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On the stability of equivariant embedding of compact CR manifolds with circle action

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Abstract We prove the stability of the equivariant embedding of compact strictly pseudoconvex CR manifolds with transversal CR circle action under circle invariant deformations of the CR structures.

Contents

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1 Introduction and statement of the main results

Let X be a compact strictly pseudoconvex CR manifold. The question of whether or not X admits a CR embedding into a complex Euclidean space has attracted a lot attention. This amounts to showing that the manifold has a sufficiently rich collection of global CR functions. It was shown by Boutet de Monvel [4] that the answer is affirmative if the dimension of X is at least five. The obstructions to constructing global CR functions lie in the first Kohn–Rossi cohomology group $H^1_b(X)$, which is finite dimensional if $\dim X \geq 5$. An essential ingredient in the embedding theorem [4] is the Hodge theory for this group, that will play an important role in the present paper, too.

In contrast, if X has dimension three, X may not be even locally embeddable, see [24, 25,30]. Furthermore, there are examples [1,5,17,31] which show that even when the CR structure on X is locally embeddable (for example, when it is real analytic), it can happen that the global CR functions on X fail to separate points of X. It was shown in [6] that, in a rather precise sense, "generic" perturbations of the standard structure on the three sphere are nonembeddable.

On the other hand, if a compact three dimensional strictly pseudoconvex CR manifold admits a transversal CR S^1-action, it was shown by Lempert [27], Epstein [12] and recently in [18,21] by using the Szegő kernel, that such CR manifolds can always be CR embedded into a complex Euclidean space.

In recent years, much progress has been made in understanding the embedding question from a deformational point of view, that is, for CR structures which lie in a small neighborhood of a fixed embedded structure, see e. g. [3,6,12,13,23,26–28,33].

There are several distinct notions of stability:

1. A CR-structure (X, J) is said to be stable provided that the entire algebra of CR functions deforms continuously under any sufficiently small embeddable deformation J'.
2. A CR-structure (X, J) is said to be stable for a class of embeddable deformations $\mathcal{F}$ provided that the entire algebra of CR-functions deforms continuously under any sufficiently small deformation $J' \in \mathcal{F}$.
3. An embedding $F : (X, J) \to \mathbb{C}^N$ is stable for a class $\mathcal{F}$ of embeddable deformations, provided that for each $J' \in \mathcal{F}$ sufficiently close to J, there is an embedding $F' : (X, J') \to \mathbb{C}^N$, so that F' is a small perturbation of F.

Notion (1) of course implies that, for any given embedding $F : (X, J) \to \mathbb{C}^N$, there is a nearby embedding $F' : (X, J') \to \mathbb{C}^N$, provided that (X, J') is embeddable and J' is sufficiently close to J. We say that two tensors are close if they are close in the C^∞ topology on the appropriate space. For the round 3-sphere the first and second notions, while not explicitly stated, already appear in Burns and Epstein [6], where it is demonstrated that the entire algebra of CR functions is stable for the class of "positive" deformations, with no requirement of S^1-invariance. This work was extended by Epstein to positive deformations of circle bundles in [12]. Lempert [27] showed that all small embeddable deformations of the round sphere are, in fact, positive. In a later paper [28] he went on to show that all small embeddable deformations of CR-structures on the boundaries of strictly pseudoconvex domains in $\mathbb{C}^2$ are stable in the strongest sense, (1), above.

In the present paper we will only use the notion (3) of stability. When X is strictly pseudoconvex, of dimension at least five, Tanaka [32] proved the stability in the sense (3), provided the dimension of the Kohn–Rossi cohomology $H^1_b(X)$ is independent of CR structure. Huang et al. [23] studied the stability of embeddings for a family of strongly pseudoconvex CR manifolds depending in a CR way on the parameters. The CR dependence on the parameters is crucial for the study of deformations of complex structures of isolated singularities. For this topic we refer the readers to [7,22,23,29] and the references therein.

On the other hand, in the case of three dimensional strictly pseudoconvex CR manifolds, Catlin and Lempert [8] showed that unstable CR embeddings exist. The CR manifolds with unstable embeddings arise as unit circle bundles in Hermitian line bundles over projective manifolds. The instability of CR embeddings is a consequence of the instability of very ampleness of line bundles.

As mentioned above, the stability of CR embeddings is closely related to the stability of the first Kohn–Rossi cohomology (see [23,32]). Recently, it was shown in [18] that a compact strictly pseudoconvex CR manifold with a locally free transversal CR S^1-action can be CR embedded into some complex Euclidean space by CR functions lying in the Fourier components with large positive frequency of the space of CR functions. Since Fourier components with large frequency of the first Kohn–Rossi cohomology vanish uniformly under S^1-invariant deformations of the CR structure (see Theorem 3.5), we can expect in analogy to [23,32] that the CR embedding established in [18] should be stable under the S^1-invariant deformations. We will prove this using an additional argument, the stability of the Szegő projector. Similar arguments can be found in a series of papers by Epstein [14–16] on relative index, where the Szegő projector also plays a central role.

Let us now formulate our main results. We refer to Sect. 2.1 for some standard notations and terminology used here. Let (X, HX, J) be a compact CR manifold of dimension $2n-1$, $n \geqslant 2$, endowed with a locally free S^1-action $S^1 \times X \to X$, $(e^{i\theta}, x) \mapsto e^{i\theta}x$ and we let T be the infinitesimal generator of the S^1-action. We assume that this S^1-action is transversal CR, that is, T preserves the CR structure $T^{1,0}X$, and T and $T^{1,0}X \oplus \overline{T^{1,0}X}$ generate the complex tangent bundle to X. Let $\overline{\partial}_b$ be the tangential Cauchy–Riemann operator on X. We denote by $\mathrm{Ker}(\overline{\partial}_b) = \{u \in C^\infty(X) : \overline{\partial}_b u = 0\}$ the space of smooth CR functions. For any $m \in \mathbb{Z}$, we define the mth Fourier component of CR functions $H^0_{b,m}(X) = \{u \in \mathrm{Ker}(\overline{\partial}_b) : Tu = imu\}$. It was shown in [18] that X can be CR embedded into complex Euclidean space by CR functions which lie in the Fourier components of CR functions with large positive frequency m. Precisely, for every $m \in \mathbb{N}$, there exist integers $\{m_j\}_{j=1}^N$ with $m_j \geq m$, $1 \leq j \leq N$, and CR functions $\{f_j\}_{j=1}^N$ with $f_j \in H^0_{b,m_j}(X)$ such the (equivariant) CR map from X to $\mathbb{C}^N$

$$\Phi : X \to \mathbb{C}^N,\ x \mapsto (f_1(x), \dots, f_N(x)), \tag{1.1}$$

is an embedding. Our goal is to show that such an embedding is stable under S^1-invariant deformations of the CR structure (cf. Definition 2.1). Our main result is the following.

Theorem 1.1 *Let (X, HX, J) be a compact connected strictly pseudoconvex CR manifold with a locally free transversal CR S^1-action. Let $\{J_t\}_{t\in(-\delta_0,\delta_0)}$ be any S^1-invariant deformation of J. Then there is a positive integer m_0 such that every CR embedding $\Phi = (\Phi_1, \dots, \Phi_N) : (X, HX, J) \to \mathbb{C}^N$ with $\Phi_j \in H^0_{b,m_j}(X)$, $m_j > m_0$, $j = 1, \dots, N$, is stable with respect to the deformation $\{J_t\}_{t\in(-\delta_0,\delta_0)}$, that is, for $|t|$ small enough there exists an S^1-equivariant CR embedding f_t of the structure J_t such that f_t converges to f as $t \to 0$ in the C^m topology for any non-negative $m \in \mathbb{Z}$.*

This paper is organized as follows. In Sect. 2, we set up notation and terminology. Section 3 is devoted to study the S^1-invariant deformation of CR structure. Furthermore, will prove the simultaneous vanishing theorem of Fourier component of Kohn-Rossi cohomology. In Sect. 4, we will be concerned with the stability of the Szegő kernel of Fourier components of Kohn–Rossi cohomology. Using the stability of Szegő kernel, we will prove Theorem 1.1 in Sect. 5.

2 Preliminaries

In this section we recall necessary notions of CR geometry, such as the canonical local coordinates of Baouendi–Rothschild–Treves, and the Tanaka–Webster connection. We also formulate a vanishing theorem for the Fourier components of Kohn–Rossi cohomology.

2.1 Set up and terminology

Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n-1$, $n \geq 2$, where $T^{1,0}X$ is a CR structure of X, that is, $T^{1,0}X$ is a subbundle of the complexified tangent bundle $\mathbb{C}TX$ of rank $n-1$ satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$ and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. There is a unique subbundle HX of TX such that $\mathbb{C}HX = T^{1,0}X \bigoplus T^{0,1}X$, i.e., HX is the real part of $T^{1,0}X \bigoplus T^{0,1}X$. Furthermore, there exists a homomorphism $J : HX \to HX$ such that $J^2 = -\mathrm{id}$, where id denotes the identity $\mathrm{id} : \mathbb{C}HX \to \mathbb{C}HX$. By complex linear extension of J to $\mathbb{C}TX$, the i-eigenspace of J is given by $T^{1,0}X = \{V \in \mathbb{C}HX : JV = iV\}$. We shall also write (X, HX, J) to denote a compact CR manifold. Let E be a smooth vector bundle over X. We use $\Gamma(E)$ to denote the space of C^∞-smooth sections of E on X.

Let (X, HX, J) be a compact CR manifold. Let $\Omega \subset \mathbb{R}$ be an open neighborhood of 0. We say that $\{J_t\}_{t\in\Omega}$ is a deformation of J if

(I) For each $t \in \Omega$, there is an endomorphism $J_t : HX \to HX$ with $J_t^2 = -\mathrm{id}$ and the i eigenspace $T_t^{1,0}X = \{U \in \mathbb{C}HX : J_t U = iU\}$ is a CR structure on X.
(II) $J_0 = J$.
(III) J_t depends smoothly on t, that is, for every $U \in HX$ and $V^* \in T^*X$ we have $\langle J_t U\,,\ V^* \rangle \in C^\infty(\Omega)$.

From now on, we assume that (X, HX, J) admits an S^1-action: $S^1 \times X \to X$, $(e^{i\theta}, x) \mapsto e^{i\theta} \circ x$. Here, we use $e^{i\theta}$ to denote the S^1-action. Let $T \in C^\infty(X, TX)$ denote the global real vector field induced by the S^1-action given as follows

$$(Tu)(x) = \frac{\partial}{\partial\theta}\left(u(e^{i\theta} \circ x)\right)\Big|_{\theta=0}, \ \ u \in C^\infty(X). \tag{2.1}$$

We say that the S^1-action $e^{i\theta}(0 \leq \theta < 2\pi)$ is CR if

$$[T, \Gamma(T^{1,0}X)] \subset \Gamma(T^{1,0}X), \tag{2.2}$$

where $[\cdot\,,\cdot]$ denotes the Lie bracket between the smooth vector fields on X. Furthermore, we say that the S^1- action is transversal if for each $x \in X$,

$$\mathbb{C}T_x X = \mathbb{C}T(x) \oplus T_x^{1,0}(X) \oplus T_x^{0,1}X. \tag{2.3}$$

From now on, we assume that the S^1-action on (X, HX, J) is transversal and CR. Let $\{J_t\}_{t\in\Omega}$ be a deformation of J, where $\Omega \subset \mathbb{R}$ is an open neighborhood $0 \in \Omega$. As before, put $T_t^{1,0}X = \{U \in \mathbb{C}HX : J_tU = iU\}$. We need

Definition 2.1 With the notations above, we say that $\{J_t\}_{t\in\Omega}$ are S^1-invariant deformations of J if $[T, \Gamma(T_t^{1,0}X)] \subset \Gamma(T_t^{1,0}X)$ for every $t \in \Omega$.

Definition 2.2 Let $f : (X, HX, J) \to \mathbb{C}^k$ be a CR embedding and let $\{J_t\}_{t\in\Omega}$ be S^1-invariant deformations of J, where $\Omega \subset \mathbb{R}$ is an open neighborhood of 0. We say that f is stable with respect to $\{J_t\}_{t\in\Omega}$ if there is a $\delta > 0$ with $[-\delta, \delta] \subset \Omega$ such that for every $t \in (-\delta, \delta)$, we can find a CR embedding $f_t : (X, HX, J_t) \to \mathbb{C}^k$ and $\lim_{t\to 0} \|f_t - f\|_{C^m(X,\mathbb{C}^k)} = 0$, for every $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Lemma 2.3 *With the notations used above, we have $L_T J = 0$ on HX, where L_T denotes the Lie derivative along the direction T.*

Proof For any $U \in \Gamma(T^{1,0}X)$, $L_T J(U) = L_T(JU) - JL_TU = \sqrt{-1}L_TU - \sqrt{-1}L_TU = 0$. Here, we have used the fact that the S^1-action is CR, that is, $L_TU \in \Gamma(T^{1,0}X)$ for any $U \in \Gamma(T^{1,0}X)$. For any $V \in \Gamma(T^{0,1}X)$, we have $L_TJ(V) = \overline{L_TJ(\overline{V})} = 0$. Since HX is the real part of $T^{1,0}X \bigoplus T^{0,1}X$, the Lemma follows. □

Since $[\Gamma(T^{1,0}X), \Gamma(T^{1,0}X)] \subset \Gamma(T^{1,0}X)$, we have $[JU, JV] - [U, V] \in C^\infty(X, HX)$ for all $U, V \in C^\infty(X, HX)$. Let ω_0 be the global real 1-form dual to T, that is,

$$\langle \omega_0, T\rangle = 1, \langle \omega_0, HX\rangle = 0. \tag{2.4}$$

Then for each $x \in X$, we define a quadratic form on HX by

$$\mathcal{L}_x(U, V) = -d\omega_0(JU, V), \forall\, U, V \in H_xX. \tag{2.5}$$

The quadratic form is called the Levi form at x. We extend $\mathcal{L}$ to $\mathbb{C}HX$ by complex linear extension. Then for $U, V \in T_x^{1,0}X$,

$$\mathcal{L}_x(U, \overline{V}) = -d\omega_0(JU, \overline{V}) = -id\omega_0(U, \overline{V}). \tag{2.6}$$

Definition 2.4 We say $T^{1,0}X$ is a strictly pseudoconvex structure and X is a strictly pseudoconvex CR manifold if the Levi form $\mathcal{L}_x$ is a positive definite quadratic form on H_xX for each $x \in X$.

In the following, we always assume that X is a compact connected strictly pseudoconvex CR manifold with a transversal CR S^1-action. It should be noted that a strictly pseudoconvex CR manifold is always a contact manifold. From (2.4), we see that ω_0 is a contact form, HX is the contact plane and T is the Reeb vector field. Using (2.5) we may extend the Levi form to a Riemannian metric g on TX, which will play a crucial role in the sequel.

Definition 2.5 Let X be a compact strictly pseudoconvex CR manifold with a transversal CR S^1-action. Let g be the Riemannian metric given by

$$g(U, V) = \mathcal{L}_x(U, V), \quad g(U, T) = g(T, U) = 0, \quad g(T, T) = 1, \tag{2.7}$$

for any $U, V \in H_xX$, $x \in X$. This is called the Webster metric on X.

The volume form associated with the Webster metric is denoted by dv_X and by direct calculation

$$dv_X = \omega_0 \wedge \frac{(d\omega_0)^{n-1}}{(n-1)!}. \tag{2.8}$$

The volume form dv_X associated with the Webster metric depends only on the contact form ω_0.

For $U, V \in T^{1,0}_x X$, we can check that $\mathcal{L}_x(U, V) = \langle\, \omega_0(x), [J\mathscr{U}, \mathscr{V}](x)\, \rangle = 0$, where $\mathscr{U}, \mathscr{V} \in \Gamma(T^{1,0}X)$ with $\mathscr{U}(x) = U$, $\mathscr{V}(x) = V$. Thus, $\mathcal{L}_x(U, \overline{V}) = -id\omega_0(U, \overline{V})$ is a positive definite Hermitian quadratic form on $T^{1,0}X$. We extend the Webster metric g to $\mathbb{C}TX$ by complex linear extension. The Webster metric g on X induces a Hermitian metric $\langle \cdot \,|\, \cdot \rangle_g$ on $\mathbb{C}TX$:

$$\langle\, U \mid V\, \rangle_g := g(U, \overline{V}), \quad U, V \in \mathbb{C}TX. \tag{2.9}$$

It is easy to check that the Webster metric is J-invariant on HX, so we have the pointwise orthogonal decomposition

$$\mathbb{C}T_x X = \mathbb{C}T(x) \oplus T^{1,0}_x(X) \oplus T^{0,1}_x X. \tag{2.10}$$

We call $\langle \cdot \,|\, \cdot \rangle_g$ the Webster Hermitian metric.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. Define the vector bundle of $(0, q)$-forms by $\Lambda^q T^{*0,1}X$. Let $D \subset X$ be an open subset. Then $\Omega^{0,q}(D)$ denotes the space of smooth sections of $\Lambda^q T^{*0,1}X$ over D.

Fix $\theta_0 \in [0, 2\pi)$. Let

$$de^{i\theta_0} : \mathbb{C}T_x X \to \mathbb{C}T_{e^{i\theta_0} \circ x} X$$

denote the differential map of $e^{i\theta_0} : X \to X$. By the property of transversal CR S^1-action, we can check that

$$\begin{aligned} de^{i\theta_0} &: T^{1,0}_x X \to T^{1,0}_{e^{i\theta_0} \circ x} X, \\ de^{i\theta_0} &: T^{0,1}_x X \to T^{0,1}_{e^{i\theta_0} \circ x} X, \\ de^{i\theta_0}&(T(x)) = T(e^{i\theta_0} \circ x). \end{aligned} \tag{2.11}$$

Let $(e^{i\theta_0})^* : \Lambda^q(\mathbb{C}T^*X) \to \Lambda^q(\mathbb{C}T^*X)$ be the pull back of $e^{i\theta_0}$, $q = 0, 1, \ldots, n-1$. From (2.11), we can check that for every $q = 0, 1, \ldots, n-1$

$$(e^{i\theta_0})^* : \Lambda^q T^{*0,1}_{e^{i\theta_0} \circ x} X \to \Lambda^q T^{*0,1}_x X. \tag{2.12}$$

For $u \in \Omega^{0,q}(X)$ we define Tu as follows:

$$(Tu)(X_1, \ldots, X_q) := \frac{\partial}{\partial\theta} \Big((e^{i\theta})^* u(X_1, \ldots, X_q) \Big)\Big|_{\theta=0}, \quad X_1, \ldots, X_q \in T^{1,0}_x X. \tag{2.13}$$

From (2.12) and (2.13), we have $Tu \in \Omega^{0,q}(X)$ for all $u \in \Omega^{0,q}(X)$. From the definition of Tu it is easy to check that $Tu = L_T u$ for $u \in \Omega^{0,q}(X)$, where $L_T u$ is the Lie derivative of u along the direction T.

Let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator. It is straightforward from (2.11) and (2.13) to see that

$$T\overline{\partial}_b = \overline{\partial}_b T \text{ on } \Omega^{0,q}(X). \tag{2.14}$$

For every $m \in \mathbb{Z}$, put $\Omega^{0,q}_m(X) := \{u \in \Omega^{0,q}(X) : Tu = imu\}$. From (2.14) we have the $\overline{\partial}_b$-complex for every $m \in \mathbb{Z}$:

$$\overline{\partial}_b : \cdots \to \Omega^{0,q-1}_m(X) \to \Omega^{0,q}_m(X) \to \Omega^{0,q+1}_m(X) \to \cdots. \tag{2.15}$$

For every $m \in \mathbb{Z}$, the mth Fourier component of Kohn–Rossi cohomology is defined as follows

$$H^q_{b,m}(X) := \frac{\operatorname{Ker} \overline{\partial}_b : \Omega^{0,q}_m(X) \to \Omega^{0,q+1}_m(X)}{\operatorname{Im} \overline{\partial}_b : \Omega^{0,q-1}_m(X) \to \Omega^{0,q}_m(X)}. \tag{2.16}$$

Definition 2.6 We say that a function $u \in C^\infty(X)$ is a Cauchy–Riemann (CR for short) function if $\overline{\partial}_b u = 0$, or in the other words, $\overline{Z}u = 0$ for all $Z \in \Gamma(T^{1,0}X)$.

For $m \in \mathbb{Z}$, when $q = 0$, $H^0_{b,m}(X)$ is a subspace of the space of CR functions which lie in the im eigenspace of T and we call $H^0_{b,m}(X)$ the mth Fourier component of the space of CR functions. The paper [20] gives asymptotic bounds for the Fourier components of the Kohn-Rossi cohomology.

2.2 Canonical local coordinates

In this work, we need the following result due to Baouendi–Rothschild–Treves.

Theorem 2.7 [2, Proposition I.2] *Let X be a compact CR manifold of* $\dim X = 2n-1$, $n \geq 2$ *with a transversal CR S^1-action. For $x_0 \in X$, there exist local coordinates $(x_1, \ldots, x_{2n-1}) = (z, \theta) = (z_1, \ldots, z_{n-1}, \theta)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n-1$, $x_{2n-1} = \theta$, defined in a small neighborhood $D = \{(z, \theta) : |z| < \varepsilon, |\theta| < \delta\}$ centered at x_0 such that*

$$\begin{aligned} T &= \frac{\partial}{\partial \theta} \\ Z_j &= \frac{\partial}{\partial z_j} + i\frac{\partial \varphi(z)}{\partial z_j}\frac{\partial}{\partial \theta}, \quad j = 1, \ldots, n-1 \end{aligned} \tag{2.17}$$

where $\{Z_j(x)\}_{j=1}^{n-1}$ form a basis of $T^{1,0}_x X$ for each $x \in D$, and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of θ.

We call D a canonical local patch, $x = (z, \theta)$ canonical local coordinates on D and $\{Z_j\}_{j=1}^{n-1}$ a canonical frame of $T^{1,0}X$ over D. On D, the contact form is given by

$$\omega_0 = d\theta - i\sum_{j=1}^{n-1} \frac{\partial \varphi(z)}{\partial z_j} dz_j + i\sum_{j=1}^{n-1} \frac{\partial \varphi(z)}{\partial \overline{z}_j} d\overline{z}_j$$

and the Levi form on $T^{1,0}X$ can be expressed as

$$\mathcal{L}_x = -id\omega_0 = 2\sum_{k,j=1}^{n-1} \frac{\partial^2 \varphi(z)}{\partial z_k \partial \overline{z}_j} dz_k \wedge d\overline{z}_j. \tag{2.18}$$

For $x \in D$, $\theta \in [0, 2\pi)$ with $e^{i\theta} \circ x \in D$, it is straightforward to see that $de^{i\theta}(Z_j(x)) = Z_j(e^{i\theta} \circ x)$ for $1 \leq j \leq n-1$.

2.3 Hermitian CR geometry

Definition 2.8 [19, Definition 1.18] Let D be an open set and let $V \in C^\infty(D, \mathbb{C}TX)$ be a vector field on D. We say that V is rigid if

$$de^{i\theta}(V(x)) = V(e^{i\theta} \circ x) \tag{2.19}$$

for any $x, \theta \in [0, 2\pi)$ satisfying $x \in D$, $e^{i\theta} \circ x \in D$.

The canonical frame $\{Z_j\}_{j=1}^{n-1}$ defined in (2.17) are rigid vector fields on the canonical local patch. Let D be an open subset of X and U be a rigid vector field on D. Then for any $\theta_0 \in [0, 2\pi)$, $de^{i\theta_0}(U)$ is still a rigid vector field on $e^{i\theta_0}D := \{e^{i\theta_0} \circ x : x \in D\}$.

Definition 2.9 [19, Definition 1.19] Let $\langle \cdot \,|\, \cdot \rangle$ be a Hermitian metric on $\mathbb{C}TX$. We say that $\langle \cdot \,|\, \cdot \rangle$ is rigid if for rigid vector fields V, W on Ω, where Ω is any open set, we have

$$\langle V(x)|W(x)\rangle = \langle (de^{i\theta}V)(e^{i\theta} \circ x)|(de^{i\theta}W)(e^{i\theta} \circ x)\rangle, \quad \forall x \in \Omega, \theta \in [0, 2\pi). \tag{2.20}$$

Lemma 2.10 *The Webster Hermitian metric $\langle \cdot \,|\, \cdot \rangle_g$ defined in (2.9) is a rigid Hermitian metric on $\mathbb{C}TX$.*

Proof Let Ω be an open subset of X and $U, V \in T^{1,0}X$ be rigid vector fields on Ω. For any $x_0 \in \Omega$, choose canonical coordinates $x = (z, \theta)$ centered at x_0 and a canonical local patch $D = \{(z, \theta) : |z| < \varepsilon, |\theta| < \delta\}$ with $\overline{D} \subset \Omega$. Let $\{Z_j\}_{j=1}^{n-1}$ be a canonical frame over D. Then on D, $U = \sum_{j=1}^{n-1} a_j(z, \theta)Z_j$ and $V = \sum_{j=1}^{n-1} b_j(z, \theta)Z_j$. Since U, V are rigid vector fields we have that on D, $\frac{\partial}{\partial\theta}a_j(z, \theta) = \frac{\partial}{\partial\theta}b_j(z, \theta) = 0$ for $1 \le j \le n-1$. Then for $|\theta| < \delta$,

$$\langle de^{i\theta}U(x_0) \,|\, de^{i\theta}V(x_0)\rangle_g = \sum_{j,k=1}^{n-1} a_j(0, 0)\overline{b}_k(0, 0) \left\langle de^{i\theta}Z_j(x_0) \,|\, de^{i\theta}Z_k(x_0)\right\rangle_g. \tag{2.21}$$

Substituting $de^{i\theta}Z_j(x_0) = \frac{\partial}{\partial z_j} + i\frac{\partial\varphi}{\partial z_j}(0)\frac{\partial}{\partial\theta}|_{(0,\theta)}$ to (2.21) we have

$$\left\langle de^{i\theta}U(x_0) \,|\, de^{i\theta}V(x_0)\right\rangle_g = \langle U(x_0) \,|\, V(x_0)\rangle_g, \quad \forall\, |\theta| < \delta. \tag{2.22}$$

Now, we claim that the above equality is also true for all $\theta \in [0, 2\pi)$. Let $0 < \delta_1 < 2\pi$ be any number such that

$$\langle de^{i\theta}U(x_0) \,|\, de^{i\theta}V(x_0)\rangle_g = \langle U(x_0) \,|\, V(x_0)\rangle_g, \forall\, 0 \le \theta < \delta_1. \tag{2.23}$$

First, we show that

$$\langle de^{i\delta_1}U(x_0) \,|\, de^{i\delta_1}V(x_0)\rangle_g = \langle U(x_0) \,|\, V(x_0)\rangle_g. \tag{2.24}$$

Set $U_1 = de^{i\delta_1}U$, $V_1 = de^{i\delta_1}V$ and $y_0 = e^{i\delta_1} \circ x_0$. Since U_1, V_1 are still rigid vector fields on $e^{i\delta_1}\Omega$, then by the same argument in the proof of (2.22), there exist $\sigma > 0$ such that

$$\langle de^{i\theta}U_1(y_0) \,|\, de^{i\theta}V_1(y_0)\rangle_g = \langle U_1(y_0) \,|\, V_1(y_0)\rangle_g, \quad \forall |\theta| < \sigma. \tag{2.25}$$

Thus, by (2.23) and (2.25) we have

$$\langle de^{i\delta_1}U(x_0) \,|\, de^{i\delta_1}V(x_0)\rangle_g = \langle de^{i(\delta_1-\frac{\sigma}{2})}U(x_0) \,|\, de^{i(\delta_1-\frac{\sigma}{2})}V(x_0)\rangle_g = \langle U(x_0) \,|\, V(x_0)\rangle_g. \tag{2.26}$$

Thus, we get the conclusion of (2.24). On the other hand, by (2.25) and (2.26) we have

$$\langle U(x_0) \,|\, V(x_0)\rangle_g = \langle de^{i(\delta_1+\varepsilon)}U(x_0) \,|\, de^{i(\delta_1+\varepsilon)}V(x_0)\rangle, \quad \forall\, \varepsilon \in (0, \sigma). \tag{2.27}$$

Thus, from (2.24) and (2.27) we have

$$\langle de^{i\theta}U(x_0) \,|\, de^{i\theta}V(x_0)\rangle = \langle U(x_0) \,|\, V(x_0)\rangle_g, \quad \forall\, 0 \le \theta < \delta_1 + \sigma. \tag{2.28}$$

From (2.24) and (2.28) we get the conclusion of the claim and the lemma follows. □

For the existence of rigid Hermitian metric on general CR manifold with S^1-action, we refer the readers to [19, Theorem 9.2].

From now on, we will fix the Webster Hermitian metric as a rigid Hermitian metric on $\mathbb{C}TX$. For convenience, we use the notation $\langle \cdot \mid \cdot \rangle$ to denote $\langle \cdot \mid \cdot \rangle_g$. The rigid Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TX$ induces by duality a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$-forms $\Lambda^q T^{*0,1}X$, $q = 0, 1 \ldots, n-1$. We shall also denote all these induced metrics by $\langle \cdot \mid \cdot \rangle$. From (2.10) we have the pointwise orthogonal decomposition:

$$\mathbb{C}T^*X = T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0 : \lambda \in \mathbb{C}\}. \tag{2.29}$$

For every $v \in \Lambda^q T^{*0,1}X$, we write $|v|^2 := \langle v|v\rangle$. Let $(\cdot \mid \cdot)$ be the L^2 inner product on $\Omega^{0,q}(X)$ induced by $\langle \cdot \mid \cdot \rangle$ and let $\|\cdot\|$ denote the corresponding norm. Then for all $u, v \in \Omega^{0,q}(X)$

$$(u|v) = \int_X \langle u|v\rangle dv_X, \tag{2.30}$$

where dv_X given in (2.8) is the volume form on X induced by the rigid Hermitian metric. As before, for $m \in \mathbb{Z}$, we denote by

$$\Omega^{0,q}_m(X) = \left\{u \in \Omega^{0,q}(X) : Tu = imu\right\} \tag{2.31}$$

the im eigenspace of T. Let $L^2_{(0,q),m}(X)$ be the completion of $\Omega^{0,q}_m(X)$ under the L^2 inner product.

Let $\overline{\partial}^*_b : \Omega^{0,q+1}(X) \to \Omega^{0,q}(X)$ be the formal adjoint of $\overline{\partial}_b$ with respect to $(\cdot \mid \cdot)$. Since the Hermitian metrics $\langle \cdot \mid \cdot \rangle$ are rigid, we can check that

$$T\overline{\partial}^*_b = \overline{\partial}^*_b T \quad \text{on } \Omega^{0,q}(X), \quad \forall q = 1, \ldots, n-1 \tag{2.32}$$

and from (2.32) we have

$$\overline{\partial}^*_b : \Omega^{0,q+1}_m(X) \to \Omega^{0,q}_m(X), \quad \forall m \in \mathbb{Z}. \tag{2.33}$$

Put

$$\Box^{(q)}_b := \overline{\partial}_b\overline{\partial}^*_b + \overline{\partial}^*_b\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q}(X).$$

Combining (2.14), (2.32) and (2.33), we have

$$T\Box^{(q)}_b = \Box^{(q)}_b T \text{ on } \Omega^{0,q}(X), \quad \forall q = 0, 1, \ldots, n-1. \tag{2.34}$$

A direct consequence of (2.34) is

$$\Box^{(q)}_b : \Omega^{0,q}_m(X) \to \Omega^{0,q}_m(X), \quad \forall m \in \mathbb{Z}. \tag{2.35}$$

We will write $\Box^{(q)}_{b,m}$ to denote the restriction of $\Box^{(q)}_b$ on $\Omega^{0,q}_m(X)$. For every $m \in \mathbb{Z}$, we extend $\Box^{(q)}_{b,m}$ to $L^2_{(0,q),m}(X)$ by

$$\Box^{(q)}_{b,m} : \mathrm{Dom}(\Box^{(q)}_{b,m}) \subset L^2_{(0,q),m}(X) \to L^2_{(0,q),m}(X), \tag{2.36}$$

where $\mathrm{Dom}(\Box^{(q)}_{b,m}) = \{u \in L^2_{(0,q),m}(X) : \Box^{(q)}_{b,m}u \in L^2_{(0,q),m}(X)$ in the sense of distribution$\}$.

The following result follows from Kohn's L^2-estimate (see Theorem 8.4.2 in [9]).

Theorem 2.11 *For every $s \in \mathbb{N}_0$, there exists a constant $C_s > 0$ such that*

$$\|u\|_{s+1} \le C_s\left(\|\Box^{(q)}_b u\|_s + \|Tu\|_s + \|u\|_s\right), \quad \forall u \in \Omega^{0,q}(X) \tag{2.37}$$

where $\|\cdot\|_s$ denotes the standard Sobolev norm of order s on X.

From Theorem 2.11, we deduce:

Theorem 2.12 *For $m \in \mathbb{Z}$ and for every $s \in \mathbb{N}_0$, there is a constant $C_{s,m} > 0$ such that*

$$\|u\|_{s+1} \le C_{s,m}\left(\|\Box^{(q)}_{b,m}u\|_s + \|u\|_s\right), \forall u \in \Omega^{0,q}_m(X). \tag{2.38}$$

According to Theorem 2.12 and a standard argument in functional analysis, we deduce the following Hodge theory for $\Box^{(q)}_{b,m}$ (see Section 3 in [10]).

Theorem 2.13 *Let $q \in \{0, 1, \dots, n-1\}$, $m \in \mathbb{Z}$. $\Box^{(q)}_{b,m} : \mathrm{Dom}(\Box^{(q)}_{b,m}) \subset L^2_{(0,q),m}(X) \to L^2_{(0,q),m}(X)$ is a self-adjoint operator. Set*

$$\mathcal{H}^q_{b,m}(X) = \left\{u \in \mathrm{Dom}(\Box^{(q)}_{b,m}) : \Box^{(q)}_{b,m}u = 0\right\}. \tag{2.39}$$

Then $\mathcal{H}^q_{b,m}(X)$ is a finite dimensional space with $\mathcal{H}^q_{b,m}(X) \subset \Omega^{0,q}_m(X)$ and the map

$$\mathcal{H}^q_{b,m}(X) \cong H^q_{b,m}(X), \quad \alpha \mapsto [\alpha], \tag{2.40}$$

is an isomorphism, where $[\alpha]$ is the cohomology class of α in $H^q_{b,m}(X)$. In particular,

$$\dim H^q_{b,m}(X) < \infty, \forall\, m \in \mathbb{Z}, \quad \forall\, 0 \le q \le n-1. \tag{2.41}$$

We call $\mathcal{H}^q_{b,m}(X)$ the harmonic space with respect to $\Box^{(q)}_{b,m}$.

2.4 Tanaka–Webster connection

Let (X, HX, J) be a compact strictly pseudoconvex CR manifold with a transversal CR S^1-action. Let T be the globally real vector field induced by the S^1-action and ω_0 be its dual form. Then it is easy to check that ω_0 is a contact form with HX as the contact structure and T, ω_0 satisfy

$$\langle\omega_0, T\rangle = 1,\ \langle\omega_0, HX\rangle = 0,\ T\rfloor d\omega_0 = 0. \tag{2.42}$$

In this section, with the notations defined above, we will review the Tanaka–Webster connection [32,34] and the notions are mainly from [11,32].

Proposition 2.14 (Proposition 3.1 in [32]) *There is a unique linear connection (Tanaka–Webster connection) denoted by $\nabla : \Gamma(TX) \to \Gamma(T^*X \otimes TX)$ satisfying the following conditions:*

1. *The contact structure HX is parallel, i.e., $\nabla_U\Gamma(HX) \subset \Gamma(HX)$ for $U \in \Gamma(TX)$.*
2. *The tensor fields $T, J, d\omega_0$ are all parallel, i.e., $\nabla T = 0, \nabla J = 0, \nabla d\omega_0 = 0$.*
3. *The torsion τ of ∇ satisfies: $\tau(U, V) = d\omega_0(U, V)T$, $\tau(T, JU) = -J\tau(T, U)$, $U, V \in C^\infty(X, HX)$.*

Recall that $\nabla J \in \Gamma(T^*X \otimes \mathscr{L}(HX, HX))$, $\nabla d\omega_0 \in \Gamma(T^*X \otimes \Lambda^2(\mathbb{C}T^*X))$ are defined by $(\nabla_U J)W = \nabla_U(JW) - J\nabla_U W$ and $\nabla_U d\omega_0(W, V) = Ud\omega_0(W, V) - d\omega_0(\nabla_U W, V) - d\omega_0(W, \nabla_U V)$ for $U \in \Gamma(TX)$, $W, V \in \Gamma(HX)$. Similarly, for any $u \in \Omega^{0,q}(X)$, we can define $\nabla u \in \Gamma(T^*X \otimes \Lambda^q(\mathbb{C}T^*X))$ in the standard way. By (1) and $\nabla J = 0$ in (2), we have $\nabla_U\Gamma(T^{1,0}X) \subset \Gamma(T^{1,0}X)$ and $\nabla_U\Gamma(T^{0,1}X) \subset \Gamma(T^{0,1}X)$ for $U \in \Gamma(TX)$. Moreover, $\nabla J = 0$ and $\nabla d\omega_0 = 0$ imply that the Tanaka–Webster connection is compatible with the Webster metric. By definition, the torsion of ∇ is given by $\tau(W, U) = \nabla_W U - \nabla_U W - [W, U]$ for $U, V \in \Gamma(TX)$ and $\tau(T, U)$ for $U \in \Gamma(HX)$ is called pseudohermitian torsion.

The existence of an S^1-action on X is not necessary in the definition of Tanaka–Webster connection. But if X admits a transversal CR S^1-action, by (2) in Proposition 2.14 and Lemma 2.16, we have $\tau(T, U) = 0$ for $U \in \Gamma(HX)$. The vanishing of pseudohermitian torsion admits an important geometric interpretation. Webster [34] proved that for a strictly pseudoconvex CR manifold, the pseudohermitian torsion vanishes if and only if the 1-parameter group of transformations of X induced by T consists of CR automorphisms.

Lemma 2.15 (Lemma 3.2 in [32]) *Let* $U \in \Gamma(T^{1,0}X)$. *Then* $\nabla_T U = L_T U + J_T U$, *where* L_T *denotes the Lie derivation and* J_T *is given by* $J_T = -\frac{1}{2} J \circ L_T J$.

Since the S^1-action on X is CR, by (2.2), the CR structure on X is invariant with respect to the S^1-action, that is, $L_T J = 0$. By Lemma 2.15 and since $L_T J = 0$, we have

Lemma 2.16 *Let X be a compact strictly pseudoconvex CR manifold with a transversal CR S^1-action. Let ∇ be the Tanaka–Webster connection on TX. Then we have*

$$J_T U = 0 \quad \text{and} \quad \nabla_T U = L_T U \text{ for } U \in \Gamma(T^{1,0}X), \tag{2.43}$$

where T denotes the induced vector field by the S^1-action and J is the CR structure tensor on X.

Since $\nabla_T u \in \Omega^{0,q}(X)$ for $u \in \Omega^{0,q}(X)$, then for any smooth sections $U_1, \ldots, U_q \in \Gamma(T^{0,1}X)$ we have

$$\begin{aligned}(\nabla_T u)(\overline{U}_1, \ldots, \overline{U}_q) &= T(u(\overline{U}_1, \ldots, \overline{U}_q)) - \sum_{j=1}^{q} u((\overline{U}_1, \ldots, \nabla_T \overline{U}_j, \ldots, \overline{U}_q)) \\ &= T(u(\overline{U}_1, \ldots, \overline{U}_q)) - \sum_{j=1}^{q} u((\overline{U}_1, \ldots, L_T \overline{U}_j, \ldots, \overline{U}_q)) \\ &= (L_T u)(\overline{U}_1, \ldots, \overline{U}_q).\end{aligned}$$

Thus, we have $\nabla_T u = L_T u$ for $u \in \Omega^{0,q}(X)$.

Let R be the curvature of Tanaka–Webster connection. Let $e_1, \ldots, e_{n-1}$ be any orthonormal basis of $T^{1,0}X$ with respect to the fixed rigid Hermitian metric $\langle \cdot \mid \cdot \rangle$, that is, $\langle e_i | e_j \rangle = \delta_{ij}$. Then the Ricci curvature operator R_* is defined by (page 34 in [32])

$$R_* U = -i \sum_{k=1}^{n-1} R(e_k, \overline{e}_k) J U, \quad U \in \Gamma(HX). \tag{2.44}$$

By duality, we can extend the Ricci operator R_* to $\Omega^{0,q}(X)$ in the following way

$$R_* u(\overline{U}_1, \ldots, \overline{U}_q) = \sum_{j=1}^{q} u(\overline{U}_1, \ldots, R_* \overline{U}_j, \ldots, \overline{U}_q) \tag{2.45}$$

for all $u \in \Omega^{0,q}(X)$ and $U_1, \ldots, U_q \in \Gamma(T^{1,0}X)$. It is straightforward to check that R_* is a self-adjoint operator with respect to the inner product $\langle \cdot \mid \cdot \rangle$ on $\Omega^{0,q}(X)$.

2.5 Pseudohermitian geometry

Let $\{Z_\alpha\}_{\alpha=1}^{n-1}$ be the canonical frame of $T^{1,0}X$ on a canonical open set D in the BRT trivialization given in Theorem 2.7. Then $\{dz_\alpha\}_{\alpha=1}^{n-1}$ is a dual frame of $\{Z_\alpha\}_{\alpha=1}^{n-1}$. Write $Z_{\overline{\alpha}} = \overline{Z_\alpha}$,

$\theta^\alpha = dz_\alpha, \theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. Then $\{\theta^\alpha\}$ is an admissible coframe on D. Then $d\omega_0 = ig_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$, where $g_{\alpha\overline{\beta}} = 2\frac{\partial^2\varphi(z)}{\partial z_\alpha \partial \overline{z}_\beta}$. Let ω_α^β be the connection form of Tanaka–Webster connection with respect to the frame $\{Z_\alpha\}_{\alpha=1}^{n-1}$. Thus, we have

$$\nabla Z_\alpha = \omega_\alpha^\beta Z_\beta, \quad \nabla Z_{\overline{\alpha}} = \omega_{\overline{\alpha}}^{\overline{\beta}} Z_{\overline{\beta}}, \nabla T = 0.$$

By direct calculation,

$$\omega_\alpha^\beta = g^{\overline{\sigma}\beta}\partial g_{\alpha\overline{\sigma}} \tag{2.46}$$

where $\{g^{\overline{\sigma}\beta}\}$ is the inverse matrix of $\{g_{\alpha\overline{\beta}}\}$. We denote by Θ_α^β the Tanaka–Webster curvature form. Since the pseudohermitian torsion vanishes, we have $\Theta_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$. It is easy to check that $\Theta_\alpha^\beta = R_{\alpha\ j\overline{k}}^{\ \beta}\theta^j \wedge \theta^{\overline{k}}$, where $R_{\alpha\ j\overline{k}}^{\ \beta}$ is the Tanaka–Webster curvature and by direct calculation

$$R_{\alpha\ j\overline{k}}^{\ \beta}\theta^j \wedge \theta^{\overline{k}} = -2g^{\overline{\sigma}\beta}\frac{\partial^4\varphi(z)}{\partial z_\alpha \partial \overline{z}_\sigma \partial z_j \partial \overline{z}_k}dz_j \wedge d\overline{z}_k - 2\frac{\partial g_{\alpha\overline{\sigma}}}{\partial z_j}\frac{\partial g^{\overline{\sigma}\beta}}{\partial \overline{z}_k}dz_j \wedge d\overline{z}_k. \tag{2.47}$$

Proposition 2.17 ([32, Theorem 5.2]) *For any $u \in \Omega^{0,q}(X)$, we have the following equalities*

$$(\Box_b^{(q)}u|u) = \|u\|_{\overline{S}}^2 - qi(\nabla_T u|u) + (R_* u|u) \tag{2.48}$$

where $\|u\|_{\overline{S}}^2 = -\int_X(\sum_{k=1}^{n-1}\langle\nabla_{e_k}\nabla_{\overline{e}_k}u|u\rangle)dv_X$. Here, $\{e_k\}_{k=1}^{n-1}$ is any orthonormal frame of $T^{1,0}X$.

From (2) in Proposition 2.14, the rigid Hermitian metric $\langle\cdot|\cdot\rangle_g$ is invariant with respect the Tanaka–Webster connection ∇. Integrating by parts, we have

$$\|u\|_{\overline{S}}^2 = -\int_X \sum_{k=1}^{n-1}\langle\nabla_{e_k}\nabla_{\overline{e}_k}u|u\rangle dv_X = \sum_{k=1}^{n-1}\int_X \langle\nabla_{\overline{e}_k}u|\nabla_{\overline{e}_k}u\rangle dv_X \geq 0. \tag{2.49}$$

As a corollary of Proposition 2.17, we have the vanishing theorem for the Fourier components of Kohn–Rossi cohomology.

Theorem 2.18 *Let X be a strictly pseudoconvex CR manifold with a locally free transversal CR S^1-action. There exists $m_0 > 0$ such that for $q \geq 1$ and any $m \in \mathbb{Z}$ with $m > m_0$, we have $H_{b,m}^q(X) = 0$.*

Proof By Lemma 2.16, we have $\nabla_T u = L_T u$. Then by (2.48) for any $u \in \Omega_m^{0,q}(X)$ we have

$$(\Box_{b,m}^{(q)}u|u) = \|u\|_{\overline{S}}^2 + qm\|u\|^2 + (R_* u|u). \tag{2.50}$$

There exists $m_0 > 0$ such that for any $m > m_0$, $m \in \mathbb{N}$ we have

$$(\Box_{b,m}^{(q)}u|u) \geq C_m\|u\|^2 \quad \text{for } u \in \Omega_m^{0,q}(X), q \geq 1. \tag{2.51}$$

This implies $\mathcal{H}_{b,m}^q(X) = 0$ for $m > m_0$, $q \geq 1$. By the Hodge isomorphism (2.40), we get the conclusion of the theorem. □

3 S^1-invariant deformation of the CR structure

Let $\{J_t\}_{t\in(-\delta,\delta)}$ be a deformation of J. As before, let $T_t^{1,0}X = \{U \in \mathbb{C}HX : J_tU = iU\}$. We also say $T_t^{1,0}X$ is a (smooth) deformation of $T^{1,0}X$. In this work, we are especially interested in the S^1-invariant deformations of CR structures. As in Definition 2.1, we introduce:

Definition 3.1 We say the smooth deformation $T_t^{1,0}X$ of $T^{1,0}X$ is S^1-invariant if for any $t \in (-\delta, \delta)$ we have $L_T J_t = 0$ or, equivalently, $[T, \Gamma(T_t^{1,0}X)] \subset \Gamma(T_t^{1,0}X)$.

We give some examples of S^1-invariant deformations of CR structures.

Example 3.2 Let $X = \mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ be the boundary of the unit ball in $\mathbb{C}^2$, and let the induced CR structure $T^{1,0}X$ be generated by $Z = \overline{z}_2 \frac{\partial}{\partial z_1} - \overline{z}_1 \frac{\partial}{\partial z_2}$. Thus, $(X, T^{1,0}X)$ forms a compact strictly pseudoconvex CR manifold. The S^1-action on X is given by

$$e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{in\theta}z_2), \quad n \in \mathbb{Z}, n > 0. \tag{3.1}$$

By direct calculation, the S^1- action given above is a locally free transversal CR S^1- action. The global vector field induced by the S^1-action on X is given by

$$T = i\left(z_1\frac{\partial}{\partial z_1} - \overline{z}_1\frac{\partial}{\partial \overline{z}_1} + nz_2\frac{\partial}{\partial z_2} - n\overline{z}_2\frac{\partial}{\partial \overline{z}_2}\right).$$

By simple calculation,

$$[T, Z] = -i(n+1)Z. \tag{3.2}$$

Let $\Phi(z,t) = \phi(z)\chi(t)$ with $\phi(z)$ and $\chi(t)$ smooth functions on X and $\mathbb{R}$ respectively. We assume that $T\phi(z) = -2i(n+1)\phi$ with ϕ a non-zero smooth function on X and $\chi(0) = 0$. This is possible because we can find a smooth function h on X such that $\int_0^{2\pi} h(e^{i\theta}z)e^{i2(n+1)\theta}d\theta \neq 0$, then we define $\phi(z) = \int_0^{2\pi} h(e^{i\theta}z)e^{i2(n+1)\theta}d\theta$. Then for each $t \in \mathbb{R}$ the deformation $T_t^{1,0}X$ of $T^{1,0}X$ is given by

$$T_t^{1,0}X = \mathrm{span}_{\mathbb{C}}\{Z + \Phi(z,t)\overline{Z}\}. \tag{3.3}$$

It is easy to check that

$$[T, Z + \Phi(z,t)\overline{Z}] = -i(n+1)(Z + \Phi(z,t)\overline{Z}) \in \Gamma(T_t^{1,0}X). \tag{3.4}$$

Thus, $T_t^{1,0}X$ is an S^1-invariant deformation of $T^{1,0}X$.

Remark 3.3 In Rossi's global non-embeddability example [1,5,17,31], a real analytic deformation of $T^{1,0}\mathbb{S}^3$ was considered. For each $t \in \mathbb{R}$, the new CR structure $T_t^{1,0}\mathbb{S}^3$ on $\mathbb{S}^3$ is generated by $Z + t\overline{Z}$. It is easy to check that this is not an S^1-invariant deformation with respect to the S^1-action given in (3.1).

Now, we assume that $\{J_t\}_{t\in(-\delta,\delta)}$ is an S^1-invariant deformation of J. Then, $\mathbb{C}HX = T_t^{1,0}X \bigoplus T_t^{0,1}X$, that is, the deformations are always horizontal. This implies that the S^1-action on X is transversal. From Definition 3.1, we know that the S^1-action on X is a transversal CR S^1-action with respect to the deformation $T_t^{1,0}X$ for $t \in (-\delta, \delta)$.

We now express $T_t^{1,0}X$ in an explicit way. Let $\{Z_j\}_{j=1}^{n-1}$ be a canonical frame of $T^{1,0}X$ defined in Theorem 2.7. Then locally we have

$$T_t^{1,0}X = \mathrm{Span}_{\mathbb{C}}\left\{Z_j + \sum_{k=1}^{n-1}\Phi_{j\overline{k}}(\cdot,t)\overline{Z}_k, \quad j = 1, \dots, n-1\right\} \tag{3.5}$$

for $|t|$ small. We may assume that (3.5) holds for all $t \in (-\delta, \delta)$. $Z_j + \sum_{k=1}^{n-1} \Phi_{j\bar{k}}(\cdot, t)\overline{Z}_k$, $1 \leq j \leq n-1$ is a basis of CR structure $T_t^{1,0}X$. Here, $\{\Phi_{j,k}(\cdot, t)\}_{1\leq j,k\leq n-1}$ is called deformation matrix and $\{\Phi_{j\bar{k}}\}_{j,k=1}^{n-1}$ are smooth functions on X which smoothly depend on $t \in (-\delta, \delta)$.

Lemma 3.4 *With the notations used above, for any $t \in (-\delta, \delta)$, we have $T\Phi_{j\bar{k}} = 0$ for $1 \leq j, k \leq n-1$.*

Proof Since the S^1-action on X is transversal and CR with respect to the CR structure J_t, we have $[T, Z_j + \sum_{k=1}^{n-1} \Phi_{j\bar{k}}\overline{Z}_k] \in \Gamma(T_t^{1,0}X)$. From Theorem 2.7, we know $[T, Z_j] = [T, \overline{Z}_j] = 0$ for $1 \leq j \leq n-1$. Then $[T, Z_j + \sum_{k=1}^{n-1} \Phi_{j\bar{k}}\overline{Z}_k] = \sum_{k=1}^{n-1} T\Phi_{j\bar{k}}\overline{Z}_k \in \Gamma(T_t^{1,0}X)$. At each point, we write $\sum_{k=1}^{n-1} T\Phi_{j\bar{k}}\overline{Z}_k = \sum_{j=1}^{n-1} c_j(Z_j + \sum_{l=1}^{n-1} \Phi_{j\bar{l}}\overline{Z}_l)$ for constants $c_l, 1 \leq l \leq n-1$. The equality implies that $c_l = 0, 1 \leq l \leq n-1$, i.e., $\sum_{k=1}^{n-1} T\Phi_{j\bar{k}}\overline{Z}_k = 0$. Since $\{\overline{Z}_l\}_{l=1}^{n-1}$ are linear independent, we have that $T\Phi_{j\bar{k}} = 0$ for $1 \leq j, k \leq n-1$. □

Associated with the CR structure tensor J_t, $t \in (-\delta, \delta)$, the Levi form on X is defined by

$$\mathcal{L}_{t,x}(U, V) = -d\omega_0(J_tU, V), \quad \forall\, U, V \in H_xX, \forall\, x \in X. \tag{3.6}$$

When δ is sufficiently small, the quadratic form $\mathcal{L}_{t,x}$ is still positive and we may assume that the CR manifold $(X, T_t^{1,0}X)$ is strictly pseudoconvex for $t \in (-\delta, \delta)$. Since the S^1-action on $(X, T_t^{1,0}X)$ is transversal and CR, using $\mathcal{L}_{t,x}$ we can define a Riemannian metric g_t on $\mathbb{C}TX$ as (2.7). As (2.9), g_t induces a rigid Hermitian metric $\langle \cdot \mid \cdot \rangle_t$ on $\mathbb{C}TX$ such that

$$T_t^{1,0}X \perp T_t^{0,1}X, \quad T \perp T_t^{1,0}X \oplus T_t^{0,1}X. \tag{3.7}$$

Denote by $T_t^{*1,0}X$ and $T_t^{*0,1}X$ the dual bundles of $T_t^{1,0}X$ and $T_t^{0,1}X$ respectively and define the vector bundle of $(0, q)$-forms by $\Lambda^q T_t^{*0,1}X$. Similarly as in Sect. 2, let $\Omega_t^{0,q}(X)$ denote the space of global smooth sections of $\Lambda^q T_t^{*0,1}X$ and for every $m \in \mathbb{Z}$, let $\Omega_{t,m}^{0,q}(X) = \{u \in \Omega_t^{0,q}(X) : Tu = imu\}$. Let $\overline{\partial}_{t,b} : \Omega_t^{0,q}(X) \to \Omega_t^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator with respect to the new CR structure $T_t^{1,0}X$. Then we still have that $T\overline{\partial}_{t,b} = \overline{\partial}_{t,b}T$ and $\overline{\partial}_{t,b,m} := \overline{\partial}_{t,b} : \Omega_{t,m}^{0,q}(X) \to \Omega_{t,m}^{0,q+1}(X)$, for every $m \in \mathbb{Z}$. Using the $\overline{\partial}_{t,b}$-complex, $\overline{\partial}_{t,b,m}$-complex on $\Omega_t^{0,q}(X)$, $\Omega_{t,m}^{0,q}(X)$ respectively, we can define the Kohn–Rossi cohomology $H_{t,b}^q(X)$ and the mth Fourier component of Kohn–Rossi cohomology $H_{t,b,m}^q(X)$ for each $m \in \mathbb{Z}$ respectively, $q = 0, 1, \ldots, n-1$. In the remainder of this section, our goal is to prove the following:

Theorem 3.5 *Let (X, HX, J) be a compact strictly pseudoconvex CR manifold of real dimension $2n-1, n \geq 2$ with a locally free transversal CR S^1-action. Let $\{J_t\}_{t\in(-\delta,\delta)}$ be a S^1-invariant deformation of J. Then there exists positive constants m_0 and $\delta_0 < \delta$ such that for $m \in \mathbb{Z}$, $m > m_0$ and $|t| < \delta_0$,*

$$\left(\Box_{t,b,m}^{(q)}u|u\right)_t \geq C_m\|u\|_t^2, \;\; u \in \Omega_{t,m}^{0,q}(X), \;\; q \geq 1, \tag{3.8}$$

where C_m is a constant independent of t. In particular, we have the simultaneous vanishing

$$H_{t,b,m}^q(X) = 0, \;\; m > m_0, \;\; |t| < \delta_0, \;\; q \geq 1. \tag{3.9}$$

Before the proof of Theorem 3.5, we first recall the harmonic theory with respect to $\{J_t\}_{t\in(-\delta,\delta)}$ on X. Let $(\cdot\,|\,\cdot)_t$ be the L^2 inner product on $\Omega_t^{0,q}(X)$ induced by the rigid Hermitian metric $\langle\cdot\,|\,\cdot\rangle_t$ and let $\|\cdot\|_t$ denote the corresponding norm. Then for all $u, v \in \Omega_t^{0,q}(X)$

$$(u|v)_t = \int_X \langle u|v\rangle_t dv_X, \tag{3.10}$$

where dv_X is the volume form on X induced by the rigid Hermitian metric $\langle\cdot\,|\,\cdot\rangle_t$. Recall that the volume $dv_X = \omega_0 \wedge \frac{(d\omega_0)^{n-1}}{(n-1)!}$ associated with $\langle\cdot\,|\,\cdot\rangle_t$ does not depend on t. Let $\overline{\partial}^*_{t,b} : \Omega_t^{0,q}(X) \to \Omega_t^{0,q-1}(X)$ be the formal adjoint of $\overline{\partial}_{t,b}$ with respect to $(\cdot\,|\,\cdot)_t$ for $t \in (-\delta,\delta)$. Since $\overline{\partial}_{t,b}T = T\overline{\partial}_{t,b}$ and the Hermitian metric $\langle\cdot\,|\,\cdot\rangle_t$ is rigid, we have $T\overline{\partial}^*_{t,b} = \overline{\partial}^*_{t,b}T$. Define $\Box^{(q)}_{t,b} = \overline{\partial}_{t,b}\overline{\partial}^*_{t,b} + \overline{\partial}^*_{t,b}\overline{\partial}_{t,b}$. From the commutation of T with $\overline{\partial}_{t,b}$, $\overline{\partial}^*_{t,b}$, we have $\Box^{(q)}_{t,b}T = T\Box^{(q)}_{t,b}$. Then $\Box^{(q)}_{t,b}$ maps $\Omega^{0,q}_{t,m}(X)$ into itself and we denote

$$\Box^{(q)}_{t,b,m} := \Box^{(q)}_{t,b}\Big|_{\Omega^{0,q}_{t,m}(X)} : \Omega^{0,q}_{t,m}(X) \to \Omega^{0,q}_{t,m}(X),$$

the restriction of $\Box^{(q)}_{t,b}$ to $\Omega^{0,q}_{t,m}(X)$. As in Sect. 2, let $L^2_{t,(0,q),m}(X)$ be the completion of $\Omega^{0,q}_{t,m}(X)$ under the L^2 inner product defined in (3.10). We extend $\Box^{(q)}_{t,b,m}$ to $L^2_{t,(0,q),m}(X)$ as in (2.36). By Hodge theory for $\Box^{(q)}_{t,b,m}$ (Theorem 2.13) there is an isomorphism $H^q_{t,b,m}(X) \cong \mathcal{H}^q_{t,b,m}(X)$, where $\mathcal{H}^q_{t,b,m}(X)$ is the kernel of $\Box^{(q)}_{t,b,m}$. Now we are going to show the simultaneous vanishing theorem for the harmonic space $\mathcal{H}^q_{t,b,m}(X)$, $q \geq 1$ and as a consequence we prove Theorem 3.5.

Proof of Theorem 3.5 Since $\{Z_{t,j} = Z_j + \Phi_{j\overline{k}}(\cdot,t)Z_{\overline{k}}\}_{j=1}^{n-1}$ is a frame of $T_t^{1,0}X$ and $\langle\cdot\,|\,\cdot\rangle_t$ depends smoothly on t , then by linear algebra argument we can find an orthonormal frame of $T_t^{1,0}X$ which depends smoothly on t. Locally, let $\{e_{t,j}\}_{j=1}^{n-1}$ be an orthonormal basis of $T_t^{1,0}X$ with respect to $\langle\cdot\,|\,\cdot\rangle_t$ depending smoothly on t and let $\{\omega_t^j\}_{j=1}^{n-1}$ be its dual basis.

Let ∇^t be the Tanaka–Webster connection with respect to $T_t^{1,0}X$ and $\langle\cdot\,|\,\cdot\rangle_t$ for any $t \in (-\delta,\delta)$. Let R^t and R^t_* be its curvature and Ricci curvature operator respectively defined as in (2.44) and (2.45). For any $u \in \Omega_t^{0,q}(X)$, then locally $u = \sum'_{|J|=q} u_J\overline{\omega}_t^J$, where $\sum'$ means that the summation is performed only over strictly increasing multi-indices. Here for a multi-index $J = \{j_1,\ldots,j_q\} \in \{1,2,\ldots,n-1\}^q$, we set $|J| = q$, $\overline{\omega}_t^J = \overline{\omega}_t^{j_1} \wedge\cdots\wedge \overline{\omega}_t^{j_q}$ and we say that J is strictly increasing if $1 \leq j_1 < \cdots < j_q \leq n-1$. By definition of R^t_*, for any strictly increasing multi-index $1 \leq k_1 < \cdots < k_q \leq n-1$

$$R^t_* u(\overline{e}_{t,k_1},\ldots,\overline{e}_{t,k_q}) = \sum_{j=1}^{q} u\left(\overline{e}_{t,k_1},\ldots,R^t_*\overline{e}_{t,k_j},\ldots,\overline{e}_{t,k_q}\right), \tag{3.11}$$

where

$$R^t_*\overline{e}_{t,k_j} = -\sum_{i=1}^{n-1} R^t(e_{t,i},\overline{e}_{t,i})\overline{e}_{t,k_j}. \tag{3.12}$$

By (2.50), for any $u \in \Omega^{0,q}_{t,m}(X)$ we have

$$\left(\Box^{(q)}_{t,b,m}u|u\right)_t = \|u\|^2_{S_t} + qm\|u\|^2_t + (R^t_*u|u)_t \tag{3.13}$$

where $\|u\|^2_{S_t} = -\sum_{i=1}^{n-1}\int\langle\nabla^t_{e_{t,i}}\nabla^t_{\overline{e}_{t,i}}u|u\rangle_t dv_X$. We claim that $(R^t_* u|u)_t \leq C\|u\|^2_t, \forall\ |t|\leq\delta$ for a constant C independent of t when δ is small. For $u = \sum'_{|J|=q} u_J\overline{\omega}^J_t \in \Omega^{0,q}_t(X)$, write $R^t_* u = \sum'_{|J|=q} u^t_J\overline{\omega}^J_t$. For any $J = \{j_1,\ldots,j_q\}$ with $1\leq j_1 < \cdots < j_q \leq n-1$ we have $u^t_J = R^t_* u(\overline{e}_{t,j_1},\ldots,\overline{e}_{t,j_q})$. Then

$$\begin{aligned}(R^t_* u|u)_t &= \sum_{j_1<\cdots<j_q}\int_X R^t_* u(\overline{e}_{t,j_1},\ldots,\overline{e}_{t,j_q})\overline{u_{j_1\cdots j_q}}dv_X\\ &= \sum_{j_1<\cdots<j_q}\sum_{l=1}^{q}\int_X u(\overline{e}_{t,j_1},\ldots,R^t_*\overline{e}_{t,j_l},\ldots,\overline{e}_{t,j_q})\overline{u_{j_1\cdots j_q}}dv_X\\ &= -\sum_{i=1}^{n-1}\sum_{j_1<\cdots<j_q}\sum_{l=1}^{q}\int_X u(\overline{e}_{t,j_1},\ldots,R^t(e_{t,i},\overline{e}_{t,i})\overline{e}_{t,j_l},\ldots,\overline{e}_{t,j_q})\overline{u_{j_1\cdots j_q}}dv_X.\end{aligned} \tag{3.14}$$

Since J_t, $\langle\cdot|\cdot\rangle_t$ and $\{Z_{t,j}\}$ depend smoothly on t, then the connection forms of ∇^t with respect to the frame $\{Z_{t,j}\}$ depend smoothly on t and as a consequence, the curvature of the ∇^t also depend smoothly on t . Thus, there exists a constant δ_0 such that for any $|t|<\delta_0$ we have

$$(R^t_* u|u)_t \leq C\|u\|^2_t \tag{3.15}$$

for some constant C independent of t. From (3.13) and (3.15), there exist a constant $m_0>0$ independent of t such that for any $m\in\mathbb{Z}, m>m_0$ we have

$$\left(\Box^{(q)}_{t,b,m}u|u\right)_t \geq C_m\|u\|^2_t,\quad \forall\ u\in\Omega^{0,q}_{t,m}(X),\ |t|<\delta_0, q\geq 1, \tag{3.16}$$

where C_m is a constant independent of t for $|t|<\delta_0$. From (3.16), we get $\mathcal{H}^q_{t,b,m}(X)=0$ for any $m\in\mathbb{Z}, m>m_0$ and $|t|<\delta_0$. By Hodge theory we get the conclusion of Theorem 3.5. □

From (3.16) we have the following.

Corollary 3.6 *Let $\lambda(t,m)$ be an eigenvalue of $\Box^{(q)}_{t,b,m}$, $1\leq q\leq n-1$. Assume that m_0,δ_0 are the same as in Theorem 3.5. Then, for any $m\in\mathbb{Z}, m>m_0$ and $|t|<\delta_0$, we have $\lambda(t,m)\geq C_m$. Here, C_m is a constant satisfying $C_1 m\leq C_m\leq C_2 m$ with C_1, C_2 independent of m and t, $|t|<\delta_0$.*

4 Stability of Szegő kernel of the Fourier components of CR functions

In this section, we assume m_0,δ_0 be the same constants as in Theorem 3.5 unless otherwise stated. Let $S_{t,m}: L^2(X)\to\mathcal{H}^0_{t,b,m}(X)$ be the orthogonal projection with respect to $(\cdot|\cdot)_t$. Since the volume form dv_X with respect to $\langle\cdot|\cdot\rangle_t$ does not depend on t, the inner product $(\cdot|\cdot)_t$ is the same as $(\cdot|\cdot)$ on the space of smooth functions on X. Let $S_{t,m}(x,y)\in C^\infty(X\times X)$ be the Schwartz kernel of $S_{t,m}$. We denote $S_m := S_{0,m}$, $S_m(x,y):=S_{0,m}(x,y)$. The goal of this section is to prove the following

Theorem 4.1 *With the notations above, assume that $m\geq m_0$. For any $k\in\mathbb{N}$ and $\varepsilon>0$ there exists $\delta_{k,\varepsilon}<\delta_0$ such that for all $t\in\mathbb{R}$ with $|t|<\delta_{k,\varepsilon}$, we have*

$$|S_{t,m}(x,y)-S_m(x,y)|_{C^k(X\times X)}<\varepsilon. \tag{4.1}$$

For $s \in \mathbb{Z}$, let $H^s(X)$ denote the Sobolev space on X of order s of functions and let $\|\cdot\|_s$ denote the standard Sobolev norm of order s with respect to $(\cdot\,|\,\cdot)$. First, we need

Lemma 4.2 *For every $m \geq m_0$ and every $s_0 \in \mathbb{N} \cup \{0\}$, there is a constant $C_{s_0,m} > 0$ independent of $t \in (-\delta_0, \delta_0)$ such that*

$$\left\|S_{t,m}u\right\|_{2s_0} \leq C_{s_0,m}\,\|u\|_{-2s_0} \quad \textit{for } u \in H^{-2s_0}(X) \textit{ and } |t| < \delta_0. \tag{4.2}$$

Proof Fix $m \geq m_0$. By Gårding's inequality, for every $s \in \mathbb{N}_0$, it is easy to see that there is a constant $C_{s,m} > 0$ independent of $t \in (-\delta_0, \delta_0)$ such that

$$\|S_{t,m}u\|_{s+2} \leq C_{s,m}\Big(\|(\Box^{(0)}_{t,b,m} - T^2)S_{t,m}u\|_s + \|S_{t,m}u\|_s\Big), \quad \forall\, u \in L^2(X). \tag{4.3}$$

From (4.3), by using induction and noticing that

$$\begin{aligned} &T^2 S_{t,m}u = -m^2 S_{t,m}u, \quad \forall u \in L^2(X),\\ &\left\|S_{t,m}u\right\| \leq \|u\|\,, \quad \forall u \in L^2(X), \end{aligned}$$

it is straightforward to see that for every $s \in \mathbb{N}$, there is a $\widetilde{C}_{s,m} > 0$ independent of t such that

$$\left\|S_{t,m}u\right\|_{2s} \leq \widetilde{C}_{s,m}\,\|u\|\,, \quad \forall u \in L^2(X). \tag{4.4}$$

From (4.4), it is straightforward to see that $S_{t,m}$ can be extended from $L^2(X)$ to $H^{-2s}(X)$ for every $s \in \mathbb{N}$. Fix $s_0 \in \mathbb{N}$ and let $u \in H^{-2s_0}(X)$, we have $(S_{t,m}u|v) = (u, S_{t,m}v)$ for $v \in L^2(X)$, where $(\cdot,\cdot)$ is the pair between $H^{-2s}(X)$ and $H^{2s}(X)$. Then

$$\left\|S_{t,m}u\right\| = \sup\left\{\left|(\,u\,,\ S_{t,m}v\,)\right| :\ v \in L^2(X), \quad (\,v\,|\,v\,) = 1\right\}. \tag{4.5}$$

Fix $v \in L^2(X)$, $(\,v\,|\,v\,) = 1$. From (4.4), we have

$$\left|(\,u\,,\ S_{t,m}v\,)\right| \leq \|u\|_{-2s_0} \cdot \left\|S_{t,m}v\right\|_{2s_0} \leq \widetilde{C}_{s_0,m}\,\|u\|_{-2s_0}\,, \tag{4.6}$$

where $\widetilde{C}_{s_0,m} > 0$ is the constant as in (4.4). From (4.6) and (4.5), we conclude that

$$\left\|S_{t,m}u\right\| \leq \widetilde{C}_{s_0,m}\,\|u\|_{-2s_0}\,, \quad \forall u \in H^{-2s_0}(X). \tag{4.7}$$

Now, from (4.4) and (4.7), we have

$$\begin{aligned} \left\|S_{t,m}u\right\|_{2s_0} &= \left\|S_{t,m}S_{t,m}u\right\|_{2s_0} \leq \widetilde{C}_{s_0,m}\left\|S_{t,m}u\right\| \\ &\leq (\widetilde{C}_{s_0,m})^2\,\|u\|_{-2s_0}\,, \quad \forall u \in H^{-2s_0}(X). \end{aligned} \tag{4.8}$$

From (4.8), the lemma follows. □

Let $N_{t,m} : L^2_m(X) \to \mathrm{Dom}\,(\Box^{(0)}_{t,b,m})$ be the partial inverse of $\Box^{(0)}_{t,b,m}$. We have

$$\begin{aligned} &\Box^{(0)}_{t,b,m}N_{t,m} + S_{t,m} = I \ \text{ on } L^2_m(X),\\ &N_{t,m}\Box^{(0)}_{t,b,m} + S_{t,m} = I \ \text{ on } \mathrm{Dom}\,(\Box^{(0)}_{t,b,m}). \end{aligned} \tag{4.9}$$

We denote $N_m := N_{0,m}$. We need

Lemma 4.3 *For every $m \geq m_0$ and every $s \in \mathbb{N}_0$, there is a constant $C_{s,m} > 0$ independent of $t \in (-\delta_0, \delta_0)$ such that*

$$\left\|N_{t,m}u\right\|_{s+2} \leq C_{s,m}\,\|u\|_s \quad \textit{for } u \in H^s(X) \bigcap L^2_m(X). \tag{4.10}$$

Proof We will prove (4.10) by induction over $s \in \mathbb{N}_0$. By Gårding's inequality, it is easy to see that there is a constant $\widetilde{C}_m > 0$ independent of t such that

$$\|N_{t,m}u\|_2 \leq \widetilde{C}_m\Big(\|(\Box^{(0)}_{t,b,m} - T^2)N_{t,m}u\| + \|N_{t,m}u\|\Big), \quad \forall u \in L^2_m(X). \tag{4.11}$$

From (4.9), we have

$$(\Box^{(0)}_{t,b,m} - T^2)N_{t,m}u = (I - S_{t,m})u + m^2 N_{t,m}u. \tag{4.12}$$

From (4.12) and Corollary 3.6, we see that there is a constant $\hat{C}_m > 0$ independent of t such that

$$\left\|N_{t,m}u\right\| + \left\|(\Box^{(0)}_{t,b,m} - T^2)N_{t,m}u\right\| \leq \hat{C}_m \left\|u\right\|, \quad \forall u \in L^2_m(X). \tag{4.13}$$

From (4.13) and (4.11), we see that (4.10) holds for $s = 0$.

We assume that (4.10) holds for some $s_0 \geq 0$. We are going to prove that (4.10) holds for s_0+1. By Gårding's inequality, it is easy to see that there is a constant $\widetilde{C}_{s_0,m} > 0$ independent of t such that

$$\begin{aligned}&\|N_{t,m}u\|_{s_0+3}\\ &\leq \widetilde{C}_{s_0,m}\Big(\|(\Box^{(0)}_{t,b,m} - T^2)N_{t,m}u\|_{s_0+1} + \|N_{t,m}u\|_{s_0+1}\Big), \quad \forall u \in H^{s_0+1}(X)\bigcap L^2_m(X).\end{aligned} \tag{4.14}$$

From (4.9), we have

$$(\Box^{(0)}_{t,b,m} - T^2)N_{t,m}u = (I - S_{t,m})u + m^2 N_{t,m}u. \tag{4.15}$$

From the proof of Lemma 4.2, we have

$$\left\|S_{t,m}u\right\|_{s_0+1} \leq \left\|S_{t,m}u\right\|_{2(s_0+1)} \leq c_{m,s_0}\left\|u\right\| \leq c_{m,s_0}\left\|u\right\|_{s_0+1}, \tag{4.16}$$

where $c_{m,s_0} > 0$ is a constant independent of t. By the induction, we have

$$\left\|N_{t,m}u\right\|_{s_0+1} \leq \left\|N_{t,m}u\right\|_{s_0+2} \leq \hat{c}_{m,s_0}\left\|u\right\|_{s_0} \leq \hat{c}_{m,s_0}\left\|u\right\|_{s_0+1}, \tag{4.17}$$

where $\hat{c}_{m,s_0} > 0$ is a constant independent of t. From (4.14), (4.15), (4.16), and (4.17), we see that (4.10) holds for $s_0 + 1$. The lemma follows. □

Let $s_1, s_2 \in \mathbb{Z}$. For a t-dependent operator $A_t : H^{s_1}(X) \to H^{s_2}(X)$, we write

$$A_t = o(t) : H^{s_1}(X) \to H^{s_2}(X), \quad t \to 0,$$

if for every $\varepsilon > 0$, there is a $\delta_1 > 0$ such that for all $|t| < \delta_1$, we have

$$\|A_t u\|_{s_2} \leq \varepsilon\,\|u\|_{s_1} \quad \text{for all } u \in H^{s_1}(X).$$

Proof of Theorem 4.1 Assume that $m \geq m_0$. From (4.9), we have

$$\begin{aligned}S_m &= (N_{t,m}\Box^{(0)}_{t,b,m} + S_{t,m})S_m\\ &= N_{t,m}(\Box^{(0)}_{t,b,m} - \Box^{(0)}_{b,m})S_m + S_{t,m}S_m.\end{aligned} \tag{4.18}$$

Note that

$$(\Box^{(0)}_{t,b,m} - \Box^{(0)}_{b,m})S_m = o(t) : H^{-s}(X) \to H^{s-2}(X), \quad \forall s \in \mathbb{N}.$$

From this observation, (4.10) and (4.18), we deduce that

$$S_m - S_{t,m}S_m = o(t) : H^{-s}(X) \to H^{s}(X), \quad \forall s \in \mathbb{N}. \tag{4.19}$$

Taking adjoints in (4.19) with respect to $(\cdot\,|\,\cdot)$, we get

$$S_m - S_m S_{t,m}^* = o(t) : H^{-s}(X) \to H^s(X), \quad \forall s \in \mathbb{N}, \tag{4.20}$$

where $S_{t,m}^*$ is the adjoint of $S_{t,m}$ with respect to $(\cdot\,|\,\cdot)$. It is clear that

$$S_{t,m} = S_{t,m}^* : H^{-s}(X) \to H^s(X), \quad \forall s \in \mathbb{N}.$$

From this observation and (4.20), we conclude that

$$S_m - S_m S_{t,m} = o(t) : H^{-s}(X) \to H^s(X), \quad \forall s \in \mathbb{N}. \tag{4.21}$$

Similarly, from (4.9), we have

$$\begin{aligned} S_{t,m} &= \left(N_m \Box_{b,m}^{(0)} + S_m\right) S_{t,m} \\ &= N_m(\Box_{b,m}^{(0)} - \Box_{t,b,m}^{(0)}) S_{t,m} + S_m S_{t,m}. \end{aligned} \tag{4.22}$$

From Lemma 4.2, it is easy to check that

$$N_m(\Box_{b,m}^{(0)} - \Box_{t,b,m}^{(0)}) S_{t,m} = o(t) : H^{-s}(X) \to H^s(X), \quad \forall s \in \mathbb{N}. \tag{4.23}$$

From (4.23) and (4.22), we deduce that

$$S_{t,m} - S_m S_{t,m} = o(t) : H^{-s}(X) \to H^s(X), \quad \forall s \in \mathbb{N}. \tag{4.24}$$

From (4.21) and (4.24), we deduce that

$$S_m - S_{t,m} = o(t) : H^{-s}(X) \to H^s(X), \quad \forall s \in \mathbb{N}. \tag{4.25}$$

From (4.25) and the Sobolev embedding theorem, Theorem 4.1 follows. □

Corollary 4.4 *There exists $\delta_1 < \delta_0$ such that for $m > m_0$, $\dim H_{t,b,m}^0(X)$ does not depend on $t \in (-\delta_1, \delta_1)$.*

Proof It is clear that

$$\left|\dim H_{t,b,m}^0(X) - \dim H_{b,m}^0(X)\right| = \left|\int_X S_{t,m}(x,x) - S_m(x,x) dv_X\right| \to 0 \tag{4.26}$$

as $t \to 0$ by Theorem 4.1. Since $\dim H_{t,b,m}^0(X)$ is an integer, for each $m > m_0$ the function $t \mapsto \dim H_{t,b,m}^0(X)$ is constant for $|t|$ is sufficiently small. □

5 Stability of equivariant embedding of CR manifolds with S^1-action

In a recent work [18, Theorem 1.2] we showed:

Theorem 5.1 *Let $(X, T^{1,0}X)$ be a compact connected strictly pseudoconvex CR manifold with a transversal CR locally free S^1-action. Then for every $m \in \mathbb{N}$, there exist integers $\{m_j\}_{j=1}^N$ with $m_j \geq m$, $1 \leq j \leq N$, and CR functions $\{f_j\}_{j=1}^N$ with $f_j \in H_{b,m_j}^0(X)$ such the S^1-equivariant CR map $\Phi : X \to \mathbb{C}^N$, $x \mapsto (f_1(x), \ldots, f_N(x))$ is an embedding.*

In this section, we choose m_0 as in Theorem 3.5. We will show that the equivariant embedding in Theorem 5.1 is stable under S^1-invariant deformations of CR structure. For $|t| < \delta_0$, set $\Phi_{t,j} = S_{t,m_j}\Phi_j$. Then $\{\Phi_{t,j}\}_{j=1}^N$ are CR functions with respect to $T_t^{1,0}X$ (or J_t). With these CR functions we define a CR map with respect to $T_t^{1,0}X$ as follows

$$\Phi_t : X \to \mathbb{C}^N, \quad x \mapsto (\Phi_{t,1}(x), \dots, \Phi_{t,N}(x)). \tag{5.1}$$

Now, we come to the following result, which implies the main result of the paper, Theorem 1.1.

Theorem 5.2 *Let (X, HX, J) be a compact connected strictly pseudoconvex CR manifold with a transversal CR S^1-action. Let $\{J_t\}_{t\in(-\delta_0,\delta_0)}$ be an S^1-invariant deformation of J. Let m_0 be as in Theorem 3.5. Let $\Phi = (\Phi_1, \dots, \Phi_N) : X \to \mathbb{C}^N$ be an equivariant CR embedding with $\Phi_j \in H^0_{b,m_j}(X)$, $m_j > m_0$, $1 \le j \le N$. Then the map Φ_t defined in (5.1) is a CR embedding when $|t|$ is sufficiently small. Moreover, for every $k \in \mathbb{N}$, $\lim_{t\to 0} \|\Phi_t - \Phi\|_{C^k(X,\mathbb{C}^N)} = 0$.*

Proof First, we prove Φ_t is an immersion for each $|t|$ is sufficiently small. Since

$$\Phi_{t,j} - \Phi_j = S_{t,m_j}\Phi_j - S_{m_j}\Phi_j = (S_{t,m_j} - S_{m_j})\Phi_j, 1 \le j \le N, \tag{5.2}$$

then by Theorem 4.1, we have $|\Phi_{t,j} - \Phi_j|_{C^1(X)}$ is sufficiently small as $|t| \to 0$ for $1 \le j \le N$. Since Φ is an immersion, i.e., the rank of the Jacobian of Φ is $2n-1$, then there exists a constant $\sigma < \delta_0$ such that for $|t| < \sigma$ the rank of the Jacobian of Φ_t is always $2n-1$, that is, Φ_t is an immersion when $|t| < \sigma$. Next, we claim that Φ_t is an injective map when t is sufficiently small. We prove this claim by seeking a contradiction. If it is not true, there exists $\varepsilon_n \to 0$ as $n \to \infty$ and two sequences of points $\{x_n\}, \{y_n\} \subset X$, $x_n \ne y_n$, for each n, such that $\Phi_{\varepsilon_n}(x_n) = \Phi_{\varepsilon_n}(y_n)$, $\forall n$. Since X is compact, we assume that $x_n \to p$ and $y_n \to q$. If $p \ne q$, letting $\varepsilon_n \to 0$ we will have $\Phi(p) = \Phi(q)$. This is a contradiction with the fact that Φ an injective map. Now, we assume that $p = q$. Then

$$|\Phi(x_n)-\Phi(y_n)| = |\Phi(x_n)-\Phi_{\varepsilon_n}(x_n)+\Phi_{\varepsilon_n}(y_n)-\Phi(y_n)| = |(\Phi-\Phi_{\varepsilon_n})(x_n)-(\Phi-\Phi_{\varepsilon_n})(y_n)|. \tag{5.3}$$

By Theorem 4.1, we have

$$\left\|\Phi_{\varepsilon_n} - \Phi\right\|_{C^1(X,\mathbb{C}^N)} \to 0 \text{ as } \varepsilon_n \to 0. \tag{5.4}$$

Then

$$|(\Phi - \Phi_{\varepsilon_n})(x_n) - (\Phi - \Phi_{\varepsilon_n})(y_n)| \le c_{\varepsilon_n}|x_n - y_n|, \quad \forall n, \tag{5.5}$$

where c_{ε_n} is a sequence of constants with $c_{\varepsilon_n} \to 0$ as $\varepsilon_n \to 0$. On the other hand, since Φ is an embedding, by implicit function theorem, there exists a constant c independent of $\{x_n\}, \{y_n\}$ such that

$$|\Phi(x_n) - \Phi(y_n)| \ge c|x_n - y_n|, \quad \text{for } n \text{ large}. \tag{5.6}$$

From (5.5) and (5.6), we get a contradiction. Thus, we get the injectivity of Φ_t for $|t|$ sufficiently small. The fact that $\|\Phi_t - \Phi\|_{C^k(X,\mathbb{C}^N)} \to 0$ for $t \to 0$ is a direct consequence of (5.2) and Theorem 4.1. □

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