}_0(X\smallsetminus K, L^k) \tag{4.40}$$

Let U be any open set with smooth boundary, $K \Subset U \Subset X$. Choose $\rho \in \mathscr{C}^\infty_0(X)$ such that $\rho = 1$ on a neighbourhood of K and $\operatorname{supp}\rho \subset U$. Applying (4.40) for $(1-\rho)u$ and using (4.7) we obtain the fundamental estimate (4.1) (with a slightly larger K). in bidigree $(n,1)$ for all $u \in \Omega^{n,1}_0(X,L^k)$. Since $\Omega^{n,1}_0(X,L^k)$ is dense in $\operatorname{Dom}(\overline{\partial}^{L^k}) \cap \operatorname{Dom}(\overline{\partial}^{L^k*}) \cap \boldsymbol{L}^{n,1}_2(X,L^k)$ by A.10 we infer that (4.1) holds true in bidigree $(n,1)$. We conclude by Theorem 4.7 that

$$\dim H^{n,0}_{(2)}(X,L^k)_0 \geqslant k^n \int_{U(\leqslant 1)} \Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n + o(k^n)\,, \quad k \longrightarrow \infty$$

Here $H^{n,0}_{(2)}(X,L^k)_0$ is the L^2–cohomology group with respect to the metric ω_L on M. But the L^2 condition for $(n,0)$–forms does not depend on the metric on M, so $H^{n,0}_{(2)}(X,L^k)_0 = H^{n,0}_{(2)}(X,L^k)$ where in the latter group the L^2 condition is with respect to an arbitrary metric on X.

(ii) Let $u \in \Omega^{0,1}_0(X\smallsetminus K, L^k)$. In order to apply the Bochner–Kodaira–Nakano formula it is necessary to consider $(0,p)$–forms with values in L^k as (n,p)–forms with values in $L^k \otimes K^*_X$. The reason are formulas (3.79a), (3.79b). If we work directly with $(0,p)$–forms and use the curvature term $\big([\sqrt{-1}\Theta(L),\Lambda]u,u\big)$, then (3.79a) shows that we cannot exploit the positivity of the eigenvalues of the curvature of L. Let $\sim\colon \Lambda^{0,q}T^*X \otimes L^k \longrightarrow \Lambda^{n,q}T^*X \otimes (L^k \otimes K^*_X)$, $\quad u \longrightarrow \widetilde{u}$ be the natural isometry, where $K^*_X = \Lambda^n TX$ is the anti-canonical bundle. Set $\widetilde{L}^k = L^k \otimes K^*_X$. By proceeding as in 3.2.2 we deduce from (3.61c), (3.60)

$$(\square^{L^k}u,u) = (\square^{\widetilde{L}^k}\widetilde{u},\widetilde{u})$$

and by Bochner–Kodaira–Nakano (B.20)

$$(\square^{\widetilde{L}^k}\widetilde{u},\widetilde{u}) = (\square^{\widetilde{L}^k}\widetilde{u},\widetilde{u}) + \big([\sqrt{-1}R^{\widetilde{L}^k},\Lambda_\omega]\widetilde{u},\widetilde{u}\big) + \big([(\nabla^{\widetilde{L}^k})',T^*]\widetilde{u},\widetilde{u}\big) - \big([(\nabla^{\widetilde{L}^k})'',\overline{T}^*]\widetilde{u},\widetilde{u}\big)$$

By using the Nakano's inequality (B.23) we obtain

$$\begin{aligned}3(\square^{\widetilde{L}^k}u,u) \geqslant 2k\big([\sqrt{-1}R^L,\Lambda_\omega]\widetilde{u},\widetilde{u}\big) + 2\big([\sqrt{-1}R^{K^*_X},\Lambda_\omega]\widetilde{u},\widetilde{u}\big) - (\|Tu\|^2 + \|T^*u\|^2 + \|\overline{T}u\|^2\\ + \|\overline{T}^*u\|^2)\end{aligned} \tag{4.41}$$

The Ricci curvature is by definition $R^{\det} = R^{K_X^*}$. The hypothesis $R^{\det}$ is bounded means that its eigenvalues with respect to ω are bounded, so (3.79b) implies

$$\big([\sqrt{-1}R^{K_X^*},\Lambda_\omega]\widetilde{u},\widetilde{u}\big) \geqslant -C_1\|\widetilde{u}\|^2 = -C_1\|u\|^2 \tag{4.42}$$

for some constant $C_1 > 0$. Since the torsion operators are also bounded, there exists $C_2 > 0$ with

$$\|Tu\|^2 + \|T^*u\|^2 + \|\overline{T}u\|^2 + \|\overline{T}^*u\|^2 \leqslant C_2\|\widetilde{u}\|^2 = C_2\|u\|^2. \tag{4.43}$$

Moreover (3.79b) entails

$$\big([\sqrt{-1}R^{L},\Lambda_\omega]\widetilde{u},\widetilde{u}\big) \geqslant C_0\|\widetilde{u}\|^2 = C_0\|u\|^2. \tag{4.44}$$

Combining (4.41)–(4.44) we get

$$3(\square^{L^k}u,u) \geqslant (2kC_0 - 2C_1 - C_2)\|u\|^2 \geqslant k\|u\|^2$$

where the last inequality holds for k sufficiently large. We can thus proceed as in the proof of (i). □

The following important special case is due to Nadel–Tsuji [**31**, Theorem 1.1]

4.31. COROLLARY (Nadel-Tsuji). *Let (X,ω) be a complete Kähler manifold with $R^{\det} \leqslant -\omega$. Then we have the following estimate:*

$$\dim H^0_{(2)}(X,K_X^k) \geqslant \tfrac{k^n}{n!}\int_X \big(\tfrac{\sqrt{-1}}{2\pi}R^{K_X}\big)^n + o(k^n)$$

where $R^{K_X} = -R^{\det}$ is the curvature of the canonical bundle K_X equipped with the metric induced from ω.

4.3.2. Hyperconcave manifolds.

4.32. DEFINITION. A complex manifold X is called *hyperconcave* or *hyper* 1*-concave* if there exists a smooth function $\varphi : X \longrightarrow (-\infty, b]$ where $b \in \mathbb{R}$, such that $X_c := \{\varphi > c\} \Subset X$ for all $c \in (-\infty, b]$ and φ is strictly plurisubharmonic outside a compact set.

Let us describe some examples.

4.33. EXAMPLE. (i) Let Y be a compact complex manifold, S a complete pluripolar set. By definition, S is complete pluripolar, if there exists a neighbourhood W of S and a strictly plurisubharmonic function $\psi : W \longrightarrow [-\infty,\infty)$ such that $S = \psi^{-1}(-\infty)$. Then $X = Y \setminus S$ is hyperconcave. Conversely, we will show in Chapter 6 that any hyperconcave manifold M is biholomorphic to a complement of a pluripolar set in a compact manifold. If $\dim X \geqslant 3$ this is a consequence of Rossi's compactification theorem.

(ii) Let X be a compact complex space with isolated singularities. Then the regular locus X_{reg} is hyperconcave. Indeed, let $\{U_\alpha\}$ be pairwise disjoint neighbourhoods of the singular points $\{p_\alpha\}$ and let $\iota_\alpha : U_\alpha \hookrightarrow \mathbb{C}^{N_\alpha}$ be holomorphic embeddings. We may assume that that the singular points are mapped to the origin, $\iota_\alpha(p_\alpha) = 0$. The function $z \longmapsto \log|z|^2$ is strictly plurisubharmonic on $\mathbb{C}^{N_\alpha}$. By taking its pullback to each U_α through ι_α we obtain a strictly plurisubharmonic function on $X_{reg} \cap (\cup U_\alpha)$ which tends to $-\infty$ at the singular points. By extending this function to X_{reg} by means of a partition of unity, we get a function as in the definition.

(iii) If X is a complete Kähler manifold of finite volume and bounded negative sectional curvature, M is hyperconcave as shown by Siu–Yau in [**41**] (see also [**31**]). Actually, this example falls in the previous case since by Corollary 6.55, X can be compactified to an algebraic space by adding finitely many points.

4.34. THEOREM. *Let X be a hyperconcave manifold carrying a line bundle (L,h) which is semi-positive outside a compact set. Then, for $k \longrightarrow \infty$*

$$\dim H^{n,0}_{(2)}(X,L^k) \geqslant \frac{k^n}{n!}\int_{X(\leqslant 1,h)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n + o(k^n)\,, \tag{4.45}$$

where the L^2 condition is with respect to h and any metric on X.

PROOF. Let us consider a proper function $\varphi : X \longrightarrow (-\infty,0)$ which is strictly plurisubharmonic outside a compact set $K \Subset X$. The fact that φ goes to $-\infty$ to the ideal boundary of X allows to construct a complete hermitian metric on X. Denote

$$\chi = -\log(-\varphi), \tag{4.46}$$

which is a smooth function on X. Note that

$$\partial\overline{\partial}\chi = \frac{\partial\overline{\partial}\varphi}{-\varphi} + \frac{\partial\varphi\wedge\overline{\partial}\varphi}{\varphi^2}$$

and

$$\frac{\partial\varphi\wedge\overline{\partial}\varphi}{\varphi^2} = \partial\chi\wedge\overline{\partial}\chi\,.$$

We can now patch $\partial\overline{\partial}\chi$ and an arbitrary hermitian metric on X by using a smooth partition of unity to get a metric ω on X such that

$$\omega = \sqrt{-1}\partial\overline{\partial}\chi = -\sqrt{-1}\partial\overline{\partial}\log(-\varphi). \tag{4.47}$$

on $X \smallsetminus K$.

Since $\sqrt{-1}\partial\overline{\partial}\varphi/(-\varphi)$ represents a metric on $X \smallsetminus K$, we get

$$|d\chi|_\omega \leqslant C. \tag{4.48}$$

Since $\chi : X \longrightarrow \mathbb{R}$ is proper, (4.48) ensures that ω is complete. Indeed, (4.48) entails that χ is Lipschitz with respect to the geodesic distance induced by ω, so any geodesic ball must be relatively compact.

Note that ω is obviously Kähler on $X \smallsetminus K$. Let us assume $\sqrt{-1}R^L \geqslant 0$ on $X \smallsetminus K$ (we stretch K if necessary). We equip L with the metric $h^L_\varepsilon = h^L\exp(-\varepsilon\chi)$ and the curvature relative to the new metric satisfies $\sqrt{-1}R^{(L,h^L_\varepsilon)} \geqslant \varepsilon\,\omega$ on $X \smallsetminus K$. We are therefore in the conditions of Theorem 4.30. Since $h_\varepsilon \gtrsim h$ there is an injective morphism

$$H^{n,0}_{(2)}(X,L^k,\omega,h_\varepsilon) \hookrightarrow H^{n,0}_{(2)}(X,L^k,\omega,h).$$

By this relation and Theorem 4.30 for the space $H^{n,0}_{(2)}(X,L^k,\omega,h_\varepsilon)$,

$$\liminf_k k^{-n}\dim H^{n,0}_{(2)}(X,L^k,\omega,h) \geqslant \frac{1}{n!}\int_{U(\leqslant 1,h_\varepsilon)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^\varepsilon_L)}\Big)^n \tag{4.49}$$

We let now $\varepsilon \longrightarrow 0$ in (4.49); since h_ε converges uniformly together with its derivatives to h on compact sets we see that we can replace h_ε with h in the right-hand side of (4.49). Now, $X(\leqslant 1,h) = X(0,h)\cup X(1,h)$. By hypothesis $X(1,h) \subset K$ and on $X(0,h)$ the integrand is positive. Hence we can let U exhaust X to get (4.45). □

Theorem 4.34 implies the first part of Theorem 1.10:

4.35. COROLLARY. *Let X be a compact complex space with ar most isolated singularities. and let (L,h^L) be a holomorphic hermitian line bundle on X_{reg} which is semi-positive in a deleted neighbourhood of X_{sing} and satisfies Demailly's condition* (1.10) *on X_{reg} (e.g. L is everywhere semi-positive and positive at one point). Then X is Moishezon.*

PROOF. We apply Theorem 4.34 for the hyperconcave manifold X_{reg} and since

$$\int_{X_{reg}(\leqslant 1)}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n=\int_{X_{reg}}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n>0$$

we deduce $\dim H^0(X_{reg},L^k\otimes K_X)\geqslant Ck^n$ for some $C>0$ and k sufficiently large. By Siegel's lemma for Andreotti pseudoconcave manifolds 4.27 and Remark 4.28 we deduce that the rank of the Kodaira map of $H^0(X_{reg},L^k\otimes K_X)$ is maximal. Corollary 4.29 entails that there exists $\dim X$ independent meromorphic functions on X_{reg}. By the Levi extension theorem we conclude that these functions extend to $\dim X$ independent meromorphic functions on X. □

Let us define the "adjoint" volume of a line bundle L over a complex manifold M by $\mathrm{vol}^*(L)=\limsup_{k\to\infty} n!\,k^{-n}\dim H^0(M,L^k\otimes K_M)$. From the proof of Theorem 4.34 we infer the following.

4.36. COROLLARY. *Let L is a line bundle over X_{reg}, where X is a compact complex space with only isolated singularities.*

(i) *If L is semipositive outside a compact set,*

$$\int_{X_{reg}(0)}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n\leqslant \mathrm{vol}^*(L)-\int_{X_{reg}(1)}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n<\infty.$$

(ii) *If L is positive on X_{reg}*

$$\int_{X_{reg}(0)}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n\leqslant \mathrm{vol}^*(L)<\infty.$$

(iii) *If $\psi : X_{reg}\longrightarrow\mathbb{R}$ is a smooth function which is psh outside a compact set,*

$$\int_{X_{reg}(0)}(\sqrt{-1}\partial\overline{\partial}\psi)^n\leqslant-\int_{X_{reg}(1)}(\sqrt{-1}\partial\overline{\partial}\psi)^n<\infty$$

where $X_{reg}(0)$ is the open set where ψ is strictly psh.

PROOF. Relation (4.45) shows the left hand-side inequality in (i), since the integral in (4.45) is the sum of two corresponding integrals taken over the sets $X_{reg}(0)$ and $X_{reg}(1)$. The latter is finite since $X_{reg}(1)$ is relatively compact by the hypothesis on the semipositivity of L. By the Serre-Siegel lemma we get also the finiteness in (i). From (i) we infer immediately (ii). To prove (iii) we apply (i) to the trivial bundle L endowed with the metric $\exp(-\psi)$ and we use the obvious fact that $\mathrm{vol}^*(L)=0$. □

4.4. Demailly's criterion for isolated singularities

Our aim is to prove now the second part of Theorem 1.10. This shows that Demailly's criterion generalizes to singular spaces with at most isolated singularities under mild growth conditions of the curvature near the singular set.

We will work on the open manifold X_{reg} and prove that it posseses a lot of meromorphic functions which extend to X by the Levi extension theorem.

In order to perform analysis on X_{reg} we introduce first a good exhaustion function and a complete metric. Let $\pi:\widetilde{X}\longrightarrow X$ be a resolution of singularities of X. Let us denote by D_i the components of the exceptional divisor. Then there exist positive integers n_i such that $D := \sum n_i D_i$ admits a smooth hermitian metric such that the induced line bundle $[D]$ is negative in a neighbourhood $\widetilde{U}$ of D (cf. [**35**]). Let us consider a canonical section s of $[D]$, i.e. $D = (s)$, and denote by $|s|^2$ the poinwise norm of s with respect to the above metric. By Lelong-Poincaré

$$\frac{\sqrt{-1}}{2\pi}\,\partial\overline{\partial}\log|s| = (\text{the current of integration on } D) - \frac{\sqrt{-1}}{2\pi}\,R^{[D]}\,. \tag{4.50}$$

Hence $\varphi = \log|s|^2$ is strictly plurisubharmonic on $\widetilde{U}\smallsetminus D$ and converges to $-\infty$ on D. By using a smooth function on $\widetilde{X}$ with compact support in $\widetilde{U}$ which equals one near D we construct a smooth function χ on $\widetilde{X}\smallsetminus D \simeq X_{reg}$ such that $\chi = -\log(-\log|s|^2)$ on $\widetilde{U}\smallsetminus D$.

With the help of the function χ we construct a complete metric on X_{reg}. For this purpose we recall first the notion of hermitian metric on a singular space. Let us consider a covering $\{U_\alpha\}$ of X and embeddings $\iota_\alpha : U_\alpha \hookrightarrow \mathbb{C}^{N_\alpha}$. A metric on X is a metric ω on X_{reg} which on every open set U_α as above is the pullback of a hermitian metric on the ambient space $\mathbb{C}^{N_\alpha}$, $\omega = \iota_\alpha^*\,\omega_\alpha$. It is constructed as usual by a partition of unity argument. Since the singularities are isolated we can assume that the metric is *distinguished*, that is, in the neighbourhood of the singular points ω_α is the euclidian metric. In particular ω is Kähler near X_{sing}. We consider then the metric $\omega_0 = A\omega + \partial\overline{\partial}\chi$ where $A > 0$ is chosen sufficiently large (to ensure that ω_0 is a metric away from the open set where $\partial\overline{\partial}\chi$ is positive definite). ω_0 is complete by the same argument as in the proof of Theorem 4.34 (see (4.48)). Note that by Corollary 4.36 the metric ω_0 has finite volume. This follows from the fact that, near X_{sing}, χ is strictly psh and ω is given by the euclidian potential.

Assume now that $L|_{U_\alpha}$ is the inverse image by ι_α of the trivial line bundle $\mathbb{C}_\alpha$ on $\mathbb{C}^{N_\alpha}$. Moreover we consider hermitian metrics $h_\alpha = e^{-\varphi_\alpha}$ on $\mathbb{C}_\alpha$ such that $\iota_\alpha^* h_\alpha = \iota_\beta^* h_\beta$ on $U_\alpha \cap U_\beta \cap X_{reg}$. The system $h^L = \{\iota_\alpha^* h_\alpha\}$ is called a hermitian metric on L. It clearly induces a hermitian metric on $L|_{X_{reg}}$. We shall allow our metrics to be singular at the singular points, that is, $\varphi_\alpha \in L^1_{loc}(\mathbb{C}^{N_\alpha})$ and φ_α is smooth outside $\iota_\alpha(X_{sing})$. The curvature current $\sqrt{-1}R^L$ is given in U_α by $\iota_\alpha^*(\sqrt{-1}\partial\overline{\partial}\varphi_\alpha)$ which on X_{reg} agrees with the curvature of the induced metric. We shall suppose in the sequel that the curvature current is dominated by the euclidian metric i.e. $\sqrt{-1}\partial\overline{\partial}\varphi_\alpha$ is bounded above and below by constant times $\omega_E = \sqrt{-1}\sum dz_j \wedge d\overline{z}_j$.

Let us consider now a neighbourhood U of the singular set. We assume that U is small enough so that there are well defined on U a potential ρ for ω and a potential φ for the curvature $\sqrt{-1}R^L$ (they are restrictions from ambient spaces). By suitably cutting-off we may define a function $\psi \in \mathscr{C}^\infty(X_{reg})$ such that

$$\psi = \chi - \varphi + A\rho \tag{4.51}$$

near X_{sing}. Remark that, since $\sqrt{-1}R^L$ is bounded above by a continuous $(1,1)$ form near X_{sing}, the potential $-\varphi$ is bounded above near the singular set. This holds true for ρ too (it is smooth) so that ψ tends to $-\infty$ at the singular set X_{sing}. Let us consider a smooth function $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\gamma(t) = \begin{cases} 0 & \text{if} \quad t \geqslant 0\,, \\ t & \text{if} \quad t \leqslant -1\,. \end{cases}$$

and the functions $\gamma_\nu : \mathbb{R} \longrightarrow \mathbb{R}$ given by $\gamma_\nu(t) = \gamma(t-\nu)$ for all positive integers ν.

Let us consider the metric $h^L_\nu = h^L \exp\big(-\gamma_\nu(\psi)\big)$ with curvature

$$\sqrt{-1}R^{(L,h^L_\nu)} = \sqrt{-1}R^{(L,h^L)} + \sqrt{-1}\gamma'_\nu(\psi)\partial\overline{\partial}\psi + \sqrt{-1}\gamma''_\nu(\psi)\partial\psi\wedge\overline{\partial}\psi\,.$$

On the set $\{\psi \leqslant -\nu-1\}$ we have $\gamma_\nu(\psi) = \psi - \nu$ so that $\gamma'_\nu(\psi) = 1$ and $\gamma''_\nu(\psi) = 0$ and therefore $\sqrt{-1}R^{(L,h^L_\nu)} = \sqrt{-1}R^{(L,h^L)} + \partial\overline{\partial}\psi$. Since ψ goes to $-\infty$ when we approach the singular set we may choose ν_0 such that for $\nu \geqslant \nu_0$ we have $\{\psi \leqslant -\nu-1\} \subset U$ where U is the neighbourhood of X_{sing} where ψ has the form (4.51). Bearing in mind the meaning of φ and ρ together with the definition of ω_0 it is straightforward that $\sqrt{-1}R^{(L,h^L_\nu)} = \omega_0$ on $\{\psi \leqslant -\nu-1\}$. By Theorem 4.30 we have for $k \longrightarrow \infty$,

$$\dim H^0(X_{reg}, L^k\otimes K_X) \geqslant \frac{k^n}{n!}\int_{U_\nu(\leqslant 1,h^L_\nu)} \big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\nu)}\big)^n + o(k^n)\,.$$

We have denoted U_ν the compact set $\{\psi \geq -\nu-2\}$. We decompose this set in $U'_\nu = \{\psi \geqslant -\nu\}$ and $U''_\nu = \{-\nu-2 \leqslant \psi \leqslant -\nu\}$ since on U'_ν we have $\gamma_\nu(\psi) = 0$ and $\sqrt{-1}R^{(L,h^L_\nu)} = \sqrt{-1}R^{(L,h)}$. We infer that

$$\int_{U'_\nu(\leqslant 1,h_\nu)} \big(\sqrt{-1}R^{(L,h^L_\nu)}\big)^n = \int_{X_{reg}(\leqslant 1,h)} \mathbf{1}_{U'_\nu}\alpha_1\cdots\alpha_n\,dv_0 \tag{4.52}$$

where $\alpha_1,\ldots,\alpha_n$ are the eigenvalues of $\sqrt{-1}R^{(L,h^L)}$ with respect to ω_0 and dv_0 is the volume form of the same metric. Our hypothesis on the domination of $\sqrt{-1}R^{(L,h^L)}$ by the euclidian metric implies that $\sqrt{-1}R^{(L,h^L)}$ is dominated by ω and by ω_0. Hence the product $\alpha_1\cdots\alpha_n$ is bounded on X_{reg}. Since $X_{reg}(\leqslant 1)$ has finite volume with respect to ω_0 the functions $|\mathbf{1}_{U'_\nu}\alpha_1\cdots\alpha_n|$ are bounded by an integrable function. On the other hand $\mathbf{1}_{U'_\nu} \longrightarrow 1$ when $\nu \longrightarrow \infty$ so that the integrals in (4.52) tend to $\int_{X_{reg}(\leqslant 1,h)}\big(\sqrt{-1}R^{(L,h^L)}\big)^n$ which is assumed to be positive.
Thus it suffices to show that the integral on the set U''_ν i.e.

$$\int_{U''_\nu(\leqslant 1,h_\nu)} \big(\sqrt{-1}R^{(L,h^L_\nu)}\big)^n$$

tends to zero as $\nu \longrightarrow \infty$. For this purpose we use the obvious bound

$$\int_{U''_\nu(\leqslant 1,h_\nu)} \big(\sqrt{-1}R^{(L,h^L_\nu)}\big)^n \leqslant \sup|\,\delta_1\cdots\delta_n|\cdot\mathrm{vol}\,(U''_\nu)$$

where $\delta_1,\ldots,\delta_n$ are the eigenvalues of $\sqrt{-1}R^{(L,h^L)}$ with respect to ω_0 and the volume is taken in the same metric. We use now the minimum-maximum principle to see that δ_1 is bounded bellow and $\delta_2,\ldots,\delta_n$ are bounded above on the set of integration $U''_\nu(\leqslant 1,h_\nu)$. For this we need the domination of $\sqrt{-1}R^{(L,h^L)}$ by ω and the boundedness of γ'_ν and γ''_ν. Since $\mathrm{vol}\,(U''_\nu) \longrightarrow 0$ as $\nu \longrightarrow \infty$ our contention follows. Hence $\dim H^0(X_{reg}, E^k\otimes K_X) \gtrsim k^n$ so that X_{reg} has n independent meromorphic functions which can be extended to X by the Levi extension theorem. The proof is finished.

4.37. REMARK. The proof of the Theorem 1.10 is based on the existence of the exhaustion function from below χ and of the complete metric ω_0 with the properties (4.48) and (4.47). These objects are specific to the case of isolated singularities. If X is a compact complex space with $\dim X_{sing} \geqslant 1$, X_{reg} does not generally possess a strictly psh exhaustion function from below. That is why for general complex spaces we need stronger hypothesis in order to obtain the fundamental L^2 estimate for $(n,1)$–forms. For example if X is a compact complex Kähler space, X_{reg} admits complete Kähler metric (Ohsawa [**33**]).

Therefore, if X_{reg} admits a semipositive line bundle which is positive at a point p, standard L^2 estimates for $\overline{\partial}$ show that $L^k \otimes K_{X_{reg}}$ gives local coordinates at p. Assuming that $\operatorname{codim} X_{sing} \geqslant 2$ it follows first that X_{reg} has a maximal number of meromorphic functions (since X_{reg} is pseudoconcave in the sense of Andreotti) and then that X is Moishezon (by the Levi extension theorem).

In the non–Kähler case we need a sort of uniform positivity condition on L near X_{sing} in order to absorb the torsion of a complete metric on X_{reg}. In this respect the hypothesis in Takayama's theorem 4.40 seem appropriate. If we want the line bundle L to be defined only on X_{reg} we can introduce the following alternative condition.

Let ω be a hermitian metric on X_{reg} induced from a resolution of singularities $\overline{X}$ of X. Assume that $\sqrt{-1}R^L \geqslant \omega$ outside a compact set of X_{reg} and that L satisfies Demailly's condition (1.10). Suppose moreover that $\operatorname{codim} X_{sing} \geqslant 2$. Then X is Moishezon. Indeed, the condition $\sqrt{-1}R^L \geqslant \omega$ shows that we can argue as in Section 4.5 and use a generalized Poincaré metric to deduce the fundamental L^2 estimate.

It would be interesting to know whether criteria as the Theorem 1.10 carry over to general complex spaces.

4.5. The Shiffman–Ji–Bonavero Criterion

In this section we study the L^2 cohomology of Zariski open sets in compact complex spaces. Using our previous results we will prove a theorem of Takayama [**43**] generalizing the Siu–Demailly criterion if $L \longrightarrow X$ is a line bundle endowed with a singular hermitian metric which is smooth outside a proper analytic set $Z \supset X_{\text{sing}}$ and defines a strictly positive current near Z.

Let us briefly describe the generalized Poincaré metric. We denote by Δ the unit disc in $\mathbb{C}$ and by $\Delta^* = \Delta \smallsetminus \{0\}$. The Poincaré metric on Δ^* is

$$\omega_P = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{|z|^2(\log|z|^2)^2} \tag{4.53}$$

More generally, on the product $(\Delta^*)^l \times \Delta^{n-l}$ we introduce the metric

$$\omega = \frac{\sqrt{-1}}{2}\sum_{k=1}^{l}\frac{dz_k\wedge d\bar{z}_k}{|z_k|^2(\log|z_k|^2)^2} + \frac{\sqrt{-1}}{2}\sum_{k=l+1}^{n} dz_k\wedge d\bar{z}_k. \tag{4.54}$$

Let us consider a compact complex manifold and let Z be a union of smooth divisors with normal crossings. For any point $p \in Z$ there exists a coordinate neighbourhood U of p isomorphic to Δ^n in which $(X \smallsetminus Z) \cap U = \{z = (z_1, \dots, z_n) : z_1 \neq 0, \dots, z_l \neq 0\}$. Such coordinates are called special. We endow $(X \smallsetminus Z) \cap U \cong (\Delta^*)^l \times \Delta^{n-l}$ with the metric (4.54). It is a metric possessing the singularity of the Poincaré metric near the punctures. We define further the generalized Poincaré metric (also called Griffiths-Carlson metric) which is a very useful tool in the analysis on Zariski open sets. It was introduced in [**9**] (see also [**48**]) and plays an important role in value distribution theory and hyperbolic geometry.

4.38. PROPOSITION. *There exists a complete metric (of finite volume) on $X \smallsetminus Z$, called the generalized Poincaré metric, which in special coordinates is equivalent to the metric* (4.54). *It has bounded torsion and Ricci curvature.*

PROOF. We write $Z = \cup Z_i$ and consider a section σ_j of the line bundle $[Z_j]$ which vanishes to first order on Z_j. Then we endow $[Z_j]$ with a hermitian metric such that the norm of σ_j satisfies $|\sigma_j| < 1$. We take an arbitrary smooth metric Θ' on X and set

$$\Theta_{\varepsilon_0} = \Theta' - \tfrac{\varepsilon_0}{2}\sqrt{-1}\sum \partial\overline{\partial}\log(-\log|\sigma_i|^2)^2. \tag{4.55}$$

In special coordinates U in which Z_i is defined by $z = 0$, $|\sigma_i|^2 = |z|^2 e^u$, $u \in \mathscr{C}^\infty(U)$. Then

$$-\frac{1}{2}\partial\overline{\partial}\log(-\log|\sigma_i|^2)^2 = \frac{1}{(u+\log|z|^2)^2}\big(\tfrac{dz}{z}+\partial u\big)\wedge\overline{\big(\tfrac{dz}{z}+\partial u\big)} - \frac{1}{u+\log|z|^2}\partial\overline{\partial}u\,.$$

The first term in the sum is semipositive. As for the second, $-1/(u+\log|z|^2) > 0$ and tends to 0 near the singular set. Since $\partial\overline{\partial}u$ is smooth, the second term is bounded below by the negative of some smooth metric on X. This argument shows that Θ_{ε_0} is a metric on $U \smallsetminus Z$ for ε_0 small enough. Taking ε_0 small enough also ensures that Θ_{ε_0} is positive on the whole $X \smallsetminus Z$. It is clear that Θ_{ε_0} and (4.54) have the same type of singularity and this shows that Θ_{ε_0} is complete and has finite volume. Let us denote by

$$\varphi = -\log\prod(-\log|\sigma_i|^2)^2 \tag{4.56}$$

The function φ is quasi–plurisubharmonic i. e. $\partial\overline{\partial}\varphi \geqslant -C\Theta'$ and $\Theta_{\varepsilon_0} = \Theta' - \frac{\varepsilon_0}{2}\sqrt{-1}\partial\overline{\partial}\varphi$. We wish to show that there exist a constant $C > 0$ such that

$$\sqrt{-1}R^{\det} > -C\Theta_{\varepsilon_0}\,,\ |T_{\varepsilon_0}| < C\,. \tag{4.57}$$

where $T_{\varepsilon_0} = [\Theta_{\varepsilon_0}, \partial\Theta_{\varepsilon_0}]$ is the torsion operator of Θ_{ε_0} and $|T_{\varepsilon_0}|$ is its norm with respect to Θ_{ε_0}. Now $\partial\Theta_{\varepsilon_0} = \partial\Theta'$ by (4.55), so it extends smoothly over $\widetilde{X}$, and thus we get the second relation of (4.57).

We turn now to the first condition of (4.57). We have

$$\Theta_{\varepsilon_0} = \Theta' + 2\sqrt{-1}\varepsilon_0\sum_i\Big(\frac{R^{[\Sigma_i]}}{\log\|\sigma_i\|_i^2} + \frac{\partial\log\|\sigma_i\|_i^2\wedge\overline{\partial}\log\|\sigma_i\|_i^2}{(\log\|\sigma_i\|_i^2)^2}\Big)\,. \tag{4.58}$$

The terms $R^{[\Sigma_i]}/\log\|\sigma_i\|_i^2$ tend to zero as we approach Σ so they can be absorbed in Θ' and do not contribute to the singularity of Θ_{ε_0} near Σ. To examine the last term let us localize to a point $x_0 \in \Sigma$. We choose special coordinates in a neighborhood U of x_0 in which Σ_j has the equation $z_j = 0$ for $j = 1,\dots,k$ and Σ_j, $j > k$, do not meet U. Then for $1 \leq i \leq k$, $\|\sigma_i\|_i^2 = u_i|z_i|^2$ for some positive smooth function u_i on U and

$$\frac{\partial\log\|\sigma_i\|_i^2\wedge\overline{\partial}\log\|\sigma_i\|_i^2}{(\log\|\sigma_i\|_i^2)^2} = \frac{dz_i\wedge d\overline{z}_i + v_i}{|z_i|^2(\log\|\sigma_i\|_i^2)^2} \tag{4.59}$$

where v_i is a smooth $(1,1)$–form on U. Without loss of generality we may assume that Θ' is the Euclidean metric on U so that Θ'^n is the Euclidean volume element. Then there exists a smooth function β such that

$$\Theta_{\varepsilon_0}^n = \left(1 + \frac{1+\beta(z)}{\prod_i|z_i|^2(\log\|\sigma_i\|_i^2)^2}\right)\Theta'^n =: \gamma(z)\Theta'^n\,. \tag{4.60}$$

and consequently

$$\sqrt{-1}R^{\det} = -\sqrt{-1}\,\partial\overline{\partial}\log\gamma(z) = -\sqrt{-1}\Big(\frac{\partial\overline{\partial}\gamma(z)}{\gamma(z)} - \frac{\partial\gamma(z)\wedge\overline{\partial}\gamma(z)}{\gamma(z)^2}\Big) \geqslant -\sqrt{-1}\,\frac{\partial\overline{\partial}\gamma(z)}{\gamma(z)}\,. \tag{4.61}$$

A brute force calculation of $-\sqrt{-1}\,\partial\overline{\partial}\gamma(z)/\gamma(z)$ and comparison to the singularities of Θ_{ε_0} given by (4.59) show that $\sqrt{-1}R^{\det} > -C\Theta_{\varepsilon_0}$ for some positive constant C. This achieves the proof of (4.57). □

4.39. THEOREM. *Let X be an n–dimensional compact manifold and let L be a holomorphic line bundle with a singular hermitian metric h^L. We assume that:*

(1) $\sqrt{-1}R^{(L,h^L)}$ *is smooth on $M := X \smallsetminus Z$ where Z is a divizor with only simple normal crossings;*
(2) $\sqrt{-1}R^{(L,h^L)}$ *is a strictly positive current in a neighbourhood of Z.*

Then,

$$\dim H^0_{(2)}(M,L^k) \geqslant \frac{k^n}{n!}\int_{M(\leqslant 1,h)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L)}\Big)^n + o(k^n), \quad k\to\infty, \tag{4.62}$$

where $H^0_{(2)}(M,L^k)$ is the space of sections of L^k which are L^2 with respect to the restrictions to M and $L|_M$ of smooth metrics on X and L.

PROOF. We use the notations from the proof of Proposition 4.38. We consider the following family of metrics on $L\restriction_M$: $h^L_\varepsilon = h^L\prod_j(-\log|\sigma_j|^2)^\varepsilon$, $\varepsilon>0$.

The curvature $\sqrt{-1}R^L$ is strictly positive in the sense of currents near Z, so there exists $A>0$ such that $\sqrt{-1}R^L \geqslant A\Theta'$ outside a compact set of $X\smallsetminus Z$. Note that

$$\sqrt{-1}R^{(L,h^L_\varepsilon)} = \sqrt{-1}R^L + \varepsilon\sqrt{-1}\partial\overline{\partial}\varphi$$

where φ has been defined in (4.54). Then

$$\begin{aligned}\sqrt{-1}R^{(L,h^L_\varepsilon)} - \delta\Theta_{\varepsilon_0} &= \sqrt{-1}R^L + (\varepsilon-\delta\varepsilon_0)\sqrt{-1}\partial\overline{\partial}\varphi - \delta\Theta' \\ &\geqslant \sqrt{-1}R^L - (\varepsilon-\delta\varepsilon_0)C\Theta' - \delta\Theta' \\ &\geqslant \big(A-(\varepsilon-\delta\varepsilon_0)C-\delta\big)\Theta'\end{aligned}$$

If ε and δ are sufficiently small, $\sqrt{-1}R^{(L,h^L_\varepsilon)} - \delta\Theta_{\varepsilon_0} \geqslant \frac{A}{2}\Theta'$, so that $\sqrt{-1}R^{(L,h^L_\varepsilon)} \geqslant \delta\Theta_{\varepsilon_0}$.

Near Z the metric h^L is locally represented by a strictly plurisubharmonic weight. Thus h^L is locally bounded below near Z and $h^L \geqslant Ch^L_*$ on X for some positive constant C and some smooth metric h^L_*. Consider the space

$$H^0_{(2)}(M,L^k)_\varepsilon := \{u\in \boldsymbol{L}^2(M,L^k,\Theta_{\varepsilon_0},h^L_\varepsilon) : \overline{\partial}^{L^k}u=0\}.$$

Since the Poincaré metric dominates the euclidian metric in local coordinates near Z, the L^2 condition with respect to the Poincaré metric implies the that the elements of $H^0_{(2)}(M,L^k)_\varepsilon$ extend holomorphically to sections of L^k, therefore $H^0_{(2)}(M,L^k)_\varepsilon \subset H^0_{(2)}(M,L^k)$. By Theorem 4.30 for $K\Subset U\Subset M$

$$\dim H^0_{(2)}(M,L^k) \geqslant \dim H^0_{(2)}(M,L^k)_\varepsilon \geqslant \int_{U(\leqslant 1,h^L_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n + o(k^n).$$

We can let $\varepsilon \longrightarrow 0$ in the right–hand side in order to replace h^L_ε with h^L. Then we can let U exhaust M to get the inequality from the statement. □

4.40. THEOREM (Takayama). *Let X be an n–dimensional reduced and irreducible compact complex space and let L be a holomorphic line bundle over X with a singular Hermitian metric h^L. Assume that the curvature current R^L is smooth on the complement of some proper analytic subset $Z\supset X_{sing}$ and that $\sqrt{-1}R^L$ is strictly positive on some neighbourhood of Z. Then $\int_{(X\smallsetminus Z)(\leqslant 1,L)}(\sqrt{-1}R^L)^n$ exists and if it is strictly positive X is a Moishezon space.*

PROOF. We show that we can reduce the proof to an application of Theorem 4.39. Let $\tau : \widehat{X} \longrightarrow X$ be a resolution of singularities of X (see [**23**]), such that $\tau^{-1}(Z) = D$, a divisor with only simple normal crossings. There exists a finite sequence of blow-ups

$$\widehat{X} = X_m \xrightarrow{\tau_m} X_{m-1} \xrightarrow{\tau_{m-1}} \cdots \xrightarrow{\tau_2} X_1 \xrightarrow{\tau_1} X_0 = X$$

such that

(a) τ_i is the blow-up along a non-singular center Y_{i-1} contained in the total transform of Z in X_{i-1}, $i \geqslant 1$

(b) the total transform of Z in $\widehat{X} = X_m$ through $\tau = \tau_m \circ \tau_{m-1} \circ \cdots \circ \tau_1$ is a divisor with only simple normal crossings.

We build next an integral Kähler current $\widehat{T}$ on $\widehat{X}$. Let $Y_0^* = \tau_1^{-1}(Y_0)$, the total transform of Y_0. Then there exists a smooth hermitian metric h_0 on the line bundle $\left[Y_0^*\right]^{-1}$ whose curvature satisfies the following conditions:

1) is strictly positive along Y_0^*,

2) is bounded on X_1,

3) vanishes outside a neighbourhood of Y_0^*.

On X_1 we consider the bundle $L_1 := \tau_1^*(L^{k_1}) \otimes [Y_0^*]^{-1}$ endowed with the metric $h^{L_1} = (h^L)^{\otimes k_1} \otimes h_0$, for $k_1 \in \mathbb{N}$. The curvature current of (L_1, h^{L_1}) is $T_1 = k_1 \tau_1^* T + \sqrt{-1} R^{([Y_0^*]^{-1}, h_0)}$. The current $\tau_1^* T$ is positive on X_1 and strictly positive on any compact set disjoint from Y_0^*. Hence, properties 1) – 3) show that for k_1 sufficiently large T_1 is a strictly positive current near $Z_1 := \tau_1^{-1}(Z)$, the integral $\int_{X_{1reg}(\leqslant 1, L_1)} (\sqrt{-1} R^{L_1})^n$ is finite if and only $\int_{X_{reg}(\leqslant 1, L)} (\sqrt{-1} R^L)^n$ is finite and the first integral is > 0 if the second is.

Continuing inductively we construct a line bundle $(\widehat{L}, h^{\widehat{L}})$ on $\widehat{X}$ with curvature current $\widehat{T}$ smooth on $\widehat{M} := \widehat{X} \smallsetminus \tau^{-1}(Z)$ and positive on Z. Note that $\tau^{-1}(Z) = \tau_m^{-1}(Z_{m-1}) = D$. By (4.62) we obtain

$$\dim H^0_{(2)}(\widehat{M}, \widehat{L}^k) \geqslant \frac{k^n}{n!} \int_{\widehat{M}(\leqslant 1, h^{\widehat{L}})} \left(\tfrac{1}{2\pi} \widehat{T}\right)^n + o(k^n), \quad k \to \infty, \tag{4.63}$$

where the L^2 condition is taken w.r.t. smooth metrics on $\widehat{M}$ and $\widehat{L}$. So actually $H^0_{(2)}(\widehat{M}, \widehat{L}^k) = H^0(\widehat{X}, \widehat{L}^k)$ by [**11**, Lemme 6.9]. Thus the integral in (4.63) is finite and with it also $\int_{(X \smallsetminus Z)(\leqslant 1, L)} (\sqrt{-1} R^L)^n$. Moreover, if the latter integral is strictly positive, we have by(4.63) that $\dim H^0(\widehat{X}, \widehat{L}^k) = O(k^n)$, for $k \to \infty$, and consequently $\widehat{L}$ is big and $\widehat{X}$ is Moishezon. Since X and $\widehat{X}$ are bimeromorphicallly equivalent, it follows that X is Moishezon, too. □

In the proof of Theorem 4.40 we cannot infer that L is big. Of course, $M = X \smallsetminus Z$ and $\widehat{M} = \widehat{X} \smallsetminus \tau^{-1}(Z)$ are biholomorphic through τ and $\widehat{L}|_{\widehat{M}} = \tau^*(L^r|_M)$, for some $r \in \mathbb{N}$. So $H^0_{(2)}(\widehat{M}, \widehat{L}^k) \hookrightarrow H^0(M, L^{rk})$ and $\dim H^0(M, L^{rk}) = O(k^n)$. But in general the sections of $H^0(M, L^{rk})$ do not extend holomorphically past Z. This happens, however, in some interesting cases.

4.41. COROLLARY. *Under the same conditions as in Theorem 4.40 assume that X is normal, $Z = X_{sing}$ and and $\int_{(X_{reg})(\leqslant 1, L)} (\sqrt{-1} R^L)^n > 0$. Then L is big and X is Moishezon.*

Indeed, in this case all sections of $H^0(M, L^k)$ extend to sections of $H^0(X, L^k)$. However, if we assume that X is smooth the situation is optimal.

4.42. COROLLARY. *Let L be a holomorphic line bundle over a compact complex manifold. Then L is big if and only if there exists a current $T \in c_1(L)$ satisfying the following* *conditions:* (i) $\operatorname{sing\,supp} T \subset Z$*, where* Z *is a proper analytic set,* (ii) T *is strictly positive in the neighbourhood of* Z *and* (iii) $\int_{(X\setminus Z)(\leqslant 1,T)} T^n > 0$.

A manifold X *is Moishezon if and only if there exists an integral current satisfying the conditions* (i)-(iii) (*in particular, if and only if it possesses an integral Kähler current satisfying* (i)).

PROOF. Assume that L is big. Then we know that X is Moishezon and by Moishezon's characterization Theorem 4.23, there exists a proper modification $\tau:\widehat{X}\to X$, with $\widehat{X}$ projective. Then $\widehat{L}=\tau^*L$ is big, since the pull-back morphism $H^0(X,L^k)\to H^0(\widehat{X},\widehat{L}^k)$ is injective. Using the existence of an ample line bundle on $\widehat{X}$ we can construct a singular hermitian metric $h^{\widehat{L}}$ on $\widehat{L}$ with $\sqrt{-1}R^{\widehat{L}}>0$ in the sense of currents (see [**15**], [**17**, Proposition 6.6. (f)], [**37**]). Indeed, one can write $\widehat{L}^r=A\otimes E$ for some r large enough, where A is ample and E is effective. On E there is a hermitian metric h^E, such that $\sqrt{-1}R^E=[\operatorname{Div} s]=[E]$, where s is a global holomorphic section of E. Then $h^{\widehat{L}}:=(h^A\otimes h^E)^{1/r}$ is a hermitian metric on $\widehat{L}$ with strictly positive curvature current $\widehat{T}$. The metric $h^L:=\tau_*h^{\widehat{L}}$ on L has curvature current $T=\tau_*\widehat{T}$, which satisfies the conditions (i) – (iii).

Conversely, we proceed as in the proof of Theorem 4.40 to construct a blow-up $\tau:\widehat{X}\to X$ such that $\tau^{-1}(Z)$ is a divisor with simple normal crossings. Moreover, we $\tau:\widehat{M}\to M$ is biholomorphic, where we set as before $\widehat{M}=\widehat{X}\setminus\tau^{-1}(Z)$ and $M=X\setminus Z$. We also construct a line bunlde $\widehat{L}$ on $\widehat{X}$ such that $\widehat{L}|_{\widehat{M}}=\tau^*(L^r|_M)$. Moreover, $\dim H^0_{(2)}(\widehat{M},\widehat{L}^k)=O(k^n)$, as $k\to\infty$, where the L^2 condition is w.r.t. smooth metrics on $\widehat{X}$ and $\widehat{L}$. Such metrics are quasi-isometric on $\widehat{M}\cong M$ and $\widehat{L}|_{\widehat{M}}\cong L^r|_M$ with smooth metrics on X and L^r. Thus, by pushing forward the elements of $H^0_{(2)}(\widehat{M},\widehat{L}^k)$ we obtain that $\dim H^0_{(2)}(M,L^{rk})=O(k^n)$, $k\to\infty$, where the L^2 condition is w.r.t. smooth metrics on X and L. Since X is smooth, [**11**, Lemme 6.9] shows again that $H^0_{(2)}(M,L^{rk})=H^0(X,L^{rk})$ and therefore L^r and L are big. □

The Shiffman-Ji-Bonavero criterion is yet another characterization of big line bundles and Moishezon manifolds, where one can drop the hypothesis (i) that the current T has singular support contained in an analytic set.

4.43. COROLLARY (Shiffman-Ji-Bonavero). *Let* L *be a holomorphic line bundle over a compact complex manifold. Then* L *is big if and only if there exists a strictly positive current* $T\in c_1(L)$.

A manifold X *is Moishezon if and only if it possesses an integral Kähler current.*

PROOF. The crucial ingredient is the approximation theorem of Demailly [**14**]. We first introduce a definition. A locally integrable function φ has analytic singularities if, locally, equals $\frac{c}{2}\log\left(\sum\lambda_j\,|f_j|^2\right)+\psi$, where f_j are holomorphic functions and, λ_j are non-negative smooth functions, whithout common zeroes, ψ is smooth and $c\in\mathbb{Q}_+$. In particular the singular set of $\partial\overline{\partial}\varphi$ is an analytic set.

4.44. THEOREM ([**14**]). *Let* T *be a closed* $(1,1)$*–current on a compact complex manifold* X*, which is bounded below by a smooth* $(1,1)$*–form* α*. There exists a sequence of Kähler currents* $T_\varepsilon=\sqrt{-1}\partial\overline{\partial}\varphi_\varepsilon+\beta$, $\varepsilon>0$*, in the same de Rham cohomology class as* T*, converging weakly to* T *as* $\varepsilon\longrightarrow 0$*, where* φ_ε *is a real function with analytic singularities and* β *is a* $\mathscr{C}^\infty$ *representative of* T.

Now, if $T \in c_1(L)$ is strictly positive, it follows from Theorem 4.44 that $T_\varepsilon \in c_1(L)$ is strictly positive for $\varepsilon > 0$ sufficiently small, and that T_ε satisfies condition (i) of Corollary 4.42. By the latter result, L is big. □

4.45. REMARK. The approach of Ji-Shiffman [**24**] is to approximate T by T_ε as above and consider a complete Kähler metric on $X \smallsetminus Z$, where $Z = \operatorname{sing\,supp} T_\varepsilon$, which allows to apply the L^2 estimates of Hörmander-Bombieri-Skoda (Theorem 4.16) in order to show that $\oplus H^0(X, L^k)$ gives local coordinates outside Z.

The proof of Bonavero [**7**] is based on his singular Morse inequalities (1.9)' and Theorem 4.44. Actually the following characterization holds:

4.46. COROLLARY (Bonavero). *Let L be a holomorphic line bundle over a compact complex manifold. Then L is big if and only if there exists a current $T \in c_1(L)$ satisfying the following conditions:* (i) $T = \sqrt{-1}\partial\overline{\partial}\varphi + \beta$, *where φ is a real function with analytic singularities and β is a $\mathscr{C}^\infty$ representative of T, and* (ii) $\int_{(X \smallsetminus Z)(\leqslant 1, T)} T^n > 0$ *where* $Z = \operatorname{sing\,supp} T$.

4.6. Holomorphic Morse Inequalities for q-convex manifolds

Let X be a q–convex manifold of dimension n and let $\varphi : X \longrightarrow \mathbb{R}$ be an exhaustion function which is q–convex outside a compact set K.

Let us consider a smooth sublevel set $X_c = \{\varphi < c\}$ such that $K \subset X_c$. We know that $\mathscr{L}_\varphi$ has at least $n - q + 1$ positive eigenvalues in the neighbourhood of ∂X_c. By [**3**, Lemma 18] we may choose a hermitian metric ω on X such that $n \inf\{\lambda_1, 0\} + \lambda_q$ is bounded from below on a neighbourhood of ∂X_c. Then [**3**, Lemma 19] shows that

$$\mathscr{L}_\varphi(u,u) \geqslant C|u|^2 \quad \text{on}\, \partial X\,, \quad u \in \Omega^{0,j}(X_c, L^k \otimes E)\,,\, j \geqslant q.$$

Let us replace the metric h^L with $h^L_\chi = he^{-\chi(\varphi)}$ for some rapidly increasing convex function χ.

We have thus

$$\int_{\partial X} \mathscr{L}_\varphi(u,u)\, dS \geqslant 0\,, \quad u \in B^{0,j}(X_c, L^k \otimes E)\,,\, j \geqslant q \tag{4.64}$$

$$\left(\sqrt{-1}R^{(L,h^L_\chi)}u, u\right) \geqslant Ck\|u\|^2\,, \quad u \in B^{0,j}(X_c, L^k \otimes E)\,, \quad \operatorname{supp} u \subset V \tag{4.65}$$

with a positive constant $C > 0$, for any $k \geqslant k_0$, with convenient k and some small neighbourhood V of ∂X_c.

Using now the Bochner–Kodaira formula with boundary term (B.25) and (4.64), (4.65) we deduce that

$$\|u\|^2 \leqslant \tfrac{C}{k}(\|\overline{\partial}^{E_k}u\|^2 + \|\overline{\partial}^{E_k *}u\|^2)\,, \quad u \in B^{0,j}(X_c, L^k \otimes E)\,, \quad \operatorname{supp} u \subset V\,,\, j \geqslant q \tag{4.66}$$

with a possibly different $C > 0$. By applying (4.66) to ζu where $u \in B^{0,j}(X_c, L^k \otimes E)$, $j \geqslant q$ and ζ is a cut–off function with $\zeta = 1$ near ∂X_c and $\operatorname{supp}\zeta \subset V$ we deduce that the fundamental estimate (4.1) holds for any $u \in B^{0,j}(X_c, L^k \otimes E)$. Since $B^{0,j}(X_c, L^k \otimes E)$ is dense in $\operatorname{Dom}(\overline{\partial}^{E_k}) \cap \operatorname{Dom}(\overline{\partial}^{E_k *})$ by Lemma A.21 we see that (4.1) is satisfied. Therefore we can apply the abstract Morse inequalities (4.19) for the spaces $H^{0,j}_{(2)}(X_c, L^k \otimes E)$, where the L^2 condition is taken with respect to the metrics ω on X_c and h^L_χ, h^E.

By the strong Hodge theorem A.26 we have $H^{0,j}_{(2)}(X_c, L^k\otimes E)\cong \mathscr{H}^{0,j}(X_c, L^k\otimes E)$ and by the representation theorem B.43 we know that

$$\mathscr{H}^{0,j}(X_c, L^k\otimes E)\cong H^{0,j}(X_c, L^k\otimes E)\cong H^j(X_c, \mathscr{O}(L^k\otimes E)).$$

Finally we obtain the following result of Th. Bouche [**8**]. His proof is based on the same principle of showing the fundamental estimate outside a compact set but he works with complete metrics.

4.47. THEOREM. *Let X be a q–convex manifold of dimension n and let (L,h^L), (E,h^E) holomorphic vector bundles of rank one and r respectively P_y.*
Then

$$\sum_{j=p}^{n}(-1)^{j-p}\dim H^j(X,\mathscr{O}(L^k\otimes E))\leqslant r\frac{k^n}{n!}\int_{X_c(\geqslant p,h^L_\chi)}\Big(\frac{\sqrt{-1}}{2\pi}R^{(L,h^L_\chi)}\Big)^n+o(k^n)$$

as $k\to\infty$, for any smooth sublevel set $X_c\supset K$, and $p\geqslant q$. If $\sqrt{-1}R^L$ is semi–positive outside a compact set K, we can replace the right–hard side integral by $\int_{X(\geqslant p,h^L)}\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n$.

In order to justify the last assertion of the theorem let us choose $d<c$ such that L is semi–positive outside X_d and $\sqrt{-1}\partial\overline{\partial}\varphi$ has $n-q+1$ positive eigenvalues.

Let us choose χ such that $\chi=0$ on $(-\infty,f)$ where $d<f<c$. The curvature $\sqrt{-1}R^{(L,h^L_\chi)}$ has then $n-q+1$ positive eigenvalues in $X_c\smallsetminus\overline{X}_d$ so that $X_c(j,h^L_\chi)\subset X_f$ for $j\geqslant q$. But on the last set $h^L_\chi=h^L$ and the assertion follows.

Instead of assuming L to be semi–positive we can assume as in [**8**] that L is l–positive outside a compact set i.e. $\sqrt{-1}R^L$ has at least $n-l+1$ positive eigenvalues. Then one can prove that the second assertion holds for $p\geqslant q+l-1$.

In the same vein one can study the growth of the cohomology groups of pseudoconvex domains and weakly 1–complete manifolds.

4.48. THEOREM ([**26**]). *Let X be a smooth, relatively compact pseudoconvex domain in a complex manifold M and let $L\longrightarrow M$ be a holomorphic line bundle which is positive in a neighbourhood of ∂X. Then*

$$\sum_{j=p}^{n}(-1)^{j-p}\dim H^{0,j}_{(2)}(X,L^k\otimes E)\leqslant r\frac{k^n}{n!}\int_{X(\geqslant p)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n)$$

for $p\geqslant 1$, where the L^2 condition is taken with respect to smooth metrics over $\overline{X}$. Moreover

$$\dim H^0(X,\mathscr{O}(L^k\otimes E))\geqslant r\frac{k^n}{n!}\int_{X(\leqslant 1)}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n). \tag{4.67}$$

In particular, if L is positive, $\dim H^0(X,\mathscr{O}(L^k))\geqslant k^n\operatorname{vol}(X)+o(k^n)$, where the volume is taken with respect to metric $\frac{\sqrt{-1}}{2\pi}R^L$.

Assume now that X is a weakly 1–complete manifold. Then X_c are smooth pseudoconvex domains for c a regular value of the exhaustion function so the previous result apply if L is positive outside a compact set $K\subset X_c$. Moreover, by [**45**, Theorem 6.2]

$$H^j(X_c,\mathscr{O}(L^k\otimes E))\cong\mathscr{H}^{0,j}(X_c,L^k\otimes E)\cong H^{0,j}_{(2)}(X,L^k\otimes E).$$

The isomorphism theorem permits therefore to state Theorem 4.48 for the cohomology groups $H^j(X,\mathscr{O}(L^k\otimes E))$, $j\geqslant 1$. This result was proved in [**8**, Th.0.2] where actually a q–positive bundle L is considered over a Kähler weakly 1–complete manifold X. The Kähler assumption was removed in [**25**] answering positively a question of Ohsawa [**32**, p.218] about the polynomial growth of degree n with respect to k of $\dim H^j(X, \mathscr{O}(L^k \otimes E))$, $j \geqslant 1$.

More importantly, (4.67) was proved by Takayama for the particular case of a sublevel set X_c of a X. By using the liberty to modify the curvature of L multiplying it with a factor $e^{-\chi(\varphi)}$ we can achieve that the integral in the right–hand side is infinite:

$$\lim_{k\to\infty} k^{-n} \dim H^0(X_c, \mathscr{O}(L^k \otimes E)) = \infty$$

if L is positive near ∂X_c.

This result toghether with the effective base point freeness methods introduced in algebraic geometry by Angehrn–Siu produce a answer to the conjecture of Nakano and Ohsawa about the embeddability of weakly 1–complete manifolds.

4.49. THEOREM ([**44**, Theorem 1.2]). *Let X be an n–dimensional weakly 1–complete manifold with a positive line bundle L. Then $L^m \otimes K_X$ is ample for $m > n(n+1)/2$. In fact, X is then embeddable into $\mathbb{P}^{2n+1}$ by a linear subsystem of $|(K_X \otimes L^m)^{\otimes(n+2)}|$ for $m > n(n+1)/2$.*

Actually, in the compact case the effective base point freeness was proved by Angehrn–Siu [**4**] with the help of the Riemann–Roch theorem, Nadel's vanishing theorem and Ohsawa–Takegoshi L^2 extension theorem. Takayama applies the same strategy replacing the Riemann–Roch theorem with (4.67).

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CHAPTER 5

Covering manifolds

5.1. Automorphic forms and the L^2–index theorem of Atiyah

5.1.1. Une observation de Poincaré. Let X be a compact Riemann surface. There is a close connection between the function theory on X and the function theory on the universal cover $\widetilde{X}$. Thanks to the uniformization theorem of Poincaré–Klein–Koebe we know that that the universal cover of a Riemann surface of genus 0 (the sphere) is a sphere, the universal cover of a Riemann surface of genus 1 (torus) is $\mathbb{C}$ and the universal cover of a Riemann surface of genus $\geqslant 2$ is the unit disc.
One of the main theorems in the theory of Riemann surfaces states that any holomorphic line bundle on X admits non-trivial meromorphic sections. In the case of tori which are quotients $\mathbb{C}/\Gamma$ where Γ is a discrete group of translations, the result follows by the theory of theta functions . These are roughly speaking functions on $\mathbb{C}$ almost periodic modulo Γ and there is a one-to-one correspondence between them and the sections of holomorphic line bundle on $\mathbb{C}/\Gamma$.
If the genus g of X is $\geqslant 2$ then the universal cover $\widetilde{X}$ is the unit disc D and $X = D/\Gamma$ where $\Gamma = \pi_1(X)$ acts on D. It is clear that Γ-invariant holomorphic or meromorphic functions on D correspond to holomorphic or meromorphic functions on X. We could try to find a non-trivial Γ-invariant meromorphic function by writting it as a quotient $m = g_1/g_2$ of two holomorphic functions on D. We have now the task to see what is the relation between the f's and the group; they are certainly not invariant, otherwise they would correspond to functions on X so they would be constants.

Poincaré solved the problem by making the following twist. Consider the canonical bundle K_D and its tensor powers $K_D^{\otimes k}$; they are trivial and a nowhere vanishing section is given by $dz^{\otimes k}$. Then any meromorphic function can be written as a quotient $m = f_1/f_2$ where f_i are Γ-invariant sections of the pluricanonical bundle $K_D^{\otimes k}$ for $k \geqslant 2$. To speak of a Γ-invariant section of a vector bundle L we need a lifting of the action of Γ on the base; if we denote by $\tilde{\gamma} : L \longrightarrow L$ the action on the bundle, then it induces a linear map $\tilde{\gamma}_z$ from L_z to $L_{\gamma z}$. A section f is called invariant if $\tilde{\gamma}_z(f(z)) = f(\gamma z)$. In our special case the action on $K_D^{\otimes k}$ is given by

$$\tilde{\gamma}(z, dz^{\otimes k}) = (\gamma z,\ \Big(\frac{d\gamma}{dz}\Big)^{-k} dz^{\otimes k}) \tag{5.1}$$

i.e. $\tilde{\gamma}_z : K_{D,z}^{\otimes k} \longrightarrow K_{D,\gamma z}^{\otimes k}$ is $f(z) \longmapsto ((\gamma^{-1})^\star f)(\gamma z)$, so that f is invariant if and only if $\gamma^\star f = f$, $\gamma \in \Gamma$, where the star indicates the pull-back of forms. Writting $f = g\, dz^{\otimes k}$ for a holomorphic function g on d we get that f is invariant if and only if

$$g(\gamma z) = \Big(\frac{d\gamma}{dz}\Big)^{-k} g(z). \tag{5.2}$$

Functions satisfying (5.2) are called *automorphic forms*. If we have two automorphic forms g_1 and g_2 it is obvious that their quotient is an invariant meromorphic function. More generally, given two Γ-invariant sections of a holomorphic line bundle with a lifting of the

action of Γ we see by using local frames that their quotient is a Γ-invariant meromorphic function.
To construct automorphic forms we use the so-called Poincaré series of order k:

$$P_k(g)(z) = \sum_{\gamma\in\Gamma} g(\gamma z)\Big(\frac{d\gamma}{dz}\Big)^k$$

where g is a bounded holomorphic function on D. It turns out that for $k \geqslant 2$ the series $P_k(g)$ is convergent and it satisfies the relation (5.2). Let us note that if $f = g\,dz^{\otimes k}$ then the Poincaré series has the form:

$$P_k(f)(z) = \sum_{\gamma\in\Gamma} \tilde{\gamma}_z^{-1} f(\gamma z).$$

Now we can use the rich function structure of D; the freedom in the choice of $g \in \mathscr{O}(D)$ allows us to find a lot of automorphic forms $P_k(g)$ for $k \geqslant 2$. If we regard them as sections of the pluricanonical line bundle $K_D^{\otimes k}$ they can be pushed down to sections of $K_X^{\otimes k}$ that separe points and tangent vectors, therefore implying that $K_X^{\otimes k}$, $k >> 2$, is very ample. This procedure can be generalized to any relatively compact open set of a Stein manifold, by replacing $\frac{d\gamma}{dz}(z)$ by the determinant of the jacobian matrix $\det J_\gamma(z)$.

5.1. THEOREM. *Let $M \Subset Y$ be an open set of a Stein manifold Y. Let $\Gamma \in \mathrm{Aut}(M)$ act properly discontinuous and freely on M such that $X = M/\Gamma$ is compact. Then the canonical bundle K_X is ample.*

For a proof see Siegel [**15**] or Kollár [**10**].

5.1.2. The L^2–index theorem. A draw-back of the method of Poincaré series is that it doesn't say anything about the existence of automorphic forms of low degree i.e. sections of $K_X^{\otimes k}$ for small k. The L^2 index theorem of Atiyah gives us the possibility of investigating this problem.

Let $(\widetilde{X}, g^{T\widetilde{X}})$ be a riemannian manifold on which a discrete group Γ acts freely and properly discontinously such that $g^{T\widetilde{X}}$ is Γ–invariant.

Let $X = \widetilde{X}/\Gamma$ be the quotient and $\pi_\Gamma : \widetilde{X} \longrightarrow X$ the canonical projection. Then $\widetilde{X}$ is a Galois covering of X of Galois group Γ. We assume $\widetilde{X}$ paracompact so that Γ will be countable. Since $g^{T\widetilde{X}}$ is Γ–invariant there exists a riemannian metric g^{TX} on X such that $\pi_\Gamma^* g^{TX} = g^{T\widetilde{X}}$. We denote by $dv_{\widetilde{X}}$ the volume form of $g^{T\widetilde{X}}$. We call U a fundamental domain of the action of Γ on $\widetilde{X}$ if the following conditions are satisfied:

a) $\widetilde{X}$ is covered by the translations of $\overline{U}$,

b) different translations of U have empty intersection and

c) $\overline{U} \smallsetminus U$ has zero measure.

Let $(\widetilde{E}, h^{\widetilde{E}})$ be a hermitian vector bundle over $\widetilde{X}$ such that the action of Γ lifts to $\widetilde{E}$. This means that for each $\gamma \in \Gamma$ there exists an automorphism $\gamma^{\widetilde{E}} : \widetilde{E} \longrightarrow \widetilde{E}$ which induces an isomorphism $\gamma_X^{\widetilde{E}} : \widetilde{E}_X \longrightarrow \widetilde{E}_{\gamma X}$ such that $(\gamma_X^{\widetilde{E}})^* h_{\gamma X}^{\widetilde{E}} = h_X^{\widetilde{E}}$. Then there exists a hermitian vector bundle (E, h^E) on X such that $(\widetilde{E}, h^{\widetilde{E}}) = (\pi_\Gamma^* E, \pi_\Gamma^* h^E)$. On the sections $\Omega_0(\widetilde{X}, \widetilde{E})$ we introduce the scalar product $(s_1, s_2) = \int_{\widetilde{X}} \langle s_1, s_2\rangle_{h^{\widetilde{E}}}\, dv_{\widetilde{X}}$ and let $L^2(\widetilde{X}, \widetilde{E})$ be the corresponding L^2 space. There is an action of Γ on $\Omega(\widetilde{X}, \widetilde{E})$ given by $L_\gamma : \Omega(\widetilde{X}, \widetilde{E}) \longrightarrow \Omega(\widetilde{X}, \widetilde{E})$,

$$L_\gamma u(x) = \gamma^{\widetilde{E}}_{\gamma^{-1}x} u(\gamma^{-1}x)\,,\; u \in \Omega(\widetilde{X}, \widetilde{E})\,,\; x \in \widetilde{X}\,.$$

By using a change of variables it is easy to check that L_γ extends to a unitary operator $L_\gamma : L^2(\widetilde{X},\widetilde{E}) \longrightarrow L^2(\widetilde{X},\widetilde{E})$.

It is easy to see that

$$L^2(\widetilde{X},\widetilde{E}) \cong L^2\Gamma \otimes L^2(U,\widetilde{E}) \cong L^2\Gamma \otimes L^2(X,E) \tag{5.3}$$

where U is a fundamental domain for the action of Γ. A basis for $L^2\Gamma$ is formed by the functions

$$\delta_\gamma(\gamma') = \begin{cases} 1 & \text{if} \quad \gamma = \gamma' \\ 0 & \text{if} \quad \gamma \neq \gamma'. \end{cases}$$

Then for $f \in L^2(\widetilde{X},\widetilde{E})$ the above identification is given by

$$f \cong (f|_{\gamma U})_\gamma \cong \sum_\gamma \delta_\gamma \otimes \tilde{\gamma}^{-1}(f|_{\gamma U})$$

which means that $L^2(U,\widetilde{E})$ is identified with those sections of $\widetilde{E}$ wich vanish outside U and for any γ we identify $\mathbb{C}\delta_\gamma \oplus L^2(U,\widetilde{E})$ with those sections wich vanish outside γU.
There is an unitary action of Γ by left translations on $L^2\Gamma$ by $l_\gamma \delta_\eta = \delta_{\gamma\eta}$. It is easy to check that actually $L_\gamma = l_\gamma \otimes \mathrm{Id}$ and that $\{L_\gamma : \gamma \in \Gamma\}$ defines a unitary action of Γ on $L^2(\widetilde{X},\widetilde{E})$.

Let $(\widetilde{F}, h^{\widetilde{F}})$ be a further Γ–invariant hermitian vector bundle over $\widetilde{X}$ and let $\widetilde{P} : \Omega(\widetilde{X},\widetilde{E}) \longrightarrow \Omega(\widetilde{X},\widetilde{F})$ be a linear differential elliptic operator.

We assume that $\widetilde{P}$ is Γ–invariant if it commutes to the action of Γ i.e. $\widetilde{P}L_\gamma = L_\gamma\widetilde{P}$. This is equivalent to the existence of a differential elliptic operator $P : \Omega(X,E) \longrightarrow \Omega(X,F)$ such that $\widetilde{P} = \pi_\Gamma^* P$.

We suppose from now on that X is *compact*. Then we can define the index of P and we wish to find a notion of index also for $\widetilde{P}$. The difficulty is that

$$\operatorname{Ker} \widetilde{P} = \{u \in L^2(\widetilde{X},\widetilde{E}) : \widetilde{P}u = 0\}$$

is in general infinite dimensional, so we cannot use the usual dimension for the definition. Nevertheless, Atiyah [**3**] showed how one can associate a positive, possibly infinite real number to this space, called the Γ–dimension. Let us call a free Hilbert Γ–module a Hilbert space of the form $L^2\Gamma \otimes \mathscr{H}$, $\mathscr{H}$ a Hilbert space, with the action $\gamma \longmapsto l_\gamma \oplus \mathrm{Id}$. For example $L^2(\widetilde{X},\widetilde{E})$ is a free Hilbert Γ–module. We call a (Hilbert) Γ–module any Γ–invariant subspace of a free Hilbert Γ–module. Note that $\operatorname{Ker}\widetilde{P}$ is a Γ–module since $\widetilde{P}$ is invariant. We shall soon take the task of defining the Γ–dimension of a Γ–module in the next Section. For the moment let us just remark that if Γ is trivial then it coincides with the usual dimension and if Γ is finite we have

$$\dim_\Gamma = \frac{1}{|\Gamma|}\dim \quad .$$

We define now the von Neumann index of $\widetilde{P}$. Let $\widetilde{P}^t : \Omega(\widetilde{X},\widetilde{F}) \longrightarrow \Omega(\widetilde{X},\widetilde{E})$ be the formal adjoint of $\widetilde{P}$, which is also Γ–invariant. We set

$$\operatorname{index}_\Gamma \widetilde{P} = \dim_\Gamma \operatorname{Ker} \widetilde{P} - \dim_\Gamma \operatorname{Ker} \widetilde{P}^t$$

We can now state Atiyah's L^2 index theorem [**3**]:

5.2. THEOREM (Atiyah). *Let $\widetilde{P} : \Omega(\widetilde{X},\widetilde{E}) \longrightarrow \Omega(\widetilde{X},\widetilde{F})$ be a Γ–invariant elliptic operator on a Galois covering $\pi_\Gamma : \widetilde{X} \longrightarrow X$ of a compact manifold $X = \widetilde{X}/\Gamma$. Then*

$$\operatorname{index}_\Gamma \widetilde{P} = \operatorname{index} P$$

where $P : \Omega(X,E) \longrightarrow \Omega(X,F)$ satisfies $\pi_\Gamma^* P = \widetilde{P}$.

Let assume in the sequel that $\widetilde{X}$ is a complex manifold and the vector bundle $\widetilde{E}$ is holomorphic. Then the Cauchy–Riemann operator $\overline{\partial}^{\widetilde{E}} : \Omega^{0,j}(\widetilde{X},\widetilde{E}) \longrightarrow \Omega^{0,j+1}(\widetilde{X},\widetilde{E})$ is Γ–invariant i.e. $L_\gamma \overline{\partial}^{\widetilde{E}} = \overline{\partial}^{\widetilde{E}} L_\gamma$: in terms of the decomposition (5.3) L_γ acts only on the first factor and $\overline{\partial}^{\widetilde{E}}$ only on the second. Recall that the formal adjoint $\vartheta^{\widetilde{E}} = -\#_{\widetilde{E^*}} \overline{\partial}^{\widetilde{E}} \#_{\widetilde{E}}$ where $\#_{\widetilde{E}} : \Omega^{0,j}(\widetilde{X},\widetilde{E}) \longrightarrow \Omega^{0,n-j}(\widetilde{X},\widetilde{E}^*)$ is the Hodge operator (B.7).

Therefore, $\vartheta^{\widetilde{E}}$ and the Kodaira–laplacian $\square^{\widetilde{E}} = \overline{\partial}^{\widetilde{E}} \vartheta^{\widetilde{E}} + \vartheta^{\widetilde{E}} \overline{\partial}^{\widetilde{E}}$ are also Γ–invariant. The Γ–index of $\square^{\widetilde{E}}$ is the Γ–Euler characteristic of $\widetilde{E}$,

$$\operatorname{index}_\Gamma(\square^{\widetilde{E}}) = \sum_{j=0}^{n} (-1)^j \dim_\Gamma \overline{H}^{0,j}_{(2)}(\widetilde{X},\widetilde{E})$$

where $\overline{H}^{0,j}_{(2)}(\widetilde{X},\widetilde{E})$ are the reduced cohomology groups (A.29). From Theorem 5.2 we obtain:

5.3. THEOREM. *Let $\widetilde{X}$ be complex manifold of dimension n with a properly discontinuous and free action of a discrete group Γ such that $X = \widetilde{X}/\Gamma$ is a compact Kähler manifold. Assume that there exist a holomorphic line bundle $\widetilde{E}$ on $\widetilde{X}$ such that the action of Γ lifts to $\widetilde{E}$ and suppose that we have fixed on $\widetilde{X}$ the pull-back of the Kähler metric from X and on $\widetilde{E}$ the pull-back of a hermitian metric on $E = \widetilde{E}/\Gamma$. Then the Euler characteristic of the adjoint bundle $K_X \otimes E$ equals the Euler Γ–characteristic of the adjoint bundle $K_{\widetilde{X}} \otimes \widetilde{E}$:*

$$\sum_{q=0}^{n} (-1)^q \dim H^q(X, K_X \otimes E) = \sum_{q=0}^{n} (-1)^q \dim_\Gamma \overline{H}^q_{(2)}(\widetilde{X}, K_{\widetilde{X}} \otimes \widetilde{E}).$$

Unfortunately the previous theorem involves higher cohomology groups. But if we make additional assumptions on the bundle E the higher cohomology vanishes:

5.4. THEOREM. *Suppose that E is a positive line bundle. Then* $\dim_\Gamma \overline{H}^0_{(2)}(\widetilde{X}, K_{\widetilde{X}} \otimes \widetilde{E}) = \dim H^0(X, K_X \otimes E)$.

PROOF. Indeed, by the Kodaira vanishing theorem $H^q(X, K_X \otimes E) = 0$ for $q \geqslant 1$ since E is positive. Moreover, $\mathscr{H}^q_{(2)}(\widetilde{X}, K_{\widetilde{X}} \otimes \widetilde{E}) = 0$ by the vanishing theorem of Andreotti-Vesentini [**2**] (see also the vanishing results from [**5**]) since $\widetilde{X}$ is a complete Kähler manifold and $\widetilde{E}$ is positive. □

The use of Theorem 5.4 is twofold. We can either deduce the existence of automorphic forms of low degree obtaining non-vanishing theorems on X or we can find holomorphic L^2 sections on $\widetilde{X}$ obtaining non-vanishing theorems on $\widetilde{X}$. As an example of the first application we have the following.

5.5. COROLLARY ([**10**]). *Let $\widetilde{X} \Subset Y$ be an open set of a Stein manifold Y and Γ a discrete group acting properly discontinuous and freely on $\widetilde{X}$. If $X = \widetilde{X}/\Gamma$ is compact then:*

i) $\dim H^0(X, K_X^{\otimes k}) \geqslant 1$ *for* $k \geqslant 2$.

ii) $\dim H^0(X, K_X^{\otimes k}) \geqslant 2$ *for* $k \geqslant 4$.

Indeed we can apply the previous theorem together with the fact that $K_{\widetilde{X}}$ is generated by its L^2 sections which shows that $\dim_\Gamma \overline{H}^0_{(2)}(\widetilde{X}, K_{\widetilde{X}}^{\otimes k}) \geqslant 1$.

The other direction is to apply Theorem 5.4 in order to get information about the existence of holomorphic sections on $\widetilde{X}$.

5.6. COROLLARY ([**10**], Theorem 16.5). *Let X be a projective manifold of dimension n and E a positive line bundle on X. Fix a positive integer $a(E)$ such that $K_X^{-1} \otimes E^{a(E)}$ is positive also. Let $p : \widetilde{X} \longrightarrow X$ a covering corresponding to a quotient Γ of $\pi_1(X)$. Then for $k \geqslant (n+2)^{(n+6)}(a(E)+n)$ we have that E^k is very ample and $\widetilde{E}^k = p^\star E^k$ is generated by its holomorphic L^2 sections. Moreover for k large enough holomorphic L^2 sections of $\widetilde{E}^k$ separate points of $\widetilde{X}$.*

This corollary has connections to the Shafarevich conjecture:

5.7. SHAFAREVICH CONJECTURE. *Let X be a smooth projective manifold and $\widetilde{X}$ its universal cover. Then there is a proper holomorphic morphism with connected fibers onto a normal Stein space.*

Since by the Remmert reduction every holomorphically convex space admits such a morphism it would be sufficient to prove that $\widetilde{X}$ is holomorphically convex (i.e. for every discrete sequence $\{x_i\}$ there exists a holomorphic function f which blows up on the sequence, $\sup|f(x_i)| = \infty$). We can introduce a weaker form, the convexity with respect to a hermitian line bundle, by replacing the function with a holomorphic section. Napier [**11**] showed that if X carries a positive line bundle E then $\widetilde{X}$ is holomorphically convex with respect to $\widetilde{E}^k$ for large k. In particular the dimension of the space of holomorphic sections is infinite for large k.

5.1.3. Definition and properties of the Γ–dimension. It is probably high time to give a definition of the Γ–dimension of a Γ–module of $L^2(\widetilde{X}, \widetilde{E})$. We follow the elementary account of Kollár [**10**]. For a general account in the natural framework of von Neumann algebras see Cohen [**6**]. Let us consider a Hilbert space G and denote by $\mathscr{B}(G)$ the algebra of bounded linear endomorphisms of G. Let us remind that there exists a function $\mathrm{Tr} : \mathscr{B}(G) \longrightarrow \mathbb{C} \cup \{\infty\}$ called trace which is linear and $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$ for $A, B \in \mathscr{B}(G)$. Moreover for a positive operator $A \geqslant 0$ (i.e. $(Ag, g) \geqslant 0$ for all $g \in G$) we have $\mathrm{Tr}\,A \geqslant 0$ and $\mathrm{Tr}\,A = 0$ if and only if $A = 0$ (for notions of operator algebras like trace see Strătilă & Zsidó [**16**]). Let us remind the definition of the trace of a positive operator. Fix an orthonormal basis $\{e_i\}$ of G and put $\mathrm{Tr}\,A := \sum (Ae_i, e_i) \in [0, \infty]$. The definition does not depend on the choice of the basis. An important case of positive operator is the orthogonal projection P on a closed subspace of $F \subset G$: choosing an orthonormal basis in F and in the orthogonal complement of F we check that

$$\mathrm{Tr}\,P = \dim F. \tag{5.4}$$

Let us also note that for any $B \in \mathscr{B}(G)$, B^*B and BB^* are positive and if B is represented by the matrix $\big(b_{ij} = (Be_i, e_j)\big)$ then $\mathrm{Tr}\,B^*B = \sum |b_{ij}|^2$ and in particular $\mathrm{Tr}\,B^*B = \mathrm{Tr}\,BB^*$. Let us take now $G = L^2\Gamma$ for a discrete group Γ. As we have already noted a basis of this space is $(\delta_\gamma)_\gamma$ and there is a unitary action of Γ on $L^2\Gamma$ given by the left translations l_γ. Let consider now the algebra $\mathscr{A}_\Gamma \subset \mathscr{B}(L^2\Gamma)$ of all operators that commute with all left translations. Let us denote the unit element of Γ by e. Then we can introduce a trace on $\mathscr{A}_\Gamma$ by

$$\mathrm{Tr}_\Gamma A = (A\delta_e, \delta_e), \quad A \in \mathscr{A}_\Gamma. \tag{5.5}$$

This definition is justified by the following. Suppose that A is given by the matrix $\big(a_{\gamma\eta} = (A\delta_\gamma, \delta_\eta)\big)$. Applying the equality $Al_\gamma = l_\gamma A$ to δ_e and using $l_\gamma \delta_e = \delta_\gamma$ we get that the matrix coefficients satisfy the relation $a_{\gamma\eta} = a_{e,\gamma^{-1}\eta}$. In order to define a trace as before we should consider $\sum (A\delta_\gamma, \delta_\gamma) = \sum a_{\gamma\gamma} = |\Gamma| a_{ee}$ which is finite only if Γ is finite or $A = 0$. Therefore we renormalize and obtain (5.5). It is easy to see that Tr_Γ satisfies the same properties as

Tr. Let us consider a left Γ–invariant subspace $F \subset L^2\Gamma$ (i.e. a Γ–module), so that the orthogonal projection P on F belongs to $\mathscr{A}_\Gamma$. Imitating (5.4) we put

$$\dim_\Gamma F = \mathrm{Tr}_\Gamma P = (P\delta_e, \delta_e).$$

The Γ–dimension has the following properties:

(1) $0 \leqslant \dim_\Gamma F \leqslant 1$.
(2) $\dim_\Gamma F = 0$ if and only if $F = 0$.
(3) $F \subset F'$ implies $\dim_\Gamma F \leqslant \dim_\Gamma F'$ with equality if and only if $F = F'$.

There is a useful formula for $\dim_\Gamma F$ in terms of an othonormal basis $\{e_i\}$ of F. We complete this basis to an ONB $\{f_i, g_j\}$ of $L^2\Gamma$ and expanding $\delta_e = \sum(\delta_e, f_i)f_i + \sum(\delta_e, g_j)g_j = \sum \overline{f_i}(e)f_i + \sum(\delta_e, g_j)g_j$ we get

$$\dim_\Gamma F = (P\delta_e, \delta_e) = (\sum \overline{f_i}(e)f_i, \delta_e) = \sum |f_i(e)|^2. \tag{5.6}$$

As an example let us consider the isomorphism $L^2(S^1) \approx L^2\mathbb{Z}$ given by $u \longmapsto (u_n)_{n\in\mathbb{Z}}$ where u_n are the Fourier coefficients with respect to athe basis $\exp(int)$. This basis correspond to the basis δ_n of $L^2\mathbb{Z}$ so the action of $\mathbb{Z}$ on $L^2(S^1)$ is given by the multiplication with $\exp(int)$. Let us consider the subspace $F_B \in L^2(S^1)$ of functions vanishing outside the measurable set B. It is a $\mathbb{Z}$–invariant set and the projection on F_B is given by $P_B f = \chi_B f$ where χ_B is the characteristic function of B. Thus

$$\dim_\mathbb{Z} F_B = (P_B\delta_0, \delta_0) = (\chi_B \cdot 1, 1) = \text{measure}(B).$$

Finally let us introduce the Γ–dimension for Γ–modules of $L^2(\widetilde{X}, \widetilde{E})$. As before we denote by $\mathscr{A}_\Gamma \subset \mathscr{B}(L^2(\widetilde{X}, \widetilde{E}))$ the algebra of all operators which commute with the action of Γ. Then to any operator $A \in \mathscr{B}(L^2(\widetilde{X}, \widetilde{E}))$ we can associate operators $a_{\gamma\eta} \in \mathscr{B}(L^2(U, \widetilde{E}))$ such that $a_{\gamma\eta}(f)$ is the projection of $A(\delta_\gamma \oplus f)$ on $\mathbb{C}\delta_\eta \oplus \mathscr{B}(L^2(U, \widetilde{E}))$. If moreover $A \in \mathscr{A}_\Gamma$ then the matrix $(a_{\gamma\eta})$ satisfies the relation $a_{\gamma\eta} = a_{e,\gamma^{-1}\eta}$ and we can define

$$\mathrm{Tr}_\Gamma A = \mathrm{Tr}\, a_{ee}.$$

Remark that if A is positive then a_{ee} is positive too so we have a formula for $\mathrm{Tr}\, a_{ee}$. If $F \in L^2(\widetilde{X}, \widetilde{E})$ is a Γ–module then the projection P on F is in $\mathscr{A}_\Gamma$ and we put

$$\dim_\Gamma F = \mathrm{Tr}_\Gamma P.$$

Imitating (5.6) we get

$$\dim_\Gamma F = \sum_i \int_U |e_i|^2 dv_{\widetilde{X}} \tag{5.7}$$

where $\{e_i\}$ is an orthonormal basis in F.

Let us introduce now the notion of Γ–morphism $A : V_1 \longrightarrow V_2$ between two Γ–modules: it is a bounded operator which commutes with the action of Γ on V_1 and V_2. We say that A is a quasi-isomorphism if $\mathrm{Ker}(A) = 0$ and $\mathrm{Im}(A)$ is dense in V_2.

5.8. PROPOSITION. *The Γ–dimension just introduced has the properties:*

(1) $0 \leqslant \dim_\Gamma F \leqslant \infty$.
(2) $\dim_\Gamma F = 0$ *if and only if* $F = 0$.
(3) $F \subset F'$ *implies* $\dim_\Gamma F \leqslant \dim_\Gamma F'$ *with equality if and only if* $F = F'$.
(4) *If* $A : V_1 \longrightarrow V_2$ *is a Γ–morphism then* $\dim_\Gamma \mathrm{Ker}(A)^\perp = \dim_\Gamma \overline{\mathrm{Im}(A)}$.
(5) *If two Γ–modules are quasi-isomorphic then* $\dim_\Gamma L_1 = \dim_\Gamma L_2$.

Let us justify the next to the last assertion. The polar decomposition $A = SW$, where $S \geqslant 0$ and W is a partial isometry (i.e. $\mathrm{Ker}(A) = \mathrm{Ker}(W)$ and $W : \mathrm{Ker}(A)^{\perp} \longrightarrow \overline{\mathrm{Im}(A)}$ is an isometry), has the property that W is a Γ–morphism too. On the other hand WW^* is the projection on $\overline{\mathrm{Im}(A)}$ and W^*W is the projection on $\mathrm{Ker}(A)^{\perp}$. Thus we have to prove that $\mathrm{Tr}_{\Gamma} WW^* = \mathrm{Tr}_{\Gamma} W^*W$. Let us consider the matrix $(w_{\gamma\eta})$ of W; since $W \in \mathscr{A}_{\Gamma}$ there exists a function $w : \Gamma \longrightarrow \mathscr{B}L^2(U,\widetilde{E})$ such that $w(\gamma^{-1}\eta) = w_{\gamma\eta}$. Then $(WW^*)_{ee} = \sum w(\gamma)w^*(\gamma)$ and

$$\mathrm{Tr}_{\Gamma} WW^* = \sum \mathrm{Tr}\, w(\gamma)w^*(\gamma) = \sum \mathrm{Tr}\, w^*(\gamma)w(\gamma) = \mathrm{Tr}_{\Gamma} W^*W.$$

The following proposition will be useful in the proof of the Morse inequalities.

5.9. PROPOSITION. *Let*

$$0 \to L_0 \to L_1 \to \dots \to L_q \to L_{q+1} \to \dots \to L_n \to 0$$

be a complex of Γ*–modules (*d_q *commutes with the action of* Γ *and* $d_{q+1}d_q = 0$*). If* $l_q = \dim_{\Gamma} L_q < \infty$ *and* $\bar{h}_q = \dim_{\Gamma} \bar{H}_q(L)$ *where*

$$\bar{H}_q(L) = \mathrm{Ker}(d_q)/\overline{\mathrm{Im}(d_{q-1})}$$

then

$$\sum_{j=1}^{q} (-1)^{q-j}\bar{h}_j \leq \sum_{j=1}^{q} (-1)^{q-j} l_j \tag{5.8}$$

for every $q = 0,1,\dots,n$ *and for* $q = n$ *the inequality becomes equality.*

The proof is the same as in the case of vector spaces of finite dimension with the single difference that we use now the property 5.8 (4) of the Γ–dimension.

5.2. Estimates of the spectrum distribution function

As before let $(\widetilde{X}, g^{T\widetilde{X}})$ be a paracompact Γ–invariant riemannian manifold, let $X = \widetilde{X}/\Gamma$ be the quotient and $\pi_{\Gamma} : \widetilde{X} \longrightarrow X$ the canonical projection. Let $(\widetilde{E}, h^{\widetilde{E}})$ be a Γ–invariant hermitian vector bundle and let (E, h^E) be a bundle on X such that $(\widetilde{E}, h^{\widetilde{E}}) = (\pi_{\Gamma}^* E, \pi_{\Gamma}^* h^E)$.

We consider an open set $Y \Subset X$ with smooth boundary and its preimage $\widetilde{Y} = \pi_{\Gamma}^{-1} Y$; Γ acts on $\widetilde{Y}$ and $\widetilde{Y}/\Gamma = Y$. In general we will decorate with tildes the preimages of objects living on the quotient. Let U be a fundamental domain of the action of Γ on $\widetilde{Y}$.

Let us consider a formally self-adjoint, strongly elliptic, positive differential operator P on $\widetilde{X}$ acting on sections of $\widetilde{E}$. Denote by $\widetilde{P}$ the Γ–invariant differential operator which is its pull-back to $\widetilde{X}$. From $\widetilde{P}$ we construct the following operators: the Friedrichs extension in $L^2(\widetilde{Y}, \widetilde{F})$ of $\widetilde{P}$ with domain $\Omega_0(\widetilde{Y}, \widetilde{F})$ and the Friedrichs extension in $L^2(U, \widetilde{F})$ of $\widetilde{P}$ with domain $\Omega_0(U, \widetilde{F})$. From now on we denote these extensions by $\widetilde{P}$ and P_U. They are closed self-adjoint positive operators. It is known that the Fridrichs extension $\widetilde{P}$ is also Γ invariant i.e. it commutes with all L_{γ}. This amounts to saying that the spectral projectors $E(\lambda, \widetilde{P})$ of $\widetilde{P}$ commute with all L_{γ}. On the other hand the Rellich lemma tells us that P_U has compact resolvent and hence discrete spectrum. We will undertake the task of comparing the distribution of the two spectra. Namely since $E(\lambda, \widetilde{P})$ is Γ invariant its image $\mathrm{Im}(E(\lambda, \widetilde{P}))$ is a Γ invariant closed subspace of the free Hilbert Γ–module $L^2\Gamma \otimes L^2(U, \widetilde{F}) \cong L^2(\widetilde{Y}, \widetilde{F})$ and we can consider its von Neumann dimension. We denote in the sequel $N_{\Gamma}(\lambda, \widetilde{P}) = \dim_{\Gamma} \mathrm{Im}\big(E(\lambda, \widetilde{P})\big)$. Similarly we consider the counting function of P_U, $N(\lambda, P_U) = \dim \mathrm{Im}\big(E(\lambda, P_U)\big)$. In order to compare $N_{\Gamma}(\lambda, \widetilde{P})$ and $N(\lambda, P_U)$ we use essentially the analysis of Shubin [**14**]. Let $\widetilde{P}$ be a Γ invariant self-adjoint positive operator on a free Γ–module $L^2\Gamma \otimes \mathscr{H}$ where $\mathscr{H}$ is Hilbert space. Then we have the following variational principle [**14**, Lemma 2.4]:

$$N_\Gamma(\lambda,\widetilde{P}) = \sup\Big\{ \dim_\Gamma L \mid L \subset \mathrm{Dom}(\widetilde{Q}), \widetilde{Q}(f,f) \leqslant \lambda \|f\|^2, \forall f \in L \Big\} \tag{5.9}$$

where L is a Γ–module and $\widetilde{Q}$ is the quadratic form of $\widetilde{P}$.

5.10. PROPOSITION (Estimate from below). *For all $\lambda \in \mathbb{R}$,*

$$N_\Gamma(\lambda,\widetilde{P}) \geqslant N(\lambda,P_U)\,. \tag{5.10}$$

PROOF. Let us denote by $\lambda_0 \leqslant \lambda_1 \leqslant \dots$ the spectrum of P_U. Let $\{e_i\}$ be an orthonormal basis of $L^2(U,\widetilde{E})$ which consists of eigenfunctions of P_U corresponding to the eigenvalues λ_i. Let $\widetilde{e}_i$ be the extension by 0 on $\widetilde{Y} \setminus \overline{U}$ of e_i. Then $\{L_\gamma \widetilde{e}_i\}$ is an orthonormal basis of $L^2(\widetilde{Y},\widetilde{E})$ and $L_\gamma \widetilde{e}_i \in \mathrm{Dom}(\widetilde{Q})$. Let Φ_λ the Γ–module spanned by the orthonormal set $\{L_\gamma \widetilde{e}_i : \lambda_i \leqslant \lambda\}$ in $L^2(\widetilde{Y},\widetilde{F})$. Then by (5.7) $\dim_\Gamma \Phi_\lambda = \sum_{\lambda_i \leqslant \lambda} 1 = N(\lambda,P_U)$. Moreover, it is easy to see that $\Phi_\lambda \subset \mathrm{Dom}(\widetilde{Q})$ and $\widetilde{Q}(f,f) \leqslant \lambda \|f\|^2$, $f \in \Phi_\lambda$, as $\mathrm{Dom}(\widetilde{Q})$ is complete in the graph norm. Thus (5.10) follows from (5.9). □

The next step is an estimate from above of $N_\Gamma(\lambda,\widetilde{P})$. We denote by $\mathrm{rank}_\Gamma T = \dim_\Gamma \overline{\mathrm{Im}(T)}$. For the following we refer to [**14**, Lemma 3.7].

5.11. LEMMA. *Let us consider the same setting as in the variational principle. Assume $T : L^2(\widetilde{Y},\widetilde{E}) \to L^2(\widetilde{Y},\widetilde{E})$ is a Γ–morphism such that $\big((\widetilde{P}+T)f,f\big) \geqslant \mu \|f\|^2$, $f \in \mathrm{Dom}(\widetilde{P})$ and $\mathrm{rank}_\Gamma T \leqslant p$. Then*

$$N_\Gamma(\mu-\varepsilon,\widetilde{P}) \leqslant p\,, \quad \forall \varepsilon > 0. \tag{5.11}$$

In order to get an estimate from above we have to enlarge a little bit the fundamental domain U and compare the counting function of $\widetilde{P}$ on $\widetilde{Y}$ to the counting function of $\widetilde{P}$ with Dirichlet boundary conditions on the enlarged domain. For $h > 0$, let $U_h = \{x \in \widetilde{Y} : d(x,U) < h\}$ where d is the distance on $\widetilde{X}$ associated to the Riemann metric on $\widetilde{X}$ and then let $U_{h,\gamma} := \gamma U_h$. Next we need a partition of unity. Let $\varphi^{(h)} \in C^\infty(\widetilde{Y})$, $\varphi^{(h)} \geqslant 0$, $\varphi^{(h)} = 1$ on $\bar{U}$ and $\mathrm{supp}\,\varphi^{(h)} \subset U_h$, $\varphi^{(h)}_\gamma = \varphi^{(h)} \circ \gamma^{-1}$. We define the function $J^{(h)}_\gamma \in C^\infty(\widetilde{Y})$ by $J^{(h)}_\gamma = \varphi^{(h)}_\gamma \big(\sum_\gamma (\varphi^{(h)}_\gamma)^2\big)^{-\frac{1}{2}}$ so that $\sum_{\gamma\in\Gamma} (J^{(h)}_\gamma)^2 = 1$. If $\widetilde{P}$ is of *order* 2, which will be assumed from now on, by [**14**, Lemma 3.1] (variant of IMS localization formula, see [**7**]),

$$\widetilde{P} = \sum_{\gamma\in\Gamma} J^{(h)}_\gamma \widetilde{P} J^{(h)}_\gamma - \sum_{\gamma\in\Gamma} \sigma_0(\widetilde{P})(dJ^{(h)}_\gamma)$$

where σ_0 is the principal symbol of $\widetilde{P}$. Since the derivative of $J^{(h)}_\gamma$ is $O(h^{-1})$ and the order of $\widetilde{P}$ is 2 we deduce that there exists $C > 0$ independent of h such that:

$$\widetilde{P} \geqslant \sum_{\gamma\in\Gamma} J^{(h)}_\gamma \widetilde{P} J^{(h)}_\gamma - \frac{C}{h^2}\mathrm{Id} \tag{5.12}$$

We let $\widetilde{P}$ act on $\mathscr{D}(U_{\gamma,h},\widetilde{F})$ and take its Friedrichs extension $P_{U_{\gamma,h}}$. Since $P_{U_{\gamma,h}}$ is positive, $P_{U_{\gamma,h}} + \lambda\, E(\lambda,P_{U_{\gamma,h}}) \geqslant \lambda \mathrm{Id}$. We define the bounded operators G_γ on $L^2(\widetilde{Y},\widetilde{F})$ given by $G_\gamma = J^{(h)}_\gamma\, \lambda\, E(\lambda,P_{U_{\gamma,h}})\, J^{(h)}_\gamma$ and $G = \sum_{\gamma\in\Gamma} G_\gamma$. Since $J^{(h)}_\gamma \widetilde{P} J^{(h)}_\gamma = J^{(h)}_\gamma P_{U_{\gamma,h}} J^{(h)}_\gamma$, (5.12) yields

$$\widetilde{P} + G \geqslant \left(\lambda - \frac{C}{h^2}\right)\mathrm{Id}. \tag{5.13}$$

5.12. LEMMA.

$$\operatorname{rank}_\Gamma G \leqslant N(\lambda, P_{U_h}) \tag{5.14}$$

PROOF. We start with the finite rank operator $\overline{G}$ on $L^2(U_h, \widetilde{F})$,

$$\overline{G} = J_e^{(h)} \lambda\, E(\lambda, P_{U_h}) J_e^{(h)}\,.$$

Then, $\operatorname{rank}\overline{G} \leqslant \operatorname{rank} E(\lambda, P_{U_h}) = N(\lambda, P_{U_h})$. Next we consider the free Γ–module $L^2\Gamma \otimes L^2(U_h, \widetilde{F})$ and the bounded Γ–invariant operator $\mathrm{Id}\otimes\overline{G}$. Then $R(\mathrm{Id}\otimes\overline{G}) = L^2\Gamma \otimes R(\overline{G})$ so that $\operatorname{rank}_\Gamma \mathrm{Id}\otimes\overline{G} = \operatorname{rank}\overline{G}$. We now identify the space $L^2\Gamma\otimes L^2(U_h, \widetilde{F})$ with $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$ by the unitary transform $K : \sum_\gamma \delta_\gamma \otimes w_\gamma \longmapsto \big(L_\gamma w_\gamma\big)_\gamma$. Thus $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$ is naturally a free Γ–module for which K is Γ invariant. We transport $\mathrm{Id}\otimes\overline{G}$ on $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$ by K and we think of it as acting on this latter space. We construct then a restriction operator $V : \bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F}) \longrightarrow L^2(\widetilde{Y}, \widetilde{F})\,,\ V\big((w_\gamma)_\gamma\big) = \sum_{\gamma\in\Gamma} w_\gamma$ which is a surjective Γ–morphism. We have also the Γ–morphism I from $L^2(\widetilde{Y}, \widetilde{F})$ to $\bigoplus_{\gamma\in\Gamma} L^2(U_{h,\gamma}, \widetilde{F})$, $I(u) = (u \restriction_{U_{h,\gamma}})_\gamma$ which is obviously bounded. With our identifications, and replacing $E(\lambda, P_{U_{\gamma,h}})$ by $L_\gamma E(\lambda, P_{U_h}) L_\gamma^{-1}$ in the definition of G_γ, we have $G = V\,(\mathrm{Id}\otimes\overline{G})\,I$. As in the case of usual dimension $\operatorname{rank}_\Gamma V\,(\mathrm{Id}\otimes\overline{G})\,I \leqslant \operatorname{rank}_\Gamma(\mathrm{Id}\otimes\overline{G})$ (see [**14**, Lemma 3.6]). Hence $\operatorname{rank}_\Gamma G \leqslant \operatorname{rank}_\Gamma(\mathrm{Id}\otimes\overline{G}) = \operatorname{rank}\overline{G} \leqslant N(\lambda, P_{U_h})\,.$ □

5.13. PROPOSITION (Estimate from above). *There is a constant $C > 0$ such that*

$$N_\Gamma(\lambda, \widetilde{P}) \leqslant N\left(\lambda + \frac{C}{h^2}, P_{U_h}\right) \quad \lambda\in\mathbb{R}, \quad h > 0\,. \tag{5.15}$$

PROOF. We obtain $N_\Gamma(\lambda, \widetilde{P}) \leqslant N\left(\lambda + \frac{C}{h^2} + \varepsilon, P_{U_h}\right)$ by Lemma 5.11, (5.13), (5.14) and let $\varepsilon \longrightarrow 0$ (the counting function is right continuous). □

We are going to apply the above results to the semi-classical asymptotics as $k \longrightarrow \infty$ of the spectral distribution function of the laplacian $k^{-1}\square^{\widetilde{E}^k}$ on $\widetilde{X}$. We let $\widetilde{E}$ and $\widetilde{G}$ be two Γ–invariant holomorphic line bundles. Let us form the Laplace–Beltrami operator $\square^{\widetilde{E}^k} = \overline{\partial}\vartheta + \vartheta\overline{\partial}$ on $(0,q)$ forms with values in $\widetilde{E}^k \otimes \widetilde{G}$. We apply the previous results for $\widetilde{P} = k^{-1}\widetilde{\square}^{\widetilde{E}^k}|_{\widetilde{Y}}$, the operator of the Dirichlet problem on $\widetilde{Y}$. Now we have to make a good choice of the parameter h. We take $h = k^{-\frac14}$ so that the derivative of the cutting off function $J_\gamma^{(h)}$ is just $O(k^{\frac14})$. Then $\sigma_0(k^{-1}\square^{\widetilde{E}^k})(dJ_\gamma^{(h)}) = k^{-1}|\overline{\partial} J_\gamma^{(h)}|^2 = O(k^{-\frac12})$. Modifying (5.12), (5.13) and (5.15) accordingly we obtain the following semi–classical estimate.

5.14. PROPOSITION. *There exists a constant $C > 0$ independent of k such that for $\lambda\in\mathbb{R}$ and $k > 0$ we have*

$$N\left(\lambda\,, \frac1k \square^{\widetilde{E}^k} \restriction_U\right) \leqslant N_\Gamma\left(\lambda\,, \frac1k \square^{\widetilde{E}^k}|_{\widetilde{Y}}\right) \leqslant N\left(\lambda + \frac{C}{\sqrt{k}}\,, \frac1k \square^{\widetilde{E}^k} \restriction_{U_{k^{-1/4}}}\right) \tag{5.16}$$

Let us fix $\varepsilon > 0$. Then $N(\lambda + \frac{C}{\sqrt{k}}\,, \frac1k \widetilde{\square}^{\widetilde{E}}) \leqslant N(\lambda+\varepsilon, \frac1k \widetilde{\square}^{\widetilde{E}} \restriction_{U_\varepsilon})$ since for sufficiently large k we have $U_{k^{-\frac14}} \subset U_\varepsilon$. So $\limsup_k k^{-n} N_\Gamma(\lambda\,, \frac1k \widetilde{\square}^{\widetilde{E}^k}|_{\widetilde{Y}}) \leqslant I^q(U_\varepsilon, \lambda+\varepsilon)$ by (5.16) and Theorem 3.15. The use of dominated convergence to make $\varepsilon \longrightarrow 0$ in the last integral yields:

5.15. THEOREM. *The spectral distribution function of* $\frac{1}{k}\widetilde{\square}^{\widetilde{E}^k}|_{\widetilde{Y}}$ *on* $L^2_{0,q}(\widetilde{Y},\widetilde{E}^k\otimes\widetilde{G})$ *with Dirichlet boundary values satisfies*

$$\limsup_k k^{-n}N_\Gamma\left(\lambda\,,\frac{1}{k}\square^{\widetilde{E}^k}|_{\widetilde{Y}}\right)\leqslant I^q(U,\lambda). \tag{5.17}$$

Moreover, there exists an at most countable set $\mathcal{N}\subset\mathbb{R}$ *such that for* λ *in* $\mathbb{R}\smallsetminus\mathcal{N}$ *the limit exists and we have equality in* (5.17).

5.3. Weak Lefschetz Theorems

5.16. THEOREM. *Let* (X,ω) *be an n–dimensional complete hermitian manifold and let* (L,h^L) *be a holomorphic hermitian line bundle. Let* $K\Subset M$ *and a constant* $C_0>0$ *such that* $\sqrt{-1}R^L\geqslant C_0\,\omega$ *on* $X\smallsetminus K$*. Let* $\pi_\Gamma:\widetilde{X}\to X$ *be a Galois covering of Galois group* Γ*,* $\widetilde{L}=\pi_\Gamma^*(L)$ *and let* U *be any open subset with smooth boundary such that* $K\Subset U\Subset X$*. Then, for* $k\longrightarrow\infty$*,*

$$\dim_\Gamma H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)\geqslant\frac{k^n}{n!}\int_{X(\leqslant1,h^L)}\left(\tfrac{\sqrt{-1}}{2\pi}R^L\right)^n+o(k^n)\,, \tag{5.18}$$

where $H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)$ *is the space of* $(n,0)$*–forms with values in* $\widetilde{L}^k$ *which are* L^2 *with respect to any metric on* $\widetilde{X}$ *and the metric* $\pi_\Gamma^*(h^L)$ *on* $\widetilde{L}$*.*

PROOF. We endow $\widetilde{X}$ and $\widetilde{L}$ with the metric $\widetilde{\omega}=\pi_\Gamma^*\omega$ and $h^{\widetilde{L}}=\pi_\Gamma^*h^L$ respectively. Then $(\widetilde{X},\widetilde{\omega})$ is complete. Let $\widetilde{K}=\pi_\Gamma^{-1}(K)$. Since π_Γ is locally biholomorphic we have $\sqrt{-1}R^{\widetilde{L}}\geqslant C_0\widetilde{\omega}$ on $\widetilde{X}\smallsetminus\widetilde{K}$.

Using the Bochner–Kodaira–Nakano formula as in the proof of Theorem 4.30 we obtain that

$$\|u\|^2\leqslant\frac{1}{k}Q^{\widetilde{L}^k}(u,u) \tag{5.19}$$

for any $u\in\Omega^{n,1}_0(\widetilde{X}\smallsetminus\widetilde{K},\widetilde{L}^k)$, where $Q^{\widetilde{L}^k}$ is the quadratic form of $\square^{\widetilde{L}^k}$. Let $W\subset V$ be two open neighbourhoods of K. Let $0\leqslant\rho\leqslant1$ be a smooth function on X such that $\rho=1$ on W and $\rho=0$ on $X\smallsetminus V$. Set $\widetilde{\rho}=\rho\circ\pi_\Gamma$. Applying (5.19) for $(1-\widetilde{\rho})u$ where $u\in\Omega^{n,1}_0(\widetilde{X},\widetilde{L}^k)$ and (4.7) we get by the density lemma of Andreotti–Vesentini A.10

$$\|u\|^2\leqslant\frac{6}{k}Q^{\widetilde{L}^k}(u,u)+2\int_X|\widetilde{\rho}u|^2\,dv_X\,,\quad u\in\operatorname{Dom}Q^{\widetilde{L}^k}\cap L^{n,1}_2(\widetilde{X},\widetilde{L}^k) \tag{5.20}$$

From (5.20) we infer that the spectral spaces corresponding to the lower part of the spectrum of $\frac{1}{k}\square^{\widetilde{L}^k}$ on $(n,1)$–forms inject into the spectral spaces of the Γ–invariant operator $\frac{1}{k}\square^{\widetilde{L}^k}_{\widetilde{V}}$ which correspond to the Dirichlet problem on $\widetilde{V}=\pi_\Gamma^{-1}(V)$. For the latter operator we know the Weyl law. We consider, analogously to (4.5) the Γ–morphism

$$\mathscr{E}^{0,1}(\lambda,\tfrac{1}{k}\square^{\widetilde{L}^k})\longrightarrow\mathscr{E}^{0,1}(6\lambda+24C_1k^{-1},\tfrac{1}{k}\square^{\widetilde{L}^k}_{\widetilde{V}})$$
$$u\longrightarrow E_{6\lambda+24C_1k^{-1}}(\frac{1}{k}\square^{\widetilde{L}^k}_{\widetilde{V}})(\widetilde{\rho}u)$$

is injective for $\lambda<\frac{1}{12}$. □

Nori [**13**] generalized the Lefschetz hypersurface theorem. Assume X and Y are smooth connected projective manifolds and Y is a hypersurface in X with positive normal bundle and $\dim Y\geqslant1$. Then the image of $\pi_1(Y)$ in $\pi_1(X)$ is of finite index. Recently, Napier and Ramachandran [**12**] proposed an analytic approach and generalized Nori's theorem showing that Y may have arbitrary codimension (but $\dim Y\geqslant1$). They use the $\overline{\partial}$–method on complete Kähler manifolds to separate the sheets of appropriate coverings. In the sequel we use the Morse inequalities to study non–necessarily Kähler manifolds. However our method requires that the image group is normal since we can deal only with Galois coverings. First we introduce the notion of formal completion (see [**4**]). Let Y be a complex analytic subspace of the manifold U and denote by $\mathscr{I}_Y$ the ideal sheaf of Y. The formal completion $\widehat{U}$ of U with respect to Y is the ringed space $(\widehat{U},\mathscr{O}_{\widehat{U}})=(Y,\operatorname{proj}\lim\mathscr{O}_U/\mathscr{I}_Y^{\nu})$. If $\mathscr{F}$ is an analytic sheaf on U we denote by $\widehat{\mathscr{F}}$ the sheaf $\widehat{\mathscr{F}}=\operatorname{proj}\lim\mathscr{F}\otimes(\mathscr{O}/\mathscr{I}_Y^{\nu})$. If $\mathscr{F}$ is coherent then $\widehat{\mathscr{F}}$ is too. Moreover by [**4**, Proposition VI.2.7] the kernel of the mapping $H^0(U,\mathscr{F})\longrightarrow H^0(\widehat{U},\widehat{\mathscr{F}})$ consists of the sections of $\mathscr{F}$ which vanish on a neighbourhood of Y. Hence for locally free $\mathscr{F}$ the map is injective.

5.17. THEOREM. *Let (X,ω) and (L,h^L) as in Theorem 5.16 and assume that the integral in* (5.18) *is positive. Let moreover Y be a connected compact complex subspace of X satisfying:* (i) *for any k, $\dim H^0(\widehat{X},\widehat{\mathscr{F}_k})<\infty$, where $\mathscr{F}_k=\mathscr{O}(L^k\otimes K_X)$,* (ii) *the image G of $\pi_1(Y)$ in $\pi_1(X)$ is normal in $\pi_1(X)$. Then G is of finite index in $\pi_1(X)$.*

PROOF. We follow the proof given in [**12**]. Since G is normal there exists a connected Galois covering $\pi:\widetilde{X}\longrightarrow X$ such that the group of deck transformations is $\Gamma=\pi_1(X)/G$. The cardinal $|\Gamma|$ equals the index of G in $\pi_1(X)$. Let $\widetilde{E}=\pi^{-1}E$. By applying Theorem 5.16, there exists $C>0$ such that for large k, $\dim_\Gamma H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)\geqslant Ck^n$. Let us choose a small open neighbourhood V of Y such that $\pi_1(Y)\longrightarrow\pi_1(V)$ is an isomorphism; so the image of $\pi_1(V)$ in $\pi_1(M)$ is G. Hence, if we denote by j the inclusion of V in X, there exists a holomorphic lifting $\widetilde{j}:V\longrightarrow\widetilde{X}$, $\pi\circ\widetilde{j}=j$. Since $\widetilde{j}$ is locally biholomorphic the pull–back map $\widetilde{j}^*:H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)\longrightarrow H^{n,0}(V,L^k)$ is injective. On the other hand $H^0(V,\mathscr{F}_k)\hookrightarrow H^0(\widehat{V},\widehat{\mathscr{F}_k})=H^0(\widehat{X},\widehat{\mathscr{F}_k})$. By (i) the latter space is finite dimensional so $\dim H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)<\infty$. We know that $\dim_\Gamma H^0_{(2)}(\widetilde{X},\widetilde{L}^k\otimes K_{\widetilde{X}})>0$ for $k>C^{-1/n}$. If Γ were infinite this would yield $\dim H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)=\infty$ which is a contradiction. Therefore $|\Gamma|<\infty$ and $\dim H^{n,0}_{(2)}(\widetilde{X},\widetilde{L}^k)\geqslant C|\Gamma|k^n\geqslant|\Gamma|$ for $k>C^{-1/n}$. Thus $|\Gamma|\leqslant\dim H^0(\widehat{X},\widehat{\mathscr{F}_k})$ for large k. ☐

5.18. REMARK. (a) By a theorem of Grothendieck [**9**], condition (i) is fulfilled if Y is locally a complete intersection with ample normal bundle N_Y (or k–ample in the sense of Sommese, $k=\dim Y-1$).
(b) Moreover, we can slightly generalize the statement of Theorem 5.17, by assuming that Y is a subset of a manifold V, where condition (i) holds, and there exists a locally biholomorphic map $\psi:V\to X$. The proof is the same, but we use the map ψ instead of the inclusion j. In particular, if Y admits an immersion in X with positive normal bundle, we can take V to be a small neigbourhood of the zero section and we obtain Theorem 1.12.
(c) We can replace condition (i) with the requirement that Y has a fundamental system of pseudoconcave neighbourhoods $\{V\}$. Then $\dim H^0(V,\mathscr{F}_k)$ is finite by [**1**]. This happens for example if Y is a smooth hypersurface and N_Y has at least one positive eigenvalue or, if Y has arbitrary codimension, if N_Y is sufficiently positive in the sense of Griffiths [**8**, Proposition 8.2].
(d) Condition (ii) is trivially satisfied if $\pi_1(Y)=0$. Thus, if X contains a simply connected subvariety satisfying either (a) or (b), $\pi_1(X)$ is finite.
(e) It follows from Remark 2.1 that, Theorem 4.1 holds for compact manifolds M, and also for Zariski open sets in Moishezon manifolds.

5.19. COROLLARY. *Let X be a Zariski open set in a compact normal Moishezon space $\overline{X}$. Let $Y \longrightarrow X_{reg}$ be a holomorphic immersion with ample normal bundle and assume that the image G of $\pi_1(Y)$ in $\pi_1(X)$ is a normal subgroup. Then G has finite index in $\pi_1(X)$.*

PROOF. Since $\overline{X}$ is normal we have an isomorphism $\pi_1(X_{reg}) \longrightarrow \pi_1(X)$, so we can replace X with X_{reg}. We can thus desingularize $\overline{X}$ and assume that it is a manifold. We consider on $\overline{X}$ a singular hermitian positive line bundle L. We modify then the proof of Theorem 4.40 in the following way. First we consider the singular support S of $\sqrt{-1}R^L$ and construct the generalized Poincaré metric on $X \smallsetminus S$. Then we consider a covering $\pi_\Gamma : \widetilde{X} \longrightarrow X$ of group Γ and apply Theorem 5.16 (or a covering version of Theorem 4.40) on the covering $\widetilde{X} \smallsetminus \pi_\Gamma^{-1}(S)$ of $X \smallsetminus S$. We obtain in this way $(n,0)$-forms on $\widetilde{X} \smallsetminus \pi_\Gamma^{-1}(S)$ which are L^2 with respect to the pull-back of the Poincaré metric on $\widetilde{X} \smallsetminus \pi_\Gamma^{-1}(S)$ and a metric on $\pi_\Gamma^{-1}(L)$ over $\widetilde{X} \smallsetminus \pi_\Gamma^{-1}(S)$ which is bounded below by a smooth metric on $\widetilde{X}$. But for $(n,0)$-forms the L^2 condition does not depend on the metric on the base manifold so we can take the L^2 condition with respect to a smooth metric on $\widetilde{X}$ and $\pi_\Gamma^{-1}(L)$. Hence these sections extend to $\widetilde{X}$ and we can apply the proof of Theorem 5.17. $\square$

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CHAPTER 6

Compactification Theorems

In this chapter we apply the ideas developped so far about the spectral gap and the Morse inequalities to the compactification problem. We wish to find sufficient conditions under which a given noncompact complex manifold can be compactified - i.e. exhibited as open or Zariski-open subset of a compact complex space (better yet, projective or Moishezon variety). By considering such a problem, one hopes to reduce the study of certain noncompact complex manifolds to that of compact ones, which are often easier to handle from the perspectives of intersection theory, analysis, and classification.

The main motivating examples are the arithmetic quotients, which are quotients X of bounded symmetric domains Ω by torsion–free arithmetic lattices $\Gamma \subset \mathrm{Aut}(\Omega)$. There are three points of view on the compactification of arithmetic quotients: algebraic, differential geometric and complex analytic.

On the algebraic side, the theory of compactification of such X was first considered by Satake [**60, 61**] who gave certain quotients of Siegel upper half spaces and other bounded symmetric domains topological compactifications. This was later extended to other bounded Baily [**7**], and Baily–Borel [**8**], based in part on the works of Satake, endowed such topological compactifications with complex structures making X into a Zariski–open subset of a (highly singular) projective–algebraic variety $\overline{X}$, called Satake-Baily-Borel compactifications. They showed this for all arithmetic quotients of bounded symmetric domains. A further succes was the toroidal compactifications of arithmetic quotients on bounded symmetric domains by Ash-Mumford-Rapopport-Tai [**6**].

Let us consider the arithmetic quotients from the point of view of differential geometry. The bounded symmetric domains Ω are equipped with Bergman metrics ω_B invariant under $\mathrm{Aut}(\Omega)$. (Ω,ω_B) are automatically Kähler-Einstein, inducing complete Kähler-Einstein metrics $\omega=\omega_{KE}$ on X. (X,ω) are automatically of finite volume (cf. [**56**]). Moreover, if we restrict our attention to ball quotients, the sectional curvature is pinched between two negative constants. This allowed Siu and Yau [**64**] to apply differential geometric methods to obtain the compactification of complete Kähler manifolds with pinched negative sectional curvature and finite volume. They therefore yield a generalization of the compactification of arithmetic quotients of rank one.

The first step in the proof Siu-Yau theorem is to show that the manifold is hyperconcave and then use results on compactifying X using complex analysis. One key point is that Siegel's theorem holds true for pseudoconcave manifolds [**1**]. Actually, Andreotti and Grauert [**2, 3**] introduced the theory of pseudoconcave manifolds for the purpose to study the arithmetic quotients. In [**2**] it was verified that certain arithmetic quotients X of the Siegel upper half planes are pseudoconcave. It was later proved that all irreducible arithmetic quotients of dimension $\geqslant 2$ are pseudoconcave (Spilker [**65**], Borel [**14**]). The primary interest of [**2**] was to give an elementary analytic proof of the extendability of meromorphic functions to Satake-Baily-Borel compactifications, but their method can also be used to embed X as open subset in certain projective varieties Z independent of the work of Satake-Baily (cf. Pjatetskii–Shapiro [**55**]).

The approach of compactifying complete Kähler manifolds by using pseudoconcavity was taken up again by Nadel-Tsuji [**51**]. Their result completed in particular the efforts started by Andreotti-Grauert [**2**] to give a complex-analytic proof that arithmetic quotients of bounded symmetric domains are biholomorphic to quasi-projective varieties.

We will be concerned here again with the complex-analytic point of view. In the case of strongly pseudoconcave manifolds X Andreotti-Tomassini [**5**] and Andreotti-Siu [**4**], based in part on an idea of Grauert, proved a general theorem for embedding X as open subsets of projective varieties Z. It is also known that general strongly pseudoconvex manifolds of dimension greater than three can be always compactified (cf. Rossi [**57**]). However, in dimension two there exists a famous conterexample of Grauert, Andreotti-Siu and Rossi [**31, 4, 57, 38, 27, 20**]. The imposibility to compactify some strongly pseudoconcave manifolds is intimately linked to the imposibility to CR-embed its boundary in the euclidian space. This is a strongly pseudoconvex CR manifold which does not have enough CR functions (solutions of the tangential $\overline{\partial}$ equation). Actually the non-solvable Lewy operator appears as the tangential $\overline{\partial}$ operator on a three dimensional strongly pseudoconvex CR manifold and the CR structutes on such manifolds are generically non-embeddable. There has been a tremendous activity about the PDE aspects of the tangential $\overline{\partial}$ operator and classification of CR structures.

Let us explain the contents of the chapter. In Section 6.1 we review the known result about the compactification of strongly pseudoconcave ends. Here we also discuss the Grauert–Andreotti–Rossi counterexamples.

Our approach is to take the model of the manifolds studied by Siu-Yau and introduce the notion of manifold with hyperconcave ends. The first result is that such ends can be always compactified [**46**], even in dimension two. This is done in Section 6.2.

It is then natural to seek conditions for a manifold with hyperconcave ends to be a Zariski open set in its compactification. Indeed, in the case of Siu-Yau one compactifies by adding one singular point to each end. The answer is provided in Section 6.3 Theorem 6.38 shows that it is sufficient for the manifold to possess a covering with Zariski-open sets whose universal cover is Stein (the condition mimics the affine cover of a projective manifold). Using this we can give a new proof of the Siu-Yau theorem and extend Nadel's compactification theorems (a sort of Kodaira characterization of projective varietes with isolated singularities) in dimension two. This is the object of Section 6.4. In Section 6.5 we prove Theorem pinched about complete Kähler manifolds with pinched negative curvature, with a strongly pseudoconvex end and finite volume away from this end. It turns out that all the other ends are hyperconcave. We include the proof, based on the holomorphic Morse inequalities, that the volume is automatically finite if the dimension is bigger than three (Napier and Ramachandran [**52**]). We also discuss some applications to ball quotients due in dimension bigger than three to Napier and Ramachandran (previously announced by Burns).

Manifolds with hyperconcave ends model arithmetic quotients of rank one. In Section 6.6 we present a proof of the compactification theorem of Nadel-Tsuji, which generalizes the compactification of arithmetic quotients of any rank. It is based in an essential way on the Morse inequalities from Theorem 4.31.

6.1. Filling strongly pseudoconvex ends

We overview here the basic results about filling strongly pseudoconvex ends of complex manifolds. We start with some piece of terminology and explain what a strongly pseudoconvex end is. For a compact subset K of a complex manifold X, an unbounded connected component of $X \setminus K$ is called an `end` of X (with respect to K). If $K_1 \subset K_2$ are two compact subsets, the number of ends with respect to K_1 is at most the number of ends with respect to K_2, so that we can define the number of ends of X. Namely, X is said to have finitely many ends if for some integer k, and for any $K \subset X$, the number of ends with respect K is at most k. The smallest such k is called the number of ends of X, and then there exists $K_0 \subset X$ such that the number of ends with respect to K_0 is precisely the number of ends of X. If no such k exists, we say that X has infinitely many ends.

In general, a manifold X is said to be `strongly pseudoconcave end` if there exists a proper, smooth function $\varphi : X \longrightarrow (c,a)$, $a \in \mathbb{R} \cup \{+\infty\}$, which is strictly plurisubharmonic on a set of the form $\{\varphi < b\}$, $b \leqslant a$. For $d < a$ we set $X_d = \{\varphi < c\}$. We call φ `exhaustion function`.

We say that a strongly pseudoconcave end can be `compactified` or `filled in` if there exists a complex space $\widehat{X}$ such that X is (biholomorphic to) an open set in $\widehat{X}$ and for any $d < a$, $(\widehat{X} \setminus X) \cup \{\varphi \leqslant d\}$ is a compact set. We will call $\widehat{X}$ somehow abusively the `compactification` of X, although it is not necessarily compact.

6.1.1. Embedding and filling. A useful device for filling an end X is first to embedd X holomorphically in the euclidian or projective space and then compactify it using the Hartogs or Harvey-Lawson phenomenon.

The following theorem is due to Rossi [**57**, Th. 3, p. 245]. Andreotti-Siu [**4**, Prop. 3.2] improved the result in different directions, e.g. they showed that it holds for normal complex spaces. The uniqueness result comes from [**4**, Cor. 3.2].

6.1. ROSSI-ANDREOTTI-SIU THEOREM. *All strongly pseudoconcave end X can be compactified provided* $\dim X \geqslant 3$. *If the exhaustion function is strictly plurisubharmonic on X, the compactification $\widehat{X}$ can be taken to be a normal Stein space with at worst isolated singularities. Two normal Stein compactifications are biholomorphic by a map which is the identitiy on X.*

Let us describe briefly the method of [**4, 57**]. Let $e < d < b$. If $\dim X \geqslant 3$, we can use the Andreotti-Grauert theory [**3**] to show that the natural restrictions $H^1(X_d, \mathscr{F}) \longrightarrow H^1(X_e, \mathscr{F})$ are isomorphisms for any coherent analytic sheaf $\mathscr{F}$ (see [**4**, Propositions 1.2-3]). Therefore, for any coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}$ whose zero set is disjoint from X_e, the natural restriction $H^0(X_d, \mathscr{O}) \longrightarrow H^0(X_d, \mathscr{O}/\mathscr{I})$ is surjective. This leads [**4**, Proposition 1.4] to the existence of many holomorphic functions that separate points, give local coordinates and have a peak at pseudoconvex boundary points of X_d. This means that we can embed the strip X_e^d into two concentric polydiscs and use Hartogs theorem [**35**, Theorem VII, D.6]:

6.2. HARTOGS THEOREM. *Let $P_1 \subset P_2$ be two concentric polydiscs in $\mathbb{C}^N$ and let $A \subset P_2 \setminus \overline{P}_1$ be an analytic set of dimension at least two. Then there exists an analytic set $\widehat{A} \subset P_2$ such that $\widehat{A} \cap (P_2 \setminus \overline{P}_1) = A$.*

Another useful idea is to try to fill in the CR manifold ∂X_e intead of the strip X_e^d. This means to solve the complex Plateu problem for which we need the Harvey-Lawson theorem. Let us first review the notion of CR manifold of hypersurface type [**19, 18, 15**].

Let X be a strongly pseudoconvex domain in a complex manifold M. From the complex structure of M, we can build on the boundary $Y = \partial X$ a partial complex structure which is called a Cauchy-Riemann or CR structure. More generally, let Y be a smooth orientable manifold of (real) dimension $(2n-1)$. A *CR structure* on Y is an $(n-1)$-dimensional complex subbundle $H_{(1,0)}Y$ of the complexified tangent bundle $T_{\mathbb{C}}Y$ such that

$$H_{(1,0)}Y \cap \overline{H_{(1,0)}Y} = \{0\},$$

and such that $H_{(1,0)}$ is integrable as a complex subbundle of $T_{\mathbb{C}}Y$ (i.e. if u and v are sections of $H_{(1,0)}Y$, the Lie bracket $[u,v]$ is still a section of $H_{(1,0)}Y$).

If Y is a CR manifold, then its Levi distribution H is the real subbundle of TY defined by $H = Re\{H_{(1,0)}Y \oplus \overline{H_{(1,0)}Y}\}$. There exists on H a complex structure J given by $J(u+\overline{u}) = \sqrt{-1}(u-\overline{u})$, with $u \in H_{(1,0)}Y$. As Y is orientable, the real line bundle $H^{\perp} \subset T^*Y$ admits a global nonvanishing section θ. The CR structure is said to be *strongly pseudoconvex* if $d\theta(.,J.)$ defines a positive definite metric on H. Notice that in this case, $\theta \wedge (d\theta)^{n-1} \neq 0$, and θ defines a real contact structure on Y.

The `tangential Cauchy-Riemann operator`, denoted $\overline{\partial}_b : \mathscr{C}^1(Y) \to H_{(1,0)}Y$, associates to a function $f \in \mathscr{C}^1(Y)$ the projection on $H_{(1,0)}Y$ of the exterior differential df. A function $f \in \mathscr{C}^1(Y)$ is called CR function if $\overline{\partial}_b f = 0$. By a CR embedding of a manifold in a complex manifold we mean an embedding whose components are CR functions. When we say that a CR manifold is a submanifold of a complex manifold, we understand that the inclusion is a CR embedding, that is, the CR structure is induced from the ambient manifold.

If $Y = \partial X$, where X is a domain in a complex manifold M, then all restrictions of holomorphic functions on M to Y are CR functions. We have also the following converse [**42**] which may be also seen as a form of Hartogs phenomenon for CR functions.

6.3. KOHN-ROSSI THEOREM. *Assume that X is smooth, relatively compact domain in a complex manifold whose Levi form of the boundary has at least one positive eigenvalue everywhere. Then any CR function defined on ∂X extends to a holomorphic function in X.*

We also need the abstract notion of complex manifold with strongly pseudoconvex boundary. Apriori, it is not a domain with boundary in a larger complex manifold.

6.4. DEFINITION. A complex manifold X with strongly pseudoconvex boundary is a real manifold with boundary, of dimension $2n$, satisfying the following conditions: (i) the interior $\operatorname{Int} X = X \setminus \partial X$ has an integrable complex structure and (ii) for each point $x \in \partial X$ there exist a neighborhood U in X, a strongly pseudoconvex domain $D \subset \mathbb{C}^n$ with smooth boundary, and a diffeomorphism h from U onto a relatively open subset $h(U)$ such that $h(\partial U) \subset \partial D$ and h is biholomorphic from $\operatorname{Int} U$ to $\operatorname{Int} h(U)$.

From this definition we infer:

6.5. CONSEQUENCE. The complex structure induces an integrable Cauchy-Riemann structure on the boundary ∂X. Moreover, if ∂X is compact, there exists a defining function $\varphi : X \to (-\infty, c]$ such that $\partial X = \{\varphi = c\}$, with the properties: (1) its Levi form is positive definite on the holomorphic tangent space of ∂X and (2) φ is strictly plurisubharmonic on $\{c_0 < \varphi < c\}$.

It follows actually from results of Heunemann [**39**, Theorem 0.2] (see also Ohsawa [**54**]) that if X is compact X can be realized as a domain with boundary in a larger complex manifold. We quote now a fundamental result [**37**].

6.6. HARVEY-LAWSON THEOREM. *Let Y be a compact strongly pseudoconvex CR submanifold of* $\mathbb{C}^N$. *Then there exists a Stein space with boundary* $S \subset \mathbb{C}^N$ *such that* $\partial S = Y$.

We say that Y **bounds** the Stein space S. Of course, we can resolve the singularities of S and obtain that Y bounds a strongly pseudoconvex complex manifold (this might not be however embeddable in the euclidian space). Conversely, if Y bounds a a strongly pseudoconvex complex manifold M, a theorem of Grauert [**30**] shows that M bounds the Remmert reduction S of M, which is a Stein space obtained by blowing down the exceptional set of M. Heunemann's theorem implies then that S can be realised as a complex space with boundary in a larger Stein space S', which can be embedded in the euclidian space by the Remmert-Bishop-Narasimhan theorem [**53**].

6.7. REMMERT-BISHOP-NARASIMHAN THEOREM. *If X is a Stein space of dimension n of finite type* $N > n$, *that is, it can be locally realized as an analytic set of dimension, the set of proper regular embedings og X in* $\mathbb{C}^{n+N}$ *is dense in the set of all holomorphic mappings of X in* $\mathbb{C}^{n+N}$ *endowed with the topology of uniform convergence.*

We see therefore that the embeddability of Y is equivalent to the bounding property of Y. In order to apply the Harvey-Lawson theorem we need conditions for a strongly pseudoconvex CR manifold to be embeddable in the euclidian space.

6.8. BOUTET DE MONVEL THEOREM ([**19**][p. 5]). *Any compact CR manifold Y of real dimension greater than three admits a CR embedding in the euclidian space.*

The proof is based on the Hodge decomposition for the Kohn-Laplacian $\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$, which is not an elliptic operator but has however a parametrix a pseudodifferential operator of type $1/2$. For this purpose Boutet de Monvel uses the microlocalisation: for each cotangent vector (x,ξ) in the cone Σ^+ of vectors on which the Levi form is positive definite, one can find a canonical transformation and an assocoated elliptic Fourier integral operator which transforms the $\overline{\partial}_b$-complex to a simple form in the neighbourhood of (x,ξ). On this form one can easily read all the relevant properties of the complex. If $\dim_{\mathbb{R}} Y = 3$, Boutet de Monvel's theorem breaks down. A counterexample is given by the boundary of the strongly pseudoconcave manifold constructed in the example of Grauert-Andreotti-Rossi. A straightforward argument can be found in Burns [**17**], where the author shows that the CR functions on S^3 equiped with the induced CR structure from the complex structure of Example 6.20 are equal at antipodal points. Therefore, CR functions cannot embed this structure in the euclidian space. Sarkis [**58**] showed that meromorphic functions do not separate antipodal points and this structures cannot be embedded in the projective space, too. A theorem of Jacobowitz-Treves [**40**] shows that a generic strongly pseudoconvex CR structure on a compact manifold is not embedable. We are led to the following beautiful result which follows from the works of Boutet de Monvel-Sjöstrand, Harvey-Lawson, Burns and Kohn [**16, 37, 15, 41**].

6.9. THEOREM. *Let Y be a compact complex CR manifold of real dimension greater than three. The following conditions are equivalent:*

(a) *Y is embeddable in the euclidian space,*
(b) *Y bounds a complex manifold,*
(c) *The tangential Cauchy-Riemann operator* $\Box_b$ *on functions of Y has closed range in* L^2.

The equivalent conditions for the bounding property are not easily verifiable. So one seeks answers to simpler questions such as the stability of an embedding. An embeddable CR structure J is called stable, if for any sufficiently close (in the $\mathscr{C}^\infty$ topology) embeddable CR structure one finds a close embedding to the embedding of J. Lempert treated this problem in [**44, 45, 43**]. (see also J. Bland and C. L. Epstein [**12**], D. M. Burns and C. L. Epstein [**18**]) His idea is to link the deformation of the CR structure on Y to the deformation of the complex structure on a strongly pseudoconcave manifold Z bounding Y. This was further developed by Epstein and Henkin [**25, 24, 26**]. In this context we have the following.

6.10. LEMPERT SEPARATION THEOREM. *Suppose a compact, strongly pseudoconvex CR manifold M bounds a strongly pseudoconvex Stein space* (*or, equivalently, a strongly pseudoconvex complex manifold*). *Then M can be realized as a smooth real hypersurface in a complex projective manifold that M divides into a strongly pseudoconvex and a strongly pseudoconcave part.*

The main ingredient is the following Nash-type approximation result.

6.11. LEMPERT APPROXIMATION THEOREM. *Assume a reduced Stein space X has only isolated singularities, and $K \subset X$ is a compact subset. Then there are an affine algebraic variety V, and a neighbourhood of K in X that is biholomorphic to an open set in V.*

The question whether the Harvey–Lawson holds for CR compact manifolds embedded in the projective space is open [**21**, Problème 1]. Even the analogue of the Rossi–Andreotti–Siu theorem is not known in the projective space. More precisely, let $\varphi : X \longrightarrow \mathbb{R}$ be a strictly plurisubharmonic function on a noncompact complex surface embedded in $\mathbb{CP}^N$.

Consider a $(1,1)$–convex–concave strip $X_c^d = \{c < \varphi < d\}$ (this means by definition that the boundary component $\{\varphi = d\}$ is strongly pseudoconvex and $\{\varphi = c\}$ is strongly pseudoconcave). Can one compactify X_c^d at the pseudoconcave end?

We know that the answer is positive if we can compactify the pseucoconvex end. This is a converse of Lempert's theorem.

6.12. ANDREOTTI THEOREM ([**1**, Théorème 6]). *Let X be an Andreotti pseudoconcave* (*in particular q–concave*) *manifold embedded in the projective space. Then X is an open set of its projective closure $\widehat{X}$.*

This is obviously a generalization of Chow's theorem. The proof is easy using the Siegel–Remmert–Thimm Theorem 4.27 for Andreotti pseudoconcave manifolds. Indeed, we know that $\dim_{\mathbb{C}} X \leqslant \dim_{\mathbb{C}} \widehat{X}$ and $\widehat{X}$ is irreducible since X is. Let $\mathscr{M}(X)$ and $\mathscr{M}(\widehat{X})$ be the fields of meromorphic functions on X and $\widehat{X}$. Since $\widehat{X}$ is minimal the restriction $\mathscr{M}(\widehat{X}) \longrightarrow \mathscr{M}(X)$ is injective. By Theorem 4.27 we have $\deg\operatorname{tr}\mathscr{M}(X) \leqslant \dim_{\mathbb{C}} X$. It follows that $\dim_{\mathbb{C}} \widehat{X} \leqslant \deg\operatorname{tr}\mathscr{M}(X) \leqslant \deg\operatorname{tr}\mathscr{M}(X) \leqslant \dim_{\mathbb{C}} X$.

We need a criterion for the embeddability of 1–concave manifolds. Let us recall that the proof of the Kodaira-Grauert embedding theorem has two parts (cf. Remark 4.19):

(i) prove that the ring $\oplus_{k\geqslant 1} H^0(X,L^k) = \mathscr{A}(X,L)$ of a positive line bundle gives local coordinates and separates points (by means of the L^2 estimates for $\overline{\partial}$).

(ii) show that the canonical map $\Phi_k : X \longrightarrow \mathbb{P}H(X,L^k)^*$ is an embedding.

Since X is compact, (ii) is an easy consequence of (i). But if X is non-compact the implication $(i) \Longrightarrow (ii)$ is in general not true. Ohsawa gave an example of a positive line bundle on a weakly 1–complete manifold which is not ample (but one can show however that (i) holds true). In general, (i) easily implies that any relatively compact subset of X embeds into the projective space. The difficulty is to obtain an embedding of all of X.

6.13. ANDREOTTI–TOMASSINI THEOREM. *Let X be a 1–concave manifold and let $L \longrightarrow X$ be a holomorphic line bundle such that $\mathscr{A}(X,L)$ gives local coordinates and separates points every where on X. Then X admits a projective embedding.*

This can be found in [**5**, Theorem 3, p.97], [**4**, Theorem 4.1] (a generalization for complex normal spaces) and [**51**, Lemma 2.1] (see also the proof of Proposition 6.52). The hypotheses of the theorem are satisfied if e.g. X is the quotient of a bounded domain $D \subset \mathbb{C}^n$ by a properly discontinous group Γ, and there exists a positive integer N such that for any $x \in D$ the isotropy group $\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}$ has order less then N.

If $\dim_{\mathbb{C}} X \geqslant 3$ in the previous theorem the assumption that the ring $\mathscr{A}(X,F)$ separates points can be dropped; one has in fact the following

6.14. THEOREM. *Let X be a 1-concave manifold with $\dim_{\mathbb{C}} X \geqslant 3$. Suppose there exists on X a holomorphic line bundle L such that the ring $\mathscr{A}(X,L)$ gives local coordinates everywhere on X. Then $\mathscr{A}(X,L)$ does also separate points of X so that X admits a projective embedding.*

In dimension two this is no longer true, as we shall see in Example 6.51, but holds for hyperconcave manifolds.

We finish with a generalization due to Epstein-Henkin [**26**] of the embedding results of Grauert, R. Narasimhan, Andreotti and Y.-T. Siu for compact, pseudoconvex and pseudoconcave two-dimensional complex spaces, respectively.

6.15. DEFINITION. A compact CR–hypersurface M_0 is called strictly CR–cobordant to a compact CR–hypersurface M_1 if there exists a complex space $\widetilde{X}$ with at most isolated singularities and a $\mathscr{C}^\infty$–strictly plurisubharmonic function ρ with at most isolated critical points on $\widetilde{X}$ such that the set $X = \{x \in \widetilde{X} : 0 < \rho(x) < 1\}$ is relatively compact, complex subspace in $\widetilde{X}$ and $\partial X = M_1 \smallsetminus M_0$.

6.16. EPSTEIN-HENKIN THEOREM ([**26**, Theorem 2]). *Let M_1 be embeddable strictly pseudoconvex CR–hypersurface. Then any (not necesarry smooth) CR–hypersurface M_0, strictly cobordant to M_1, is also embeddable.*

6.17. COROLLARY ([**26**, Corollary 2]). *If under the hypothesis of Theorem 6.16 a complex space X defines complex cobordism between M_1 and M_0, then X is embeddable in some $\mathbb{C}^N$.*

6.1.2. The Grauert-Andreotti-Rossi example. If $\dim X = 2$, the Andreotti-Grauert theory cannot be applied. Theorem 6.1 is no longer true if $\dim X = 2$, as shown in the counterexample of Grauert-Andreotti-Rossi [**31, 4, 57**]. They are obtained as finite coverings of small neighbourhoods of the boundaries of Stein manifolds of dimension 2. The basic lemma for construction of non-fillable holes is [**4**, Proposition 7.1, p. 263]. We call henceforth a non-ramified covering simply covering.

6.18. LEMMA. *Let V be a relatively compact simply connected Stein domain in a complex manifold of dimension $\geqslant 2$. Let U be a neighbourhood of bV and let $p : W \longrightarrow U$ be a nontrivial finite cover of U. Then W cannot be compactified. If $\pi_1(bV)$ has proper subgroups of finite index, such coverings do exist.*

PROOF. Indeed, assume that $\widehat{W}$ is a completion of W. Then $\widehat{W}\setminus W$ has a strongly pseudoconvex boundary. By the Hartogs' extension theorem for functions [**35**, Theorem VII, D 4] we obtain an extension $\widehat{p}:\widehat{W}\longrightarrow V$. The map $\widehat{p}$ must be ramified, otherwise $\widehat{p}$ would be a nontrivial cover of the simply connected manifold V. The ramification set R is analytic and contained in $V\setminus U$, thus a finite set. Since V is simply connected, non–singular and of dimension $\geqslant 2$, $V\setminus R$ is simply connected. Therefore $\widehat{W}\setminus\widehat{p}^{-1}(R)$ is an irreducible covering of $V\setminus R$. But this is possible only if p is trivial. This contradiction shows that $\widehat{W}$ cannot possibly exist.

Now, if $\pi_1(bV)$ contains non-trivial subgroups of finite index, this remains true for $\pi_1(U)$ where U is any small neighbourhood which retracts on bV. This means that U has non-trivial finite coverings. □

Note that, if $n\geqslant 3$, $\pi_1(bV)=\pi_1(V)=\{0\}$, by a Morse theoretic argument of Andreotti-Frankel, so examples when Lemma 6.18 is not empty can occur only for $\dim V=2$.

6.19. EXAMPLE (Andreotti-Siu [**4**, p. 267–70]). Let $K\subset\mathbb{P}^3$ be a Kummer surface with 16 isolated, non-degenerate, canonical singular points. K is isomorphic to a quotient $T/\{id,\tau\}$, where T is an algebraic torus and τ is an involution ($\tau^2=id$) with 16 fixed points. Therefore K_{reg} admits as double covering the torus T minus 16 points.

There exists a 1-real-parameter family of algebraic surfaces $\{K_\varepsilon\}$ such that for $\varepsilon\neq 0$, K_ε is non–singular and $K_0=K$. There exists a manifold $\mathscr{K}\subset\mathbb{R}\times\mathbb{P}^3$ such that $K_\varepsilon=\mathrm{pr}_{\mathbb{R}}^{-1}(\varepsilon)$, where $\mathrm{pr}_{\mathbb{R}}$ is the projection on the first factor.

If $G\subset\mathbb{P}^3$ is the union of 16 small neighbourhoods of the singular points in $\mathbb{P}^3$, $\mathscr{K}\setminus\mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ is differentiably trivial near 0. This implies that $K_\varepsilon\setminus\mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ is diffeomorphic to $K_0\setminus\mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ for small ε.

Consider now a singular point p of the Kummer surface K and B be a small ball around p in $\mathbb{P}^3$. Then $V_\varepsilon=K_\varepsilon\cap\mathrm{pr}_{\mathbb{P}^3}^{-1}(B)$ are simply connected Stein spaces, which for $\varepsilon\neq 0$ are non-singular. From the preceding paragraph if follows that, from small $\varepsilon\neq 0$, ∂V_ε is diffeomorphic to ∂V, so that small neighbourhoods U_ε of ∂V_ε have differentiable double cover small concentric shells W_ε in the neighbourhood of a fixed point of the involution $\tau:T\longrightarrow T$.

On W_ε we take the induced complex structure. By Lemma 6.18 the holes of W_ε cannot be filled.

6.20. EXAMPLE. This example appeared in Rossi [**57**, p. 252–6] (being attributed to Andreotti), Andreotti-Siu [**4**, p. 262–70] (where credit is given to Grauert) and Grauert [**31**, p. 273]. It is constructed by the same principle as above, but it is more spectacular. It provides complex structures on a ball minus a point, actually on $\mathbb{P}^2\setminus\{[1:0:0]\}$, which are not fillable.

Let Q_ε be the family of quadrics in $\mathbb{P}^3$ given in the homogeneous coordinates $[w_0:w_1:w_2:w_3]$ by the equation $w_3(w_3+\varepsilon w_0)=w_1w_2$. For $\varepsilon\neq 0$ they are non-singular. There exists an application $\Phi:\mathbb{P}^2\setminus\{[1:0:0]\}\longrightarrow V\setminus A$, see [**4**, (1), p. 265], where A is a real analytic sphere, such that Φ is a two-sheeted differentiable ramified covering. We can use Φ to induce a new complex structure on $\mathbb{P}^2\setminus\{[1:0:0]\}$, so that Φ becomes holomorphic. By a variant of Lemma 6.18 for ramified coverings, we see that $\mathbb{P}^2\setminus\{[1:0:0]\}$ with the new structure cannot be compactified.

6.2. Compactification of hyperconcave ends

We already studied hyperconcave manifolds in Section 4.3.2. A generalization of this concept is as follows.

6.21. DEFINITION. A manifold X of dimension $\geqslant 2$ is a hyperconcave end if there exists a proper, smooth function $\varphi : X \longrightarrow (-\infty, a)$, $a \in \mathbb{R} \cup \{+\infty\}$, which is strictly plurisubharmonic on a set of the form $\{\varphi < b\}$, $b \leqslant a$. For $d < c < a$ we set $X_c = \{\varphi < c\}$ and $X_d^c = \{d < \varphi < c\}$.

The regular part of a variety with isolated singularities or the complement of a compact completely pluripolar set of a strongly plurisubharmonic function in a complex manifold have hyperconcave ends. Of course, hyperconcave manifolds have hyperconcave ends.

Our goal is to compactify hyperconcave ends at $\{\varphi = -\infty\}$ (to fill the hole at $-\infty$).

6.22. THEOREM. *Any hyperconcave end X can be compactified, i.e., there exists a complex space $\widehat{X}$ such that X is (biholomorphic to) an open set in $\widehat{X}$ and for any $d < a$, $(\widehat{X} \smallsetminus X) \cup \{\varphi \leqslant d\}$ is a compact set. More specifically, if φ is strictly plurisubharmonic on the whole X, $\widehat{X}$ can be chosen a normal Stein space with at worst isolated singularities.*

Theorem 6.22 identifies a large class of strongly pseudoconcave ends which can be compactified even in dimension two.

In the rest of the section we will prove Theorem 6.22. The idea of proof is to analytically embed small strips $X_{c-\delta}^c$, for c in a neighbourhood of minus infinity, into the difference of two concentric polydiscs. Then apply the Hartogs extension theorem to extend the image to an analytic set which will provide the compactification. To obtain the embedding we follow the stategy of Grauert and Kohn for the solution of the Levi problem. Namely, we solve the L^2 $\overline{\partial}$-Neumann for $(0,1)$-forms on domains X_c with strongly pseudoconvex boundary $\{\varphi = c\}$ endowed with a complete metric at minus infinity. Instead of using the finiteness of the sheaf cohomology, which is not available, we prove the finiteness of L^2 Dolbeault cohomology $H_{(2)}^{0,1}(X_c)$ which in turn implies the existence of peak holomorphic functions at each point of the boundary $\{\varphi = c\}$.

6.2.1. Existence of peak functions. As in Section 4.3.2 we consider the smooth function $\chi = -\log(-\varphi)$. We set $\omega = \sqrt{-1}\partial\overline{\partial}\chi = -\sqrt{-1}\partial\overline{\partial}\log(-\varphi)$. Note that $\partial\overline{\partial}\chi = \partial\overline{\partial}\varphi/(-\varphi) + (\partial\varphi \wedge \overline{\partial}\varphi)/\varphi^2$ and $(\partial\varphi \wedge \overline{\partial}\varphi)/\varphi^2 = \partial\chi \wedge \overline{\partial}\chi$. Since $\sqrt{-1}\partial\overline{\partial}\varphi/(-\varphi)$ represents a metric on X_0, we get the Donnelly-Fefferman condition:

$$|\partial\chi|_\omega \leqslant 1\,. \tag{6.1}$$

Since $\chi : X_0 \longrightarrow \mathbb{R}$ is proper, (6.1) also ensures that ω is complete. Let $c < 0$ be a regular value of φ. The metric ω is complete at the pseudoconcave end of X_c and extends smoothly over the boundary ∂X_c.

We wish to derive the fundamental estimate for $(0,1)$-forms on X_c. For this goal we look first at the minus infinity end and use the Berndtsson-Siu trick [**10, 63**]. Roughly speaking, it uses the negativity of the trivial line bundle, thus avoiding the problems raised by the control of the Ricci curvature of ω at $-\infty$. Let us denote by $\Omega_0^{0,q}(X_c)$ the space of smooth $(0,q)$-forms with compact support in X_c. Let $\vartheta = -*\partial*$ be the formal adjoint of $\overline{\partial}$ with respect to the scalar product $(u,v) = \int_{X_c} \langle u,v\rangle\, dv_\omega$, where $\langle u,v\rangle = \langle u,v\rangle_\omega$ and $dv_\omega = \omega^n/n!\,$.

6.23. LEMMA. *For any $v \in \Omega_0^{0,1}(X_c)$ we have $\|v\|^2 \leqslant 8\big(\|\overline{\partial}v\|^2 + \|\vartheta v\|^2\big)$.*

PROOF. On the trivial bundle $L = X_c \times \mathbb{C}$ we introduce the auxilliary hermitian metric $e^{\chi/2}$. Let be ϑ_χ the formal adjoint of $\overline{\partial}$ with respect to the scalar product $(u,v)_\chi = \int_{X_c} \langle u,v\rangle e^{\chi/2}\, dv_\omega$. Then $\vartheta_\chi = e^{-\chi/2}\,\vartheta\, e^{\chi/2}$. We apply the Bochner-Kodaira-Nakano formula for $u \in \mathscr{C}_c^{0,1}(X_c)$:

$$\int_{X_c} \big\langle [\sqrt{-1}\partial\overline{\partial}(-\chi/2), \Lambda_\omega]\, u, u\big\rangle e^{\chi/2}\, dv_\omega \leqslant \int_{X_c} \big(|\overline{\partial}u|^2 + |\vartheta_\chi u|^2\big)\, e^{\chi/2}\, dv_\omega\,, \tag{6.2}$$

where Λ_ω represents the contraction with ω and $[A,B] = AB - (-1)^{\deg A\cdot \deg B} BA$ is the graded commutator of the operators A, B. The idea is to substitute $v = u\, e^{\chi/4}$. It is readily seen that

$$|\overline{\partial}u|^2 e^{\chi/2} \leqslant 2|\overline{\partial}v|^2 + \tfrac{1}{8}|\overline{\partial}\chi|^2|v|^2\,, \quad |\vartheta_\chi u|^2 e^{\chi/2} \leqslant 2|\vartheta v|^2 + \tfrac{1}{8}|\partial\chi|^2|v|^2\,. \tag{6.3}$$

Moreover $\big\langle [\sqrt{-1}\partial\overline{\partial}(-\chi/2), \Lambda_\omega]u,u\big\rangle e^{\chi/2} = \big\langle [\sqrt{-1}\partial\overline{\partial}(-\chi/2), \Lambda_\omega]v,v\big\rangle$. In general, for a (p,q)–form α we have the identity $\langle [\omega,\Lambda_\omega]\alpha,\alpha\rangle = (p+q-n)|\alpha|^2$, where $n = \dim X$. Taking into account that $\omega = \sqrt{-1}\partial\overline{\partial}\chi$ and that v is a $(0,1)$–form, we obtain

$$\big\langle [\sqrt{-1}\partial\overline{\partial}(-\chi/2), \Lambda_\omega]u,u\big\rangle e^{\chi/2} = \tfrac{n-1}{2}|v|^2 \geqslant \tfrac{1}{2}|v|^2\,. \tag{6.4}$$

By (6.2), (6.3), (6.4) and (6.1) we obtain

$$\frac{1}{2}\int_{X_c} |v|^2\, dv_\omega \leqslant 2\int_{X_c} \big(|\overline{\partial}v|^2 + |\vartheta v|^2\big)\, dv_\omega + \frac{1}{4}\int_{X_c} |v|^2\, dv_\omega\,. \tag{6.5}$$

This immediately implies Lemma 6.23 for elements $v \in \Omega_0^{0,1}(X_c)$. □

Let $\eta : (-\infty,0) \longrightarrow \mathbb{R}$ be a smooth function such that $\eta(t) = 0$ on $(-\infty,-2]$, $\eta'(t) > 0$, $\eta''(t) > 0$ on $(-2,0)$. Let us introduce the scalar product

$$(u,v)_{\eta(\varphi)} = \int_{X_c} \langle u,v\rangle\, e^{-\eta(\varphi)}\, dv_\omega\,, \tag{6.6}$$

the corresponding norm $\|\cdot\|_{\eta(\varphi)}$ and L^2 spaces, denoted $L_2^{0,q}(X_c,\eta(\varphi))$. Let $\Omega_0^{0,q}(\overline{X}_c)$ be the space of smooth $(0,q)$-forms with compact support in $\overline{X}_c$. Consider the maximal closed extension of $\overline{\partial}$ to $L_2^{0,q}(X_c,\eta(\varphi))$ and let $\overline{\partial}^*_{\eta(\varphi)}$ be its the Hilbert-space adjoint. Note that $L_2^{0,q}(X_c,\eta(\varphi)) = L_2^{0,q}(X_c)$ and that the two norms are equivalent. We denote by ϑ_η the formal adjoint of $\overline{\partial}$ with respect to the scalar product (6.6). Then $\vartheta_\eta = \vartheta + i\big(\partial\eta(\varphi)\big)$, where $i(\cdot)$ represents the interior product. Let $\sigma(\vartheta,df) = *\partial f \wedge *$ be the symbol of ϑ, calculated on the cotangent vector df. It is clear that $\sigma(\vartheta_\eta,df) = \sigma(\vartheta,df)$ does not depend on η. We introduce the spaces $B^{0,q} = \{\,\alpha \in \Omega_c^{0,q}(\overline{X}_c) : \sigma(\vartheta,d\varphi)\alpha = 0 \text{ on } \partial X_c\}$ (cf. (A.19)), where $\sigma(\vartheta,d\varphi) = *\partial\varphi\wedge *$ be the symbol of ϑ, calculated on the cotangent vector $d\varphi$. Integration by parts [**28**, Propositions 1.3.1–2] yields $\operatorname{Dom}\overline{\partial}^*_\eta \cap \mathscr{C}_c^{0,q}(\overline{X}_c) = B^{0,q}$, $\quad \overline{\partial}^*_\eta = \vartheta_\eta$ on $B^{0,q}$,

6.24. LEMMA. *The space $B^{0,q}$ is dense in* $\operatorname{Dom}\overline{\partial} \cap \operatorname{Dom}\overline{\partial}^*_\eta$ *in the graph norm*

$$u \longmapsto \big(\|u\|^2_{\eta(\varphi)} + \|\overline{\partial}u\|^2_{\eta(\varphi)} + \|\overline{\partial}^*u\|^2_{\eta(\varphi)}\big)^{1/2}\,.$$

PROOF. We use first the idea from A.10 in order to reduce the proof to the case of a compactly supported form u. The completeness of the metric ω implies the existence of a sequence $\{a_\nu\}_\nu \subset \mathscr{C}_c^\infty(\overline{X}_c)$, such that $0 \leqslant a_\nu \leqslant 1$, $a_{\nu+1} = 1$ on $\operatorname{supp} a_\nu$, $|da_\nu| \leqslant 1/\nu$ for every $\nu \geqslant 1$ and $\{\operatorname{supp} a_\nu\}_\nu$ exhaust $\overline{X}_c$. Indeed, consider a smooth function $\rho : \mathbb{R} \longrightarrow [0,1]$

such that $\rho = 0$ on a neighbourhood of $(-\infty,-2]$, $\rho = 1$ on a neighbourhood of $[-1,\infty)$ and $0 \leqslant \rho' \leqslant 2$. Then $a_\nu = \rho(\chi/2^{\nu+1})$ satisfies the conditions above.

Let $u \in \operatorname{Dom}(\overline{\partial}) \cap \operatorname{Dom}(\overline{\partial}^*)$. Then $a_\nu u \in \operatorname{Dom}(\overline{\partial}) \cap \operatorname{Dom}(\overline{\partial}^*)$ and

$$\|\overline{\partial}(a_\nu u) - a_\nu \overline{\partial} u\|_\eta = O(1/\nu)\|u\|_\eta\,,$$
$$\|\overline{\partial}^*(a_\nu u) - a_\nu \overline{\partial}^* u\|_\eta = O(1/\nu)\|u\|_\eta\,.$$

Hence $\{a_\nu u\}$ converges to u in the graph norm. So to prove the assertion we can start with a form u having compact support in $\overline{X}_c$. But then the approximation in the graph norm follows from the Friedrichs theorem on the identity of weak and strong derivatives (see Lemma A.21). □

We confine next our attention to the fundamental estimate on X_c.

6.25. LEMMA. *If η grows sufficiently fast, there exists a constant $C > 0$ such that*

$$\|u\|^2_{\eta(\varphi)} \leqslant C\Big(\|\overline{\partial} u\|^2_{\eta(\varphi)} + \|\overline{\partial}^*_{\eta(\varphi)} u\|^2_{\eta(\varphi)} + \int_K |u|^2 e^{-\eta(\varphi)}\, dv_\omega\Big)\,, \tag{6.7}$$

for any $u \in \operatorname{Dom}\overline{\partial} \cap \operatorname{Dom}\overline{\partial}^*_{\eta(\varphi)} \subset L^{0,1}_2(X_c,\eta(\varphi))$, where $K = \{-3 \leqslant \varphi \leqslant -3/2\}$.

PROOF. We give the trivial line bundle $L = X_c \times \mathbb{C}$ the metric $e^{-\eta(\varphi)}$. Let $u \in B^{0,1}$, $\operatorname{supp} u \subset \{-3 \leqslant \varphi\}$.

Let us use a form of the Bochner-Kodaira formula introduced by Andreotti-Vesentini and Griffiths. The curvature of the hermitian bundle $(L, e^{-\eta(\varphi)})$ is denoted R^L. It is a $(1,1)$-form on X, $R^L = \sum \theta_{\alpha\beta}\, dz^\alpha \wedge d\overline{z}^\beta$, where $\theta_{\alpha\beta} = \partial_{z^\alpha}\partial_{\overline{z}^\beta}\eta(\varphi)$. Let θ^μ_β be the curvature tensor with the first index raised. Let $u = \sum u_\lambda\, d\overline{z}^\lambda$ be a $(0,1)$–form on X_c. We define the $(0,1)$–form $R^L u = \sum \theta^\mu_\lambda u_\mu\, d\overline{z}^\lambda$. We also introduce the Ricci curvature $R^{\det} = -R^{K_X}$, where K_X is the canonical bundle of X.

By (B.25) we have for any $u \in B^{0,1}$ we have:

$$\|\overline{\partial} u\|^2_{\eta(\varphi)} + \|\overline{\partial}^* u\|^2_{\eta(\varphi)} = \big\|\overline{\nabla} u\big\|^2_{\eta(\varphi)} + \big(R^L u, u\big)_{\eta(\varphi)} + \big(R^{\det} u, u\big)_{\eta(\varphi)} + \int_{\partial X_c} \mathscr{L}(u,u) e^{-\eta(\varphi)}\, dS \tag{6.8}$$

where $\mathscr{L}$ is the Levi operator (B.24) and $\overline{\nabla}$ denotes the covariant derivative in the $(0,1)$–direction.

Since ∂X_c is pseudoconvex, $\mathscr{L}(u,u) \geqslant 0$ for all $u \in B^{0,1}$. On $\{-3 \leqslant \varphi \leqslant -1\}$, $R^{\det}$ is bounded and independent of η, so there exists a constant $R > 0$ such that $\langle R^{\det} u, u\rangle \geqslant -R|u|^2$, pointwise, for any u with $\operatorname{supp} u \subset \{-3 \leqslant \varphi\}$. On the other hand, we can use the strict plurisubharmonicity of φ to choose a sufficiently increasing η (replace η with $\tau\eta$ for $\tau \gg 1$) such that $\langle R^L u, u\rangle \geqslant (R+1)|u|^2$, pointwise on $\{(-3/2) \leqslant \varphi\}$, for any u. Since $\langle R^L u, u\rangle \geqslant 0$ everywhere, we obtain from (6.8)

$$\|u\|^2_{\eta(\varphi)} \leqslant \|\overline{\partial} u\|^2_{\eta(\varphi)} + \|\overline{\partial}^*_{\eta(\varphi)} u\|^2_{\eta(\varphi)} + (R+1)\int_{\{-3\leqslant\varphi\leqslant(-3/2)\}} |u|^2 e^{-\eta(\varphi)}\, dV_\omega\,,$$
$$u \in B^{0,1}\,,\ \operatorname{supp} u \subset \{-3 \leqslant \varphi\}\,. \tag{6.9}$$

Let $u \in B^{0,1}$. We choose a cut-off function $\rho_1 \in \mathscr{C}^\infty(X_c)$ such that $\operatorname{supp}\rho_1 = \{-3 \leqslant \varphi\}$, $\rho_1 = 1$ on $\{-2 \leqslant \varphi\}$. Set $\rho_2 = 1 - \rho_1$. On $\operatorname{supp}\rho_2$, η vanishes, therefore $\overline{\partial}^*_{\eta(\varphi)}(\rho_2 u) = \vartheta(\rho_2 u)$. Upon applying (6.23) for $\rho_2 u$ we get

$$\|\rho_2 u\|^2_{\eta(\varphi)} \leqslant 8\big(\|\overline{\partial}(\rho_2 u)\|^2_{\eta(\varphi)} + \|\overline{\partial}^*_{\eta(\varphi)}(\rho_2 u)\|^2_{\eta(\varphi)}\big)\,, \quad u \in B^{0,1}\,. \tag{6.10}$$

The estimate (6.9) for $\rho_1 u$ and (6.10) together with standard inequalities deliver (6.7) for elements $u \in B^{0,1}$. By Lemma 6.24, estimate (6.7) holds for all forms $u \in \operatorname{Dom}\overline{\partial} \cap \operatorname{Dom}\overline{\partial}^*_{\eta(\varphi)} \subset L_2^{0,1}(X_c, \eta(\varphi))$. □

In the sequel we fix a function η as in Lemma 6.25. Then the fundamental estimate (6.7) implies the solution of the L^2 $\overline{\partial}$-Neumann problem. Consider the complex of closed, densely defined operators

$$T = \overline{\partial} : L_2^{0,0}(X_c, \eta(\varphi)) \xrightarrow{T=\overline{\partial}} L_2^{0,1}(X_c, \eta(\varphi)) \xrightarrow{S=\overline{\partial}} L_2^{0,2}(X_c, \eta(\varphi)),$$

and the Gaffney extension (cf. Section A.2.4)

$$\operatorname{Dom}\Box = \{u \in \operatorname{Dom} S \cap \operatorname{Dom} T^* : Su \in \operatorname{Dom} S^*,\ T^*u \in \operatorname{Dom} T\},$$
$$\Box u = S^*Su + TT^*u \quad \text{for } u \in \operatorname{Dom}\Box.$$

Remark that $\Box$ is an extension of the operator $\overline{\partial}\vartheta_{\eta(\varphi)} + \vartheta_{\eta(\varphi)}\overline{\partial}$ defined on $\{u \in B^{0,q} : \overline{\partial}u \in B^{0,q+1}\}$. It is actually its Friedrichs extension, which can be seen with a similar argument as in Proposition A.22. Once the fundamental estimate (6.7) is established we deduce the strong Hodge decomposition from Theorem A.26:

6.26. THEOREM. *The operator T has closed range and* $\operatorname{Range}(T)$ *has finite codimension in* $\operatorname{Ker}(S)$. *If $f \in \operatorname{Range} T$, there is a unique solution $u \perp \operatorname{Ker} T$ of the equation $Tu = f$ given by $u = \overline{\partial}^*_{\eta(\varphi)} Gf$, where G is the green operator; if f is smooth in X_c so is u.*

Theorem 6.26 can be regarded as a variant 'with boundary' of the vanishing theorem of Donelly–Fefferman [**22**], which asserts that $H^{p,q}_{(2)}(X) = 0$ for $p+q \neq \dim X$, where (X, ω) is a complete Kähler manifold, where the Kähler form ω admits a global potential satisfying (6.1).

By solving the $\overline{\partial}$–equation we construct peak functions at each point of ∂X_c.

6.27. COROLLARY. *Let $p \in \partial X_c$ and f be a holomorphic function on a neigbourhood of p such that $\{f = 0\} \cap \overline{X}_c = \{p\}$. Then for every m big enough, there is a function $g \in \mathscr{O}(X_c) \cap \mathscr{C}^\infty(\overline{X}_c \smallsetminus \{p\})$, a smooth function Φ on a neigbourhood V of p and constants $a_1, \cdots, a_{m-1}$ such that*

$$g = \frac{1 + a_{m-1}f + \cdots + a_1 f^{m-1}}{f^m} + \Phi$$

on $V \cap \Omega$. In particular, we have $\lim_{z \to p} |g(z)| = \infty$.

PROOF. We will apply the last theorem for a domain $X_{c+\delta} = \{\varphi < c + \delta\}$ with $\delta > 0$ small enough. Let U be a small neighbourhood of p where f is defined. Pick $\psi \in \mathscr{C}_0^\infty(U)$ such that $\psi = 1$ on a neighbourhood V of p. Set

$$h_m = \psi / f^m \text{ on } U \text{ and } 0 \text{ on } X \smallsetminus U$$

and

$$v_m = 0 \text{ on } V \text{ and } \overline{\partial}h_m \text{ on } X \smallsetminus V.$$

Observe that v_m belongs to $\Omega_c^{0,1}(\overline{X}_{c+\delta})$ for δ small enough. Moreover, we have $\overline{\partial}v_m = 0$ on $X_{c+\delta}$. Fix a such δ and now apply Theorem 6.26 for $X_{c+\delta}$. By this theorem, the codimension of $\operatorname{Range} T$ in $\operatorname{Ker} S$ is finite. For every m big enough, there are constants $a_1, \cdots, a_{m-1}$ such that $v = v_m + a_{m-1}v_{m-1} + \cdots + a_1 v_1$ belongs to $\operatorname{Range} T$. Then there is $\Phi' \in \mathscr{C}^{0,0}(X_{c+\delta})$ such that $\overline{\partial}\Phi' = -v$. Set $h = h_m + a_{m-1}h_{m-1} + \cdots + a_1 h_1$ and $g = h + \Phi'$. We have $\overline{\partial}g = 0$ on $X'_c \smallsetminus \{f = 0\}$. Then $g \in \mathscr{O}(X_c) \cap \mathscr{C}^\infty(\overline{X}_c \smallsetminus \{p\})$. The function Φ in the corollary is equal to Φ' on V. Thus it is smooth on V. The proof is completed. □

By using the estimates in local Sobolev norms near the boundary points, we can prove as in Folland-Kohn [**28**] that N maps $L_2^{0,1}(X_c,\eta(\varphi))\cap\Omega^{0,1}(\overline{X}_c)$ into itself. We could repeat then the solution of the Levi problem as given in [**28**, Theorem 4.2.1], in order to find holomorphic peak functions, for each boundary point. However, we propose in Corollary 6.27, a simpler proof for the existence of peak functions, which doesn't involve the regularity up to the boundary of the $\overline{\partial}$-Neumann problem.

6.2.2. The embedding. In this section we prove Theorem 6.22, using the results from Section 6.2.1 and the method of [**4**].

6.28. PROPOSITION. *Let c be a regular value of φ. Then for $\delta>0$ small enough we have:*

(a) *The holomorphic functions on X_c separate points on $X^c_{c-\delta}$,*
(b) *The holomorphic functions on X_c give local coordinates on $X^c_{c-\delta}$, and*
(c) *for any $d\in(c-\delta,c)$ there exists $d^*\in(d,c)$, such that the holomorphically convex hull of X_d, with respect to the algebra of holomorphic functions on X_c, is contained in X_{d^*}.*

6.29. REMARK. If $\dim X\geqslant 3$, an analogous statement to Proposition 6.28 was proved in [**4**, Proposition 1.4], using the Andreotti-Grauert theory, as explained at the beginning of §6.2.1.

Proposition 6.28 will be the consequence of the following two lemmas. We can assume that $\partial X_{c-\varepsilon}$ is smooth for $\varepsilon>0$ small enough. Choose a projection π from a neighbourhood of ∂X_c into ∂X_c. We will denote by (x,ε) the point of $\partial X_{c-\varepsilon}$ whose projection is $x\in\partial X_c$.

6.30. LEMMA. *Let x_1, x_2 be two different points in ∂X_c. Then there are two neighbourhoods V_1, V_2 of x_1, x_2 and $\nu=\nu(x_1,x_2)>0$ such that the holomorphic functions of X_c separate $V_1\times(0,\nu]$ and $V_2\times(0,\nu]$.*

PROOF. This is a direct corollary from the existence of a function holomorphic in X_c, and $\mathscr{C}^\infty$ in $\overline{X}_c\smallsetminus\{x_1\}$ which tends to ∞ at x_1. □

6.31. LEMMA. *Let x be a point of ∂X_c. Then there are a neighbourhood V of x and $\tau=\tau(x)>0$ such that the holomorphic functions in X_c give local coordinates for $V\times(0,\tau]$.*

PROOF. Without loss of generality and in order to simplify the notations, we consider the case $n=2$. Choose a local coordinates system such that $x=0$ and locally $X_c\subset\{|z_1-1/2|^2+|z_2|^2<1/4\}$. We now apply Corollary 6.27 for functions $f_1(z)=z_1$ and $f_2(z)=z_1(1-z_2)$. Denote by g_1, g_2 the holomorphic functions constructed by this corollary for a number m big enough. We can also construct the analogue functions if we replace m by $m+1$. Denote by g_1' and g_2' these new functions.

Let $G:X_c\longrightarrow\mathbb{C}^4$ given by $G=(g_1,g_2,g_1',g_2')$. We will prove that G gives local coordinates. Set

$$I(z)=\left(\frac{z_1}{z_3},1-\frac{z_2z_3}{z_1z_4}\right).$$

Let W be a small neigbourhood of 0. By Corollary 6.27, the map $I\circ G$ is defined on $W\cap X_c$ and can be extended to a smooth function on W. Moreover, on W we have

$$I\circ G(z)=\Big(z_1+O(z_1^2),z_2+O(z_1)\Big).$$

Then $I\circ G$ gives an immersion of $W\cap X_c$ in $\mathbb{C}^2$, whenever W is small enough. In consequence, G gives coordinates on $W\cap X_c$. □

PROOF OF PROPOSITION 6.28. We cover $\partial X_c \times \partial X_c$ by a finite family of open sets of the form $V_1 \times V_2$ (from Lemma 6.30) and the form $V \times V$ (from Lemma 6.31). We have a finite family of ν and τ. Then properties (a) and (b) hold for every δ smaller than these τ and ν. Property (c) is an immediate consequence of Corollary 6.27. □

PROOF OF THEOREM 6.22. First let us remark that the assertion (i) is a consequence of (ii), so we shall prove only the latter. We assume therefore that the function $\varphi : X \longrightarrow (-\infty, a)$ is strictly plurisubharmonic everywhere.

The proof of the compactification statement for $\dim X \geqslant 3$ in [**4**, Proposition 3.2] uses only the assertions (a), (b) and (c) of Proposition 6.28, so we just have to follow it. For the readers' convenience we give here the details. The main tool is the Hartogs extension phenomenon.

Let c and δ as in Proposition 6.28 and choose $d \in (c-\delta, c)$. By Proposition 6.28, (c), the holomorphically convex hull of $\{\varphi \leqslant d\}$, with respect to $\mathscr{O}(X_c)$, is contained in X_{d^*}, for some $d^* \in (d, c)$. We can therefore find a finite number of open sets U_i, and functions $f_i \in \mathscr{O}(X_c)$, $1 \leqslant i \leqslant k$, such that

$$\{\varphi = d^*\} \subset \cup_{i=1}^k U_i\,, \quad |\, f_i \upharpoonright_{U_i} | > 1\,, \quad |\, f_i \upharpoonright_{\{\varphi \leqslant d\}} | < 1/2\,. \tag{6.11}$$

By Proposition 6.28, (a) and (b), we can find $f_{k+1}, \cdots, f_l \in \mathscr{O}(X_c)$ which separate points and give local coordinates on $\{d \leqslant \varphi \leqslant d^*\}$. Whithout lost of generality we can assume that $|\, f_i \upharpoonright_{\{\varphi \leqslant d\}} | < 1/2$.

Consider the map $\alpha : X_c \longrightarrow \mathbb{C}^l$, $\alpha(x) = (f_1(x), \cdots, f_l(x))$. We shall use the notation

$$P_\varepsilon = \Big\{ z \in \mathbb{C}^l \,:\, |z_i| < \varepsilon\,,\, 1 \leqslant i \leqslant l \Big\}\,, \quad 0 < \varepsilon \leqslant 1\,.$$

From relation (6.11) we deduce

$$\alpha\big(\{\varphi \leqslant d\}\big) \subset P_{1/2}\,, \quad \alpha\big(\{\varphi = d^*\}\big) \cap P_1 = \emptyset\,. \tag{6.12}$$

Denote $G = P_1 \smallsetminus \overline{P}_{1/2}$ and set $H = \alpha^{-1}(G) \cap X_d^{d^*}$. It is clear that $\alpha(H)$ is a complex submanifold of G, for α is a proper injective immersion. Since $\alpha(H)$ has dimension at least 2, it follows from the Hartogs phenomenon 6.2 that we can find an $\varepsilon \in [1/2, 1]$, such that $\alpha(H) \cap (P_1 \smallsetminus \overline{P}_\varepsilon)$ can be extended to an analytic subset V of P_1.

We can glue the topological spaces $X_c \smallsetminus \alpha^{-1}(\overline{P}_\varepsilon)$ and V along $H \smallsetminus \alpha^{-1}(\overline{P}_\varepsilon)$ using the identification given by the holomorphic map α. Hence, we obtain a complex space $\widetilde{X}_c$, such that $X_c \smallsetminus \alpha^{-1}(\overline{P}_\varepsilon)$ and V are open subsets of $\widetilde{X}_c$.

We show next that $\widetilde{X}_c$ is a Stein space. It is enough to construct a strictly plurisubharmonic exhaustion function. For this purpose, observe first that f_i, $1 \leqslant i \leqslant l$, extend naturally to functions $\widetilde{f}_i$, by setting $\widetilde{f}_i = z_i$ on V. Choose $0 < \delta_1 < \delta_2 < \delta$ and a cut-off function ρ which equals 0 on $\{\varphi \leqslant c - \delta_2\}$ and 1 on $\{\varphi \geqslant c - \delta_1\}$. Consider moreover a smooth increasing convex function λ defined on $(-\infty, c)$, such that $\lim_{t \to c} \lambda(t) = +\infty$. Then for a sufficiently large constant $A > 0$, the function

$$\psi = A \sum_{i=1}^{l} |\, \widetilde{f}_i|^2 + \rho \lambda(\varphi)$$

is a strictly plurisubharmonic exhaustion function on $\widetilde{X}_c$.

We have thus found for every $c < a$ a Stein space $\widetilde{X}_c$, such that the strip X_c^a is (biholomorphic to) an open subset of $\widetilde{X}_c$. By passing to the normalization, we may assume that $\widetilde{X}_c$ is normal. From [**4**, Corollary 3.2], we deduce that for any $c^* < a$ the normal Stein spaces $\widetilde{X}_c$ and $\widetilde{X}_{c^*}$ are biholomorphic and the biholomorphism is the identity on the smaller strip. We can drop the subscript c and denote by $\widetilde{X}$ the common Stein completion. By letting $c \longrightarrow -\infty$ we obtain that X itself is biholomorphic to an open set of $\widetilde{X}$. The proof is complete. $\square$

6.32. REMARK. We can use another argument than Proposition 6.28 to show Theorem 6.22. Firstly, we can find a CR embedding $\Psi : \partial X_c \longrightarrow \mathbb{C}^N$, using Sarkis [**59**, Corollaire 4.13]. The latter result asserts that a compact strictly pseudoconvex 3-dimensional CR manifold is embeddable in the euclidian space provided it is embeddable in the projective space and possesses a non-constant CR function. In our case these conditions are fullfiled. Using the complete metric (4.47) and the positivity of the trivial line bundle, it is easy to see that the $(n,0)$-forms embed ∂X_c in the projective space. By Corollary 6.27 and Sarkis' theorem the embedability of ∂X_c in some $\mathbb{C}^N$ follows.

Secondly, we apply the Harvey-Lawson theorem 6.6 to find a Stein space $S \subset \mathbb{C}^N$ which bounds $\Psi(\partial X_c)$, show that Ψ extends to a holomorphic map $\widehat{\Psi} : X_c \longrightarrow S$, injective near ∂X_c, and finally infer from here that X can be compactified.

The existence of peak holomorphic functions in Corollary 6.27 affords however the simpler and more elementary proof based on Proposition 6.28.

6.33. REMARK (Generalization of Theorem 6.22). Theorem 6.22 holds also for normal complex spaces with isolated singularities. These are the only allowed normal singularities in dimension 2.

Indeed, let X be a hyperconcave end with isolated normal singularities. Note that Definition 6.21 makes sense also for complex spaces. Let a_i denote the singular points and choose functions φ_i with pairwise disjoint compact supports, such that φ_i is strictly plurisubharmonic in a neighbourhood of a_i and $\lim_{z\to a_i} = -\infty$. Using the function $\widehat{\varphi} = \varphi + \sum \varepsilon_i \varphi_i$, with ε_i small enough, we see that X_{reg} is a hyperconcave end. By Theorem 6.22 we get a normal Stein compactification Y of X_{reg}.

Take $\{V_i\}$ pairwise disjoint Stein neighbourhoods of $\{a_i\}$. Then $V_i \smallsetminus \{a_i\} \subset X_{reg}$ and a normal Stein compactification of $V_i \smallsetminus \{a_i\}$ is V_i. Using the uniqueness of a normal Stein compactification [**4**, Corollary 3.2] we infer that the V_i are disjointly embedded in Y. Therefore, Y is also a compactification of X.

In particular, the singular set of a hyperconcave end with only isolated singularities must be finite.

6.34. REMARK (Complex cobordism). Another point of view on Theorem 6.22 is to consider the sets $Y_c = \{\varphi = c\}$, which, for regular values c of φ, are compact strongly pseudoconvex CR manifolds of dimension 3. Following Epstein and Henkin [**26**], we call two CR manifolds Y_1 and Y_2 of dimension 3 strictly complex cobordant, if there exists a complex manifold X with boundary, such that $\partial X = Y_1 \cup Y_2$ and there exists a strictly plurisubharmonic function ρ on X so that Y_1 and Y_2 are two non-critical level sets of ρ. They show then [**26**, Theorem 1] that if Y_1 bounds a complex manifold, also the components of Y_2 bound complex manifolds.

Theorem 6.22 can be rephrased by saying that, if a compact strongly pseudoconvex CR manifold Y is strictly complex cobordant to $-\infty$, the manifold Y bounds a strongly pseudoconvex compact manifold. In particular, Y is embeddable in $\mathbb{C}^N$, for some N.

Note that, by the example of Grauert, Andreotti-Siu and Rossi, there exist compact strongly pseudoconvex CR manifolds of dimension 3 which do not bound a complex manifold and are not embeddable in $\mathbb{C}^N$. This is in contrast to theorem 6.8.

6.2.3. Comparison with the Grauert-Andreotti-Rossi example. We want now to compare the compactification result with the examples of Grauert, Andreotti-Siu and Rossi. We call as before a non-ramified covering simply covering. An immediate consequence of Theorem 6.22 is the following.

6.35. COROLLARY. *Let V be a Stein manifold of $\dim V \geqslant 2$. Let K be a compact completely pluripolar set, $K = \varphi^{-1}(-\infty)$ where φ is a strictly plurisubharmonic function defined on a neighbourhood U of K, smooth on $U \smallsetminus K$. Then any finite covering of $V \smallsetminus K$ can be compactified to a strongly pseudoconvex space.*

PROOF. $V \smallsetminus K$ is a hyperconcave end and any finite covering of a hyperconcave end is also a hyperconcave end. □

Corollary 6.35 is in stark contrast to the examples of non-compactifiable pseudoconcave ends of Examples 6.19 and 6.20. They are obtained as finite coverings of small neighbourhoods of the boundaries of Stein manifolds of dimension 2. Such coverings have 'big' holes which cannot be filled, whereas 'small', i.e. completely pluripolar holes can always be. Moreover, if $\dim V = 2$ and $K \Subset V$ is completely pluripolar, it follows from Corollary 6.35 and Lemma 6.18 that $\pi_1(V \smallsetminus K)$ doesn't have proper subgroups of finite index.

It is obvious that the manifolds W_ε constructed in Example 6.19 are not hyperconcave ends. Also, $\mathbb{P}^2 \smallsetminus \{[1:0:0]\}$ with the new complex structure constructed in Example 6.20 cannot be a hyperconcave end, for if it were, $V \smallsetminus A$ would be one too (we average the values of a defining function on the two sheets). So A would be completely pluripolar, which is a plain contradiction.

6.2.4. Embedding of Sasakian $3-$manifolds. We explain first some well known facts about Sasakian manifolds. Let X be a strictly pseudoconvex CR manifold, with compactible complex structure J, and compactible contact form θ. This allows us to define a Riemannian metric g_θ on X given by

$$g_\theta(.,.) = d\theta(.,J.) + \theta(.)\theta(.).$$

Let R be the Reeb vector field associated to θ, defined by

$$i_R\theta = 1, \; i_R d\theta = 0.$$

Associated to the data $(X, \theta, R, J, g_\theta)$, there is a canonical connection ∇ on TX, called the *Tanaka-Webster connection* (see Tanaka [**66**] and Webster [**69**]), which is the unique affine connection on TX such that

- $\nabla g_\theta = 0$, $\nabla J = 0$, $\nabla\theta = 0$.
- For any u, v in the Levi distribution H, the torsion T of ∇ satisfies $T(u,v) = d\theta(u,v)R$ and $T(R,Ju) = JT(R,u)$

In particular, the torsion of the Tanaka-Webster connection cannot vanish identically. However, we have the following definition

6.36. DEFINITION. A strictly pseudoconvex manifold is called a Sasakian manifold if the torsion of its Webster connection in the direction of the Reeb vector field vanishes, i.e. $T(R,.) = 0$ with the notations above.

Examples of compact Sasakian manifolds are the unit sphere in $\mathbb{C}^n$ or the Heisenberg nilmanifold (see Urakawa [**68**]). Sasakian 3-manifolds were classified up to diffeomorphism by H. Geiges [**29**]. They were further studied by F. Belgun in [**9**]: every Sasakian $3-$manifold is obtained as a deformation of some standard model (see [**9**] for more details).

In [**11**], O. Biquard and M. Herzlich consider a class of manifolds which are modelled on the complex unit ball, and are thus called asymptotically complex hyperbolic. This construction will allow us to get an embedding theorem for Sasakian manifolds. Let us first recall what an asymptotically complex hyperbolic manifold is.

Let X be a $(2m-1)-$dimensional compact manifold, $m \geqslant 2$. We assume that X has a strongly pseudoconvex CR-structure. Let θ be a compatible contact form, and J a compatible almost complex structure. $\gamma(.,.) := d\theta(.,J.)$ is then a metric on the contact distribution. Following Biquard and Herzlich [**11**], we endow $M := (0,\infty)\times X$ with the metric

$$g = dr^2 + e^{-2r}\theta^2 + e^{-r}\gamma. \tag{6.13}$$

Actually, in [**11**], the authors consider the metric $dr^2 + e^{2r}\theta^2 + e^r\gamma$ on M, but the reason for our choice will become clear later.

We can extend the almost complex structure to act on the whole tangent bundle as follows. Consider the Reeb vector field R and define

$$J\partial_r = e^r R,$$

where ∂_r is the unit vector field in the r direction. g is then a Hermitian metric with respect to J, i.e. J is an $g-$isometry. The fundamental $2-$form associated to g is

$$\omega = d(e^{-r}\theta). \tag{6.14}$$

Although ω is a closed form, in this general setting, $((0,\infty)\times X, g)$ is not necessarily a Kähler manifold, because J is not necessarily an integrable almost complex structure. Indeed, the proof of [**11**, Proposition 3.1] shows that J is integrable if and only if the torsion of the Webster connection of (X,θ) in the direction of the Reeb vector field vanishes identically, i.e. if and only if X is a Sasakian manifold (see Definition 6.36).

Assume now that X will be a Sasakian manifold, so that (M,g) is a Kähler manifold. Then

$$\omega = -2\sqrt{-1}\partial\overline{\partial} r, \tag{6.15}$$

so that $-r : M \to (-\infty, 0)$ is a proper smooth strictly plurisubharmonic function. (this explains the choice of sign of the r variable for g in (6.13): with our choice, M has a strongly pseudoconvex boundary, whereas with the choice of [**11**] M has a strongly pseudoconcave boundary). Therefore M is a hyperconcave end, and the following theorem is a direct consequence of Theorem 6.22:

6.37. THEOREM ([**47**]). *Let X be a Sasakian manifold of dimension at least* 3*. Then there is a CR embedding of X in $\mathbb{C}^N$ for some integer N.*

6.3. Compactification by adding finitely many points

The present section is devoted to proving sufficient conditions for the set $\widehat{X} \smallsetminus X$ to be analytic. If this is the case, it can be actually blown down to a finite set, due to the existence of a strongly pseudoconvex neighbourhood.

6.38. THEOREM. *Let X be a hyperconcave end and let $\widehat{X}$ be a smooth completion of X. Assume that X can be covered by Zariski-open sets which are uniformized by Stein manifolds. Then $\widehat{X} \smallsetminus X$ is the union of a finite set D' and an exceptional analytic set which can be blown down to a finite set D. Each connected component of X_c, for sufficiently small c, can be analytically compactified by one point from $D' \cup D$. If X itself has a Stein cover, $D' = \emptyset$ and D consists of the singular set of the Remmert reduction of $\widehat{X}$.*

In order to prove Theorem 6.38 we consider first the particular case when the compactification $\widehat{X}$ is a Stein space.

We begin with some preparations. Let V by a complex manifold. We say that V satisfies the *Kontinuitätssatz* if for any smooth family of closed holomorphic discs $\overline{\Delta}_t$ in V indexed by $t \in [0,1)$ such that $\cup b\Delta_t$ lies on a compact subset of V, then $\cup\overline{\Delta}_t$ lies on a compact subset of V. It is clear that every Stein manifold satisfies the Kontinuitätssatz, using the strictly plurisubharmonic exhaustion function and the maximum principle. Moreover, if the universal cover of V is Stein then V satisfies Kontinuitätssatz since we can lift the family of discs to the universal cover.

Let F be a closed subset of V. We say that F is *pseudoconcave* if $V \setminus F$ satisfies the local Kontinuitätssatz in V, i.e. for every $x \in F$ there is a neigbourhood W of x such that $W \setminus F$ satisfies the Kontinuitätssatz. Observe that the finite union of pseudoconcave subsets is pseudoconcave and every complex hypersurface is pseudoconcave.

We have the following proposition which implies the Theorem 6.38.

6.39. PROPOSITION. *Let $\widehat{X}$ be a Stein space with isolated singularities S and K a completely pluripolar compact subset of $\widehat{X}$ which contains S. Assume that $X = \widehat{X} \setminus K$ can be covered by Zariski-open sets which satisfy the local Kontinuitätssatz in $\widehat{X} \setminus S$. Then K is a finite set. If $X = \widehat{X} \setminus K$ satisfies the local Kontinuitätssatz, $K = S$.*

PROOF. We can suppose that $\widehat{X}$ is a subvariety of a complex space $\mathbb{C}^N$. Let B be a ball containing K such that $bB \cap \widehat{X}$ is transversal. By hypothesis, we can choose a finite family of Zariski-open sets $V_1, \ldots, V_k$ which are uniformized by Stein manifolds and $\cap F_i$ is empty near bB, where $F_i = X \setminus V_i$. Observe that F_i is an analytic subset of X, $\overline{F}_i \subset F_i \cup K$. Since $F_i \cup (K \setminus S)$ is pseudoconcave in $\widehat{X} \setminus S$, F_i have no component of codimension $\geqslant 2$. By Hartogs theorem, if $n = \dim X > 2$, there is a complex subvariety $\widehat{F_i}$ of $\widehat{X}$ which contains F_i. This is also a consequence of Harvey-Lawson theorem [**37**]. We will prove this property for the case $n = 2$. Set $F = \cup F_i$.

Observe that $\Gamma = F \cap bB$ is an analytic real curve. The classical Wermer theorem [**70**] says that $\mathrm{hull}(\Gamma) \setminus \Gamma$ is an analytic subset of pure dimension 1 of $\mathbb{C}^N \setminus \Gamma$ where $\mathrm{hull}(\Gamma)$ is the polynomial hull of Γ. By uniqueness theorem, $\mathrm{hull}(\Gamma) \subset \widehat{X}$. Since S is finite, we have $\mathrm{hull}(\Gamma \cup S) = \mathrm{hull}(\Gamma) \cup S$. Set $F' = (F \cup K) \cap \overline{B}$ and $F'' = \mathrm{hull}(\Gamma) \cup S$.

6.40. LEMMA (In the case $n = 2$). *We have $F' \subset F''$.*

PROOF. Assume that $F' \not\subset F''$. Then there are a point $p \in F'$ and a polynomial h on U such that $\sup_{F''} |h| < \sup_{F'} |h| = |h(p)|$. Set $r = h(p)$. By maximum principle, we have $h^{-1}(r) \cap F' \subset K \setminus S$. In particular, we have $p \in K \setminus S$. Recall that $F' \setminus S$ is pseudoconcave in $\widehat{X}' = \widehat{X} \cap B \setminus S$. We will construct a smooth family of discs which does not satisfy the Kontinuitätssatz. This gives a contradiction. The construction is trivial if p is isolated in F'. We assume that p is not isolated. By using a small perturbation of h, we can suppose that $h(p)$ is not isolated in $h(F')$.

Set $\Sigma' = h(F')$ and $\Sigma'' = h(F'')$. Then Σ' (resp. Σ'') is included in the closed disk (resp. open disk) of center 0 and of radius $|r|$. The holomorphic curves $\{h = \text{const}\}$ define a holomorphic foliation, possibly singular, of $\widehat{X}'$. The difficulty is that the fibre $\{h = r\}$ can be singular at p. Denote by T the set of critical values of h in $h(B)$. Then T is finite.

Denote also Θ the unbounded component of $\mathbb{C} \setminus (\Sigma'' \cup T)$. It is clear that Σ' meets Θ. This property is stable for every small pertubation of the polynomial h. Since K is a completely pluripolar, $K \cap h^{-1}(a)$ is a polar subset of $h^{-1}(a)$ for every $a \in \mathbb{C}$.

Choose a point $b\in\Theta$ such that $0<\mathrm{dist}(b,\Sigma')<\mathrm{dist}(b,\Sigma''\cup T)$ and $a\in\Sigma'$ such that $\mathrm{dist}(a,b)=\mathrm{dist}(b,\Sigma')$. We have $a\notin\Sigma''\cup T$. Replacing b by a point of the interval (a,b) we can suppose that $\mathrm{dist}(a,b)<\mathrm{dist}(a',b)$ for every $a'\in\Sigma''\smallsetminus\{a\}$. Fix a point $q\in F'$ such that $h(q)=a$. Set $\delta_1=|a-b|$. Since $a\notin T$, we can choose a local coordinates system (z_1,z_2) of an open neigbourhood W of q in $\widehat{\Omega}'$ such that $z_1=h(z)-b$, $q=(a-b,0)$ and $\{(z_1,z_2),\ |z_1|<\delta_1+\delta_2,|z_2|<2\}\subset W$ with $\delta_2>0$ small enough. We can choose a W which does not meet F'' and is small as we want.

Let L be the complex line $\{z_1=a-b\}$. By maximum principle, $K'=F'\cap L$ is equal to $K\cap L$. Then K' is a polar subset of L. This implies that the length of K' is equal to 0. Thus, for almost every $s\in(0,2)$ the circle $\{|z_2|=s\}\cap L$ does not meet K'. Without lost of generality, we can suppose that K' does not meet $\{|z_2|=1\}\cap L$. Now we define the disk $\overline{\Delta}_t$ by

$$\overline{\Delta}_t=\{z_1=(a-b)t,|z_2|\leq 1\}$$

for $t\in[0,1)$. This smooth familly of discs does not verify the Kontinuitätssatz for $W\smallsetminus F'$. □

Now, denote by $\widehat{F_i}$ the smaller hypersuface of $\widehat{X}$ which contains F_i. Set $\widehat{F}=\cup\widehat{F_i}$. If $n=2$ we have $F\cup K\subset\widehat{F}$. This is also true for $n>2$. It is sufficient to apply the last lemma for linear slices of $\widehat{X}$.

6.41. LEMMA. *Let L be a pseudoconcave subset of a complex manifold V. If L is included in a hypersurface L' of V then L is itself a hypersurface of V.*

PROOF. Observe that L is not included in a subvariety of codimension ≥ 2 of V. Assume that L is not a hypersurface of V. Then there is a point p in $\mathrm{Reg}\,L'$ which belongs to the boundary of L in L'. Choose a local coordinates system $(z_1,\ldots,z_n)$ of a neigbourhood W of p such that W contains the unit polydisk Δ^n, $p\in\Delta^n$ and $L'\cap W=\{z_1=0\}\cap W$. We can suppose that $0\notin L$ and we can choose W small as we want.

Let $\pi:\Delta^n\longrightarrow\Delta^{n-1}$ be the projection on the last $n-1$ coordinates. Let $q\in L^*=\pi(L\cap\Delta^n)$ such that $\mathrm{dist}(0,L^*)=\mathrm{dist}(0,q)$. Consider the smooth family of discs given by $\overline{\Delta}_t=\{z=(z_1,z''):|z_1|<1/2\,,z''=tq\}$. This family does not verify the Kontinuitätssatz in $W\smallsetminus L$. □

We can end the proof of Proposition 6.39. We know that $(F_i\cup K)\smallsetminus S$ is pseudoconcave in $\widehat{X}\smallsetminus S$ and $F_i\cup K\subset\widehat{F}$. By Lemma 6.41, $(F_i\cup K)\smallsetminus S$ is a hypersurface of $\widehat{X}\smallsetminus S$. By Remmert-Stein theorem, any analytic set can be extended through a point, so $F_i\cup K$ is a hypersurface of $\widehat{X}$. Then $F_i\cup K\subset\widehat{F_i}$ since $\cap F_i=\varnothing$. We deduce that K is included in $\cap\widehat{F_i}$ which is analytic and bounded subset of $\mathbb{C}^N$. Therefore K must be a finite set. □

6.42. REMARK. The Proposition 6.39 holds for K not pluripolar. For this case, the proof is more complicated. Using another submersion of $\widehat{X}$ given by the map $z\longmapsto(h(z),h(z)+\varepsilon z_1,\cdots,h(z)+\varepsilon z_N)$, we can suppose that $R=\max_{K\smallsetminus S}|z|>\max_{F''}|z|$. Let $q\in bB_R\cap(K\smallsetminus S)$ where B_R is is a ball of center 0 and radius R. Using a small affine change of coordinates, we can suppose that $bB_R\cap\widehat{X}$ is transversal at q. We then construct easily a family of discs close to $T_q(bB_R)\cap\widehat{X}$, which does not satisfy the Kontinuitätssatz, where $T_q(bB_R)$ is the complex tangent space of bB_R at q.

PROOF OF THEOREM 6.38. Let X be a hyperconcave end such that the exhaustion function φ is overall plurisubharmonic. Let $\widehat{X}$ be a smooth completion of X. Then $\widehat{X}\smallsetminus X$ has a strictly pseudoconvex neighbourhood V. Based on Remmert's reduction theory, Grauert [**30**, Satz 3, p.338] showed that there exists a maximal analytic set A of V. Moreover, by [**30**, Satz 5, p.340] there exists a normal Stein space V' with at worst isolated singularities, a discrete set $D \subset V'$ and a proper holomorphic map $\pi : V \longrightarrow V'$, biholomorphic between $V \smallsetminus A$ and with $V' \smallsetminus D$ and $\pi(A) = D$. That is, A can be blown down to the finite set D. Of course, $V'_{sing} \subset D$.

The maximum principle for φ implies $A \subset \widehat{X} \smallsetminus X$. Let $\psi : V' \longrightarrow [-\infty, \infty)$ be given by $\psi = \varphi \circ \pi^{-1}$ on $V' \smallsetminus D$ and $\psi = -\infty$ on $\pi(\widehat{X} \smallsetminus X)$. Then ψ is a strictly plurisubharmonic function on V' and $\pi(\widehat{X} \smallsetminus X)$ is its pluripolar set. By Proposition 6.39, $\pi(\widehat{X} \smallsetminus X)$ is a finite set. Therefore $\widehat{X} \smallsetminus X$ consists of A and possibly a finite set D'. If X has a Stein cover, it follows from the Kontinuitätssatz that $\pi(\widehat{X} \smallsetminus X) = V'_{sing}$. Therefore $D' = \emptyset$ and $D = V'_{sing}$. □

6.43. REMARK. If in Theorem 6.38 we suppose only that X admits a Zariski-open dense set which is uniformized by a Stein manifold, we can prove in the same way, that $\widehat{X} \smallsetminus X$ is included in a hypersurface of $\widehat{X}$, i.e. X contains a Zariski-open dense set of $\widehat{X}$.

6.4. Extension of Nadel's theorems

We are in the position to extend the theorems of Nadel [**50**] to dimension two. If X is a hyper 1-concave manifold, it follows from [**1**] that the meromorphic function field $\mathscr{K}(X)$ has transcendence degree over $\mathbb{C}$ less or equal than $\dim X$. If the transcendence degree equals $\dim X$, that is, if there exist $\dim X$ algebraically independent meromorphic functions over $\mathbb{C}$, we say that X is *Moishezon*. We have the following characterization.

6.44. PROPOSITION. *A hyper 1-concave manifold X is Moishezon if and only if X is biholomorphic to an open set of a compact Moishezon space $\widehat{X}$. A sufficient condition for X to be Moishezon is to admit a semipositive line bundle which is positive at one point.*

PROOF. By Theorem 6.22 there exist a compact complex space $\widehat{X}$, such that $X \subset \widehat{X}$. Moreover $\widehat{X} \smallsetminus X$ is a pluripolar set. Let us remark that, due to the existence of a Stein neighbourhood of the set $\widehat{X} \smallsetminus X$, all meromorphic functions on X extend uniquely to $\widehat{X}$. Hence $\mathscr{K}(\widehat{X}) = \mathscr{K}(X)$, which implies the first part of the proposition. The second part is the content of [**67**, Corollary 3.2]. □

We extend now Nadel's main result [**50**, Theorem 0.1] to dimension 2.

6.45. PROPOSITION. *Let X be a connected manifold of dimension $n \geqslant 2$. Assume that:*

(i) *X is hyper 1-concave.*
(ii) *X is Moishezon.*
(iii) *X can be covered by Zariski-open sets which can be uniformized by Stein manifolds.*

Then X can be compactified by adding finitely many points to a compact Moishezon space.

PROOF. By conditions (i) and (iii) and Theorem 6.38, we can find a compact complex space $\widehat{X}$, with at worst isolated singularities, such that X is an open set of $\widehat{X}$ and $\widehat{X} \smallsetminus X$ is finite. Since $\mathscr{K}(\widehat{X}) = \mathscr{K}(X)$, $\widehat{X}$ is itself Moishezon. □

Note that Proposition 6.45 implies that a manifold satisfying (i)-(iii) has finite topological type.

The next result characterizes, along the lines of Kodaira, those non-compact manifolds of dimension $n \geqslant 2$ that can be compactified by adding finitely many points and that admit quasiprojective algebraic structure.

It corresponds to [**50**, Theorem 0.2], where the case $n \geqslant 3$ is considered. We have formulated condition (ii) below more geometrically. In [**50**] the corresponding condition is that the ring $\oplus_{k>0} H^0(X, E^k)$ gives local coordinates and separates points of X. Note also that the next result answers [**48**, Problem 1] for the case $q = 0$.

6.46. PROPOSITION. *Let X be a connected manifold of dimension $n \geqslant 2$. The following conditions are necessary and sufficient for X to be a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many points.*

(i) *X is hyper 1-concave.*
(ii) *X admits a positive line bundle E.*
(iii) *X can be covered by Zariski-open sets which can be uniformized by Stein manifolds.*

PROOF. The necessity of conditions (i) and (ii) is obvious, while the necessity of (iii) follows from a theorem of Griffiths [**34**, Theorem I].

For the sufficiency, we need a variant of the embedding theorem of Andreotti-Tomassini 6.13.

6.47. LEMMA. *Let X be a hyperconcave manifold and E be a positive line bundle on X. Then X is biholomorphic to an open set of a projective algebraic manifold.*

PROOF. If φ denotes the exhaustion function of X, $\sqrt{-1}\big(R^E + \partial\overline{\partial}(-\log(-\varphi))\big)$, $A \gg 1$, is a complete Kähler metric on X. Using the L^2 estimates with singular weights for positive line bundles (Theorem 4.16), we obtain that $\oplus_{k>0} H^0(X, E^k \otimes K_X)$ separates points and gives local coordinates everywhere on X. Repeating the proof of [**51**, Lemma 2.1] we obtain an embedding in the projective space and then conclude by Theorem 6.12. ☐

Let $\widehat{X}$ be a projective compactification given by Lemma 6.47. Then Theorem 6.38 implies that $\widehat{X} \smallsetminus X$ is an analytic set. Proposition 6.46 is proved. ☐

6.48. REMARK. We can obtain a stronger version of [**51**, Theorem 0.1] in dimension two. Nadel and Tsuji use the following terminology. A manifold X of dimension n is called very strongly $(n-2)$-concave if there exists a $\mathscr{C}^2$ function $\psi : X \longrightarrow \mathbb{R}$ such that $\{\psi > c\} \Subset X$, for all $c \in \mathbb{R}$, and outside a compact set ψ is plurisubharmonic and $\sqrt{-1}\partial\overline{\partial}\psi$ has at least 2 positive eigenvalues. If $\dim X = 2$, this notion coincides with hyperconcavity. Nadel and Tsuji prove the following theorem: a complete Kähler manifold (X, ω), $\dim X = n$, satisfying the conditions

(i) $\mathrm{Ric}(\omega) < 0$,
(ii) X is very strongly $(n-2)$-concave,
(iii) The universal covering of X is Stein,

is biholomorphic to a quasiprojective manifold. By Proposition 6.46 we can, if $\dim X = 2$, remove the requirement that X is complete Kähler and replace (i) with the existence of a positive line bundle.

Let $X = M/\Gamma$ be an irreducible arithmetic quotient of dimension $n \geqslant 2$. The proof of Borel [**13**] shows that X is very strongly $(n-2)$-concave. Thus, Proposition 6.46 gives, in dimension two, a generalization of the fact that arithmetic quotients can be compactified, with a completely complex-analytic proof.

6.49. REMARK. A compact Moishezon space $\widehat{X}$ with isolated singularities such that $\widehat{X} \smallsetminus X$ is finite needs not be either projective or algebraic in the sense of Weil. See the example of Grauert [**30**, p. 365–6].

6.50. REMARK. We can obtain the following version of Proposition 6.45. Namely, a manifold X, $\dim X = 2$, satisfying the two conditions,

(i)' X is hyperconcave and
(ii)' X can be covered by Zariski-open sets which can be uniformized by bounded domains of holomorphy in $\mathbb{C}^n$,

can be compactified by adding finitely many points to a compact Moishezon space. Indeed, we show first as in [**50**, p. 187] that X is Moishezon, using the result of Mok-Yau on the existence of complete Kähler-Einstein metrics on bounded domains of holomorphy, as well as the L^2-estimates for $\overline{\partial}$. Then we resort to Proposition 6.45.

Note that in [**50**, Theorem 0.3] it is shown that a manifold X as above, with $\dim X > 2$, has moreover the stucture of an abstract algebraic variety.

We close the section with an extension of Theorem 6.14. Generalizing the Andreotti-Tomassini theorem, Andreotti-Siu [**4**, Theorem 7.1] show that a strongly 1-concave manifold X of $\dim X \geqslant 3$ can be embedded in the projective space, if it admits a line bundle E such that $\oplus_{k>0} H^0(X, E^k)$ gives local coordinates on a sufficiently large compact of X. The proof is based on techniques of extending analytic sheaves. Moreover, the result breaks down in dimension 2 as the following example shows.

6.51. EXAMPLE. We use Example 6.19 and its notations. The strongly 1-concave manifold $Y_\varepsilon = K_\varepsilon \smallsetminus \mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ has as differential double covering the torus minus 16 small balls around the fixed points of the involution. Denote X_ε this new manifold. With the induced complex structure from Y_ε, X_ε is a strongly 1-concave, non-compactifiable manifold. If on X_ε we consider the pull-back E_ε of the hyperplane line bundle on Y_ε, $\oplus_{k>0} H^0(X_\varepsilon, E_\varepsilon^k)$ gives local coordinates everywhere. But X_ε is not embeddable in the projective space, for if it were, we could compactify it by Theorem 6.13.

We show in the next proposition that, if we impose the condition of hyperconcavity, such phenomenon cannot occur. Here φ and b have the same meaning as in Definition 6.21.

6.52. PROPOSITION. *Let X be a hyperconcave manifold of dimension $n \geqslant 2$. Let c be a real number such that $c < b$. Assume there is a line bundle E over $X' = \{\varphi > c\}$ such that the ring $\oplus_{k>0} H^0(X', E^k)$ gives local coordinates on X'. Then X is biholomorphic to an open subset of a projective manifold.*

PROOF. By Theorem 6.22, X is an open subset of a variety $\widehat{X}$ with isolated singularities. Moreover $\widehat{X} \smallsetminus X'$ is a Stein space.

Replacing c by a c' such that $c < c' < b$ we can suppose that there are holomorphic sections $s_0, \ldots, s_m$ of $H^0(X', E^k)$ which give local coordinates of X' where k is big enough. We can define a holomorphic map $\pi : X' \longrightarrow \mathbb{P}^m$ by

$$\pi(z) = [s_0(z) : \cdots : s_m(z)].$$

Then π gives a local immersion of X' in $\mathbb{P}^m$. Since $\widehat{X} \smallsetminus X'$ is embeddable in an euclidian space, a theorem of Dolbeault-Henkin-Sarkis [**21**], [**58**] implies that π can be extended to a meromorphic map from $\widehat{X}$ into $\mathbb{P}^m$.

Denote by Z the set consisting of the singular points of $\widehat{X}$, the points of indeterminacy of π and the critical points of π. Then Z is a compact analytic subset of $\widehat{X} \smallsetminus X'$. Since $\widehat{X} \smallsetminus X'$ is Stein space, Z is a finite set. The map π gives local immersion of $\widehat{X} \smallsetminus Z$ in $\mathbb{P}^m$.

Let H be the canonical line bundle of $\mathbb{P}^m$ and set $L = \pi^*(H)$. Then L is a positive line bundle of $\widehat{X} \smallsetminus Z$. In particular L is positive on $X \smallsetminus Z$ and by a theorem of Shiffman [**62**] extends to a positive line bundle on X. By Lemma 6.47, X is biholomorphic to an open subset of a projective manifold. □

6.5. Compactification of manifolds with pinched negative curvature

Our goal is to prove the following generalization of the Siu-Yau theorem in the case when a strongly pseudoconvex end is allowed.

6.53. THEOREM ([**47**]). *Let X be a connected complex manifold with compact strongly pseudoconvex boundary and of complex dimension $n \geqslant 2$. Assume that* $\operatorname{Int} X$ *is endowed with a complete Kähler metric with pinched negative curvature, such that away from a neighborhood of ∂X, the volume of X is finite. Then*

(1) *∂X is embeddable in some $\mathbb{C}^N$.*
(2) *There exists a compact strongly pseudoconvex domain D in a smooth projective variety and an embedding $h : X \to D$ which is a biholomorphism between* $\operatorname{Int} X$ *and $h(\operatorname{Int} X)$, $h(\partial X) = \partial D$, and $D \smallsetminus h(X)$ is an exceptional analytic set which can be blown down to a finite set of singular points.*

PROOF. We show that all the ends of X, with the exeption of the end corresponding to ∂X, are hyperconcave. Indeed, let $E_1, \dots, E_m$ be the cusps of $X \smallsetminus U$, where U is a neighbourhood of ∂X. We fix some end E_j and consider the associated Busemann function $r : E_j \to (0, \infty)$. It follows from [**64**, Proposition 1] that $-r : E_j \to (-\infty, 0)$ is a strictly plurisubharmonic proper function (note that for the Busemann function, Siu and Yau use the opposite sign convention). From Theorem 6.22 we deduce that there exists a Stein space S with boundary and an embedding of X as an pen set in S, such that $\partial S = \partial X$.

In particular, the Kohn-Rossi theorem 6.3 shows that every holomorphic function defined in a neighbourhood of ∂X extends to a holomorphic function on X. As a by-product we obtain that ∂X is connected.

Using [**39**, Theorem 0.2] (see also Ohsawa [**54**]) the space S can be embedded as a domain with boundary in a larger Stein space S' such that ∂S is a hypersurface in S'. By the embedding theorem of Remmert-Bishop-Narasimhan [**53**], S' admits a proper holomorphic embedding in $\mathbb{C}^N$ for some N. Restricting this embedding to $\partial S = \partial X$ we obtain the conclusion (1).

We prove now point (2). Note that by applying theorems 6.22 and 6.38 we can deduce (2) for some strongly pseudoconvex domain D, but we cannot say directly that this domain is an open set of a projective manifold. Therefore we proceed as follows. We start by glueing S' to a pseudoconcave projective manifold. Lempert's approximation theorem 6.11 allows to assume that the Stein space S' constructed before is an open set in an affine algebraic variety, hence also in a projective variety Y. We use now the notations from the proof of Consequence 6.5. Let $\varepsilon \in (0, c)$ be sufficiently small. We set $W = \{\varepsilon - \delta < \varphi \leqslant c\}$ and glue the manifolds X and $(Y \smallsetminus S) \cup W$ along W. The resulting manifold will be denoted by $\widehat{X}$. Hence X is a domain with compact strongly pseudoconvex boundary in $\widehat{X}$.

Since S' is an affine space in some $\mathbb{C}^N$, we can regard the embedding of $W' = \{\varepsilon - \delta < \varphi < \varepsilon\}$ in Y as a map with values in $\mathbb{C}^N$. Now X can be compactified to a compact strongly pseudoconvex domain, so the extension theorem of Kohn-Rossi, applied to the components of this embedding, show that the embedding extends to a holomorphic map from X to

$\mathbb{C}^N \subset \mathbb{P}^N$. Pulling back the hyperplane line bundle of $\mathbb{P}^N$ through this map, we obtain a line bundle $E \to \widehat{X}$ which is semi-positive on $\widehat{X}$ and positive on $(X \setminus S') \cup W$.

A partition of unity argument delivers a Hermitian metric on $\widehat{X}$ which agrees with the original metric ω of $\operatorname{Int} X$ on say $\{\varphi < \varepsilon\}$. With respect to this metric the canonical bundle of X is positive on $\{\varphi < \varepsilon\}$. Hence, the bundle $L = E^k \otimes K_{\widehat{X}}$ is positive on $\widehat{X}$ for k sufficiently large. Moreover, the curvature $\sqrt{-1}R^L$ of L dominates ω on $\{\varphi < \varepsilon\}$ and therefore $\sqrt{-1}R^L$ is a complete metric on $\widehat{X}$. The L_2 estimates of Hörmander-Andreotti-Vesentini from Theorem 4.16 produce sections of $\oplus_\nu H^0(\widehat{X}, L^\nu \otimes K_{\widehat{X}})$ that separate points and give local coordinates on $\widehat{X}$.

On the other hand the manifold $\widehat{X}$ is hyper 1-concave (we use again the plurisuperharmonicity of the Busemann function on each cusp). By Lemma 6.47 we find a smooth compactification $\widetilde{X} \subset \mathbb{P}^N$ of $\widehat{X}$ and therefore of X. The desired projective strongly pseudoconvex domain is $D = \widetilde{M} \setminus (X \setminus S')$. By [**30**, Satz 3, p.338] there exists a maximal, nowhere discrete analytic set A of D (the exceptional analytic set [**30**, Definition 3, p. 341]) and by [**30**, Satz 5, p.340] there exists a Remmert reduction $\pi : D \to D'$, which blows down A to a discrete set of points. The set $D \setminus M = \widetilde{M} \setminus M$ is a pluripolar set, namely the set where the plurisubharmonic function $-r$ takes the value $-\infty$. By the maximum principle for plurisubharmonic functions, $A \subset D \setminus M$. The set $\pi(D \setminus M)$ is also a pluripolar set, and $D'_{sing} \subset \pi(A) \subset \pi(D \setminus M)$. By Wu's theorem [**32**], any simply connected complete Kähler manifold of nonpositive sectional curvature is Stein. Hence the universal covering of M is Stein. It is then shown in [**64**, § 4] (using the Schwarz-Pick Lemma of Yau) and in Proposition 6.39 (using Wermer's theorem) that $D'_{sing} = \pi(D \setminus M)$.

Therefore $D \setminus M = A$ is an exceptional analytic set in the sense of Grauert and by blowing down this exceptional set we obtain the set D'_{sing}. Actually, each end $E_1, \dots, E_m$ of M can be compactified with one point of the singular set $D'_{sing} = \{x_1, \dots, x_m\}$. Moreover, by the uniqueness of the Stein completion from Theorem 6.1 we see that D' and S' coincide. □

6.54. REMARK. It follows from the proof that each end of X, exept that corresponding to ∂X, can be analytically compactified by adding one singular point. Theorem 6.53 also holds true if we assume that the complete Hermitian metric is defined on X (including the boundary). However, in the case of quotients in Corollary 6.56 the induced Hermitian metric is defined only in the interior of X.

The first application of Theorem 6.53 is the classical theorem of Siu–Yau, which is the particular case when $\partial X = \varnothing$.

6.55. COROLLARY (Siu-Yau). *Let X be a complete Kähler manifold of finite volume and bounded negative sectional curvature. If* $\dim X \geqslant 2$, *X is biholomorphic to a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many singular points.*

As a second application we study some quotients of the unit complex ball B^n in $\mathbb{C}^n$ which where considered by Burns and Napier-Ramachandran [**52**, Theorem 4.2].

6.56. COROLLARY. *Let Γ be a torsion-free discrete group of automorphisms of the unit ball B^n in $\mathbb{C}^n$, $n \geqslant 2$, and let $X = B^n/\Gamma$. Assume that the limit set Λ is a proper subset of ∂B^n and that the quotient $(\partial B^n \setminus \Lambda)/\Gamma$ has a compact component A. Let E be the end of x corresponding to A and assume that $X \setminus E$ has finite volume. Then A is embeddable in* *some $\mathbb{C}^N$ and X can be compactified to a strongly pseudoconvex domain in a projective variety by adding an exceptional analytic set.*

PROOF. As is well known, the limit set Λ is the set of accumulation points of any orbit $\Gamma\cdot x$, $x\in B^n$, and is a closed Γ-invariant subset of the sphere at infinity ∂B^n. The complement $\partial B^n\smallsetminus\Lambda$ is precisely the set of points at which Γ acts properly discontinuously, and the space $X\cup(\partial B^n\smallsetminus\Lambda)/\Gamma$ is a manifold with boundary $(\partial B^n\smallsetminus\Lambda)/\Gamma$ (see for example [**23**, §10]). A is a compact subset of this boundary, hence there is a neighborhood E of A in X which is diffeomorphic to the product $A\times(0,1)$. It follows that E is an end of X, because A is compact and connected. Actually, E is a strongly pseudoconvex end, in the sense that its boundary A at infinity is strictly pseudoconvex. Since $X=B^n/\Gamma$ is a complete manifold with sectional curvature pinched between -4 and -1, Corollary 6.56 is an immediate consequence of Theorem 6.53. □

The third application is to actually establish the equivalence between the finite volume condition and the embeddability of the boundary.

6.57. COROLLARY. *Let X be a connected complex manifold with compact strongly pseudoconvex boundary and of complex dimension $n\geqslant 2$. Assume that X is endowed with a complete Kähler metric with pinched negative curvature. The following assertions are equivalent :*

(a) *∂X is embeddable in some $\mathbb{C}^N$.*
(b) *X has finite volume away from a neighbourhood of ∂X.*

If one of the equivalent conditions (a) or (b) holds true, X can be compactified to a strongly pseudoconvex domain in a projective variety by adding an exceptional analytic set.

PROOF. The implication $(b)\Rightarrow(a)$ was proved in Theorem 6.53. We wish to prove $(a)\Rightarrow(b)$. Indeed, once ∂X is assumed to be embeddable, we follow the second part of the proof of Theorem 6.53. The difficulty is now that we do not know apriori that X can be compactified.

Since $\partial X=\{\varphi=c\}$ is embeddable it follows from the Epstein-Henkin Theorem 6.16 that also $\{\varphi=\varepsilon\}$ is embeddable for $\varepsilon\in(0,c)$. Using the Harvey-Lawson theorem we fill in $\{\varphi=\varepsilon\}$ and we compactify the stip $\{\varepsilon<\varphi<c\}$ to an affine Stein space S, which can be realized as a Stein domain with boundary in a bigger Stein space. Lempert approximation theorem entails that we can assume that S is a domain in an affine variety. We extend now the embedding of $\{\varepsilon<\varphi<c\}$ in $S\subset\mathbb{C}^N$ to X by using the following Hartogs type result:

6.58. PROPOSITION ([**52**, Prop. 4.4]). *Let (M,ω) be a connected complete Hermitian manifold of dimension $n>1$ and let $X\subset M$ be a domain with nonempty smooth compact strongly pseudoconvex boundary. Assume that the restriction $\omega|_X$ of ω to X is Kähler. Suppose f is a holomorphic function on $U\cap X$ for some neighbourhood U of ∂X in X. Then there exists a holomorphic function h on M such that $h=f$ near ∂X. In particular, ∂X is connected.*

We can repeat the proof of Theorem 6.53 and construct the positive holomorphic line bundle L on the manifold $\widehat{X}$. By applying Theorem 4.34 we obtain

$$\dim H^{n,0}_{(2)}(\widehat{X},L^k)\geqslant\frac{k^n}{n!}\int_{\widehat{X}}\Big(\tfrac{\sqrt{-1}}{2\pi}R^L\Big)^n+o(k^n)\,,\tag{6.16}$$

Since $\dim H^{n,0}_{(2)}(\widehat{X}, L^k) < \infty$ ($\widehat{X}$ is hyperconcave) and the curvature $\sqrt{-1}R^L$ of L dominates ω on $\{\varphi < \varepsilon\}$, where $\varepsilon < c$, we deduce that (X,ω) has finite volume away from a neighbourhood of ∂M. □

The assertion $(a) \Rightarrow (b)$ is [**52**, Theorem 4.1]. We infer the following result anounced by D. M. Burns and proved by T. Napier and M. Ramachandran [**52**, Theorem 4.2]:

6.59. THEOREM (Burns, Napier-Ramachandran). *Let Γ be a torsion-free discrete group of automorphisms of the unit ball B in $\mathbb{C}^n$ with $n \geqslant 3$ and let $X = B/\Gamma$. Assume that the limit set Λ is a proper subset of ∂B and that the quotient $(\partial B \smallsetminus \Lambda)/\Gamma$ has a compact component A. Then X has only finitely many ends, all of which, except for the unique end corresponding to A, are cusps. In fact, X is diffeomorphic to a compact manifold with boundary and can be compactified.*

The proof is based on [**52**, Theorem 4.1] which shows that the finite volume hypothesis of Corollary 6.56 is automatically satisfied in the case $n \geqslant 3$. The presence of the strongly pseudoconvex boundary forces the volume to be finite, since ∂X is then embeddable by 6.8, having real dimension at least 5.

If $n = 2$ we have to assume the volume to be finite in order to obtain the embedding of the boundary. It is interesting to ask whether Burns' theorem holds also in dimension 2 or, equivalently, whether the compact strongly pseudoconvex component of a set $(\partial B \smallsetminus \Lambda)/\Gamma$ is embeddable for all torsion-free discrete groups of automorphisms of the unit ball B in $\mathbb{C}^2$.

6.6. Nadel-Tsuji compactification theorem

In this section we discuss briefly a generalization of the theorem of Siu-Yau, namely a geometric proof of the compactification of arithmetic quotients of arbitrary rank. We will call a complex manifold of dimension n very strongly $(n-q)$–pseudoconcave if there exists a $\mathscr{C}^2$ function $\psi : X \to \mathbb{R}$ such that $\{\psi > b\}$ is compact for any $b \in \mathbb{R}$ and outside a compact set ψ is weakly plurisubharmonic and its Levi form has at least q positive eigenvalues. Note that in the case $q = n$ we recover the class of hyperconcave manifolds.

6.60. THEOREM (Nadel–Tsuji [**51**]). *Let (X,ω) be a complete Kähler manifold of dimension n of negative Ricci curvature. Assume that X is uniformized by a Stein manifold and that X is very strongly $(n-2)$–pseudoconcave. Then, X is biholomrphic to a quasi–projective variety.*

The first step is to generalize the Andreotti-Tomassini embeding theorem to the present context [**51**, Lemma 2.1]. There exists therefore a holomorphic embedding $F : X \longrightarrow Z$ of X onto a non–singular projective–algebraic variety Z. We will henceforth identify X with its image under F. The goal is to show that $Z \smallsetminus X$ is an analytic set of Z. The main toll are the Morse inequalities of Theorem 4.31 and existence theorems for complete Kähler–Einstein metrics on bounded domains of holomophy and certain quasi–projective varieties. Let us briefly explain the proof. $Z \smallsetminus X$ contains at most a finite number of irreducible hypersurfaces D_i of Z. Denoting the union of D_i by D we will show that $X = Z \smallsetminus D$. By removing some hypersurface V in Z and using existence theorems for complete Kähler–Einstein metrics [**49**, Main Theorem] one can show that both $W = X \smallsetminus V$ and $W' = (Z \smallsetminus D) \smallsetminus V$ admit Kähler–Einstein metrics of Ricci curvatures $\equiv -1$. Denote the Kähler forms by ω and ω' resp. From Yau's Ahlfors–Schwarz lemma [] for volume forms we have

$\omega^n \geqslant \omega'^n$. To show that $W = W'$ and hence $X = Z \smallsetminus D$ it suffices to show that $\omega^n \equiv \omega'^n$. In fact, since Kähler–Einstein metrics are determined by their volume forms this would imply $\omega \equiv \omega'$ and hence $W = W'$.

Denote by K the canonical line bundle and by $[D]$ the divisor line bundle. We relate the volumes of (W,ω) and (W,ω') to the asymptotic growth of the dimensions of the spaces of L^2 holomorphic sections $H^0_{(2)}(W,(K\otimes[D]\otimes[V])^k)$ and $H^0_{(2)}(W',(K\otimes[D]\otimes[V])^k)$. As pseudoconcavity implies extension theorems for holomorphic sections of line bundles from W to W' the asymptotic growth of these spaces is the same. On the one hand for the quasi–projective variety W' the asymptotic growth of $H^0_{(2)}(W',(K\otimes[D]\otimes[V])^k)$ is determined by the volume of (W',ω'). On the other hand, Theorem 4.31 gives a lower bound for the asymptotic growth of $H^0_{(2)}(W,(K\otimes[D]\otimes[V])^k)$ by the volume of (W,ω). Equating the two asymptotic rates yields the inequality $\mathrm{vol}(X,\omega) \leqslant \mathrm{vol}(X',\omega')$. Combined with $\omega^n \geqslant \omega'^n$ this yields the desired identity $\omega^n \equiv \omega'^n$ and hence $X = Z \smallsetminus D$.

It should be possible to give a proof of the Nadel-Tsuji theorem without recourse to the existence of Kähler-Einstein metrics by using complex-analytic tools, as we did in the previous sections for the case of hyperconcave manifolds.

6.7. Embedding of strongly q-concave manifolds

Let us consider the problem of embedding a q–concave manifold in a projective space or more generally in a Moishezon manifold.

The first basic result in this direction is the Andreotti Theorem 6.12, which shows that the embeddability implies the compactification of the manifold, which is a generalization of Chow's algebraicity theorem. An analytic criterion for the embeddability is given by the Andreotti-Tomassini embedding Theorem 6.13. We wish to discuss now a more *intrinsic* characterisation of q–concave manifolds. The general problem is the following:

6.61. PROBLEM. *Let X be a q–concave manifold and let $L \longrightarrow X$ be a positive line bundle. Find sufficient conditions for X to be an open set of a projective or Moishezon manifold. Find examples of non–algebraic X (if any).*

It is easy to see that hyper 1–concave manifolds possesing a positive line bundle are projectively embeddable. Moreover, when X admits a positively embedded (i.e. with positive normal bundle) smooth compact divisor Z, from the rigidity theorem of Griffiths [**33**] we infer that global sections in high tensor powers of the the associated bundle $[Z]$ embed a small neighbourhood V of Z in the projective space. In particular X has a maximal number of independent meromorphic functions, since the meromorphic functions extend from V to X.

In general, the difficulty consists in applying the Bochner–Kodaira–Nakano formula for solving $\overline{\partial}$. Indeed, there is a conflict of signs between the negativity of the Levi form of the exhaustion function and the positivity of the curvature. If we want to modify the curvarure of L by multiplying the metric with a suitable weight we lose the positivity. Therefore, we cannot solve the $\overline{\partial}$–equation directly but rather use a quantitative version of the $\overline{\partial}$–method, namely the holomorphic Morse inequalities.

We will prove an existence criterion giving a lower bound for $\dim H^0(X,L^k)$ in terms of geometric data such as the Levi form of ∂X and the curvature of L. As a corollary we see that, roughly speaking, if the volume of X in the metric $\sqrt{-1}R^L$ exceeds the volume of ∂X times a constant expressing the size of the Levi form and of the curvature $\sqrt{-1}R^L$ near the boundary, the ring $\mathscr{A}(X,L)$ contains local coordinates for each point outside a proper analytic set of X. An important feature of our estimate is the presence of a negative boundary term which expresses the obstruction to finding holomorphic sections. We need some preparations and notations in order to state the result. Let X be a q–concave manifold with exhaustion function φ. If ∂X_c is smooth the Levi form of ∂X_c has at least $n-q-1$ negative eigenvalues (since the defining function for X_c is $c-\varphi$). Therefore the following setting may be considered.

Let $D \Subset X$ be a smooth domain in a complex manifold X such that the Levi form of ∂D has at least 2 negative eigenvalues. Then we can choose a defining function φ for D which is smooth on $\overline{D}$, $D = \{\varphi < 0\}$ and $\partial\overline{\partial}\varphi$ has at least 3 negative eigenvalues. We can in fact modify a defining function in order to get an extra negative eigenvalue in the complex normal direction to ∂D. In the following we keep the function φ fixed.

Let us explain why we need at least two negative eigenvalues of the Levi form restricted to the boundary, or at least 3 positive eigenvalues for $\sqrt{-1}\partial\overline{\partial}\varphi$. Our method is based on L^2 estimates for $(0,1)$–forms on D which imply the finiteness of the first cohomology group $H^1(X,F)$ for holomorphic vector bundles F over X. By the Andreotti–Grauert theory we have $\dim H^p(X,F) < \infty$ for $p \leqslant n-(q+1)-1 = n-q-2$ and $\dim H^p(X,F) = \infty$ for $p = n-q-1$. Therefore we have to impose $n-q-1 > 1$ i.e. $n-q > 2$.

We introduce a hermitian metric $\omega = \omega_\varphi$ in the neighbourhood of $\overline{D}$ such that in a neighbourhood V of ∂D the following property holds:

6.62. PROPERTY. The first 3 eigenvalues of $\sqrt{-1}\partial\overline{\partial}\varphi$ with respect to ω are at most $-2n+3$ and all others are at most 1.

Finally set dS_L for the volume form of ∂D in the induced metric from $\sqrt{-1}R^L$ and $|d\varphi|_L$ for the norm of $d\varphi$ in the metric associated to $\sqrt{-1}R^L$.

We can state the estimate for the dimension of the space holomorphic sections on the concave domain D.

6.63. EXISTENCE CRITERION. *Let $D \Subset X$ be a smooth domain in a complex manifold X such that the Levi form of ∂D possesses at least* 2 *negative eigenvalues. Let L be a holomorphic line bundle on X which is assumed to be positive on a neighbourhood of $\overline{D}$. Then*

$$\liminf_{k\longrightarrow\infty} k^{-n}\dim H^0(D,L^k) \geqslant \int_D \left(\frac{\sqrt{-1}}{2\pi}R^L\right)^n - C(\varphi,L)\int_{\partial D}\frac{dS_L}{|d\varphi|_L} \tag{6.17}$$

The constant $C(\varphi,L)$ depends explicitely on the curvature of L and on the Levi form $\sqrt{-1}\partial\overline{\partial}\varphi$ (cf. (6.18)).

PROOF. We proceed as follows. In a first instance we find the fundamental L^2 estimate for the $(0,1)$–forms with values in L^k. Then following the proof of Theorem 4.2 we compare the spectrum of the Laplace operator on D (for a complete metric) with the spectrum of the Dirichlet problem over a smaller domain $D(\varepsilon/2)$ which is a set of points of D at distance less than $\sqrt{\varepsilon/2}$ times a certain constant from ∂D (see (6.28) for the precise definition). On $D(\varepsilon/2)$ we can use Demailly's spectral formula and get a lower bound for the dimension of the space of sections in L^k for large k. We shall need the full strength of Demailly's result since the curvature of the changed metric has negative eigenvalues. In the last step we apply the results to metrics which approximate the positive metric on L in the interior of the manifold. In the process of approximation the set where the curvature has a negative part concentrates to the boundary ∂D and is responsible for the negative boundary term in the final estimate of the Existence Criterion.

We begin by setting some notations and defining the constant $C(\varphi,L)$.

Let η a hermitian metric on X, Φ a real $(1,1)$–form and K a compact set in X. We set:

$$M_\eta(\Phi,K)=\sup_{x\in K}\sup_{v\in T_xX\smallsetminus\{0\}}\frac{\Phi(v,v)}{\eta(v,v)}\,,$$

the supremum over K of the highest eigenvalue of Φ with respect to η. In hindsight to our previous situation denote:

$$\begin{aligned}
&M_L(\varphi)=M_{\sqrt{-1}R^L}(\sqrt{-1}\partial\overline{\partial}\varphi,\overline{D})\\
&M_L(-\varphi)=M_{\sqrt{-1}R^L}(-\sqrt{-1}\partial\overline{\partial}\varphi,\overline{D})\\
&M_\omega(L)=M_\omega(\sqrt{-1}R^L,\overline{D})\\
&M'_\omega(L)=1+2(n-1)M_\omega(\sqrt{-1}R^L,\overline{D})\\
&M_L(\partial\varphi)=M_{\sqrt{-1}R^L}(\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi,\partial D)
\end{aligned}$$

which represent the relative size of the respective $(1,1)$–forms. We also put:

$$\begin{aligned}
&C_1=\sqrt{2M_L(-\varphi)M'_\omega(L)}-1\\
&C_2=2M_L(-\varphi)M'_\omega(L)-1\\
&C_3=2M_L(\varphi)M'_\omega(L)+1\\
&C_4=2M'_\omega(L)M_L(\partial\varphi)
\end{aligned}$$

The definition of $C(\varphi,L)$ is then

$$C(\varphi,L)=(2\pi)^{-n}C_1C_2C_3^{n-2}C_4\,. \tag{6.18}$$

Let $\gamma_1\leqslant\gamma_2\leqslant\cdots\leqslant\gamma_n$ be the eigenvalues of $\sqrt{-1}\partial\overline{\partial}\varphi$ with respect to ω. We have chosen ω such that (see Property 6.62) in a neighbourhood V of ∂D,

$$\Gamma_1\leqslant\Gamma_2\leqslant\Gamma_3\leqslant-2n+3\,, \tag{6.19a}$$

$$\Gamma_n\leqslant1\,. \tag{6.19b}$$

Let $\chi:(-\infty,0)\longrightarrow\mathbb{R}$, $\chi(t)=t^{-2}$. We consider the complete metric:

$$\omega_0=\omega+\chi(\varphi)\partial\varphi\wedge\overline{\partial}\varphi \tag{6.20}$$

which grows as φ^{-2} in the normal direction to ∂D. Along the fibers of L we introduce the metric:

$$h^L_\varepsilon=h^L\exp\left(-\varepsilon\int_{\inf\varphi}^{\varphi}\chi(t)\,dt\right) \tag{6.21}$$

where h^L is the given metric on L (for which $\sqrt{-1}R^L$ is positive). The curvature of h^L_ε is

$$\sqrt{-1}R^{(L,h^L_\varepsilon)}=\sqrt{-1}R^L+\sqrt{-1}\varepsilon\chi(\varphi)\partial\overline{\partial}\varphi+\sqrt{-1}\varepsilon\chi'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi$$

We evaluate the eigenvalues of $\sqrt{-1}R^{(L,h^L_\varepsilon)}$ with respect to ω_0 with the goal to apply the Bochner–Kodaira formula. Denote by $\Gamma^0_1\leqslant\Gamma^0_2\leqslant\cdots\leqslant\Gamma^0_n$ the eigenvalues of $\sqrt{-1}\partial\overline{\partial}\varphi$ and $\Gamma^\varepsilon_1\leqslant\Gamma^\varepsilon_2\leqslant\cdots\leqslant\Gamma^\varepsilon_n$ the eigenvalues of $\sqrt{-1}\varepsilon\chi(\varphi)\partial\overline{\partial}\varphi+\sqrt{-1}\varepsilon\chi'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi$ with respect to ω_0. The minimum–maximum principle yields

$$\Gamma_1\leqslant\Gamma^0_1\leqslant\Gamma_2\leqslant\Gamma^0_2\leqslant\Gamma_3\leqslant-2n+3 \qquad \text{by (6.19a)}\,, \tag{6.22a}$$

$$\Gamma^0_3<0 \qquad \text{since } \Gamma_3<0\,, \tag{6.22b}$$

$$\Gamma^0_j\leqslant\max\{\Gamma_n,0\}\leqslant1 \quad \text{for } 4\leqslant j\leqslant n\,, \qquad \text{by (6.19b)}\,. \tag{6.22c}$$

on V. It is also easy to see that the highest eigenvalue of $\sqrt{-1}\chi'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi$ with respect to ω_0 satisfies

$$\sup_{v\in T_xX\smallsetminus\{0\}}\frac{\sqrt{-1}\chi'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi(v,v)}{\omega_0(v,v)}\leqslant\chi(\varphi)\,,\quad\text{for all }x\in D\,. \tag{6.23}$$

By (6.23) we have

$$\Gamma^\varepsilon_j\leqslant\varepsilon\chi(\varphi)(\Gamma^0_j+1)$$

and therefore,

$$\begin{aligned}&\Gamma^\varepsilon_1\leqslant\Gamma^\varepsilon_2\leqslant(-2n+4)\varepsilon\chi(\varphi) &&\text{by (6.22a)}\,,\\ &\Gamma^\varepsilon_3\leqslant\varepsilon\chi(\varphi) &&\text{by (6.22b)}\,,\\ &\Gamma^\varepsilon_j\leqslant2\varepsilon\chi(\varphi)\quad\text{for }4\leqslant j\leqslant n\,, &&\text{by (6.22c)}\,.\end{aligned}$$

Summing up we obtain

$$\Gamma^\varepsilon_2+\cdots+\Gamma^\varepsilon_n\leqslant-\varepsilon\chi(\varphi)\,. \tag{6.24}$$

This sum will appear in the Bochner–Kodaira formula and carries the information about the concavity of D.

We also have to estimate the eigenvalues of $\sqrt{-1}R^L$ with respect to ω_0. We denote by $\alpha_1\leqslant\alpha_2\leqslant\cdots\leqslant\alpha_n$ the eigenvalues of $\sqrt{-1}R^L$ with respect to ω and by $\alpha^0_1\leqslant\alpha^0_2\leqslant\cdots\leqslant\alpha^0_n$ the eigenvalues of $\sqrt{-1}R^L$ with respect to ω_0. It is straightforward that

$$\alpha^0_n\leqslant\alpha_n\leqslant M_\omega(E)<\infty\quad\text{on }V\,. \tag{6.25}$$

Since the torsion operator of ω_0 with respect to ω_0 are bounded by a constant $A>0$ (depending only on ω_0), the Bochner–Kodaira formula (B.20), (3.70) assumes the following form:

$$\begin{aligned}&\tfrac{3}{2}\left(\|\overline{\partial}^{L^k}u\|^2+\|\overline{\partial}^{L^k*}u\|^2\right)\\&\qquad\geqslant\int_D\left[-k(\Gamma^\varepsilon_2+\cdots+\Gamma^\varepsilon_n)-k(\alpha^0_2+\cdots+\alpha^0_n)-A\chi(\varphi)\right]|u|^2\,dV\end{aligned} \tag{6.26}$$

for any compactly supported $(0,1)$–form in D with values in L^k. The volume form is taken with respect to ω_0 and the norms are with respect to ω_0 on D and h^L_ε on L. The inequalities (6.24), (6.25) and (6.26) entail

$$\tfrac{3}{2}\left(\|\overline{\partial}^{L^k}u\|^2+\|\overline{\partial}^{L^k*}u\|^2\right)\geqslant\int_D[-k(n-1)M_\omega(L)+k\varepsilon\chi(\varphi)-A\chi(\varphi)]\,|u|^2\,dV \tag{6.27}$$

for any compactly supported $(0,1)$–form in D with values in L^k and support in V. We use now the term $k\varepsilon\chi(\varphi)$ to absorb the negative terms in the left–hand side of (6.27). We introduce the following notation:

$$D(\varepsilon)=\left\{x\in D\,:\,\varphi(x)<-\sqrt{\varepsilon/M'_\omega(L)}\right\}\,. \tag{6.28}$$

We may assume that V contains the set $\complement D(\varepsilon)$ (for ε small enough). In the set $\complement D(\varepsilon)$ we have $\varepsilon\chi(\varphi)\geqslant M'_\omega(L)$ and if we choose $k\geqslant2A\varepsilon^{-1}$ we get

$$-k(n-1)M_\omega(L)+k\varepsilon\chi(\varphi)-A\chi(\varphi)\geqslant\frac{k}{2}$$

so that (6.27) yields

$$3\left(\|\overline{\partial}^{L^k}u\|^2+\|\overline{\partial}^{L^k*}u\|^2\right)\geqslant k\int_D|u|^2\,dV\,,\quad\operatorname{supp}u\Subset\complement D(\varepsilon)\,,\;k\geqslant2A\varepsilon^{-1} \tag{6.29}$$

Since the metric ω_0 is complete we deduce that (6.29) holds true for any $(0,1)$–form $u \in \operatorname{Dom}\overline{\partial} \cap \operatorname{Dom}\overline{\partial}^{L^k *}$ with support in $\complement D(\varepsilon)$ (by the density lemma of Andreotti–Vesentini A.10).

Having obtained the fundamental estimate (6.29) we follow the proof of the abstract Morse inequalities Theorem (4.6). We fill in the details since we need the precise output to be able to make $\varepsilon \longrightarrow 0$.

By (4.13), for λ sufficintly small and k sufficintly large,

$$\dim H^0(D, L^k \otimes K_X) + N^1\left(\lambda, \square^{L^k}\right) \geqslant N^0\left(\lambda, \square^{L^k}\right). \tag{6.30}$$

Thus, we have to estimate $N^1(\lambda, \square^{L^k})$ from above and then $N^0\left(\lambda, \square^{L^k}\right)$ from below.

Following Theorem 4.2 we show that the essential spectrum of $\square^{L^k}$ on $(0,1)$–forms does not contain the open interval $(0, 1/24)$ and we can compare the counting function on this interval with the counting function of the same operator considered with Dirichlet boundary conditions on the domain $D(\varepsilon/2)$ (introduced in (6.28)) and denoted $\square^{L^k}_{D(\varepsilon/2)}$. In particular $N^1\left(\lambda, \square^{L^k}\right)$ is finite dimensional for $\lambda < 1/24$. Let $\rho_\varepsilon \in \mathscr{C}^\infty(D)$ such that $\rho_\varepsilon = 0$ on a closed neighbourhood of $D(\varepsilon)$ and $\rho_\varepsilon = 1$ on $\complement D(\varepsilon/2)$. Denote $C_\varepsilon = 6 \sup |d\rho_\varepsilon|^2 < \infty$. The constant depends on ε (which is fixed) but not on k. Then for k sufficiently large the operator $\square^{L^k}$ on $(0,1)$–forms has discrete spectrum in $(0, 1/24)$ and

$$N^1\left(\lambda, \square^{L^k}\right) \leqslant N^1\left(24\lambda + 16C_\varepsilon k^{-1}, \square^{L^k}_{D(\varepsilon/2)}\right), \quad \text{for } \lambda \in (0, \varepsilon/2). \tag{6.31}$$

We obtain now a lower estimate for $N^0\left(\lambda, \square^{L^k}\right)$. For $\lambda < 1/24$ and sufficiently large k the following relation holds :

$$N^0(\lambda, \square^{L^k}) \geqslant N^0(\lambda, \square^{L^k}_{D(\varepsilon/2)}). \tag{6.32}$$

The asymptotic behaviour of the spectrum distribution function for the Dirichlet problem has been determined explicitely in Theorem 3.15. There exists a function $\overline{\nu}^j_\varepsilon(\mu, x) = \overline{\nu}^j_{R^{(L,h^L_\varepsilon)}}(\mu, x)$ (cf. (3.90)), depending on the eigenvalues of the curvature of (L, h^L_ε), which is bounded on compact sets of D and right continuous in μ such that for any $\mu \in \mathbb{R}$

$$\limsup_{k \longrightarrow \infty} k^{-n} N^j\left(\mu, \widetilde{\square}^{\widetilde{E}^k}|_{\widetilde{Y}}\right) \leqslant \tfrac{1}{n!} \int_{D(\varepsilon/2)} \overline{\nu}^j_\varepsilon(\mu, x)\, dv_X. \tag{6.33}$$

Moreover there exists an at most countable set $\mathscr{D}_\varepsilon \subset \mathbb{R}$ such that for μ outside $\mathscr{D}_\varepsilon$ the limit of the left–hand side expression exists and we have equality in (6.33).
For $\lambda < (1/24)$ and sufficiently large k we have

$$\dim H^0(D, L^k) \geqslant N^0\left(\lambda, \square^{L^k}\right) - N^1\left(\lambda, \square^{L^k}\right) \tag{6.34}$$

For $\lambda < (1/24)$ and λ outside $\mathscr{D}_\varepsilon$ we apply (6.33) and (6.32):

$$\lim_{k \longrightarrow \infty} k^{-n} N^0(\lambda, \square^{L^k}) \geqslant \tfrac{1}{n!} \int_{D(\varepsilon/2)} \overline{\nu}^0_\varepsilon(\lambda, x)\, dv_X. \tag{6.35}$$

On the other hand given $\delta > 0$ we learn from (6.31) that for large k

$$\begin{aligned} N^1\left(\lambda, \square^{L^k}\right) &\leqslant N^1\left(24\lambda + 16C_\varepsilon k^{-1}, \widetilde{\square}^{\widetilde{E}^k}|_{\widetilde{Y}}\right) \\ &\leqslant N^1\left(24\lambda + \delta, \widetilde{\square}^{\widetilde{E}^k}|_{\widetilde{Y}}\right) \end{aligned}$$

hence

$$\limsup_{k\longrightarrow\infty} k^{-n}N^1(\lambda,\square^{L^k}) \leqslant \tfrac{1}{n!}\int_{D(\varepsilon/2)} \overline{\nu}^1_\varepsilon(24\lambda+\delta,x)\,dV(x)\,.$$

and after letting k go to infinity we can also let δ go to zero. Using these remarks we see that for all but a countable set of λ we have

$$\liminf_{k\longrightarrow\infty} k^{-n}\dim H^0(D,L^k) \geqslant \tfrac{1}{n!}\int_{D(\varepsilon/2)} \left[\overline{\nu}^0_\varepsilon(\lambda,x)-\overline{\nu}^1_\varepsilon(24\lambda,x)\right]dV(x)$$

In the latter estimate we may let $\lambda \longrightarrow 0$ (through values outside the exeptional countable set) and this yields, by the formulas in (3.87) for the right–hand side

$$\liminf_{k\longrightarrow\infty} k^{-n}\dim H^0(D,L^k) \geqslant \tfrac{1}{n!}\int_{D(\varepsilon/2)(\leqslant 1,h_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n \tag{6.36}$$

The set $D(\varepsilon/2)(\leqslant 1,h_\varepsilon)$ is the set of points in $D(\varepsilon/2)$ where $\sqrt{-1}R^{(L,h^L_\varepsilon)}$ is non–degenerate and has at most one negative eigenvalue. Thus $D(\varepsilon/2)(\leqslant 1,h_\varepsilon)$ splits in two sets: the set $D(\varepsilon/2)(0,h_\varepsilon)$ where $\sqrt{-1}R^{(L,h^L_\varepsilon)}$ is positive definite and the set $D(\varepsilon/2)(1,h_\varepsilon)$ where $\sqrt{-1}R^{(L,h^L_\varepsilon)}$ is non–degenerate and has exactly one negative eigenvalue. The integral in (6.36) splits accordingly into one positive and one negative term:

$$\liminf_{k\longrightarrow\infty} k^{-n}\dim H^0(D,L^k) \geqslant \tfrac{1}{n!}\int_{D(\varepsilon/2)(0,h^L_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n + \tfrac{1}{n!}\int_{D(\varepsilon/2)(1,h^L_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n \tag{6.37}$$

Our next task is to make $\varepsilon \longrightarrow 0$ in (6.37). For $\varepsilon \longrightarrow 0$ the metrics h_ε converges uniformly to the metric h of positive curvature on every compact set of D. So on any compact of D we recover the integral of R^L. On the other hand $D(\varepsilon/2)$ exhausts D and the sets $D(\varepsilon/2)(1,h^L_\varepsilon)$ concentrate to the boundary ∂D.
Let us fix a compact set $K\subset D$. For sufficiently small ε we have $K\subset D(\varepsilon/2)$ and

$$\int_{D(\varepsilon/2)(0,h_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n \geqslant \int_{K(0,h_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n$$

We have $h^L_\varepsilon \longrightarrow h^L$ on L in the $\mathscr{C}^\infty$–topology. Since $K(0,h)=K$ letting $\varepsilon\longrightarrow 0$ in the previous inequality yields

$$\liminf_{\varepsilon\longrightarrow 0}\int_{K(\varepsilon/2)(0,h_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n \geqslant \int_L \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L)}\Big)^n \tag{6.38}$$

Let us study the more delicate second integral in (6.37). For this goal we fix on D the ground metric $\omega_L = R^L$. This choice will simplify our computations. We denote by $\lambda^\varepsilon_1 \leqslant \lambda^\varepsilon_2 \leqslant \cdots \leqslant \lambda^\varepsilon_n$ the eigenvalues of $\sqrt{-1}R^{(L,h^L_\varepsilon)}$ with respect to ω_L. Then the integral we study is

$$I_\varepsilon = \tfrac{1}{n!}\int_{D(\varepsilon/2)(1,h_\varepsilon)} \Big(\tfrac{\sqrt{-1}}{2\pi}R^{(L,h^L_\varepsilon)}\Big)^n = \tfrac{1}{(2\pi)^n}\int_{S(\varepsilon)} \lambda^\varepsilon_1\lambda^\varepsilon_2\cdots\lambda^\varepsilon_n\;\omega^n_L/n!$$

where the integration set is

$$S(\varepsilon) := D(\varepsilon/2)(1,h_\varepsilon) = \{x\in D(\varepsilon/2)\,:\,\lambda^\varepsilon_1(x) < 0 < \lambda^\varepsilon_2(x)\}$$

We find an upper bound for $|I_\varepsilon|$ so we determine upper bounds for $|\lambda^\varepsilon_1|$, $|\lambda^\varepsilon_2|$, …, $|\lambda^\varepsilon_n|$ on $S(\varepsilon)$. Since λ^ε_1 is negative on $S(\varepsilon)$ we have to obtain a lower bound for this eigenvalue. By the min-max principle

$$\lambda_1^\varepsilon(x) = \min_{v\in T_xD} \frac{\Big[R^{(L,h^L)} + \sqrt{-1}\varepsilon\chi(\varphi)\partial\overline{\partial}\varphi + \sqrt{-1}\varepsilon\chi'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi\Big](v)}{R^L(v)}\,.$$

We use now $\sqrt{-1}\varepsilon\chi'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi(v)\geqslant 0$. Moreover, since $\lambda_1^\varepsilon(x)<0$ we have

$$\min_{v\in T_xD}\frac{\sqrt{-1}\partial\overline{\partial}\varphi(v)}{R^L(v)}<0\,,\quad \min_{v\in T_xD}\frac{\sqrt{-1}\partial\overline{\partial}\varphi(v)}{R^L(v)} = -\max_{v\in T_xD}\frac{-\sqrt{-1}\partial\overline{\partial}\varphi(v)}{R^L(v)}\,.$$

Hence

$$\lambda_1^\varepsilon \geqslant 1-\varepsilon\chi(\varphi)M_L(-\varphi)\quad \text{on } S(\varepsilon)\,. \tag{6.39}$$

The inequality (6.39) gives information about the size of $S(\varepsilon)$. Indeed, $\lambda_1^\varepsilon<0$ and (6.39) entail $\varphi > -\sqrt{\varepsilon M_L(-\varphi)}$. Thus the integration set is contained in a 'corona' of size $\sqrt{\varepsilon}$:

$$S(\varepsilon)\subset D(\varepsilon/2)\bigcap\Big\{x\in D:\,\varphi(x)>-\sqrt{\varepsilon M_L(-\varphi)}\Big\}\,. \tag{6.40}$$

Since $\varepsilon\chi(\varphi)<2M'_\omega(L)$ on $D(\varepsilon/2)$ (see (6.28)) we deduce the final estimate for the first eigenvalue:

$$|\lambda_1^\varepsilon|\leqslant 2M_L(-\varphi)M'_\omega(L)-1=:C_2\quad\text{on } S(\varepsilon)\,. \tag{6.41}$$

We examine now the eigenvalues λ_j^ε for $j=2,\dots,n-1$. The min–max principle yields:

$$\lambda_j^\varepsilon\leqslant 1+\varepsilon\chi(\varphi)M_L(\varphi)+\min_{\substack{F\subset T_xD\\ \dim F=j}}\max_{v\in F}\frac{\sqrt{-1}\varepsilon\chi'(\varphi)\partial\varphi\wedge\overline{\partial}\varphi(v)}{R^L(v)}\,.$$

The minimum in the last expression is 0 and is attained on some space $F\subset\ker\partial\varphi$. Therefore we get:

$$|\lambda_j^\varepsilon|\leqslant 1+2M'_\omega(L)M_L(\varphi)=:C_3\quad\text{on } S(\varepsilon)\text{ for } j=2,\dots,n-1\,. \tag{6.42}$$

The highest eigenvalue satisfies the estimate:

$$\lambda_n^\varepsilon\leqslant 1+\varepsilon\chi(\varphi)M_L(\varphi)+\varepsilon\chi'(\varphi)\max_{v\in T_xD}\frac{\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi(v)}{R^L(v)}\,.$$

The inequalities: $\varepsilon\chi(\varphi)<2M'_\omega(L)$ and $\varepsilon\chi'(\varphi)\leqslant(2M'_\omega(L))^{3/2}\varepsilon^{-1/2}$ hold on $D(\varepsilon/2)$ (the last one since $\chi'(\varphi)=-\varphi^{-3}$). We introduce the short notation:

$$M_L^\varepsilon(\partial\varphi)=M_{\sqrt{-1}R^L}(\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi,K_\varepsilon)\,,$$

where $K_\varepsilon:=\overline{D}\smallsetminus\Big\{x\in D:\,\varphi(x)>-\sqrt{\varepsilon M_L(-\varphi)}\Big\}$. It is clear that $M_L^\varepsilon(\partial\varphi)$ converges to $M_L(\partial\varphi)$ for $\varepsilon\longrightarrow 0$. With this notation,

$$|\lambda_n^\varepsilon|\leqslant 1+2M'_\omega(L)M_L(\varphi)+\varepsilon^{-1/2}(2M'_\omega(L))^{3/2}M_L^\varepsilon(\partial\varphi)\quad\text{on } S(\varepsilon)\,. \tag{6.43}$$

At this point we may return to $|I_\varepsilon|$ and use the obvious inequality

$$|I_\varepsilon|\leqslant(2\pi)^{-n}\,\mathrm{Vol}_{R^L}(S(\varepsilon))\sup_{S(\varepsilon)}|\lambda_1^\varepsilon|\,|\lambda_2^\varepsilon|\cdots|\lambda_n^\varepsilon|$$

where Vol_{R^L} represents the volume with respect to the metric $\sqrt{-1}R^L$. We need to find a bound only for the volume. Taking into account (6.40),

$$\mathrm{Vol}_{R^L)}(S(\varepsilon)) \leqslant \sqrt{\varepsilon}\left(\sqrt{M_L(-\varphi)} - \sqrt{(2M'_\omega(L))^{-1}}\right) \times$$
$$\times \sup\left\{\int_{\{\varphi=c\}} \frac{dS_L}{|d\varphi|_L} : c \in \left[-\sqrt{\varepsilon M_L(-\varphi)}, -\sqrt{\varepsilon(2M'_\omega(L))^{-1}}\right]\right\} \quad (6.44)$$

Relations (6.43) and (6.44) yield:

$$\limsup_{\varepsilon\longrightarrow 0} \mathrm{Vol}_{R^L}(S(\varepsilon)) \sup_{S(\varepsilon)} |\lambda_n^\varepsilon|$$
$$\leqslant \left(\sqrt{2M'_\omega(L)M_L(-\varphi)} - 1\right) 2M'_\omega(L) M_L(\partial\varphi) \int_{\partial D} \frac{dS_L}{|d\varphi|_L}$$
$$= C_1 C_4 \int_{\partial D} \frac{dS_L}{|d\varphi|_L}$$

Using (6.41) and (6.42) we conclude

$$\limsup_{\varepsilon\longrightarrow 0} |I_\varepsilon| \leqslant (2\pi)^{-n} C_1 C_2 C_3^{n-2} C_4 \int_{\partial D} \frac{dS_L}{|d\varphi|_L} \quad (6.45)$$

We are ready to let $\varepsilon \longrightarrow 0$ in (6.37) and we use (6.38) and (6.45). In (6.38) we can further let the compact K exhaust D. This proves (6.17) and with it the Existence Criterion. □

6.7.1. Perturbation of line bundles. As application of the existence theorem we prove a stability property for certain q–concave manifolds. Let us consider the complement X of a sufficiently small neighbourhood of a submanifold of codimension $\geqslant 3$ in a projective manifold. Assume that we perform a small perturbation of the complex stucture of X such that along a (not necessaraly compact) smooth divisor the structure remains unchanged. Then the resulting manifold still has a maximal number of meromorphic functions. If moreover the canonical bundle is positive, any small perturbation suffices for the result to hold.

Let M be a compact complex manifold and $A \subset M$ of dimension q. Then $M \smallsetminus A$ is $(q+1)$–concave. It is well known (see [**1**]) that for a q–concave manifold X ($q \leqslant n-1$) the transcedence degree $\deg\mathrm{tr}\,\mathscr{K}(X)$ of the meromorphic function field is at most the complex dimension of X. In analogy to the corresponding notion for compact manifolds we say that a q–concave manifold is *Moishezon* if $\deg\mathrm{tr}\,\mathscr{K}(X) = \dim_{\mathbb{C}} X$.

Let us consider now a projective manifold M, a submanifold $A \subset M$ and the concave manifold $X := M \smallsetminus A$. Our aim is to study to what extent small deformations of the sublevel sets X_c for small values of $c > \inf\varphi$ (i.e. for X_c close to X) give rise to concave Moishezon manifolds. As a matter of fact we may consider small neighbourhoods V of A, which means that $X_c \subset M \smallsetminus V$ for small $c > \inf\varphi$. Then $M \smallsetminus V$ is pseudoconcave in the sense of Andreotti and the notion of Moishezon manifold still makes sense (see [**1**]).

6.64. STABILITY THEOREM. *Let M be a compact projective manifold and let Z be an ample smooth divisor. Let $A \subset M$ be a complex submanifold of codimension at least* 3*. Then for any sufficiently small neighbourhood V of A and for any sufficiently small deformation of the complex structure of $M \smallsetminus V$ leaving $T(Z)$ invariant, the manifold $M \smallsetminus V$ with the new structure is a pseudoconcave Moishezon manifold. If the canonical bundle K_M is positive, the statement holds for any small enough perturbation.*

An immediate consequence is the following.

6.65. COROLLARY. *Let M be a compact projective manifold and let let Z be an ample smooth divisor. Let $A \subset M$ be a complex submanifold of codimension at least* 3. *Then for any sufficiently small neighbourhood V of A and for any deformation of the complex structure of M which is sufficiently small on $M \setminus V$ and leaves $T(Z)$ invariant, the manifold M with the new structure is Moishezon.*

In this section we discuss the relation between the perturbation of the complex structure of a line bundle and the perturbation of the complex structure on the base manifold. This requires a glance to the corresponding section of Lempert's article [**44**]. Let us consider a compact complex manifold $Y = (Y, \mathscr{I})$ with boundary endowed with a complex structure $\mathscr{I}$. Let Z be a smooth divisor in Y. Denote as usual by $[Z]$ the associated line bundle. We are interested in the effect of a small perturbation of $\mathscr{I}$ on Y on the complex structure of $[Z]$ or of the canonical bundle K_Y over a compact set $D \Subset Y$. This will suffice for the proof of the Stability Theorem. Indeed, denote by L a positive line bundle on a concave manifold Y and assume that for a small perturbation $\mathscr{I}'$ of $\mathscr{I}$ there exists a perturbation L' of L such that the curvature forms of L and L' are close on a sublevel set D. Then the right hand–side terms in (6.17) calculated for $\mathscr{I}$ and $\mathscr{I}'$ are also close. If one is positive so is the other and both manifolds D and D' (and therefore Y and Y') are Moishezon.

Let us remark that not every perturbation of the complex structure on Y lifts to a perturbation of $[Z]$. We need the hypothesis that the tangent space $T(Z)$ is $\mathscr{I}'$ invariant. Then Z is a divisor in the new manifold $Y' = (Y, \mathscr{I}')$ and we consider the associated bundle $[Z]'$. Of course any perturbation of $\mathscr{I}$ lifts to a perturbation of the canonical line bundle.

The next Lemma is a "small perturbation" of Lemma 4.1 of Lempert [**44**]. In the latter a compact divisor $Z \subset \operatorname{Int} Y$ is considered whereas in our case we deal with a divisor which may cut the boundary. However, since we are interested in the effect of the perturbation just on a compact set the proof is the same. We use the $\mathscr{C}^\infty$ topology on the spaces of tensors defined on Y and also on spaces of restrictions of tensors to compact subsets of Y. We say that two tensors are close when they are close in the $\mathscr{C}^\infty$ topology.

6.66. LEMMA. *Let $(Y, \mathscr{I})$ be a compact complex manifold, Z a smooth divisor in Y and $D \Subset Y$. There exists a finite covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of D and a multiplicative cocycle $\{g_{\alpha\beta} \in \mathscr{O}_{\mathscr{I}}(\overline{U}_\alpha \cap \overline{U}_\beta) \,:\, \alpha, \beta \in A\}$ defining the bundle $L = [Z]$ in the vicinity of D, with the following property. If $\mathscr{I}'$ is another complex structure on Y close to $\mathscr{I}$ such that $T(Z)$ rests $\mathscr{I}'$ invariant, the bundle L' determined by Z in the structure $\mathscr{I}'$ can be defined in the vicinity of D by the cocycle $\{g'_{\alpha\beta} \in \mathscr{O}_{\mathscr{I}'}(\overline{U}_\alpha \cap \overline{U}_\beta) \,:\, \alpha, \beta \in A\}$ such that $g'_{\alpha\beta}$ will be as close as we please to $g_{\alpha\beta}$ on $\overline{U}_\alpha \cap \overline{U}_\beta$ assuming $\mathscr{I}'$ and $\mathscr{I}$ are sufficiently close.*

PROOF. We remind for the sake of completeness the construction of the cocycles. For every point of $Y \cap \overline{D}$ there exists an open neighbourhood U in Y and a $\mathscr{I}$–biholomorphism ψ_U of some neighbourhood of $\overline{U}$ into $\mathbb{C}^n$, $n = \dim Y$, such that $\psi_U(U)$ is the unit polydisc and $\psi_U(Z) \subset \{z \in \mathbb{C}^n \,:\, z_1 = 0\}$. Let $\{U_\alpha\}_{1 \leqslant \alpha \leqslant m}$ be a finite covering consisting of sets U as above and for each α denote by ψ_α the corresponding biholomorphism. We select further an open set $U_0 \Subset Y \setminus Z$ such that $\mathscr{U} = \{U_\alpha\}_{0 \leqslant \alpha \leqslant m}$ is a covering of $\overline{D}$. For every $1 \leqslant \alpha \leqslant m$ we select a smooth strictly pseudoconvex Stein domain $U^*_\alpha \supset U_\alpha$ such that ψ_α is biholomorphic in the neighbourhood of U^*_α. Set moreover $U^*_0 = U_0$. We construct a cocycle defining $L = [Z]$ in the open set $\cup_\alpha U^*_\alpha$ as follows. First define functions g_α such that g_0 is identically 1 on U_0 and $g_\alpha = z_1 \circ \psi_\alpha$ for $\alpha \geqslant 1$. The bundle L is defined in the vicinity of D by the $\mathscr{I}$ holomorphic multiplicative cocycle $\{g_{\alpha\beta}\}$ where $g_{\alpha\beta} = g_\alpha / g_\beta$. Note that $g_{\alpha\beta}$ is holomorphic on a neighbourhood of $\overline{U}^*_\alpha \cap \overline{U}^*_\beta \supset \overline{U}_\alpha \cap \overline{U}_\alpha$.

Let $\mathscr{I}'$ be a complex structure as in the statement. Then Z is a complex hypersurface in the new structure and defines a line bundle L'. We describe next the cocycle of L'. The hypothesis on the sets U^*_α allows the use of a theorem of Hamilton [**36**] for U^*_α. The theorem asserts that for a small perturbation $\mathscr{I}'$ of the complex structure on a neighbourhood of $\overline{U}^*_\alpha$ there is a $\mathscr{I}'$ biholomorphism ψ'_α of a neighbourhood of $\overline{U}^*_\alpha$ into $\mathbb{C}^n$ close to ψ_α. As shown in [**44**] we can even assume $\psi'_\alpha(Z) \subset \{z \in \mathbb{C}^n : z_1 = 0\}$. Set g'_0 to be identically 1 on U_0 and $g'_\alpha = z_1 \circ \psi'_\alpha$ for $\alpha \geqslant 1$. Then put $g'_{\alpha\beta} = g'_\alpha / g'_\beta$. Since ψ_α and ψ'_α are close, g'_α is $\mathscr{I}'$ holomorphic on a neighbourhood of $\overline{U}^*_\alpha$ and $g'_{\alpha\beta}$ is $\mathscr{I}'$ holomorphic on a neighbourhood of $\overline{U}^*_\alpha \cap \overline{U}^*_\beta$. The cocycle $\{g'_{\alpha\beta}\}$ defines L' in the open set $\cup_\alpha U^*_\alpha$.

The functions g_α and g'_α are close on $\overline{U}_\alpha$. We can now repeat the arguments from [**44**] to show that $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are also close on $\overline{U}_\alpha \cap \overline{U}_\beta$. □

6.67. LEMMA. *Let $(Y, \mathscr{I})$, Z and $D \Subset Y$ be as in the preceding Lemma. Assume that $[Z]$ is endowed with a hermitian metric h. If $\mathscr{I}'$ is another complex structure on Y close to $\mathscr{I}$, leaving $T(Z)$ invariant, there exists a hermitian metric h' on the line bundle $[Z]'$ near D such that the curvature form $R^{[Z]'}$ will be as close as we please to $R^{[Z]}$ on D assuming $\mathscr{I}'$ and $\mathscr{I}$ are sufficiently close.*

PROOF. We can define a smooth bundle isomorphism $[Z] \longrightarrow [Z]'$ in the vicinity of D by resolving the smooth additive cocycle $\log(g'_{\alpha\beta}/g_{\alpha\beta})$ in order to find smooth functions f_α, close to 1 on a neighbourhood of $\overline{U}_\alpha$ such that $g'_{\alpha\beta} = f_\alpha g_{\alpha\beta} f_\beta^{-1}$. Then the isomorphism between $[Z]$ and $[Z]'$ is defined by $f = \{f_\alpha\}$. The metric h is given in terms of the covering $\mathscr{U}$ by a collection $h = \{h_\alpha\}$ of smooth strictly positive functions satisfying the relation $h_\beta = h_\alpha |g_{\alpha\beta}|$. We define a hermitian metric $h' = \{h'_\alpha\}$ on $[Z]'$ by $h'_\alpha = h_\alpha |f_\alpha^{-1}|$; h'_α is close to h_α on D. The curvature form of $[Z]'$ has the form

$$\frac{\sqrt{-1}}{2\pi} R^{[Z]'} = \frac{1}{4\pi} d \circ \mathscr{I}' \circ d\, (\log h'_\alpha)\,.$$

Therefore, when $\mathscr{I}'$ is sufficiently close to $\mathscr{I}$, $\frac{\sqrt{-1}}{2\pi} R^{[Z]'}$ is close to $\frac{\sqrt{-1}}{2\pi} R^{[Z]}$ on D. □

In the same vein we study the perturbation of the canonical bundle.

6.68. LEMMA. *Let $(Y, \mathscr{I})$ and $D \Subset Y$ be as above. Assume K_Y is endowed with a hermitian metric h. If $\mathscr{I}'$ is another complex structure on Y close to $\mathscr{I}$, there exists a hermitian metric h' on $K_{Y'}$ near D such that the curvature form $R^{K_{Y'}}$ will be as close as we please to R^{K_Y} on D assuming $\mathscr{I}'$ and $\mathscr{I}$ are sufficiently close.*

PROOF. We find as before a finite covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of $\overline{D}$ and biholomorphisms ψ_α defined in a neighbourhood of $\overline{U}_\alpha$ which map U_α onto the unit polydisc in $\mathbb{C}^n$. For every $\alpha \in A$ we select a smooth strictly pseudoconvex Stein domain $U^*_\alpha \supset U_\alpha$ such that ψ_α is biholomorphic in the neighbourhood of U^*_α. The canonical bundle K_Y is defined in the vicinity of D by $g_{\alpha\beta} = \det\left(\partial\psi_\alpha/\partial\psi_\beta\right) = \det\left(\partial\left(\psi_\alpha \circ \psi_\beta^{-1}\right)/\partial w\right)$ which is $\mathscr{I}$–holomorphic on a neighbourhood of $\overline{U}^*_\alpha \cap \overline{U}^*_\beta \supset \overline{U}_\alpha \cap \overline{U}_\alpha$. Here w are the canonical coordinates on $\mathbb{C}^n$. We apply as before Hamilton's theorem and obtain $\mathscr{I}'$ biholomorphisms ψ'_α in a neighbourhood of $\overline{U}^*_\alpha$ into $\mathbb{C}^n$ close to ψ_α.

The cononical bundle $K_{Y'}$ is defined in the vicinity of D by $g'_{\alpha\beta} = \det\left(\partial\psi'_\alpha/\partial\psi'_\beta\right)$. Since ψ'_α is close to ψ_α we see that $g'_{\alpha\beta}$ is close to $g_{\alpha\beta}$ on $\overline{U}_\alpha \cap \overline{U}_\alpha$. By repeating the arguments in the proof of Lemma 6.67 we conclude. □

6.7.2. The Stability Theorem. In this section we prove the Stability Theorem. Let us consider a compact manifold M, $\dim M = n$, and a complex submanifold A of dimension q. Then $X = M \smallsetminus A$ is $(q+1)$–concave. Let us remind the construction of an exhaustion function. Select a finite covering $\hat{\mathscr{U}} = \{U_\alpha\}_{\alpha \geqslant 1}$ of A with coordinate domains such that if the coordinates in U_α are $z_\alpha = (z_\alpha^1, z_\alpha^2, \cdots, z_\alpha^n)$ we have $A \cap U_\alpha = \{z \in U_\alpha : z_\alpha^{q+1} = \cdots = z_\alpha^n = 0\}$. Set $\varphi_\alpha(z) = \sum_{q+1}^n |z_\alpha^j|^2$. Choose a relatively compact open set $U_0 \Subset M \smallsetminus A$ such that $\mathscr{U} = \{U_0\} \cup \hat{\mathscr{U}} = \{U_\alpha\}_{\alpha \geqslant 0}$ is a covering of M and set $\varphi_0 \equiv 1$ on U_0. Let $\{\rho_\alpha\}_{\alpha \geqslant 0}$ be a partition of unity subordinated to $\mathscr{U}$. Define $\varphi = \varphi_A = \sum_{\alpha \geqslant 0} \rho_\alpha \varphi_\alpha$. The function φ enjoys the following properties:

(1) $\varphi \in \mathscr{C}^\infty(M), A = \{\varphi = 0\}$ and $\varphi \geqslant 0$.
(2) For any $c > 0$ we have $\{\varphi > c\} \Subset M \smallsetminus A$.
(3) $\partial\overline{\partial}\varphi = \sum_\alpha \left(\rho_\alpha \partial\overline{\partial}\varphi_\alpha + \varphi_\alpha \partial\overline{\partial}\rho_\alpha + \partial\rho_\alpha \wedge \overline{\partial}\varphi_\alpha + \partial\varphi_\alpha \wedge \overline{\partial}\rho_\alpha\right)$ where $\partial\overline{\partial}\varphi_\alpha = 2\sum_{q+1}^n dz_\alpha^j \wedge d\overline{z}_\alpha^j$.

For $z \in A$, $\partial\overline{\partial}\varphi(z) = \sum_\alpha \rho_\alpha(z)\partial\overline{\partial}\varphi_\alpha(z)$ has $n-q$ positive eigenvalues. Hence $\partial\overline{\partial}\varphi$ has $n-q$ positive eigenvalues in a neighbourhood of A. Moreover $\partial\overline{\partial}\varphi$ is positive semidefinite on A. Let us construct a hermitian metric on M which is "small" in the normal direction to A (near A) and "large" in the tangential direction to A. We can consider on each U_α the metric $\delta^{-1}\sum_1^q dz_\alpha^j \wedge d\overline{z}_\alpha^j + \delta \sum_{q+1}^n dz_\alpha^j \wedge d\overline{z}_\alpha^j$, $(\delta > 0)$, and then patch these metrics together with the partition of unity to obtain a metric ω_δ on M. Let $\gamma_1^\delta \leqslant \gamma_2^\delta \leqslant \cdots \leqslant \gamma_n^\delta$ be the eigenvalues of $\sqrt{-1}\partial\overline{\partial}\varphi$ with respect to ω_δ. For δ sufficiently small there exists a neighbourhood U_δ of A such that on U_δ, $\gamma_j^\delta \geqslant -O(\delta)$ for $j = 1, \ldots, q$ and $\gamma_j^\delta \geqslant O(\delta^{-1})$ for $j = q+1, \ldots, n$. Therefore we can choose δ such that on U_δ, $\gamma_j^\delta \geqslant -1$ for $j = 1, \ldots, q$ and $\gamma_j^\delta \geqslant 2n-3$ for $j = q+1, \ldots, n$.

Let us consider now the domains $X_c = \{\varphi > c\}$ for $c > 0$ sufficiently small. If $\operatorname{codim} A \geqslant 3$ the domains X_c admit as definition function $c - \varphi$ whose complex hessian has 3 negative eigenvalues in the vicinity of ∂X_c. If M possesses a positive line bundle we are in the conditions of the Existence Criterion. Note that the metric ω_δ satisfies Property 6.62 for all X_c with c sufficiently small. For technical reasons we construct a metric ω as follows. Consider the real part g_δ of the hermitian metric ω_δ. Thus g_δ is a riemannian metric on M. Take a hermitian metric ω whose real part g satisfies $g(u,v) = g_\delta(u,v) + g_\delta(\mathscr{I}u, \mathscr{I}v)$ $(u, v \in \mathbb{C} \otimes T(M))$ where $\mathscr{I}$ is the complex structure of M. If δ is sufficiently small ω still satisfies Property 6.62. From now on we fix such a metric ω on M. The constants $M'_\omega(L)$ are calculated with respect to this metric.

6.69. LEMMA. *Assume that M is a projective manifold and L is a positive line bundle over M. Let A be a submanifold with* $\operatorname{codim} A \geqslant 3$. *Then for sufficiently small regular values $c > 0$ we have*

$$\int_{X_c} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n > C(c-\varphi, L) \int_{\partial X_c} \frac{dS_L}{|d\varphi|_L} \tag{6.46}$$

where $C(c-\varphi, L)$ has been introduced in (6.18).

PROOF. Remark first that the constant $C(c-\varphi, L)$ converges to 0 for $c \longrightarrow 0$. Indeed, $\partial\overline{\partial}(c-\varphi) = -\partial\overline{\partial}\varphi$ so the constants $M_L(c-\varphi)$, $M_L(\varphi - c)$ and $M'_\omega(L)$ are bounded for c running in a compact interval since $\partial\overline{\partial}\varphi$ and L are defined over all M. We observe further that $d\varphi(z) \longrightarrow 0$ when $z \longrightarrow A$ (in fact $d\varphi \restriction_A = 0$). Hence $M_L(\partial(c-\varphi) \wedge \overline{\partial}(c-\varphi), \partial X_c)$

converges to 0 (and with it $C(c-\varphi)$) when c goes to 0. Examine now the term

$$\int_{\partial X_c} \frac{dS_L}{|d\varphi|_L}\,.$$

Although $|d\varphi_L| \longrightarrow 0$ for $z \longrightarrow A$ this integral goes to 0 too for $c \longrightarrow 0$. Indeed, since A has codimension $\geqslant 3$ we have

$$\int_{\partial X_c} dS_L = \int_{\{\varphi=c\}} dS_L = O(c^5)\,, \quad c \longrightarrow 0\,.$$

On the other hand for a regular value c of φ,

$$\left|d\varphi\restriction_{\partial X_c}\right| = O(c)\,, \quad c \longrightarrow 0\,.$$

We infer

$$\int_{\partial X_c} \frac{dS_L}{|d\varphi|_L} = O(c^4)\,, \quad c \longrightarrow 0\,.$$

for regular values c of φ. In conclusion the boundary integral in (6.46) goes to 0 as $c \longrightarrow 0$. The domain integral in (6.46) being bounded from below by a positive constant the Lemma follows. □

At this stage we can prove the Stability Theorem. Let us consider a smooth domain $Y := X_c$ for c small enough such that condition (6.46) holds. Let $\mathscr{I}'$ be a new complex structure on Y which leaves $T(Z)$ invariant, for an ample smooth divisor Z on M. We apply Lemma 6.67 for the manifold Y and a smooth relatively compact set $\overline{D}$ where $D := X_d$, $d > c$, such that (6.46) still holds on X_d. By hypothesis the bundle L carries a hermitian metric with positive curvature. Lemma 6.67 shows that there exists a hermitian metric h' on the bundle L' near D such that R^L and $R^{L'}$ are as close as we please in the $\mathscr{C}^\infty$ topology on $\overline{D}$ if $\mathscr{I}$ and $\mathscr{I}'$ are sufficiently close. In particular $R^{L'}$ is positive near $\overline{D}$. Note that a defining function for D' is still $d-\varphi$ and its complex hessian will have 3 negative eigenvalues in the vicinity of $\partial D'$ for a small perturbation of the complex structure.

Thus we can apply the Existence Criterion for D' and L'. In order to calculate the constant $C(d-\varphi, L')$ we construct first a metric ω' on Y in the following way. The metric ω determines a riemannian metric g on Y which was chosen such that $g(u,v) = g_\delta(u,v) + g_\delta(\mathscr{I}u, \mathscr{I}v)$ for $u,v \in \mathbb{C}\otimes T(M)$. We consider then a hermitian metric ω' on Y' with real part g' where $g'(u,v) = g_\delta(u,v) + g_\delta(\mathscr{I}'u, \mathscr{I}'v)$ for $u,v \in \mathbb{C}\otimes T(M)$. The metric ω' satisfies the Property 6.62 with respect to the defining function $d-\varphi$ of D', provided $\mathscr{I}$ and $\mathscr{I}'$ are sufficiently close. Therefore the constants $M_{L'}(d-\varphi)$, $M_{L'}(\varphi-d)$, $M_{\omega'}(L')$ and $M_{L'}(\partial(d-\varphi), \partial D')$ are close to the corresponding constants $M_L(d-\varphi)$, $M_L(\varphi-d)$, $M_\omega(L)$ and $M_L(\partial(d-\varphi), \partial D)$ respectively. This entails that $C(d-\varphi, L')$ is close to $C(d-\varphi, L)$.

It is also clear that $\int_{D'}\left(\frac{\sqrt{-1}}{2\pi}R^{L'}\right)^n$ and $\int_{\partial D'} dS_{L'}/|d\varphi|_{L'}$, are close to the corresponding integrals on D and ∂D of $\frac{\sqrt{-1}}{2\pi}\sqrt{-1}R^L$ and $dS_{L'}/|d\varphi|_{L'}$. Therefore

$$\int_{D'}\left(\frac{\sqrt{-1}}{2\pi}R^{L'}\right)^n > C(d-\varphi, L')\int_{\partial D'} \frac{dS_{L'}}{|d\varphi|_{L'}} \tag{6.46$'$}$$

By the Existence Criterion

$$\dim H^0(D', L'^k) \gtrsim k^n \tag{6.47}$$

for large k and thus D' and so Y' are Moishezon, provided $\mathscr{I}$ and $\mathscr{I}'$ are sufficiently close. An entirely analogous argument takes care of the case of perturbation of the canonical bundle K_Y. This proves the Stability Theorem.

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CHAPTER 7

Generalized Bergman kernels on symplectic manifolds

In this Chapter we wish to prove the asymptotic expansion of the generalized Bergman kernels as stated in Theorems 1.21 and 1.22. The method is to first use the spectral gap of the renormalized Bochner-Laplace operator and the finite propagation speed of solutions of hyperbolic equations to localize the problem. Then we combine the Sobolev norm estimates and a formal power series trick, and in this way, we compute the coefficients of the expansion (cf. (7.78), (7.68)).

The asymptotic expansion of the Bergman kernel of the spin^c Dirac operator was obtained in [**18**]. We will adapt the method used there to our situation. One of the difficulties of the analysis of the renormalized Bochner–Laplacian is that there are small eigenvalues (cf. Corollary 7.2). In the case of the spin^c Dirac operator the only small eigenvalue of the operator is zero when $k \longrightarrow \infty$, which permits to obtain the *full off-diagonal* asymptotic expansion (cf. [**18**, Theorem 3.18]). In the current situation, we have small eigenvalues and we are interested to prove the *near diagonal* expansion of the generalized Bergman kernels. This result is enough for most of applications.

Let us provide a short road-map of the chapter. The first section is devoted to the proof of the existence of the spectral gap in Corollary 7.2. Then we shall sketch the ideas of the proof of Theorems 1.21 and 1.22 in Section 7.2. The full details are available in the recent preprint [**?, 24**]. In Section 7.3, we explain some applications of our results. Among others, we give a symplectic version of the convergence of the induced Fubini-Study metric [**33**], and we show how to handle the first-order pseudo-differential operator D_b of Boutet de Monvel and Guillemin [**13**], which was studied extensively by Shiffman and Zelditch [**28**], and the operator $\overline{\partial}+\overline{\partial}^*$ when X is Kähler but $\mathbf{J} \neq J$. We include also generalizations for non-compact or singular manifolds and as a consequence we obtain an unified treatment of the convergence of the induced Fubini–Study metric, the holomorphic Morse inequalities and the characterization of Moishezon spaces.

7.1. The spectral gap of the Dirac and Bochner-Lapace operators

Our first task is to define a generalization for the space of holomorphic sections from the case of complex manifolds. For this purpose we shall exhibit the spectral gap of the Bochner-Laplacian. The results of this section are taken from [**?**].

7.1.1. Statement of the results. Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the prequantization property:

$$\frac{\sqrt{-1}}{2\pi}R^L = \omega \tag{7.1}$$

where $R^L = (\nabla^L)^2$ is the curvature of (L,∇^L). In different applications it is also necessary to consider a Hermitian vector bundle (E,h^E) on X with Hermitian connection ∇^E and curvature R^E. Let g^{TX} be a Riemannian metric on X and $\mathbf{J}: TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u,v) = g^{TX}(\mathbf{J}u,v) \quad \text{for} \quad u,v \in TX. \tag{7.2}$$

Let J be an almost complex structure which is separately compatible with g^{TX} and ω, i.e. $g^{TX}(J\cdot,J\cdot) = g^{TX}(\cdot,\cdot)$, $\omega(J\cdot,J\cdot) = \omega(\cdot,\cdot)$ and $\omega(\cdot,J\cdot)$ defines a metric on TX (for example, $\mathbf{J}(-\mathbf{J}^2)^{-\frac{1}{2}}$ verifies these conditions; see [**25**, p.61]). Then J commutes with $\mathbf{J}$.

We introduce the Levi-Civita connection ∇^{TX} on (TX,g^{TX}) with its curvature R^{TX} and scalar curvature r^X. Let $\nabla^X J \in T^*X \otimes \mathrm{End}(TX)$ be the covariant derivative of J induced by ∇^{TX}. We introduce the induced Bochner-Laplacian acting on $\mathscr{C}^\infty(X,L^k\otimes E)$:

$$\Delta^{L^k\otimes E} = (\nabla^{L^k\otimes E})^*\nabla^{L^k\otimes E} = -\sum_i [(\nabla^{L^p\otimes E}_{e_i})^2 - \nabla^{L^k\otimes E}_{\nabla^{TX}_{e_i}e_i}] \tag{7.3}$$

where $\{e_i\}_i$ is an orthonormal frame of (TX,g^{TX}). The spectrum of $\Delta^{L^k\otimes E}$ drifts to the right at linear rate in k as $k\to\infty$. Thus we do not have any analog of the space of harmonic forms. That's why we renormalize in the following manner. We fix a smooth Hermitian section Φ of $\mathrm{End}(E)$ on X. Set

$$\tau(x) = -\pi\,\mathrm{Tr}_{|TX}[J\mathbf{J}] = \frac{\sqrt{-1}}{2}R^L(e_j,Je_j) > 0, \tag{7.4}$$

$$\mu_0 = \inf_{u\in T_xX,\,x\in X} \sqrt{-1}R^L_x(u,Ju)/|u|^2_{g^{TX}} > 0, \tag{7.5}$$

and define the *renormalized Bochner-Laplacian*:

$$\Delta_{k,\Phi} = \Delta^{L^k\otimes E} - k\tau + \Phi. \tag{7.6}$$

In order to study this operator we construct canonically a spin^c Dirac operator D_k acting on $\Omega^{0,\bullet}(X,L^k\otimes E) = \bigoplus_{q=0}^n \Omega^{0,q}(X,L^k\otimes E)$, the direct sum of spaces of $(0,q)$–forms with values in $L^k\otimes E$. We have the following vanishing theorem:

7.1. THEOREM. *There exists $C>0$ such that for $k\in\mathbb{N}$, the spectrum of D_k^2 is contained in the set $\{0\}\cup(2k\mu_0 - C,+\infty)$. Set $D_k^- = D_k\restriction_{\Omega^{0,odd}}$, then for k large enough, we have*

$$\ker D_k^- = \{0\}. \tag{7.7}$$

As a simple corollary, $\ker D_k^- = \{0\}$ for k large enough, where $D_k^- = D_k\restriction_{\Omega^{0,\mathrm{odd}}}$, which is the vanishing result of [**8**, Theorem 2.3], [**15**, Theorem 3.2]. If A is any operator, we denote by $\mathrm{Spec}(A)$ the spectrum of A.

7.2. COROLLARY. *There exists a constant $C_L>0$ (which can be estimated precisely by using the $\mathscr{C}^0$-norms of R^{TX}, R^E, R^L, $\nabla^X J$ and Φ) such that the spectrum of the Schrödinger operator $\Delta_{k,\Phi}$ is contained in the union $[-C_L,C_L]\cup[2k\mu_0 - C_L,+\infty)$:*

$$\mathrm{Spec}\,\Delta_{k,\Phi} \subset [-C_L,C_L]\cup[2k\mu_0 - C_L,+\infty[. \tag{7.8}$$

where C_L is a positive constant independent of k. For k large enough, the number d_k of eigenvalues on the interval $[-C_L,C_L]$ satisfies

$$d_k = \langle \mathrm{ch}(L^k\otimes E)\,\mathrm{Td}(TX),[X]\rangle. \tag{7.9}$$

In particular $d_k \sim k^n(\mathrm{rank}\,E)\,\mathrm{vol}_\omega(X)$.

In the case E is a trivial line bundle, Corollary 7.2 is the main result of Guillemin and Uribe [**21**, Theorem 2]. In [**21**] it was established that $d_k \sim k^n \operatorname{vol}_\omega(X)$. When $\mathbf{J} = J$, Borthwick and Uribe [**8**, p. 854] got the precise value d_k, for large enough k, in this case. The idea in [**8, 15, 16, 21**] is to first reduce the problem to a problem on the unitary circle bundle of L^*, then apply Melin's inequality [**22**, Theorem 22.3.2] to show that $\Delta_{k,\Phi}$ is semi–bounded from below. In order to prove [**21**, Theorem 2], they apply the analysis of Toeplitz structures of Boutet de Monvel-Guillemin [**13**]. We provide a simple proof based on a direct application of Lichnerowicz formula.

For the interesting applications of [**21**, Theorem 2], we refer the reader to Borthwick and Uribe [**8, 10, 11**]. For the related topic of geometric quantization see [**26, 34**].

This section is organized as follows. In Section 7.1.2, we recall the construction of the spin^c Dirac operator and prove our main technical result, Theorem 7.7. In Section 7.1.3, we prove Theorem 7.1 and Corollary 7.2. In Section 7.1.4, we generalize our result to the L^2 case. In particular, we obtain a new proof of [**16**, Theorem 2.6].

7.1.2. The Lichnerowicz formula. Let $TX^c = TX \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the tangent bundle. The almost complex structure J induces a splitting $TX^c = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $P^{0,1} = \frac{1}{2}(1-\sqrt{-1}J)$, $P^{0,1} = \frac{1}{2}(1+\sqrt{-1}J)$ be the natural projections from TX^c onto $T^{(1,0)}X$, respectively $T^{(0,1)}X$.

Accordingly, we have a decomposition of the complexified cotangent bundle: $T^*X^c = T^{(1,0)*}X \oplus T^{(0,1)*}X$. The exterior algebra bundle decomposes as $\Lambda T^*X^c = \oplus_{p,q}\Lambda^{p,q}$, where $\Lambda^{p,q} := \Lambda^{p,q}T^*X^c = \Lambda^p(T^{(1,0)*}X) \otimes \Lambda^q(T^{(0,1)*}X)$.

Let $\nabla^{1,0}$ and $\nabla^{0,1}$ be the canonical hermitian connections on $T^{(1,0)}X$ and $T^{(0,1)}X$ respectively:

$$\nabla^{1,0} = P^{1,0}\nabla^{TX}P^{1,0},$$
$$\nabla^{0,1} = P^{0,1}\nabla^{TX}P^{0,1}.$$

Set $A_2 = \nabla^{TX} - \left(\nabla^{1,0} \oplus \nabla^{0,1}\right) \in T^*X \otimes \operatorname{End}(TX)$ which satisfies $JA_2 = -A_2 J$.

Let us recall some basic facts about the spin^c Dirac operator on an almost complex manifold [**23**, Appendix D]. The fundamental $\mathbb{Z}_2$ spinor bundle induced by J is given by $\Lambda^{0,\bullet} = \Lambda^{\text{even}}(T^{(0,1)*}X) \oplus \Lambda^{\text{odd}}(T^{(0,1)*}X)$. For any $v \in TX$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0} \in T^{(0,1)*}X$ be the metric dual of $v_{1,0}$. Then $\mathbf{c}(v) = \sqrt{2}(\overline{v}^*_{1,0} \wedge -i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda^{0,\bullet}$, where $\wedge$ and i denote the exterior and interior product respectively.

Formally, we may think

$$\Lambda^{0,\bullet} = S(TX) \otimes \left(\det T^{(1,0)}X\right)^{1/2},$$

where $S(TX)$ is the spinor bundle of the possibly non–existent spin structure on TX, and $\left(\det T^{(1,0)}X\right)^{1/2}$ is the possibly non–existent square root of $\det T^{(1,0)}X$.

Moreover, by [**23**, pp. 397–398], ∇^{TX} induces canonically a Clifford connection on $\Lambda^{0,\bullet}$. Formally, let $\nabla^{S(TX)}$ be the Clifford connection on $S(TX)$ induced by ∇^{TX}, and let $\nabla^{\det}$ be the connection on $(\det T^{(1,0)}X)^{1/2}$ induced by $\nabla^{1,0}$. Then

$$\nabla^{\text{Cliff}} = \nabla^{S(TX)} \otimes \operatorname{Id} + \operatorname{Id} \otimes \nabla^{\det}.$$

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$. Then

$$e_{2j} = \tfrac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j-1} = \tfrac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j), \quad j = 1, \dots, n, \tag{7.10}$$

form an orthonormal frame of TX. Let $\{w^j\}_{j=1}^n$ be the dual frame of $\{w_j\}_{j=1}^n$. Let Γ be the connection form of $\nabla^{1,0} \oplus \nabla^{0,1}$ in local coordinates. Then $\nabla^{TX} = d + \Gamma + A_2$. By [**23**, Theorem 4.14, p.110], the Clifford connection ∇^{Cliff} on $\Lambda^{0,\bullet}$ has the following local form:

$$\begin{aligned}\nabla^{\text{Cliff}} &= d + \tfrac{1}{4}\sum_{i,j}\big\langle(\Gamma + A_2)e_i, e_j\big\rangle \mathbf{c}(e_i)\mathbf{c}(e_j)\\ &= d + \sum_{l,m}\Big\{\big\langle \Gamma w_l, \overline{w}_m\big\rangle \overline{w}^l \wedge i_{\overline{w}_m} + \\ &\qquad \tfrac{1}{2}\big\langle A_2 w_l, w_m\big\rangle i_{\overline{w}_l} i_{\overline{w}_m} + \tfrac{1}{2}\big\langle A_2 \overline{w}_l, \overline{w}_m\big\rangle \overline{w}^l \wedge \overline{w}^m \wedge \Big\}.\end{aligned} \tag{7.11}$$

Let $\nabla^{L^k\otimes E}$ be the connection on $L^k \otimes E$ induced by ∇^L, ∇^E. Let $\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}$ be the connection on $\Lambda^{0,\bullet} \otimes L^k \otimes E$,

$$\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} = \nabla^{\text{Cliff}} \otimes \text{Id} + \text{Id} \otimes \nabla^{L^k\otimes E}. \tag{7.12}$$

Along the fibers of $\Lambda^{0,\bullet} \otimes L^k \otimes E$, we consider the pointwise scalar product $\langle\cdot,\cdot\rangle$ induced by g^{TX}, h^L and h^E. Let dv_X be the riemannian volume form of (TX, g^{TX}). The L^2–scalar product on $\Omega^{0,\bullet}(X, L^k \otimes E)$, the space of smooth sections of $\Lambda^{0,\bullet} \otimes L^k \otimes E$, is given by

$$(s_1, s_2) = \int_X \langle s_1(x), s_2(x)\rangle\, dv_X(x). \tag{7.13}$$

We denote the corresponding norm with $\|\cdot\|$.

7.3. DEFINITION. The spin^c Dirac operator D_k is defined by

$$D_k = \sum_{j=1}^{2n} \mathbf{c}(e_j)\nabla_{e_j}^{\Lambda^{0,\bullet}\otimes L^k\otimes E} : \Omega^{0,\bullet}(X, L^k \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^k \otimes E). \tag{7.14}$$

D_k is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, L^k \otimes E)$, which interchanges $\Omega^{0,\text{even}}(X, L^k \otimes E)$ and $\Omega^{0,\text{odd}}(X, L^k \otimes E)$. We denote

$$D_k^+ = D_k \restriction_{\Omega^{0,\text{even}}}, \quad D_k^- = D_k \restriction_{\Omega^{0,\text{odd}}}. \tag{7.15}$$

Let $R^{T^{(1,0)}X}$ be the curvature of $\big(T^{(1,0)}X, \nabla^{1,0}\big)$. Let

$$\begin{aligned}\omega_d &= -\sum_{l,m} R^L(w_l, \overline{w}_m)\,\overline{w}^m \wedge i_{\overline{w}_l},\\ \tau(x) &= \sum_j R^L(w_j, \overline{w}_j).\end{aligned} \tag{7.16}$$

Remark that by (7.2), at $x \in X$, there exists $\{w_i\}_{i=1}^n$ an orthogonal basis of $T^{(1,0)}X$, such that $\mathbf{J} = \sqrt{-1}\,\text{diag}(a_1(x), \cdots, a_n(x)) \in \text{End}(T^{(1,0)}X)$, and $a_i(x) > 0$ for $i \in \{1, \cdots, n\}$. So

$$\begin{aligned}\omega_d &= -2\pi \sum_l a_l(x)\,\overline{w}^l \wedge i_{\overline{w}_l},\\ \tau(x) &= 2\pi \sum_l a_l(x).\end{aligned} \tag{7.17}$$

The following Lichnerowicz formula is crucial for us.

7.4. THEOREM. *The square of the Dirac operator satisfies the equation:*

$$D_k^2 = \left(\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}\right)^* \nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} - 2k\omega_d - k\tau + \tfrac{1}{4}K + \mathbf{c}(R), \tag{7.18}$$

where K is the scalar curvature of (TX, g^{TX}), and

$$\mathbf{c}(R) = \sum_{l<m}\left(R^E + \tfrac{1}{2}\operatorname{Tr}\left[R^{T^{(1,0)}X}\right]\right)(e_l, e_m)\,\mathbf{c}(e_l)\,\mathbf{c}(e_m).$$

PROOF. By Lichnerowicz formula [**2**, Theorem 3.52], we know that

$$D_k^2 = \left(\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E}\right)^* \nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} + \tfrac{1}{4}K + \mathbf{c}(R) + k\sum_{l<m} R^L(e_l, e_m)\,\mathbf{c}(e_l)\,\mathbf{c}(e_m). \tag{7.19}$$

Now, we identify R^L with a purely imaginary antisymmetric matrix $-2\pi\sqrt{-1}\mathbf{J} \in \operatorname{End}(TX)$ by (7.2). As $\mathbf{J} \in \operatorname{End}(T^{(1,0)}X)$, by [**2**, Lemma 3.29], we get (7.18). □

7.5. REMARK. Let $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ be a Clifford module. Then it was observed by Braverman [**15**, §9] that, with the same proof of [**2**, Proposition 3.35], there exists a vector bundle W on X such that $\mathscr{E} = \Lambda^{0,\bullet} \otimes W$ as a $\mathbb{Z}_2$–graded Clifford module.

As a simple consequence of Theorem 7.4, we recover the statement on the drift of spectrum of the metric Laplacian first proved by Guillemin–Uribe [**21**, Theorem 1], (see also [**8**, Theorem 2.1], [**15**, Theorem 4.4]), by passing to the circle bundle of L^* and applying Melin's inequality [**22**, Theorems 22.3.2–3].

7.6. COROLLARY. *There exists $C > 0$ such that for $k \in \mathbb{N}$, the Bochner–Laplacian $\Delta^{L^k\otimes E} = \left(\nabla^{L^k\otimes E}\right)^* \nabla^{L^k\otimes E}$ on $\mathscr{C}^\infty(X, L^k \otimes E)$ satisfies :*

$$\Delta^{L^k\otimes E} - k\tau \geqslant -C. \tag{7.20}$$

PROOF. By (7.18), for $s \in \mathscr{C}^\infty(X, L^k \otimes E)$,

$$\|D_k s\|^2 = \|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} s\|^2 - k(\tau(x)s, s) + \left(\left(\tfrac{1}{4}K + \mathbf{c}(R)\right)s, s\right). \tag{7.21}$$

From (7.11), we infer that

$$\left\|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} s\right\|^2 = \left\|\nabla^{L^k\otimes E} s\right\|^2 + \left\|\sum_{l,m}\langle A_2\overline{w}_l, \overline{w}_m\rangle\, \overline{w}^l \wedge \overline{w}^m \wedge s\right\|^2.$$

and therefore there exists a constant $C > 0$ not depending on k such that

$$0 \leqslant \|D_k s\|^2 \leqslant \left\|\nabla^{L^k\otimes E} s\right\|^2 - k(\tau(x)s, s) + C\|s\|^2 = \left((\Delta_k - k\tau(x))s, s\right) + C\|s\|^2.$$

□

7.7. THEOREM. *There exists a constant $C > 0$ such that for any $k \in \mathbb{N}$ and any section $s \in \Omega^{>0}(X, L^k \otimes E) = \bigoplus_{q\geqslant 1}\Omega^{0,q}(X, L^k \otimes E)$,*

$$\|D_k s\|^2 \geqslant (2k\mu_0 - C)\|s\|^2. \tag{7.22}$$

PROOF. By (7.18), for $s \in \Omega^{0,\bullet}(X, L^k \otimes E)$,

$$\|D_k s\|^2 = \{\|\nabla^{\Lambda^{0,\bullet}\otimes L^k\otimes E} s\|^2 - k(\tau(x)s, s)\} - 2k(\omega_d s, s) + \left(\left(\tfrac{1}{4}K + \mathbf{c}(R)\right)s, s\right). \tag{7.23}$$

We consider now $s \in \mathscr{C}^\infty(X, L^k \otimes E')$, where $E' = E \otimes \Lambda^{0,\bullet}$. Estimate (7.20) becomes

$$\left\|\nabla^{L^k\otimes E'} s\right\|^2 - k(\tau(x)s, s) \geqslant -C\|s\|^2. \tag{7.24}$$

If $s \in \Omega^{>0}(X, L^k \otimes E)$, the second term of (7.23), $-2k(\omega_d s, s)$ is bounded below by $2k\mu_0\|s\|^2$. While the third term of (7.23) is $O(\|s\|^2)$. The proof of (7.22) is completed. □

7.1.3. Applications of Theorem 7.7. We give now the proofs of the result stated in Section 7.1.1.

PROOF OF THEOREM 7.1. By (7.22), we get immediately (7.7). For the rest, we use the trick of the proof of Mckean–Singer formula.

Let $\mathscr{H}_\mu$ be the spectral space of D_k^2 corresponding to the interval $(0,\mu)$. Let $\mathscr{H}_\mu^+$, $\mathscr{H}_\mu^-$ be the intersections of $\mathscr{H}_\mu$ with the spaces of forms of even and odd degree respectively. Then $\mathscr{H}_\mu = \mathscr{H}_\mu^+ \oplus \mathscr{H}_\mu^-$. Since D_k^+ commutes with the spectral projection, we have a well defined operator $D_k^+ : \mathscr{H}_\mu^+ \longrightarrow \mathscr{H}_\mu^-$ which is obviously injective. But estimate (7.22) implies that $\mathscr{H}_\mu^- = 0$ for every $\mu < 2k\lambda - C$, hence also $\mathscr{H}_\mu^+ = 0$, for this range of μ. Thus $\mathscr{H}_\mu = 0$, for $0 < \mu < 2k\mu - C$. The proof of our theorem is completed. □

PROOF OF COROLLARY 7.2. Without loss of generality we may assume $\Phi = 0$. Let $P_k : \Omega^{0,\bullet}(X, L^k \otimes E) \longrightarrow \mathscr{C}^\infty(X, L^k \otimes E)$ be the orthogonal projection. For $s \in \Omega^{0,\bullet}(X, L^k \otimes E)$, we will denote $s_0 = P_k s$ its 0 – degree component. We will estimate $\Delta_{k,0}$ on $P_k(\ker D_k^+)$ and $(\ker D_k^+)^\perp \cap \mathscr{C}^\infty(X, L^k \otimes E)$.

In the sequel we denote with C all positive constants independent of k, although there may be different constants for different estimates. From (7.21), there exists $C > 0$ such that for $s \in \mathscr{C}^\infty(X, L^k \otimes E)$,

$$\big| \|D_k s\|^2 - (\Delta_{k,0} s, s) \big| \leqslant C \|s\|^2 . \tag{7.25}$$

Theorem 7.1 and (7.25) show that there exists $b > 0$ such that for $k \in \mathbb{N}$,

$$(\Delta_{k,0} s, s) \geqslant (2k\lambda - b)\|s\|^2 , \quad \text{for } s \in \mathscr{C}^\infty(X, L^k \otimes E) \cap (\ker D_k^+)^\perp . \tag{7.26}$$

We focus now on elements from $P_k(\ker D_k^+)$, and assume $s \in \ker D_k$. Set $s' = s - s_0 \in \Omega^{>0}(X, L^k \otimes E)$. By (7.23), (7.24),

$$-2k(\omega_d s, s) \leqslant C \|s\|^2 . \tag{7.27}$$

We obtain thus [**8**, Theorem 2.3] (see also [**9**], [**15**, Theorem 3.13]) for $k \gg 1$,

$$\|s'\| \leqslant C k^{-1/2} \|s_0\| . \tag{7.28}$$

(from (7.28), they got $\operatorname{Ker} D_k^- = 0$ for $k \gg 1$, as $s_0 = 0$ if $s \in \ker D_k^-$). In view of (7.23) and (7.28),

$$\|\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E} s\|^2 - k(\tau(x) s_0, s_0) \leqslant C \|s_0\|^2 . \tag{7.29}$$

By (7.11),

$$\nabla^{\Lambda^{0,\bullet} \otimes L^k \otimes E} s = \nabla^{L^k \otimes E} s_0 + A_2' s_2 + \alpha , \tag{7.30}$$

where s_2 is the component of degree 2 of s, A_2' is a contraction operator comming from the middle term of (7.11), and $\alpha \in \Omega^{>0}(X, L^k \otimes E)$. By (7.29), (7.30), we have

$$\big\| \nabla^{L^k \otimes E} s_0 + A_2' s_2 \big\|^2 - k(\tau(x) s_0, s_0) \leqslant C \|s_0\|^2 , \tag{7.31}$$

and by (7.28), (7.31),

$$\big\| \nabla^{L^k \otimes E} s_0 \big\|^2 \leqslant C k \|s_0\|^2 , \tag{7.32}$$

By (7.28) and (7.32), we get

$$\begin{aligned} \big\| \nabla^{L^k \otimes E} s_0 + A_2' s_2 \big\|^2 &\geqslant \big\| \nabla^{L^k \otimes E} s_0 \big\|^2 - 2 \big\| \nabla^{L^k \otimes E} s_0 \big\| \big\| A_2' s_2 \big\| \\ &\geqslant \big\| \nabla^{L^k \otimes E} s_0 \big\|^2 - C \|s_0\|^2 . \end{aligned} \tag{7.33}$$

Thus, (7.31) and (7.33) yield

$$\big\| \nabla^{L^k \otimes E} s_0 \big\|^2 - k(\tau(x) s_0, s_0) \leqslant C \|s_0\|^2 . \tag{7.34}$$

By (7.20) and (7.34), there exists a constant $a>0$ such that

$$\big|\big(\Delta_{k,0}s,s\big)\big|\leqslant a\|s\|^2\,,\quad s\in P_k(\ker D_k^+)\,. \tag{7.35}$$

By (7.28), we know that for $k\gg 1$, $P_k:\ker D_k^+\longrightarrow P_k(\ker D_k^+)$ is bijective, and

$$\mathscr{C}^\infty(X,L^k\otimes E)=P_k(\ker D_k^+)\oplus(\ker D_k^+)^\perp\cap\mathscr{C}^\infty(X,L^k\otimes E)\,. \tag{7.36}$$

The proof is now reduced to a direct application of the minimax principle for the operator $\Delta_{k,0}$. It is clear that (7.26) and (7.35) still hold for elements in the Sobolev space $W^1(X,L^k\otimes E)$, which is the domain of the quadratic form $Q_k(f)=\big\|\nabla^{L^k\otimes E}f\big\|^2-k(\tau(x)f,f)$ associated to $\Delta_{k,0}$. Let $\lambda_1^k\leqslant\lambda_2^k\leqslant\cdots\leqslant\lambda_j^k\leqslant\cdots$ ($j\in\mathbb{N}$) be the eigenvalues of $\Delta_{k,0}$. Then, by the minimax principle A.30,

$$\lambda_j^k=\min_{F\subset \operatorname{Dom}Q_k}\;\max_{f\in F\,,\,\|f\|=1}Q_k(f)\,. \tag{7.37}$$

where F runs over the subspaces of dimension j of $\operatorname{Dom}Q_k$.

By (7.35) and (7.37), we know $\lambda_j^k\leqslant a$, for $j\leqslant\dim\ker D_k^+$. Moreover, any subspace $F\subset\operatorname{Dom}Q_k$ with $\dim F\geqslant\dim\ker D_k^++1$ contains an element $0\neq f\in F\cap(\ker D_k^+)^\perp$. By (7.26), (7.37), we obtain $\lambda_j^k\geqslant 2k\mu_0-b$, for $j\geqslant\dim\ker D_k^++1$.

By Theorem 7.1 and Atiyah–Singer theorem [**1**],

$$\dim\ker D_k^+=\operatorname{index}D_k^+=\langle\operatorname{ch}(L^k\otimes E)\operatorname{Td}(TX),[X]\rangle \tag{7.38}$$

where $\operatorname{Td}(TX)$ is the Todd class of an almost complex structure compatible with ω. The index is a polynomial in k of degree n and of leading term $k^n(\operatorname{rank}E)\operatorname{vol}_\omega(X)$, where $\operatorname{vol}_\omega(X)$ is the symplectic volume of X. □

7.8. REMARK. If (X,ω) is Kähler and if L, E are holomorphic vector bundles, then $D_k=\sqrt{2}\big(\overline{\partial}+\overline{\partial}^*\big)$ where $\overline{\partial}=\overline{\partial}^{L^k\otimes E}$. D_k^2 preserves the $\mathbb{Z}$–grading of $\Omega^{0,\bullet}$. By using the Bochner–Kodaira–Nakano formula, Bismut and Vasserot [**6**, Theorem 1.1] proved Theorem 7.7. As $\overline{\partial}:(\ker D_k^+)^\perp\cap\mathscr{C}^\infty(X,L^k\otimes E)\longrightarrow\Omega^{0,1}(X,L^k\otimes E)$ is injective, we infer

$$2\big\|\overline{\partial}s\big\|^2\geqslant(2k\mu_0-C_L)\|s\|^2,\ \text{for } s\in(\ker D_k^+)^\perp\cap\mathscr{C}^\infty(X,L^k\otimes E). \tag{7.39}$$

By Lichnerowicz formula [**6**, (21)], $2\overline{\partial}^*\overline{\partial}=\Delta_{k,0}+\frac{1}{4}K+\mathbf{c}(R)$ on $\mathscr{C}^\infty(X,L^k\otimes E)$, and Corollary 7.2 follows immediately.

7.9. REMARK. As in [**7**], we assume that (L,h^L,∇^L) is a positive Hermitian vector bundle, i.e. the curvature R^L is an $\operatorname{End}(L)$–valued $(1,1)$–form, and for any $u\in T^{(1,0)}X\smallsetminus\{0\}$, $s\in L\smallsetminus\{0\}$, $\langle R^L(u,\overline{u})s,\overline{s}\rangle>0$. Let $S^k(L)$ be the k^{th} symmetric tensor power of L. Then if we replace L^k in Sections 7.1.2, 7.1.3 by $S^k(L)$, or by the irreducible representations of L, which are associated with the weight ka (where a is a given weight), when k tends to $+\infty$, the techniques used above still apply.

7.1.4. The spin^c Dirac operator on a covering manifold. We extend in this section our results to covering manifolds and refer to Chapter 5 for the necessary background about elliptic operators on covering manifolds and Γ–dimension.

Let $\widetilde{X}$ be a paracompact smooth manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ with a compact quotient $X=\widetilde{X}/\Gamma$. Let $\pi_\Gamma:\widetilde{X}\longrightarrow X$ be the projection. Assume that there exists a Γ–invariant pre–quantum line bundle $\widetilde{L}$ on $\widetilde{X}$ and a Γ–invariant connection $\nabla^{\widetilde{L}}$ such that $\widetilde{\omega}=\frac{\sqrt{-1}}{2\pi}(\nabla^{\widetilde{L}})^2$ is non–degenerate. We endow $\widetilde{X}$ with a Γ–invariant Riemannian metric $g^{T\widetilde{X}}$. Let $\widetilde{J}$ be an Γ-invariant almost complex structure on $T\widetilde{X}$ which is separately compatible with $\widetilde{\omega}$ and $g^{T\widetilde{X}}$. Then $\widetilde{J}$, $g^{T\widetilde{X}}$, $\widetilde{\omega}$, $\widetilde{J}$, $\widetilde{L}$, $\widetilde{E}$ are the pull-back of the corresponding objects in Section 7.1.2 by the projection $\pi_\Gamma : \widetilde{X} \to X$. Let Φ be a smooth Hermitian section of $\mathrm{End}(E)$, and $\widetilde{\Phi} = \Phi \circ \pi_\Gamma$. Then the renormalized Bochner-Laplacian $\widetilde{\Delta}_{k,\widetilde{\Phi}}$ is

$$\widetilde{\Delta}_{k,\widetilde{\Phi}} = \Delta^{\widetilde{L}^k\otimes\widetilde{E}} - k\,(\tau\circ\pi_\Gamma) + \widetilde{\Phi} \tag{7.40}$$

which is an essentially self–adjoint operator. Following Section 7.1.2, we introduce the Γ–invariant spin^c Dirac operator $\widetilde{D}_k$ on $\Omega^{0,\bullet}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$ and the Γ–invariant Laplacian $\Delta^{\widetilde{L}^k\otimes\widetilde{E}} = \big(\nabla^{\widetilde{L}^k\otimes\widetilde{E}}\big)^*\nabla^{\widetilde{L}^k\otimes\widetilde{E}}$ on $\mathscr{C}^\infty(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$. Let $\widetilde{D}_k^+$ and $\widetilde{D}_k^-$ be the restrictions of $\widetilde{D}_k$ to $\boldsymbol{L}_2^{0,\mathrm{even}}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$ and $\boldsymbol{L}_2^{0,\mathrm{odd}}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$, respectively.

7.10. PROPOSITION. *There exists a constant $C>0$ such that for $k\in\mathbb{N}$, $\widetilde{\Delta}_{k,\widetilde{\Phi}} \geqslant -C$ on $\boldsymbol{L}^2(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$.*

PROOF. By applying Lichnerowicz formula (7.18) for $s\in\mathscr{C}_0^\infty(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$, we obtain as in the proof of Corollary 7.6, that there exists $C>0$ such that $\big(\widetilde{\Delta}_{k,\widetilde{\Phi}}s,s\big)\geqslant -C\|s\|^2$. Since the metric $g^{T\widetilde{X}}$ is complete, this is valid for any $s\in\mathrm{Dom}(\widetilde{\Delta}_{k,\widetilde{\Phi}})$. □

In the same vein, we can generalize Theorem 7.7.

7.11. THEOREM. *There exists $C>0$ such that for $k\in\mathbb{N}$ and any $s\in\mathrm{Dom}(\widetilde{D}_k)$ with vanishing degree zero component,*

$$\|\widetilde{D}_k s\|^2 \geqslant (2k\lambda - C)\,\|s\|^2\,. \tag{7.41}$$

As an immediate application of the estimate (7.41) for the Dirac operator and Remark 7.5, we get the following asymptotic vanishing theorem which is the main result in [**16**, Theorem 2.6].

7.12. COROLLARY. $\ker\widetilde{D}_k^- = \{0\}$ *for large enough k.*

We have also an analogue of Theorem 7.1.

7.13. COROLLARY. *There exists $C>0$ such that for $k\in\mathbb{N}$, the spectrum of $\widetilde{D}_k^2$ is contained in the set $\{0\}\cup(2k\lambda - C,+\infty)$.*

PROOF. The proof of Theorem 7.1 does not use the fact that the spectrum is discrete. Therefore it applies in this context, too. □

We study now the spectrum of the Γ–invariant Bochner-Laplacian $\widetilde{\Delta}_{k,\widetilde{\Phi}}$.

7.14. COROLLARY. *The spectrum of $\widetilde{\Delta}_{k,\widetilde{\Phi}}$ is contained in the union $(-C_L,C_L)\cup(2k\mu_0 - C_L,+\infty)$, where C_L and μ_0 are the same positive constants as in Corollary 7.2. For large enough k, the Γ–dimension d_k of the spectral space $E\big([-C_L,C_L],\widetilde{\Delta}_{k,\widetilde{\Phi}}\big)$ corresponding to $(-C_L,C_L)$ satisfies $d_k = \langle \mathrm{ch}(L^k\otimes E)\,\mathrm{Td}(X),[X]\rangle$. In particular $d_k\sim k^n(\mathrm{rank}\,E)\,\mathrm{vol}_\omega(X)$.*

PROOF. By repeating the proof of Corollary 7.2, we get estimates (7.26) and (7.35) for smooth elements with compact support. Lemma 5.2 yields then

$$\big|\big(\widetilde{\Delta}_{k,\widetilde{\Phi}}s,s\big)\big| \leqslant a\|s_0\|^2\,,\quad s\in\mathrm{Dom}(\widetilde{\Delta}_{k,\widetilde{\Phi}})\cap P_k(\ker\widetilde{D}_k^+)\,, \tag{7.42a}$$

$$(\widetilde{\Delta}_{k,\widetilde{\Phi}}s,s)\geqslant(2k\lambda-b)\|s\|^2\,,\quad s\in\mathrm{Dom}(\widetilde{\Delta}_{k,\widetilde{\Phi}})\cap(\ker\widetilde{D}_k^+)^\perp\,. \tag{7.42b}$$

Recall that P_k represents the projection $\boldsymbol{L}_2^{0,\bullet}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})\longrightarrow\boldsymbol{L}_2^{0,0}(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$. Since the curvatures of all our bundles are Γ–invariant, estimate (7.28) extends to the covering context with the same proof. In particular, $P_k:\ker\widetilde{D}_k^+\longrightarrow P_k(\ker\widetilde{D}_k^+)$ is bijective, $P_k\restriction_{\ker\widetilde{D}_k^+}$ and its inverse are bounded. So $P_k(\ker\widetilde{D}_k^+)$ is closed. By Proposition 5.8, (4),

$$\dim_\Gamma\ker\widetilde{D}_k^+=\dim_\Gamma P_k(\ker\widetilde{D}_k^+)\,. \tag{7.43}$$

As in (7.36), we have

$$\mathrm{Dom}(\widetilde{\Delta}_{k,\widetilde{\Phi}})=P_k(\ker\widetilde{D}_k^+)\oplus(\ker\widetilde{D}_k^+)^\perp\cap\mathrm{Dom}(\widetilde{\Delta}_{k,\widetilde{\Phi}})\,. \tag{7.44}$$

We use now a suitable form of the minimax principle from [**29**, Lemma 2.4] (cf. also (5.9)):

$$N_\Gamma(\mu,\widetilde{\Delta}_{k,\widetilde{\Phi}})=\sup\{\dim_\Gamma V\,:\,V\subset\mathrm{Dom}\,\widetilde{\Delta}_{k,\widetilde{\Phi}}\,;\,\big(\widetilde{\Delta}_{k,\widetilde{\Phi}}f,f\big)\leqslant\mu\|f\|^2\,,\forall\,f\in V\} \tag{7.45}$$

where V runs over the Γ–modules of $L_2(\widetilde{X},\widetilde{L}^k\otimes\widetilde{E})$.
By (7.41), (7.42a) and (7.45), we get

$$N_\Gamma(C_L,\widetilde{\Delta}_{k,\widetilde{\Phi}})\geqslant\dim_\Gamma\ker\widetilde{D}_k^+\,. \tag{7.46}$$

Let us consider $\mu<2k\mu_0-C_L$. We prove that

$$N_\Gamma(\mu,\widetilde{\Delta}_{k,\widetilde{\Phi}})\leqslant\dim_\Gamma\ker\widetilde{D}_k^+\,. \tag{7.47}$$

Let $V\subset\mathrm{Dom}(\widetilde{\Delta}_{k,\widetilde{\Phi}})$ be an arbitrary Γ–module with $\big(\widetilde{\Delta}_{k,\widetilde{\Phi}}u,u\big)\leqslant\mu\|u\|^2$. If $\dim_\Gamma V>\dim_\Gamma\ker\widetilde{D}_k^+$, by Proposition 5.8, (4) and (7.44), there exists $0\neq v\in V\cap(\ker D_k^+)^\perp$, which in view of (7.42b) is a contradiction. Therefore $\dim_\Gamma V\leqslant\dim_\Gamma\ker\widetilde{D}_k^+$. By (7.45), we get (7.47).

Relations (7.46) and (7.47) entail that the function $N_\Gamma(\mu,\widetilde{\Delta}_{k,\widetilde{\Phi}})$ is constant in the interval $\mu\in[C_L,2k\mu_0-C_L)$ and equal to $\dim_\Gamma\ker\widetilde{D}_k^+$. Enlarging a bit C_L if necessary, we see that the spectrum of $\widetilde{\Delta}_{k,\widetilde{\Phi}}$ is indeed contained in $[-C_L,C_L]\cup[2k\mu_0-C_L,+\infty)$, and the Γ–dimension d_k of the spectral space $E\big([-C_L,C_L],\widetilde{\Delta}_{k,\widetilde{\Phi}}\big)$ equals $\dim_\Gamma\ker\widetilde{D}_k^+$.

By Corollary 7.12, $\dim_\Gamma\ker\widetilde{D}_k^+=\mathrm{index}_\Gamma\widetilde{D}_k^+$. Moreover, Atiyah's L^2 index Theorem (cf. [**?**, Theorem 3.8] or 5.2) shows that $\mathrm{index}_\Gamma\widetilde{D}_k^+=\mathrm{index}\,D_k^+$. By (7.38), the proof is achieved. □

7.2. Asymptotic of the Bergman kernel

We use now the existence of the spectral gap of the renormalized Bochner-Laplacian in order to define the Bergman kernels.

7.15. DEFINITION. Let us denote by $\mathscr{H}_k$, the span of eigensections of $\Delta_{k,\Phi}=\Delta^{L^k\otimes E}-k\tau+\Phi$ corresponding to eigenvalues in $[-C_L,C_L]$. Let $P_{\mathscr{H}_k}$ be the orthonormal projection from $\mathscr{C}^\infty(X,L^k\otimes E)$ onto $\mathscr{H}_k$. The smooth kernel of $(\Delta_{k,\Phi})^qP_{\mathscr{H}_k}$, $q\geqslant0$ (where $(\Delta_{k,\Phi})^0=1$), with respect to $dv_X(x')$ is denoted $P_{q,k}(x,x')$ and is called a *generalized Bergman kernel* of $\Delta_{k,\Phi}$.

The kernel $P_{q,k}(x,x')$ is a section of $\pi_1^*(L^k\otimes E)\otimes\pi_2^*(L^k\otimes E)^*$ over $X\times X$, where π_1 and π_2 are the projections of $X\times X$ on the first and second factor. Let $\{S_i^k\}_{i=1}^{d_k}$ be any orthonormal basis of $\mathscr{H}_k$ with respect to the inner product (7.13) such that $\Delta_{k,\Phi}S_i^k = \lambda_{i,k}S_i^k$. Using these notations we can write

$$P_{q,k}(x,x') = \sum_{i=1}^{d_k} \lambda_{i,k}^q S_i^k(x)\otimes (S_i^k(x'))^* \in (L^k\otimes E)_x \otimes (L^k\otimes E)^*_{x'}. \tag{7.48}$$

Since $L_x^k \otimes (L_x^k)^*$ is canonically isomorphic to $\mathbb{C}$, the restriction of $P_{q,k}$ to the diagonal $\{(x,x): x\in X\}$ can be identified to $B_{q,k}\in \mathscr{C}^\infty(X, E\otimes E^*) = \mathscr{C}^\infty(X,\mathrm{End}(E))$.

7.2.1. Localization of the problem. We will first show how to *localize* the problem. Let a^X be the injectivity radius of (X, g^{TX}). We fix $\varepsilon \in (0, a^X/4)$. We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. Then the map $T_xX \ni Z \to \exp_x^X(Z)\in X$ is a diffeomorphism from $B^{T_xX}(0,\varepsilon)$ on $B^X(x,\varepsilon)$ for $\varepsilon \leqslant a^X$. From now on, we identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ for $\varepsilon\leqslant a^X$.

Let $f:\mathbb{R}\to[0,1]$ be a smooth even function such that $f(v)=1$ for $|v|\leqslant \varepsilon/2$, and $f(v)=0$ for $|v|\geqslant\varepsilon$. Set

$$F(a) = \Big(\int_{-\infty}^{+\infty} f(v)dv\Big)^{-1}\int_{-\infty}^{+\infty} e^{iva}f(v)dv. \tag{7.49}$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathscr{S}(\mathbb{R})$ and $F(0)=1$. Let $\widetilde{F}$ be the holomorphic function on $\mathbb{C}$ such that $\widetilde{F}(a^2)=F(a)$. The restriction of $\widetilde{F}$ to $\mathbb{R}$ lies in the Schwartz space $\mathscr{S}(\mathbb{R})$. Then there exists $\{c_j\}_{j=1}^\infty$ such that for any $p\in\mathbb{N}$, the function

$$F_p(a) = \widetilde{F}(a) - \sum_{j=1}^{k} c_j a^j \widetilde{F}(a), \tag{7.50}$$

verifies

$$F_p^{(i)}(0) = 0 \quad \text{for any } 0 < i \leqslant p. \tag{7.51}$$

7.16. PROPOSITION. *For any $l,m\in\mathbb{N}$, there exists $C_{l,m}>0$ such that for $k\geqslant 1$*

$$\Big| F_l\big(\tfrac{1}{\sqrt{k}}\Delta_{k,\Phi}\big)(x,x') - P_{0,k}(x,x')\Big|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m} k^{-\frac{l}{2}+2(2m+2n+1)}. \tag{7.52}$$

Here the $\mathscr{C}^m$ norm is induced by ∇^L and ∇^E.

Using (7.49), (7.50) and the finite propagation speed [**17**, §7.8], [**32**, §4.4], it is clear that for $x,x'\in X$, $F_l\big(\frac{1}{\sqrt{k}}\Delta_{k,\Phi}\big)(x,\cdot)$ only depends on the restriction of $\Delta_{k,\Phi}$ to $B^X(x,\varepsilon k^{-\frac14})$, and $F_l\big(\frac{1}{\sqrt{k}}\Delta_{k,\Phi}\big)(x,x')=0$, if $d(x,x')\geqslant \varepsilon k^{-\frac14}$. This means that the asymptotic of $\Delta_{k,\Phi}^q P_{\mathscr{H}_k}(x,\cdot)$ when $k\to+\infty$, modulo $\mathscr{O}(k^{-\infty})$ (i.e. terms whose $\mathscr{C}^m$ norm is $\mathscr{O}(k^{-l})$ for any $l,m\in\mathbb{N}$), only depends on the restriction of $\Delta_{k,\Phi}$ to $B^X(x,\varepsilon k^{-\frac14})$. In particular, the asymptotic of $P_{q,k}(x_0,x')$ as $k\to\infty$ is localized on a neighborhood of x_0.

7.2.2. Rescaling and Taylor expansion of the rescaled operator. Thus we can *translate our analysis* from X to the manifold $\mathbb{R}^{2n}\simeq T_{x_0}X =: X_0$. For $Z\in B^{T_{x_0}X}(0,\varepsilon)$ we identify L_Z, E_Z and $(L^k\otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^k\otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^k\otimes E}$ along the curve $\gamma_Z:[0,1]\ni u\to \exp_{x_0}^X(uZ)$. Let $\{e_i\}_i$ be an oriented orthonormal basis of $T_{x_0}X$, and let $\{e^i\}_i$ be its dual basis.

For $\varepsilon>0$ small enough, we will extend the geometric objects from $B^{T_{x_0}X}(0,\varepsilon)$ to $\mathbb{R}^{2n}\simeq T_{x_0}X$ (here we identify $(Z_1,\cdots,Z_{2n})\in\mathbb{R}^{2n}$ to $\sum_i Z_ie_i\in T_{x_0}X$) such that $\Delta_{k,\Phi}$ is the restriction of a renormalized Bochner-Laplacian on $\mathbb{R}^{2n}$ associated to a Hermitian line bundle with positive curvature. In this way, we replace X by $\mathbb{R}^{2n}$.

At first, we denote by L_0, E_0 the trivial bundles with fiber L_{x_0}, E_{x_0} on $X_0 = \mathbb{R}^{2n}$. We still denote by ∇^L, ∇^E, h^L etc. the connections and metrics on L_0, E_0 on $B^{T_{x_i}X}(0,4\varepsilon)$ induced by the above identification. Then h^L, h^E is identified to the constant metrics $h^{L_0} = h^{L_{x_0}}$, $h^{E_0} = h^{E_{x_0}}$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = 1 \ \text{ if } \ |v| < 2; \quad \rho(v) = 0 \ \text{ if } |v| > 4. \tag{7.53}$$

Let $\varphi_\varepsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the map defined by $\varphi_\varepsilon(Z) = \rho(|Z|/\varepsilon)Z$. Then $\Phi_0 = \Phi \circ \varphi_\varepsilon$ is a smooth self-adjoint section of $\mathrm{End}(E_0)$ on X_0. Let $g^{TX_0}(Z) = g^{TX}(\varphi_\varepsilon(Z))$, $J_0(Z) = J(\varphi_\varepsilon(Z))$ be the metric and complex structure on X_0. Set $\nabla^{E_0} = \varphi_\varepsilon^* \nabla^E$. Then ∇^{E_0} is the extension of ∇^E on $B^{T_{x_0}X}(0,\varepsilon)$. If $\mathscr{R} = \sum_i Z_i e_i = Z$ denotes radial vector field on $\mathbb{R}^{2n}$, we define the Hermitian connection ∇^{L_0} on (L_0, h^{L_0}) by

$$\nabla^{L_0}|_Z = \varphi_\varepsilon^* \nabla^L + \frac{1}{2}(1-\rho^2(|Z|/\varepsilon))R^L_{x_0}(\mathscr{R},\cdot). \tag{7.54}$$

Then we calculate easily that its curvature $R^{L_0} = (\nabla^{L_0})^2$ is

$$\begin{aligned} R^{L_0}(Z) &= \varphi_\varepsilon^* R^L + \frac{1}{2} d\Big((1-\rho^2(|Z|/\varepsilon))R^L_{x_0}(\mathscr{R},\cdot)\Big) \\ &= \Big(1-\rho^2(|Z|/\varepsilon)\Big)R^L_{x_0} + \rho^2(|Z|/\varepsilon)R^L_{\varphi_\varepsilon(Z)} \\ &\quad - (\rho\rho')(|Z|/\varepsilon)\frac{Z_i e^i}{\varepsilon |Z|} \wedge \big[R^L_{x_0}(\mathscr{R},\cdot) - R^L_{\varphi_\varepsilon(Z)}(\mathscr{R},\cdot)\big]. \end{aligned} \tag{7.55}$$

Thus R^{L_0} is positive in the sense of (7.4) for ε small enough, and the corresponding constant μ_0 for R^{L_0} is bigger than $\frac{1}{2}\mu_0$. From now on, we fix ε as above.

Let $\Delta^{X_0}_{k,\Phi_0}$ be the renormalized Bochner-Laplacian on X_0 associated to the above data by (7.6). Observe that R^{L_0} is uniformly positive on $\mathbb{R}^{2n}$, so by the argument in the proof of Corollary 7.2, we know that (7.8) still holds for $\Delta^{X_0}_{k,\Phi_0}$. Especially, there exists $C_{L_0} > 0$ such that

$$\operatorname{Spec} \Delta^{X_0}_{k,\Phi_0} \subset [-C_{L_0}, C_{L_0}] \cup [k\mu_0 - C_{L_0}, +\infty[. \tag{7.56}$$

Let S_L be an unit vector of L_{x_0}. Using S_L and the above discussion, we get an isometric $E_0 \otimes L_0^k \simeq E_{x_0}$.

7.17. DEFINITION. Let $P_{0,\mathscr{H}_k}$ be the spectral projection of $\Delta^{X_0}_{k,\Phi_0}$ from $\mathscr{C}^\infty(X_0, L_0^k \otimes E_0) \simeq \mathscr{C}^\infty(X_0, E_{x_0})$ corresponding to the interval $[-C_{L_0}, C_{L_0}]$, and let $P^0_{0,q,k}(x,x') \in \mathrm{End}(E_{x_0})$, $(x, x' \in X_0)$ $(q \geqslant 0)$ be the smooth kernels of $P_{0,q,k} = (\Delta^{X_0}_{k,\Phi_0})^q P_{0,\mathscr{H}_k}$ (we set $(\Delta^{X_0}_{k,\Phi_0})^0 = 1$) with respect to the volume form $dv_{X_0}(x')$.

7.18. PROPOSITION. *For any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $x, x' \in B^{T_{x_0}X}(0,\varepsilon)$,*

$$\big|(P_{0,q,k} - P_{q,k})(x,x')\big|_{\mathscr{C}^m} \leqslant C_{l,m} k^{-l}. \tag{7.57}$$

PROOF. Using (7.49) and (7.56), we know that for $x, x' \in B^{T_{x_0}X}(0,\varepsilon)$,

$$\Big|F_k\big(\tfrac{1}{\sqrt{k}}\Delta_{k,\Phi}\big)(x,x') - P_{0,0,k}(x,x')\Big|_{\mathscr{C}^m} \leqslant C_{k,m} k^{-\frac{k}{2}+2(m+n+1)}. \tag{7.58}$$

Thus from (7.52) and (7.58) for k big enough, we infer (7.57) for $q=0$; Now from the definition of $P_{0,q,k}$ and $P_{q,k}$, we get (7.57) from the relation $\nabla^{L^k\otimes E}_{e_j} = \nabla_{e_j} + k\Gamma^L(e_j) + \Gamma^E(e_j)$ and (7.57) for $q=0$. □

Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_{X_0}(Z) = \kappa(Z) dv_{TX}(Z). \tag{7.59}$$

with $\kappa(0)=1$. Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U, and set $\partial_i = \nabla_{e_i}$. If $\alpha = (\alpha_1, \cdots, \alpha_{2n})$ is a multi-index, set $Z^\alpha = Z_1^{\alpha_1}\cdots Z_{2n}^{\alpha_{2n}}$. We also denote by $(\partial^\alpha R^L)_{x_0}$ the tensor $(\partial^\alpha R^L)_{x_0}(e_i,e_j) = \partial^\alpha (R^L(e_i,e_j))_{x_0}$. Denote by $t = \frac{1}{\sqrt{k}}$. For $s\in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$ and $Z\in\mathbb{R}^{2n}$, set

$$\begin{aligned} &(S_t s)(Z) = s(Z/t), \quad \nabla_t = tS_t^{-1}\kappa^{\frac12}\nabla^{L_0^k\otimes E_0}\kappa^{-\frac12}S_t, \\ &\mathscr{L}_t = S_t^{-1}\,\tfrac{1}{k}\,\kappa^{\frac12}\,\Delta^{X_0}_{k,\Phi_0}\,\kappa^{-\frac12}\,S_t. \end{aligned} \tag{7.60}$$

7.19. THEOREM. *There exist polynomials $\mathscr{A}_{i,j,r}$ (resp. $\mathscr{B}_{i,r}$, $\mathscr{C}_r$) ($r\in\mathbb{N}, i,j\in\{1,\cdots,2n\}$) in Z with the following properties:*

– their coefficients are polynomials in R^{TX} (resp. R^{TX}, R^L, R^E, Φ) and their derivatives at x_0 up to order $r-2$ (resp. $r-1$, r, $r-1$, r),

– $\mathscr{A}_{i,j,r}$ is a monomial in Z of degree r, the degree in Z of $\mathscr{B}_{i,r}$ (resp. $\mathscr{C}_r$) has the same parity with $r-1$ (resp. r),

– if we denote by

$$\mathscr{O}_r = \mathscr{A}_{i,j,r}\nabla_{e_i}\nabla_{e_j} + \mathscr{B}_{i,r}\nabla_{e_i} + \mathscr{C}_r, \tag{7.61}$$

then

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^m t^r\mathscr{O}_r + \mathscr{O}(t^{m+1}). \tag{7.62}$$

and there exists $m'\in\mathbb{N}$ such that for any $k\in\mathbb{N}$, $t\leqslant 1$ the derivatives of order $\leqslant k$ of the coefficients of the operator $\mathscr{O}(t^{m+1})$ are dominated by $Ct^{m+1}(1+|Z|)^{m'}$. Moreover

$$\begin{aligned} &\mathscr{L}_0 = -\sum_j\Big(\nabla_{e_j} + \frac12 R^L_{x_0}(Z,e_j)\Big)^2 - \tau_{x_0}, \qquad (7.63)\\ &\mathscr{O}_1(Z) = -\frac23(\partial_j R^L)_{x_0}(\mathscr{R},e_i)Z_j\Big(\nabla_{e_i} + \frac12 R^L_{x_0}(\mathscr{R},e_i)\Big) - \frac13(\partial_i R^L)_{x_0}(\mathscr{R},e_i) - (\nabla_{\mathscr{R}}\tau)_{x_0},\\ &\mathscr{O}_2(Z) = \frac13\left\langle R^{TX}_{x_0}(\mathscr{R},e_i)\mathscr{R},e_j\right\rangle_{x_0}\Big(\nabla_{e_i} + \frac12 R^L_{x_0}(\mathscr{R},e_i)\Big)\Big(\nabla_{e_j} + \frac12 R^L_{x_0}(\mathscr{R},e_j)\Big)\\ &\quad + \Big[\frac23\left\langle R^{TX}_{x_0}(\mathscr{R},e_j)e_j,e_i\right\rangle_{x_0} - \Big(\frac12\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!} + R^E_{x_0}\Big)(\mathscr{R},e_i)\Big]\Big(\nabla_{e_i} + \frac12 R^L_{x_0}(\mathscr{R},e_i)\Big)\\ &\quad - \frac14\nabla_{e_i}\Big(\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!}(\mathscr{R},e_i)\Big) - \frac19[(\partial_j R^L)_{x_0}(\mathscr{R},e_i)Z_j]^2\\ &\quad - \frac1{12}\Big[\mathscr{L}_0, \left\langle R^{TX}_{x_0}(\mathscr{R},e_i)\mathscr{R},e_i\right\rangle_{x_0}\Big] - \sum_{|\alpha|=2}(\partial^\alpha\tau)_{x_0}\frac{Z^\alpha}{\alpha!} + \Phi_{x_0}. \end{aligned}$$

$\{\mathscr{L}_t\}$ is a family of self-adjoint differential operators with coefficients in $\mathrm{End}(E)_{x_0}$. We denote by $\mathscr{P}_{0,t} : (\mathscr{C}^\infty(X_0,E_{x_0}), \|\quad\|_0) \to (\mathscr{C}^\infty(X_0,E_{x_0}), \|\quad\|_0)$ the spectral projection of $\mathscr{L}_t$ corresponding to the interval $[-C_{L_0}t^2, C_{L_0}t^2]$. Let $\mathscr{P}_{q,t}(Z,Z') = \mathscr{P}_{q,t,x_0}(Z,Z')$,

$(Z, Z' \in X_0,\ q \geqslant 0)$ be the smooth kernel of $\mathscr{P}_{q,t} = (\mathscr{L}_t)^q \mathscr{P}_{0,t}$ with respect to $dv_{TX}(Z')$. We can view $\mathscr{P}_{q,t,x}(Z,Z')$ as a smooth section of $\pi^* \operatorname{End}(E)$ over $TX \times_X TX$, where $\pi : TX \times_X TX \to X$, by identifying a section $S \in \Gamma(TX \times_X TX, \pi^* \operatorname{End}(E))$ with the family $(S_x)_{x \in X}$, where $S_x = S|_{\pi^{-1}(x)}$. Given a trivialization $\varphi : TX|_U \to U \times \mathbb{R}^{2n}$ and $v, v' \in \mathbb{R}^{2n}$, $x \mapsto S_x(\varphi^{-1}(v), \varphi^{-1}(v'))$ is a section of $\operatorname{End}(E)$ over U, so we can define the $\mathscr{C}^s$-norm $|S(\varphi^{-1}(v), \varphi^{-1}(v'))|_{\mathscr{C}^s(X, \operatorname{End}(E))}$ of this section.

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$. By (7.8),

$$\mathscr{P}_{q,t} = \frac{1}{2\pi i} \int_\delta \lambda^q (\lambda - \mathscr{L}_t)^{-1} d\lambda. \tag{7.64}$$

From (7.8), we can apply the techniques in [**18**] which is inspired from [**5**, §11] to get the following key estimate.

7.20. THEOREM. *There exist smooth sections $F_{q,r}$ of $\operatorname{End}(E_{x_0})$ such that for $k, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $t \in]0,1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

$$\sup_{|\alpha|,|\alpha'| \leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(\mathscr{P}_{q,t} - \sum_{r=0}^{k} F_{q,r} t^r \Big)(Z,Z') \right|_{\mathscr{C}^{m'}(X, \operatorname{End}(E))} \leqslant C t^k. \tag{7.65}$$

Recall that $P_{0,q,k}(x,x')$ was defined in Definition 7.17. By (7.60), for $Z, Z' \in \mathbb{R}^{2n}$,

$$P_{0,q,p}(Z,Z') = t^{-2n-2q}\, \kappa^{-\frac{1}{2}}(Z)\, \mathscr{P}_{q,t}(Z/t, Z'/t)\, \kappa^{-\frac{1}{2}}(Z'). \tag{7.66}$$

By (7.57), (7.66), Proposition 7.18, Theorems 7.20, we get the following *near diagonal expansion* of the Bergman kernel:

7.21. THEOREM. *For $l, m, m' \in \mathbb{N}$, $l \geqslant 2q$, $\sigma > 0$, there exists $C > 0$ such that if $k \geqslant 1$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leq \sigma/\sqrt{k}$,*

$$\begin{aligned} \sup_{|\alpha|+|\alpha'| \leqslant m} \Bigg| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \Big(\frac{1}{k^n} P_{q,k}(Z,Z') \\ - \sum_{r=2q}^{l} F_{q,r}(\sqrt{k} Z, \sqrt{k} Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') k^{-\frac{r}{2}+q} \Big) \Bigg|_{\mathscr{C}^{m'}(X)} \leqslant C k^{-\frac{l-m}{2}+q}. \end{aligned} \tag{7.67}$$

To complete the proof the Theorem 7.2, we finally prove $F_{q,r} = 0$ for $r < 2q$. Moreover, (7.65) and (7.66) yield

$$b_{q,r}(x_0) = F_{q,2r+2q}(0,0). \tag{7.68}$$

Let us check our formulas with the help of the Atiyah-Singer formula. For k large enough we have from (7.9)

$$\begin{aligned} \dim \mathscr{H}_k = d_k &= \int_X \operatorname{ch}(L^k \otimes E) \operatorname{Td}(TX) \\ &= \operatorname{rank}(E) \int_X \frac{c_1(L)^n}{n!} k^n + \int_X \Big(c_1(E) + \frac{\operatorname{rk}(E)}{2} c_1(TX) \Big) \frac{c_1(L)^{n-1}}{(n-1)!} k^{n-1} + \mathscr{O}(k^{n-2}), \end{aligned} \tag{7.69}$$

where $\operatorname{ch}(\cdot), c_1(\cdot), \operatorname{Td}(\cdot)$ are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles (TX is a complex vector bundle with complex structure J). Let $P^{1,0} = \frac{1}{2}(1 - \sqrt{-1} J)$ be the natural projection from $TX \otimes_\mathbb{R} \mathbb{C}$ onto $T^{(1,0)}X$.

Then $\nabla^{1,0} = P^{1,0}\nabla^{TX}P^{1,0}$ is a Hermitian connection on $T^{(1,0)}X$, and the Chern-Weil representative of $c_1(TX)$ is $c_1(T^{(1,0)}X, \nabla^{1,0}) = \frac{\sqrt{-1}}{2\pi}\operatorname{Tr}_{|T^{(1,0)}X}(\nabla^{1,0})^2$. Then

$$(\nabla^{1,0})^2 = P^{1,0}\Big[R^{TX} - \frac{1}{4}(\nabla^X J)\wedge(\nabla^X J)\Big]P^{1,0}. \tag{7.70}$$

Thus if $J = \mathbf{J}$, then by (7.70),

$$\Big\langle c_1(T^{(1,0)}X, \nabla^{1,0}), \omega\Big\rangle = \frac{1}{4\pi}\Big(r^X + \frac{1}{2}|\nabla^X J|^2\Big). \tag{7.71}$$

Therefore, by integrating over X the expansion (1.21) for $k=1$ we obtain (7.69), so (1.22) is compatible with (7.69).

7.2.3. Evaluation of $F_{q,r}$. The almost complex structure J induces a splitting $T_{\mathbb{R}}X\otimes_{\mathbb{R}}\mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. We choose $\{w_i\}_{i=1}^n$ an orthonormal basis of $T^{(1,0)}_{x_0}X$, such that

$$-2\pi\sqrt{-1}\mathbf{J}_{x_0} = \operatorname{diag}(a_1,\cdots,a_n) \in \operatorname{End}(T^{(1,0)}_{x_0}X). \tag{7.72}$$

We use the orthonormal basis $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1,\ldots,n$ of $T_{x_0}X$ to introduce the normal coordinates as in Section 7.1.3. In what follows we will use the complex coordinates $z = (z_1,\cdots,z_n)$, thus $Z = z + \overline{z}$, and $w_i = \sqrt{2}\frac{\partial}{\partial z_i}$, $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial \overline{z}_i}$. It is very useful to introduce the creation and annihilation operators b_i, b_i^+,

$$b_i = -2\tfrac{\partial}{\partial z_i} + \frac{1}{2}a_i\overline{z}_i\,, \quad b_i^+ = 2\tfrac{\partial}{\partial \overline{z}_i} + \frac{1}{2}a_i z_i\,, \quad b = (b_1,\cdots,b_n)\,. \tag{7.73}$$

Now there are second order differential operators $\mathscr{O}_r$ whose coefficients are polynomials in Z with coefficients as polynomials in R^{TX}, $R^{\det}$, R^E, R^L and their derivatives at x_0, such that

$$\mathscr{L}_t = \mathscr{L}_2^0 + \sum_{r=1}^{\infty}\mathscr{O}_r t^r, \quad \text{with } \mathscr{L}_2^0 = \sum_i b_i b_i^+\,. \tag{7.74}$$

By proceeding as in [**31**], we obtain

7.22. THEOREM. *The spectrum of the restriction of $\mathscr{L}_2^0$ on $L^2(\mathbb{R}^{2n})$ is given by $\Big\{2\sum_{i=1}^n \alpha_i a_i : \alpha_i \in \mathbb{R}\Big\}$ and an orthogonal basis of the eigenspace of $2\sum_{i=1}^n \alpha_i a_i$ is given by*

$$b^{\alpha}\big(z^{\beta}\exp\big(-\frac{1}{4}\sum_i a_i|z_i|^2\big)\big)\,, \quad \text{with } \beta \in \mathbb{R}^n\,. \tag{7.75}$$

Let $N^{\perp}$ be the orthogonal space of $N = \ker\mathscr{L}_2^0$ in $(L^2(\mathbb{R}^{2n}, E_{x_0}), \|\quad\|_0)$. Let P^N, $P^{N^{\perp}}$ be the orthogonal projections from $L^2(\mathbb{R}^{2n}, E_{x_0})$ onto N, $N^{\perp}$, respectively. Let $P^N(Z,Z')$ be the smooth kernel of the operator P^N with respect to $dv_{TX}(Z)$. From (7.75), we get

$$P^N(Z,Z') = \frac{1}{(2\pi)^n}\prod_{i=1}^n a_i\ \exp\Big(-\frac{1}{4}\sum_i a_i\big(|z_i|^2 + |z_i'|^2 - 2z_i\overline{z}_i'\big)\Big). \tag{7.76}$$

Now for $\lambda \in \delta$, we solve the following formal power series on t, with $g_r(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N)$, $f_r^{\perp}(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}), N^{\perp})$,

$$(\lambda - \mathscr{L}_t)\sum_{r=0}^{\infty}\Big(g_r(\lambda) + f_r^{\perp}(\lambda)\Big)t^r = \operatorname{Id}_{L^2(\mathbb{R}^{2n}, E_{x_0})}\,. \tag{7.77}$$

From (7.64), (7.77), we claim that

$$F_{q,r} = \frac{1}{2\pi i}\int_{\delta} \lambda^q g_r(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\delta} \lambda^q f_r^{\perp}(\lambda)d\lambda. \tag{7.78}$$

From Theorem 7.22, (7.78), the key observation $P^N\mathscr{O}_1P^N = 0$, and the residue formula, we can get $F_{q,r}$ by using the operators $(\mathscr{L}_2^0)^{-1}$, P^N, $P^{N^\perp}$, $\mathscr{O}_i(i \leqslant r)$. This gives us a way to compute $b_{q,r}$ in view of Theorem 7.22 and (7.68). Especially, for $q > 0, r < 2q$,

$$\begin{aligned}
&F_{0,0} = P^N, \quad F_{q,r} = 0, \qquad (7.79)\\
&F_{q,2q} = (P^N\mathscr{O}_2P^N - P^N\mathscr{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1P^N)^qP^N,\\
&F_{0,2} = (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1P^N - (\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_2P^N\\
&\qquad + P^N\mathscr{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp}\mathscr{O}_1(\mathscr{L}_2^0)^{-1}P^{N^\perp} - P^N\mathscr{O}_2(\mathscr{L}_2^0)^{-1}P^{N^\perp}.
\end{aligned}$$

In fact, $\mathscr{L}_2^0$ and $\mathscr{O}_r$ are formal adjoints with respect to $\| \quad \|_0$, thus in $F_{0,2}$, we only need to compute the first two terms, as the last two terms are their adjoint. This simplifies the computation in Theorem 1.22.

7.3. Applications

In this Section, we discuss various applications of our results. In Section 7.3.1, we study the density of states function of $\Delta_{k,\Phi}$. In Section 7.3.2, we explain how to handle the first-order pseudo-differential operator D_b of Boutet de Monvel and Guillemin [**13**] which was studied extensively by Shiffman and Zelditch [**28**]. In Section 7.3.3, we prove a symplectic version of the convergence of the Fubini-Study metric of an ample line bundle [**33**]. In Section 7.3.4, we show how to handle the operator $\overline{\partial} + \overline{\partial}^*$ when X is Kähler but $\mathbf{J} \neq J$. Finally, in Sections 7.3.5, 7.3.6, we establish some generalizations for non-compact or singular manifolds.

7.3.1. Density of states function. Let (X, ω) be a compact symplectic manifold of real dimension $2n$ and (L, ∇^L, h^L) is a pre-quatum line bundle as in Section 7.1.1. Assume that E is the trivial bundle $\mathbb{C}$, $\Phi = 0$ and $\mathbf{J} = J$. The latter means, by (7.2), that g^{TX} is the Riemannian metric associated to ω and J. We denote by $\mathrm{vol}(X) = \int_X \frac{\omega^n}{n!}$ the Riemannian volume of (X, g^{TX}). Recall that d_k is defined in (7.9) (see also (7.69)).

Our aim is to describe the asymptotic distribution of the energies of the bound states as k tends to infinity. We define the spectrum counting function of $\Delta_k := \Delta_{k,0}$ by $N_k(\lambda) = \#\{i : \lambda_{i,k} \leqslant \lambda\}$ and the spectral density measure on $[-C_L, C_L]$ by

$$\nu_k = \frac{1}{d_k}\frac{d}{d\lambda}N_k(\lambda), \quad \lambda \in [-C_L, C_L]. \tag{7.80}$$

Clearly, ν_k is a sum of Dirac measures supported on $\mathrm{Spec}\,\Delta_k \cap [-C_L, C_L]$. Set

$$\rho : X \longrightarrow \mathbb{R}, \quad \rho(x) = \frac{1}{24}|\nabla^X J|^2. \tag{7.81}$$

7.23. THEOREM. *The weak limit of the sequence $\{\nu_k\}_{k\geqslant 1}$ is the direct image measure $\rho_*\Big(\frac{1}{\mathrm{vol}(X)}\frac{\omega^n}{n!}\Big)$, that is, for any continuous function $f \in \mathscr{C}([-C_L, C_L])$, we have*

$$\lim_{k\to\infty}\int_{-C_L}^{C_L} f\,d\nu_k = \frac{1}{\mathrm{vol}(X)}\int_X (f\circ\rho)\,\frac{\omega^n}{n!}\,. \tag{7.82}$$

PROOF. By (7.48), we have for $q \geqslant 1$ (now E is trivial): $B_{q,k}(x) = \sum_{i=1}^{d_k} \lambda_{i,k}^q |S_i^k(x)|^2$, which yields by integration over X,

$$\frac{1}{d_k}\int_X B_{q,k}\, dv_X = \frac{1}{d_k}\sum_{i=1}^{d_k} \lambda_{i,k}^q = \int_{-C_L}^{C_L} \lambda^q\, d\nu_k(\lambda)\,, \tag{7.83}$$

since S_i^k have unit L^2 norm. On the other hand, (7.69), (1.21) entail for $k \to \infty$,

$$\begin{aligned}\frac{1}{d_k}\int_X B_{q,k}\, dv_X &= \frac{k^n}{d_k}\int_X b_{q,0}\, dv_X + \frac{\mathscr{O}(k^{n-1})}{d_k}\\ &= \frac{1}{\mathrm{vol}(X)}\int_X \rho^q\, dv_X + \mathscr{O}(k^{-1}).\end{aligned} \tag{7.84}$$

We infer from (7.83)-(7.84) that (7.82) holds for $f(\lambda) = \lambda^q$, $q \geqslant 1$. Since this is obviously true for $f(\lambda) \equiv 1$, too, we deduce it holds for all polynomials. Upon invoking the Weierstrass approximation theorem, we get (7.82) for all continuous functions on $[-C_L, C_L]$. This achieves the proof. □

7.24. REMARK. A function ρ satisfying (7.82) is called spectral density function. Its existence and uniqueness were demonstrated by Guillemin-Uribe [**21**]. As for the explicit formula of ρ, the paper [**11**] is dedicated to its computation. Our formula (7.81) is different from [**11**, Theorem 1.2]. [1]

An interesting corollary of (7.81) and (7.82) is the following result which was first stated in [**11**, Cor. 1.3].

7.25. COROLLARY. *The spectral density function is identically zero iff (X,ω,J) is Kähler.*

7.26. REMARK. Theorem 7.23 can be slightly generalized. Assume namely that $\mathbf{J} = J$ and E is a Hermitian vector bundle as in Section 7.1.1 such that $R^E = \eta \otimes \mathrm{Id}_E$, $\Phi = \varphi\, \mathrm{Id}_E$, where η is a 2-form and φ a real function on X. Then there exists a spectrum density function satisfying (7.82) given by

$$\rho : X \longrightarrow \mathbb{R}, \quad \rho(x) = \frac{1}{12}|\nabla^X J|^2 + \frac{\sqrt{-1}}{2}\eta(e_j, Je_j) + \varphi\,. \tag{7.85}$$

The proof is similar to the previous one, as $\mathrm{Tr}_{E_x} B_{q,k}(x) = \sum_{i=1}^{d_k} \lambda_{i,k}^q |S_i^k(x)|^2$.

7.3.2. Almost-holomorphic Szegö kernels. We use the notations and assumptions from Section 7.3.1, especially, we assume $\mathbf{J} = J$.

Let $Y = \{u \in L^*, |u|_{h^{L^*}} = 1\}$ be the unit circle bundle in L^*. Then the smooth sections of L^k can be identified to the smooth functions

$$\mathscr{C}^\infty(Y)_k = \{f \in \mathscr{C}^\infty(Y,\mathbb{C});\, f(ye^{\sqrt{-1}\theta}) = e^{\sqrt{-1}k\theta} f(y) \text{ for } e^{\sqrt{-1}\theta} \in S^1, y \in Y\},$$

where $ye^{\sqrt{-1}\theta}$ is the S^1 action on Y.

The connection ∇^L on L induces a connection on the S^1-principal bundle $\pi : Y \to X$, and let $T^H Y \subset TY$ be the corresponding horizontal bundle. Let $g^{TY} = \pi^* g^{TX} \oplus d\theta^2$ be the metric on $TY = T^H Y \oplus TS^1$, with $d\theta^2$ the standard metric on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let Δ_Y be the

[1] In [**11**, (3.7)], the leading term of G_{0j} should be $\kappa^{-1/2} b_j^{(1)}$ which was missed therein, as the principal terms of $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial y^j}$ are ∂_0, $T_j^l \partial_l$ by [**11**, equation after (3.11)]. Now, from [**11**, (3.5)], $b_j^{(1)}$ is $\frac{1}{2}\langle Jz, T_j^l \partial_l\rangle$. Thus $\mathscr{L}_0$ in [**11**, (3.8)] is incorrect.

Bochner-Laplacian on (Y, g^{TY}), then by construction, it commutes with the generator ∂_θ of the circle action, and so it commutes with the horizontal Laplacian

$$\Delta_h = \Delta_Y + \partial_\theta^2, \tag{7.86}$$

then Δ_h on $\mathscr{C}^\infty(Y)_k$ is identical with Δ^{L^k} on $\mathscr{C}^\infty(X, L^k)$ (cf. [**10**, §2.1]).

In [**13**, Lemma 14.11, Theorem A 5.9], [**14**], [**21**, (3.13)], they construct a self-adjoint second-order pseudodifferential operator Q on Y such that

$$V = \Delta_h + \sqrt{-1}\tau\partial_\theta - Q \tag{7.87}$$

is a self-adjoint pseudodifferential operator of order zero on Y, and V, Q commute with the S^1-action. The orthogonal projection Π onto the kernel of Q is called the *Szegö projector* associated with the almost CR manifold Y. In fact, the Szegö projector is not unique or canonically defined, but the above construction defines a canonical choice of Π modulo smoothing operators. In the complex case, the construction produces the usual Szegö projector Π.

We denote the operators on $\mathscr{C}^\infty(X, L^k)$ corresponding to Q, V, Π by Q_k, V_k, Π_k, especially, $V_k(x,y) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}p\theta} V(xe^{\sqrt{-1}\theta}, y)d\theta$. Then by (7.87),

$$Q_k = \Delta^{L^k} - k\tau - V_k. \tag{7.88}$$

By [**21**, §4], there exists $\mu_1 > 0$ such that for k large,

$$\mathrm{Spec}\, Q_k \subset \{0\} \cap [\mu_1 k, +\infty[. \tag{7.89}$$

Since the operator V_k is uniformly bounded in k, naturally, from (7.8), (7.69), we get

$$\dim\ker Q_k = d_k = \int_X \mathrm{Td}(TX)\,\mathrm{ch}(L^k). \tag{7.90}$$

Now we explain how to study the Szegö projector Π_k [2]. This can be done from our point of view. Recall $\widetilde{F}$ is the function defined after (7.49). Let $\Pi_k(x,x')$, $\widetilde{F}(Q_k)(x,x')$ be the smooth kernels of Π_k, $\widetilde{F}(Q_k)$ with respect to the volume form $dv_X(x')$.

Note that V_k is a 0-order pseudodifferential operator on X induced from a 0-order pseudodifferential operator on Y. Thus from (7.88), (7.89), we have the analogue of [**18**, Proposition 3.1]: for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $k \geqslant 1$,

$$|\widetilde{F}(Q_k)(x,x') - \Pi_k(x,x')|_{\mathscr{C}^m(X\times X)} \leqslant C_{l,m} k^{-l}. \tag{7.91}$$

By finite propagation speed [**32**, §4.4], we know that $\widetilde{F}(Q_k)(x,x')$ only depends on the restriction of Q_k to $B^X(x,\varepsilon)$, and is zero if $d(x,x') \geqslant \varepsilon$. It transpires that the asymptotic of $\Pi_k(x,x')$ as $k \to \infty$ is localized on a neighborhood of x. Thus we can translate our analysis from X to the manifold $\mathbb{R}^{2n} \simeq T_{x_0}X =: X_0$ as in Section 7.2.2, especially, we extend ∇^L to a Hermitian connection ∇^{L_0} on $(L_0, h^{L_0}) = (X_0 \times L_{x_0}, h^{L_{x_0}})$ on $T_{x_0}X$ in such a way so that we still have positive curvature R^{L_0}; in addition $R^{L_0} = R^L_{x_0}$ outside a compact set.

Now, by using a micro-local partition of unity, one can still construct the operator Q^{X_0} as in [**13**, Lemma 14.11, Theorem A 5.9], [**14**], [**21**, (3.13)], such that V^{X_0} differs from V by a smooth operator in a neighborhood of 0. On X_0, and Q^{X_0} still verifies (7.89). Thus we can work on $\mathscr{C}^\infty(X_0, \mathbb{C})$ as in Section 7.2. We rescale then the coordinates as in (7.60). The $V_k^{X_0}$ is a 0-order pseudodifferential operator on X_0 induced from a 0-order pseudodifferential

[2] As Professor Sjöstrand pointed out to us, in general, $\Pi_k - P_{0,k}$ is not $\mathscr{O}(k^{-\infty})$ as $k \to \infty$, where $P_{0,k}$ is the smooth kernel of the operator $\Delta_{0,k}$ (Definition 7.15). This can also be seen from the presence of a contribution coming from Φ in the expression (1.21) of the coefficient $b_{0,2}$.

operator on Y_0. This guarantees that the operator rescaled from $V_k^{X_0}$ will have the similar expansion as (7.74) with leading term $t^2 R_2$ in the sense of pseudo-differential operators.

From (7.90) and [**18**, (3.89)], similar to the argument in [**18**, Theorem 3.18], we can also get the full off diagonal expansion for Π_k, which is an extension of [**28**, Theorem 1], where the authors obtain (7.92) for $|Z|,|Z'| \leqslant C/\sqrt{k}$ with $C > 0$ fixed. More precisely, recalling that P^N is the Bergman kernel of $\mathscr{L}_0$ as in

(7.76) we have:

7.27. THEOREM. *There exist polynomials $\mathbf{j}_r(Z,Z')$ $(r \geqslant 0)$ of Z,Z' with the same parity with r, and $\mathbf{j}_0 = 1$, $C'' > 0$ such that for any $k,m,m' \in \mathbb{N}$, there exist $N \in \mathbb{N}, C > 0$ such that for $\alpha,\alpha' \in \mathbb{Z}^{2n}$, $|\alpha|+|\alpha'| \leqslant m$, $Z,Z' \in T_{x_0}X$, $|Z|,|Z'| \leqslant \varepsilon$, $x_0 \in X$, $k > 1$,*

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \left(\frac{1}{p^n} \Pi_k(Z,Z') - \sum_{r=0}^{k} (\mathbf{j}_r P^N)(\sqrt{k}Z, \sqrt{k}Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') k^{-r/2} \right) \right|_{\mathscr{C}^{m'}(X)}$$
$$\leqslant C k^{-(k+1-m)/2} (1 + |\sqrt{k}Z| + |\sqrt{k}Z'|)^N \exp(-\sqrt{C''\mu_1}\sqrt{k}|Z-Z'|). \quad (7.92)$$

The term $\kappa^{-\frac{1}{2}}$ in (7.92) comes from the conjugation of the operators as in (7.67), $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$ -norm for the parameter $x_0 \in X$, and we use the trivializations from Section 7.2. We leave the details to the interested reader.

7.3.3. Symplectic version of Kodaira Embedding Theorem. Let (X,ω) be a compact symplectic manifold of real dimension $2n$ and let (L,∇^L,h^L) be a pre-quantum line bundle and let g^{TX} be a Riemannian metric on X as in Section 7.1.1.

Recall that $\mathscr{H}_k \subset \mathscr{C}^\infty(X,L^k)$ is the span of those eigensections of $\Delta_k = \Delta^{L^k} - \tau k$ corresponding to eigenvalues from $[-C_L,C_L]$. We denote by $\mathbb{P}\mathscr{H}_k^*$ the projective space associated to the dual of $\mathscr{H}_k$ and we identify $\mathbb{P}\mathscr{H}_k^*$ with the Grassmannian of hyperplanes in $\mathscr{H}_k$. The *base locus* of $\mathscr{H}_k$ is the set $\mathrm{Bl}(\mathscr{H}_k) = \{x \in X : s(x) = 0 \text{ for all} s \in \mathscr{H}_k\}$. As in algebraic geometry, we define the Kodaira map

$$\begin{aligned} \Phi_k : X \smallsetminus \mathrm{Bl}(\mathscr{H}_k) &\longrightarrow \mathbb{P}\mathscr{H}_k^* \\ \Phi_k(x) &= \{s \in \mathscr{H}_k : s(x) = 0\} \end{aligned} \quad (7.93)$$

which sends $x \in X \smallsetminus \mathrm{Bl}(\mathscr{H}_k)$ to the hyperplane of sections vanishing at x. Note that $\mathscr{H}_k$ is endowed with the induced L^2 product (7.13) so there is a well–defined Fubini–Study metric g_{FS} on $\mathbb{P}\mathscr{H}_k^*$ with the associated form ω_{FS}.

7.28. THEOREM. *Let (L,∇^L) be a pre–quantum line bundle over a compact symplectic manifold (X,ω). The following assertions hold true*:

(i) *For large k, the Kodaira maps $\Phi_k : X \longrightarrow \mathbb{P}\mathscr{H}_k^*$ are well defined.*

(ii) *The induced Fubini–Study metric $\frac{1}{k}\Phi_k^*(\omega_{FS})$ converges in the $\mathscr{C}^\infty$ topology to ω ; for any $l \geqslant 0$ there exists $C_l > 0$ such that*

$$\left| \frac{1}{k}\, \Phi_k^*(\omega_{FS}) - \omega \right|_{\mathscr{C}^l} \leqslant \frac{C_l}{k}. \quad (7.94)$$

(iii) *For large k the Kodaira maps Φ_k are embeddings.*

7.29. REMARK. 1) Assume that X is Kähler and L is a holomorphic bundle. Then Δ_k is the twice the Kodaira-Laplacian and $\mathscr{H}_k$ coincides with the space $H^0(X,L^k)$ of holomorphic sections of L^k. Then (i) and (iii) are simply the Kodaira embedding theorem. Assertion (ii) is due to Tian [**33**, Theorem A] as an answer to a conjecture of Yau. In [**33**] the case $l = 2$ is considered and the left–hand side of (7.94) is estimated by $C_l/\sqrt{k}$. Ruan

[**27**] proved the $\mathscr{C}^\infty$ convergence and improved the bound to C_l/k. Both papers use the peak section method, based on L^2–estimates for $\overline{\partial}$. A proof for $l=0$ using the heat kernel appeared in Bouche [**12**]. Finally, Zelditch deduced (ii) from the asymptotic expansion of the Szegö kernel [**35**].

2) Borthwick and Uribe [**10**, Theorem 1.1], Shiffman and Zelditch [**28**, Theorems 2, 3] prove a different symplectic version of [**33**, Theorem A]. Instead of $\mathscr{H}_k$, they use the space $H^0_J(X,L^k)$ (cf. [**10**, p.601], [**28**, §2.3]) of 'almost holomorphic sections' proposed by Boutet de Monvel and Guillemin [**13**], [**14**].

PROOF. Let us first give an alternate description of the map Φ_k which relates it to the Bergman kernel. Let $\{S^k_i\}_{i=1}^{d_k}$ be any orthonormal basis of $\mathscr{H}_k$ with respect to the inner product (7.13). Once we have fixed a basis, we obtain an identification $\mathscr{H}_k\cong\mathscr{H}_k^*\cong\mathbb{C}^{d_k}$ and $\mathbb{P}\mathscr{H}_k^*\cong\mathbb{CP}^{d_k-1}$. Consider the commutative diagram.

$$\begin{array}{ccc} X\smallsetminus \mathrm{Bl}(\mathscr{H}_k) & \xrightarrow{\Phi_k} & \mathbb{P}\mathscr{H}_k^* \\ \big\downarrow \mathrm{Id} & & \big\downarrow \cong \\ X\smallsetminus \mathrm{Bl}(\mathscr{H}_k) & \xrightarrow{\widetilde{\Phi}_k} & \mathbb{CP}^{d_k-1} \end{array} \tag{7.95}$$

Then

$$\Phi_k^*(\omega_{FS})=\widetilde{\Phi}_k^*\Big(\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^{d_k}|w_j|^2\Big), \tag{7.96}$$

where $[w_1,\dots,w_{d_k}]$ are homogeneous coordinates in $\mathbb{CP}^{d_k-1}$. To describe $\widetilde{\Phi}_k$ in a neighborhood of a point $x_0\in X\smallsetminus\mathrm{Bl}(\mathscr{H}_k)$, we choose a local frame e_L of L and write $S^k_i=f^k_i e_L^{\otimes k}$ for some smooth functions f^k_i. Then

$$\widetilde{\Phi}_k(x)=[f^k_1(x);\dots;f^k_{d_k}(x)], \tag{7.97}$$

and this does not depend on the choice of the frame e_L.

(i) Let us choose an unit frame e_L of L. Then $|S^k_i|^2=|f^k_i|^2|e_L|^{2k}=|f^k_i|^2$, hence

$$B_{0,k}=\sum_{i=1}^{d_k}|S^k_i|^2=\sum_{i=1}^{d_k}|f^k_i|^2.$$

Since $b_{0,0}>0$, the asymptotic expansion (1.21) shows that $B_{0,k}$ does not vanish on X for k large enough, so the sections $\{S^k_i\}_{i=1}^{d_k}$ have no common zeroes. Therefore Φ_k and $\widetilde{\Phi}_k$ are defined on all X.

(ii) Let us fix $x_0\in X$. We identify a small geodesic ball $B^X(x_0,\varepsilon)$ to $B^{T_{x_0}X}(0,\varepsilon)$ by means of the exponential map and consider a trivialization of L as in Section 7.2.2, i.e. we trivialize L by using an unit frame $e_L(Z)$ which is parallel with respect to ∇^L along $[0,1]\ni u\to uZ$ for $Z\in B^{T_{x_0}X}(0,\varepsilon)$. We can express the Fubini–Study metric as

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_{j=1}^{d_k}|w_j|^2\Big)=\frac{\sqrt{-1}}{2\pi}\left[\frac{1}{|w|^2}\sum_{j=1}^{d_k}dw_j\wedge d\overline{w}_j-\frac{1}{|w|^4}\sum_{j,k=1}^{d_k}\overline{w}_jw_k\,dw_j\wedge d\overline{w}_k\right],$$

and therefore, from (7.97),

$$\Phi_k^*(\omega_{FS})(x_0) = \frac{\sqrt{-1}}{2\pi}\left[\frac{1}{|f^k|^2}\sum_{j=1}^{d_k} df_j^k \wedge d\overline{f_j^k} - \frac{1}{|f^k|^4}\sum_{j,k=1}^{d_k} \overline{f_j^k} f_k^k\, df_j^k \wedge d\overline{f_k^k}\right](x_0)$$

$$= \frac{\sqrt{-1}}{2\pi}\big[f^k(x_0,x_0)^{-1} d_x d_y f^k(x,y) - f^k(x_0,x_0)^{-2} d_x f^k(x,y)\wedge d_y f^k(x,y)\big]|_{x=y=x_0}, \quad (7.98)$$

where $f^k(x,y) = \sum_{i=1}^{d_k} f_i^k(x)\overline{f_i^k}(y)$ and $|f^k(x)|^2 = f^k(x,x)$. Since

$$P_{0,k}(x,y) = f^k(x,y) e_L^k(x)\otimes e_L^k(y)^*,$$

thus $P_{0,k}(x,y)$ is $f^k(x,y)$ under our trivialization of L. By (7.67), we obtain

$$\frac{1}{k}\,\Phi_k^*(\omega_{FS})(x_0) = \frac{\sqrt{-1}}{2\pi}\big[\frac{1}{F_{0,0}} d_x d_y F_{0,0} - \frac{1}{F_{0,0}^2} d_x F_{0,0}\wedge d_y F_{0,0}\big](0,0)$$
$$- \frac{\sqrt{-1}}{2\pi}\frac{1}{\sqrt{k}}\big[\frac{1}{F_{0,0}^2}(d_x F_{0,1}\wedge d_y F_{0,0} + d_x F_{0,0}\wedge d_y F_{0,1})\big](0,0) + \mathscr{O}\big(1/k\big).$$

Using again (7.76), (7.79), we obtain

$$\frac{1}{k}\,\Phi_k^*(\omega_{FS})(x_0) = \frac{\sqrt{-1}}{4\pi}\sum_{j=1}^{n} a_j dz_j\wedge d\overline{z}_j|_{x_0} + \mathscr{O}\big(k^{-1}\big) = \omega(x_0) + \mathscr{O}\big(1/k\big),$$

and the convergence takes place in the $\mathscr{C}^\infty$ topology with respect to $x_0\in X$.

(iii) Since X is compact, we have to prove two things for k sufficiently large: (a) Φ_k are immersions and (b) Φ_k are injective. We note that (a) follows immediately from (7.94).

To prove (b) let us assume the contrary, namely that there exists a sequence of distinct points $x_k\neq y_k$ such that $\Phi_k(x_k) = \Phi_k(y_k)$. Relation (7.95) implies that $\widetilde{\Phi}_k(x_k) = \widetilde{\phi}_k(y_k)$, where $\widetilde{\Phi}_k$ is defined by any particular choice of basis.

The key observation is that Theorem 7.21 ensures the existence of a sequence of *peak sections* at each point of X. The construction goes like follows. Let $x_0\in X$ be fixed. Since Φ_k is point base free for large k, we can consider the hyperplane $\Phi_k(x_0)$ of all sections of $\mathscr{H}_k$ vanishing at x_0. We construct then an orthonormal basis $\{S_i^k\}_{i=1}^{d_k}$ of $\mathscr{H}_k$ such that the first d_k-1 elements belong to $\Phi_k(x_0)$. Then $S_{d_k}^k$ is a unit norm generator of the orthogonal complement of $\Phi_k(x_0)$, and will be denoted by $S_{x_0}^k$. This is a peak section at x_0. We note first that $|S_{x_0}^k(x_0)|^2 = B_{0,k}(x_0)$ and $P_{0,k}(x,x_0) = S_{x_0}^k(x)\otimes S_{x_0}^k(x_0)^*$ and therefore

$$S_{x_0}^k(x) = \frac{1}{B_{0,k}(x_0)} P_{0,k}(x,x_0)\cdot S_{x_0}^k(x_0). \quad (7.99)$$

From (7.67) we deduce that for a sequence $\{r_k\}$ with $r_k\to 0$ and $r_k\sqrt{k}\to\infty$,

$$\int_{B(x_0,r_k)} |S_{x_0}^k(x)|^2\, dv_X(x) = 1 - \mathscr{O}(1/k)\,, \quad \text{for } k\to\infty. \quad (7.100)$$

Relation (7.100) explains the term ‘peak section’: when k grows, the mass of $S_{x_0}^k$ concentrates near x_0. Since $\Phi_k(x_k) = \Phi_k(y_k)$ we can construct as before the peak section $S_{x_k}^k = S_{y_k}^k$ as the unit norm generator of the orthogonal complement of $\Phi_k(x_k) = \Phi_k(y_k)$. We fix in the sequel such a section which peaks at both x_k and y_k.

We consider the distance $d(x_k,y_k)$ between the two points x_k and y_k. By passing to a subsequence we have two possibilities: either $\sqrt{k}d(x_k,y_k)\to\infty$ as $k\to\infty$ or there exists a constant $C>0$ such that $d(x_k,y_k)\leqslant C/\sqrt{k}$ for all k.

Assume that the first possibility is true. For large k, we learn from relation (7.100) that the mass of $S^k_{x_k} = S^k_{y_k}$ (which is 1) concentrates both in neighborhoods $B(x_k, r_k)$ and $B(y_k, r_k)$ with $r_p = d(x_k, y_k)/2$ and approaches therefore 2 if $k \to \infty$. This is a contradiction which rules out the first possibility.

To exclude the second possibility we follow [**28**]. We identify as usual $B^X(x_k, \varepsilon)$ to $B^{T_{x_k}X}(0, \varepsilon)$ so the point y_k gets identified to $Z_k/\sqrt{k}$ where $Z_k \in B^{T_{x_k}X}(0,C)$. We define then

$$f_k : [0,1] \longrightarrow \mathbb{R}\,, \quad f_k(t) = \frac{|S^k_{x_k}(tZ_k/\sqrt{k})|^2}{B_{0,k}(tZ_k/\sqrt{k})}\,. \tag{7.101}$$

We have $f_k(0) = f_k(1) = 1$ (again because $S^k_{x_k} = S^k_{y_k}$) and $f_k(t) \leqslant 1$ by the definition of the generalized Bergman kernel. We deduce the existence of a point $t_k \in (0,1)$ such that $f''_k(t_k) = 0$. Equations (7.67), (7.99), (7.101) imply the estimate

$$f_k(t) = e^{-\frac{t^2}{4}\sum_j a_j |z_{k,j}|^2}\big(1 + g_k(tZ_k)/\sqrt{k}\big) \tag{7.102}$$

and the $\mathscr{C}^2$ norm of g_k over $B^{T_{x_k}X}(0,C)$ is uniformly bounded in k. From (7.102), we infer that $|Z_k|_0^2 := \frac{1}{4}\sum_j a_j |z_{k,j}|^2 = \mathscr{O}(1/\sqrt{k})$. Using a limited expansion $e^x = 1 + x + x^2\varphi(x)$ for $x = t^2|Z_k|_0^2$ in (7.102) and taking derivatives, we obtain $f''_k(t) = -2|Z_k|_0^2 + \mathscr{O}(|Z_k|_0^4) + \mathscr{O}(|Z_k|_0^2/\sqrt{k}) = (-2 + \mathscr{O}(1/\sqrt{k}))|Z_k|_0^2$. Evaluating at t_k we get $0 = f''_k(t_k) = (-2 + \mathscr{O}(1/\sqrt{k}))|Z_k|_0^2$, which is a contradiction since by assumption $Z_k \neq 0$. This finishes the proof of (iii). □

7.30. REMARK. Let us point out complementary results which are analogues of [**10**, (1.3)–(1.5)] for the spaces $\mathscr{H}_k$. Computing as in (7.98) the pull-back $\Phi_k^* h_{FS}$ of the Hermitian metric $h_{FS} = g_{FS} - \sqrt{-1}\,\omega_{FS}$ on $\mathbb{P}\mathscr{H}_k^*$, we get the similar inequality to (7.94) for g_{FS} and $\omega(\cdot, J\cdot)$. Thus, Φ_k are asymptotically symplectic and isometric. Moreover, arguing as in [**10**, Proposition 4.4] we can show that Φ_k are 'nearly holomorphic' :

$$\frac{1}{k}\,\|\partial\Phi_k\| = 1 + \mathscr{O}(1/k)\,, \quad \frac{1}{k}\,\|\overline{\partial}\Phi_k\| = \mathscr{O}(1/k)\,, \tag{7.103}$$

uniformly on X, where $\|\cdot\|$ is the operator norm.

7.3.4. Holomorphic case revisited. In this Section, we assume that (X, ω, J) is Kähler and the vector bundles E, L are holomorphic on X, and ∇^E, ∇^L are the holomorphic Hermitian connections on (E, h^E), (L, h^L), moreover, $\frac{\sqrt{-1}}{2\pi}R^L = \omega$. Let g^{TX} be any Riemannian metric on TX compatible with J. But we assume that $\mathbf{J} \neq J$ in (7.2). Set

$$\Theta(X,Y) = g^{TX}(JX, Y). \tag{7.104}$$

Then the 2-form Θ need not to be closed (the convention here is different to [**3**, (2.1)] by a factor -1). We denote by $T^{(1,0)}X$, $T^{(0,1)}X$ the holomorphic and anti-holomorphic tangent bundles as in Section 7.2. Let $\{e_i\}$ be an orthonormal frame of (TX, g^{TX}).

Let $\overline{\partial}^{E_k *}$ be the formal adjoint of the Cauchy-Riemann operator $\overline{\partial}^{E_k}$ on the Dolbeault complex $\Omega^{0,\bullet}(X, L^k \otimes E)$ with the scalar product induced by g^{TX}, h^L, h^E as in (7.13). Set $D_k = \sqrt{2}(\overline{\partial}^{E_k} + \overline{\partial}^{E_k *})$. Then $D_k^2 = 2(\overline{\partial}^{E_k}\overline{\partial}^{E_k *} + \overline{\partial}^{E_k *}\overline{\partial}^{E_k})$ preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, L^k \otimes E)$. Then for k big enough,

$$\ker D_k = \ker D_k^2 = H^0(X, L^k \otimes E). \tag{7.105}$$

Here D_k is not a spin^c Dirac operator on $\Omega^{0,\bullet}(X,L^k\otimes E)$, and D_p^2 is not a renormalized Bochner–Laplacian as in (7.6). Now we explain how to put it in the frame of our work.

7.31. THEOREM. *The smooth kernel of the orthogonal projection from* $\mathscr{C}^\infty(X,L^k\otimes E)$ *on* $\ker D_k^2$, *has a full off–diagonal asymptotic expansion analogous to* (7.92) *with* $\mathbf{j}_0 = \det_{\mathbb{C}}\mathbf{J}$ *as* $k\to\infty$.

PROOF. As pointed out in 7.8, by [**6**, Theorem 1], there exist $\mu_0, C_L>0$ such that for any $k\in\mathbb{N}$ and any $s\in\Omega^{>0}(X,L^k\otimes E) := \bigoplus_{q\geqslant 1}\Omega^{0,q}(X,L^k\otimes E)$,

$$\|D_k s\|_{L^2}^2 \geqslant (2k\mu_0 - C_L)\|s\|_{L^2}^2. \tag{7.106}$$

Moreover $\operatorname{Spec} D_k^2\subset\{0\}\cup[2k\mu_0-C_L,+\infty[$.

Let S^{-B} denote the 1-form with values in antisymmetric elements of $\operatorname{End}(TX)$ which is such that if $U,V,W\in TX$,

$$\langle S^{-B}(U)V,W\rangle = -\frac{\sqrt{-1}}{2}\Big((\partial-\overline{\partial})\Theta\Big)(U,V,W). \tag{7.107}$$

The Bismut connection ∇^{-B} on TX is defined by

$$\nabla^{-B} = \nabla^{TX} + S^{-B}. \tag{7.108}$$

Then by [**4**, Prop. 2.5], ∇^{-B} preserves the metric g^{TX} and the complex structure of TX. Let $\nabla^{\det}$ be the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$ with its curvature $R^{\det}$. Then these two connections induce naturally an unique connection on $\Lambda(T^{*(0,1)}X)$, and with the connections ∇^L,∇^E, we get a connection ∇^{-B,E_k} on $\Lambda(T^{*(0,1)}X)\otimes L^k\otimes E$. Let Δ^{-B,E_k} be the Laplacian on $\Lambda(T^{*(0,1)}X)\otimes L^k\otimes E$ induced by ∇^{-B,E_p} as in (7.3). For any $v\in TX$ with decomposition $v=v_{1,0}+v_{0,1}\in T^{(1,0)}X\oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0}\in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then $c(v)=\sqrt{2}(\overline{v}^*_{1,0}\wedge - i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior product respectively. We define a map ${}^c:\Lambda(T^*X)\to C(TX)$, the Clifford bundle of TX, by sending $e^{i_1}\wedge\cdots\wedge e^{i_j}$ to $c(e_{i_1})\cdots c(e_{i_j})$ for $i_1<\cdots<i_j$. Then we can formulate [**4**, Theorem 2.3] as following,

$$D_k^2 = \Delta^{-B,E_k} + \frac{r^X}{4} + {}^c(R^E + kR^L + \frac{1}{2}R^{\det}) + \frac{\sqrt{-1}}{2}{}^c(\overline{\partial}\partial\Theta) - \frac{1}{8}|(\partial-\overline{\partial})\Theta|^2. \tag{7.109}$$

We use now the connection ∇^{-B,E_k} instead of ∇^{E_k} in [**18**, §2]. Then by (7.106), (7.109), everything goes through perfectly well and as in [**18**, Theorem 3.18], so we can directly apply the result in [**18**] to get the *full off-diagonal* asymptotic expansion of the Bergman kernel. As the above construction preserves the $\mathbb{Z}$-grading on $\Omega^{0,\bullet}(X,L^k\otimes E)$, we can also directly work on $\mathscr{C}^\infty(X,L^k\otimes E)$. □

7.3.5. Generalizations to non-compact manifolds. Let (X,Θ) be a Kähler manifold and (L,h^L) be a holomorphic Hermitian line bundle over X. As in Section 7.3.4, let $R^L,R^{\det}$ be the curvatures of the holomorphic Hermitian connections $\nabla^L,\nabla^{\det}$ on L, $\det(T^{(1,0)}X)$, and let $J^L\in\operatorname{End}(TX)$ such that $\frac{\sqrt{-1}}{2\pi}R^L(\cdot,\cdot)=\Theta(J^L\cdot,\cdot)$. The space of holomorphic sections of L^k which are L^2 with respect to the norm given by (7.13) is denoted by $H^0_{(2)}(X,L^k)$. Choose an orthonormal basis $(S_i^k)_{i\geqslant 1}$ of $H^0_{(2)}(X,L^k)$. For each local holomorphic frame e_L we have $S_i^k = f_i^k e_L^{\otimes k}$ for some local holomorphic functions f_i^k. As shown in [**33**, Lemma 4.1] the series $\sum_{i\geqslant 1}|f_i^k(x)|^2$ converges uniformly on compact sets (together with all its derivatives) to a smooth function. It follows that $B_k(x)=\sum_{i\geqslant 1}|S_i^k(x)|^2 =$

$\sum_{i\geqslant 1}|f_i^k(x)|^2|e_L^{\otimes k}|^2$ is also a well defined smooth function. We have the following generalization of Theorem 1.21.

7.32. THEOREM. *Assume that* (X,Θ) *is a complete Kähler manifold. Suppose that there exist* $\varepsilon>0\,,C>0$ *such that one of the following assumptions holds true*:

$$\sqrt{-1}R^L>\varepsilon\Theta,\ \sqrt{-1}R^{\det}>-C\Theta. \tag{7.110}$$

$$L=\det(T^{*(1,0)}X),\ h^L \textit{ is induced by } \Theta \textit{ and } \sqrt{-1}R^{\det}<-\varepsilon\Theta. \tag{7.111}$$

Then there exist coefficients $b_r\in\mathscr{C}^\infty(X)\,,r\in\mathbb{N}$, *with* $b_0=(\det J^L)^{1/2}$ *such that for any compact set* $K\subset X$, *any* $m,l\in\mathbb{N}$, *there exists* $C_{k,l,K}>0$ *such that for* $k\in\mathbb{N}$,

$$\Big|\frac{1}{k^n}B_k(x)-\sum_{r=0}^{k}b_r(x)k^{-r}\Big|_{\mathscr{C}^l(K)}\leqslant C_{m,l,K}\,k^{-m-1}. \tag{7.112}$$

PROOF. By the argument in Section 7.2, if the Kodaira–Laplacian $\square^{L^k}=\frac12\Delta_k:=\frac12\Delta_{k,0}$ acting on sections of L^k has a spectral gap as in (7.8), then we can localize the problem, and we get directly (7.112). Observe that $D_k^2|_{\Omega^{0,\bullet}}=\Delta_k$. In general, on a non-compact manifold, we define a self-adjoint extension of D_k^2 by

$$\operatorname{Dom} D_k^2=\big\{u\in\operatorname{Dom}\overline{\partial}^{L^k}\cap\operatorname{Dom}\overline{\partial}^{L^k*}:\overline{\partial}^{L^k}u\in\operatorname{Dom}\overline{\partial}^{L^k*},\overline{\partial}^{L^k*}u\in\operatorname{Dom}\overline{\partial}^{L^k}\big\},$$
$$D_k^2u=2(\overline{\partial}^{L^k}\overline{\partial}^{L^k*}+\overline{\partial}^{L^k*}\overline{\partial}^{L^k})u,\quad \text{for } u\in\operatorname{Dom}D_p^2.$$

The quadratic form associated to D_k^2 is the form Q_k given by

$$\begin{aligned}\operatorname{Dom} Q_k&:=\operatorname{Dom}\overline{\partial}^{L^k}\cap\operatorname{Dom}\overline{\partial}^{L^k*}\\ Q_k(u,v)&=2\big\langle\overline{\partial}^{L^k}u,\overline{\partial}^{L^k}v\big\rangle+2\big\langle\overline{\partial}^{L^k*}u,\overline{\partial}^{L^k*}v\big\rangle,\quad u,v\in\operatorname{Dom}Q_k.\end{aligned} \tag{7.113}$$

In the previous formulas $\overline{\partial}^{L^k}$ is the weak maximal extension of $\overline{\partial}^{L^k}$ to L^2 forms and $\overline{\partial}^{L^k*}$ is its Hilbert space adjoint. We denote by $\Omega_0^{0,\bullet}(X,L^k)$ the space of smooth compactly supported forms and by $L_2^{0,\bullet}(X,L^k)$ the corresponding L^2-completion.

Under one of the hypotheses (7.110) or (7.111) there exists $\mu>0$ such that for k large enough

$$Q_k(u)\geqslant\mu k\|u\|^2,\quad u\in\operatorname{Dom}Q_k\cap L_2^{0,q}(X,L^k)\ \text{for } q>0. \tag{7.114}$$

Indeed, the estimate holds for $u\in\Omega_0^{0,q}(X,L^k)$ since the Bochner-Kodaira formula [**2**, Prop. 3.71] reduces to $Q_k(u)\geqslant 2\big\langle(kR^L+R^{\det})(w_i,\overline{w}_j)\overline{w}^j\wedge i_{\overline{w}_i}u\,,u\big\rangle$, for $u\in\Omega_0^{0,q}(X,L^k)$, where $\{w_i\}$ is an orthonormal frame of $T^{(1,0)}X$. But this implies (7.114) in general, since $\Omega_0^{0,\bullet}(X,L^k)$ is dense in $\operatorname{Dom}Q_k$ with respect to the graph norm, as the metric is complete.

Next, consider $f\in\operatorname{Dom}\Delta_k\cap L_2^{0,0}(X,L^k)$ and set $u=\overline{\partial}^{L^k}f$. It follows from the definition of the Laplacian and (7.114) that

$$\|\Delta_k f\|^2=2\big\langle\overline{\partial}^{L^k*}u,\overline{\partial}^{L^k*}u\big\rangle=Q_k(u)\geqslant\mu k\|u\|^2=\mu k\big\langle\Delta_k f\,,f\big\rangle. \tag{7.115}$$

This clearly implies $\operatorname{Spec}(\Delta_k)\subset\{0\}\cup[k\mu,\infty[$ for large k. ☐

Theorem 7.32 permits an immediate generalization of Tian's convergence theorem. Tian [**33**, Theorem 4.1] already proved a non–compact version for convergence in the $\mathscr{C}^2$ topology and convergence rate $1/\sqrt{k}$. Another easy consequence are holomorphic Morse inequalities for the space $H^0_{(2)}(X,L^k)$.

Observe that the quantity $\sum_{i\geqslant 1}|f_i^k(x)|^2$ is not globally defined, but the current

$$\omega_k = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_{i\geqslant 1}|f_i^k(x)|^2\Big) \tag{7.116}$$

is well defined globally on X. Indeed, since $R^L = -\partial\overline{\partial}\log|e_L|^2_{h^L}$ we have

$$\frac{1}{k}\omega_k - \frac{\sqrt{-1}}{2\pi}R^L = \frac{\sqrt{-1}}{2\pi k}\partial\overline{\partial}\log B_k\,. \tag{7.117}$$

If $\dim H^0_{(2)}(X,L^k)<\infty$ we have by (7.93) that $\omega_k=\Phi_k^*(\omega_{FS})$ where Φ_k is defined as in (7.93) with $\mathscr{H}_k$ replaced by $H^0_{(2)}(X,L^k)$.

7.33. COROLLARY. *Assume one of the hypotheses* (7.110) *or* (7.111) *holds. Then*:

(a) *for any compact set $K\subset X$ the restriction $\omega_k|_K$ is a smooth $(1,1)$-form for sufficiently large k; moreover, for any $l\in\mathbb{N}$ there exists a constant $C_{l,K}$ such that*

$$\Big|\frac{1}{k}\omega_k - \frac{\sqrt{-1}}{2\pi}R^L\Big|_{\mathscr{C}^l(K)} \leqslant \frac{C_{l,K}}{k}\,;$$

(b) *the Morse inequalities hold in bidegree $(0,0)$*:

$$\liminf_{k\longrightarrow\infty} k^{-n}\dim H^0_{(2)}(X,L^k) \geqslant \frac{1}{n!}\int_X\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n. \tag{7.118}$$

In particular, if $\dim H^0_{(2)}(X,L^k)<\infty$, the manifold (X,Θ) has finite volume.

PROOF. Due to (7.112), B_k doesn't vanish on any given compact set K for k sufficiently large. Thus, (a) is a consequence of (7.112) and (7.117).

Part (b) follows from Fatou's lemma, applied on X with the measure

$\Theta^n/n!$ to the sequence $k^{-n}B_k$ which converges pointwise to $(\det J^L)^{1/2}=\big(\frac{\sqrt{-1}}{2\pi}R^L\big)^n/\Theta^n$ on X. □

The inequality (7.118) was also obtained in (4.39). Under hypothesis (7.111) it represents Theorem 4.31.

Another generalization is a version of Theorem 1.21 for covering manifolds. We retain the notaitons from 7.1.4. It is shown in Corollary 7.14 that the spectrum of the renormalized Bochner-Laplacian satisfies

$$\operatorname{Spec}\widetilde{\Delta}_{k,\widetilde{\Phi}} \subset [-C_L,C_L]\cup[2k\mu_0-C_L,+\infty[\,, \tag{7.119}$$

where C_L is the same constant as in Section 7.1.1 and μ_0 is introduced in (7.5). Let $\widetilde{\mathscr{H}_k}$ be the eigenspace of $\widetilde{\Delta}_{k,\widetilde{\Phi}}$ with the eigenvalues in $[-C_L,C_L]$:

$$\widetilde{\mathscr{H}_k} = \operatorname{Range}E\big([-C_L,C_L],\widetilde{\Delta}_{k,\widetilde{\Phi}}\big)\,, \tag{7.120}$$

where $E(\cdot,\widetilde{\Delta}_{k,\widetilde{\Phi}})$ is the spectral measure of $\widetilde{\Delta}_{k,\widetilde{\Phi}}$. From Corollary 7.14, the von Neumann dimension of $\widetilde{\mathscr{H}_k}$ equals $d_k=\dim\mathscr{H}_k$. Finally, we define the generalized Bergman kernel $\widetilde{P}_{q,k}$ of $\widetilde{\Delta}_{k,\widetilde{\Phi}}$ as in Definition 7.15. Unlike most of the objects on $\widetilde{X}$, $\widetilde{P}_{q,k}$ is not Γ–invariant.

By (7.119) and the proof of Proposition 7.16, the analogue of (7.52) still holds on any compact set $K\subset\widetilde{X}$. By the finite propagation speed as the end of Section 7.2.1, we have:

7.34. THEOREM. *We fix* $0<\varepsilon_0<\inf_{x\in X}\{$*injectivity radius of* $x\}$. *For any compact set* $K\subset\widetilde{X}$ *and* $m,l\in\mathbb{N}$, *there exists* $C_{m,l,K}>0$ *such that for* $x,x'\in K$, $p\in\mathbb{N}$,

$$\begin{aligned}&\Big|\frac{1}{k^n}\widetilde{P}_{q,k}(x,x')-P_{q,k}(\pi_\Gamma(x),\pi_\Gamma(x'))\Big|_{\mathscr{C}^l(K\times K)}\leqslant C_{m,l,K}\,k^{-m-1}, \quad \text{if } d(x,x')<\varepsilon_0,\\ &\Big|\frac{1}{k^n}\widetilde{P}_{q,k}(x,x')\Big|_{\mathscr{C}^l(K\times K)}\leqslant C_{m,l,K}\,k^{-m-1}, \quad \text{if } d(x,x')\geqslant\varepsilon_0.\end{aligned} \tag{7.121}$$

Especially, $\widetilde{P}_{q,k}(x,x)$ *has the same asymptotic expansion as* $B_{q,k}(\pi_\Gamma(x))$ *in Theorem 1.21 on any compact set* $K\subset\widetilde{X}$.

7.35. REMARK. Theorem 7.34 works well for coverings of non-compact manifolds. Let $(X,\widetilde{\Theta})$ be a complete Kähler manifold, $(\widetilde{L},\widetilde{h}^L)$ be a holomorphic line bundle on X and let $\pi_\Gamma:\widetilde{X}\to X$ be a Galois covering of $X=\widetilde{X}/\Gamma$. Let $\widetilde{\Theta}$ and $(\widetilde{L},h^{\widetilde{L}})$ be the inverse images of Θ and (L,h^L) through π_Γ. If (X,Θ) and (L,h^L) satisfy one of the conditions (7.110) or (7.111), $(\widetilde{X},\widetilde{\Theta})$ and $(\widetilde{L},h^{\widetilde{L}})$ have the same properties. We obtain therefore as in (7.118) (by integrating over a fundamental domain):

7.36. COROLLARY. *Assume one of the hypotheses 7.110 or 7.111 holds. Then*

$$\liminf_{k\longrightarrow\infty}k^{-n}\dim_\Gamma H^0_{(2)}(\widetilde{X},\widetilde{L}^k)\geqslant\frac{1}{n!}\int_X\Big(\frac{\sqrt{-1}}{2\pi}R^L\Big)^n. \tag{7.122}$$

where $\dim_\Gamma$ *is the von Neumann dimension of the* Γ*–module* $H^0_{(2)}(X,L^k)$.

Note that in Theorem 5.16 we obtain Morse inequalities on covering manifolds for $(n,0)$–forms.

7.3.6. Singular polarizations. Let X be a compact complex manifold. A *singular Kähler metric* on X is a closed, strictly positive $(1,1)$-current ω.

If the cohomology class of ω in $H^2(X,\mathbb{R})$ is integral, there exists a holomorphic line bundle (L,h^L), endowed with a singular Hermitian metric, such that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$ in the sense of currents. We call (L,h^L) a *singular polarization* of ω. If we change the metric h^L, the curvature of the new metric will be in the same cohomology class as ω. In this case we speak of a polarization of $[\omega]\in H^2(X,\mathbb{R})$. Our purpose is to define an appropriate notion of polarized section of L^k, possibly by changing the metric of L, and study the associated Bergman kernel.

First recall that a Hermitian metric h^L is called *singular* if it is given in local trivialization by functions $e^{-\varphi}$ with $\varphi\in L^1_{\rm loc}$. The curvature current R^L of h^L is well defined and given locally by the currents $\partial\overline{\partial}\varphi$.

By the approximation theorem of Demailly [**20**, Theorem 1.1], we can assume that h^L is smooth outside a proper analytic set $\Sigma\subset X$. Let $\pi:\widetilde{X}\longrightarrow X$ be a resolution of singularities such that $\pi:\widetilde{X}\smallsetminus\pi^{-1}(\Sigma)\longrightarrow X\smallsetminus\Sigma$ is biholomorphic and $\pi^{-1}(\Sigma)$ is a divisor with only simple normal crossings. Let $g_0^{T\widetilde{X}}$ be an arbitrary smooth J-invariant metric on $\widetilde{X}$ and $\Theta'(\cdot,\cdot)=g_0^{T\widetilde{X}}(J\cdot,\cdot)$ the corresponding $(1,1)$-from. The generalized Poincaré metric on $X\smallsetminus\Sigma=\widetilde{X}\smallsetminus\pi^{-1}(\Sigma)$ is defined in (4.55) by

$$\Theta_{\varepsilon_0}=\Theta'-\varepsilon_0\sqrt{-1}\sum_i\partial\overline{\partial}\log(-\log\|\sigma_i\|_i^2)^2, \quad 0<\varepsilon_0\ll 1 \text{ fixed}, \tag{7.123}$$

where $\pi^{-1}(\Sigma)=\cup_i\Sigma_i$ is the decomposition into irreducible components Σ_i of $\pi^{-1}(\Sigma)$ and each Σ_i is non-singular; σ_i are sections of the associated holomorphic line bundle $[\Sigma_i]$ which vanish to first order on Σ_i, and $\|\sigma_i\|_i$ is the norm for a smooth Hermitian metric on $[\Sigma_i]$ such that $\|\sigma_i\|_i < 1$.

We can construct as in the proof of Theorem 4.40 (cf. [**30**, §4]) a singular Hermitian line bundle $(\widetilde{L}, h^{\widetilde{L}})$ on $\widetilde{X}$ which is strictly positive and $\widetilde{L}|_{\widetilde{X}\smallsetminus\pi^{-1}(\Sigma)} \cong \pi^*(L^{k_0})$, for some $k_0 \in \mathbb{N}$. We introduce on $L|_{X\smallsetminus\Sigma}$ the metric $(h^{\widetilde{L}})^{1/k_0}$ whose curvature extends to a strictly positive $(1,1)$–current on $\widetilde{X}$. Set

$$h^L_\varepsilon = (h^{\widetilde{L}})^{1/k_0} \prod_i (-\log\|\sigma_i\|_i^2))^\varepsilon, \quad 0 < \varepsilon \ll 1, \tag{7.124}$$

$$H^0_{(2)}(X\smallsetminus\Sigma, L^k) = \big\{u \in L^{0,0}_2(X\smallsetminus\Sigma, L^k, \Theta_{\varepsilon_0}, h^L_\varepsilon) : \overline{\partial}^{L^k} u = 0\big\}. \tag{7.125}$$

The space $H^0_{(2)}(X\smallsetminus\Sigma, L^k)$ is the space of L^2-holomorphic sections relative to the metrics Θ_{ε_0} on $X\smallsetminus\Sigma$ and h^L_ε on $L|_{X\smallsetminus\Sigma}$. Since $(h^{\widetilde{L}})^{1/k_0}$ is bounded away from zero (having plurisubharmonic weights), the L^2 condition with respect to the Poincaré metric imply that the elements of $H^0_{(2)}(X\smallsetminus\Sigma, L^k)$ extend holomorphically to sections of L^k over X. The subspace $H^0_{(2)}(X\smallsetminus\Sigma, L^k) \subset H^0(X, L^k)$ is the space of polarized sections of L^k.

7.37. COROLLARY. *Let (X,ω) be a compact complex manifold with a singular Kähler metric with integral cohomology class. Let (L,h^L) be a singular polarization of $[\omega]$ with strictly positive curvature current having singular support along a proper analytic set Σ. Then the Bergman kernel of the space of polarized sections* (7.125) *has the asymptotic expansion* (7.112).

PROOF. We first remark that by the localization argument in Section 7.2.1, Theorem 7.31 has a noncompact version analogous to Theorem 7.32, provided we can prove the existence of the spectral gap of the Kodaira-Laplacian $\square^{L^k}$. We will show that this is the case for the non–Kähler Hermitian manifold $(X\smallsetminus\Sigma, \Theta_{\varepsilon_0})$ equipped with the Hermitian bundle $(L|_{X\smallsetminus\Sigma}, h^L_\varepsilon)$. By applying the generalized Bochner-Kodaira-Nakano formula of Demailly [**19**, Theorem 0.3] as in [**6**, Theorem 1], we see that the existence of the spectral gap follows if we show that there exists constants $\eta > 0, C > 0$ such that

$$\sqrt{-1}R^{(L|_{X\smallsetminus\Sigma}, h^L_\varepsilon)} > \eta\Theta_{\varepsilon_0}, \ \sqrt{-1}R^{\det} > -C\Theta_{\varepsilon_0}, \ |T_{\varepsilon_0}| < C. \tag{7.126}$$

where $T_{\varepsilon_0} = [\Theta_{\varepsilon_0}, \partial\Theta_{\varepsilon_0}]$ is the torsion operator of Θ_{ε_0} and $|T_{\varepsilon_0}|$ is its norm with respect to Θ_{ε_0}. But (7.126) follows from Proposition 4.38. □

7.38. REMARK. (a) Corollary 7.37 gives an alternative proof of the characterization of Moishezon manifolds given by the Shiffman-Ji-Bonavero Criterion (cf. also Takayama [**30**]), discussed in detail in Section 4.5. Indeed, any Moishezon manifold possesses a strictly positive singular polarization (L,h^L). Conversely, Corollary 7.37 entails a weaker form of Theorem 4.39 where we suppose that the curvature $\sqrt{-1}R^L$ is positive on the whole manifold X. As in Section 4.5 this implies Corollarry 4.43.

(b) The results of this section hold also for reduced compact complex spaces X possessing a holomorphic line bundle L with singular Hermitian metric h^L having positive curvature current. This is just a matter of desingularizing X. As space of polarized sections we obtain $H^0_{(2)}(X\smallsetminus\Sigma, L^k)$ where Σ is an analytic set containing the singular set of X. We obtain thus a new proof of of a particular case of Theorem 4.40.

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APPENDIX A

Elliptic differential operators

The purpose of this Appendix is to collect some basic facts about the theory of differential operators on manifolds in the form they are used in the text. General references for the subject are Hörmander [6, 8], Narasimhan [10] and Taylor [13, 14].

A.1. Functional spaces

A.1.1. Basic notations. Let M be a differentiable manifold and E be a vector bundle over M. Let U be an open set of M. We denote by $\Omega(U,E)$ the space of smooth sections of E over U. The correspondence $U \longrightarrow \Omega(U,E)$ defines a sheaf denoted $\Omega(E)$. If $E = \Lambda^l T^*M$ and F is a vector bundle over M we set $\Omega(U,\Lambda^l T^*M \otimes F) = \Omega^l(U,F)$. If F is trivial we omit it from the notation. The space of smooth sections with compact support in U is denoted by $\Omega_0(U,F)$. Accordingly we have the notation $\Omega_0^l(U,F)$. We keep the traditional $\Omega^0(U,\mathbb{C}) = \mathscr{C}^\infty(U)$ and $\Omega_0^0(U) = \mathscr{C}_0^\infty(U)$.

A.1.2. The canonical measure on a riemannian manifold. Let M be a riemannian manifold, endowed with a riemannian metric $g = g^{TM}$. There exists a unique positive measure on M determined by g, $v_M : \mathscr{C}_0(M) \longrightarrow \mathbb{R}$ so that for any chart $(U,x^1,\ldots,x^n)$

$$\int \varphi\, dv_M = \int_{\mathbb{R}^n} \varphi(x^1,\ldots,x^n)\sqrt{|\det(g_{ij})|}\, dx^1 \ldots dx^n \tag{A.1}$$

for all $\varphi \in \mathscr{C}_0(M)$, $\operatorname{supp}\varphi \subset U$. The assertion is proved as follows. For any chart $(U,x^1,\ldots,x^n)$ we define $v_{U,M} : \mathscr{C}_0(U) \longrightarrow \mathbb{R}$ by the formula (A.1). The change of variable formula shows that for another chart $(W,y^1,\ldots,y^n)$ the functionals $v_{U,M}$ and $v_{W,M}$ are equal on $\mathscr{C}_0(U\cap W)$. By using a partition of unity we show the existence of a functional v_M on $\mathscr{C}_0(M)$, whose restriction to $\mathscr{C}_0(U)$ is $v_{U,M}$ for any chart $(U,x^1,\ldots,x^n)$. The uniqueness is clear.

If M is orientable, the canonical measure is given by integration against the volume form determined by g. This is the unique n-form ω on M such that for any positively oriented chart $(U,x^1,\ldots,x^n)$, $\omega\restriction_U = \sqrt{\det(g_{ij})}\, dx^1 \wedge \ldots \wedge dx^n$. In this case we denote $\omega = dv_M$.

A.1.3. The L_2 spaces. Let $E \longrightarrow M$ be a vector bundle, endowed with a hermitian metric. We introduce a global scalar product by

$$(\alpha,\beta) = \int_M \langle \alpha,\beta\rangle\, dv_M\,, \quad \alpha,\beta \in \Omega_0(M,E). \tag{A.2}$$

Let $L_2(M,E)$ be the completion of $\Omega_0(M,E)$ under this scalar product. In the case of $E = \Lambda^l T^*M$ we denote $L_2(M,\Lambda^l T^*M)$ by $L_2^l(M)$. We also introduce the space $L_2(M,E,\mathrm{loc})$ of locally L_2-integrable sections. It is a Fréchet space, with seminorms given by integration on compact sets.

A.1.4. The topology of $\Omega(M,E)$ and $\Omega_0(M,E)$. There exists a unique Frechet space topology on $\Omega(M,E)$ such that the linear one–to–one map $\Omega(M,E) \to \prod_{x\in M} E_x$ is continuous. Moreover, the functionals $f \mapsto \|f\|_{K,\nu} := \sum_{|\alpha|\leqslant\nu} \sup|D^\alpha f|$, K compact set in a coordinate chart $(U,x^1,\ldots,x^n)$, $\nu \in \mathbb{N}$, form a fundamental system of continuous seminorms.

For a compact set $K \subset M$ we denote by $\Omega_K(M,E)$ the subspace of sections with compact support in K. We endow $\Omega_0(M,E)$ with the inductive limit topology of the spaces $\Omega_{K_\nu}(M,E)$, $\nu \in \mathbb{N}$ where $\{K_\nu\}_\nu$ is an exhaustion with compact sets of M.

A.1.5. Currents on manifolds. Let M be a manifold and $E \to M$ a vector bundle. The space of continuous linear functionals on $\Omega_0(M,E)$ is denoted by $\Omega_0'(M,E)$ and the space of continuous linear functionals on $\Omega(M,E)$ is denoted by $\Omega'(M,E)$. Of course, if M is compact, $\Omega'(M,E) = \Omega_0'(M,E)$. If M is endowed with a riemannian metric g and E is hermitian, $L_2(M,E,\mathrm{loc})$ is embedded in $\Omega_0'(M,E)$ by

$$\varphi \mapsto (\alpha,\varphi) = \int \langle \alpha,\varphi\rangle dv_M,\ \varphi \in \Omega_0(M,E),\ \alpha \in L_2(M,E,\mathrm{loc})$$

If $T \in \Omega_0'(M,E)$ we can define its restriction $T|_U \in \Omega_0'(U,E)$ to an open set, as well as the multiplication with a smooth function. $\Omega_0'(M,E)$ is a sheaf of $\mathscr{C}^\infty$–modules, where $\mathscr{C}^\infty$ is the sheaf of smooth functions on M.

Let $(U,x^1,\dots,x^n)$ be a coordinate system such that $E|_U$ is trivial. Let $(e_1,\dots,e_r)$ be a frame in $E|_U$. If $T \in \Omega_0'(M,E)$, $T|_U$ has a unique representation

$$T|_U = T_1 e_1 + \dots + T_r e_r \tag{A.3}$$

with distributions $T_j \in \Omega_0'(U)$. Thus, there exists a sheaf isomorphism

$$\Omega_0'(M,E) \cong \Omega(E) \otimes_{\mathscr{C}^\infty} \Omega_0'(M)\,,$$

where $\Omega(E)$ is the sheaf of smooth sections in E. If we denote by $\widehat{U}$ the image of U in $\mathbb{R}^n$ we define the distributions $T_j(x^1,\dots,x^n)$ on $\widehat{U} \subset \mathbb{R}^n$ by composing with the chart diffeomorphism. We identify in this way $T|_U$ with a vector of r distributions on $\widehat{U}$.

A.1.6. Sobolev spaces. More generally, we need to introduce Sobolev norms on sections of E.

A current $T \in \Omega_0'(M,E)$ is said to be in the Sobolev space $W_s(M,E;\mathrm{loc})$ if for any coordinate system $(U,x^1,\dots,x^n)$ such that $E|_U$ is trivial, the distributions $T_j(x^1,\dots,x^n) \in W_s(\widehat{U},\mathrm{loc})$, where T_j are defined by (A.3). The topology on $W_s(M,E;\mathrm{loc})$ is defined by the seminorms

$$T \longrightarrow \sum_{i=1}^r \|\varphi T_j(x^1,\dots,x^n)\|_s,\ \varphi \in \mathscr{C}_0^\infty(M),\ \mathrm{supp} \subset U$$

The definition is correct, due to the diffeomorphism invariance of Sobolev spaces, [**6**, Theorem 2.6.1]. We have then the following result on the regularity properties of Sobolev spaces. Let $\Omega_k(M,E)$ represent the space of sections of E of class $\mathscr{C}^k$.

A.1. THEOREM (Sobolev embedding). *For $s > n/2$, $n = \dim M$, we have a continuous injection $W_{s+k}(M,E;\mathrm{loc}) \subset \Omega_k(M,E)$.*

We need also the following Sobolev spaces. Let K be a compact set in M. Assume that E is hermitian. We set

$$W_s(K,E) = \{T \in W_s(M,E;\mathrm{loc}) : \mathrm{supp}\, T \subset K\}.$$

If M is a compact manifold, $W_s(M,E;\mathrm{loc}) = W_s(M,E)$. The induced topology on $W_s(K,E)$ can be defined by a hermitian scalar product which makes it a Hilbert space. Let $\{U_\nu\}$ be a finite open covering of K and $\{\varphi_\nu\}$ be a subordinated smooth partition of unity. We assume that U_ν are coordinate charts and $E|_{U_\nu}$ are trivial. On each U_ν we choose an orthonormal frame $(e_1^\nu,\dots,e_r^\nu)$. We have the representation $T|_{U_\nu} = T_1^\nu e_1^\nu + \dots + T_r^\nu e_r^\nu$, where T_j^ν can be thought as distributions on $\widehat{U}_\nu \subset \mathbb{R}^n$. We set

$$\|T\|_s = \Big(\sum_\nu \sum_{j=1}^r \|\varphi_\nu T_j^\nu\|_s^2\Big)^{1/2}.$$

A.2. RELLICH'S LEMMA ([**13**, Proposition 4.4]). *The inclusion $W_s(K,E) \hookrightarrow W_t(K,E)$ is compact for $s > t \geqslant 0$.*

Let us define

$$W_{s,0}(K,E) = \text{the closure of}\, \Omega_0(K,E)\, \text{in}\, W_s(K,E)$$

If $s \in \mathbb{Z}_+$, $W_{s,0}(K,E) = W_s(K,E)$ [**13**, p. 291]. As before, we set $W_m^l(M) = W_m(M,\Lambda^l T^*M)$.

A.1.7. Differential operators. We refer to Narasimhan [**10**] for elementary properties of differential operators $P:\Omega(M,E)\longrightarrow\Omega(M,F)$ acting between the (sheaves of) sections of two vector bundles E, F on M.

Let $P:\Omega(M,E)\longrightarrow\Omega(M,F)$ be a differential operator acting between sections of two hermitian vector bundles. The *formal adjoint* P^t of P is a differential operator $P^t:\Omega(M,F)\longrightarrow\Omega(M,E)$ of the same order as P, satisfying

$$(P\alpha,\beta)=(\alpha,P^t\beta),\quad \alpha\in\Omega_0(M,E),\beta\in\Omega(M,F).$$

We use the notation P^t in order to distinguish the formal adjoint P^t from the Hilbert space adjoint P^*. Note that P is symmetric if and only if $P=P^t$.

We extend the operator P to an operator

$$P:\Omega_0'(M,E)\longrightarrow\Omega_0'(M,F)$$

by setting

$$P\alpha(\beta)=\alpha(P^t\beta)\quad,\alpha\in\Omega_0'(M,E),\beta\in\Omega(M,F)$$

If $\alpha\in L^2(M,E)$, $\beta\in L^2(M,F)$, the relation $P\alpha=\beta$ in distribution sense means

$$(\beta,\varphi)=(\alpha,{}^tP\varphi)\quad,\varphi\in\Omega_0(M,F).$$

Let $P:\Omega(M,E)\longrightarrow\Omega(M,F)$ a m–order differential operator. If $(e_1,\dots,e_r)$ and $(f_1,\dots,f_q)$ are local frames of $E|_U$ and $F|_U$ over an open set $U\subset M$, any section $s\in\Omega(U,E)$ can be written $s=\sum s^\nu e_\nu$ and P has the form

$$Ps=\sum(P^\mu_\nu s^\nu)f_\mu$$

where P^μ_ν are scalar differential operators. The symbol of P is a form on T^*_xM whith values in $\mathrm{Hom}(E_x,F_x)$, for each $x\in M$. Let $\xi\in T^*M,\eta\in E$. If $f\in\mathscr{C}^\infty(M)$, $s\in\Omega(M,E)$ satisfy $df(x)=\xi$, $f(x)=0$, $s(x)=\eta$.

We set $\sigma_P(x,\xi):E_x\longrightarrow F_x$, $\sigma_P(x,\xi)\eta=\frac{1}{m!}P(f^ms)(x)$. We have $\sigma_{P\circ Q}(x,\xi)=\sigma_P(x,\xi)\circ\sigma_Q(x,\xi)$ for two operators P an Q.

A.3. DEFINITION. A differential operator P is said to be *elliptic* if $\sigma_P(x,\xi)$ is injective for all $\xi\in T^*_xM\setminus\{0\}$ and $x\in M$.

We introduce the more general notion of *elliptic complex*. Let M be a riemannian manifold and $E^\bullet=\oplus_{i=0}^nE^i$ be a graded hermitian vector bundle. Let $D:\Omega(M,E^\bullet)\longrightarrow\Omega(M,E^\bullet)$ be a graded differential operator of order 1 and degree 1 i.e. $D:\Omega(M,E^i)\longrightarrow\Omega(M,E^{i+1})$. We assume that $D^2=0$, that is, we have a complex

$$0\longrightarrow\Omega(M,E^0)\longrightarrow\Omega(M,E^1)\longrightarrow\dots\longrightarrow\Omega(M,E^n)\longrightarrow0 \tag{A.4}$$

We say that the complex (A.4) is *elliptic* if

$$\Delta=\Delta_D=DD^t+D^tD \tag{A.5}$$

is an elliptic operator.

A.4. EXAMPLE: THE DE RHAM COMPLEX. Let $d:\Omega^l(M)\longrightarrow\Omega^{l+1}(M)$ be the the exterior derivative (de Rham operator). In this case $E^\bullet=\Lambda^\bullet T^*M$, $D=d$. Then $\sigma_d(x,\xi)=\xi\wedge$. The formal adjoint d^t is traditionally denoted δ. We have $\delta:\Omega^{l+1}(M)\longrightarrow\Omega^l(M)$ and $\sigma_\delta(x,\xi)=i_\xi$. Define $\Delta=d\delta+\delta d$ the Laplace–Beltrami operator. Therefore $\sigma_\Delta(x,\xi)=|\xi|^2$ and Δ is elliptic. It is also easy to see that the Witten deformation $D=d_t$ defined in (2.1) has the same symbol as d and the Witten laplacian Δ_t defined in (2.8) is elliptic.

A.5. EXAMPLE: THE CAUCHY-RIEMANN COMPLEX. Assume M is a complex hermitian manifold and F is a holomorphic hermitian vector bundle on M. Let $\overline{\partial}:\Omega^{p,q}T^*M\otimes F\longrightarrow\Omega^{p,q+1}T^*M\otimes F$ be the Cauchy-Riemann operator so that $E^\bullet=\Lambda^{p,\bullet}T^*M\otimes F$, $D=\overline{\partial}$. Then $\sigma_{\overline{\partial}}(x,\xi)=\xi_{0,1}\wedge$ where $\xi_{0,1}$ is the $(0,1)$–part of ξ. Let $\vartheta:\Omega^{p,q}(M)\longrightarrow\Omega^{p,q-1}(M)$ be the formal adjoint of $\overline{\partial}$. Then $\sigma_\vartheta(x,\xi)=i_{\xi_{0,1}}$.

We set $\Delta'' = \overline{\partial}\vartheta + \vartheta\overline{\partial}$. This is the $\overline{\partial}$–Laplacian. Its symbol is $\sigma_{\Delta''}(x,\xi) = |\xi_{0,1}|^2$. Thus Δ'' are elliptic.

A.1.8. Regularity.

A.6. GÅRDING'S INEQUALITY ([**15**, II ,Theorem 8.1]). *Let P be an elliptic second order differential operator acting on sections of a vector bundle E over a manifold M. For any compact $K \subset M$ there exists a constant $C > 0$ such that for all $u \in \Omega_0(K,E)$ we have*

$$\|u\|_1^2 \leqslant C((Pu,u) + \|u\|_0^2). \tag{A.6}$$

We quote now the regularity theorem [**15**, III, Corollary 1.5], [**12**, Corollary 7.1, p.54].

A.7. REGULARITY THEOREM. *Let P be an elliptic second order differential operator acting on sections of a vector bundle E over a manifold M. Assume $u \in \Omega_0'(M,E)$ satisfies $Pu \in W_m(M,E;\text{loc})$. Then $u \in W_{m+2}(M,E;\text{loc})$. In particular, P is hypoelliptic, that is $Pu \in \Omega(M,E)$ implies $u \in \Omega(M,E)$.*

This is a consequence of the fact that P admits a parametrix (i. e. an inverse modulo smoothing operators) a pseudodifferential operator of order -2. The same argument leads to the following variant of the Gårding inequality. For any compact $K \subset M$ there exists a constant $C > 0$ such that for any $u \in W_0(K,E)$ we have

$$\|u\|_{m+2} \leqslant C_m(\|Pu\|_m + \|u\|_0)\,. \tag{A.7}$$

A.2. Selfadjointness

Our aim is to study the L_2-cohomology on non-compact spaces by using harmonic and spectral theory. For this purpose we need to understand the self-adjoint extensions of the Laplace operators. We encounter two situations:

- (i) M is endowed with a complete metric, e.g. M is compact. Then the Laplace operator is essentially self-adjoint and there exists only one self-adjoint extension.
- (ii) the metric of M is incomplete, e.g. M has a non-empty boundary. Then there are in general more then one self-adjoint extension. We will use in this case the *Friedrichs extension*.

We present the basic definitions and introduce the quadratic forms in A.2.1. The essential self-adjointness on complete manifolds is proved in A.2.2. In A.2.3 we define the Friedrichs extension and as particular case we disccuss the Dirichlet and Neumann boundary conditions. For the purpose of Hodge theory we introduce also a self-adjoint extension called the *Gaffney extension* in A.2.4.

A.2.1. Basic definitions. Let M be a riemannian manifold and E be a hermitian vector bundle. Assume now that $P : \Omega(M,E) \longrightarrow \Omega(M,E)$ is an elliptic symmetric differential operator.

In general, at least two possibilities present themselves to extend P as a closed densely defined operator on $L^2(M,E)$. The first choice is to consider the closure $\overline{P}$ whose graph is the closure of the graph of $P : \Omega_0(M,E) \longrightarrow L_2(M,E)$. This is the *minimal extension* of P denoted also P_{min}. The second is to consider the *maximal extension*

$$\text{Dom}(P_{\max}) = \{\alpha \in L^2(M,E) : P\alpha \in L^2(M,E)\}$$

and for $\alpha \in \text{Dom}(P_{\max})$ set $P_{max}\alpha = P\alpha$ (where $P\alpha$ is computed in the sense of distributions). Clearly, $\text{Dom}(P_{\min}) \subset \text{Dom}(P_{\max})$ and $P_{\max}$ is an extension of $P_{\min}$. Let us denote by P^* the Hilbert space adjoint of $P : \Omega_0(M,E) \longrightarrow L_2(M,E)$. It is easy to see that $P^* = P_{\max}$. If P' is a self–adjoint extension of P, we have $P_{\min} \subset P' \subset P_{\max}$. P is called *essentialy self-adjoint* if $\overline{P} = P^*$ (equivallently $P_{\min} = P_{\max}$). We can already prove that elliptic operators are essentially self-adjoint on compact manifolds.

A.8. THEOREM. *Let $P : \Omega(M,E) \longrightarrow \Omega(M,E)$ be an elliptic symmetric operator on a compact manifold M. Then $P_{\min} = P_{\max}$ and P is essentially selfadjoint.*

PROOF. If $\alpha \in \mathrm{Dom}(P_{\max})$, the regularity theorem A.7 implies $\alpha \in W_2(M,E)$. Since $W_2(M,E) \subset \mathrm{Dom}(P_{\min})$, we get $P_{\min} = P_{\max}$. Moreover, it is easy to see that $P^* = P_{\max}$, where P^* is the Hilbert space adjoint of P. Thus $\overline{P} = P^*$, so P is essentially selfadjoint. □

Therefore P has a unique selfadjoint extension, namely its closure, denoted $P : W_2(M,E) \longrightarrow L_2(M,E)$. If $P : \Omega(M,E) \longrightarrow \Omega(M,E)$ is positive, like Δ, its closure is also positive.

We shall work in general with the quadratic form associated to an operator rather than with the operator directly. We intoduce here this useful tool. For an exhaustive study of quadratic forms see [**9**, Chapter 6], [**11**, VIII.6] (and historical notes at p. 307), [**2**, Chapter 4].

Let H be a complex Hilbert space. A quadratic form is a sesquilineare map $Q : \mathrm{Dom}(Q) \times \mathrm{Dom}(Q) \longrightarrow \mathbb{C}$ where $\mathrm{Dom}(Q)$ is a dense linear subset of H. Q is said *semibounded* if $Q(u,u) \geqslant C\|u\|^2$ for $u \in \mathrm{Dom}(Q)$ and positive if $Q(u,u) \geqslant 0$ for $u \in \mathrm{Dom}(Q)$. A semibounded quadratic form $Q : \mathrm{Dom}(Q) \times \mathrm{Dom}(Q) \longrightarrow \mathbb{C}$, $\mathrm{Dom}(Q) \subset H$, is called *closed* if $(\mathrm{Dom}(Q), \|\cdot\|_Q)$ is complete, where $\|f\|_Q = (Q(f) + \|f\|^2)^{1/2}$, $f \in \mathrm{Dom}(Q)$. There exist a basic correspondence between semibounded closed quadratic forms and semibounded self-adjoint operators which is described as follows. For the sake of simplicity we may assume that the lower bound is $c = 0$.

A.9. PROPOSITION. *To a positive self-adjoint operator F we associate the quadratic form Q given by*

$$\mathrm{Dom}(Q_F) = \mathrm{Dom}(F^{1/2}) \text{ and } Q_F(u,v) = (F^{1/2}u, F^{1/2}v)\,,\ u,v \in \mathrm{Dom}(Q_F). \tag{A.8}$$

Then

$$\begin{aligned}\mathrm{Dom}(F) = \{u \in \mathrm{Dom}(Q) : &\text{there exists } v \in H \text{ with} \\ &Q(u,w) = (v,w) \text{ for all } w \in \mathrm{Dom}(Q)\}\end{aligned} \tag{A.9}$$

Conversely, given a positive closed quadratic form Q there exists a positive self– adjoint operator F such that (A.8) *holds i. e. $Q = Q_F$. In particular, the domain of F is described by* (A.9).

For the proofs we refer to [**2**, Lemma 4.4.1] and [**2**, Theorem 4.4.2].

A.2.2. Selfadjointness on complete manifolds. Let $D : \mathscr{E}(M,E^\bullet) \longrightarrow \mathscr{E}(M,E^\bullet)$ be an elliptic complex (A.4). We consider the weak maximal extension

$$D = D_{\max} : L_2(M,E^\bullet) \longrightarrow L_2(M,E^\bullet)\,. \tag{A.10}$$

$\mathrm{Dom}(D)$ consists of elements u such that Du calculated in distributional sense is in L^2. We obtain in this way a complex of closed, densely defined operators, i.e. $\mathrm{Im}(D_{\max}) \subset \mathrm{Ker}(D_{\max})$. Since we mainly work with the operator $D_{\max}$, we will drop the subscript and write simply D instead of $D_{\max}$. Let $D^t = D^t_{\max}$ be the maximal extension of the formal adjoint of D and D^* be the Hilbert-space adjoint of D.

In order to study the domain of our operators we use the following fundamental lemma of Andreotti–Vesentini [**1**, Lemma 4,p. 92–3]. For an operator P the *graph-norm* is defined by $\mathrm{Dom}(P) \ni u \longmapsto \|u\| + \|Pu\|$.

A.10. ANDREOTTI–VESENTINI LEMMA. *Assume (M, g^{TM}) is complete. Then $\Omega_0(M,E^\bullet)$ is dense in* $\mathrm{Dom}(D)$, $\mathrm{Dom}(D^t)$, $\mathrm{Dom}(D) \cap \mathrm{Dom}(D^t)$ *in the graph norms of D, D^t and $D + D^t$, respectively.*

PROOF. We first reduce the proof to the case of a compactly supported form u. The completeness of the metric ω implies the existence of a sequence $\{a_\mu\}_\mu \subset \mathscr{C}^\infty_c(M)$, such that $0 \leqslant a_\mu \leqslant 1$, $a_{\mu+1} = 1$ on $\mathrm{supp}\, a_\mu$, $|da_\mu| \leqslant 1/\mu$ for every $\mu \geqslant 1$ and $\{\mathrm{supp}\, a_\mu\}_\mu$ exhaust M. To construct this sequence we first construct an exhaustive function $a : M \longrightarrow \mathbb{R}$ with $|da| < 1$. This is done by smoothing the distance to a point (we can assume that M is connected). Next, consider a smooth function $\rho : \mathbb{R} \longrightarrow [0,1]$ such that $\rho = 0$ on a neighbourhood of $(-\infty, -2]$, $\rho = 1$ on a neighbourhood of $[-1, \infty)$ and $0 \leqslant \rho' \leqslant 2$. Then $a_\mu = \rho(a/2^{\mu+1})$ satisfies the conditions above.

Let $u \in \mathrm{Dom}(D) \cap \mathrm{Dom}(D^t)$. Then $a_\mu u \in \mathrm{Dom}(D) \cap \mathrm{Dom}(D^t)$ and

$$\|D(a_\mu u) - a_\mu Du\| = O(1/\mu)\|u\|\,,$$
$$\|D^t(a_\mu u) - a_\mu D^t u\| = O(1/\mu)\|u\|\,.$$

Hence $\{a_\mu u\}$ converges to u in the graph norm. So to prove the assertion we can start with a form u having compact support in M. But then the approximation in the graph norm follows from the Friedrichs theorem on the identity of weak and strong derivatives [**7**, Proposition 1.2.4]:

A.11. FRIEDRICH'S LEMMA. *Let $Pf = \sum a_k \partial f/\partial x_k + bf$ be a differential operator of order* 1 *on an open set $\Omega \subset \mathbb{R}^n$, with coefficients $a_k \in \mathscr{C}^1(\Omega)$, $b \in \mathscr{C}^0(\Omega)$. Then for any $v \in L^2(\mathbb{R}^n)$ with compact support in Ω we have*

$$\lim_{\varepsilon \longrightarrow 0} \|P(v * \rho_\varepsilon) - (Pv) * \rho_\varepsilon\|_{L^2} = 0.$$

The proof of the Andreotti-Vesentini lemma is achieved. □

As a by-product of Friedrichs lemma we obtain also:

A.12. COROLLARY (Integration by parts). *Assume that $u \in L_2(X, E^q, \mathrm{loc})$, $v \in L_2(X, E^{q+1}, \mathrm{loc})$ and $Du \in L_2(X, E^{q+1}, \mathrm{loc})$, $D^t v \in L_2(X, E^{q-1}, \mathrm{loc})$. Suppose also that u and v have compact support. Then $(Du, v) = (u, D^t v)$.*

PROOF. We may assume that u, v have support in a trivialization patch diffeomorphic to $\mathbb{R}^m$. We denote $w_\varepsilon = w * \rho_\varepsilon$. We have by Friedrichs lemma:

$$\begin{aligned}(Du,v) &= \lim_{\varepsilon \longrightarrow 0}((Du)_\varepsilon, v_\varepsilon) = \lim_{\varepsilon \longrightarrow 0}(Du_\varepsilon, v_\varepsilon) = \lim_{\varepsilon \longrightarrow 0}(u_\varepsilon, D^t v_\varepsilon)\\ &= \lim_{\varepsilon \longrightarrow 0}(u_\varepsilon, (D^t v)_\varepsilon) = (u, D^t v).\end{aligned} \tag{A.11}$$

□

A.13. COROLLARY. *If M is complete, $D^* = D^t$, $(D^t)^* = D$.*

PROOF. It is clear that from definitions $D^* \subset D^t$. Conversely if $u \in \mathrm{Dom}(D^t)$ there exist $u_\mu \in \Omega_0(M, E^\bullet)$ with $u_\mu \longrightarrow u\,,\quad D^t u_\mu \longrightarrow D^t u$. Then

$$(Dw, u_\mu) = (w, D^t u_\mu)\,, \quad w \in \mathrm{Dom}(D)$$

by the definition of D. The limit of this equalities for $\mu \longrightarrow \infty$ gives

$$(Dw, u) = (w, D^t u)\,, \quad w \in \mathrm{Dom}(D)$$

i.e. $u \in \mathrm{Dom}(D^*)$ and $D^* u = D^t u$. □

Of utmost importance is the self–adjointness of the laplacian (A.5). For simplicity we denote $\Delta = \Delta_D$.

A.14. COROLLARY. *Δ is essentially self–adjoint. The quadratic form associated to its unique selfadjoint extension is the form Q given by* $\mathrm{Dom}(Q) \doteq \mathrm{Dom}(D) \cap \mathrm{Dom}(D^*)$ *and*

$$Q(u,v) \doteq (Du, Dv) + (D^* u, D^* v)\,, \quad u, v \in \mathrm{Dom}(Q) \tag{A.12}$$

In particular,

$$\mathrm{Ker}\,\Delta_{\max} = \mathrm{Ker}\,D \cap \mathrm{Ker}\,D^* \tag{A.13}$$

PROOF. We will show that $\Delta_{\max}$ is selfadjoint. Since $\Delta_{\max} = \Delta^*$ this implies

$$\overline{\Delta} = \Delta^{**} = (\Delta_{\max})^* = \Delta_{\max} = \Delta^*\,,$$

which means that Δ is essentially self-adjoint.

Let $u \in \mathrm{Dom}(\Delta_{\max})$. Since Δ is elliptic we have $u \in W_2(M, E^\bullet, \mathrm{loc})$ by Gårding's inequality. Thus $D^t u, Du \in L_2(M, E^\bullet, \mathrm{loc})$ and by Corollary A.12 we can integrate by parts if u is multiplied with a smooth compactly supported function. Let a_μ be the family of functions defined in the proof of the Andreotti-Vesentini lemma A.10. By following [**3**] we obtain:

$$\begin{aligned}\|a_\mu Du\|^2+\|a_\mu D^t u\| &= (a_\mu^2 Du, Du)+(u, D(a_\mu^2 D^t u))\\ &= (D(a_\mu^2)u, Du)+(u, a_\mu^2 DD^t u)-2(a_\mu da_\mu\wedge u, Du)+2(u, a_\mu da_\mu\wedge D^t u)\\ &= (a_\mu^2 u, \Delta u)-2(da_\mu\wedge u, a_\mu Du)+2(u, da_\mu\wedge(a_\mu D^t u))\\ &\leqslant (a_\mu^2 u, \Delta_{\max} u)+2^{-\mu}(2\|a_\mu Du\|\|u\|+2\|a_\mu D^t u\|\|u\|)\\ &\leqslant (a_\mu^2 u, \Delta_{\max} u)+2^{-\mu}(\|a_\mu Du\|^2+\|a_\mu D^t u\|^2+2\|u\|^2).\end{aligned}$$

We get therefore

$$\|a_\mu Du\|^2+\|a_\mu D^t u\|^2\leqslant\frac{1}{1-2^{-\mu}}((a_\mu^2 u, \Delta_{\max} u)+2^{1-\mu}\|u\|^2).$$

By letting $\mu\longrightarrow\infty$, we obtain $\|Du\|^2+\|D^t u\|^2\leqslant(u, \Delta_{\max} u)$, in particular Du, $D^t u$ are in $L_2(M, E^\bullet)$. This implies

$$(u, \Delta_{\max} v)=(Du, Dv)+(D^t u, D^t v)\quad u, v\in\operatorname{Dom}\Delta_{\max}, \tag{A.14}$$

because the equality holds for $a_\mu u$ and v, and because we have $a_\mu u\longrightarrow u$, $D(a_\mu u)\longrightarrow Du$ and $D^t(a_\mu u)\longrightarrow D^t u$ in L^2. An analogous calculations show that the right-hand side of (A.14) equals $(\Delta_{\max} u, v)$. Thus

$$(u, \Delta_{\max} v)=(\Delta_{\max} u, v), \quad u, v\in\operatorname{Dom}(\Delta_{\max}) \tag{A.15}$$

which means that $\Delta_{\max}\subset(\Delta_{\max})^*$. But $\Delta_{\max}$ is the maximal extension of Δ so that $\Delta_{\max}=(\Delta_{\max})^*$. □

A.2.3. Friedrichs extension. In general, if M is not complete, the symmetric elliptic operator $P:\Omega_0(M, E)\longrightarrow L_2(M, E)$ is not essentially self–adjoint. In this case, we have to choose one self–adjoint extension and a particulary useful one is the Friedrichs extension [**2**, p. 86]. In order to exhibit the Friedrichs extension we use quadratic forms.

Let $P:\operatorname{Dom}(P)\subset H_1\longrightarrow H_2$ be a positive linear operator, where H_1 and H_2 are Hilbert spaces. (A positive operator is also symmetric due to the polarization formula.) The associated quadratic form Q_P is defined by

$$\operatorname{Dom}(Q_P)=\operatorname{Dom}(P),\ Q_P(u, v):=(Pu, v),\ u, v\in\operatorname{Dom}(Q_P).$$

By [**2**, Theorem 4.4.5] Q_P is closable, i. e. there exists a positive closed form Q extending Q_P . Let us take the smallest closed positive form $\overline{Q}_P$ with this property. Note that

$$\operatorname{Dom}(\overline{Q}_P)\,\text{is the completion of}\,\operatorname{Dom}(Q_P)=\operatorname{Dom}(P)\,\text{for the norm}\,\|\cdot\|_{Q_P}. \tag{A.16}$$

that is, $u\in\operatorname{Dom}(\overline{Q}_P)$ if and only if there exists $(u_\nu)\subset\operatorname{Dom}(Q_P)$ so that $\|u_\nu-u\|\to 0$ and $Q_P(u_\nu-u_\mu)\to 0$ for $\nu, \mu\to\infty$. In this case $\overline{Q}_P(u)=\lim Q_P(u_\nu)$.

A.15. DEFINITION. The positive self-adjoint operator F with $Q_F=\overline{Q}_P$ as defined in (A.9) is an extension of P, called the *Friedrichs extension*.

Due to the one–to–one correspondence between closed quadratic forms and self–adjoint operators, we will practically work only with the the closure $\overline{Q}_P$.

A.16. EXAMPLE: THE DIRICHLET PROBLEM. Assume M is a compact manifold with boundary. Let $P:\Omega_0(M, E)\longrightarrow L_2(M, E)$ be a positive elliptic operator. The Friedrichs extension of P is called the operator P with Dirichlet boundary conditions. If $P=\Delta$ is given by (A.5), its quadratic form is

$$Q_\Delta(u, v)=(Du, Dv)+(D^* u, D^* v), \quad u, v\in\Omega_0(M, E) \tag{A.17}$$

Arguing as in Proposition A.19 we obtain

$$\operatorname{Dom}\overline{Q}_\Delta=W_{1,0}(M, E). \tag{A.18}$$

A.17. EXAMPLE: THE $\overline{\partial}$-NEUMANN PROBLEM. Let M be a relatively compact smooth domain in a complex manifold X and $F \longrightarrow X$ be a holomorphic vector bundle. Let $M = \{x \in X : r(x) < 0\}$ where r is a smooth function on X which has non–vanishing gradient on ∂M. We denote

$$B^{0,q}(M,F) = \{u \in \Omega^{0,q}(\overline{M},F) \,:\, \partial r \wedge *u = 0 \text{ on } \partial M\}\,. \tag{A.19}$$

Integration by parts ([**4**, Propositions 1.3.1–2]) shows that

$$(\overline{\partial}u, v) = (u, \vartheta v) \quad \text{for } u \in \Omega^{0,q}(\overline{M},F),\, v \in B^{0,q}(M,F) \tag{A.20}$$

We consider the operator

$$\begin{aligned} \operatorname{Dom}(P) &\doteq \left\{u \in B^{0,q}(M,F) \,:\, \overline{\partial}u \in B^{0,q+1}(M,F)\right\} \\ Pu &= \overline{\partial}\vartheta u + \vartheta\overline{\partial}u\,, \quad \text{for } u \in \operatorname{Dom}(P)\,. \end{aligned} \tag{A.21}$$

which by (A.20) is positive. An extension of the associated form Q_P is

$$\operatorname{Dom}(Q) \doteq B^{0,q}(M,F)\,, \quad Q(u,v) \doteq (\overline{\partial}u, \overline{\partial}v) + (\vartheta u, \vartheta v)\,. \tag{A.22}$$

It is easy to see that Q is closable and its closure is the form $\overline{Q}$ with

$$\begin{aligned} \operatorname{Dom}(\overline{Q}) = \big\{u \in L_2^{0,q}(M,F) \,:\, \exists\, (u_\mu) \subset B^{0,q}(M,F) \text{ so that } \lim u_\mu = u \\ \text{and } (\|\overline{\partial}u_\mu\|),\, (\|\vartheta u_\mu\|) \text{ are Cauchy sequences}\big\} \\ \overline{Q}u = \lim\left(\|\overline{\partial}u_\mu\|^2 + \|\vartheta u_\mu\|^2\right), \quad \text{for } u \in \operatorname{Dom}(\overline{Q})\,. \end{aligned} \tag{A.23}$$

The self-adjoint extension of P given by the Friedrichs method is called the $\overline{\partial}$-Neumann laplacian.

A.2.4. The Gaffney extension. We return to the complex of weak maximal extensions (A.10). We describe a self-adjoint extension of the D-laplacian which is very convenient for the formulation of the Hodge decomposition. It was introduced by Gaffney in [**5**] and it coincides, of course, with the unique self-adjoint extension in the case of a complete metric and also with the Friedrichs extension in the case of the $\overline{\partial}$-Neumann problem.

Let D^* be the Hilbert-space adjoint of D.

A.18. PROPOSITION. *Let (M, g^{TM}) be a riemannian manifold and $(\Omega^\bullet(M), D)$ be an elliptic complex. The operator defi ned by*

$$\begin{aligned} \operatorname{Dom}(\Delta) &:= \{u \in \operatorname{Dom}(D) \cap \operatorname{Dom}(D^*) \,:\, Du \in \operatorname{Dom}(D^*)\,,\, D^*u \in \operatorname{Dom}(D)\}\,, \\ \Delta u &= DD^*u + D^*Du \;\; \textit{for } u \in \operatorname{Dom}(\Delta)\,. \end{aligned} \tag{A.24}$$

is a self-adjoint extension of the D-laplacian, called the Gaffney extension. The quadratic form associated to Δ is the form Q given by

$$\begin{aligned} \operatorname{Dom}(Q) &:= \operatorname{Dom} D \cap \operatorname{Dom} D^* \\ Q(u,v) &= (Du, Dv) + (D^*u, D^*v)\,, \quad u, v \in \operatorname{Dom}(Q) \end{aligned} \tag{A.25}$$

PROOF. For a proof of the self-adjointness of Δ we refer to [**5**], [**4**, Proposition 1.3.8]. We determine now the quadratic form of Δ. First remark that Q is a closed form. Indeed, if $\{u_\mu\}$ is a Cauchy sequence in the norm $\|\cdot\|_Q$, there exist elements u, α, β aus L_2 such that $u_\mu \to u$, $\overline{\partial}u_\mu \to \alpha$, $\vartheta u_\mu \to \beta$. Since $\overline{\partial}u_\mu \to \overline{\partial}u$ in distribution sense, $\alpha = \overline{\partial}u$ and hence $\overline{\partial}u \in L_2$. Since $u_\mu \in \operatorname{Dom}(D^*)$ we have $(Dw, u_\mu) = (w, D^*u_\mu)$ for any $w \in \operatorname{Dom}(D)$. By passing to the limit we get $(Dw, u) = (w, \beta)$, thus $u \in \operatorname{Dom}(D^*)$ and $D^*u = \beta$.

Let F be the unique self-adjoint, positive operator associated to Q. The domain of F is given by (A.9). For $u \in \operatorname{Dom}(\Delta)$ it is clear that $u \in \operatorname{Dom}(Q)$ and u satisfi es (A.9) with $w = \Delta u$. Therefore $\Delta \subset F$ and, since both operators are self-adjoint, $F = \Delta$. □

Note that by the definition of the Gaffney extension

$$\operatorname{Ker}\Delta \doteq \{u \in \operatorname{Dom}\Delta : \Delta u = 0\} = \operatorname{Ker} D \cap \operatorname{Ker} D^* \tag{A.26}$$

and call it the space of harmonic elements of $L_2(M,E^\bullet)$.

It is useful to formulate an interior regularity result for elements of the domain of the quadratic form of an elliptic operator.

A.19. PROPOSITION. *The domain of the quadratic form* (A.25) *satisfies* $\operatorname{Dom}(\overline{Q}_D) \subset W_1(M,E,\mathrm{loc})$.

PROOF. Let $K \subset M$ be a compact set, $\xi \in \mathscr{C}_0^\infty$ with $\xi = 1$ on K. Let $u \in \operatorname{Dom}(Q)$. It easy to see that $\xi u \in \operatorname{Dom}(D)$ and $\xi u \in \operatorname{Dom}(D^*)$. Without loss of generality, we can replace u by ξu and assume thus that u has compact support in M. Let L be a compact which contains $\operatorname{supp} u$ in its interior. By Friedrichs lemma, there exists a sequence $(u_\mu) \subset \Omega_0(M,E)$ such that $u_\mu \longrightarrow u$, $Du_\mu \longrightarrow Du$, $D^*u_\mu \longrightarrow D^*u$ in L_2. By Garding's ineguality A.6 applied to $u_\mu - u_\nu$ we obtain that (u_μ) is a Cauchy sequence in $W_{1,0}(L,E^\bullet)$ and therefore $u \in W_{1,0}(L,E^\bullet)$. □

A.20. EXAMPLE: THE $\overline{\partial}$-NEUMANN PROBLEM. We describe now the Gaffney extension in the case of the $\overline{\partial}$-Neumann problem. Let $D = \overline{\partial}$ be the weak maximal extension of the Cauchy-Riemann operator and let $\overline{\partial}^*$ be its Hilbert-space adjoint. Integration by parts ([**4**, Propositions 1.3.1–2]) shows that

$$B^{0,q}(M,F) = \Omega^{0,q}(\overline{M},F) \cap \operatorname{Dom}(\overline{\partial}^*), \quad \overline{\partial}^* = \vartheta \text{ on } B^{0,q}(M,F). \tag{A.27}$$

From (A.27) follows that the restriction to $\Omega^{0,q}(\overline{M},F)$ of the Gaffney extension (A.24) for $D = \overline{\partial}$ is exactly the operator (A.21). Moreover, we have an analogon to the Andreotti-Vesentini lemma A.10.

A.21. LEMMA ([**7**, Proposition 1.2.4]). $\Omega(\overline{M},F)$ *is dense in* $\operatorname{Dom}(\overline{\partial})$ *in the graph-norm of* $\overline{\partial}$ *and* $B^{0,q}(M,F)$ *is dense in* $\operatorname{Dom}(\overline{\partial}^*)$ *and in* $\operatorname{Dom}(\overline{\partial}) \cap \operatorname{Dom}(\overline{\partial}^*)$ *in the graph-norms of* $\overline{\partial}^*$ *and* $\overline{\partial} + \overline{\partial}^*$*, respectively.*

The proof is again based on the Friedrichs lemma A.11, but a more delicate convolution process in the tangential direction is required.

A.22. PROPOSITION. *The* $\overline{\partial}$*-Neumann laplacian i.e. the Friedrichs extension of* (A.21) *coincides with the Gaffney extension* (A.24) *of the* $\overline{\partial}$*-laplacian.*

PROOF. By Proposition A.9, Definition A.15 and Proposition A.18, it suffices to show that the quadratic forms (A.23) and (A.25) are the same. But this results immediately from Lemma A.21. □

A.3. Hodge Decomposition

Let (M, g^{TM}) be a riemannian manifold and $E^\bullet = \oplus_{i=0}^n E^i$ be a graded hermitian vector bundle. Let $D : \mathscr{E}(M,E^\bullet) \longrightarrow \mathscr{E}(M,E^\bullet)$ be an elliptic complex and consider the complex of closed, densely defined, weakly maximal extensions (A.10).

The ordinary L^2 cohomology of (A.10) is defined to be

$$H^q_{(2)}(M,E) \doteq \operatorname{Ker} D \cap L_2(M,E^q) \Big/ \operatorname{Range} D \cap L_2(M,E^q) \tag{A.28}$$

The cohomology $H^\bullet_{(2)}(M,E)$ is denoted in case of the de Rham complex Example A.4 by $H^\bullet_{(2)}(M) = H^\bullet_{dR,(2)}(M)$, called the L^2 de Rham cohomology of M. In case of Example A.5 it is denoted $H^{p,\bullet}_{(2)}(M,F)$ and called the L^2 Dolbeault cohomology of (M,F).

If we are want the cohomology group to inherit the Hilber-space stucture, we introduce the reduced L^2 cohomology of (A.10), defined to be

$$\overline{H}^q_{(2)}(M,E) \doteq \operatorname{Ker} D \cap L_2(M,E^q) \Big/ [\operatorname{Range} D \cap L_2(M,E^q)] \tag{A.29}$$

where $[V]$ denotes the closure of the space V. We take the closure in order to make sure that $\overline{H}^q_{(2)}(M,E)$ is a Hilbert space.

We link the L^2 cohomology to the theory of elliptic operators via the Hodge decomposition for the Gaffney extension Δ (A.24) of the Laplace–Beltrami operator. We denote by

$$\mathscr{H}^\bullet(M,E) \doteq \operatorname{Ker}\Delta = \operatorname{Ker}D \cap \operatorname{Ker}D^* \tag{A.30}$$

and call it the space of harmonic elements of $L_2(M,E^\bullet)$.

A.23. WEAK HODGE DECOMPOSITION. *The following weak Hodge decomposition holds:*

$$\begin{aligned} L_2(M,E^\bullet) &= \mathscr{H}^\bullet(M,E) \oplus [\operatorname{Im}D] \oplus [\operatorname{Im}D^*]\,, \\ \operatorname{Ker}D &= \mathscr{H}^\bullet(M,E) \oplus [\operatorname{Im}D]\,. \end{aligned}$$

In particular we have an isomorphism:

$$\overline{H}^q_{(2)}(M,E) \cong \mathscr{H}^q(M,E)\,. \tag{A.31}$$

A general condition for the range of D to be closed and for the finiteness of the L^2 cohomology is as follows.

A.24. PROPOSITION ([**7**, Theorem 1.1.1-3]). (*i*) *A necessary and sufficient condition for* $\operatorname{Im}(D) \cap L_2(M,E^q)$ *and* $\operatorname{Im}(D^*) \cap L_2(M,E^q)$ *to be closed is that there exists a positive constant C such that*

$$\|u\|^2 \leqslant C(\|Du\|^2 + \|D^*u\|^2), \quad u \in \operatorname{Dom}D \cap \operatorname{Dom}D^* \cap L_2(M,E^q)\,, u \perp \mathscr{H}^q(M,E) \tag{A.32}$$

(*ii*)*Assume that from every sequence $u_k \in \operatorname{Dom}(D) \cap \operatorname{Dom}(D^*) \cap L_2(M,E^q)$ with $\|u_k\|$ bounded and $Du_k \longrightarrow 0$ in $L_2(M,E^{q+1})$, $D^*u_k \longrightarrow 0$ in $L^q_2(M,E^{q-1})$ one can select a strongly convergent subsequence. Then both* $\operatorname{Im}(D) \cap L_2(M,E^q)$, $\operatorname{Im}(D^*) \cap L_2(M,E^q)$ *are closed. Moreover, $\mathscr{H}^q(M,E)$ is finite dimensional and $H^q_{(2)}(M,E) \cong \mathscr{H}^q(M,E)$.*

Practically we deal with an estimate which will imply the strong Hodge decomposition.

A.25. DEFINITION. We say that the fundamental estimate holds in degree q if there exist a compact set $K \subset M$ and $C > 0$ such that

$$\|u\|^2 \leqslant C\left(\|Du\|^2 + \|D^*u\|^2 + \int_K |u|^2\,dv_M\right), \quad u \in \operatorname{Dom}(D) \cap \operatorname{Dom}(D^*) \cap L_2(M,E^q)\,. \tag{A.33}$$

If M is hermitian and $E^q = \Omega^{p,q}T^*M \otimes F$, we say that the fundamental estimate holds in bidegree (p,q).

A.26. THEOREM. *Assume that the fundamental estimate holds in degree q.*

(i) *The operators D on $L_2(M,E^{q-1})$ and Δ on $L_2(M,E^q)$ have closed range and we have the strong Hodge decomposition:*

$$L_2(M,E^q) = \mathscr{H}^q(M,E) \oplus \operatorname{Im}DD^* \oplus \operatorname{Im}D^*D\,, \tag{A.34}$$

$$\operatorname{Ker}D \cap L_2(M,E^q) = \mathscr{H}^q(M,E) \oplus \operatorname{Im}D \cap L_2(M,E^q)\,. \tag{A.35}$$

(ii) *There exists a bounded operator G on $L_2(M,E^q)$, called the Green operator, such that $\Delta G = G\Delta = \operatorname{Id} - P_{\mathscr{H}}$, $P_{\mathscr{H}}G = GP_{\mathscr{H}} = 0$, where $P_{\mathscr{H}}$ is the orthogonal projection on $\mathscr{H}^q$.*

(iii) *If $f \in \operatorname{Im}(D) \cap L_2(M,E^q)$, the unique solution $u \perp \operatorname{Ker}T \cap L_2(M,E^{q-1})$ of the equation $Du = f$ is given by $u = D^*Gf$.*

(iv) *The operator G maps $L_2(M,E^q) \cap \Omega(M,E^q)$ into itself.*

PROOF. Consider a sequence $\{u_k\} \subset \operatorname{Dom}D \cap \operatorname{Dom}D^* \cap L_2(M,E^q)$ with $\{\|u_k\|\}$ bounded and $\|D^*u_k\| + \|Du_k\| \longrightarrow 0$, for $k \longrightarrow \infty$.

Let ξ be a smooth, compactly supported function on $\operatorname{Int}M$, such that $\xi = 1$ on K. Hence

$$Q(\xi u_k, \xi u_k) + \|\xi u_k\|^2 = \|D(\xi u_k)\|^2 + \|D^*(\xi u_k)\|^2 + \|\xi u_k\|^2 \tag{A.36}$$

is also bounded. Let $L = \operatorname{supp}\xi$ and consider the Sobolev space $W_{1,0}(L,E^q)$ with norm $\|\cdot\|_1$. By A.19, $\xi u_k \in W_{1,0}(L,E^q)$. Gårding's inequality (A.6) shows that $(\|\xi u_k\|_1)$ is bounded.

By the Rellich's Lemma A.2, the inclusion $\big(W_{1,0}(L,E^q), \|\cdot\|_1\big) \hookrightarrow \big(L_2(M,E^q), \|\cdot\|\big)$ is compact. We can select therefore a convergent subsequence in $L_2(M,E^q)$, denoted also $\{\xi u_k\}$. Since $\xi = 1$ on K, it follows that $\{u_k|_K\}$ converges in $\|\cdot\|$. By estimate (A.33), this entails that $\{u_k\}$ converges in $L_2(M,E^q)$.

Proposition A.24 implies that (A.32) holds and $\mathscr{H}^q$ is fi nite dimensional. From (A.32) we infer that

$$\|f\| \leqslant C\|\Delta f\|\,, \quad f \in \operatorname{Dom}\Delta,\ f \perp \operatorname{Ker}\Delta\,. \tag{A.37}$$

Therefore Δ has closed range. Since Δ is self-adjoint we have

$$L_2(M,E^q) = \operatorname{Im}\Delta \oplus \operatorname{Ker}\Delta = \operatorname{Im}(DD^*) \oplus \operatorname{Im}(D^*D) \oplus \mathscr{H}^q(M,E)\,.$$

By (A.37) there exists a bounded inverse G of Δ on $\operatorname{Im}\Delta$. We extend G to $L_2(M,E^q)$ by setting $G = 0$ on $\mathscr{H}^q(M,E)$. We obtain thus a bounded operator G on $L_2(M,E^q)$, bounded by the constant C from (A.37), satisfying $\operatorname{Ker}(G) = \mathscr{H}^q(M,E)$ and $\operatorname{Im}(G) = \operatorname{Im}(\Delta)$. It is now easy to check that (i), (ii) and (iii) hold true.

Finally, assertion (iv) follows from Proposition A.7 (the interior regularity for the elliptic operator Δ). □

Let us say what the previous theory gives if the manifold M is supposed to be compact. Then Δ is essentially self–adjoint by A.8 or A.14, and the unique self–adjoint extension may be described as the Gaffney extension. The fundamental estimate (A.33) holds in all degrees (just take $K = M$). Theorem A.26 implies that the strong Hodge decomposition holds in all degrees. Since $\Omega(M,E^\bullet) \subset L_2(M,E^\bullet)$ we can restrict the strong Hodge decomposition to $\Omega(M,E^\bullet)$ we obtain:

$$\begin{gathered}\Omega(M,E^\bullet) = \mathscr{H}^\bullet(M,E) \oplus D\Omega(M,E^\bullet) \oplus D^t\Omega(M,E^\bullet)\\ u = DD^tGu + D^tDGu + P_{\mathscr{H}}u\end{gathered} \tag{A.38}$$

Note that G and $P_{\mathscr{H}}$ map smooth forms to smooth forms.

The cohomology of the complex (A.4) is denoted by $H^\bullet(M,E)$. From (A.38) it folows that the map.

$$\begin{aligned} H^\bullet(M,E) &\longrightarrow \mathscr{H}^\bullet(M,E)\\ [u] &\longrightarrow P_{\mathscr{H}}u\end{aligned} \tag{A.39}$$

is an isomorphism. The classical formulation of (A.39) is that each cohomology class contains a unique harmonic form. We deduce also

$$H^\bullet(M,E) \cong \mathscr{H}^\bullet(M,E) \cong H^\bullet_{(2)}(M,E)\,. \tag{A.40}$$

We call $H^\bullet(M,E)$ the de Rham cohomology in the case of the de Rham complex A.4 and the Dolbeault cohomology in the case of the Cauchy-Riemann complex A.5.

A.4. Spectral properties

The point of extending a differential operator to a selfadjoint one is to study its spectral properties. Let A be a closed operator on a Hilbert space H. We say that a complex number λ lies in the *resolvent set* of A if $\lambda - A$ is a bijection of $\operatorname{Dom}(A)$ onto H with a bounded inverse. Note that by the closed–graph theorem if $\lambda - A : \operatorname{Dom}(A) \longrightarrow H$ is a bijection, the inverse is automatically bounded.

The *spectrum* of A, denoted $\sigma(A)$, is the complement in $\mathbb{C}$ of the resolvent set. We shall study only self–adjoint operators, whose spectrum is always a non–empty set of the real line. Let $E : \operatorname{Bor}(\mathbb{R}) \longrightarrow \mathscr{L}(H)$ be a spectral measure, where $\operatorname{Bor}(\mathbb{R})$ is the family of Borel sets in $\mathbb{R}$ (for the definition of a spectral measure, see [**11**, Vol I, p.263]). If $f:\mathbb{R}\longrightarrow\mathbb{C}$ is a bounded Borel function we can define the integral

$$\int_{\mathbb{R}} f(t)\,dE(t)\in\mathscr{L}(H)$$

using the usual pattern of defining the integral for step–functions first and then writling the integral of a general bounded Borel function f as the limit of the integrals of a sequence of step–functions, which converge uniformly to f. If $E:\mathrm{Bor}(\mathbb{R})\longrightarrow\mathscr{L}(H)$ is a spectral measure we can define the associated scalar measures on $\mathbb{R}$, by setting $\mathrm{Bor}(\mathbb{R})\ni B\longrightarrow(E(B)u,v)$, for each $(u,v)\in H\times H$. For any bounded Borel function $f:\mathbb{R}\longrightarrow\mathbb{C}$ we have then

$$\Big(\Big(\int_{\mathbb{R}} f(t)\,dE(t)\Big)u,v\Big)=\int_{\mathbb{R}} f(t)\,d(E(t)u,v)$$

We have the following fundamental result [**11**, Theorem VIII.6], [**2**, Theorem 2.5.5].

A.27. SPECTRAL THEOREM. *Each self–adjoint operator A has a unique spectral measure E such that*

$$\mathrm{Dom}(A)=\Big\{u\in H:\int_{\mathbb{R}} t^2\,d(E(t)u,u)<\infty\Big\}$$

and

$$Au=\lim_{k\longrightarrow\infty}\Big(\int_{\mathbb{R}}\chi_{[-k,k]}(t)t\,dE(t)\Big)u=:\Big(\int_{\mathbb{R}} t\,dE(t)\Big)u,\quad u\in\mathrm{Dom}(A).$$

We can actually define the integral of a general Borel function with respect to E and obtain a Borel functional calculus. Namely, we set

$$\int_{\mathbb{R}} f(t)\,dE(t):=\lim_{k\longrightarrow\infty}\int_{\mathbb{R}}\chi_{[-k,k]}(t)f(t)\,dE(t)$$

and define the closed, densely defined operator $f(A)$ by

$$\mathrm{Dom}\,f(A)=\Big\{u\in H:\int|f(t)|^2\,d(E(t)u,u)<\infty\Big\}$$

$$f(A)u=\lim_{k\longrightarrow\infty}\Big(\int_{\mathbb{R}}\chi_{[-k,k]}(t)f(t)\,dE(t)\Big)u$$

In particular

$$(f(A)u,v)=\int_{\mathbb{R}} f(t)\,d(E(t)u,v).$$

Let Q be the quadratic form associated to A. We deduce that

$$\begin{aligned}\mathrm{Dom}(Q)&=\Big\{u\in H:\int_{\mathbb{R}}|t|\,d(E(t)u,u)<\infty\Big\},\\ Q(u,v)&=\int_{\mathbb{R}} t\,d(E(t)u,v),\quad u,v\in\mathrm{Dom}(Q).\end{aligned}\tag{A.41}$$

A.28. DEFINITION. Let us define the *spectral resolution* associated to A by $(E_\lambda)_{\lambda\in\mathbb{R}}$ where $E_\lambda=E((-\infty,\lambda])$. When we want to stress the dependence on A we note $E_\lambda(A)$. We set $\mathscr{E}(\lambda)=\mathscr{E}(\lambda,A):=\mathrm{Im}\,E_\lambda(A)$. The *spectrum counting function* of A is defined as

$$N(\lambda):=\dim\mathrm{Im}\,E_\lambda=\dim\mathscr{E}(\lambda)$$

The most important tool for estimating and for comparing the eigenvalues of different operators is the variational principle or minmax principle. The simplest but very useful form of the variational principle is the following

A.29. GLAZMAN'S LEMMA. *The spectrum conting function of a semibounded self–adjoint operator A satisfies the variational formula*

$$N(\lambda)=\sup\{\dim F: F\,closed\subset\mathrm{Dom}(Q)\,,\,Q(u,u)\leqslant\lambda\|u\|^2,\forall u\in F\}\tag{A.42}$$

where Q is the quadratic form of A.

PROOF. Assume that $u \in \operatorname{Im} E_\lambda$. For any $B \in \operatorname{Bor}(\mathbb{R})$

$$(E(B)u,u) = (E(B)E((-\infty,\lambda])u,u) = (E(B\cap(-\infty,\lambda])u,u)$$

so formula (A.41) entails

$$Q(u,u) = \int_{-\infty}^{\lambda} t\,d(E(t)u,u) \leqslant \lambda(E((-\infty,\lambda])u,u) = \lambda\|u\|^2$$

Hence $Q(u,u) \leqslant \lambda\|u\|^2$ for all $u \in \operatorname{Im} E_\lambda$ and $N(\lambda)$ does not exceed the right–hand side of (A.42). Consider a closed linear space $F \subset \operatorname{Dom}(Q)$ such that $Q(u,u) \leqslant \lambda\|u\|^2$ for all $u \in L$. We show that $E_\lambda : F \longrightarrow \operatorname{Im} E_\lambda$ is injective. If $u \in \operatorname{Ker} E_\lambda = \operatorname{Im} E((\lambda,+\infty))$ we have

$$Q(u,u) = \int_{(\lambda,+\infty)} t\,d(E(t)u,u) > \lambda(E((\lambda,+\infty))u,u) = \lambda\|u\|^2$$

if $u \neq 0$. Thus any $u \in F \cap \operatorname{Ker} E_\lambda$ must vanish. We infer that $\dim F \leqslant \dim \operatorname{Im} E_\lambda = N(\lambda)$. Formula (A.42) is established. □

Let A be a semibounded self–adjoint operator and let Q_A be the associated closed quadratic form. We consider the sequence $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_j \leqslant \ldots$ according to the formula

$$\lambda_j = \inf_{F\subset \operatorname{Dom}(Q_A)} \sup_{f\in F,\|f\|=1} Q_A(f,f) \tag{A.43}$$

where F runs through the j–dimensional subspaces of $\operatorname{Dom}(Q_A)$.

A.30. VARIATIONAL PRINCIPLE (compact resolvent case). *Assume that the injection* $(\operatorname{Dom}(Q_A),\|\cdot\|_{Q_A}) \hookrightarrow (H,\|\cdot\|)$ *is a compact operator. Then A has a discrete spectrum and the numbers* λ_j *satisfy* $\lim_{j\longrightarrow\infty}\lambda_j = \infty$*, and coincide with the eigenvalues of A written in increasing order and repeated according to multiplicity.*

In the presence of the essential spectrum the situation is more complicated.

A.31. VARIATIONAL PRINCIPLE. *Let us define the bottom of the essential spectrum as* $\inf \sigma_{\text{ess}}(A)$ *if* $\sigma_{\text{ess}}(A) \neq \varnothing$ *and* $+\infty$ *if* $\sigma_{\text{ess}}(A) = \varnothing$*. Then for each fixed j, either a) there are at least j eigenvalues (counted according to multiplicity) below the bottom of the essential spectrum and* λ_j *is the j–th eigenvalue or b)* λ_j *is the bottom of the essential spectrum, in which case* $\lambda_j = \lambda_{j+1} = \lambda_{j+2} = \ldots$ *and there are at most* $j-1$ *eigenvalues (countig multiplicity) below* λ_j.

For the proofs we refer to [**11**, Vol IV, p.76-78], [**2**, Cap. 4.5]. Let us mention other alternative definitions of λ_j:

$$\lambda_j = \sup_{F\subset \operatorname{Dom}(Q_A)} \inf_{f\in F,\|f\|=1} Q_A(f) \tag{A.44}$$

where F runs over the $\|\cdot\|_{Q_A}$–closed $(j-1)$–codimensional subspaces of $\operatorname{Dom}(Q_A)$. Moreover, in (A.43) we can let F run through j-dimensional subspaces of a care form for Q_A or through subspaces of $\operatorname{Dom}(A)$ and the value of λ_j does not change.

We wish to prove the well-known fact that the essential spectrum is stable under compact perturbations. Consider an elliptic complex (A.4) and the corresponding Laplace operator Δ defined on a manifold M. Let $K \subset M$ be a compact set. We denote also by Δ and $\Delta_{M\smallsetminus K}$ the Friedrichs extensions of Δ restricted to $\Omega_0(M,E)$ and $\Omega_0(M\smallsetminus K,E)$, respectively.

A.32. PROPOSITION (decomposition principle). *In the notation as above, the Laplace–Beltrami operators* Δ *and* $\Delta_{M\setminus K}$ *have the same essential spectrum.*

PROOF. If A is a densely defined self–adjoint operator on a Hilbert space, then the essential spectrum $\sigma_{\text{ess}}(A)$ may be defined as the set $\lambda \in \mathbb{R}$ for which there exists a noncompact sequence $\{f_n\}_{n\in\mathbb{N}}$ in the domain of A with

$$\|f_n\| = 1 \quad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\longrightarrow\infty}\|(A-\lambda\,\mathrm{Id})f_n\| = 0.$$

Any part of such a sequence, from which it is impossible to extract a convergent subsequence, is called a characteristic sequence for (A,λ). Let φ be a smooth compactly supported nonnegative function on M which is equal to one on a neighborhood of K. If $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal characteristic sequence for (Δ,λ) for some $\lambda\geqslant 0$, then we set $g_n = f_{2n}-f_{2n-1}\,(n\geqslant 1)$. We see that $\{g_n\}_{n\in\mathbb{N}}$ is noncompact and that $\lim_{n\longrightarrow\infty}\|(\Delta-\lambda\,\mathrm{Id})g_n\|=0$. The Rellich lemma implies that $\{\varphi g_n\}_{n\in\mathbb{N}}$ is compact, since φ is compactly supported. Moreover, by passing to a subsequence of $\{f_n\}_{n\in\mathbb{N}}$, if necessary, we may assume that $g_n\longrightarrow 0$ in the E–valued first Sobolev space $W_1(U,E)$, where U is a relatively compact neighborhood of the support of φ. Then $\lim_{n\longrightarrow\infty}\|(\Delta_{M\setminus K}-\lambda\,\mathrm{Id})(1-\varphi)g_n\|=0$, and consequently

$$\{\widetilde{g}_n\}_{n\in\mathbb{N}}\quad\text{with}\quad \widetilde{g}_n := \frac{(1-\varphi)g_n}{\|(1-\varphi)g_n\|_M}$$

is a characteristic sequence for $(\Delta_{M\setminus K},\lambda)$. So $\sigma_{\mathrm{ess}}(\Delta)\subset\sigma_{\mathrm{ess}}(\Delta_{M\setminus K})$. We trivially have $\sigma_{\mathrm{ess}}(\Delta_{M\setminus K})\subset\sigma_{\mathrm{ess}}(\Delta)$. □

We end with some words about elliptic operators on compact manifolds. Let $P: W_2(M,E)\longrightarrow L_2(M,E)$ be a self–adjoint, positive elliptic operator on a compact manifold M.

Consider the operator $P+I: W_2(M,E)\longrightarrow L_2(M,E)$. It is also a selfadjoint operator. Since $\|(P+I)\alpha\|\geqslant\|\alpha\|$ we can easily verify that $P+I$ is injective, has closed and dense range. Thus, $P+I: W_2(M,E)\longrightarrow L_2(M,E)$ is a bijective map so there exists $(P+I)^{-1}: L_2(M,E)\longrightarrow W_2(M,E)$. Consider $i: W_2(M,E)\longrightarrow L_2(M,E)$ the inclusion. The Rellich theorem says that i is a compact operator. It follows that

$$T: i\circ(P+I)^{-1}: L_2(M,E)\longrightarrow L_2(M,E)$$

if a compact operator with $\|T\|\leqslant 1$. Moreover, $T=T^*$ since $P+I$ is selfadjoint. From the spectral theory of selfadjoint compact operators we infer that $L_2(M,E)$ has an orthonormal basis $\{u_j\}$ consisting of eigen functions of T:

$$Tu_j=\mu_j u_j\quad, u_j\in W_2(M,E).$$

such that each μ_j appears only finitely many times in the sequence $(\mu_j)_j$. Moreover $0<\mu_j\leqslant 1$ and we can order μ_j so that $\mu_j\searrow 0$ as $j\to\infty$.

We deduce that $\{u_j\}$ is an ONB of eigen vectors for P it self:

$$Pu_j=\lambda_j u_j$$

with $\lambda_j=\frac{1}{\mu_j}-1$, so $\lambda_j\geqslant 0$, $\lambda_j\nearrow\infty$ as $j\to\infty$. By A.1.8 $u_j\in W_2(M,E)$ and the regularity theorem, we see $u_j\in W_4(M,E)$. Proceeding by induction we get $u_j\in W_2^m(M,E)$ for all $m\in\mathbb{N}$. By the Sobolev embedding theorem $u_j\in\Omega(M,E)$.

A.33. THEOREM. *Let $P: W_2(M,E)\longrightarrow L_2(M,E)$ be a selfadjoint positive elliptic operator of order* 2.

- *a) We can find an ONB $\{u_j\}_{j\geqslant 0}$ for $L_2(M,E)$ of eigenvectors, $Pu_j=\lambda_j u_j$*
- *b) The eigenvectors u_j are smooth and $\lim_{j\to\infty}\lambda_j=\infty$.*

A.34. COROLLARY. *$P: W_2(M,E)\longrightarrow L_2(M,E)$ is Fredholm and $\operatorname{Ker}P\subset\Omega(M,E)$.*

PROOF. We have $\operatorname{Ker}P=E(0)$ so $\dim\operatorname{Ker}P<\infty$ and $\operatorname{Ker}P\subset\Omega(M,E)$. Moreover $R(P)=\bigoplus_{\lambda_j>0}E(\lambda_j)$ is closed and $R(P)^\perp=\operatorname{Ker}P$ so $\operatorname{codim}R(P)<\infty$. □

We define finally the Green operator. Set $G: L_2(M,E)\longrightarrow L_2(M,E)$ where

$$Gu_j=\begin{cases}0 & \lambda_j=0,\\ \frac{1}{\lambda_j} & \lambda_j>0\end{cases}\tag{A.45}$$

G is bounded and $\|G\|\leqslant 1/\min\{\lambda_j|\lambda_j>0\}$. We denote $\mathscr{H}=\operatorname{Ker}P$ and $P_{\mathscr{H}}$ the orthogonal projection on $\mathscr{H}$. Thus

$$PG=I-P_{\mathscr{H}}.\tag{A.46}$$

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APPENDIX B

Elements of analytic and hermitian geometry

B.1. Hermitian geometry

B.1.1. Hermitian metrics on manifolds. Let X be a complex manifold. A *hermitian metric* on X is a smooth family $h^{TX} = \{h^{TX}_x\}_{x\in X}$, where h^{TX}_x is a hermitian metric on the fiber T_xX for each $x \in X$. Let $(z_1,\dots,z_n)$ local coordinates on X. Then h^{TX} has the form

$$h = \sum h_{jk} dz_j \otimes d\overline{z}_k,\ (h_{ij}) \in M_n(\mathbb{C}),\ h_{jk} = h(\partial/\partial x_j, \partial/\partial x_k) \tag{B.1}$$

where h_{jk} are smooth functions on the chart domain. The real part $g^{TX} = \operatorname{Re} h^{TX}$ is therefore a riemannian metric on X, compatible to the complex structure. The Kähler form $\omega = -\operatorname{Im} h^{TX}$ is a real $(1,1)$-form on X, given in local coordinates by

$$\omega = \frac{\sqrt{-1}}{2} \sum h_{jk} dz_j \wedge d\overline{z}_k \tag{B.2}$$

Moreover, the volume form of the riemannian metric g^{TX} satisfies

$$dv_X = \frac{\omega^n}{n!}\,. \tag{B.3}$$

B.1. DEFINITION. The metric h^{TX} (or sometimes even g^{TX}) is called *Kähler metric* if ω is d-closed, $d\omega = 0$.

Let $F \to X$ be a complex vector bundle on X. On each fiber there exists a complex structure, so we can define, as for TX, a *hermitian metric* on F as a smooth family $h^F = \{h^F_x\}_{x\in X}$, where h^F_x is a hermitian metric on the fiber F_x for each $x \in X$. For two sections v, w of F over an open set U we have a function on U defined by the pointwise scalar product:

$$\langle v,w\rangle(x) := \langle v,w\rangle_{h^F}(x) := \langle v(x), w(x)\rangle_{h^F_x} = h^F_x(v(x), w(x))\,. \tag{B.4}$$

If we (E,h^E), (F,h^F) are two hermitian vector bundles, then we define a hermitian metric on $E\otimes F$, by $h^{E\otimes F} := h^E \otimes h^F$. We are particulary interested in the following particular case. Suppose X is endowed with a hermitian metric h^{TX} (with real part $g^{TX} = \operatorname{Re} h^{TX}$) and that (F,h^F) is a hermitian vector bundle on X. The bundle $\Lambda^{p,q}T^*X$ has a hermitian metric $h^{\Lambda^{p,q}T^*X}$. The metric on $\Lambda^{p,q}T^*X \otimes F$ is then $h^{\Lambda^{p,q}T^*X} \otimes h^F$.

B.2. NOTATION. The space of smooth sections of $\Lambda^{p,q}T^*X \otimes F$ over an open set U is denoted $\Omega^{p,q}(U,F)$. An element of this space is called an E-valued (p,q)-form. The sheaf $U \longmapsto \Omega^{p,q}(U,F)$ is denoted $\Omega^{p,q}(F)$. The space of smooth sections of $\Lambda^{p,q}T^*X \otimes F$ with compact support in open set U is denoted $\Omega^{p,q}_0(U,F)$.

Let us introduce a global scalar product on $\Omega^{p,q}_0(X,F)$ by setting

$$(v,w) = \int_X \langle v,w\rangle\, dV_X \quad v,w \in \Omega^{p,q}_0(X,F)\,. \tag{B.5}$$

where for simplicity $\langle v,w\rangle = \langle v,w\rangle_{h^{\Lambda^{p,q}T^*X}\otimes h^F}$ and dV_X is given by (B.3). The completion of $\Omega^{p,q}_0(X,F)$ with respect to (B.5) is denoted $L^{p,q}_2(X,F)$.

We can restate (B.5) in terms of the star operator which we introduce now. First, let us view h^F as an element of $\operatorname{Hom}_{\mathbb{C}}(F, \overline{F}^*)$. Then

$$\langle v,w\rangle_{h^F} = h^F(v)(\overline{w}) = \overline{h^F(w)}(v)\,. \tag{B.6}$$

We define

$$\begin{aligned} \#_E : \Lambda^{p,q}T^*X \otimes F &\longrightarrow \Lambda^{n-p,n-q}T^*X \otimes F^* \\ \#_E(\alpha \otimes f) &= *\overline{\alpha} \otimes \overline{h^F(f)} \end{aligned} \tag{B.7}$$

Now, there is a natural duality, denoted $\wedge$, between $\Lambda^{p,q}T^*X \otimes F$ and $\Lambda^{s,t}T^*X \otimes F^*$ given by combining the wedge product with the natural duality $F \times F^* \longrightarrow \mathbb{C}$:

$$\begin{aligned} \Lambda^{p,q}T^*X \otimes F \times \Lambda^{s,t}T^*X \otimes F^* &\longrightarrow \Lambda^{p+s,q+t}T^*X \\ (\alpha \otimes f, \beta \otimes \xi) &\longmapsto (\alpha \otimes f) \wedge (\beta \otimes \xi) := \alpha \wedge \beta \otimes \xi(f) \end{aligned} \tag{B.8}$$

Using (B.8) we obtain

$$v \wedge \#_E w = \langle v, w \rangle \, dV_X \,, \quad v, w \in \Lambda^{p,q}T^*X \otimes F \,. \tag{B.9}$$

B.1.2. Local potentials of Kähler metrics and the $\partial\overline{\partial}$– Lemma. Let (X, h^{TX}) be a hermitian manifold, $\omega = \operatorname{Re} h^{TX}$.

B.3. DEFINITION. A real valued function $\varphi : U \longrightarrow \mathbb{R}, U$ open set in X, is called an *local potential* of the metric h^{TX}, if $\sqrt{-1}\partial\overline{\partial}\varphi = \omega$ on U.

B.4. LEMMA ($\partial\overline{\partial}$–Lemma). *Let ω be a smooth d–closed $(1,1)$–form on a manifold X. Then for each point there exist a neighbourhood U of x and a smooth function $\varphi : U \longrightarrow \mathbb{R}$ such that $\sqrt{-1}\partial\overline{\partial}\varphi = \omega$ on U.*

PROOF. (compare [**22**]). Choose U a ball in a local coordinate system. By the Poincare lemma we can construct a real 1–form α with $d\alpha = \omega$ in U. We decompose $\alpha = \beta + \overline{\beta}$, with β of bidegree $(1,0)$. Comparing bi–degrees, we get $\partial\beta = 0, \overline{\partial\beta} = 0$ and $\overline{\partial}\beta + \partial\overline{\beta} = \omega$. By Dolbeault lemma, there exists a smooth function ψ with $\overline{\partial}\psi = \overline{\beta}$. Hence $\partial\overline{\psi} = \beta$ and

$$\omega = \overline{\partial}\beta + \partial\overline{\beta} = \overline{\partial}\partial\overline{\psi} + \partial\overline{\partial}\psi = \partial\overline{\partial}(\psi - \overline{\psi}) = \sqrt{-1}\partial\overline{\partial}(2\operatorname{Im}\psi)$$

We choose $\varphi = 2\operatorname{Im}\psi$. □

B.5. COROLLARY. *A hermitian metric G^{TX} is Kähler if and only if G^{TX} admits local potentials in the neighbourhood of each point.*

B.1.3. The Current associated to a plurisubharmonic function. Let X be a complex manifold and $\varphi \in L^1(X, loc)$ be a plurisubharmonic function. Then $T = \sqrt{-1}\partial\overline{\partial}\varphi$ is a d–closed positive current on X. If φ is strictly plurisubharmonic, T is a strictly positive current.

B.6. LEMMA ($\partial\overline{\partial}$–Lemma for currents). *Let T be a closed positive $(1,1)$–current. Then for every point of X there exists a neighbourhood U of x and a plurisubharmonic function $\varphi \in L^1(U, loc)$ such that $T = \sqrt{-1}\partial\overline{\partial}\varphi$.*

PROOF. We follow the proof of the $\partial\overline{\partial}$–lemma for smooth forms. Since the Poincare and Dolbeault lemmata hold for currents too, we get as before a distribution u on U such that $\sqrt{-1}\partial\overline{\partial}u = T$. Since T is positive, it follows that u is represented by a function $\varphi \in L^1(U, loc)$, which is obviously plurisubharmonic. □

B.7. NOTE. We can replace T with a strictly positive current in the previous statement. The solution φ will be then strictly plurisubharmonic.

B.1.4. Curvature form. Let $F \to X$ be a hermitian holomorphic vector bundle endowed with the hermitian metric $h = h^F$. There exists a unique connection ∇^F compatible with the complex structure and the hermitian metric, [**10**, p. 75] [**5**, p. 304], called the *Chern connection* .

The *curvature operator* $(\nabla^F)^2$ is a bundle morphism $F \longrightarrow \Lambda^{1,1}T^*X \otimes F$ given by the multiplication with the *curvature matrix* $R^F \in \Omega^{1,1}(\mathrm{End}(F))$. Since we are concerned only with the positivity of line bundles we describe the curvature matrix only for this case. If $\operatorname{rank} F = 1$, $\mathrm{End}(F)$ is trivial and R^F is canonically identifi ed to a $(1,1)$–form on X, such that $\sqrt{-1}R^F$ is real.

Let $\vartheta : F|_U \longrightarrow U \times \mathbb{C}$ be a trivialisation of F and let $e(x) = \vartheta^{-1}(x,1)$, $x \in U$, be the corresponding holomorphic frame.

The hermitian metric is represented by the smooth function $h : U \longrightarrow \mathbb{R}_+$, $h(x) = |e(x)|^2_h$. It is useful to denote $h = e^{-\varphi}$, where $\varphi \in \mathscr{C}^\infty(U,\mathbb{R})$ is called a weight of h.

The curvature has the form

$$R^F = -\partial\overline{\partial}\log h = \partial\overline{\partial}\varphi \quad \text{on } U.$$

This is a global form. If (U_ν) is a covering of X such that $F|_{U_\nu}$ is trivial and $e_\nu \in \Gamma(U_\nu, F)$ are holomorphic frames, we have a cocycle $c_{\mu\nu} \in \mathscr{O}^*(U_\mu \cap U_\nu)$ such that $e_\nu = c_{\mu\nu}e_\mu$ on $U_\mu \cap U_\nu$. There fore $h_\nu = |c_{\mu\nu}|^2 h_\mu$ and since $c_{\mu\nu}$ is holomorphic, $\partial\overline{\partial}\log h_\mu = \partial\overline{\partial}\log h_\nu$ on $U_\mu \cap U_\nu$.

B.1.5. Bochner-Kodaira-Nakano formula. Let (X, h^{TX}) be a hermitian manifold and let $\omega = \mathrm{Re}\, h^{TX}$ be its K¨ahler form. Let $F \to X$ be a hermitian holomorphic vector bundle endowed with the hermitian metric $h = h^L$. We defi ne the Lefschetz operator,

$$\omega\wedge : \Lambda^{p,q}T^*X \otimes F \longrightarrow \Lambda^{p+1,q+1}T^*X \otimes F\,, \tag{B.10}$$

which is the exterior multiplication with ω, acting trivially on the F-component. It is a bundle morphism having as adjoint with respect to the fi berwise scalar product the map

$$\Lambda : \Lambda^{p,q}T^*X \otimes F \longrightarrow \Lambda^{p-1,q-1}T^*X \otimes F\,, \quad \Lambda u = (-1)^{\deg u}\,\overline{*\omega \wedge *\overline{u}} \tag{B.11}$$

that is, $\langle \omega \wedge v, w\rangle = \langle v, \Lambda w\rangle$ for any elements $v \in \Lambda^{p,q}T^*X \otimes F$, $w \in \Lambda^{p+1,q+1}T^*X \otimes F$. Of course, $\omega\wedge$ and Λ are also formal adjoints for the integrated scalar product (B.5).

Let ∇^F be the Chern connection of F. It can be extended to the sheaf $\Omega^\bullet(F)$ by forcing the Leibniz rule:

$$\nabla^F : \Omega^\bullet(F) \longrightarrow \Omega^{\bullet+1}(F)\,, \quad \nabla^F(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\deg\alpha}\,\alpha \wedge \nabla^F s \tag{B.12}$$

for α a form and s a section of F over some open set U of X.

We have a decomposition after bi-degree

$$\begin{gathered}\nabla^F = (\nabla^F)' + (\nabla^F)''\,,\\ (\nabla^F)' : \Omega^{\bullet,\bullet}(F) \longrightarrow \Omega^{\bullet+1,\bullet}(F)\,, \quad (\nabla^F)'' : \Omega^{\bullet,\bullet}(F) \longrightarrow \Omega^{\bullet,\bullet+1}(F)\,.\end{gathered} \tag{B.13}$$

The fact that F is holomorphic implies that there exist operators

$$\partial : \Omega^{\bullet,\bullet}(F) \longrightarrow \Omega^{\bullet+1,\bullet}(F) \quad \overline{\partial} : \Omega^{\bullet,\bullet}(\overline{F}) \longrightarrow \Omega^{\bullet,\bullet+1}(\overline{F})\,.$$

The Chern connection is characterized by the properties

$$(\nabla^F)'' = \overline{\partial}\,, \quad (\nabla^F)' = (h^F)^{-1}\,\partial\, h^F \tag{B.14}$$

where h^F is considered as an element of $\mathrm{Hom}(F, \overline{F}^*)$.

We introduce the operators

$$\begin{gathered}(\nabla^F)''^* = -\#_{F^*}\,\overline{\partial}\,\#_F\,, \quad (\nabla^F)''^* : \Omega^{\bullet,\bullet}(F) \longrightarrow \Omega^{\bullet,\bullet-1}(F)\\ (\nabla^F)'^* = -\overline{*}\,\partial\,\overline{*}\,, \quad (\nabla^F)'^* : \Omega^{\bullet,\bullet}(F) \longrightarrow \Omega^{\bullet-1,\bullet}(F)\end{gathered} \tag{B.15}$$

which are the formal adjoints of $(\nabla^F)''$ and $(\nabla^F)'$:

$$\big((\nabla^F)''u,v\big) =(u,(\nabla^F)''^{*}v), \quad u\in\Omega_0^{p,q}(F), v\in\Omega_0^{p,q+1}(F)$$
$$\big((\nabla^F)'u,v\big) =(u,(\nabla^F)'^{*}v), \quad u\in\Omega_0^{p,q}(F), v\in\Omega_0^{p-1,q}(F)$$

For two graded operators A, B acting on $\Omega^{\bullet}(F)$ we define the graded commutator (or graded Lie bracket) as

$$[A,B]=AB+(-1)^{\deg A\cdot\deg B}BA. \tag{B.16}$$

Let us introduce the laplacians

$$\Delta' =[(\nabla^F)',(\nabla^F)'^{*}] \tag{B.17a}$$
$$\Delta'' =[(\nabla^F)'',(\nabla^F)''^{*}] \tag{B.17b}$$

and the *torsion operator*

$$T=[\Lambda,\partial\omega] \tag{B.18}$$

We have the following generalization of the usual Kähler identities in the presence of torsion. For the proof we refer to [**19**], [**3**] or [**5**, Ch. VII].

B.8. KÄHLER IDENTITIES. *We have the commutation relations*

$$[\Lambda,(\nabla^F)''] =-\sqrt{-1}\big((\nabla^F)'^{*}+T^{*}\big) \tag{B.19a}$$
$$[\Lambda,(\nabla^F)'] =\sqrt{-1}\big((\nabla^F)''^{*}+\overline{T}^{*}\big) \tag{B.19b}$$

The identities (B.19a), (B.19b) and the Jacobi identity yield:

B.9. BOCHNER-KODAIRA-NAKANO IDENTITY.

$$\Delta''_F=\Delta'+[\sqrt{-1}R^F,\Lambda]+[(\nabla^F)',T^{*}]-[(\nabla^F)'',\overline{T}^{*}] \tag{B.20}$$

By using repeatedly the Jacobi identity Demailly obtains the following useful reformulation:

B.10. COROLLARY ([**3**]). *The operator* $\Delta'_{F,tor}:=[(\nabla^F)'+T,(\nabla^F)'^{*}+\overline{T}^{*}]$ *is a positive and formally selfadjoint operator with the same principal symbol as* Δ'_F. *Moreover,*

$$\Delta''_F=\Delta'_{F,tor}+[\sqrt{-1}R^F,\Lambda]+S \tag{B.21}$$

where S is an 0*-th order operator depending only on the torsion of the metric* ω. *Namely,*

$$S=\left[\Lambda,\left[\Lambda,\frac{\sqrt{-1}}{2}\partial\overline{\partial}\omega\right]\right]-[\partial\omega,(\partial\omega)^{*}]\,. \tag{B.22}$$

Combining (B.20) with the inequality of the geometric and arithmetic means and the fact that $(\Delta'u,u)\geqslant 0$ we obtain:

B.11. THEOREM (Nakano's Inequality). *For any* $u\in\Omega_0^{p,q}(X,F)$,

$$\frac{3}{2}(\Delta''u,u)\geqslant\big([\sqrt{-1}R^F,\Lambda]u,u\big)-\frac{1}{2}(\|Tu\|^2+\|T^{*}u\|^2+\|\overline{T}u\|^2+\|\overline{T}^{*}u\|^2) \tag{B.23}$$

Next let us use a form of the Bochner–Kodaira formula introduced by Andreotti–Vesentini [**2**] and Griffiths [**11**]. Let M be a smooth, relatively compact domain in a complex manifold. Let us assume that there exists hermitian metric on X, Kähler near ∂M, written in local coordinates z^{α} as a smooth positive definite matrix $(g_{\alpha\beta})$. Let us consider a holomorphic kermitiam vector bundle (G,h^G) in a neighbourhood of $\overline{M}$ and let $R^G=\sum\theta_{\alpha\beta}\,dz^{\alpha}\wedge d\overline{z}^{\beta}$ be its curvature, where $\theta_{\alpha\beta}=-\partial_{z^{\alpha}}\partial_{\overline{z}^{\beta}}\log h^G$. Let θ^{μ}_{β} be the curvature tensor with the first index raised. Let $u=\frac{1}{q!}\sum u_{\lambda_1\dots\lambda_q}\,d\overline{z}^{\lambda_1}\wedge\cdots\wedge d\overline{z}^{\lambda_q}\otimes f$ be a G–valued $(0,q)$–form on X. We define the $(0,q)$–form $R^Gu=\frac{1}{q!}\sum\theta^{\mu}_{\lambda_1}u_{\mu\lambda_2\dots\lambda_q}\,d\overline{z}^{\lambda_1}\wedge\cdots\wedge d\overline{z}^{\lambda_q}\otimes f$.

We define next the Levi operator (see [**11**, p.418]). First let us remark that we can choose a Γ–invariant defining function r for M such that $|\partial r|=1$ in a neighbourhood of ∂M, with respect to the hermitian metric on X. Let us pick, near a boundary point of M, an orthonormal frame $\omega^1,\dots,\omega^n$

for the bundle of $(1,0)$–forms, such that $\omega^n = \partial r$. We have $\partial\overline{\partial} r = -\overline{\partial}\omega^n = \sum l_{\alpha\beta}\, \omega^{\alpha}\wedge\overline{\omega}^{\beta}\,, \quad (1 \leqslant \alpha,\beta \leqslant n)\,.$

B.12. DEFINITION. The Levi form of ∂M is the restriction of $\partial\overline{\partial} r$ to the holomorphic tangent bundle of ∂M; it is given in the dual frame of $\omega^1,\dots,\omega^{n-1}$ by the matrix $(l_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant n-1}$. M is pseudoconvex if the Levi form is everywhere positive semi–definite.

For a $(0,q)$–form written locally $u = \frac{1}{q!}\sum u_{\alpha_1\dots\beta_q}\,\overline{\omega}^{\alpha_1}\wedge\cdots\wedge\overline{\omega}^{\alpha_q}\otimes f$ where f is an orthonormal frame in G, we set

$$\mathscr{L}(u,u) = \tfrac{1}{(q-1)!}\sum l_{\alpha\beta}\, u_{\alpha\beta_1\dots\beta_{q-1}}\,\overline{u}_{\beta\beta_1\dots\beta_{q-1}} \quad (1 \leqslant \alpha,\beta \leqslant n)\,.$$

The form $u \in B^{0,q}(M,G)$ if and only if $u_{\alpha_1\dots\alpha_q} = 0$ for $n \in \{\alpha_1,\dots,\alpha_q\}$. Therefore for $u \in B^{0,q}(M,G)$ the summation restricts over $1 \leqslant \alpha,\beta \leqslant n-1$ and

$$\mathscr{L}(u,u) = \tfrac{1}{(q-1)!}\sum l_{\alpha\beta}\, u_{\alpha\beta_1\dots\beta_{q-1}}\,\overline{u}_{\beta\beta_1\dots\beta_{q-1}} \geqslant 0\,. \tag{B.24}$$

Finally, let $\overline{\nabla}$ denote the covariant derivative in the $(0,1)$–direction.

B.13. LEMMA. *Assume that the Γ–invariant metric on X is Kähler in a Γ–invariant neighbourhood U of ∂M. Then for any $u \in B^{0,q}(M,G)$ with support in U we have*

$$Q(u,u) = \left\|\overline{\nabla} u\right\|^2 + \left(R^G u,u\right) + \left(\mathrm{Ric}\,u,u\right) + \int_{\partial M}\mathscr{L}(u,u)\,dS \tag{B.25}$$

PROOF. This formula was given by Griffiths [**11**, p. 429, (7.14)]. □ □

B.2. Positivity concepts

B.2.1. Plurisubharmonic functions.

B.14. DEFINITION. A function $\varphi : U \longrightarrow \mathbb{R}$ on a complex manifold is called *strictly plurisubharmonic* if

$$\partial\overline{\partial}\varphi(u,\overline{u}) > 0,\ u \in T^{1,0}X \smallsetminus \{0\}. \tag{B.26}$$

φ is called *plurisubharmonic* if we have just $\geqslant$ in (B.26).

B.2.2. Positive forms. We define here the notion of positivity for $(1,1)$–forms. For a thorough discussion see [**5**, III.1].

B.15. DEFINITION AND THEOREM (Strictly positive form). Let α be a real $(1,1)$–form on the complex manifold U. Then the following assertions are equivalent:

(*i*) For all $u \in T^{1,0}U \smallsetminus \{0\}$ we have $\alpha(u,\overline{u}) > 0$,
(*ii*) If α has the local form $\alpha = \sqrt{-1}\sum\alpha_{\mu\nu}(z)\,dz_\mu\wedge d\overline{z}_\nu$, the hermitian matrix $(\alpha_{\mu\nu}(z))$ is positive definite for all z.
(*iii*) α is the Kähler form of a hermitian metric on TU.

If they are satisfied, α is called *strictly positive*.

Let $\varphi : U \longrightarrow \mathbb{R}$ is smooth. Then φ is strictly plurisubharmonic if and only if $\sqrt{-1}\partial\overline{\partial}\varphi$ is strictly positive.

B.16. DEFINITION AND THEOREM (Positive form). Let α be a real $(1,1)$–form on the complex manifold U. Then the following assertions are equivalent:

(*i*) For all $u \in T^{1,0}U \smallsetminus \{0\}$ we have $\alpha(u,\overline{u}) \geqslant 0$,
(*ii*) If α has the local form $\alpha = \sqrt{-1}\sum\alpha_{\mu\nu}(z)\,dz_\mu\wedge d\overline{z}_\nu$, the hermitian matrix $(\alpha_{\mu\nu}(z))$ is positive semi-definite for all z.

If they are satisfied, α is called *positive*.

B.2.3. Positive line bundles. We introduce the important notion of positive line bundles. Some observations are in order. We use a differential-geometric notion of positivity, which goes back to Kodaira. Thus one could speak about Kodaira-positivity. Since this is our primary definition we say simply "positive line bundle" instead of "positive line bundle in the sense of Kodaira".

There are many notions of positivity. On the differential-geometric side we have the Nakano and Griffiths positivity. On the function-theoretical side, we have the approach of Grauert, which we'll describe below. And we have also the algebraic-geometric concept of ampleness of Grothendieck and Hartshorne. We refer the reader to [**22, 5, 12**]. All this notions are equivalent for line bundles over complex manifolds.

B.17. DEFINITION (Positive line bundle). The hermitian line bundle L is called *positive* if the real $(1,1)$–form $\sqrt{-1}\Theta(L,h)$ is strictly positive. This is equivalent to saying that the local weights φ of the curvature form $\sqrt{-1}\Theta(L,h)$ are strictly plurisubharmonic. We say that L is *semi–positive*, if $\sqrt{-1}\Theta(L,h)$ is a positive form. This is equivalent to saying that the local weights φ of the curvature form $\sqrt{-1}\Theta(L,h)$ plurisubharmonic.

B.18. THEOREM (Kodaira). *A line bundle is positive if only if the image of its Chern class $c_1(L) \in H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{R})$ is represented by a positive $(1,1)$– form.*

One direction is simple, since $\frac{\sqrt{-1}}{2\pi}R^L$ represents in $H^2(X,\mathbb{R})$ the de Rham class of the image of $c_1(L)$ trough the inclusion $\mathbb{Z} \to \mathbb{R}$. For the other direction, one has to construct the hermitian metric h^L, see [**5**, p. 308], [**10**, p. 148], [**17**]

B.2.4. Projective algebraic spaces. We call a compact complex space X *projective algebraic*, if there exists a holomorphic embedding $\psi : X \longrightarrow \mathbb{P}^N$. This means that X is biholomorphic to a compact analytic subset $A \subset \mathbb{P}^N$. By a theorem of Chow, A is an algebraic variety so X has a natural algebraic structure. Let X be a projective compact manifold, $\psi : X \longrightarrow \mathbb{P}^N$. Then the pull–back $\psi^*\mathscr{O}(1)$ of hyperplane is positive. Kodaira established that this property characterises the projective manifolds.

B.19. THEOREM. *Let X be a compact manifold and $L \to X$ be a positive line bundle. Then for large k the maps $\Phi_k : X \longrightarrow \mathbb{P}H^0(X,L^k)^*$ are embeddings. Thus, X is projective if and only if it admits a positive line bundle.*

B.2.5. Positive bundles on complex spaces. We define now the positivity notion for complex spaces. There exists a classical definition of Grauert, which will be decribed below. However, since we use methods of potential theory, we formulate our definitions in terms of curvature. We need first to extend the notions of line bundle, hermitian metric to complex spaces.

B.20. DEFINITION. Let X be a complex space. A function $\varphi : X \longrightarrow \mathbb{R}$ is said to be strictly plurisubharmonic if for any point $x \in X$ there exists a local chart $\tau : U \longrightarrow \widehat{U} \subset \mathbb{C}^N$ of $X, x \in U$, and a strictly plurisubharmonic function $\widehat{\varphi} \in \mathscr{C}^\infty(\widehat{U},\mathbb{R})$ such that $\varphi|_U = \widehat{\varphi} \circ \tau$.

In a similar manner we define the notion of function of class $\mathscr{C}^k$. These definitions do not depend on the choice of local chart [**9**, p. 335].

B.21. DEFINITION. A line bundle L over a complex space X is a complex space together with a holomorphic map $\pi : L \longrightarrow X$ which satisfy the local triviality axiom: for any point $x \in X$ there exists an open set $U \ni x$ and a biholomorphic map $\varphi_U : L|_U \longrightarrow U \times \mathbb{C}$ which is linear on the fibers.

Let $\tau : U \hookrightarrow \widehat{U} \subset \mathbb{C}^N$ be a local chart of X such that $L|_U$ is trivial.We obtain then a local chart on L by setting

$$\tau_L : L|_U \longrightarrow U \times \mathbb{C} \longrightarrow \widehat{U} \times \mathbb{C} \tag{B.27}$$

B.22. DEFINITION. Let $L \to X$ be a line bundle. A hermitian metric h on L is a system of hermitian products $\{h_x\}_{x\in X}$ on the fibers of L which varies smoothly with $x \in X$. This means that there exists a covering of L with local charts $\tau : L|_U \longrightarrow \widehat{U} \times \mathbb{C}$ as in (B.27), and smooth hermitian metrics on $\widehat{U} \times \mathbb{C}$ such that $h|_{L|_U} = \widehat{h} \circ \tau$.

As before, the definition is independent on the choice of local charts. We can describe the hermitian metric in terms of a cocycle.

Exactly as in the case of line bundles over manifolds, a line bundle can be defined by a cocycle. Let $\{U_\nu\}$ be a covering such that $L|_{U_\nu}$ is trivial and let $e_\nu : U_\nu \longrightarrow L$ be holomorphic frames. Then there exist holomorphic functions $c_{\mu\nu} : U_\mu \cap U_\nu \longrightarrow \mathbb{C}^*$ such that $c_\mu = c_{\mu\nu} e_\nu$. The system $(c_{\mu\nu})$ forms a cocycle which defines L.

In this language, a hermitian metric is a system $h_\nu : U_\nu \longrightarrow \mathbb{R}^*_+$ of positive smooth functions such that $h_\mu = |c_{\mu\nu}|^2 h_\nu$. This follows by setting $h_\mu := |e_\mu|^2_h$

B.23. DEFINITION. The hermitian holomorphic line bundle (L, h^L) is called *positive* if $-\log h_\nu$ is a strictly plurisubharmonic function on U_ν for all ν.

By the observation after Definition B.17 this is equivalent to the definition we gave if X were a manifold. Note that the locally defined $(1,1)$–forms $-\sqrt{-1}\partial\overline{\partial}\log h_\nu$ patch together and give a globally defined smooth $(1,1)$–form $\sqrt{-1}R^L$ on X_{reg}.

For the moment we want to make the connection with the definition of Grauert. Let (L, h^L) be a hermitian holomorphic line bundle over the complex space X. The dual line bundle L^* is described by the cocycle $(g^{-1}_{\mu\nu})$, where $(g_{\mu\nu}), g_{\mu\nu} \in \mathscr{O}^*(U_\mu \cap U_\nu)$ is the cocycle of L. The hermitian metric h^L induces a hermitian metric h^{L^*} on L^*, given locally by the system (h^{-1}_ν), if h^L is represented by (h_ν). We define $\rho : L^* \longrightarrow \mathbb{C}, \rho(v) = |v|^2_{h^{L^*}}$. Then $\rho : L^*|_{U_\nu} \longrightarrow \mathbb{C}$ has the form $\rho(v) = |we^*_\nu|_{h^*} = |w|^2 h^{-1}_\nu$, for $v = we^*_\nu$, with e^*_ν the dual of e_ν. It follows $\log\rho(v) = \log|w|^2 - \log h_\nu$. Since the function $\mathbb{C} \ni w \to \log|w|$ is strictly plurisubharmonic if follows that $\log\rho$ is smooth and strictly plurisubharmonic on $L^* \smallsetminus \{\text{zero section}\}$ if L is positive. Let us denote by

$$T = \{v \in L^* : |v|_{h^{L^*}} < 1\} \tag{B.28}$$

T is called the Grauert tube of L^*. We introduce now the notion of pseudoconvexity adapted to the general situation of complex spaces. In the case of manifolds see Definition B.34.

B.24. DEFINITION. Let G be an open set in a copmplex space. G is called strictly pseudoconvex, if for any boundary point $x \in \partial G$ we can find a neighbourhood U an a strictly plurisubharmonic $\varphi : U \longrightarrow \mathbb{R}$ such that $U \cap G = \{x \in U : \varphi(x) < 0\}$.

We deduce the following

B.25. PROPOSITION. *If (L, h^L) is positive, the Grauert tube $T \subset L^*$ is strongly pseudoconvex.*

PROOF. Indeed, we write $T = \{v \in L^* : \log\rho(v) < 0\}$ and use that $\log\rho$ is strictly plurisubharmonic on $L^* \smallsetminus \{\text{zero section}\}$. □

If X is a manifold, the converse is also true as shown in [**9**, Satz. 62,p. 341]

B.26. DEFINITION. A line bundle E over a complex space X is called Grauert– negative if the zero section of E has a strongly pseudoconvex neighbourhood $V \subset E$. The bundle E is called Grauert–positive if E^* is Grauert–negative.

We used here [**9**, Def 1,p. 342], where general vector bundles are considered. The terminology of Grauert is "weakly negative (positive)". We can therefore reformulate the Proposition B.25.

B.27. COROLLARY. *If (L,h) is positive, L is Grauert– positive.*

We quote now Grauert's generalization of Kodaira's embedding theorem.

B.28. THEOREM (Grauert). *A compact complex space X is projective algebraic if and only if it carries a Grauert– positive bundle.*

B.3. Differential forms and currents on complex spaces

In this paragraph, we let X be an n–dimensional paracompact complex space.

We follow the definition of smooth differential forms and currents by Fujiki [7]. The sheaf Ω^r (resp. $\Omega^{p,q}$) of germs of $\mathscr{C}^\infty$–r–forms (resp. $\mathscr{C}^\infty$–(p,q)– forms) with direct sum decomposition $\Omega^r = \oplus_{p+q=r}\Omega^{p,q}$ and the differentials $d : \Omega^r \longrightarrow \Omega^{r+1}$ (resp. $\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}$ and $\overline{\partial} : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}$) with $d = \partial + \overline{\partial}$ is locally defined as follows and globally defined by gluing them.

When X is a subspace of a domain V in $\mathbb{C}^l = \mathbb{C}^l(z_1,\dots,z_l)$ with the ideal sheaf $\mathscr{I} = \mathscr{I}_X$. We define $\Omega = \mathscr{E}^0_X$ by $\Omega = \mathscr{E}_V/(\mathscr{I} + \overline{\mathscr{I}})\mathscr{E}_V$, where $\overline{\mathscr{I}} = \{\overline{f}; f \in \mathscr{I}\}$, $\overline{f}$ being the complex conjugate of f. Next define the $\mathscr{E}_V$–submodule $\widetilde{\Omega}_X$ of $\mathscr{E}^1_V$ by

$$\widetilde{\Omega}_X = \sum \mathscr{I}\mathscr{E}_V dz_\alpha + \sum \overline{\mathscr{I}}\mathscr{E}_V d\overline{z}_\beta + \mathscr{E}_V d\mathscr{I} + \mathscr{E}_V d\overline{\mathscr{I}},$$

where $\mathscr{E}_V d\mathscr{I} = \{\sum h_\nu dg_\nu; h_\nu \in \mathscr{E}_V \text{ and } g_\nu \in \mathscr{I}\}$ and similary for $\mathscr{E}_V d\overline{\mathscr{I}}$. Then put

$$\Omega^r = \mathscr{E}^r_V/(\widetilde{\Omega}_X \wedge \Omega^{r-1})$$

for $r \geqslant 1$. These naturally form as Ω–graded algebra $\mathscr{E}^\bullet_Y$. Further, define the $\mathscr{E}_V$–submodules $\Omega^{p,q}(p+q=r)$ of Ω^r by $\mathscr{E}^{p,q}_{X,x} = \{\psi \in \mathscr{E}^r_{X,x};$ there exists a $\widetilde{\psi} \in \mathscr{E}^{p,q}_{V,x}$ inducing $\psi\}$. Then is immediate to see that we have a direct sum decomposition $\Omega^r = \oplus_{p+q=r}\Omega^{p,q}$. Moreover, the usual differential d (resp. ∂ and $\overline{\partial}$) on $\mathscr{E}^r_V$ (resp. $\mathscr{E}^{p,q}_V$) induces the one on Ω^r (resp. $\Omega^{p,q}$) with $d = \partial + \overline{\partial}$. On the other hand, the natural complex conjugation on $\mathscr{E}^r_V$ induces a $\mathbb{C}$–antilinear involution on Ω^r. In particular, we can define the real form on X as those left fixed by this involution. Morphisms of complex spaces $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ induce a natural pull–back homomorphism $f^* : \mathscr{E}^\bullet_Y \longrightarrow \mathscr{E}^\bullet_X$ and satisfy $f^* \circ g^* = (g \circ f)^*$.

We let $\Omega^r_0(X)$ denote the space of smooth r–forms on X with compact support endowed with $\mathscr{C}^\infty$–topology.

We define the space $\Omega^r(X)'$ of r–currents on X as the vector space of complex–valued contiuous linear functionals on $\Omega^{2n-r}_0(X)$ with the $\mathscr{C}^\infty$–topology. The differential $d : \Omega^r(X)' \longrightarrow \Omega^{r+1}(X)'$ is defined by $dT(\varphi) = (-1)^{r+1}T(d\varphi)$ for $T \in \Omega^r(X)'$ and $\varphi \in \Omega^{2n-r-1}_0(X)$. By gluing them, we can define the sheaf $\Omega^{r\prime}$ of germs of r–currents on X and $d : \Omega^{r\prime} \longrightarrow \Omega^{r+1\prime}$. We also denote by $\Omega^{p,q}_0(X)$ the space of smooth (p,q)–forms on X with compact support. The $\mathscr{C}^\infty$–topology of $\Omega^{p,q}_0(X)$, the space $\Omega^{p,q}(X)'$ of (p,q)–currents, the sheaves $\Omega^{p,q\prime}$ and $\partial : \Omega^{p,q}(X)' \longrightarrow \Omega^{p+1,q}(X)'$, $\partial : \Omega^{p,q\prime} \longrightarrow \Omega^{p+1,q\prime}$, $\overline{\partial} : \Omega^{p,q}(X)' \longrightarrow \Omega^{p,q+1}(X)'$, $\overline{\partial} : \Omega^{p,q\prime} \longrightarrow \Omega^{p,q+1\prime}$ with $d = \partial + \overline{\partial}$ are also defined as above and as in the case of usual complex manifolds.

By the discussion above we get complexes of sheaves on X:

$$(\Omega^\bullet, d) : \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \cdots$$

and

$$(\Omega^{\bullet\prime}, d) : \Omega^{0\prime} \longrightarrow \Omega^{1\prime} \longrightarrow \Omega^{2\prime} \longrightarrow \cdots$$

Note that the sheaves Ω^r and $\Omega^{r\prime}$, $(r \geqslant 0)$ are fine, but in general, $(\Omega^\bullet, d)$ and $(\Omega^{\bullet\prime}, d)$ are not resolutions of $\mathbb{C}$ (or $\mathbb{R}$) on X. There are natural homomorphismus of complexes of sheaves

$$\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{C} \longrightarrow (\Omega^\bullet, d) \longrightarrow (\Omega^{\bullet\prime}, d)$$

which induce homomorphisms of hypercohomology groups

$$H^*(X,\mathbb{Z}) \longrightarrow H^*(X,\mathbb{R}) \longrightarrow H^*(X,\Omega^\bullet) \longrightarrow H^*(X,\Omega^{\bullet\prime}). \tag{B.29}$$

By the fineness of Ω^r and $\Omega^{r\prime}(r \geqslant 0)$, the canonical edge homomorphisms $H^*(\Gamma(X,\Omega^\bullet)) \longrightarrow H^*(X,\Omega^\bullet)$ and $H^*(\Gamma(X,\Omega^{\bullet\prime})) \longrightarrow H^*(X,\Omega^{\bullet\prime})$ are isomorphisms.

The *singular support* of a current $T \in \Omega^{p,q\prime}$ is defined as the smallest subset S of X such that T is a smooth form on $X \setminus S$.

A real $\mathscr{C}^\infty$–(p,p)–form ξ on X is *strictly positive* (resp. *semipositive*) if there exists an open covering $\mathscr{U} = \{U_\alpha\}$ of X with, for each α, an embedding $j_\alpha : U_\alpha \longrightarrow V_\alpha$ of U_α into a subdomain

V_α in $\mathbb{C}^{l_\alpha}$ and a $\mathscr{C}^\infty$ strictly positive (resp. semipositive) (p,p)–form ξ_α on V_α in the usual sense such that $j_\alpha^*\xi_\alpha = \xi|_{U_\alpha}$.

A (p,p)–current T is *real* if $T=\overline{T}$ in the sense that $\overline{T}(\varphi)=T(\overline{\varphi})$ for all $\varphi\in\Omega_0^{n-p,n-p}(X)$, and a real current T is *positive* when $(\sqrt{-1})^{p^2}T(\psi\wedge\overline{\psi})\geqslant 0$ for all $\psi\in\Omega_0^{n-p,0}(X)$.

A real (p,p)–current T on X is *strictly positive* if there exists a strictly positive $\mathscr{C}^\infty$–(p,p)–form ω^p on X such that $T-\omega^p$ is a positive current on X. T is said to be *strictly positive* at a point $x\in X$ if there exists a neighbourhood U of x such that $T|_U$ ia a strictly positive current on U.

A real $(1,1)$–current ω on X is said to be a *Kähler current* (cf. [**16**]) if it is d–closed and strictly positive on X. A d–closed $(1,1)$–current or a d–closed $\mathscr{C}^\infty$–$(1,1)$–form is said to be *integral* if its hypercohomology class is in the image of $H^2(X,\mathbb{Z})$ under the map in (B.29).

Let $\pi: L\longrightarrow X$ be a holomorphic line bundle over X. A *singular Hermitian metric* h^L on L is a map $h^L: L\longrightarrow[-\infty,+\infty]$ which is given in any local trivialization $\tau:\pi^{-1}(U)\longrightarrow U\times\mathbb{C}$ by $h^L(v)=|\tau(v)|e^{-\psi_U(\pi(v))}$ for $v\in\pi^{-1}(U)$, where $\psi_U\in L^1(U,loc)$. The *curvature current* of (L,h^L) is the d–closed $(1,1)$–current $\sqrt{-1}R^L)$ given by $\sqrt{-1}R^L)=\sqrt{-1}\partial\overline{\partial}\psi_U$ on U, which is independent of the choice of the local trivialisation.

B.4. Pseudoconvex and pseudoconcave manifolds

Manifolds satisfying convexity conditions are very important in complex analysis. This point will be made clear in the sequel. Let us just mention at the outset that domains of holomorphy in $\mathbb{C}^n$ (natural domains of existence of holomorphic functions) are characterized by the pseudoconvexity property. Convexity in complex analysis is introduced imposing conditions on the complex hessian of an exhaustion function. These functions can be viewed as Morse functions in the complex setting. They permit the use of the powerful methods of cohomology theory and the main applications are: finiteness, vanishing and isomorphism theorems, extension of analytic objects, algebraicity of the meromorphic function field, filling of holes of complex manifolds and many others.

B.4.1. Basic notions. We review first the basic facts about analytic convexity. Let U be an open subset of $\mathbb{C}^n$ and $\varphi: U\longrightarrow\mathbb{R}$ be a smooth function. Assume for simplicity that $0\in U$ and consider the Taylor expansion of φ at 0:

$$\varphi(z)=\varphi(0)+2\,\mathrm{Re}\Big[\sum_{j=1}^n\frac{\partial\varphi}{\partial z_j}(0)z_j+\sum_{j,k=1}^n\frac{\partial^2\varphi}{\partial z_j\partial z_k}(0)z_jz_k\Big]+\sum_{j,k=1}^n\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}(0)z_j\overline{z}_k+O(|z|^3)$$

The quadratic form

$$\mathscr{L}_\varphi^{(0)}(v,v)=\partial\overline{\partial}\varphi(0)(v,v)=\sum_{j,k=1}^n\frac{\partial^2\varphi}{\partial z_j\partial\overline{z}_k}(0)v_j\overline{v}_k$$

is called the Levi form of φ at 0. A biholomorphic change of variables near 0 acts on $\mathscr{L}_\varphi(0)$ by a linear change of variables given by the Jacobi metrix of the transformation. This implies that the numbers of positive and negative eigenvalues of the Levi form at a point do not depend on the choice of local coordinates.

Let us introduce the notion of convexity due to Andreotti–Grauert [**1**].

B.29. DEFINITION. (i) A manifold X of complex dimension n is called q–convex $(1\leqslant q\leqslant n)$ if there exists a smooth function $\varphi: X\longrightarrow[a,b)$, $a\in\mathbb{R}$, $b\in\mathbb{R}\cup\{+\infty\}$ such that $X_c=\{\varphi<c\}\Subset X$ for all $c\in[a,b)$ and the Levi form $\partial\overline{\partial}\varphi$ has at least $n-q+1$ positive eigenvalues outside a compact set K.

(ii) A manifold X is called q–complete if it is q–convex with $K=\varnothing$.

(iii) A manifold X of complex dimension n is called q–concave $(1\leqslant q\leqslant n)$ if there exists a smooth functions $\varphi: X\longrightarrow(a,b]$, $a\in\mathbb{R}\cup\{-\infty\}$, $b\in\mathbb{R}$ such that $X_c=\{\varphi>c\}\Subset X$ for all $c\in(a,b]$ and $\partial\overline{\partial}\varphi$ has at least $n-q+1$ positive eigenvalues outside a compact set.

B.30. EXAMPLE. A manifold is 1–complete if it admits a strictly plurisubharmonic exhaustion function. Recall the following.

B.31. DEFINITION. A complex manifold is called Stein, if the the global holomorphic functions (α) separate points, (β) give local holomorphic coordinates everywhere and (γ) blow up on any disctrete sequence (holomorphic convexity).

It is easy to see that a Stein manifold is 1–complete. The problem whether the converse is true was known as the Levi problem on complex manifolds. It was solved affi rmatively by Hans Grauert [**8**].

B.32. THEOREM. *A manifold is Stein if and only if it admits a strictly plurisubharmonic exhaustion function.*

Grauert's proof uses the so called bumping lemma. Further proofs are based on existence theorems for the $\overline{\partial}$–equations are due to Hörmander [**15**], Kohn [**6**], Demailly [**4**]

The analytic convexity of a manifold is determined by the behaviour of the Levi form of an exhaustion function on the analytic tangent space of sublevel sets. Let D be a relatively compact domain with smooth boundary in a complex manifold X. Let $\rho \in \mathscr{C}^\infty(U)$ defi ned on an open neighbourhood U of $\overline{D}$ such that $D = \{x \in X : \rho(x) < 0\}$ and $d\rho \neq 0$ on ∂D. We say that ρ is a defi ning function of D. The analytic tangent space to ∂D at $x \in \partial D$ is given by $T^{1,0}(\partial D) = \{v \in T^{1,0}(X) : \partial\rho(n) = 0\}$. The defi nition does not depend on the choice of ρ.

B.33. LEMMA. *The number of positive and negative eigenvalues of the Levi form restricted to the analytic tangent plane is independent of the chice of local holomorphic coordinates and defi ning function at a point x.*

For the proof we refer to [**21**].

B.34. DEFINITION. The domain D is called *strongly pseudoconvex* if the Levi form restricted to the analytic tangent space is positive defi nite. D is called *(weakly) pseudoconvex* if the Levi form restricted to the analytic tangent space is positive semi-defi nite.

B.35. EXAMPLE. Let X be a compact manifold and let $(L,h^L) \longrightarrow X$ be a positive line bundle. We consider the Grauert tube (B.28) $T = \{v \in L^* : |v|_{h^{L^*}} < 1\}$. If we denote by $\rho = |v|^2_{h^{L^*}} - 1$ then $\partial\overline{\partial}\rho|_{T^{1,0}(\partial T)} = \pi^*(R^L)|_{T^{1,0}(\partial T)}$ where $\pi : T \longrightarrow X$ is the projection. Thus $\partial\overline{\partial}\rho$ is positive defi nite on the analytic tangential space. Then T is a strictly pseudoconvex domain called the Grauert tube.

Assume that D is strongly pseudoconvex. By replacing ρ with $e^{Ap} - 1$ for $A \gg 1$, we can achieve that the defi ning function has positive Levi form on the whole tangent space $T^{1,0}_x(X)$, $x \in \partial D$ and therefore we can assume that ρ is strictly plurisubharmonic in a neigbourhood of ∂D. It follows that D is a 1–convex manifold. Indeed, the function $\varphi : D \longrightarrow \mathbb{R}$, $\varphi = \frac{1}{\rho^2}$ is strictly plurisubharmonic exhaustion function. Conversely, the smooth sublevel sets of a 1-convex manifold are strongly pseudoconvex.

An important related notion which is also very natural is the following.

B.36. DEFINITION. A manifold X is called *weakly* 1*-complete* or *weakly pseudoconvex* if there exist a plurisubharmonic exhaustion function.

The notion was introduced by S. Nakano [**18**] in order to solve the problem of the inverse of the monoidal transformation. Any 1-convex (and therefore any compact or Stein) manifold is weakly 1-complete. A proper modifi cations of a weakly 1-complete manifold is again weakly 1-complete. In particular a holomorphically complete manifold is weakly 1-complete.

It follows immediately from defi nitions that if X is q–convex (q–concave), X_c is q–convex (q–concave) for c such that the exceptional set is contained in X_c. If X is q–convex, we can choose $b = +\infty$ and φ to be an exhaustion function. This is achieved by composing φ with a convex rapidly function $\lambda : [a,b) \longrightarrow \mathbb{R}$, for example $\lambda(t) = \frac{1}{(t-b)^2}$, and use Remark(*). This is not true for the q–concave case, that us we cannot always choose φ to be an exhaustion function from below. The complex manifold $\mathbb{P}^1 \setminus S^1$, where S^1 is the unit circle in $\mathbb{C}$ canonically embedded, is 1–concave but we cannot take $Q = -\infty$ in the definition. If this were the case, the function φ would be strictly plurisubharmonic in a neighbourhood of S^1 and $S^1 = \{\varphi = -\infty\}$ i.e. S^1 would be pluripolar. But this is a contradiction. 1–concave manifolds for which $a = -\infty$ are called hyperconcave and will be studied in the sequel.

B.37. EXAMPLE. Let X be a compact complex space with isolated singularities. Then X_{reg} is 1–concave and actually hyperconcave. If $p \in X_{\text{sing}}$ we consider a local chart $\tau : U \hookrightarrow \mathbb{C}^N$, $p \in U$, $\tau(p) = 0$. Let $\varphi_p : U \setminus \{p\} \longrightarrow \mathbb{R}$, $\varphi_p(x) = \log|\tau(x)|^2$ which is the pullback of the s plurisubharmonic function $\log|z|^2$ to $U \setminus \{p\}$. Note that $\varphi_p(x) \longrightarrow -\infty$ as $x \longrightarrow p$. By patching togheter the functions φ_p with the help of a partition of unity we obtain a function $\varphi : X_{\text{reg}} \longrightarrow (-\infty, b]$ such that $\{\varphi > c\} \Subset X_{\text{reg}}$ for all $c \leqslant b$ and φ is strictly plurisubharmonic outside a compact set of X_{reg}.

B.38. EXAMPLE. More generally, let X be a compact complex space and let Z be a an analytic subset containing X_{sing}. If $\dim Z = q$, then $X \setminus Z$ is $(q+1)$–concave [**20**].

B.39. EXAMPLE. Let (E, h^E) be a holomorphic hermitian vector bundle of rank r over a compact manifold X of dimension n. Assume that $\Theta(E, h^E)$ has signature (s,t) i.e. for any $e \in E$, $e \neq 0$ the hermitian form $\langle \sqrt{-1}(E, h^E)e, e\rangle$ on $T^{1,0}$ has s positive and t negative eigenvalues. Let us define consider the function $\varphi : E \longrightarrow \mathbb{R}$, $\varphi(v) = |v|^2_{h^E}$. The Levi form $\partial\overline{\partial}\varphi$ restricted to the analytic tangent space $T^{1,0}(\partial E_c)$, where $E_c = \{v \in E : \varphi(v) < c\} \quad (c > 0)$, has $t + r - 1$ positive and s negative eigenvalues. ([**1**, 23], [**10**, p .426]). Replacing φ with $e^{A\varphi}$, $A \gg 1$, we can gain one more positive eigenvalue in the exterior normal direction to the sublevel sets on any compact set. It follows that E_c is $(n - t + 1)$–convex. In particular, if E is negative in the sense of Griffiths i.e. $t = n$ then E_c is 1–convex for $c > 0$. If, on the contrary E is Griffiths positive i.e. $s = n$, $t = 0$, E_c is $(r+1)$–concave.

B.40. EXAMPLE. Let Z be a hypersurface of a compact manifold X and let N_Z be the normal bundle of Z in X. Assume that N_Z is endowed with a hermitian metric such that $\sqrt{-1}\Theta(N_Z)$ has signature (s,t), where $s + t = \dim Z = n$. By the previous example, for each $c > 0$, the subset $(N_Z)_c = \{v \in N_Z : |v|_{h^{N_Z}} < c\}$ is $(n - t + 1)$–convex and $(n - s + 2)$– concave. Let $\psi : N_Z \longrightarrow \mathbb{R}$ such that ψ is a defining function of $(N_Z)_c$ and $\partial\overline{\partial}\psi$ has $(t+1)$ positive and s negative eigenvalues in a neighbourhood of $\partial(N_Z)_c$. One can extend ψ to a function on $L = [Z]$. Moreover, by considering the canonical section $\sigma \in H^0(X, [Z])$ we can define the function $\eta = \psi \circ \sigma$. By [**11**, Proposition 8.3], the Levi form $\partial\overline{\partial}\eta$ of η has $(t+1)$ positive and s negative eigenvalues on a neighbourhood of $W = \{x \in X : \eta(x) < c\}$. Assume for example that N_Z is positive i.e. $s = n$, $t = 0$. Then Z has a 1-concave neighbourhood in X.

We recall now some basic facts of the Andreotti–Grauert theory [**1**], [**13**] complemented with similar results for weakly 1–complete manifolds.

B.41. THEOREM (Finiteness theorem). *Let X be an n–dimensional complex manifold and $E \longrightarrow X$ be a holomorphic vector bundle.*

(i) *If X is q–convex* (*resp. q–concave*), $\dim H^j(X, E) < \infty$ *for* $j \geqslant q$ (*resp.* $j \leqslant n - q - 1$)

(ii) *If X is weakly* 1*–complete and $L \longrightarrow X$ is a line bundle which is positive outside a compact set then* $\dim H^j(X, L^k \otimes E) < \infty$ *for* $j \geqslant 1$ *and k sufficiently large.*

The proof of Andreotti and Grauert is sheaf–theoretic and makes use of the "bumping lemma". It works on complex spaces too. On the analytic side the approach is to represent the sheaf cohomology

$$H^j(X, \mathscr{O}(E)) \cong H^{0,j}_{(2)}(X, E, \text{loc}) := \frac{\{u \in L^{0,j}_2(X, E, \text{loc}) : \overline{\partial}^E u = 0\}}{\{u \in L^{0,j}_2(X, E, \text{loc}) : u = \overline{\partial}^E v,\, v \in L^{0,j-1}_2(X, E, \text{loc})\}} \tag{B.30}$$

Indeed, let us consider the fine sheaf

$$U \longrightarrow W^{0,j}(U, E) = \{u \in L^{0,j}_2(U, E, \text{loc}) : \overline{\partial}^E u \in L^{0,j+1}(U, E, \text{loc})\}.$$

By the Dolbeault–Grothendieck lemma we have a resolution of sheaves

$$0 \longrightarrow \mathscr{O}(E) \longrightarrow W^{0,0}(E) \xrightarrow{\overline{\partial}^E} W^{0,1}(E) \xrightarrow{\overline{\partial}^E} \dots \xrightarrow{\overline{\partial}^E} W^{0,n}(E) \longrightarrow 0$$

and (B.30) follows from the abstract de Rham theorem. There are now two points of view. One is due to Andreotti–Vesentini and consists in introducing a complete hermitian metric on a sublevel set X_c of X and on the fibers of the bundles which represents the cohomology of X_c: i.e. such that the natural morphism

$$H^{0,j}_{(2)}(X_c,E) \longrightarrow H^{0,j}_{(2)}(X_c,E,\mathrm{loc}) \tag{B.31}$$

or

$$H^{0,j}_{(2)}(X_d,L^k\otimes E) \longrightarrow H^{0,j}(X_d,L^k\otimes E)\,, \quad j\geqslant 1\,,k\geqslant 1 \tag{B.32}$$

are isomorphism. The fundamental estimate, which implies the finiteness of the L^2–cohomology, is derived using the modified metrics. By the L^2 Hodge theory A.26 each class of cohomology is represented by a harmonic form with respect to the complete metric. In order to study the cohomology of X, consider the restriction morphisms for $c > d$ and $K \subset X_d$:

Using the representation (B.31), (B.32) and a Runge type approximation theorem one has the following:

B.42. THEOREM (Isomorphism theorem). *The morphisms* (B.32) *are isomorphisms for* $j \geqslant q$ (*resp.* $j \leqslant n-q-1$) *if* X *is* q*– convex* (*resp.* q*– concave*). *The morphisms* (B.32) *are isomorphisms for* $j \geqslant 1$ *and* $k \gg 1$ *under the hypotheses of Theorem B.41(ii).*

The proofs of Theorems B.41 and B.42 using the method described above and technical elements from [**14**] can be found in [**19**].

Another point of view, due to Kohn and Hörmander is to represent the cohomology of X_c by smooth forms up to the boundary. First, by the Dolbeault isomorphism

$$H^j(X_c,\Omega^0(E)) \cong \frac{\{u\in\Omega^{0,j}(X_c,E):\overline{\partial}^E u=0\}}{\overline{\partial}^E\Omega^{0,j-1}(X_c,E)} =: H^{0,j}(X_c,E)$$

Introduce also the space of harmonic forms of the $\overline{\partial}$–Neumann problem:

$$\mathscr{H}^{0,j}(X_c,E)=\{u\in\Omega^{0,j}(\overline{X}_c):u\in\mathrm{Dom}(\overline{\partial}^E)\cap\mathrm{Dom}(\overline{\partial}^{E*})\,,\overline{\partial}^E u=0\,,\overline{\partial}^{E*}u=0\}$$

By definition, such harmonic forms satisfy boundary conditions on ∂X_c since they belong to $\mathrm{Dom}(\overline{\partial}^{E*})$. We have then:

B.43. THEOREM (Representation theorem). (i) *If* X *is* q*– convex* (*resp.* q*– concave*) *then the canonical morphism* $\mathscr{H}^{0,j}(X_c,E) \longrightarrow H^{0,j}(X_c,E)\,, u\to[u]$ *is an isomorphism for* $j\geqslant q$ (*resp.* $j\leqslant n-q-1$).

(ii) *If* X *is weakly* 1*– complete and* L *is positive outside a compact set, the canonical morphism* $\mathscr{H}^{0,j}(X_c,L^k\otimes E) \longrightarrow H^{0,j}(X_c,L^k\otimes E)$ *is an isomorphism for* $j\geqslant 1$ *and* $k\gg 1$.

For the proof of (i) see [**6**, Theorem 4.3.1] and for (ii) [**23**, Theorem 6.2].

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