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Geometric quantization on CR manifolds

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Let X be a compact connected orientable Cauchy–Riemann (CR) manifold with the action of a compact Lie group G. Under natural pseudoconvexity assumptions we show that the CR Guillemin–Sternberg map is an isomorphism at the level of Sobolev spaces of CR functions, modulo a finite-dimensional subspace. As application we study this map for holomorphic line bundles which are positive near the inverse image of 0 by the momentum map. We also show that "quantization commutes with reduction" for Sasakian manifolds.

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1. Introduction and Statement of the Main Results

The famous geometric quantization conjecture of Guillemin and Sternberg [19] states that for a compact pre-quantizable symplectic manifold admitting a Hamiltonian action of a compact Lie group, the principle of "quantization commutes with reduction" holds. This conjecture was first proved independently by Meinrenken [43] and Vergne [53] for the case where the Lie group is abelian, and by Meinrenken [44] in the general case, then Tian–Zhang [52] gave a purely analytic proof in general case with various generalizations, see [54] for a survey and complete references on this subject. In the case of a non-compact symplectic manifold M with a compact Lie group action G, this question was solved by Ma–Zhang [39, 40] as a solution to a conjecture of Vergne in her ICM 2006 plenary lecture [55], see [36] for a survey. Paradan [47] gave a new proof, cf. also the recent work [24]. A natural choice for the quantum spaces of a compact symplectic manifold is the kernel of the Dirac operator.

In [38], Ma–Zhang established the asymptotic expansion of the G-invariant Bergman kernel for a positive line bundle L over a compact symplectic manifold M and by using the asymptotic expansion of G-invariant Bergman kernel, they could establish the "quantization commutes with reduction" theorem when the power of the line bundle L is high enough.

On a compact Kähler manifold M endowed with a prequantum line bundle L, a natural choice for the Hilbert space of quantum states is the space $H^0(M, L^m)$ of holomorphic sections of the tensor powers L^m. The family of quantum spaces $H^0(M, L^m)$ indexed by $m \in \mathbb{N}$ plays an essential role in geometric quantization and the semi-classical limit $m \to \infty$ allows to recover the classical mechanics of the phase space M.

One can wrap up the family of spaces $H^0(M, L^m)$, $m \in \mathbb{N}$, as subspaces of a single Hilbert space by considering the S^1-bundle $X \subset L^*$, which is a strictly pseudoconvex CR manifold and identifying $H^0(M, L^m)$ with the S^1 isotypes of m-equivariant CR functions on X. The Hilbert space sum $\widehat{\oplus}_{m\in\mathbb{N}} H^0(M, L^m)$ can be identified to the space $H^0_b(X)$ of L^2 CR functions on X and the sum of the Bergman projections B_m on $H^0(M, L^m)$ can be identified to the Szegő projector S on $H^0_b(X)$. A fundamental fact is that the asymptotic behavior of B_m is encoded in the singularities of the Szegő kernel $S(\cdot, \cdot)$. We can thus think of X as the quantizing principal bundle of M and of the space of L^2 CR functions as the quantum space of X. In the presence of a G-action on M, which lifts to an action on L, we have an induced G-action on X and on $H^0_b(X)$.

The quantization of strictly pseudoconvex or more generally contact manifolds via the Szegő projector or its generalizations was developed by Boutet de Monvel and Guillemin [7] and can be applied to the Kähler quantization by using the above construction. In this paper, we study the quantization of CR manifolds and the principle of "quantization commutes with reduction". For this purpose we develop a G-invariant Fourier integral operator calculus which will be used to obtain the asymptotics of the G-invariant Szegő kernel. Our Theorem 1.2 is new even for compact strictly pseudoconvex CR manifolds and our results pertain also to Sasakian manifolds (see Theorem 1.3). Sasakian geometry is an important odd-dimensional counterpart of Kähler geometry. It is known that irregular Sasakian manifolds admit a compact CR torus action (see [23, Sec. 3] or Remark 2.2) and the study of G-equivariant CR functions on a Sasakian manifold is important in Sasaki geometry. We refer to [1, 9, 16, 17, 57] for the fundamentals of contact and Sasakian reduction and examples.

An important difference between the CR setting and the Kähler/symplectic setting is that the quantum spaces in the case of a compact Kähler/symplectic manifolds are finite-dimensional, whereas for the compact strictly pseudoconvex CR manifolds that we consider the quantum spaces consist of CR functions and are infinite-dimensional. In [20, Theorem 3.6], Guillemin–Sternberg proposed a version of quantization for symplectic cones associated with a homogeneous moment map. Again, in contrast to our Theorem 1.2, their invariant spaces are finite-dimensional.

We now formulate the main results. We refer to Sec. 2 for some notations and terminology used here. Let $(X, T^{1,0}X)$ be a compact orientable CR manifold of dimension $2n+1$, $n \geq 1$, where $T^{1,0}X$ denotes the CR structure of X. Let $HX \subset TX$ be the associated Levi distribution with complex structure $J \in \mathrm{End}(HX)$ and let $\omega_0 \in \mathscr{C}^\infty(X, T^*X)$ be a non-vanishing real 1-form annihilating HX, called characteristic 1-form.

Let G be a d-dimensional compact Lie group with Lie algebra $\mathfrak{g}$ acting on X by preserving J and ω_0. Let $\mu : X \to \mathfrak{g}^*$ be the associated moment map $\mu : X \to \mathfrak{g}^*$ (cf. (2.27)). We will mainly work in the following setting.

Assumption 1.1. The G-action preserves the complex structure J on HX and the characteristic 1-form ω_0, it is free on $\mu^{-1}(0)$, and one of the following conditions are fulfilled:

(i) $\dim X \geq 5$ and the Levi form of X is positive definite near $\mu^{-1}(0)$.
(ii) $\dim X = 3$, the Levi form of X is positive definite everywhere and $\overline{\partial}_b$ has closed range in L^2 on X.

Due to Lemma 2.5, Assumption 1.1 implies that 0 is a regular value of μ, hence $\mu^{-1}(0)$ is a d-codimensional submanifold of X. Let

$$Y := \mu^{-1}(0), \quad X_G := \mu^{-1}(0)/G. \tag{1.1}$$

The space X_G is called the CR reduction. Under our hypotheses, if $\dim X_G \geq 3$, X_G is a strictly pseudoconvex CR manifold with characteristic 1-form (in this case also contact form) $\omega_{0,G}$ induced canonically by ω_0, see (2.38). If $\dim X_G = 1$, then each of the finitely many components of X_G is diffeomorphic to a circle.

Let $\overline{\partial}_b : \mathscr{C}^\infty(X) \to \Omega^{0,1}(X)$ and if $\dim X_G \geq 3$, let $\overline{\partial}_{b,X_G} : \mathscr{C}^\infty(X_G) \to \Omega^{0,1}(X_G)$ be the tangential Cauchy–Riemann operators on X and X_G, respectively. We extend $\overline{\partial}_b$ and $\overline{\partial}_{b,X_G}$ to L^2 spaces by taking their weak maximal extension, see (2.13). We consider the spaces of L^2 CR functions

$$H^0_b(X) := \left\{u \in L^2(X) : \overline{\partial}_b u = 0\right\}, \quad H^0_b(X_G) := \left\{u \in L^2(X_G) : \overline{\partial}_{b,X_G} u = 0\right\}. \tag{1.2}$$

If $\dim X_G = 1$, so X_G is a finite union of circles, we set $H^0_b(X_G)$ to be the direct sum of the Hardy spaces of the components, that is, the L^2 subspaces of functions with vanishing Fourier coefficients of negative degree. The common feature of the spaces $H^0_b(X_G)$ for $\dim X_G \geq 3$ and $\dim X_G = 1$ is the fact that they are boundary values of holomorphic functions in a filling of X_G by a complex manifold (see Sec. 2.4).

Then $H^0_b(X)$ is a (possible infinite-dimensional) G-representation, its G-invariant part is the G-invariant L^2 CR functions on X,

$$H^0_b(X)^G := \left\{u \in H^0_b(X) : h^* u = u, \text{ for any } h \in G\right\}. \tag{1.3}$$

For every $s \in \mathbb{R}$ let $H^s(X)$ and $H^s(X_G)$ denote the Sobolev spaces of X and X_G of order s and let $(\,\cdot\,,\,\cdot\,)_s$ and $(\,\cdot\,,\,\cdot\,)_{X_G,s}$ be the inner products on $H^s(X)$ and $H^s(X_G)$, respectively (see (5.3)). For every $s \in \mathbb{R}$ put

$$H^0_b(X)_s := \left\{u \in H^s(X) : \overline{\partial}_b u = 0 \text{ in the sense of distributions}\right\}. \tag{1.4}$$

We define $H^0_b(X_G)_s$ and $H^0_b(X)^G_s$ in the same way. If $\dim X_G = 1$, we set $H^0_b(X_G)_s$ to be the direct sum of the Hardy–Sobolev spaces of the components, that is, the subspaces of $H^s(S^1)$ of distributions with vanishing Fourier coefficients of negative degree.

Let $\iota : Y \to X$ be the natural inclusion and let $\iota^* : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(Y)$ be the pull-back by ι. Let $\iota_G : \mathscr{C}^\infty(Y)^G \to \mathscr{C}^\infty(X_G)$ be the natural identification. Let

$$\sigma_G : H^0_b(X)^G \cap \mathscr{C}^\infty(X)^G \to H^0_b(X_G), \quad \sigma_G = \iota_G \circ \iota^*. \tag{1.5}$$

The map (1.5) is well defined, see the construction of the CR reduction in Sec. 2.3. The map σ_G does not extend to a bounded operator on L^2, so it is necessary to consider its extension to Sobolev spaces. From Theorem 5.3, σ_G extends by density to a bounded operator

$$\sigma_G = \sigma_{G,s} : H^0_b(X)^G_s \to H^0_b(X_G)_{s-\frac{d}{4}}, \quad \text{for every } s \in \mathbb{R}. \tag{1.6}$$

This operator can be thought as a Guillemin–Sternberg map in the CR setting. It maps the "first quantize and then reduce" space (the space of G-invariant Sobolev CR functions on X) to the "first reduce and then quantize" space (the space of Sobolev CR functions on X_G). Indeed, from the point of view of quantum mechanics, the Hilbert space structures play an essential role. It is natural, then, to investigate the extent to which the CR Guillemin–Sternberg map is Fredholm. The following main result of this work gives a CR analogue of the fact that the canonical reduction map in the symplectic context is an isomorphism for sufficiently large tensor powers of the prequantum line bundle (cf. [38, Theorem 0.9] and Theorem 1.4).

Theorem 1.2. *Let X be a compact orientable CR manifold and let G be a compact Lie group acting on X such that the G-action preserves J and ω_0 and Assumption* 1.1 *holds. Suppose that $\overline{\partial}_{b,X_G}$ has closed range in L^2. Then, for every $s \in \mathbb{R}$, the CR Guillemin–Sternberg map* (1.6) *is Fredholm. Actually,* $\operatorname{Ker}\sigma_{G,s}$ *and* $(\operatorname{Im}\sigma_{G,s})^\perp$ *are finite-dimensional subspaces of $\mathscr{C}^\infty(X) \cap H^0_b(X)^G$ and $\mathscr{C}^\infty(X_G) \cap H^0_b(X_G)$, respectively,* $\operatorname{Ker}\sigma_{G,s}$ *and the index* $\dim \operatorname{Ker}\sigma_{G,s} - \dim(\operatorname{Im}\sigma_{G,s})^\perp$ *are independent of s.*

We note that $(\operatorname{Im}\sigma_{G,s})^\perp$ is given by

$$(\operatorname{Im}\sigma_{G,s})^\perp := \left\{ u \in H^0_b(X_G)_{s-\frac{d}{4}} : (\, \sigma_{G,s} v \,, u\,)_{X_G, s-\frac{d}{4}} = 0, \text{ for every } v \in H^0_b(X)^G_s \right\}. \tag{1.7}$$

Under Assumption 1.1(i) the hypothesis that $\dim X \geq 5$ is used in order to have local subelliptic Sobolev estimates on the set where the Levi form is positive definite (Theorem 3.10) and leads to the fact that the G-invariant Kohn Laplacian (3.26) has closed range in L^2. Note also that the Kohn Laplacian on strictly pseudoconvex CR manifolds of dimension greater than or equal to five has always closed range in L^2 but this is not true for all three-dimensional strictly pseudoconvex CR manifolds (a detailed discussion about the closed range of $\overline{\partial}_b$ in L^2 can be found in Sec. 2.4). In the case when $\dim X = 3$ we will state in Theorem 5.7 a version of Theorem 1.2 under weaker hypotheses as Assumption 1.1(ii), namely that X is pseudoconvex of finite type and $\overline{\partial}_{b,X}$ has closed range in L^2.

We turn now our attention to Sasakian manifolds. Let $(X, T^{1,0}X)$ be a compact connected Sasakian manifold (see Sec. 2.2), i.e. $(X, T^{1,0}X)$ is a compact connected strictly pseudoconvex CR manifold and we can fix a contact form ω_0 and a Reeb vector field R such that $i_R\omega_0 = 1$, $i_R d\omega_0 = 0$ and the flow associated with R preserves $T^{1,0}X$ (cf. Remarks 2.2 and 2.3). Assume that the action of the compact Lie group G on X verifies Assumption 1.1. Moreover, we assume that

$$R \text{ is } G\text{-invariant.} \tag{1.8}$$

By Remark 2.2, the flow associated with R preserves HX, J and the natural metric g_{ω_0} on TX, in particular, R is a Killing vector field. Since X is compact, this implies the flow associated with R generates a compact torus $\mathbb{T}$-action on X and this $\mathbb{T}$-action commutes with the G-action. Thus it naturally induces a $\mathbb{T}$-action on X_G and the generator R induces the Reeb vector field $\widehat{R}$ on X_G. Since the flow associated with R preserves the CR structure $T^{1,0}X$ and commutes with G, it follows that the flow associated with $\widehat{R}$ preserves the CR structure $T^{1,0}X_G$. Thus, $(X_G, T^{1,0}X_G)$ is a strictly pseudoconvex CR manifold with a CR Reeb vector field $\widehat{R}$. By (2.17) and Remark 2.3, X_G is also a compact Sasakian manifold.

Now $H^0_b(X)^G$ and $H^0_b(X_G)$ are both $\mathbb{T}$-Hilbert spaces, thus we have the decomposition of Hilbert spaces via the weight $\alpha \in \mathbb{T}^* (\simeq \mathbb{Z}^{\dim \mathbb{T}})$ of $\mathbb{T}$-action:

$$H^0_b(X)^G = \oplus_{\alpha\in\mathbb{T}^*} H^0_{b,\alpha}(X)^G, \quad H^0_b(X_G) = \oplus_{\alpha\in\mathbb{T}^*} H^0_{b,\alpha}(X_G). \tag{1.9}$$

We refer to [23] for further applications of this weight decomposition. Both $H^0_{b,\alpha}(X)^G$ and $H^0_{b,\alpha}(X_G)$ are finite-dimensional subspaces of $\mathscr{C}^\infty(X)^G$ and $\mathscr{C}^\infty(X_G)$, respectively, as subspaces of the eigenspaces of the elliptic operators $\overline{\partial}^*_{b,X}\overline{\partial}_{b,X} - R^2$, $\overline{\partial}^*_{b,X_G}\overline{\partial}_{b,X_G} - \widehat{R}^2$, respectively, of eigenvalues $|\alpha(R)|^2$. From the definition (1.5) of the map σ_G we see that

$$\sigma_G R u = \widehat{R}\sigma_G u, \quad \text{for any } u \in H^0_b(X)^G, \tag{1.10}$$

and hence σ_G maps $H^0_{b,\alpha}(X)^G$ to $H^0_{b,\alpha}(X_G)$. From this observation, Theorem 1.2 and the fact that $\overline{\partial}_b$ has closed range in L^2 on Sasakian manifolds (see [42], also Sec. 2.4), we deduce:

Theorem 1.3 (quantization commutes with reduction for Sasakian manifolds). *Let X be a compact connected Sasakian manifold with a CR Reeb vector field R. Suppose that X admits a compact Lie group action G which preserves the complex structure J on HX and the characteristic* 1*-form ω_0, it is free on $\mu^{-1}(0)$, and the Reeb vector field R is G-invariant. Then with the exception of finitely many $\alpha \in \mathbb{T}^*$ the map*

$$\sigma_G : H^0_{b,\alpha}(X)^G \to H^0_{b,\alpha}(X_G) \tag{1.11}$$

is an isomorphism.

We now apply Theorem 1.2 to the case of complex manifolds. Let (L, h^L) be a Hermitian holomorphic line bundle over a connected compact complex manifold (M, J) with $\dim_{\mathbb{C}} M = n$, where J denotes the complex structure of TM and h^L is a Hermitian metric of L. We denote by R^L the Chern curvature of (L, h^L). We assume that G acts holomorphically on (M, J), and that the action lifts to a holomorphic action on L. We assume further that h^L is preserved by the G-action. Then R^L is a G-invariant form. Let $\tilde{\mu} : M \to \mathfrak{g}^*$ be the moment map defined by the Kostant formula (6.1).

Assume that $0 \in \mathfrak{g}^*$ is regular and the action of G on $\tilde{\mu}^{-1}(0)$ is free. If iR^L is positive near $\tilde{\mu}^{-1}(0)$, then the analogue of the Marsden–Weinstein reduction holds. More precisely, the complex structure J on M induces a complex structure J_G on $M_G := \tilde{\mu}^{-1}(0)/G$, for which the line bundle $L_G := L/G$ is a holomorphic line bundle over M_G. For $m \in \mathbb{N}$, let $H^0(M, L^m)^G$ denote the space of G-invariant holomorphic sections with values in L^m.

Let $X := \{v \in L^* : |v|^2_{h^{L^*}} = 1\}$ be the circle bundle of L^*, where h^{L^*} is the Hermitian metric on L^* induced by h^L. Let $e^{i\theta}$ be the natural S^1-action on the fibers of X. The action of G on M lifts to a CR action on X. For every $m \in \mathbb{N}$, put

$$H^0_{b,m}(X)^G := \{u \in H^0_b(X)^G : (e^{i\theta})^* u = e^{im\theta} u \text{ on } X, \text{ for every } e^{i\theta} \in S^1\}. \tag{1.12}$$

It is easy to check that for every $m \in \mathbb{N}$ there are canonical isomorphisms

$$H^0_{b,m}(X)^G \cong H^0(M, L^m)^G, \quad H^0_{b,m}(X_G) \cong H^0(M_G, L^m_G). \tag{1.13}$$

On account of (1.13), Theorem 1.2 (under Assumption 1.1 (i)) and the correspondence between the curvature iR^L and the Levi form of X (cf. Section 2.2) we deduce:

Theorem 1.4. *Let M be a compact connected complex manifold*, $\dim_{\mathbb{C}} M \geq 2$, *and (L, h^L) be a Hermitian holomorphic line bundle over M. Let G be a compact Lie group acting holomorphically on M and whose action lifts to (L, h^L). Suppose that iR^L is positive near $\tilde{\mu}^{-1}(0)$ and G acts freely on $\tilde{\mu}^{-1}(0)$. Then for m large enough*, *the canonical map between $H^0(M, L^m)^G$ and $H^0(M_G, L^m_G)$ by restriction is an isomorphism*, *in particular*

$$\dim H^0(M, L^m)^G = \dim H^0(M_G, L^m_G). \tag{1.14}$$

If L is positive on the whole M, this canonical isomorphism between $H^0(M, L^m)^G$ and $H^0(M_G, L^m_G)$ was constructed in [19, 58] for $m = 1$ (see also for the metric aspect of this isomorphism [38, (0.27), Corollary 4.13] for m large enough). This implies that the map σ_G in Theorem 1.2 is actually an isomorphism in the case of the circle bundle X of L^*. Thus we expect in many situations (in particular, if additionally X is strictly pseudoconvex), that σ_G in Theorem 1.2 is an isomorphism. Theorem 1.4 gives a version of the result of [19, 58] for large enough powers of L by requiring the positivity of iR^L only on $\tilde{\mu}^{-1}(0)$. In the case $\dim_{\mathbb{C}} M = 1$, hence $\dim X = 3$, Assumption 1.1(ii) corresponds to L being positive everywhere on M, so an application of Theorem 1.2 does not bring anything new. We will give in Theorem 6.1 a version for almost complex manifolds of Theorem 1.4.

In the rest of Introduction we explain some technical aspects, in particular some results on the G-invariant Szegő projection, necessary to establish Theorem 1.2.

We introduce a G-invariant Hermitian metric $g = g^{\mathbb{C}TX}$ on X as in Lemma 2.7 which fixes the L^2 spaces on X. The *G-invariant Szegő projection* is the orthogonal projection

$$S_G : L^2(X) \to H^0_b(X)^G \tag{1.15}$$

with respect to $(\,\cdot\,,\,\cdot\,)$. The *G-invariant Szegő kernel* $S_G(x, y) \in \mathscr{D}'(X \times X)$ is the distribution kernel of S_G. In Theorem 3.25, we will prove that S_G is a complex Fourier integral operator and in Theorem 3.27, we will show the regularity property

$$S_G : \mathscr{C}^\infty(X) \to H^0_b(X)^G \cap \mathscr{C}^\infty(X). \tag{1.16}$$

From (1.16) we conclude that $H^0_b(X)^G \cap \mathscr{C}^\infty(X)$ is dense in $H^0_b(X)^G$.

Let $S_{X_G}: L^2(X_G) \to H^0_b(X_G)$ be the orthogonal projection with respect to $(\cdot\,,\cdot)_{X_G}$ (cf. Convention 2.8). It follows from general results [8, 27] that if $\overline{\partial}_{b,X_G}$ has closed range in L^2, then S_{X_G} is a Fourier integral operator with complex phase and also a pseudodifferential operator on X_G. In particular,

$$S_{X_G}: \mathscr{C}^\infty(X_G) \to H^0_b(X_G) \cap \mathscr{C}^\infty(X_G) \tag{1.17}$$

which implies that $H^0_b(X_G)\cap\mathscr{C}^\infty(X_G)$ is dense in $H^0_b(X_G)$ if $\overline{\partial}_{b,X_G}$ has closed range in L^2.

We consider the linear map

$$\mathcal{R}_x: \underline{\mathfrak{g}}_x \to \underline{\mathfrak{g}}_x, \quad u \mapsto \mathcal{R}_x u, \quad \langle \mathcal{R}_x u, v\rangle = d\omega_0(u, Jv)_x. \tag{1.18}$$

For $x \in Y$ we denote by $Y_x = \{h.x : h \in G\}$ the G-orbit of Y; then Y_x is a d-dimensional submanifold of X. The G-invariant Hermitian metric g induces a volume form dv_{Y_x} on Y_x. Put

$$f_G(x) = |\det \mathcal{R}_x|^{-\frac{1}{4}}\sqrt{V_{\rm eff}\,(x)} \in \mathscr{C}^\infty(Y)^G \quad \text{with } V_{\rm eff}\,(x) := \int_{Y_x} dv_{Y_x}, \tag{1.19}$$

where $\mathscr{C}^\infty(Y)^G$ denotes the space of G-invariant smooth functions on $Y = \mu^{-1}(0)$.

Let $E: \mathscr{C}^\infty(X_G) \to \mathscr{C}^\infty(X_G)$ be a classical elliptic pseudodifferential operator with principal symbol $p_E(x,\xi) = |\xi|^{-d/4}$. Let

$$\sigma: H^0_b(X)^G \cap \mathscr{C}^\infty(X)^G \to H^0_b(X_G), \quad \sigma = S_{X_G} \circ E \circ \iota_G \circ f_G \circ \iota^*. \tag{1.20}$$

It turns out that the operator σ in (1.20) is bounded, see Corollary 4.16, and thus extends by density to a bounded operator

$$\sigma: H^0_b(X)^G \to H^0_b(X_G). \tag{1.21}$$

We have encoded in the definition (1.20) some corrections in order to obtain good analytic properties of σ. One correction is the multiplication with the function f_G from (1.19); this reflects the need to reconcile the volume forms on $\mu^{-1}(0)$ and on X_G. The multiplication by f_G changes the CR character of the result, therefore the need to project back to the CR space by S_{X_G}. Here comes the role of E, which is more subtle (see also Remark 1.6). Ideally, the map σ should be unitary. But we can content ourselves to require that $\sigma^*\sigma$ is "microlocally close" to S_G, where $\sigma^*: H^0_b(X_G) \to \mathscr{D}'(X)$ is the adjoint of σ. In other words, we want $\sigma^*\sigma$ to be a complex Fourier integral operator with the same phase, the same order and the same leading symbol as S_G. To achieve this, we need to take E to be a classical elliptic pseudodifferential operator with principal symbol $p_E(x,\xi) = |\xi|^{-d/4}$.

The main technical result of this work is the following.

Theorem 1.5. *Under the assumption of Theorem* 1.2, *the map σ is Fredholm. Actually,* $\operatorname{Ker}\sigma$ *and* $(\operatorname{Im}\sigma)^\perp$ *are finite-dimensional subspaces of* $\mathscr{C}^\infty(X) \cap H^0_b(X)^G$ *and* $\mathscr{C}^\infty(X_G) \cap H^0_b(X_G)$, *respectively.*

Remark 1.6. Note that the definition of σ depends on the choice of the elliptic pseudodifferential operator E. We actually show that for any classical elliptic pseudodifferential operator E with the same principal symbol $p_E(x,\xi) = |\xi|^{-d/4}$, the map $\sigma : H^0_b(X)^G \to H^0_b(X_G)$ is Fredholm. Up to lower order terms of E, the map σ is a canonical choice. The elliptic pseudodifferential operator E corresponds to the power $m^{-d/4}$ in the isomorphism map between $H^0(M, L^m)^G$ and $H^0(M_G, L^m_G)$ in complex case. Here we use the same notations as in the discussion after Theorem 1.3. More precisely, Ma–Zhang [38, Theorem 0.10] showed that the map

$$\sigma_m : H^0(M, L^m)^G \to H^0(M_G, L^m_G), \quad \sigma_m = m^{-d/4} B^m_{M_G} \circ \iota_G \circ f_G \circ \iota^*, \tag{1.22}$$

is an asymptotic isometry if m is large enough, where ι_G, ι^* and $f_G \in \mathscr{C}^\infty(M)^G$ are defined as in the discussion before (1.20) and $B^m_{M_G} : L^2(M_G, L^m_G) \to H^0(M_G, L^m_G)$ is the orthogonal projection. When we change $m^{-d/4}$ in (1.22) to any m-depend function with order $m^{-d/4} + O(m^{-\frac{d}{4}-1})$, we still have an isomorphism between $H^0(M, L^m)^G$ and $H^0(M_G, L^m_G)$ for m large. Moreover, in view of Theorem 1.2, to get L^2 isomorphism it makes sense to take an elliptic pseudodifferential operator E of order $-\frac{d}{4}$.

Remark 1.7. (i) In this work, we do not assume that $\overline{\partial}_b$ has closed range in L^2 on X. We will show in Sec. 3.2 that under the assumption that the Levi form is positive on $Y = \mu^{-1}(0)$, the G-invariant Kohn Laplacian has closed range in L^2 and this is enough to obtain a full asymptotic expansion for the G-invariant Szegő kernel $S_G(x,y)$ (see Theorem 3.25). In order to show that the G-invariant Kohn Laplacian has closed range in L^2 we need the hypothesis that the dimension of X is greater than or equal to five.

(ii) The asymptotic expansion for S_G is also a new result. In [28], Hsiao and Huang obtained an asymptotic expansion for S_G under the assumption that $\overline{\partial}_b$ has closed range in L^2 on X. In [28], Hsiao and Huang established "quantization commutes with reduction" results for CR manifolds with S^1-action. The spaces considered in [28] are finite-dimensional. To handle the infinite-dimensional case, we need to develop a new kind of calculus of complex Fourier integral operators.

(iii) In [19, Theorem A6], Guillemin and Sternberg showed that the underlying canonical relation of S_G is given by $\{(hx, x) : x \in \mu^{-1}(0), h \in G\}$ under the assumption that X is strictly pseudoconvex. From our result about S_G we can also deduce Guillemin and Sternberg's result and also generalize this result to some class of non-strictly pseudoconvex cases.

(iv) If we assume that the CR manifold X is strictly pseudoconvex of dimension greater or equal to five, the proof is shorter. In this case, from the result of Boutet de Monvel-Sjöstrand [8], we see that the Szegő projection is a complex Fourier integral operator on X and by using integration by parts, we can easily see that S_G is smoothing away $\mu^{-1}(0)$. Hence, we do not need Secs. 3.1–3.3. For some class of CR manifolds, the Levi form is just positive near $\mu^{-1}(0)$ (see

Example (1) in Sec. 2.5), therefore it is natural to work with the assumption that the Levi form is just positive near $\mu^{-1}(0)$ since we cannot apply the result of Boutet de Monvel–Sjöstrand directly.

This paper is organized as follows. In Sec. 2, we review some facts on CR and Sasakian manifolds, CR reduction and Szegő kernel. In Sec. 3, we establish asymptotic expansion for the G-invariant Szegő kernel. In Sec. 4, we study the distribution kernel of the map σ in (1.20) and establish Theorem 1.5. In Sec. 5, we establish Theorem 1.2. In Sec. 6, we establish a version for almost complex manifolds of Theorem 1.4.

Notations. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural numbers and set $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, $\mathbb{R}_+ = [0, \infty)$. We use standard notations about distributions and Sobolev spaces on manifolds, as in [30, 37]. In this paper we will systematically use the correspondence between operators A and their kernels $A(\cdot, \cdot) = A(x, y)$ via the Schwartz kernel theorem [26, Theorems 5.2.1, 5.2.6], [37, Theorem B.2.7]. For two distributions u, v we write $u \equiv v$ if $u - v$ is a smooth function. For two operators A, B, we write $A \equiv B$ if their Schwartz kernels satisfy $A(\cdot, \cdot) \equiv B(\cdot, \cdot)$, equivalently, if $A - B$ is a smoothing operator.

In the whole paper we will denote by G a compact Lie group, by $\mathfrak{g}$ its Lie algebra, and by $d\mu$ the Haar measure on G with $\int_G d\mu(h) = 1$. If E is a complex representation of G, we denote by E^G the G-trivial component of E.

We denote by $\operatorname{Spec} A$ the spectrum of an operator A. For a real vector space/bundle V we denote by $\mathbb{C}V = V \otimes_{\mathbb{R}} \mathbb{C}$ the associated complexified vector space/bundle.

2. Preliminaries

In this section, we explain some basic facts on CR and Sasakian manifolds, CR reduction and Szegő kernel.

2.1. *CR manifolds and CR functions*

Let $(X, T^{1,0}X)$ be a compact, connected and orientable Cauchy–Riemann (CR) manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X, that is, $T^{1,0}X$ is a complex vector sub-bundle of rank n of the complexified tangent bundle $\mathbb{C}TX$, satisfying

$$T^{1,0}X \cap T^{0,1}X = \{0\}, \quad [\mathcal{V}, \mathcal{V}] \subset \mathcal{V}, \text{ with } T^{0,1}X = \overline{T^{1,0}X}, \mathcal{V} = \mathscr{C}^\infty(X, T^{1,0}X). \tag{2.1}$$

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. Define the vector bundle of $(0, q)$-forms by

$$T^{*0,q}X := \Lambda^q\, T^{*0,1}X. \tag{2.2}$$

The Levi distribution (or holomorphic tangent space) HX of the CR manifold X is the real part of $T^{1,0}X \oplus T^{0,1}X$, i.e. the unique sub-bundle HX of TX such that

$$\mathbb{C}HX = T^{1,0}X \oplus T^{0,1}X. \tag{2.3}$$

Let $J : HX \to HX$ be the complex structure given by $J(u + \overline{u}) = iu - i\overline{u}$, for every $u \in T^{1,0}X$. If we extend J complex linearly to $\mathbb{C}HX$ we have

$$T^{1,0}X = \{V \in \mathbb{C}HX : JV = iV\}. \tag{2.4}$$

Thus the CR structure $T^{1,0}X$ is determined by the Levi distribution and we shall also write (X, HX, J) to denote the CR manifold $(X, T^{1,0}X)$.

The annihilator $(HX)^0 \subset T^*X$ of HX is called the characteristic conormal bundle of the CR manifold. Since X is orientable, the characteristic bundle $(HX)^0$ is a trivial real line sub-bundle. We fix a global frame of $(HX)^0$, that is, a real non-vanishing 1-form $\omega_0 \in \mathscr{C}^\infty(X, T^*X)$ such that $(HX)^0 = \mathbb{R}\omega_0$, called characteristic 1-form. We have

$$\langle\, \omega_0(x), u \,\rangle = 0, \quad \text{for any } u \in H_xX, \quad x \in X. \tag{2.5}$$

Then by (2.1), the restriction of $d\omega_0$ on HX is a $(1,1)$-form. The Levi form $\mathscr{L}_x = \mathscr{L}_x^{\omega_0}$ of X at $x \in X$ associated to ω_0 is the symmetric bilinear map

$$\mathscr{L}_x : H_xX \times H_xX \to \mathbb{R}, \quad \mathscr{L}_x(u, v) = \frac{1}{2} d\omega_0(u, Jv), \quad \text{for } u, v \in H_xX. \tag{2.6}$$

It induces a Hermitian symmetric map

$$\mathscr{L}_x : T_x^{1,0}X \times T_x^{1,0}X \to \mathbb{C}, \quad \mathscr{L}_x(U, \overline{V}) = \frac{1}{2i} d\omega_0(U, \overline{V}), \quad \text{for } U, V \in T_x^{1,0}X. \tag{2.7}$$

A CR manifold X is said to be strictly pseudoconvex if there exists a characteristic 1-form ω_0 such that for every $x \in X$ the Levi form $\mathscr{L}_x^{\omega_0}$ is positive definite. In this case ω_0 is a contact form and the Levi distribution HX is a contact structure.

Back to the general case, given a characteristic 1-form ω_0, let $T \in \mathscr{C}^\infty(X, TX)$ be a vector field, called characteristic vector field, such that

$$\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T \tag{2.8}$$

and

$$i_T\, \omega_0 = 1. \tag{2.9}$$

Let $g^{\mathbb{C}TX}$ be a Hermitian metric on $\mathbb{C}TX$ such that the decomposition (2.8) is orthogonal. For $u, v \in \mathbb{C}TX$ we denote by $\langle u, v\rangle_g$ the inner product given by $g^{\mathbb{C}TX}$ and for $u \in \mathbb{C}TX$, we write $|u|_g^2 := \langle u, u\rangle_g$.

The determinant of the Levi form $\mathscr{L}_x$ at $x \in X$ with respect to $g^{\mathbb{C}TX}$ is defined by

$$\det \mathscr{L}_x = \lambda_1(x) \dots \lambda_n(x), \tag{2.10}$$

where $\lambda_1(x), \dots, \lambda_n(x)$, are the eigenvalues of $\mathscr{L}_x$ as Hermitian form on $T_x^{1,0}X$ with respect to the inner product $\langle\, \cdot\,, \cdot\,\rangle_g$ on $T_x^{1,0}X$.

The Hermitian metric $g^{\mathbb{C}TX}$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0,q)$ forms $T^{*0,q}X$, $q=1,2,\ldots,n$. We shall also denote the inner product given by these metrics by $\langle\,\cdot\,,\cdot\,\rangle_g$. The metric $g^{\mathbb{C}TX}$ induces a Riemannian metric g^{TX} on TX and g^{TX} induces in turn a Riemannian volume form $dv=dv(x)$ on X and a distance function $d(\cdot,\cdot)$ on X.

The natural global L^2 inner product $(\,\cdot\,,\,\cdot\,)$ on $\Omega^{0,q}(X)$ induced by $dv(x)$ and $\langle\,\cdot\,,\,\cdot\,\rangle_g$ is given by

$$(\,u,v\,) := \int_X \langle\, u(x), v(x)\,\rangle_g \, dv(x), \quad u,v\in\Omega^{0,q}(X). \tag{2.11}$$

We denote by $(L^2_{(0,q)}(X),(\,\cdot\,,\,\cdot\,))$ the completion of $\Omega^{0,q}(X)$ with respect to $(\,\cdot\,,\,\cdot\,)$ and denote $\|\cdot\|$ the corresponding L^2 norm. We set $L^2(X):=L^2_{(0,0)}(X)$.

Let $\overline{\partial}_b:\Omega^{0,q}(X)\to\Omega^{0,q+1}(X)$ be the tangential CR operators on X which is the composition of the exterior differential d and the projection $\pi^{0,q+1}:\Lambda^{q+1}(\mathbb{C}T^*X)\to T^{*0,q+1}X$. We consider the weak maximal extension of $\overline{\partial}_b$ to L^2 spaces as follows:

$$\begin{aligned}\operatorname{Dom}\overline{\partial}_b &= \Big\{u\in L^2_{(0,q)}(X):\overline{\partial}_b u\in L^2_{(0,q+1)}(X)\Big\},\\ \overline{\partial}_b:\operatorname{Dom}\overline{\partial}_b &\ni u\mapsto \overline{\partial}_b u\in L^2_{(0,q+1)}(X),\end{aligned} \tag{2.12}$$

where $\overline{\partial}_b u$ is defined in the sense of distributions. The space of L^2 CR functions on X is given by

$$H^0_b(X):=\Big\{u\in L^2(X)=L^2_{(0,0)}(X):\overline{\partial}_b u=0\Big\}. \tag{2.13}$$

Since differential operators are continuous on distributions, $H^0_b(X)$ is a closed subspace of $L^2(X)$. The *Szegő projection* is the orthogonal projection

$$S:(L^2(X),(\,\cdot\,,\,\cdot\,))\to H^0_b(X). \tag{2.14}$$

The *Szegő kernel* $S(x,y)\in\mathscr{D}'(X\times X)$ is the distribution kernel of S.

2.2. *Sasakian manifolds*

We recall here some facts about Sasakian manifolds, cf. [9, 17]. Recently, the subject of Sasakian geometry generated a great deal of interest due to the study of existence of Sasaki–Einstein metrics, and more generally, Sasakian metrics of constant scalar curvature, see for example [13].

Let (X,HX,J) be an orientable strictly pseudoconvex CR manifold of dimension $2n+1$ and let ω_0 be a contact form whose Levi form (2.6) is positive definite, hence $g^{HX}=d\omega_0(\cdot,J\cdot)$ defines a J-invariant metric on HX. The Reeb vector field R associated to the contact form ω_0 is the vector field on X defined by

$$i_R\,\omega_0=1,\quad i_R\,d\omega_0=0. \tag{2.15}$$

From (2.15), $0=d\omega_0(R,u)=-\omega_0([R,u])$ for any $u\in\mathscr{C}^\infty(X,HX)$, thus we get

$$[R,\mathscr{C}^\infty(X,HX)]\subset\mathscr{C}^\infty(X,HX). \tag{2.16}$$

We define a Riemannian metric g_{ω_0} on X by $g_{\omega_0}(\cdot,\cdot) = d\omega_0(\cdot, J\cdot)+\omega_0(\cdot)\omega_0(\cdot)$. Associated to the data $(X,\omega_0,R,J,g_{\omega_0})$, called contact metric manifold, there is a canonical connection ∇ on TX, called the Tanaka–Webster connection (see Tanaka [51] and Webster [56]), that is the unique affine connection on TX such that

- $\nabla g_{\omega_0} = 0$, $\nabla J = 0$, $\nabla d\omega_0 = 0$ and ∇ preserves the decomposition $TX = HX \oplus \mathbb{R}R$.
- For any u, v in the Levi distribution HX, the torsion T_∇ of ∇ satisfies $T_\nabla(u,v) = d\omega_0(u,v)R$ and $T_\nabla(R,Ju) = -JT_\nabla(R,u)$.

The torsion of the Tanaka–Webster connection in the direction of the Reeb vector field, $\tau\colon u \mapsto T_\nabla(R,u)$, is called pseudo-Hermitian torsion of ∇. We see that $\nabla R = 0$ and thus $\nabla_R = \tau + \mathcal{L}_R$, where $\mathcal{L}$ denotes the Lie derivative. By using $\nabla J = 0$ we deduce that (cf. [51, Lemma 3.2(3)])

$$2J\tau u = (\mathcal{L}_R J)u \quad \text{for any } u \in HX. \tag{2.17}$$

Definition 2.1. A contact metric manifold $(X,\omega_0,R,J,g_{\omega_0})$ is called a Sasakian manifold if the pseudo-Hermitian torsion of its Tanaka–Webster connection vanishes: $\tau = T_\nabla(R,\cdot) = 0$.

By (2.17), $\tau = 0$ is equivalent to $\mathcal{L}_R J = 0$ and by [9, Corollary 6.5.11] this is equivalent to the fact that the contact metric structure on X is normal. Thus the definition above is equivalent to the definition of Sasaki manifolds given in [9, Definition 6.5.13] (which in turn is equivalent to the metric cone $(C(X) = \mathbb{R}_+ \times X, dr^2 + r^2 g_{\omega_0})$ being a Kähler manifold [9, Definition 6.5.15]).

Sasakian manifolds can be classified in three categories based on the properties of the Reeb foliation $\mathcal{F}_R$ consisting of the orbits of the Reeb field (see [9, Definition 6.1.25]). If the orbits of the Reeb field are all closed, then the Reeb field R generates a locally free, isometric S^1-action on (X, g_{ω_0}) and the Reeb foliation is called quasi-regular. If this S^1-action is free, then the Reeb foliation is said to be regular. If $\mathcal{F}_R$ is not quasi-regular, it is said to be irregular. In this case, the flow associated with R generates a transversal CR $\mathbb{R}$-action on X. We say that the $\mathbb{R}$-action η on X is CR transversal if $HX \oplus \mathbb{R}\eta_X = TX$, where η_X denotes the infinitesimal generator field of the $\mathbb{R}$-action.

If $\mathcal{F}_R$ is quasi-regular, then by the structure theorem [9, Theorem 7.1.3] the quotient space $M := X/\mathcal{F}_R = X/S^1$ is a Kähler orbifold and the quotient map $\pi\colon X \to M$ an orbifold Riemannian submersion. Moreover, X is the total space of a principal S^1-bundle over M with connection 1-form ω_0; there exists an integral Kähler form ω on M such that the curvature $d\omega_0$ of ω_0 is the pullback by the quotient map of ω: $d\omega_0 = \pi^*\omega$. If $\mathcal{F}_R$ is regular, then (M,ω) is a Hodge manifold and the assertions above are the content of the Boothby–Wang theorem [5] (cf. also [22]).

Remark 2.2. For a Sasakian manifold $(X,\omega_0,R,J,g_{\omega_0})$, from (2.16), the flow associated with R preserves HX. Moreover, it follows from Definition 2.1 and (2.17) that $\mathcal{L}_R J = 0$. Combining with (2.15), we get $\mathcal{L}_R g_{\omega_0} = 0$, i.e. the Reeb vector field is a Killing vector field on (X, g_{ω_0}).

Remark 2.3. Let $(X, T^{1,0}X)$ be a strictly pseudoconvex CR manifold. Then X is Sasakian if and only if there is a global vector field $T \in \mathscr{C}^\infty(X, TX)$ such that

$$[T, \mathscr{C}^\infty(X, T^{1,0}X)] \subset \mathscr{C}^\infty(X, T^{1,0}X) \tag{2.18}$$

and $T^{1,0}X \oplus T^{0,1}X$ and T generate the complex tangent bundle of X (cf. [9, Proposition 6.4.8], [46, Theorem 1.2]). In fact, if X is Sasakian, we can take $T = R$ by (2.17). For the inverse direction, we define $\omega_0 \in \Omega^1(X)$ by $i_T\omega_0 = 1$, $\omega_0|_{HX} = 0$. Now for any $V \in \mathscr{C}^\infty(X, T^{1,0}X)$ we get by (2.18) that $(i_T d\omega_0)(V) = -\omega_0([T, V]) = 0$. Thus ω_0 verifies (2.15). Again by (2.18) we have $\mathcal{L}_T J = 0$, thus X is Sasakian by (2.17).

Related to Theorem 1.4 and motivated by the structure of quasi-regular Sasakian manifolds let us consider now the case of a circle bundle associated to a Hermitian holomorphic line bundle. Let (L, h^L) be a Hermitian holomorphic line bundle over a connected compact complex manifold (M, J). Let h^{L^*} be the Hermitian metric on L^* induced by h^L. Let

$$X := \big\{ v \in L^* : |v|^2_{h^{L^*}} = 1 \big\} \tag{2.19}$$

be the circle bundle of L^* (Grauert tube); it is isomorphic to the S^1 principal bundle associated to L. Since X is a hypersurface in the complex manifold L^*, it has a CR structure inherited from the complex structure of L^* by setting $T^{1,0}X = TX \cap T^{1,0}L^*$.

In this situation, S^1 acts on X by fiberwise multiplication, denoted $(x, e^{i\theta}) \mapsto xe^{i\theta}$. A point $x \in X$ is a pair $x = (p, \lambda)$, where λ is a linear functional on L_p, the S^1 action is $xe^{i\theta} = (p, \lambda)e^{i\theta} = (p, e^{i\theta}\lambda)$.

On X we have a globally defined vector field ∂_θ, the generator of the S^1 action. The span of ∂_θ defines a rank one subbundle $T^V X \cong TS^1 \subset TX$, the vertical subbundle of the fibration $\pi : X \to M$. Moreover (2.8) holds for $T = \partial_\theta$.

For $m \in \mathbb{Z}$ the space $\mathscr{C}^\infty(X, L^m)$ of smooth sections of L^m can be identified to the space m-equivariant smooth functions

$$\mathscr{C}^\infty(X)_m = \{ f \in \mathscr{C}^\infty(X, \mathbb{C}) : f(xe^{i\theta}) = e^{im\theta} f(x), \text{ for } e^{i\theta} \in S^1, x \in X \}.$$

by

$$\mathscr{C}^\infty(M, L^m) \ni s \mapsto f \in \mathscr{C}^\infty(X)_m, \quad f(x) = f(p, \lambda) = \lambda^{\otimes m}(s(p)), \tag{2.20}$$

where $\lambda^m = \lambda^{\otimes m}$ for $m \geq 0$ and $\lambda^m = (\lambda^{-1})^{\otimes(-m)}$ for $m < 0$. Through the identification (2.20) holomorphic sections correspond to CR functions

$$H^0(M, L^m) \cong H^0_{b,m}(X) := \{ f \in \mathscr{C}^\infty(X)_m : \overline{\partial}_b f = 0 \}. \tag{2.21}$$

We construct now a Riemannian metric on X. Let g^{TM} be a J-invariant metric on M. The Chern connection ∇^L on L induces a connection on the S^1-principal bundle $\pi : X \to M$, and let $T^H X \subset TX$ be the corresponding horizontal bundle.

Let $g^{TX} = \pi^* g^{TM} \oplus \frac{d\theta^2}{4\pi^2}$ be the metric on $TX = T^H X \oplus TS^1$, with $d\theta^2$ the standard metric on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Pertaining to g^{TX} we construct the L^2 inner product $(\cdot,\cdot)_X$ given by (2.11) on X. The metric g^{TM} induces a Riemannian volume form dv_M on M, which together with the fiber metric h^{L^m} gives rise to an L^2 inner product $(\cdot,\cdot)_m$ on $\mathscr{C}^\infty(X, L^m)$. Then the isomorphism (2.20) becomes an isometry $(\mathscr{C}^\infty(M, L^m), (\cdot,\cdot)_m) \cong (\mathscr{C}^\infty(X)_m, (\cdot,\cdot)_X)$ and accordingly an isometry $L^2(M, L^m) \cong L^2(X)_m$, where the latter space is the completion of $(\mathscr{C}^\infty(X)_m, (\cdot,\cdot)_X)$. Moreover, (2.20) induces an isometry

$$(H^0(M, L^m), (\cdot,\cdot)_m) \cong (H^0_{b,m}(X), (\cdot,\cdot)_X). \tag{2.22}$$

The S^1-action gives rise to a Fourier decomposition $L^2(X) \cong \widehat{\bigoplus}_{m\in\mathbb{Z}} L^2(X)_m$ and this induces the following decomposition at the level of CR functions:

$$H^0_b(X) \cong \widehat{\bigoplus}_{m\in\mathbb{Z}} H^0_{b,m}(X) \cong \widehat{\bigoplus}_{m\in\mathbb{Z}} H^0(M, L^m). \tag{2.23}$$

Let ω_0 be the connection 1-form on X associated to the Chern connection ∇^L. Then $\omega_0(\partial_\theta) = 1$, thus (2.8) and (2.9) are fulfilled and $T = \partial_\theta$ is a characteristic vector field on X and ω_0 is a characteristic 1-form for the CR structure on X. Moreover,

$$d\omega_0 = \pi^*(iR^L), \tag{2.24}$$

where R^L is the curvature of ∇^L. On account of (2.6) X is strictly pseudoconvex at $x \in X$ if and only if (L, h^L) is positive at $\pi(x) \in M$. In particular, if (L, h^L) is positive on M, X is a strictly pseudoconvex CR manifold, ω_0 is a contact form and ∂_θ is the associated Reeb vector field.

Assume now (L, h^L) is positive on M. We claim that X is a Sasakian manifold. In fact, the S^1-action is fiberwise multiplication on L^*, thus S^1 acts holomorphically on L^* and preserves $T^{1,0}L^*$. This means that the S^1-action preserves $T^{1,0}X = TX \cap T^{1,0}L^*$, i.e. $\mathcal{L}_{\partial_\theta} J = 0$, hence X is Sasakian by (2.17). Since the S^1-action is free, X is a regular Sasakian manifold. Conversely, any compact regular Sasakian manifold is CR-isomorphic to a S^1-fibration associated with a positive line bundle on a projective manifold by the Boothby–Wang theorem [5], [9, Theorem 6.1.26], [22].

Note also that if (L, h^L) is positive on M then $H^0(X, L^m) = 0$ for $m < 0$ by the Kodaira vanishing theorem, so the decomposition (2.23) becomes

$$H^0_b(X) \cong \widehat{\bigoplus}_{m\in\mathbb{N}} H^0_{b,m}(X) \cong \widehat{\bigoplus}_{m\in\mathbb{N}} H^0(M, L^m). \tag{2.25}$$

Note further that X is (weakly) pseudoconvex, that is, the Levi form is positive semidefinite on X if and only if the curvature iR^L is semi-positive on M. Moreover, X has finite type if and only if R^L vanishes to finite order at any point of M, cf. [31], [41, Proposition 11]. A CR manifold X is said to be of finite type if at each point $p \in X$ if the space T_pX is generated by vectors of the form $[V_1, [V_2, [\ldots [V_{k-1}, V_k] \ldots]]](p)$, where $V_1, \ldots, V_k$, $k \geq 2$, are sections of HX.

2.3. *CR reduction*

We refer to [1, 9, 17] for the fundamentals of contact geometry and examples. In this subsection, we extend the well-known symplectic reduction of Kähler manifolds to CR manifolds. We refer to [1, 9, 16, 17, 57] for constructions and examples of contact and Sasakian reduction. Our construction is a CR analogue of [48, §2.1].

Let (X, HX, J) be a compact connected and orientable CR manifold of dimension $2n+1$, $n \geq 1$, and let ω_0 be a characteristic 1-form.

Let G be a d-dimensional compact Lie group with Lie algebra $\mathfrak{g}$. We assume that G acts smoothly on X and that the G-action preserves J and ω_0.

For any $\xi \in \mathfrak{g}$, we denote $\xi_X(x) = \frac{\partial}{\partial t} \exp(-t\xi)x\big|_{t=0}$ the vector field on X induced by ξ. For $x \in X$, set

$$\underline{\mathfrak{g}}_x = \operatorname{Span}\{\xi_X(x) : \xi \in \mathfrak{g}\}. \tag{2.26}$$

Definition 2.4. The moment map associated to the characteristic 1-form ω_0 is the map $\mu : X \to \mathfrak{g}^*$ defined by

$$\langle \mu(x), \xi \rangle = \omega_0(\xi_X(x)), \quad x \in X, \quad \xi \in \mathfrak{g}. \tag{2.27}$$

The moment map is G-equivariant, i.e. for $x \in X$, $h \in G$, we have

$$\mu(h.x) = \operatorname{Ad}_h^* \mu(x). \tag{2.28}$$

Relation (2.28) implies that G acts on $\mu^{-1}(0)$. In fact, for any $\xi \in \mathfrak{g}$, we have

$$\begin{aligned} \langle \mu(h.x), \xi \rangle &= \omega_0(\xi_X(h.x))) = \omega_0(dh(\operatorname{Ad}_{h^{-1}} \xi)_X))_{h.x} \\ &= (h^*\omega_0)((\operatorname{Ad}_{h^{-1}} \xi)_X)_x = \omega_0((\operatorname{Ad}_{h^{-1}} \xi)_X)_x \\ &= \langle \mu(x), \operatorname{Ad}_{h^{-1}} \xi \rangle = \langle \operatorname{Ad}_h^* \mu(x), \xi \rangle. \end{aligned} \tag{2.29}$$

Lemma 2.5. *If G acts freely on $\mu^{-1}(0)$ and the Levi form is positive on $\mu^{-1}(0)$, then 0 is a regular value of μ.*

Proof. Observe first that since ω_0 is G-invariant, (2.5) yields that HX is an G-equivariant sub-bundle of TX, thus

$$[\xi_X, U] \in \mathscr{C}^\infty(X, HX) \quad \text{for any } U \in \mathscr{C}^\infty(X, HX), \xi \in \mathfrak{g}. \tag{2.30}$$

Now (2.5) and (2.27) entail

$$\xi_X(x) \in HX \quad \text{if } x \in \mu^{-1}(0). \tag{2.31}$$

Let $U_x \in H_xX$ that we extend to a section U of HX near x. Then (2.5) and (2.30) yield

$$\begin{aligned} d\omega_0(U, \xi_X)_x &= U(\omega_0(\xi_X)) - \xi_X(\omega_0(U)) - \omega_0([U, \xi_X]) \\ &= U_x\langle \mu, \xi \rangle = \langle U(\mu)_x, \xi \rangle. \end{aligned} \tag{2.32}$$

If $d\mu_y : HX \to \mathfrak{g}^*$ were not surjective for some $y \in \mu^{-1}(0)$, there would exist $\xi \in \mathfrak{g}$ such that $\langle Y(\mu)_y, \xi\rangle = 0$ for any $Y \in H_yX$. This is a contradiction since $d\omega_0$ is nondegenerate on H_yX and $0 \neq \xi_{X,y} \in H_yX$ by (2.31). Thus $d\mu_y : HX \to \mathfrak{g}^*$ is surjective for $y \in \mu^{-1}(0)$. □

Set as in (1.1) $X_G = Y/G$ with $Y = \mu^{-1}(0)$. Let $\iota : Y \to X$ be the natural injection and let $\pi : Y \to X_G = Y/G$ be the natural projection.

Theorem 2.6. *If G acts freely on $Y = \mu^{-1}(0)$ and the Levi form is positive on $\mu^{-1}(0)$, then the reduced space $X_G = Y/G$ is a strictly pseudoconvex manifold with contact form $\omega_{0,G}$ satisfying $\iota^*\omega_0 = \pi^*\omega_{0,G}$. Moreover, we can choose the characteristic vector field T (cf.* (2.8), (2.9)) *such that $T|_Y \in \mathscr{C}^\infty(Y, TY)$ and T is G-invariant.*

Proof. By Lemma 2.5, $\mu^{-1}(0)$ is a smooth manifold. Since G acts freely on Y, X_G is a compact manifold. The positivity of the Levi form on $\mu^{-1}(0)$ means that

$$g^{HX} = d\omega_0(\cdot, J\cdot) \tag{2.33}$$

is a J-invariant and G-equivariant metric on HX on a neighborhood of $Y = \mu^{-1}(0)$.

Since G acts freely on Y, the vector spaces $\underline{\mathfrak{g}}_x$ defined in (2.26) form a vector bundle $\underline{\mathfrak{g}}$ near $\mu^{-1}(0)$. We denote $\underline{\mathfrak{g}}_Y = \underline{\mathfrak{g}}|_Y$. Then $\underline{\mathfrak{g}}_Y \subset TY \cap HX$ by (2.31).

For $x \in \mu^{-1}(0)$, by (2.31), (2.32), and the fact that $d\omega_0(\cdot, J\cdot)$ is a metric on H_xX we have that $d\mu|_{TY} = 0$ and $d\mu|_{J\underline{\mathfrak{g}}_x} \to \mathfrak{g}^*$ is surjective. Since $\dim Y + \dim \underline{\mathfrak{g}} = \dim TX$, we have

$$J\underline{\mathfrak{g}}|_Y \oplus TY = TX|_Y. \tag{2.34}$$

From (2.34) and $J\underline{\mathfrak{g}}|_Y \subset HX$, we know $\omega_0(TY) \neq 0$. Thus $TY \cap HX$ is a codimension 1 sub-bundle of TY, and

$$\frac{TY}{TY \cap HX} = \left.\frac{TX}{HX}\right|_Y. \tag{2.35}$$

From (2.35), we can choose the vector field T in (2.8) such that $T|_Y \in \mathscr{C}^\infty(Y, TY)$ and T is G-invariant.

Let T^HY be the orthogonal complement of $\underline{\mathfrak{g}}_Y$ in $TY \cap HX$ with respect to g^{HX}. By (2.32), (2.33), for any $U \in TY \cap HX$, $\xi \in \mathfrak{g}$, as $J\underline{\mathfrak{g}}_Y \subset HX$, we have $g^{HX}(U, J\xi_X) = -d\omega_0(U, \xi_X) = 0$. This means that $TY \cap HX$ is orthogonal with $J\underline{\mathfrak{g}}_Y$ with respect to g^{HX}. Thus we have the G-equivariant orthogonal decomposition on Y,

$$TY \cap HX = T^HY \oplus \underline{\mathfrak{g}}_Y, \quad HX|_Y = T^HY \oplus \underline{\mathfrak{g}}_Y \oplus J\underline{\mathfrak{g}}|_Y, \tag{2.36}$$

note that the second equation is from the dimension counting argument as $TY \cap HX$ is a codimension 1 sub-bundle of TY. Thus from (2.36) and the metric g^{HX} on

$HX|_Y$ is J-invariant, we get

$$JT^HY = T^HY = (TY \cap HX) \cap J(TY \cap HX). \tag{2.37}$$

By (2.27) $\iota^*\omega_0$ is a G-invariant horizontal 1-form on Y, thus there exists a unique 1-form $\omega_{0,G} \in \Omega^1(X_G)$ such that

$$\iota^*\omega_0 = \pi^*\omega_{0,G}. \tag{2.38}$$

We now define the Levi distribution on X_G by

$$HX_G := \ker \omega_{0,G}. \tag{2.39}$$

From (2.35), (2.36), $d\pi : \mathbb{R}T \oplus T^HY \to TX_G$ is bijective, and we get the isomorphism $\mathbb{R}T \oplus T^HY \simeq \pi^*TX_G$. Thus $d\pi$ maps T^HY onto HX_G and this gives an isomorphism $T^HY \simeq \pi^*HX_G$. Thus for $U \in H_yX_G$, we take $x \in \pi^{-1}(y)$ and $U^H \in T^HY$ the lift of U, then by (2.37), we define $J_G \in \operatorname{End}(HX_G)$ by

$$(J_GU)^H = JU^H. \tag{2.40}$$

From (2.38), we have

$$\iota^*d\omega_0 = \pi^*d\omega_{0,G}. \tag{2.41}$$

Thus from (2.37), $\iota^*d\omega_0(\cdot, J\cdot)$ is positive and G-invariant on T^HY implies that $d\omega_{0,G}(\cdot, J_G\cdot)$ is positive and J_G-invariant on HX_G. We verify now that

$$T^{1,0}X_G = \left\{u - \sqrt{-1}J_Gu : \, u \in HX_G\right\}. \tag{2.42}$$

defines a CR structure on X. For $U, V \in \mathscr{C}^\infty(X_G, HX_G)$, from (2.40),

$$(U - \sqrt{-1}J_GU)^H = U^H - \sqrt{-1}JU^H \in \mathscr{C}^\infty(Y, T^{1,0}X \cap \mathbb{C}TY), \tag{2.43}$$

thus by (2.1), $[U^H - \sqrt{-1}JU^H, V^H - \sqrt{-1}JV^H] \in \mathscr{C}^\infty(Y, T^{1,0}X \cap \mathbb{C}TY)$. By (2.36), $T^{1,0}X \cap \mathbb{C}TY = \{v - \sqrt{-1}Jv : v \in T^HY\}$. Thus there exists $W \in \mathscr{C}^\infty(X_G, HX_G)$ such that

$$[U^H - \sqrt{-1}JU^H, V^H - \sqrt{-1}JV^H] = W^H - \sqrt{-1}JW^H. \tag{2.44}$$

From (2.43), (2.44), we obtain

$$\begin{aligned}[U - \sqrt{-1}J_GU, V - \sqrt{-1}J_GV] &= d\pi[U^H - \sqrt{-1}JU^H, V^H - \sqrt{-1}JV^H] \\ &= W - \sqrt{-1}J_GW. \end{aligned} \tag{2.45}$$

i.e. $[\mathscr{C}^\infty(X_G, T^{1,0}X_G), \mathscr{C}^\infty(X_G, T^{1,0}X_G)] \subset \mathscr{C}^\infty(X_G, T^{1,0}X_G)$. Let us finally note that (2.35) shows that we can choose the characteristic vector field T (cf. (2.8), (2.9)) such that $T|_Y \in \mathscr{C}^\infty(Y, TY)$ and T is G-invariant. The proof of Theorem 2.6 is completed. □

In the rest of this paper we will work under Assumption 1.1.

Lemma 2.7. *Under Assumption* 1.1 *there is a* G*-invariant Hermitian metric* $g = g^{\mathbb{C}TX}$ *on* $\mathbb{C}TX$ *so that*

(i) $T^{1,0}X$ *is orthogonal to* $T^{0,1}X$,
(ii) $\underline{\mathfrak{g}}$ *is orthogonal to* $HY \cap JHY$ *at every point of* Y,
(iii) $\langle T, T\rangle_g = 1$,
(iv) T *is orthogonal to* $T^{1,0}X \oplus T^{0,1}X$,

where on Y, $HY := HX \cap TY$.

Proof. This follows from the proof of Theorem 2.6. Let U be a G-invariant neighborhood of Y so that the Levi form is positive definite on U. Then the metric $g^{HX} = d\omega_0(\cdot, J\cdot)$ is a J-invariant and G-equivariant metric on HX and we have the orthogonal decomposition (2.36) on Y. Now we extend the metric g^{HX} from U to X as a J-invariant and G-equivariant metric on HX by a partition of unity argument. Thus we can take g^{TX} on $TX = \mathbb{R}T \oplus HX$ as the direct sum metric on (HX, g^{HX}) and $(\mathbb{R}T, \langle T, T\rangle = 1)$. □

Convention 2.8. From now on we fix a G-invariant Hermitian metric $g = g^{\mathbb{C}TX}$ on $\mathbb{C}TX$ so that (i)–(iv) in Lemma 2.7 hold. This metric induces natural Hermitian metrics $\langle\,\cdot\,,\,\cdot\,\rangle_{X_G}$ on $\mathbb{C}TX_G$ and $\mathbb{C}T^*X_G$. As in (2.11) we define the L^2 inner products and spaces induced by g on X and X_G by $(L^2_{(0,q)}(X), (\,\cdot\,,\,\cdot\,))$ and $(L^2_{(0,q)}(X_G), (\,\cdot\,,\,\cdot\,)_{X_G})$.

2.4. *Closed range in L^2 for $\overline{\partial}_b$ and Szegő projections*

The property of closed range in L^2 for $\overline{\partial}_b$ in (2.12) plays an important role in CR geometry. It follows from the works of Boutet de Monvel [6], Boutet de Monvel–Sjöstrand [8], Harvey–Lawson [21], Burns [10] and Kohn [33] that the conditions below are equivalent for a compact strictly pseudoconvex CR manifold X, $\dim_{\mathbb{R}} X \geq 3$:

(a) X is embeddable in the Euclidean space $\mathbb{C}^N$, for N sufficiently large;
(b) X bounds a strictly pseudoconvex complex manifold;
(c) The tangential CR operator $\overline{\partial}_b\colon \operatorname{Dom}\overline{\partial}_b \subset L^2(X) \to L^2_{(0,1)}(X)$ on functions has closed range.

If X is a compact strictly pseudoconvex CR manifold of dimension greater than or equal to five, then X satisfies condition (a), by the embedding theorem of Boutet de Monvel [6]. However, there are examples of non-embeddable compact strictly pseudoconvex CR manifolds of dimension three given by Grauert *et al.* [2, 18, 49], see also Colţoiu–Tibăr [14]. In fact this happens for arbitrarily small perturbations of the standard CR structure on the unit sphere in $\mathbb{C}^2$. For these examples the closed range in L^2 property fails.

Assume that condition (b) is satisfied and let M be a strictly pseudoconvex complex manifold such that $\partial M = X$. If u is continuous on $\overline{M}$ and holomorphic on M, then $u|_X$ satisfies the tangential CR equations $\overline{\partial}_b(u|_X) = 0$. Conversely, by [34] any smooth function u on X satisfying $\overline{\partial}_b u = 0$ admits a smooth extension $\widetilde{u}$ to $\overline{M}$ which is holomorphic in M. In this sense the space $H^0_b(X) \cap \mathscr{C}^\infty(X)$ is the space of boundary values of holomorphic functions $\mathcal{O}(M) \cap \mathscr{C}^\infty(\overline{M})$. Note also that the Hardy space $H^0_b(S^1)$ consists of boundary values of holomorphic functions on the unit disc in $\mathbb{C}$ cf. [50, Theorem 17.10]. This is the unifying feature of our definition of $H^0_b(X)$ for strictly pseudoconvex X and $X = S^1$.

There are important classes of embeddable compact strictly pseudoconvex three-dimensional CR manifolds (for which $\overline{\partial}_b$ has thus closed range in L^2) carrying interesting geometric structures such as

- transverse CR S^1-actions [4, 15, 35],
- conformal structures [3],
- Sasakian structures (transverse CR $\mathbb{R}$-actions) [42],

If X is a compact strictly pseudoconvex CR manifold and $\overline{\partial}_b$ has closed range in L^2, Boutet de Monvel–Sjöstrand [8] showed that $S(x, y)$ is a Fourier integral operator with complex phase. In particular, $S(x, y)$ is smooth outside the diagonal of $X \times X$ and there is a precise description of the singularity on the diagonal $x = y$, where $S(x, x)$ has a certain asymptotic expansion. Hsiao [27, Theorem 1.2] generalized Boutet de Monvel–Sjöstrand's result to $(0, q)$ forms when the Levi form is non-degenerate and Kohn Laplacian for $(0, q)$ forms has closed range in L^2. If the Levi form is degenerate (for example X is weakly pseudoconvex), Hsiao and Marinescu [30, Theorem 1.14] showed that the Szegő projector S is a complex Fourier integral operator on the subset where the Levi form is positive definite if $\overline{\partial}_b$ has closed range in L^2.

Let

$$\overline{\partial}^*_b : \operatorname{Dom} \overline{\partial}^*_b \subset L^2_{(0,1)}(X) \to L^2(X) \tag{2.46}$$

be the Hilbert space adjoint of $\overline{\partial}_b$ in the L^2 space with respect to $(\,\cdot\,,\,\cdot\,)$. Let $\Box_b$ denote the (Gaffney extension) of the Kohn Laplacian on functions given by

$$\begin{aligned} &\operatorname{Dom} \Box_b = \{u \in L^2(X) : u \in \operatorname{Dom} \overline{\partial}_b,\ \overline{\partial}_b u \in \operatorname{Dom} \overline{\partial}^*_b\}, \\ &\Box_b u = \overline{\partial}^*_b \overline{\partial}_b u \quad \text{for } u \in \operatorname{Dom} \Box^{(0)}_b. \end{aligned} \tag{2.47}$$

By a result of Gaffney, $\Box_b$ is a positive self-adjoint operator (see [37, Proposition 3.1.2]). In particular, the spectrum $\operatorname{Spec} \Box_b$ of $\Box_b$ is contained in $[0, \infty)$. For a Borel set $B \subset \mathbb{R}$ we denote by $E(B)$ the spectral projection of $\Box_b$ corresponding to the set B, where E is the spectral measure of $\Box_b$. For $\lambda \geq 0$, we set

$$H^0_{b,\leq\lambda}(X) := \operatorname{Im} E((-\infty, \lambda]) \subset L^2(X), \tag{2.48}$$

and let

$$S_{\le\lambda} : L^2(X) \to H^0_{b,\le\lambda}(X), \tag{2.49}$$

be the orthogonal projection with respect to the product $(\cdot\,,\,\cdot)$ and let

$$S_{\le\lambda}(x,y) \in \mathscr{D}'(X \times X) \tag{2.50}$$

denote the distribution kernel of $S_{\le\lambda}$. For $\lambda = 0$, we write $S := S_{\le 0}$, $S(x,y) := S_{\le 0}(x,y)$.

Without the assumption that the range of $\overline{\partial}_b$ is closed, $\operatorname{Ker}\overline{\partial}_b$ could be trivial and therefore it is natural to consider the spectral projection $S_{\le\lambda}$ for $\lambda > 0$.

Theorem 2.9 ([30, Theorem 1.5]). *For any $\lambda > 0$ the spectral projector $S_{\le\lambda}$ is a complex Fourier integral operator on the subset where the Levi form is positive definite.*

Theorems 3.22 and 3.24 are more detailed statements of this result. Since we do not assume that $\overline{\partial}_b$ has closed range in L^2 on X, Theorem 2.9 plays an important role in this work. We only assume that the Levi form is non-degenerate on Y and we will show in Theorems 3.17 and 3.19 that the G-invariant tangential CR $\overline{\partial}_{b,G}$ has closed range in $L^2(X)^G$ and we have

$$S_G(x,y) = \int_G S_{\le\lambda_0}(x, h\circ y)d\mu(h) \tag{2.51}$$

for some $\lambda_0 > 0$, where $d\mu(h)$ is the Haar measure on G with $\int_G d\mu(h) = 1$. From (2.51), we can apply Theorem 2.9 to study S_G without a closed range in L^2 assumption on $\overline{\partial}_b$ on X.

2.5. *Examples*

We give here some simple but non-trivial examples.

(1) Let $X := \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^4 + |z_2|^2 + |z_3|^2 = 1\}$. Then X is a weakly pseudoconvex CR manifold of dimension five, and X admits a S^1-action

$$S^1 \times X \to X, \quad e^{i\theta}\cdot(z_1,z_2,z_3) = (e^{-i\theta}z_1, e^{i\theta}z_2, e^{i\theta}z_3).$$

Let $\xi_X \in \mathscr{C}^\infty(X,TX)$ be the vector field on X induced by $\xi = -\frac{\partial}{\partial\theta}$ in the Lie algebra of S^1, then

$$\xi_X = -iz_1\frac{\partial}{\partial z_1} + i\overline{z}_1\frac{\partial}{\partial\overline{z}_1} + \sum_{j=2}^{3}\left(iz_j\frac{\partial}{\partial z_j} - i\overline{z}_j\frac{\partial}{\partial\overline{z}_j}\right).$$

Let $\omega_0 := J(dr)$, where $r := |z_1|^4 + |z_2|^2 + |z_3|^2 - 1$ and J is the complex structure map on $T\mathbb{C}^3$. It is straightforward to calculate that

$$\omega_0 = 2iz_1\overline{z}_1^2dz_1 - 2iz_1^2\overline{z}_1d\overline{z}_1 + \sum_{j=2}^{3}(i\overline{z}_jdz_j - iz_jd\overline{z}_j)$$

and

$$\langle\, \omega_0\,,\, \xi_X \,\rangle = 4|z_1|^4 - 2|z_2|^2 - 2|z_3|^2.$$

Hence,

$$\mu^{-1}(0) = \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_2|^2 + |z_3|^2 = \frac{2}{3}, |z_1|^4 = \frac{1}{3}\right\}.$$

Thus, X is strictly pseudoconvex near $\mu^{-1}(0)$. Since X is strictly pseudoconvex near $\mu^{-1}(0)$ and S^1 acts freely on X, from Lemma 2.5, zero is a regular value of the moment map μ. Note that $\mu^{-1}(0)/S^1 \cong S^3$. From our main results, we see that modulo some finite-dimensional subspaces of smooth functions, the space of S^1 invariant CR functions on X is isomorphic to the space of CR functions on S^3.

(2) Let

$$X := \Bigg\{ (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 : (|z_5|^4 + |z_6|^2) \times \left(\sum_{j=1}^{4} |z_j|^2 + z_1 z_3 + z_2 z_4 + \overline{z}_1 \overline{z}_3 + \overline{z}_2 \overline{z}_4\right) = 1 \Bigg\}.$$

Then, X admits a $G := S^1 \times SU(2)$ action:

$$\begin{aligned}
&(e^{i\theta}, g) \cdot z = (w_1, w_2, \ldots, w_6),\\
&(w_1, w_2)^t := g(z_1, z_2)^t, \quad (w_3, w_4)^t := \overline{g}(z_3, z_4)^t, \quad (w_5, w_6) = (e^{-i\theta} z_5, e^{i\theta} z_6),\\
&\qquad g \in SU(2), \quad e^{i\theta} \in S^1, \quad z \in X,
\end{aligned}$$

where z^t denotes the transpose of z. Then X is a weakly pseudoconvex CR manifold and we set $\omega_0 := J(dr)$, where

$$r := (|z_5|^4 + |z_6|^2) \left(\sum_{j=1}^{4} |z_j|^2 + z_1 z_3 + z_2 z_4 + \overline{z}_1 \overline{z}_3 + \overline{z}_2 \overline{z}_4\right) - 1$$

and J is the complex structure map on $T\mathbb{C}^6$. As in Example (1) we can check that if $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mu^{-1}(0)$, then $z_5 \neq 0$. Thus, X is strictly pseudoconvex near $\mu^{-1}(0)$. Since X is strictly pseudoconvex near $\mu^{-1}(0)$ and $S^1 \times SU(2)$ acts freely on X, from Lemma 2.5, zero is a regular value of the moment map μ. Since X is weakly pseudoconvex, it is difficult to understand the Szegő kernel. But from our main results, we see that the G-invariant Szegő kernel is a complex Fourier integral operator.

3. *G*-Invariant Szegő Kernel Asymptotics

In this section, we will establish an asymptotic expansion of the G-invariant Szegő kernel. From now on, we work under Assumption 1.1 and use the same notations as in Secs. 1 and 2. We do not assume that $\overline{\partial}_{b,X_G}$ has closed range in L^2.

3.1. *Subelliptic estimates for G-invariant smooth functions away Y*

In this section, we estimate the Szegő kernel outside Y. The manifold X is supposed to have arbitrary dimension ≥ 3. Let

$$L_1, \ldots, L_N \in \mathscr{C}^\infty(X, HX), \quad N \in \mathbb{N}^*,$$

such that for any $x \in X$, $\{L_1(x), \ldots, L_N(x)\}$ span $H_x X$. Let $s \in \mathbb{N}^*$. For $u \in \mathscr{C}^\infty(X)$, we define

$$|||\, u|||_s := \sum_{\nu=1}^{s} \sum_{1 \leq j_1, \ldots, j_\nu \leq N} \|L_{j_1} L_{j_2} \ldots L_{j_\nu} u\| + \|u\|. \tag{3.1}$$

Theorem 3.1. *There exists $C > 0$ such that for all $u \in \mathscr{C}^\infty(X)$,*

$$|||\, u|||_1^2 \leq C((\,\Box_b u, u\,) + |(\,Tu, u\,)| + \|u\|^2). \tag{3.2}$$

Proof. The proof uses the same method as the proof of [11, Theorem 8.3.5], so we only sketch the proof for the convenience of the reader. Let $D \Subset X$ be a small open set and let $\{Z_1, \ldots, Z_n\} \in \mathscr{C}^\infty(D, T^{1,0}X)$ be an orthonormal frame of $T^{1,0}X$ on D. Let $u \in \mathscr{C}^\infty(D)$. We have

$$(\,\Box_b u, u\,) = \|\overline{\partial}_b u\|^2 = \sum_{j=1}^{n} \|\overline{Z}_j u\|^2. \tag{3.3}$$

For every $j = 1, \ldots, n$, by using integration by parts, we have

$$\begin{aligned}\|Z_j u\|^2 &= (\,Z_j u, Z_j u\,) = (\,Z_j^* Z_j u, u\,) = (\,-\overline{Z}_j Z_j u, u\,) + O(\|Z_j u\|\|u\|)\\ &= (\,-Z_j \overline{Z}_j u, u\,) + (\,[Z_j, \overline{Z}_j]u, u\,) + O(\|Z_j u\|\|u\|)\\ &= \|\overline{Z}_j u\|^2 + (\,[Z_j, \overline{Z}_j]u, u\,) + O(\|Z_j u\|\|u\|) + O(\|\overline{Z}_j u\|\|u\|),\end{aligned} \tag{3.4}$$

where Z_j^* is the formal adjoint of Z_j. From (3.3) and (3.4), we get (3.2) for every $u \in \mathscr{C}_c^\infty(D)$. By using a partition of unity we get (3.2) for every $u \in \mathscr{C}^\infty(X)$. ☐

Fix $x_0 \notin Y$. By definition of Y, we can find a vector field $V \in \mathscr{C}^\infty(X, \underline{\mathfrak{g}})$ such that $\omega_0(V) \neq 0$ in an open neighborhood D of x_0 with $\overline{D} \cap Y = \emptyset$. Then,

$$L := V - \omega_0(V)T \in HX. \tag{3.5}$$

In the rest of this subsection we fix the neighborhood D as above and let

$$\chi, \widetilde{\chi}, \chi_1 \in \mathscr{C}_0^\infty(D), \quad \widetilde{\chi} = 1 \text{ near } \operatorname{supp}\chi \text{ and } \chi_1 = 1 \text{ near } \operatorname{supp}\widetilde{\chi}. \tag{3.6}$$

Let $u \in \mathscr{C}^\infty(X)^G$. From (3.5) and since that $V(u) = 0$ and $\omega_0(V) \neq 0$ on D, we have

$$\begin{aligned}Tu &= \frac{-1}{\omega_0(V)} L(u) \quad \text{on } D,\\ T(\chi u) &= (T\chi)u + \chi Tu = (T\chi)u + \chi \frac{-1}{\omega_0(V)} L(u).\end{aligned} \tag{3.7}$$

From (3.7), we deduce that there exists $C>0$ such that for all $u\in\mathscr{C}^\infty(X)^G$,

$$\|T(\chi u)\|\leq C(|||\chi u\,|||_1+\|\chi_1 u\|). \tag{3.8}$$

For $k\in\mathbb{N}^*$, $U\subset X$ an open set, let $\mathscr{D}^k(U)$ be the set of differential operators which can be written as a linear combination of operators as $W_1\circ\cdots\circ W_j$ with $W_1,\ldots,W_j\in\mathscr{C}^\infty(U,\mathbb{C}HX)$, $1\leq j\leq k$.

Lemma 3.2. *With the notations above, fix $V_1,\ldots,V_s\in\mathscr{C}^\infty(X,\mathbb{C}HX)$, $s\in\mathbb{N}^*$, then there exist $V_{1,1}\in\mathscr{C}^\infty(X,\mathbb{C}HX)$, $Q_1\in\mathscr{D}^{s-1}(X)$, $Q_2\in\mathscr{D}^s(X)$, $a,b\in\mathscr{C}^\infty(X)$, such that*

$$TV_1=V_1T+V_{1,1}+a(x)T,\quad \textit{if } s=1, \tag{3.9}$$

$$TV_1\ldots V_s=V_1\ldots V_sT+Q_1T+Q_2+b(x)T,\quad \textit{if } s\geq 2. \tag{3.10}$$

Proof. We first prove (3.9). Note that

$$TV_1=V_1T+[T,V_1]. \tag{3.11}$$

We have $[T,V_1]=\widetilde{V}_{1,1}+a(x)T$, where $\widetilde{V}_{1,1}\in\mathscr{C}^\infty(X,\mathbb{C}HX)$ and $a(x)\in\mathscr{C}^\infty(X)$. From this observation and (3.11), we get (3.9).

We now prove (3.10). Let $s=2$. By the argument after (3.11), we have

$$\begin{aligned} TV_1V_2&=V_1TV_2+[T,V_1]V_2\\ &=V_1TV_2+(\widetilde{V}_{1,1}+a(x)T)V_2. \end{aligned} \tag{3.12}$$

From (3.9) and (3.12), we get (3.10) for $s=2$.

Assume that the claim (3.10) holds for $s=s_0$ for some $s_0\geq 2$. We are going to prove that the claim (3.10) holds for $s=s_0+1$. From the argument after (3.11), we have

$$\begin{aligned} TV_1\ldots V_{s_0+1}&=V_1TV_2\ldots V_{s_0+1}+[T,V_1]V_2\ldots V_{s_0+1}\\ &=V_1TV_2\ldots V_{s_0+1}+(\widetilde{V}_{1,1}+a(x)T)V_2\ldots V_{s_0+1}. \end{aligned} \tag{3.13}$$

From (3.13) and the induction assumption we get the claim (3.10) for $s=s_0+1$. The lemma follows. ∎

Theorem 3.3. *Fix $s\in\mathbb{N}^*$. Let $V_1,\ldots,V_s\in\mathscr{C}^\infty(D,\mathbb{C}TX)$. Then there exists $Q\in\mathscr{D}^s(D)$ such that we have for every $u\in\mathscr{C}^\infty(X)^G$,*

$$V_1\ldots V_su=Qu\quad \textit{on } D. \tag{3.14}$$

Proof. From (3.7), we see that (3.14) holds for $s=1$. Assume that (3.14) holds for $s=s_0$. We are going to prove that (3.14) holds for $s=s_0+1$. By induction assumption we only need to assume that $V_1=T$ and $V_j\in\mathscr{C}^\infty(D,\mathbb{C}HX)$,

$j = 2, 3, \dots, s_0 + 1$. From (3.9) and (3.10), there exist $\widetilde{V}_{1,1} \in \mathscr{C}^\infty(D, \mathbb{C}HX)$, $Q_1 \in \mathscr{D}^{s-1}(D)$, $Q_2 \in \mathscr{D}^s(D)$ such that

$$\begin{aligned} TV_2 &= V_2T + \widetilde{V}_{1,1} + a(x)T, \quad \text{if } s_0 = 1, \\ TV_2 \dots V_{s_0+1} &= V_2 \dots V_{s_0+1}T + Q_1T + Q_2 + b(x)T, \quad \text{if } s_0 \geq 2, \end{aligned} \tag{3.15}$$

where $a, b \in \mathscr{C}^\infty(D)$. From (3.7) and (3.15), we get (3.14). □

From Theorem 3.3 and (3.6) we deduce:

Corollary 3.4. *Let $s \in \mathbb{N}^*$ and $V_1, \dots, V_s \in \mathscr{C}^\infty(D, \mathbb{C}TX)$. Then, there exists $C > 0$ such that for all $u \in \mathscr{C}^\infty(X)^G$ we have*

$$\begin{aligned} \|(V_1 \dots V_s)\chi u\| &\leq C(|||\chi u|||_s + |||\chi_1 u|||_{s-1}), \\ \|\chi(V_1 \dots V_s)u\| &\leq C(|||\chi u|||_s + |||\chi_1 u|||_{s-1}). \end{aligned} \tag{3.16}$$

For $s \in \mathbb{Z}$, let $\|\cdot\|_s$ denote the standard Sobolev norm of order s on X. From Corollary 3.4 and (3.6), we deduce:

Corollary 3.5. *For every $s \in \mathbb{N}$ there exists $C_s > 0$ such that for any $u \in \mathscr{C}^\infty(X)^G$,*

$$\|\chi u\|_s \leq C_s\, |||\chi_1 u|||_s.$$

Theorem 3.6. *For every $s \in \mathbb{N}$, there exists $C_s > 0$ such that for any $u \in \mathscr{C}^\infty(X)^G$ we have*

$$|||\chi u|||_{s+1}^2 \leq C_s(|||\chi\Box_b u|||_s^2 + |||\chi_1 u|||_s^2). \tag{3.17}$$

Proof. We prove (3.17) by induction over s. From (3.2), there exists $C > 0$ such that for any $u \in \mathscr{C}^\infty(X)^G$ we have

$$|||\chi u|||_1^2 \leq C((\Box_b(\chi u), \chi u) + |(T(\chi u), \chi u)| + \|\chi u\|^2). \tag{3.18}$$

Now by (3.6),

$$(\Box_b(\chi u), \chi u) = (\chi\Box_b u, \chi u) + (\chi[\Box_b, \chi]u, \chi_1 u). \tag{3.19}$$

From (3.7), (3.18), (3.19), and some elementary computation, we get (3.17) for $s = 0$. We now assume that (3.17) holds for every $s < k$ and $k \geq 1$. We will prove that (3.17) holds for $s = k$. Let $Z_1, \dots, Z_{k+1} \in \mathscr{C}^\infty(X, \mathbb{C}HX)$. By (3.2), there exist $C_0, C > 0$ such that for any $u \in \mathscr{C}^\infty(X)^G$, we have

$$\begin{aligned} &\|Z_1 \dots Z_{k+1}(\chi u)\|^2 \\ &\quad \leq C_0\, |||Z_2 \dots Z_{k+1}(\chi u)|||_1^2 \\ &\quad \leq C\Big((\Box_b Z_2 \dots Z_{k+1}(\chi u), Z_2 \dots Z_{k+1}(\chi u)) + \|Z_2 \dots Z_{k+1}(\chi u)\|^2 \\ &\qquad + |(TZ_2 \dots Z_{k+1}(\chi u), Z_2 \dots Z_{k+1}(\chi u))|\Big). \end{aligned} \tag{3.20}$$

We have

$$\begin{aligned}
&(\Box_b(Z_2\dots Z_{k+1})\chi u,(Z_2\dots Z_{k+1})\chi u)\\
&\quad=(Z_2\dots Z_{k+1}\chi\Box_b u+[\Box_b,Z_2\dots Z_{k+1}]\chi u\\
&\qquad+Z_2\dots Z_{k+1}[\Box_b,\chi]u,Z_2\dots Z_{k+1}\chi u)\\
&\quad=(Z_2\dots Z_{k+1}\chi\Box_b u,Z_2\dots Z_{k+1}\chi u)\\
&\qquad+(\chi_1 Z_3\dots Z_{k+1}[\Box_b,\chi]u,Z_2^*Z_2\dots Z_{k+1}\chi u)\\
&\qquad+([\Box_b,Z_2\dots Z_{k+1}]\chi u,Z_2\dots Z_{k+1}\chi u),
\end{aligned}\tag{3.21}$$

where Z_2^* denotes the adjoint of Z_2 and $Z_2^* = -Z_2 +$ zero-order term. From (3.16) and (3.21) we see that there exists $C > 0$ such that

$$\begin{aligned}
&|(\Box_b Z_2\dots Z_{k+1}(\chi u),Z_2\dots Z_{k+1}(\chi u))|\\
&\quad\leq C\left(|||\chi\Box_b u\,|||_k^2+\frac{1}{\varepsilon}\,|||\,\chi_1 u\,|||_k^2+\varepsilon\,|||\,\chi u|||_{k+1}^2\right),
\end{aligned}\tag{3.22}$$

for every $\varepsilon > 0$. Similarly, from (3.16), there exists $\widehat{C} > 0$ such that

$$|(TZ_2\dots Z_{k+1}(\chi u),Z_2\dots Z_{k+1}(\chi u))|\leq\widehat{C}\left(\frac{1}{\varepsilon}\,|||\,\chi_1 u\,|||_k^2+\varepsilon\,|||\,\chi u|||_{k+1}^2\right),\tag{3.23}$$

for every $\varepsilon > 0$. From (3.20), (3.22) and (3.23), we conclude that (3.17) holds for $s = k$ by applying (3.17) for $s < k$ for the term $\|Z_2\dots Z_{k+1}(\chi u)\|^2$ in (3.20). The theorem follows. □

From Corollary 3.5 and Theorem 3.6 we get:

Theorem 3.7. *For every $s \in \mathbb{N}$, there is $C_s > 0$ such that for any $u \in \mathscr{C}^\infty(X)^G$,*

$$\|\chi u\|_{s+1}^2\leq C_s(\|\chi_1\Box_b u\|_s^2+\|\chi_1 u\|_s^2),\tag{3.24}$$

where $\chi,\chi_1 \in \mathscr{C}_0^\infty(D)$ are as in (3.6).

3.2. *Closed range property for the G-invariant Kohn Laplacian*

In this section, we will work in the setting of Assumption 1.1(i). Under these hypotheses we prove subelliptic estimates, regularity and the closed range in L^2 property for the G-invariant Kohn Laplacian.

Let $\overline{\partial}_b^*:\mathrm{Dom}\,\overline{\partial}_b^*\subset L^2_{(0,q+1)}(X)\to L^2_{(0,q)}(X)$ be the Hilbert space adjoint of $\overline{\partial}_b$ in (2.12) with respect to $(\,\cdot\,,\,\cdot\,)$ in (2.11). The operators $\overline{\partial}_b$, $\overline{\partial}_b^*$ commute with

G-action, thus we can define $\overline{\partial}_{b,G}$ and $\overline{\partial}^*_{b,G}$ with

$$\operatorname{Dom}\overline{\partial}_{b,G} := \operatorname{Dom}\overline{\partial}_b \cap L^2_{(0,q)}(X)^G, \quad \operatorname{Dom}\overline{\partial}^*_{b,G} := \operatorname{Dom}\overline{\partial}^*_b \cap L^2_{(0,q+1)}(X)^G, \tag{3.25}$$

and $\overline{\partial}^*_{b,G} : \operatorname{Dom}\overline{\partial}^*_{b,G} \to L^2_{(0,q)}(X)^G$ is the Hilbert space adjoint of $\overline{\partial}_{b,G}$. Let $\Box^{(q)}_{b,G}$ denote the (Gaffney extension) of the G-invariant Kohn Laplacian given by

$$\begin{aligned}\operatorname{Dom}\Box^{(q)}_{b,G} = \{u \in L^2_{(0,q)}(X)^G : u \in \operatorname{Dom}\overline{\partial}_{b,G} \cap \operatorname{Dom}\overline{\partial}^*_{b,G},\\ \overline{\partial}_{b,G}u \in \operatorname{Dom}\overline{\partial}^*_{b,G},\, \overline{\partial}^*_{b,G}u \in \operatorname{Dom}\overline{\partial}_{b,G}\},\\ \Box^{(q)}_{b,G}u = \overline{\partial}_{b,G}\overline{\partial}^*_{b,G}u + \overline{\partial}^*_{b,G}\overline{\partial}_{b,G}u \quad \text{for } u \in \operatorname{Dom}\Box^{(q)}_{b,G}.\end{aligned} \tag{3.26}$$

Lemma 3.8. *Let* $u \in \operatorname{Dom}\overline{\partial}_{b,G} \cap L^2_{(0,q)}(X)^G$. *Then* $\overline{\partial}_{b,G}u \in \operatorname{Dom}\overline{\partial}_{b,G}$ *and* $\overline{\partial}^2_{b,G}$ $u = 0$.

Proof. By Friedrichs' lemma [11, Appendix D], there is a sequence $\{u_j\}_{j=1}^\infty \subset \Omega^{0,q}(X)^G$ such that $u_j \to u$ in $L^2_{(0,q)}(X)^G$ as $j \to \infty$ and $\overline{\partial}_{b,G}u_j \to \overline{\partial}_{b,G}u$ in $L^2_{(0,q+1)}(X)^G$ as $j \to \infty$. Let $v \in \Omega^{0,q+2}(X)^G$. We have

$$(\, \overline{\partial}_{b,G}u \,,\, \overline{\partial}^*_{b,G}v \,) = \lim_{j\to\infty} (\, \overline{\partial}_{b,G}u_j \,,\, \overline{\partial}^*_{b,G}v \,) = \lim_{j\to\infty} (\, \overline{\partial}^2_{b,G}u_j \,,\, v \,) = 0. \tag{3.27}$$

Hence, $\overline{\partial}_{b,G}u \in \operatorname{Dom}\overline{\partial}_{b,G}$ and $\overline{\partial}^2_{b,G}u = 0$. □

Lemma 3.9. *The operator* $\Box^{(q)}_{b,G} : \operatorname{Dom}\Box^{(q)}_{b,G} \subset L^2_{(0,q)}(X)^G \to L^2_{(0,q)}(X)^G$ *is closed.*

Proof. Let $\{(f_k, \Box^{(q)}_{b,G}f_k) \in L^2_{(0,q)}(X)^G \times L^2_{(0,q)}(X)^G : f_k \in \operatorname{Dom}\Box^{(q)}_{b,G}\}_{k=1}^\infty$ with

$$\lim_{k\to\infty} f_k = f, \quad \lim_{k\to\infty} \Box^{(q)}_{b,G}f_k = h \quad \text{in } L^2_{(0,q)}(X)^G. \tag{3.28}$$

By definition, to check that $\Box^{(q)}_{b,G}$ is a closed operator, we need to show that $f \in \operatorname{Dom}\Box^{(q)}_{b,G}$ and $\Box^{(q)}_{b,G}f = h$. Since $f_k \in \operatorname{Dom}\Box^{(q)}_{b,G}$, for each k, we have

$$\begin{aligned}\|\overline{\partial}^*_{b,G}(f_j - f_k)\|^2 &\le \|\overline{\partial}^*_{b,G}(f_j - f_k)\|^2 + \|\overline{\partial}_{b,G}(f_j - f_k)\|^2\\ &= (\, \Box^{(q)}_{b,G}(f_j - f_k) \,,\, f_j - f_k \,).\end{aligned} \tag{3.29}$$

From (3.28) and (3.29), $\{\overline{\partial}^*_{b,G}f_k\}_{k=1}^\infty$ is a Cauchy sequence in L^2, hence $\lim_{k\to\infty} \overline{\partial}^*_{b,G}f_k = h_1$ in $L^2_{(0,q-1)}(X)^G$, for some $h_1 \in L^2_{(0,q-1)}(X)^G$.

Let $v \in \operatorname{Dom}\overline{\partial}_{b,G} \cap L^2_{(0,q-1)}(X)^G$. We have

$$(\, f \,,\, \overline{\partial}_{b,G}v \,) = \lim_{k\to\infty} (\, f_k \,,\, \overline{\partial}_{b,G}v \,) = \lim_{k\to\infty} (\, \overline{\partial}^*_{b,G}f_k \,,\, v \,) = (\, h_1 \,,\, v \,). \tag{3.30}$$

Hence, $f \in \operatorname{Dom} \overline{\partial}^*_{b,G}$ and $\overline{\partial}^*_{b,G} f = \lim_{k\to\infty} \overline{\partial}^*_{b,G} f_k$. Similarly, we can repeat the procedure above and show that $f \in \operatorname{Dom} \overline{\partial}_{b,G}$ and $\overline{\partial}_{b,G} f = \lim_{k\to\infty} \overline{\partial}_{b,G} f_k$.

Now, we show that $\overline{\partial}_{b,G} f \in \operatorname{Dom} \overline{\partial}^*_{b,G}$. From Lemma 3.8, we know that

$$(\,\overline{\partial}^*_{b,G}\,\overline{\partial}_{b,G} u\,,\,\overline{\partial}_{b,G}\,\overline{\partial}^*_{b,G} v\,) = 0, \text{ for every } u, v \in \operatorname{Dom} \Box^{(q)}_{b,G}. \tag{3.31}$$

From (3.31), we have

$$\|\overline{\partial}^*_{b,G}\,\overline{\partial}_{b,G}(f_j - f_k)\|^2 + \|\overline{\partial}_{b,G}\,\overline{\partial}^*_{b,G}(f_j - f_k)\|^2 = \|\Box^{(q)}_{b,G}(f_j - f_k)\|^2. \tag{3.32}$$

From (3.28), (3.32), $\{\overline{\partial}^*_{b,G}\overline{\partial}_{b,G} f_j\}_{j=1}^{\infty}$ is a Cauchy sequence in L^2, hence $\lim_{k\to\infty} \overline{\partial}^*_{b,G}\overline{\partial}_{b,G} f_k = h_2$ in $L^2_{(0,q)}(X)^G$, for some $h_2 \in L^2_{(0,q)}(X)^G$ and thus, $\overline{\partial}_{b,G} f \in \operatorname{Dom} \overline{\partial}^*_{b,G}$ and $\overline{\partial}^*_{b,G}\,\overline{\partial}_{b,G} f = \lim_{k\to\infty} \overline{\partial}^*_{b,G}\,\overline{\partial}_{b,G} f_k$. Similarly, we can repeat the process above and show that

$$\overline{\partial}^*_{b,G} f \in \operatorname{Dom} \overline{\partial}_{b,G}, \quad \overline{\partial}_{b,G}\,\overline{\partial}^*_{b,G} f = \lim_{k\to\infty} \overline{\partial}_{b,G}\,\overline{\partial}^*_{b,G} f_k. \tag{3.33}$$

Hence, $f \in \operatorname{Dom} \Box^{(q)}_{b,G}$ and $\Box^{(q)}_{b,G} f = \lim_{k\to\infty} \Box^{(q)}_{b,G} f_k = h$. The lemma follows. □

Since the dimension of X is greater or equal to five, we can repeat Kohn's method [32] (see also the proof of [11, Theorem 8.3.5]) and deduce the following subelliptic estimates:

Theorem 3.10. *Under Assumption* 1.1(i), *let* $\eta, \eta_1 \in \mathscr{C}^\infty(X)$ *such that* $\eta = 1$ *near* Y, $\eta_1 = 1$ *near* $\operatorname{supp} \eta$ *and the Levi form is positive near* $\operatorname{supp} \eta_1$. *Then for every* $s \in \mathbb{N}$ *there is* $C_s > 0$ *such that for any* $u \in \Omega^{0,1}(X)$,

$$\|\eta u\|^2_{s+1} \le C_s(\|\eta_1 \Box^{(1)}_b u\|^2_s + \|\eta_1 u\|^2_s). \tag{3.34}$$

Repeating the proof of Theorem 3.7 with minor changes we get:

Theorem 3.11. *Let* $\gamma, \gamma_1 \in \mathscr{C}^\infty(X)$ *with* $\gamma_1 = 1$ *near* $\operatorname{supp} \gamma$ *and* $\operatorname{supp} \gamma_1 \cap Y = \emptyset$. *For every* $s \in \mathbb{N}$, *there exists* $C_s > 0$ *such that for any* $u \in \Omega^{0,1}(X)^G$,

$$\|\gamma u\|^2_{s+1} \le C_s(\|\gamma_1 \Box^{(1)}_{b,G} u\|^2_s + \|\gamma_1 u\|^2_s). \tag{3.35}$$

From Theorems 3.10, 3.11 and by using a partition of unity we obtain:

Theorem 3.12. *Assume that Assumption* 1.1(i) *holds. Then for every* $s \in \mathbb{N}$ *and every* $\gamma, \gamma_1 \in \mathscr{C}^\infty(X)$ *with* $\gamma_1 = 1$ *near* $\operatorname{supp} \gamma$, *there is* $C_s > 0$ *such that for any* $u \in \Omega^{0,1}(X)^G$,

$$\|\gamma u\|^2_{s+1} \le C_s(\|\gamma_1 \Box^{(1)}_{b,G} u\|^2_s + \|\gamma_1 u\|^2_s). \tag{3.36}$$

For every $s \in \mathbb{Z}$, let $H^s_{(0,q)}(X)^G$ be the completion of $\Omega^{0,q}(X)^G$ with respect to $\|\cdot\|_s$. From Theorem 3.12, we can repeat the technique of elliptic regularization (see the proof of [11, Theorem 8.4.2]) and conclude:

Theorem 3.13. *Under Assumption* 1.1(i) *let* $u \in \operatorname{Dom} \Box^{(1)}_{b,G}$ *and let* $\mathcal{U}$ *be an open set of* X. *Let* $\Box^{(1)}_{b,G} u = v \in L^2_{(0,1)}(X)^G$. *If* $v|_{\mathcal{U}}$ *is smooth, then* $u|_{\mathcal{U}}$ *is smooth.*

Let $\tau, \tau_1 \in \mathscr{C}^\infty(X)^G$ *with* $\tau_1 = 1$ *near* $\operatorname{supp} \tau$. *If* $\tau_1 v \in H^s_{(0,1)}(X)^G$, *for some* $s \in \mathbb{N}$, *then* $\tau u \in H^{s+1}_{(0,1)}(X)^G$ *and there is* $C_s > 0$ *independent of* u, v, *such that*

$$\|\tau u\|_{s+1} \leq C_s(\|\tau_1 \Box^{(1)}_{b,G} u\|_s + \|\tau_1 u\|_s). \tag{3.37}$$

Theorem 3.14. *If Assumption* 1.1(i) *holds, then the operator* $\Box^{(1)}_{b,G} : \operatorname{Dom} \Box^{(1)}_{b,G} \subset L^2_{(0,1)}(X)^G \to L^2_{(0,1)}(X)^G$ *has closed range.*

Proof. We claim that there is $c > 0$ such that

$$\|\Box^{(1)}_{b,G} u\| \geq c\|u\|, \text{ for all } u \in \operatorname{Dom} \Box^{(1)}_{b,G},\ u \perp \operatorname{Ker} \Box^{(1)}_{b,G}. \tag{3.38}$$

Suppose that the claim is not true. We can find $u_j \in \operatorname{Dom} \Box^{(1)}_{b,G}$, $u_j \perp \operatorname{Ker} \Box^{(1)}_{b,G}$, $\|u_j\| = 1$, $j = 1, 2, \ldots$, such that

$$\|\Box^{(1)}_{b,G} u_j\| \leq \frac{1}{j}\|u_j\|, \quad j = 1, 2, \ldots. \tag{3.39}$$

From Theorem 3.13, (3.39) and Rellich's lemma in functional analysis, we can find $\{j_s\}_{s=1}^\infty \subset \mathbb{N}$, $1 \leq j_1 < j_2 < \ldots$, $\lim_{s\to\infty} j_s = \infty$, such that $\lim_{s\to\infty} \|u_{j_s} - u\| = 0$, for some $u \in L^2_{(0,1)}(X)^G$. It is obvious that $\|u\| = 1$ and $u \perp \operatorname{Ker} \Box^{(1)}_{b,G}$. Let $g \in \operatorname{Dom} \overline{\partial}_{b,G} \bigcap L^2(X)^G$. We have

$$\begin{aligned} |(\, u, \overline{\partial}_{b,G} g\,)| &= \lim_{s\to\infty} |(\, u_{j_s}, \overline{\partial}_{b,G} g\,)| = \lim_{s\to\infty} |(\, \overline{\partial}^*_{b,G} u_{j_s}, g\,)| \\ &\leq \lim_{s\to\infty} \|\overline{\partial}^*_{b,G} u_{j_s}\| \|g\| \leq \lim_{s\to\infty} \|\Box^{(1)}_{b,G} u_{j_s}\|^{\frac{1}{2}} \|g\| \leq \lim_{s\to\infty} \frac{1}{\sqrt{j_s}} \|g\| = 0. \end{aligned}$$

Hence, $u \in \operatorname{Dom} \overline{\partial}^*_{b,G}$ and $\overline{\partial}^*_{b,G} u = 0$. Let $f \in L^2_{(0,2)}(X)^G$. We have

$$\begin{aligned} |(\, u, \overline{\partial}^*_{b,G} f\,)| &= \lim_{s\to\infty} |(\, u_{j_s}, \overline{\partial}^*_{b,G} f\,)| = \lim_{s\to\infty} |(\, \overline{\partial}_{b,G} u_{j_s}, f\,)| \\ &\leq \lim_{s\to\infty} \|\overline{\partial}_{b,G} u_{j_s}\| \|f\| \leq \lim_{s\to\infty} \|\Box^{(1)}_{b,G} u_{j_s}\|^{\frac{1}{2}} \|f\| \leq \lim_{s\to\infty} \frac{1}{\sqrt{j_s}} \|f\| = 0. \end{aligned}$$

Hence, $u \in \operatorname{Dom} \overline{\partial}_{b,G}$ and $\overline{\partial}_{b,G} u = 0$. We have proved that $u \in \operatorname{Ker} \Box^{(1)}_{b,G}$. But $u \perp \operatorname{Ker} \Box^{(1)}_{b,G}$, we get a contradiction. The claim (3.38) follows. From (3.38), we get the lemma. $\square$

Let $N^{(1)}_G : L^2_{(0,1)}(X)^G \to \operatorname{Dom} \Box^{(1)}_{b,G}$ be the partial inverse of $\Box^{(1)}_{b,G}$ and let $S^{(1)}_G : L^2_{(0,1)}(X)^G \to \operatorname{Ker} \Box^{(1)}_{b,G}$ be the Szegő projection, i.e. the orthogonal projection onto $\operatorname{Ker} \Box^{(1)}_{b,G}$ with respect to $(\cdot\,, \cdot\,)$. We have

$$\begin{aligned} &\Box^{(1)}_{b,G} N^{(1)}_G + S^{(1)}_G = I \quad \text{on } L^2_{(0,1)}(X)^G, \\ &N^{(1)}_G \Box^{(1)}_{b,G} + S^{(1)}_G = I \quad \text{on } \operatorname{Dom} \Box^{(1)}_{b,G}. \end{aligned} \tag{3.40}$$

From Theorem 3.13 we deduce:

Theorem 3.15. *Assume that Assumption* 1.1(i) *holds. Then for every* $s \in \mathbb{Z}$ *we can extend* $N_G^{(1)}$ *to a continuous operator* $N_G^{(1)} : H^s_{(0,1)}(X)^G \to H^{s+1}_{(0,1)}(X)$*. Moreover,* $\operatorname{Ker} \Box^{(1)}_{b,G}$ *is a finite-dimensional subspace of* $\Omega^{0,1}(X)^G$*.*

Let

$$\mathcal{P}_G^{(q)} : L^2_{(0,q)}(X) \to L^2_{(0,q)}(X)^G, \tag{3.41}$$

be the orthogonal projection with respect to $(\,\cdot\,,\,\cdot\,)$. It is not difficult to see that for every $s \in \mathbb{Z}$ we can extend $\mathcal{P}_G^{(q)}$ to $H^s_{(0,q)}(X)$ and $\mathcal{P}_G^{(q)} : H^s_{(0,q)}(X) \to H^s_{(0,q)}(X)^G$ is continuous. We extend $N_G^{(1)}$ and $S_G^{(1)}$ to $H^s_{(0,1)}(X)$ by

$$N_G^{(1)}u := N_G^{(1)}\mathcal{P}_G^{(1)}u, \quad S_G^{(1)}u := S_G^{(1)}\mathcal{P}_G^{(1)}u, \quad \text{for } u \in H^s_{(0,1)}(X), \quad s \in \mathbb{Z}. \tag{3.42}$$

From Theorem 3.15 we see that $S_G^{(1)}$ is smoothing and

$$N_G^{(1)} : H^s_{(0,1)}(X) \to H^{s+1}_{(0,1)}(X)^G \text{ is continuous for every } s \in \mathbb{Z}. \tag{3.43}$$

Theorem 3.16. *Under Assumption* 1.1(i), *let* $\tau, \tau_1 \in \mathscr{C}^\infty(X)^G$ *with* $\operatorname{supp}\tau \cap \operatorname{supp}\tau_1 = \emptyset$*. Then* $\tau N_G^{(1)}\tau_1$ *is smoothing.*

Proof. Let $\widetilde{\tau}, \hat{\tau} \in \mathscr{C}^\infty(X)^G$ with $\widetilde{\tau} = 1$ near $\operatorname{supp}\hat{\tau}$, $\hat{\tau} = 1$ near $\operatorname{supp}\tau$ and $\operatorname{supp}\widetilde{\tau} \cap \operatorname{supp}\tau_1 = \emptyset$. Let $v \in L^2_{(0,1)}(X)^G$ and put $\widetilde{\tau}N_G^{(1)}\tau_1 v = u \in H^1_{(0,1)}(X)^G$. From (3.40), we have

$$\begin{aligned}
\Box^{(1)}_{b,G}u &= \Box^{(1)}_{b,G}\widetilde{\tau}N_G^{(1)}\tau_1 v \\
&= \widetilde{\tau}\,\Box^{(1)}_{b,G}N_G^{(1)}\tau_1 v + [\Box^{(1)}_{b,G}, \widetilde{\tau}]N_G^{(1)}\tau_1 v \\
&= \widetilde{\tau}(I - S_G^{(1)})\tau_1 v + [\Box^{(1)}_{b,G}, \widetilde{\tau}]N_G^{(1)}\tau_1 v \\
&= -\widetilde{\tau}S_G^{(1)}\tau_1 v + [\Box^{(1)}_{b,G}, \widetilde{\tau}]N_G^{(1)}\tau_1 v.
\end{aligned} \tag{3.44}$$

Since $S_G^{(1)}$ is smoothing, $-\widetilde{\tau}S_G^{(1)}\tau_1 v \in \mathscr{C}^\infty(X)^G$. Since $\widetilde{\tau} = 1$ near $\operatorname{supp}\hat{\tau}$, $\hat{\tau}[\Box^{(1)}_{b,G}, \widetilde{\tau}]N_G^{(1)}\tau_1 v = 0$. From this observation, we deduce that

$$\hat{\tau}\Box^{(1)}_{b,G}u \in \mathscr{C}^\infty(X)^G. \tag{3.45}$$

Fix $s \in \mathbb{N}$. From (3.37) and (3.44), there exist $C_s, \widetilde{C}_s > 0$ such that for any $v \in L^2_{(0,1)}(X)^G$, we have

$$\begin{aligned}
\|\tau N_G^{(1)}\tau_1 v\|_{s+1} = \|\tau u\|_{s+1} &\leq C_s(\|\hat{\tau}\Box^{(1)}_{b,G}u\|_s + \|\hat{\tau}u\|_s) \\
&\leq \widetilde{C}_s(\|-\hat{\tau}S_G^{(1)}\tau v\|_s + \|\hat{\tau}N_G^{(1)}\tau_1 v\|_s).
\end{aligned} \tag{3.46}$$

Take $s = 1$ in (3.46), from Theorem 3.15 and note that $S_G^{(1)}$ is smoothing, we conclude that $\|\tau N_G^{(1)}\tau_1 v\|_2 \leq C\|v\|$. We have proved that for any $\gamma, \gamma_1 \in \mathscr{C}^\infty(X)^G$

with $\operatorname{supp}\gamma\cap\operatorname{supp}\gamma_1=\emptyset$, we have

$$\gamma N_G^{(1)}\gamma_1 : L^2_{(0,1)}(X)^G \to H^2_{(0,1)}(X)^G \text{ is continuous.} \tag{3.47}$$

Take $s=2$ in (3.46), from (3.47) and note that $S_G^{(1)}$ is smoothing, we conclude that there exists $\hat{C}>0$ such that for any $v\in L^2_{(0,1)}(X)^G$, $\|\tau N_G^{(1)}\tau_1 v\|_3\leq \hat{C}\|v\|$. Continuing in this way, we conclude that for any $\gamma,\gamma_1\in\mathscr{C}^\infty(X)^G$ with $\operatorname{supp}\gamma\cap\operatorname{supp}\gamma_1=\emptyset$, we have

$$\gamma N_G^{(1)}\gamma_1 : L^2_{(0,1)}(X)^G \to H^s_{(0,1)}(X)^G \text{ is continuous, for every } s\in\mathbb{N}^*. \tag{3.48}$$

By taking adjoint in (3.48), we deduce that for any $\gamma,\gamma_1\in\mathscr{C}^\infty(X)^G$ with $\operatorname{supp}\gamma\cap\operatorname{supp}\gamma_1=\emptyset$, we have

$$\gamma N_G^{(1)}\gamma_1 : H^{-s}_{(0,1)}(X)^G \to L^2_{(0,1)}(X)^G \text{ is continuous, for every } s\in\mathbb{N}^*. \tag{3.49}$$

Now, let $v\in H^{-s_0}_{(0,1)}(X)^G$, $s_0\in\mathbb{N}^*$, and put $\widetilde{\tau}N_G^{(1)}\tau_1 v=u\in H^{-s_0+1}_{(0,1)}(X)^G$. Let $v_j\in\Omega^{0,1}(X)^G$, $j=1,2,\ldots$, with $v_j\to v$ in $H^{-s_0}_{(0,1)}(X)^G$ as $j\to\infty$. Taking $s=0$ in (3.46), we deduce from (3.49) and the fact that $S_G^{(1)}$ is smoothing that there exists $C>0$ such that for any $h\in L^2_{(0,1)}(X)^G$,

$$\|\tau N_G^{(1)}\tau_1 h\|_1\leq C\|h\|_{-s_0}. \tag{3.50}$$

From (3.50), we have

$$\lim_{j,k\to\infty}\|\tau N_G^{(1)}\tau_1(v_j-v_k)\|_1\leq C\|v_j-v_k\|_{-s_0}=0. \tag{3.51}$$

From (3.51), we see that $\{\tau N_G^{(1)}\tau_1 v_j\}_{j\geq 1}$ is a Cauchy sequence in $H^1_{(0,1)}(X)^G$. Since $\tau N_G^{(1)}\tau_1 v_j\to\tau N_G^{(1)}\tau_1 v$ in $H^{-s_0+1}_{(0,1)}(X)^G$ as $j\to\infty$ (see Theorem 3.15), we deduce that $\tau N_G^{(1)}\tau_1 v\in H^1_{(0,1)}(X)^G$ and $\|\tau N_G^{(1)}\tau_1 v\|_1\leq C\|v\|_{-s_0}$. We have proved that for any $\gamma,\gamma_1\in\mathscr{C}^\infty(X)^G$ with

$$\operatorname{supp}\gamma\cap\operatorname{supp}\gamma_1=\emptyset,$$

we have

$$\gamma N_G^{(1)}\gamma_1 : H^{-s_0}_{(0,1)}(X)^G \to H^1_{(0,1)}(X)^G \text{ is continuous.} \tag{3.52}$$

Again, take $s=1$ in (3.46) , from (3.52) and note that $S_G^{(1)}$ is smoothing, we conclude that there exists $C>0$ such that for any $h\in L^2_{(0,1)}(X)^G$,

$$\|\tau N_G^{(1)}\tau_1 h\|_2\leq C\|h\|_{-s_0}. \tag{3.53}$$

From (3.53), we can repeat the argument above and deduce that $\tau N_G^{(1)}\tau_1 v\in H^2_{(0,1)}(X)^G$ and $\|\tau N_G^{(1)}\tau_1 v\|_2\leq C\|v\|_{-s_0}$. Continuing in this way, we conclude that for any $\gamma,\gamma_1\in\mathscr{C}^\infty(X)^G$ with $\operatorname{supp}\gamma\cap\operatorname{supp}\gamma_1=\emptyset$, we have

$$\gamma N_G^{(1)}\gamma_1 : H^{-s_0}_{(0,1)}(X)^G \to H^s_{(0,1)}(X)^G \text{ is continuous, for every } s\in\mathbb{N}. \tag{3.54}$$

The theorem follows. ∎

We return to the case of functions. As before, let $S_G : L^2(X) \to \operatorname{Ker} \Box^{(0)}_{b,G} = H^0_b(X)^G$ be the G-invariant Szegő projection. In view of Theorem 3.14, we can repeat the proof of [27, Proposition 6.15, p. 56] and deduce the following.

Theorem 3.17. *Under Assumption* 1.1(i) *the operator* $\Box^{(0)}_{b,G} : \operatorname{Dom} \Box^{(0)}_{b,G} \subset L^2(X)^G \to L^2(X)^G$ *has closed range and*

$$S_G = \mathcal{P}^{(0)}_G - \overline{\partial}^*_{b,G} N^{(1)}_G \overline{\partial}_{b,G} \mathcal{P}^{(0)}_G \quad \text{on } L^2(X). \tag{3.55}$$

Theorem 3.18. *Let* $\tau, \tau_1 \in \mathscr{C}^\infty(X)^G$ *with* $\operatorname{supp} \tau \cap \operatorname{supp} \tau_1 = \emptyset$. *If Assumption* 1.1(i) *holds, then* $\tau S_G \tau_1$ *is smoothing.*

Proof. By (3.41) and (3.55) we have

$$\tau S_G \tau_1 = -\tau \overline{\partial}^*_{b,G} N^{(1)}_G \overline{\partial}_{b,G} \tau_1 = -\tau \overline{\partial}^*_{b,G} \widetilde{\tau} N^{(1)}_G \widetilde{\tau}_1 \overline{\partial}_{b,G} \tau_1, \tag{3.56}$$

where $\widetilde{\tau}, \widetilde{\tau}_1 \in \mathscr{C}^\infty(X)^G$ with $\widetilde{\tau} = 1$ near $\operatorname{supp} \tau$, $\widetilde{\tau}_1 = 1$ near $\operatorname{supp} \tau_1$, $\operatorname{supp} \widetilde{\tau} \cap \operatorname{supp} \widetilde{\tau}_1 = \emptyset$. In view of Theorem 3.16, we see that $\widetilde{\tau} N^{(1)}_G \widetilde{\tau}_1$ is smoothing. From this observation and (3.56), the theorem follows. $\square$

Let $d\mu = d\mu(h)$ be the Haar measure on G with $\int_G d\mu(h) = 1$. We also need the following:

Theorem 3.19. *Under Assumption* 1.1(i) *there exists* $c_0 > 0$ *such that for any* $\lambda \in (0, c_0)$ *such that*

$$S_G(x, y) = \int_G S_{\leq \lambda}(x, h.y) d\mu(h) \quad \text{on } X \times X,$$

where $S_{\leq \lambda}$ *is the spectral projection given by* (2.49).

Proof. Since $\Box^{(0)}_{b,G} : \operatorname{Dom} \Box^{(0)}_{b,G} \subset L^2(X)^G \to L^2(X)^G$ has closed range, we can define the partial inverse

$$N^{(0)}_G : L^2(X)^G \to \operatorname{Dom} \Box^{(0)}_{b,G}$$

of $\Box^{(0)}_{b,G}$ as follows: Let $u \in L^2(X)^G$. Since $\operatorname{Im} \Box^{(0)}_{b,G}$ is closed in $L^2(X)^G$, we have the orthogonal decomposition

$$u = v_0 + v_1, \quad v_0 \in \operatorname{Im} \Box^{(0)}_{b,G}, \quad v_1 \perp \operatorname{Im} \Box^{(0)}_{b,G}.$$

There is a unique $\beta \in \operatorname{Dom} \Box^{(0)}_{b,G}$, $\beta \perp \operatorname{Ker} \Box^{(0)}_{b,G}$, such that $\Box^{(0)}_{b,G} \beta = v_0$. Define $N^{(0)}_G u := \beta$. Since $\Box^{(0)}_{b,G}$ is self-adjoint, $v_1 \in \operatorname{Ker} \Box^{(0)}_{b,G}$. Thus, we have the Hodge decomposition:

$$\Box^{(0)}_{b,G} N^{(0)}_G + S_G = I \text{ on } L^2(X)^G,$$

$$N^{(0)}_G \Box^{(0)}_{b,G} + S_G = I \text{ on } \operatorname{Dom} \Box^{(0)}_{b,G}.$$

Then, $N^{(0)}_G : L^2(X)^G \to \operatorname{Dom} \Box^{(0)}_{b,G}$ is a linear operator. We claim that

$$N^{(0)}_G : L^2(X)^G \to \operatorname{Dom} \Box^{(0)}_{b,G} \text{ is a closed operator.} \tag{3.57}$$

Let $\{(f_k, N^{(0)}_G f_k) \in L^2(X)^G \times L^2(X)^G : k \in \mathbb{N}\}$, with $\lim_{k\to\infty} f_k = f$ in $L^2(X)^G$ and $\lim_{k\to\infty} N^{(0)}_G f_k = h$ in $L^2(X)^G$. For every $k \in \mathbb{N}$, write

$$f_k = \Box^{(0)}_{b,G} g_k + \xi_k, \quad g_k \in \operatorname{Dom} \Box^{(0)}_{b,G}, \quad g_k \perp \operatorname{Ker} \Box^{(0)}_{b,G}, \quad \xi_k \perp \operatorname{Im} \Box^{(0)}_{b,G}. \tag{3.58}$$

From (3.58) and the definition of $N^{(0)}_G$, we see that

$$N^{(0)}_G f_k = g_k, \quad \lim_{k\to\infty} g_k = h \quad \text{in } L^2(X)^G. \tag{3.59}$$

Since $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence, $\{\Box^{(0)}_{b,G} g_k\}_{k=1}^\infty$ and $\{\xi_k\}_{k=1}^\infty$ are Cauchy sequences. Hence,

$$\lim_{k\to\infty} \Box^{(0)}_{b,G} g_k = \eta, \quad \lim_{k\to\infty} \xi_k = \xi \quad \text{in } L^2(X)^G, \tag{3.60}$$

for some $\eta, \xi \in L^2(X)^G$, $\xi \perp \operatorname{Im} \Box^{(0)}_{b,G}$, and we have the orthogonal decomposition

$$f = \eta + \xi. \tag{3.61}$$

From (3.59) and (3.60), we see that $(g_k, \Box^{(0)}_{b,G} g_k) \to (h, \eta)$ in $L^2(X)^G \times L^2(X)^G$ as $k \to \infty$. Since $\Box^{(0)}_{b,G}$ is a closed operator, we conclude that $h \in \operatorname{Dom} \Box^{(0)}_{b,G}$, $h \perp \operatorname{Ker} \Box^{(0)}_{b,G}$, and $\Box^{(0)}_{b,G} h = \eta$. From this observation and (3.61), we get the orthogonal decomposition

$$f = \Box^{(0)}_{b,G} h + \xi$$

and hence $N^{(0)}_G f = h$. The claim (3.57) follows.

Since $N^{(0)}_G$ is a closed operator defined on the Banach space $L^2(X)^G$, by the closed graph theorem, $N^{(0)}_G$ is a continuous operator. Hence, there is $c_0 > 0$ such that

$$\|N^{(0)}_G \beta\| \leq \frac{1}{c_0} \|\beta\|, \quad \text{for every } \beta \in L^2(X)^G. \tag{3.62}$$

Let $\beta = \Box^{(0)}_{b,G} u$ in (3.62), where $u \in \operatorname{Dom} \Box^{(0)}_{b,G}$, $u \perp \operatorname{Ker} \Box^{(0)}_{b,G}$, we get

$$\|\Box^{(0)}_{b,G} u\| \geq c_0 \|u\|, \quad \text{for every } u \in \operatorname{Dom} \Box^{(0)}_{b,G},\ u \perp \operatorname{Ker} \Box^{(0)}_{b,G}. \tag{3.63}$$

Fix $0 < \lambda < c_0$. We claim that with the notation (3.41), we have

$$S_G = S_{\leq\lambda} \circ \mathcal{P}^{(0)}_G \quad \text{on } L^2(X). \tag{3.64}$$

If the claim is not true, we can find a $u \in H^0_{b,\leq\lambda}(X) \cap L^2(X)^G$ with $u \perp \operatorname{Ker} \Box^{(0)}_{b,G}$, $\|u\| = 1$, where $H^0_{b,\leq\lambda}(X)$ is given by (2.48). Since $u \in H^0_{b,\leq\lambda}(X)$, we have

$\|\Box^{(0)}_{b,G}u\| \leq \lambda\|u\| < c_0\|u\|$. From this observation and (3.63) we get a contradiction. The claim (3.64) follows and this yields the theorem. □

3.3. *G-invariant Szegő kernel asymptotics away Y*

Let $S_G(x,y) \in \mathscr{D}'(X \times X)$ be the distribution kernel of S_G. For any subset A of X, put $GA := \{g \cdot x : x \in A, g \in G\}$.

Theorem 3.20. *Under Assumption* 1.1, *let* $\tau \in \mathscr{C}^\infty(X)$ *with* $\operatorname{supp}\tau \cap Y = \emptyset$. *Then* τS_G *and* $S_G\tau$ *are smoothing operators.*

Proof. First we work under Assumption 1.1(i) as in the previous section. Let $v \in L^2(X)$. Take $v_j \in \mathscr{C}^\infty(X)$, $j \in \mathbb{N}^*$, such that $\|v_j - v\| \to 0$ as $j \to \infty$. We have $\|S_G v_j - S_G v\| \to 0$ as $j \to \infty$. By (3.43) and (3.55), we see that

$$S_G v_j \in \mathscr{C}^\infty(X), \quad \text{for every } j \in \mathbb{N}^*. \tag{3.65}$$

For every j, put $u_j := S_G v_j$. By Corollary 3.5 and (3.17), it is straightforward to see that for every $s \in \mathbb{N}$ there exists $C_s > 0$ such that for χ in (3.6),

$$\|\chi u_j\|_s \leq C_s, \quad \text{for every } j \in \mathbb{N}^*. \tag{3.66}$$

From (3.66) we deduce that $\chi S_G v \in H^s(X)$ for every $s \in \mathbb{N}$ and

$$\chi S_G : L^2(X) \to H^s(X) \text{ is continuous, for every } s \in \mathbb{N}. \tag{3.67}$$

By using a partition of unity we conclude that for any $\tau, \tau_1 \in \mathscr{C}^\infty(X)^G$ with $\operatorname{supp}\tau \cap Y = \emptyset$, $\operatorname{supp}\tau_1 \cap Y = \emptyset$, we have

$$\tau S_G : L^2(X) \to H^s(X) \text{ is continuous, for every } s \in \mathbb{N} \tag{3.68}$$

and hence

$$S_G\tau_1 : H^{-s}(X) \to L^2(X) \text{ is continuous, for every } s \in \mathbb{N}. \tag{3.69}$$

From (3.68) and (3.69), we conclude that

$$\tau S_G \tau_1 = (\tau S_G) \circ (S_G \tau_1) : H^{-s}(X) \to H^s(X) \text{ is continuous, for every } s \in \mathbb{N}, \tag{3.70}$$

and hence $\tau S_G \tau_1$ is smoothing.

Since G acts on Y, it is not difficult to see that there is a small neighborhood W of Y such that $\operatorname{supp}\tau \cap GW = \emptyset$. Let $\gamma_0, \gamma_1 \in \mathscr{C}^\infty(X)^G \cap \mathscr{C}^\infty_0(GW)$ with $\gamma_0 = 1$ near $\operatorname{supp}\gamma_1$, $\gamma_1 = 1$ near Y. For $i = 0, 1$, put $\tau_i := 1 - \gamma_i$; then $\tau_i \in \mathscr{C}^\infty(X)^G$ with

$\operatorname{supp}\tau_i \cap Y = \emptyset$. Moreover, it is straightforward to check that

$$\tau_0 = 1 \text{ on } \operatorname{supp}\tau, \quad \operatorname{supp}\tau_0 \cap \operatorname{supp}(1-\tau_1) = \emptyset. \tag{3.71}$$

From (3.71) follows

$$\tau S_G = \tau\tau_0 S_G = \tau\tau_0 S_G \tau_1 + \tau\tau_0 S_G(1-\tau_1). \tag{3.72}$$

By (3.70) $\tau_0 S_G \tau_1$ is smoothing. In view of Theorem 3.18 and (3.71), we see that

$$\tau\tau_0 S_G(1-\tau_1) \text{ is smoothing.} \tag{3.73}$$

From (3.72) and (3.73), we get that τS_G is smoothing and hence $S_G\tau$ is smoothing.

Let us now work under the Assumption 1.1(ii). By [8], [30, Theorem 1.14] the Szegő projector S is a Fourier integral operator with complex phase on X. We have $S_G = S \circ \mathcal{P}_G^{(0)}$. Since S, $\mathcal{P}_G^{(0)}$ map smooth functions to smooth functions, S_G maps smooth functions to smooth functions. Thus, (3.65) still holds. Now,

$$\tau\tau_0 S_G(1-\tau_1) = \tau\tau_0 S \circ \mathcal{P}_G^{(0)}(1-\tau_1) = \tau\tau_0 S(1-\tau_1) \circ \mathcal{P}_G^{(0)}. \tag{3.74}$$

Since S is smoothing away the diagonal, $\tau\tau_0 S \circ (1-\tau_1)$ is smoothing. From this observation and (3.74), we conclude that $\tau\tau_0 S_G(1-\tau_1)$ is smoothing. Thus, (3.73) still holds. Since (3.65) and (3.73) hold so by repeating the proof above we conclude also in this case. □

3.4. *G-invariant Szegő kernel asymptotics near Y*

In this section, X has arbitrary dimension ≥ 3. We first recall the definition of the Hörmander symbol spaces. Let $D \subset X$ be a local coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$.

Definition 3.21. For $m \in \mathbb{R}$, $S^m_{1,0}(D \times D \times \mathbb{R}_+)$ is the space of all $a(x,y,t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+)$ such that, for all compact $K \Subset D \times D$ and all $\alpha, \beta \in \mathbb{N}^{2n+1}$, $\gamma \in \mathbb{N}$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta \partial_t^\gamma a(x,y,t)| \leq C_{\alpha,\beta,\gamma}(1+|t|)^{m-\gamma}, \quad (x,y,t) \in K \times \mathbb{R}_+, \quad t \geq 1.$$

Put

$$S^{-\infty}(D \times D \times \mathbb{R}_+) := \bigcap_{m\in\mathbb{R}} S^m_{1,0}(D \times D \times \mathbb{R}_+).$$

Let $a_j \in S^{m_j}_{1,0}(D \times D \times \mathbb{R}_+)$, $j = 0, 1, 2, \ldots$ with $m_j \searrow -\infty$, as $j \to \infty$. Then there exists $a \in S^{m_0}_{1,0}(D \times D \times \mathbb{R}_+)$ unique modulo $S^{-\infty}$, such that

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{1,0}(D \times D \times \mathbb{R}_+) \quad \text{for all } k \in \mathbb{N}^*.$$

If a and a_j have the properties above, we write $a \sim \sum_{j=0}^\infty a_j$ in $S^{m_0}_{1,0}(D \times D \times \mathbb{R}_+)$. We write

$$s(x,y,t) \in S^m_{\mathrm{cl}}(D \times D \times \mathbb{R}_+) \tag{3.75}$$

if $s(x,y,t) \in S^m_{1,0}(D \times D \times \mathbb{R}_+)$ and

$$s(x,y,t) \sim \sum_{j=0}^{\infty} s_j(x,y) t^{m-j} \quad \text{in } S^m_{1,0}(D \times D \times \mathbb{R}_+),$$
$$s_j(x,y) \in \mathscr{C}^\infty(D \times D), \quad \text{for any } j \in \mathbb{N}. \tag{3.76}$$

Let $W_1 \subset \mathbb{R}^{N_1}$, $W_2 \subset \mathbb{R}^{N_2}$ be open sets. We can also define $S^m_{1,0}(W_1 \times W_2 \times \mathbb{R}_+)$, $S^m_{\mathrm{cl}}(W_1 \times W_2 \times \mathbb{R}_+)$ and asymptotic sum in the similar way.

By Theorem 2.9 (cf. [30, Theorem 1.5]) the spectral projector $S_{\leq\lambda}$ is for every $\lambda > 0$ a complex Fourier integral operator on the subset where the Levi form is positive definite. We give here a detailed description of the spectral kernel.

Theorem 3.22 ([30, Theorem 4.7]). *Fix $\lambda > 0$. Let $D \subset X$ be a local coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. Assume that the Levi form of X is positive definite at every point of D. Then,*

$$S_{\leq\lambda}(x,y) \equiv \int_0^\infty e^{i\varphi(x,y)t} s(x,y,t)dt \quad \text{on } D \times D,$$

with a symbol $s(x,y,t) \in S^n_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$ such that the coefficient s_0 of the expansion (3.76) *is given by*

$$s_0(x,x) = \frac{1}{2}\pi^{-n-1}|\det \mathscr{L}_x|, \quad x \in D, \tag{3.77}$$

where $\det \mathscr{L}_x$ is the determinant of the Levi form, see (2.10), *and the phase function φ satisfies*

$$\begin{aligned} &\varphi \in \mathscr{C}^\infty(D \times D), \quad \operatorname{Im}\varphi(x,y) \geq 0, \\ &\varphi(x,x) = 0, \quad \varphi(x,y) \neq 0 \quad \text{if } x \neq y, \\ &d_x\varphi(x,y)|_{x=y} = -d_y\varphi(x,y)|_{x=y} = \lambda(x)\omega_0(x), \quad \lambda(x) > 0, \\ &\varphi(x,y) = -\overline{\varphi}(y,x). \end{aligned} \tag{3.78}$$

Moreover, let $X' := \{x \in X : \text{the Levi form is non-degenerate at } x\}$. Then, $S_{\leq\lambda}$ is smoothing away the diagonal on $X' \times X'$.

Remark 3.23. With the same notations used in Theorem 3.22, let

$$\Lambda_\varphi := \{(x, td_x\varphi, y, td_y\varphi) \in T^*D \times T^*D : \varphi(x,y) = 0, t > 0\}.$$

From (3.78), we see that

$$\Lambda_\varphi = \{(x, \lambda\omega_0(x), x, -\lambda\omega_0(x)) \in T^*D \times T^*D : x \in D, \lambda > 0\}.$$

Hence, the canonical relation C_φ of $S_{\leq\lambda}$ is given by

$$C_\varphi = \{(x, \lambda\omega_0(x), x, \lambda\omega_0(x)) \in T^*D \times T^*D : x \in D, \lambda > 0\}.$$

The following result describes the phase function in local coordinates (see [27, Chap. 8 of part I]). For a given point $p \in D$, let $\{W_j\}_{j=1}^n$ be an orthonormal frame of $T^{1,0}X$ in a neighborhood of p such that the Levi form is diagonal at p, i.e. $\mathscr{L}_p(W_j, \overline{W}_s) = \delta_{j,s}\mu_j$, $j,s = 1,\dots,n$, where $\delta_{j,s} = 1$ if $j = s$, $\delta_{j,s} = 0$ if $j \neq s$. We take local coordinates $x = (x_1,\dots,x_{2n+1})$ defined on some neighborhood of p such that $\omega_0(p) = -dx_{2n+1}$, $x(p) = 0$ and

$$W_j = \frac{\partial}{\partial z_j} - i\mu_j\overline{z}_j\frac{\partial}{\partial x_{2n+1}} - c_j x_{2n+1}\frac{\partial}{\partial x_{2n+1}} + \sum_{k=1}^{2n} a_{j,k}(x)\frac{\partial}{\partial x_k} + O(|x|^2), \quad j = 1,\dots,n, \tag{3.79}$$

where $z_j = x_{2j-1} + ix_{2j}$, $c_j \in \mathbb{C}$, $a_{j,k}(x)$ is $\mathscr{C}^\infty$, $a_{j,k}(x) = O(|x|)$, for every $j = 1,\dots,n$, $k = 1,\dots,2n$. Set $y = (y_1,\dots,y_{2n+1})$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1,\dots,n$.

Theorem 3.24. *With the same notations and assumptions used in Theorem* 3.22 *we have for the phase function φ in some neighborhood of* $(0,0)$,

$$\operatorname{Im}\varphi(x,y) \geq c\sum_{j=1}^{2n}|x_j - y_j|^2, \quad c > 0, \tag{3.80}$$

$$\begin{aligned}\varphi(x,y) = {} & -x_{2n+1} + y_{2n+1} + i\sum_{j=1}^{n}|\mu_j||z_j - w_j|^2 \\ & + \sum_{j=1}^{n}\Big(i\mu_j(\overline{z}_j w_j - z_j\overline{w}_j) + c_j(-z_j x_{2n+1} + w_j y_{2n+1}) \\ & + \overline{c}_j(-\overline{z}_j x_{2n+1} + \overline{w}_j y_{2n+1})\Big) + (x_{2n+1} - y_{2n+1})f(x,y) \\ & + O(|(x,y)|^3),\end{aligned} \tag{3.81}$$

where f is smooth and satisfies $f(0,0) = 0$, $f(x,y) = \overline{f}(y,x)$. Moreover, we can take the phase φ so that

$$\overline{\partial}_{b,x}\varphi(x,y) \text{ vanishes to infinite order at } x = y. \tag{3.82}$$

Furthermore, for any $\varphi_1(x,y) \in \mathscr{C}^\infty(D \times D)$, if φ_1 satisfies (3.78), (3.80), (3.81) *and* (3.82), *then there is a function $h(x,y) \in \mathscr{C}^\infty(D \times D)$ with $h(x,x) \neq 0$, for every $x \in D$, such that $\varphi(x,y) - h(x,y)\varphi_1(x,y)$ vanishes to infinite order at $x = y$.*

For the next result we recall that the map $\mathcal{R}_x$ and the function V_{eff} were defined in (1.18) and (1.19). We denote by $\lambda'_1(x),\dots,\lambda'_d(x)$ the eigenvalues of $\mathcal{R}_x$ with respect to the G-invariant Hermitian metric g and we define the determinant of $\mathcal{R}_x$ by

$$\det\mathcal{R}_x = \lambda'_1(x)\dots\lambda'_d(x). \tag{3.83}$$

Theorem 3.25. *Under the assumptions of Theorem* 1.2, *let $p \in Y$, let U be an open neighborhood of p and let $x = (x_1,\dots,x_{2n+1})$ be local coordinates defined in U. Then,*

$$S_G(x,y) \equiv \int_0^\infty e^{i\Phi(x,y)t}a(x,y,t)dt \quad \text{on } U \times U \tag{3.84}$$

with a symbol $a(x,y,t) \in S^{n-\frac{d}{2}}_{\mathrm{cl}}(U \times U \times \mathbb{R}_+)$ *such that the coefficient* a_0 *in its expansion* (3.76) *satisfies*

$$a_0(x,x) = 2^{d-1}\frac{1}{V_{\mathrm{eff}}(x)}\pi^{-n-1+\frac{d}{2}}|\det \mathcal{R}_x|^{-\frac{1}{2}}|\det \mathscr{L}_x|, \quad x \in U \cap Y, \tag{3.85}$$

and the phase function Φ *satisfies*

$$\begin{aligned} &\Phi(x,y) \in \mathscr{C}^\infty(U \times U), \quad \operatorname{Im}\Phi(x,y) \geq 0, \\ &d_x\Phi(x,x) = -d_y\Phi(x,x) = \lambda(x)\omega_0(x), \quad \lambda(x) > 0, \quad x \in U \cap Y. \end{aligned} \tag{3.86}$$

Moreover, there exists $C \geq 1$ *such that for all* $(x,y) \in U \times U$,

$$\begin{aligned} &|\Phi(x,y)| + \operatorname{Im}\Phi(x,y) \leq C(\inf\{d^2(g\cdot x,y) : g \in G\} + d^2(x,Y) + d^2(y,Y)), \\ &|\Phi(x,y)| + \operatorname{Im}\Phi(x,y) \geq \frac{1}{C}(\inf\{d^2(g\cdot x,y) : g \in G\} + d^2(x,Y) + d^2(y,Y)), \\ &Cd^2(x,Y) \geq \operatorname{Im}\Phi(x,x) \geq \frac{1}{C}d^2(x,Y), \quad x \in U, \end{aligned} \tag{3.87}$$

and $\Phi(x,y)$ *satisfies* (3.93), (3.94) *and* (3.95) *below.*

Proof. If Assumption 1.1(i) holds, then by Theorem 3.20 we can localize the study of the G-invariant Szegő kernel S_G to Y and from Theorems 3.19 and 3.22, we repeat the proof of [28, Theorem 1.5] and conclude. If Assumption 1.1(ii) holds, we know by [8], [30, Theorem 1.14] that the Szegő projector S is a Fourier integral operator with complex phase on X. Repeating the argument from [28, Theorem 1.5] we conclude. □

Remark 3.26. With the same notations used in Theorem 3.25, let

$$\Lambda_\Phi := \{(x, td_x\Phi, y, td_y\Phi) \in T^*X \times T^*X : \Phi(x,y) = 0, t > 0\}.$$

From (3.87), we see that

$$\Lambda_\Phi = \{(x, \lambda\omega_0(x), g\cdot x, -\lambda\omega_0(x)) \in T^*X \times T^*X : x \in Y, g \in G, \lambda > 0\}.$$

Hence, the canonical relation C_Φ of S_G is given by

$$C_\Phi = \{(x, \lambda\omega_0(x), g\cdot x, \lambda\omega_0(x)) \in T^*X \times T^*X : x \in Y, g \in G, \lambda > 0\}.$$

As applications of Theorems 3.18, 3.20 and 3.25, we establish the following regularity property for S_G.

Theorem 3.27. *Under the assumptions of Theorem* 1.2 *we have*

$$S_G : \mathscr{C}^\infty(X) \to H^0_b(X)^G \cap \mathscr{C}^\infty(X).$$

In particular, $H^0_b(X)^G \cap \mathscr{C}^\infty(X)$ *is dense in* $H^0_b(X)^G$.

Proof. Let U be an open coordinate patch of X and let $u \in \mathscr{C}^\infty_0(U)$. If $U \cap Y = \emptyset$, we see in view of Theorem 3.20 that $S_Gu \in \mathscr{C}^\infty(X)$. Assume now $U \cap Y \neq \emptyset$. By

Theorem 3.25 S_G is a Fourier integral operator with complex phase on U and hence $S_Gu \in \mathscr{C}^\infty(U)$. We only need to show that S_Gu is smooth outside U. Let $x_0 \notin U$. If $x_0 \notin Y$, by Theorem 3.20 again we deduce that S_Gu is smooth near x_0. Now, we suppose that $x_0 \in Y$.

Case I: $Gx_0 \cap U = \emptyset$. We can find $\tau, \tau_1 \in \mathscr{C}^\infty(X)^G$ with $\operatorname{supp}\tau \cap \operatorname{supp}\tau_1 = \emptyset$, $\tau \equiv 1$ near x_0, $\tau_1 \equiv 1$ near $\operatorname{supp} u$. We have $\tau S_G u = \tau S_G \tau_1 u$. In view of Theorem 3.18 under Assumption 1.1(i) or by the fact that S_G is a Fourier integral operator under Assumption 1.1(ii), we see that $\tau S_G \tau_1$ is smoothing and we deduce that $\tau S_G \tau_1 u \in \mathscr{C}^\infty(X)$. In particular, S_Gu is smooth near x_0.

Case II: $Gx_0 \cap U \neq \emptyset$. There is a $\hat{g} \in G$ such that $\hat{g} \cdot x_0 \in U$. Since $S_Gu \in \mathscr{C}^\infty(U)$, S_Gu is smooth near $\hat{g} \cdot x_0$. Since S_Gu is G-invariant, S_Gu is smooth near x_0.

We have thus proved that $S_Gu \in \mathscr{C}^\infty(X)$. By using a partition of unity we conclude. □

Let e_0 be the identity element in G. Fix $p \in Y$. It was shown in [28, Theorem 3.6] that there exist local coordinates $v = (v_1, \ldots, v_d)$ on G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$ (until further notice, we will identify the element $h \in V$ with $v(h)$), local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood $U = U_1 \times U_2$ of p with $0 \leftrightarrow p$, where $U_1 \subset \mathbb{R}^d$ is a neighborhood of $0 \in \mathbb{R}^d$, $U_2 \subset \mathbb{R}^{2n+1-d}$ is an open neighborhood of $0 \in \mathbb{R}^{2n+1-d}$ and a smooth function $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathscr{C}^\infty(U_2, U_1)$ with $\gamma(0) = 0 \in \mathbb{R}^d$ such that for $(v_1, \ldots, v_d) \in V$, $(x_{d+1}, \ldots, x_{2n+1}) \in U_2$,

$$
\begin{aligned}
&(v_1, \ldots, v_d).(\gamma(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}) \\
&\qquad = (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n+1}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}), \\
&Y \cap U = \{x_{d+1} = \ldots = x_{2d} = 0\}, \quad \underline{\mathfrak{g}} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right\}, \\
&J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_{d+j}} + a_j(x)\frac{\partial}{\partial x_{2n+1}}, \quad \text{on } Y \cap U, \text{ for } 1 \leq j \leq d,
\end{aligned}
\tag{3.88}
$$

where $a_j(x)$ are smooth functions on $Y \cap U$, independent of $x_1, \ldots, x_{2d}$, x_{2n+1}, $a_j(0) = 0$, and $T_p^{1,0}X = \operatorname{span}\{Z_1, \ldots, Z_n\}$ with

$$
\begin{aligned}
&Z_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{d+j}}\right)(p), \quad \text{for } j = 1, \ldots, d, \\
&Z_j = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\right)(p), \quad \text{for } j = d+1, \ldots, n,
\end{aligned}
\tag{3.89}
$$

$$
\langle\, Z_j, Z_k \,\rangle_g = \delta_{j,k}, \quad \mathscr{L}_p(Z_j, \overline{Z}_k) = \mu_j\delta_{j,k}, \quad \text{for } j, k = 1, 2, \ldots, n,
$$

$$
\begin{aligned}
-\omega_0(x) = (1 + O(|x|))dx_{2n+1} &+ \sum_{j=1}^{d} 4\mu_j x_{d+j} dx_j + \sum_{j=d+1}^{n} 2\mu_j x_{2j} dx_{2j-1} \\
&- \sum_{j=d+1}^{n} 2\mu_j x_{2j-1} dx_{2j} + \sum_{j=d+1}^{2n} b_j x_{2n+1} dx_j + O(|x|^2),
\end{aligned}
\tag{3.90}
$$

where $b_{d+1},\dots,b_{2n}\in\mathbb{R}$. Put

$$x''=(x_{d+1},\dots,x_{2n+1}),\quad \widehat{x}''=(x_{d+1},\dots,x_{2d}),\quad \mathring{x}''=(x_{d+1},\dots,x_{2n}).\tag{3.91}$$

Theorem 3.28 ([28, Theorem 1.11]). *The phase function $\Phi(x,y)$ appearing in the expression of the Szegő kernel* (3.84), (3.86) *is independent of $(x_1,\dots,x_d)$ and $(y_1,\dots,y_d)$. Hence,*

$$\Phi(x,y)=\Phi((0,x''),(0,y''))=:\Phi(x'',y'').\tag{3.92}$$

Moreover, there exists $c>0$ such that

$$\operatorname{Im}\Phi(x'',y'')\geq c(|\widehat{x}''|^2+|\widehat{y}''|^2+|\mathring{x}''-\mathring{y}''|^2),\quad \textit{for } (0,x''),(0,y'')\in U,\tag{3.93}$$

and there exists a smooth function $g(x,y)\in\mathscr{C}^\infty(U\times U,T^{*0,1}X)$ such that

$$\begin{aligned}&\overline{\partial}_{b,x}\Phi(x'',y'')-g(x,y)\Phi(x'',y'')\ \textit{vanishes to}\\&\qquad\textit{infinite order on}\ \operatorname{diag}((Y\cap U)\times(Y\cap U)),\end{aligned}\tag{3.94}$$

and with the same $\mu_j,b_{d+1},\dots,b_{2n}\in\mathbb{R}$ as in (3.90) *we have*

$$\begin{aligned}\Phi(x'',y'')=&-x_{2n+1}+y_{2n+1}+2i\sum_{j=1}^{d}|\mu_j|y_{d+j}^2+2i\sum_{j=1}^{d}|\mu_j|x_{d+j}^2\\&+i\sum_{j=d+1}^{n}|\mu_j||z_j-w_j|^2+\sum_{j=d+1}^{n}i\mu_j(\overline{z}_jw_j-z_j\overline{w}_j)\\&+\sum_{j=1}^{d}(-b_{d+j}x_{d+j}x_{2n+1}+b_{d+j}y_{d+j}y_{2n+1})\\&+\sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}-ib_{2j})(-z_jx_{2n+1}+w_jy_{2n+1})\\&+\sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}+ib_{2j})(-\overline{z}_jx_{2n+1}+\overline{w}_jy_{2n+1})\\&+(-x_{2n+1}+y_{2n+1})f(x'',y'')+O(|(x'',y'')|^3),\end{aligned}\tag{3.95}$$

where $z_j=x_{2j-1}+ix_{2j}$, $w_j=y_{2j-1}+iy_{2j}$, for $j=d+1,\dots,n$, and $f(x'',y'')\in\mathscr{C}^\infty(U\times U)$ with $f(0,0)=0$.

Remark 3.29. The phase function $\Phi(x'',y'')$ is not unique. For example, we can replace $\Phi(x'',y'')$ by $\Phi(x'',y'')r(x'',y'')$, where $r(x'',y'')\in\mathscr{C}^\infty(U\times U)$, $r(0,0)=1$. In [28, Theorem 5.2], the first author and Huang characterized the phase Φ. Since $\partial_{y_{2n+1}}\Phi(0,0)\neq 0$, the Malgrange preparation theorem [26, Theorem 7.57] implies that

$$\Phi(x'',y'')=g(x'',y'')(y_{2n+1}+\widehat{\Phi}(x'',\mathring{y}''))$$

in some neighborhood of $(0,0)$, where $g(x'',y''), \widehat{\Phi}(x'',\mathring{y}'') \in \mathscr{C}^\infty(U\times U)$. We can replace $\Phi(x'',y'')$ by $y_{2n+1}+\widehat{\Phi}(x'',\mathring{y}'')$. From now on, we assume that

$$\Phi(x'',y'') = y_{2n+1} + \widehat{\Phi}(x'',\mathring{y}''), \quad \widehat{\Phi}(x'',\mathring{y}'') \in \mathscr{C}^\infty(U\times U). \tag{3.96}$$

It is straightforward to check that the phase $\Phi(x'',y'')$ satisfies (3.86), (3.87), (3.93), and

$$\overline{\partial}_{b,x}\Phi(x'',y'') \text{ vanishes to infinite order at } \mathrm{diag}((Y\cap U)\times(Y\cap U)) \tag{3.97}$$

and with the same notations as in (3.95) we have

$$\begin{aligned}\Phi(x'',y'') = {} & -x_{2n+1} + y_{2n+1} + 2i\sum_{j=1}^{d}|\mu_j|y_{d+j}^2 + 2i\sum_{j=1}^{d}|\mu_j|x_{d+j}^2 \\ & + i\sum_{j=d+1}^{n}|\mu_j||z_j-w_j|^2 + \sum_{j=d+1}^{n} i\mu_j(\overline{z}_jw_j - z_j\overline{w}_j) \\ & + \sum_{j=1}^{d}(-b_{d+j}x_{d+j}x_{2n+1} + b_{d+j}y_{d+j}x_{2n+1}) \\ & + \sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}-ib_{2j})(-z_jx_{2n+1}+w_jx_{2n+1}) \\ & + \sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}+ib_{2j})(-\overline{z}_jx_{2n+1}+\overline{w}_jx_{2n+1}) \\ & + O(|(x'',\mathring{y}'')|^3). \end{aligned} \tag{3.98}$$

4. The Distribution Kernels of the Maps σ and $\sigma^*\sigma$

In this section, we will study the map σ defined in (1.20) and prove Theorem 1.5. We assume throughout that $\dim X_G \geq 3$ and $\overline{\partial}_{b,X_G}$ has closed range in L^2. The case when $\dim X_G = 1$ will be treated separately.

Let $\iota^*:\mathscr{C}^\infty(X)\to\mathscr{C}^\infty(Y)$ be the pull-back of the inclusion $\iota:Y\to X$. Let $\iota_G:\mathscr{C}^\infty(Y)^G\to\mathscr{C}^\infty(X_G)$ be the natural identification. Let $S_{X_G}:L^2(X_G)\to H_b^0(X_G)$ be the orthogonal projection with respect to $(\,\cdot\,,\,\cdot\,)_{X_G}$ (cf. Convention 2.8). By Theorem 3.27 and (1.17) (here we use that $\overline{\partial}_{b,X_G}$ has closed range in L^2), we can extend σ to $\mathscr{C}^\infty(X)$ by

$$\begin{gathered}\sigma:\mathscr{C}^\infty(X)\to H_b^0(X_G)\cap\mathscr{C}^\infty(X_G)\subset\mathscr{C}^\infty(X_G), \\ \sigma = S_{X_G}\circ E\circ\iota_G\circ f_G\circ\iota^*\circ S_G.\end{gathered} \tag{4.1}$$

Let $\sigma^*:\mathscr{C}^\infty(X_G)\to\mathscr{D}'(X)$ be the formal adjoint of σ. We will show in Theorem 4.12 that σ^* actually maps $\mathscr{C}^\infty(X_G)$ into $H_b^0(X)^G\cap\mathscr{C}^\infty(X)^G$.

In this section, we will study the distribution kernels of σ and $\sigma^*\sigma$. We explain briefly the role of the operator E in (4.1). To prove our main result, we need to show that $\sigma^*\sigma$ is "microlocally close" to S_G. In other words, we want $\sigma^*\sigma$ to be a complex Fourier integral operator with the same phase, the same order and the same leading symbol as S_G. To achieve this we need to take E to be a classical elliptic pseudodifferential operator with principal symbol $p_E(x,\xi) = |\xi|^{-d/4}$, see also Remark 1.6.

This section is organized as follows. In Sec. 4.1, we develop the calculus of Fourier integral operators of G-Szegő type. In Sec. 4.2, we study the distribution kernels of σ and $\sigma^*\sigma$ and prove Theorem 1.5.

4.1. *Calculus of complex Fourier integral operators*

Let $p \in Y$ and $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in the discussion before Theorem 3.28 defined in an open neighborhood U of p. From now on, we change x_{2n+1} to $x_{2n+1} - \sum_{j=1}^{d} a_j(x)x_{d+j}$, where $a_j(x)$ are as in (3.88). With this new local coordinates $x = (x_1, \ldots, x_{2n+1})$ on $Y \cap U$ we have

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_{d+j}}, \quad \text{for } j = 1, 2, \ldots, d. \tag{4.2}$$

Moreover, it is clear that $\Phi(x,y)+\sum_{j=1}^{d} a_j(x)x_{d+j}-\sum_{j=1}^{d} a_j(y)y_{d+j}$ satisfies (3.98). Note that $a_j(x)$ is a smooth function on $Y \cap U$, independent of $x_1, \ldots, x_{2d}$, x_{2n+1} and $a_j(0) = 0$, $j = 1, \ldots, d$. We may assume that $U = \Omega_1 \times \Omega_2 \times \Omega_3$, where $\Omega_1 \subset \mathbb{R}^d$, $\Omega_2 \subset \mathbb{R}^d$ are open neighborhoods of $0 \in \mathbb{R}^d$, $\Omega_3 \subset \mathbb{R}^{2n+1-2d}$ is an open neighborhood of $0 \in \mathbb{R}^{2n+1-2d}$. From now on, we identify Ω_2 with

$$\{(0, \ldots, 0, x_{d+1}, \ldots, x_{2d}, 0, \ldots, 0) \in U : (x_{d+1}, \ldots, x_{2d}) \in \Omega_2\},$$

Ω_3 with $\{(0, \ldots, 0, x_{2d+1}, \ldots, x_{2n+1}) \in U : (x_{2d+1}, \ldots, x_{2n+1}) \in \Omega_3\}$, $\Omega_2 \times \Omega_3$ with

$$\{(0, \ldots, 0, x_{d+1}, \ldots, x_{2n+1}) \in U : (x_{d+1}, \ldots, x_{2n+1}) \in \Omega_2 \times \Omega_3\}.$$

For $x = (x_1, \ldots, x_{2n+1})$, we set

$$\begin{aligned} &x' = (x_1, \ldots, x_d), \\ &x'' = (x_{d+1}, \ldots, x_{2n+1}), \quad \mathring{x}'' = (x_{d+1}, \ldots, x_{2n}), \\ &\widehat{x}'' = (x_{d+1}, \ldots, x_{2d}), \quad \underline{x}'' = (x_{2d+1}, \ldots, x_{2n+1}). \end{aligned} \tag{4.3}$$

From now on, we identify

$$\begin{aligned} &x'' \text{ with } (0, \ldots, 0, x_{d+1}, \ldots, x_{2n+1}) \in U, \\ &\widehat{x}'' \text{ with } (0, \ldots, 0, x_{d+1}, \ldots, x_{2d}, 0, \ldots, 0) \in U, \\ &\underline{x}'' \text{ with } (0, \ldots, 0, x_{2d+1}, \ldots, x_{2n+1}) \in U. \end{aligned} \tag{4.4}$$

Since G acts freely on Y, we take Ω_2 and Ω_3 small enough so that if $x, x_1 \in \Omega_2 \times \Omega_3$ and $x \neq x_1$, then

$$g \cdot x \neq x_1, \quad \text{for all } g \in G. \tag{4.5}$$

Recall that we take Φ so that (3.96), (3.97), (3.98) hold. Put

$$\Phi^*(x, y) := -\overline{\Phi}(y, x). \tag{4.6}$$

From (3.97) and notice that for $j = 1, \ldots, d$, $x \in Y$, we have $\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_{d+j}} \in T^{0,1}_x X$ and $\frac{\partial}{\partial x_j}\Phi(x, y) = \frac{\partial}{\partial y_j}\Phi^*(x, y) = 0$, we conclude that for $j = 1, \ldots, d$,

$$\begin{gathered}\frac{\partial}{\partial x_{d+j}}\Phi(x,y)\bigg|_{x_{d+1}=\ldots=x_{2d}=0} \quad \text{and} \quad \frac{\partial}{\partial y_{d+j}}\Phi^*(x,y)\bigg|_{y_{d+1}=\ldots=y_{2d}=0}\\ \text{vanish to infinite order at } \mathrm{diag}((Y \cap U) \times (Y \cap U)).\end{gathered} \tag{4.7}$$

Let

$$\begin{aligned} G_j(x, y) &:= \frac{\partial}{\partial y_{d+j}}\Phi^*(x, y)|_{y_{d+1}=\ldots=y_{2d}=0}, \\ H_j(x, y) &:= \frac{\partial}{\partial x_{d+j}}\Phi(x, y)|_{x_{d+1}=\ldots=x_{2d}=0}. \end{aligned}$$

Put

$$\begin{aligned} \Phi_1(x, y) &:= \Phi^*(x, y) - \sum_{j=1}^{d} y_{d+j}G_j(x, y), \\ \Phi_2(x, y) &:= \Phi(x, y) - \sum_{j=1}^{d} x_{d+j}H_j(x, y). \end{aligned} \tag{4.8}$$

Then for $j = 1, 2, \ldots, d$,

$$\frac{\partial}{\partial y_{d+j}}\Phi_1(x,y)\bigg|_{y_{d+1}=\ldots=y_{2d}=0} = \frac{\partial}{\partial x_{d+j}}\Phi_2(x,y)\bigg|_{x_{d+1}=\ldots=x_{2d}=0} = 0, \tag{4.9}$$

and

$$\begin{aligned} &\Phi^*(x, y) - \Phi_1(x, y) \text{ vanishes to infinite order on } \mathrm{diag}((Y \cap U) \times (Y \cap U)), \\ &\Phi(x, y) - \Phi_2(x, y) \text{ vanishes to infinite order on } \mathrm{diag}((Y \cap U) \times (Y \cap U)). \end{aligned} \tag{4.10}$$

We also write $u = (u_1, \ldots, u_{2n+1})$ to denote the local coordinates of U. For any smooth function $h \in \mathscr{C}^\infty(U)$, let $\widetilde{h} \in \mathscr{C}^\infty(U^{\mathbb{C}})$ be an almost analytic extension of

h (see [45, Sec. 1]). Let ϑ be a local coordinate of $\mathbb{R}$. Let

$$F(\widetilde{x},\widetilde{y},\widetilde{u''},\widetilde{\vartheta}\,) := \widetilde{\Phi}_1(\widetilde{x},\widetilde{u''}) + \widetilde{\vartheta}\,\widetilde{\Phi}_2(\widetilde{u''},\widetilde{y}). \tag{4.11}$$

We consider the following two systems for $j = 1, 2, \ldots, 2n - 2d + 1$ and $j = 1, 2, \ldots, 2n - d + 1$, respectively,

$$\begin{aligned} &\frac{\partial F}{\partial \widetilde{\vartheta}}(\widetilde{x},\widetilde{y},\widetilde{\underline{u}''},\widetilde{\vartheta}\,) = \widetilde{\Phi}_2(\widetilde{\underline{u}''},\widetilde{y}) = 0,\\ &\frac{\partial F}{\partial \widetilde{u}_{2d+j}}(\widetilde{x},\widetilde{y},\widetilde{\underline{u}''},\widetilde{\vartheta}\,) = \frac{\partial \widetilde{\Phi}_1}{\partial \widetilde{y}_{2d+j}}(\widetilde{x},\widetilde{\underline{u}''}) + \widetilde{\vartheta}\frac{\partial \widetilde{\Phi}_2}{\partial \widetilde{x}_{2d+j}}(\widetilde{\underline{u}''},\widetilde{y}) = 0, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\frac{\partial F}{\partial \widetilde{\vartheta}}(\widetilde{x},\widetilde{y},\widetilde{u''},\widetilde{\vartheta}\,) = \widetilde{\Phi}_2(\widetilde{u''},\widetilde{y}) = 0,\\ &\frac{\partial F}{\partial \widetilde{u}_{d+j}}(\widetilde{x},\widetilde{y},\widetilde{u''},\widetilde{\vartheta}\,) = \frac{\partial \widetilde{\Phi}_1}{\partial \widetilde{y}_{d+j}}(\widetilde{x},\widetilde{u''}) + \widetilde{\vartheta}\frac{\partial \widetilde{\Phi}_2}{\partial \widetilde{x}_{d+j}}(\widetilde{u''},\widetilde{y}) = 0, \end{aligned} \tag{4.13}$$

where $\widetilde{\underline{u}''} = (0, \ldots, 0, \widetilde{u}_{2d+1}, \ldots, \widetilde{u}_{2n+1})$, $\widetilde{u''} = (0, \ldots, 0, \widetilde{u}_{d+1}, \ldots, \widetilde{u}_{2n+1})$. Here we always use $\frac{\partial}{\partial \widetilde{x}_k}$ for first variable, $\frac{\partial}{\partial \widetilde{y}_k}$ for second variable. From (3.96) and (4.9), we can take $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ so that for every $j = 1, 2, \ldots, d$,

$$\frac{\partial \widetilde{\Phi}_1}{\partial \widetilde{y}_{d+j}}(\widetilde{x},\widetilde{u''}) = \frac{\partial \widetilde{\Phi}_2}{\partial \widetilde{x}_{d+j}}(\widetilde{u''},\widetilde{y}) = 0, \quad \text{if } \widetilde{u}_{d+1} = \ldots = \widetilde{u}_{2d} = 0, \tag{4.14}$$

and $\widetilde{\widetilde{\Phi}}_1, \widetilde{\widetilde{\Phi}}_2 \in \mathscr{C}^\infty(U^{\mathbb{C}} \times U^{\mathbb{C}})$ such that

$$\widetilde{\Phi}_1(\widetilde{x},\widetilde{y}) = -\widetilde{x}_{2n+1} + \widetilde{\widetilde{\Phi}}_1(\widetilde{\mathring{x}}'',\widetilde{y''}), \quad \widetilde{\Phi}_2(\widetilde{x},\widetilde{y}) = \widetilde{y}_{2n+1} + \widetilde{\widetilde{\Phi}}_2(\widetilde{x''},\widetilde{\mathring{y}}''), \tag{4.15}$$

where $\widetilde{\mathring{x}}'' = (0, \ldots, 0, \widetilde{x}_{d+1}, \ldots, \widetilde{x}_{2n}, 0)$, $\widetilde{\mathring{y}}'' = (0, \ldots, 0, \widetilde{y}_{d+1}, \ldots, \widetilde{y}_{2n}, 0)$.

By (3.86), (4.6), (4.7) and $\underline{x}'' \in Y$, we know that $d_x\Phi(x,x), d_y\Phi(x,x)$ are real for $x \in Y$, and for $j = 1, 2, \ldots, 2n - d$,

$$\begin{aligned} \frac{\partial \widetilde{\Phi}_1}{\partial \widetilde{y}_{d+j}}(\underline{x}'',\underline{x}'') &= -\frac{\partial \overline{\Phi}}{\partial \widetilde{x}_{d+j}}(\underline{x}'',\underline{x}'') = -\frac{\partial \Phi}{\partial x_{d+j}}(\underline{x}'',\underline{x}'')\\ &= \frac{\partial \Phi}{\partial y_{d+j}}(\underline{x}'',\underline{x}'') = \frac{\partial \Phi_2}{\partial y_{d+j}}(\underline{x}'',\underline{x}''), \end{aligned}$$

$$\frac{\partial \widetilde{\Phi}_2}{\partial \widetilde{y}_{d+j}}(\underline{x}'',\underline{x}'') + \frac{\partial \widetilde{\Phi}_2}{\partial \widetilde{x}_{d+j}}(\underline{x}'',\underline{x}'') = 0.$$

Note that by (3.87), $\Phi(x,x) = 0$ for $x \in Y$. Combining it with the above equation, we get

$$\begin{aligned} &\widetilde{\Phi}_2(\underline{x}'',\underline{x}'') = 0,\\ &\frac{\partial \widetilde{\Phi}_1}{\partial \widetilde{y}_{d+j}}(\underline{x}'',\underline{x}'') + \widetilde{\vartheta}\frac{\partial \widetilde{\Phi}_2}{\partial \widetilde{x}_{d+j}}(\underline{x}'',\underline{x}'')\big|_{\widetilde{\vartheta}=1} = 0, \quad j = 1, 2, \ldots, 2n - d, \end{aligned} \tag{4.16}$$

and the Hessians

$$F_{\widetilde{\vartheta},\widetilde{\underline{u}''}}(0,0,0,1)=\left.\begin{pmatrix}\dfrac{\partial^2 F}{\partial^2\widetilde{\vartheta}} & \dfrac{\partial^2 F}{\partial\widetilde{\vartheta}\partial\widetilde{\underline{u}''}}\\ \dfrac{\partial^2 F}{\partial\widetilde{\underline{u}''}\partial\widetilde{\vartheta}} & \dfrac{\partial^2 F}{\partial^2\widetilde{\underline{u}''}}\end{pmatrix}\right|_{(0,0,0,1)} \tag{4.17}$$

and

$$F_{\widetilde{\vartheta},\widetilde{u''}}(0,0,0,1)=\left.\begin{pmatrix}\dfrac{\partial^2 F}{\partial^2\widetilde{\vartheta}} & \dfrac{\partial^2 F}{\partial\widetilde{\vartheta}\partial\widetilde{u''}}\\ \dfrac{\partial^2 F}{\partial\widetilde{u''}\partial\widetilde{\vartheta}} & \dfrac{\partial^2 F}{\partial^2\widetilde{u''}}\end{pmatrix}\right|_{(0,0,0,1)} \tag{4.18}$$

are non-singular. Moreover, from (3.98), we calculate

$$\begin{aligned}\det\left(\frac{1}{2\pi i}F_{\widetilde{\vartheta},\widetilde{\underline{u}''}}\right)(0,0,0,1)&=\frac{2^{2n-2d-2}}{\pi^{2n-2d+2}}(|\mu_{d+1}|\dots|\mu_n|)^2,\\ \det\left(\frac{1}{2\pi i}F_{\widetilde{\vartheta},\widetilde{\underline{u}''}}\right)(0,0,0,1)&=\frac{2^{2n-2}}{\pi^{2n-d+2}}(|\mu_1|\dots|\mu_d|)(|\mu_{d+1}|\dots|\mu_n|)^2.\end{aligned} \tag{4.19}$$

Hence, near (p,p) and $\widetilde{\vartheta}=1$, we can solve (4.12) and (4.13) and the solutions are unique. Let

$$\begin{aligned}\widetilde{\underline{u}''}&=\alpha(x,y)=(\alpha_{2d+1}(x,y),\dots,\alpha_{2n+1}(x,y))\in\mathscr{C}^\infty(U\times U,\mathbb{C}^{2n-2d+1}),\\ \widetilde{\vartheta}&=\gamma(x,y)\in\mathscr{C}^\infty(U\times U,\mathbb{C}),\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\widetilde{u''}&=\beta(x,y)=(\beta_{d+1}(x,y),\dots,\beta_{2n+1}(x,y))\in\mathscr{C}^\infty(U\times U,\mathbb{C}^{2n-d+1}),\\ \widetilde{\vartheta}&=\delta(x,y)\in\mathscr{C}^\infty(U\times U,\mathbb{C})\end{aligned} \tag{4.21}$$

be the solutions of (4.12) and (4.13), respectively. From (4.14), it is easy to see that

$$\begin{aligned}\beta(x,y)&=(\beta_{d+1}(x,y),\dots,\beta_{2n+1}(x,y))\\ &=(0,\dots,0,\alpha_{2d+1}(x,y),\dots,\alpha_{2n+1}(x,y)),\\ \gamma(x,y)&=\delta(x,y).\end{aligned} \tag{4.22}$$

From (4.22), we see that the value of $\widetilde{\Phi}_1(x,\widetilde{\underline{u}''})+\widetilde{\vartheta}\widetilde{\Phi}_2(\widetilde{\underline{u}''},y)$ at critical points $\widetilde{\underline{u}''}=\alpha(x,y)$, $\widetilde{\vartheta}=\gamma(x,y)$ is equal to the value of $\widetilde{\Phi}_1(x,\widetilde{u''})+\widetilde{\vartheta}\widetilde{\Phi}_2(\widetilde{u''},y)$ at critical points

$\widetilde{u''} = \beta(x,y)$, $\widetilde{\vartheta} = \delta(x,y)$. Put

$$\begin{aligned}\Phi_3(x,y) &:= \widetilde{\Phi}_1(x,\alpha(x,y)) + \gamma(x,y)\widetilde{\Phi}_2(\alpha(x,y),y)\\ &= \widetilde{\Phi}_1(x,\beta(x,y)) + \delta(x,y)\widetilde{\Phi}_2(\beta(x,y),y).\end{aligned} \tag{4.23}$$

Then $\Phi_3(x,y)$ is a complex phase function, $\operatorname{Im}\Phi_3(x,y) \geq 0$ and $\Phi_3(x,y) = \Phi_3(x'',y'')$. It is easy to check that

$$\begin{aligned}d_x\Phi_3(x,x) &= -d_y\Phi_3(x,x) = d_x\Phi(x,x)\\ &= -d_y\Phi(x,x) = -\lambda(x)\omega_0(x), \quad \lambda(x)>0, \quad \text{for } x\in U\cap Y.\end{aligned} \tag{4.24}$$

From now on we take U small enough so that the Levi form is positive on U and

$$d_x\Phi_3(x,y)\neq 0, \quad d_y\Phi_3(x,y)\neq 0, \quad \text{for every } (x,y)\in U\times U \tag{4.25}$$

and $a_0(x,y)\neq 0$, for every $(x,y)\in U\times U$, where $a_0(x,y)\in\mathscr{C}^\infty(U\times U)$ is as in (3.85).

Fix an open neighborhood $\mathcal{U}\Subset U$ of p with $\widehat{\Omega}_2\times\widehat{\Omega}_3\subset\mathcal{U}$, where $\widehat{\Omega}_2\Subset\Omega_2\subset\mathbb{R}^d$ is an open neighborhood of $0\in\mathbb{R}^d$ and $\widehat{\Omega}_3\Subset\Omega_3\subset\mathbb{R}^{2n+1-2d}$ is an open neighborhood of $0\in\mathbb{R}^{2n+1-2d}$.

Theorem 4.1. *The phase functions Φ and Φ_3 are equivalent on $\mathcal{U}$, that is, for any $b_1\in S^{n-\frac{d}{2}}_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$ there exist $\hat{b}_1, b_2\in S^{n-\frac{d}{2}}_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$ such that*

$$\begin{aligned}\int_0^\infty e^{i\Phi(x,y)t}b_1(x,y,t)dt &\equiv \int_0^\infty e^{i\Phi_3(x,y)t}b_2(x,y,t)dt \quad \text{on } \mathcal{U}\times\mathcal{U},\\ \int_0^\infty e^{i\Phi(x,y)t}\hat{b}_1(x,y,t)dt &\equiv \int_0^\infty e^{i\Phi_3(x,y)t}b_1(x,y,t)dt \quad \text{on } \mathcal{U}\times\mathcal{U}.\end{aligned} \tag{4.26}$$

Proof. We consider the kernel $(S_G\circ S_G)(\cdot,\cdot)$ on U. Let $\mathcal{U}\Subset U_1\Subset U$ be open neighborhoods of p. Let $\chi(x'')\in\mathscr{C}^\infty_0(\Omega_2\times\Omega_3)$. By (4.5) we can extend $\chi(x'')$ to

$$W := \{g\cdot x : g\in G, \quad x\in\Omega_2\times\Omega_3\}$$

by $\chi(g\cdot x'') := \chi(x'')$, for every $g\in G$. Assume that $\chi=1$ on some neighborhood of $\overline{U}_1$. Let $\chi_1\in\mathscr{C}^\infty_0(U)$ with $\chi_1=1$ on some neighborhood of $\overline{U}_1$ and $\operatorname{supp}\chi_1\subset\{x\in X:\ \chi(x)=1\}$. We have

$$\chi_1S_G\circ S_G = \chi_1S_G\chi\circ S_G + \chi_1S_G(1-\chi)\circ S_G. \tag{4.27}$$

Let us first consider $\chi_1S_G(1-\chi)\circ S_G$. We have

$$\begin{aligned}(\chi_1S_G(1-\chi))(x,u) &= \chi_1(x)\int_G S_{\leq\lambda_0}(x,g\cdot u)(1-\chi(u))d\mu(g)\\ &= \chi_1(x)\int_G S_{\leq\lambda_0}(x,u)(1-\chi(g^{-1}\cdot u))d\mu(g),\end{aligned} \tag{4.28}$$

where $\lambda_0>0$ is a small constant as in Theorem 3.19. If $g^{-1}\cdot u\notin\{x\in X:\chi(x)=1\}$, since $\operatorname{supp}\chi_1\subset\{x\in X:\chi(x)=1\}$ and $\chi(x)=\chi(g\cdot x)$, for every $g\in G$, $x\in X$, we conclude that $u \notin \operatorname{supp} \chi_1$. From this observation and since that $S_{\leq \lambda_0}$ is smoothing away the diagonal on GU (see Theorem 3.22), we deduce that $\chi_1 S_G(1-\chi)$ is smoothing and hence

$$\chi_1 S_G(1-\chi) \circ S_G \equiv 0 \quad \text{on } X \times X. \tag{4.29}$$

From (4.27) and (4.29), we get

$$\chi_1 S_G \circ S_G \equiv \chi_1 S_G \chi \circ S_G \quad \text{on } X \times X. \tag{4.30}$$

From Theorem 3.25 and using that $S_G^* = S_G$, where S_G^* is the adjoint of S_G with respect to $(\cdot\,,\cdot)$, we obtain that on U,

$$\begin{aligned}
&(\chi_1 S_G \chi \circ S_G)(x,y)\\
&\quad\equiv \int_{\Omega_2\times\Omega_3}\int_0^\infty\int_0^\infty e^{i\Phi^*(x,u'')t+i\Phi(u'',y)s}\chi_1(x)a^*(x'',u'',t)V_{\text{eff}}(u'')\\
&\qquad\times\chi(u'')a(u'',y'',s)dsdtdv(u'')\\
&\quad\equiv \int_{\Omega_2\times\Omega_3}\int_0^\infty\int_0^\infty e^{i\Phi_1(x,u'')t+i\Phi_2(u'',y)s}\chi_1(x)b(x,u'',t)\\
&\qquad\times\chi(u'')c(u'',y,s)dsdtdv(u'') \quad \text{(here we used (4.10))}\\
&\quad\equiv \int_{\Omega_2\times\Omega_3}\int_0^\infty\int_0^\infty e^{i(\Phi_1(x,u'')+\Phi_2(u'',y)\vartheta)t}\\
&\qquad\times\chi_1(x)b(x,u'',t)\chi(u'')c(u'',y,t\vartheta)td\vartheta dtdv(u''),
\end{aligned} \tag{4.31}$$

where $s=\vartheta t$, $d\mu(g)dv(u'')=dv(x)$ on U, $a^*(x'',u'',t)=\overline{a}(u'',x'',t)$ and

$$\begin{aligned}
&b(x,y,t), c(x,y,t) \in S^{n-\frac{d}{2}}_{\text{cl}}(U\times U\times\mathbb{R}_+),\\
&b_0(x,x)\neq 0,\ c_0(x,x)\neq 0, \text{ for any } x\in U\cap Y \text{ (cf. Notation (3.76))}.
\end{aligned} \tag{4.32}$$

We apply the complex stationary phase formula of Melin–Sjöstrand [45] to carry out the $dv(u'')d\vartheta$ integration in (4.31). This yields the existence of a symbol $d(x,y,t)\in S^{n-\frac{d}{2}}_{\text{cl}}(U\times U\times\mathbb{R}_+)$ with the expansion $d(x,y,t)\sim\sum_{j=0}^\infty t^{n-\frac{d}{2}-j}d_j(x,y)$ in $S^{n-\frac{d}{2}}_{1,0}(U\times U\times\mathbb{R}_+)$ (cf. (3.76)) with $d_0(x,x)\neq 0$ for $x\in U\cap Y$ and

$$(\chi_1 S_G\chi\circ S_G)(x,y)\equiv\int_0^\infty e^{i\Phi_3(x,y)t}d(x,y,t)dt \quad \text{on } U\times U. \tag{4.33}$$

From (4.30), (4.33), we deduce that

$$\int_0^\infty e^{i\Phi_3(x,y)t}d(x,y,t)dt\equiv\int_0^\infty e^{i\Phi(x,y)t}\chi_1(x)a(x,y,t)dt \quad \text{on } U\times U. \tag{4.34}$$

Note that Φ_3 and Φ are independent of x' and y'. By the Malgrange preparation theorem we may assume that Φ_3 and Φ have the form

$$\begin{aligned}
\Phi_3(x'',y'')&=y_{2n+1}+h(x'',\mathring{y}''),\\
\Phi(x'',y'')&=y_{2n+1}+h_1(x'',\mathring{y}''),
\end{aligned}$$

where $\hat{\Phi}_3(x'', \mathring{y}'')$, $\hat{\Phi}(x'', \mathring{y}'') \in \mathscr{C}^\infty(U \times U)$. From (4.34), we can repeat the proof of [28, Theorem 5.2] and deduce that $\Phi_3(x,y) - \Phi(x,y)$ vanishes to infinite order at $\operatorname{diag}(Y \times Y)$. Since $\Phi_3(x,y)$ and $\Phi(x,y)$ are independent of x', y', we conclude that $\Phi_3(x,y) - \Phi(x,y)$ vanishes to infinite order on the underlying canonical relation

$$\{(x, g\cdot x) : x \in Y, g \in G\} \cap (U \times U).$$

Hence, Φ and Φ_3 are equivalent on $\mathcal{U}$. For the convenience of the reader, we sketch briefly the idea of the proof. We will use some semi-classical notations as in the proof of [28, Theorem 5.2]. Suppose that $U = U' \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, U' is an open set of $\mathbb{R}^{2n}$. Let $\tau \in \mathscr{C}^\infty_c((-\varepsilon,\varepsilon))$, $\tau \geq 0$, $\tau \equiv 1$ on $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. For each $k > 0$, we consider the distributions (see the proof of [28, Theorem 5.2])

$$A_k : u \mapsto \int\int_0^\infty e^{i(y_{2n+1}+h(x'',\mathring{y}''))t - iky_{2n+1}} d(x,y,t)\tau(y_{2n+1})u(\mathring{y})dydt,$$

$$B_k : u \mapsto \int\int_0^\infty e^{i(y_{2n+1}+h_1(x'',\mathring{y}''))t - iky_{2n+1}} \tau(x)\chi_1(x)a(x,y,t)\tau_1\tau(y_{2n+1})u(\mathring{y})dydt,$$

for $u \in \mathscr{C}^\infty_0(U')$, where $\mathring{y} = (y_1, \ldots, y_{2n})$. By using the stationary phase formula of Melin–Sjöstrand [45], we can show that (cf. the proof of [29, Theorem 3.12]) A_k and B_k are smoothing operators and

$$\begin{aligned}
A_k(x,\mathring{y}) &\equiv e^{ikh(x'',\mathring{y}'')}g(x,\mathring{y},k) + O(k^{-\infty}),\\
B_k(x,\mathring{y}) &\equiv e^{ikh_1(x'',\mathring{y}'')}p(x,\mathring{y},k) + O(k^{-\infty}),\\
g(x,\mathring{y},k), p(x,\mathring{y},k) &\in S^{n-\frac{d}{2}}_{\mathrm{cl}}(U \times U'),\\
g(x,\mathring{y},k) &\sim \sum_{j=0}^\infty g_j(x,\mathring{y})k^{n-\frac{d}{2}-j} \quad \text{in } S^{n-\frac{d}{2}}_{\mathrm{cl}}(U\times U'),\\
p(x,\mathring{y},k) &\sim \sum_{j=0}^\infty p_j(x,y')k^{n-\frac{d}{2}-j} \quad \text{in } S^{n-\frac{d}{2}}_{\mathrm{cl}}(U\times U'),\\
g_j(x,\mathring{y}), p_j(x,\mathring{y}) &\in \mathscr{C}^\infty(U\times U'), \quad j = 0,1,\ldots,\\
g_0(x_0,\mathring{x}_0) &\neq 0.
\end{aligned}$$

Since

$$\int_0^\infty e^{i(y_{2n+1}+h(x'',\mathring{y}''))t} d(x,y,t)dt - \int_0^\infty e^{i(y_{2n+1}+h_1(x'',\mathring{y}''))t}\chi_1(x)a(x,y,t)dt$$

is smoothing, by using integration by parts with respect to y_{2n+1}, it is easy to see that $A_k - B_k = O(k^{-\infty})$ (see [29, Sec. 3]). From this observation, we can repeat the argument as in the discussion after [28, (5.2)] and deduce that $h - h_1$ vanishes to infinite order at $\operatorname{diag}(Y \times Y)$. Since $\Phi_3(x,y)$ and $\Phi(x,y)$ are independent of x', y', we conclude that $\Phi_3(x,y) - \Phi(x,y)$ vanishes to infinite order on the underlying canonical relation $\{(x, g\cdot x) : x \in Y, g \in G\} \cap (U \times U)$. □

The following two theorems follow from the proof of Theorem 4.1, (4.10), (4.23), the proof of (4.34), the complex stationary phase formula of Melin–Sjöstrand [45].

Theorem 4.2. *Consider the Fourier integral operators*

$$A(x,y)=\int_0^{\infty} e^{i\Phi(x,y)t}a(x,y,t)dt, \quad B(x,y)=\int_0^{\infty} e^{i\Phi(x,y)t}b(x,y,t)dt,$$

with symbols $a(x,y,t)\in S^k_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$ *and* $b(x,y,t)\in S^\ell_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$*. Consider* $\chi(x'')\in\mathscr{C}^\infty_0(\widehat{\Omega}_2\times\widehat{\Omega}_3)$*. Then, we have*

$$\int A(x,u'')\chi(u'')B(u'',y)dv(u'')\equiv\int_0^{\infty} e^{i\Phi(x,y)t}c(x,y,t)dt \quad \text{on } \mathcal{U}\times\mathcal{U},$$

with $c(x,y,t)\in S^{k+\ell-(n-\frac{d}{2})}_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$*. For* $x\in Y\cap U$ *we have*

$$c_0(x,x)=2^{-n-\frac{d}{2}+1}\pi^{n-\frac{d}{2}+1}|\det\mathscr{L}_x|^{-1}|\det\mathcal{R}_x|^{\frac{1}{2}}a_0(x,x)b_0(x,x)\chi(x''), \tag{4.35}$$

where $\det\mathcal{R}_x$ *is the determinant* $\mathcal{R}_x$ *cf.* (3.83)*. Moreover, if there are* $N_1,N_2\in\mathbb{N}^*$, $C>0$*, such that for all* $x_0\in Y\cap\mathcal{U}$,

$$|a_0(x,y)|\le C|(x,y)-(x_0,x_0)|^{N_1}, \quad |b_0(x,y)|\le C|(x,y)-(x_0,x_0)|^{N_2}, \tag{4.36}$$

then there exists $\widehat{C}>0$ *such that for all* $x_0\in Y\cap\mathcal{U}$

$$|c_0(x,y)|\le\widehat{C}|(x,y)-(x_0,x_0)|^{N_1+N_2}. \tag{4.37}$$

Proof. By (4.10) and since Φ, Φ_1, Φ_2 and Φ^* are independent of (x',y'), $\Phi-\Phi_2$, $\Phi^*-\Phi_1$ vanish to infinite order on the underlying canonical relation

$$\{(g\cdot x,x): x\in Y, g\in G\}.$$

Hence, we can change the phase of A to Φ_1 and the phase of B to Φ_2. Thus,

$$\begin{aligned}&\int A(x,u'')\chi(u'')B(u'',y)dv(u'')\\&\quad\equiv\int dt\int_0^{\infty} ds\int_0^{\infty} e^{i\Phi_1(x,u'')t+i\Phi_2(u'',y)s}e(x,u'',t)f(u'',y,s)\chi(u'')du'',\end{aligned}$$

for some $e\in S^k_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$, $f\in S^\ell_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$ such that the leading terms e_0, f_0 of e, f, respectively, satisfy (4.36). From (4.23), the proof of (4.34) and the complex stationary phase formula of Melin–Sjöstrand [45], we get

$$\begin{aligned}&\int A(x,u'')\chi(u'')B(u'',y)dv(u'')\\&\quad\equiv\int_0^{\infty} dt\int\int_0^{\infty} e^{it(\Phi_1(x,u'')+\gamma\Phi_2(u'',y))}te(x,u'',t)f(u'',y,\gamma t)\chi(u'')du''d\gamma\\&\quad\equiv\int_0^{\infty} e^{i\Phi(x,y)t}c(x,y,t)dt \quad \text{on } \mathcal{U}\times\mathcal{U},\end{aligned}$$

with $c(x,y,t)\in S^{k+\ell-(n-\frac{d}{2})}_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$ (as we integrate over u'' and γ, whose total dimension is $2n-d+2$, we need to lower the order by $n+1-d/2$; however, due to the factor t in the integral we gain one order, thus we get the order $k+l-(n-d/2))$ and

$$c_0(x,y) = g(x,y)\widetilde{e}_0(x,\beta(x,y))\widetilde{f}_0(\beta(x,y),y), \tag{4.38}$$

for some smooth function g, where $\beta(x,y)$ is as in (4.21), $\widetilde{e}_0$, $\widetilde{f}_0$ denote almost analytic extensions of e_0, f_0, respectively. Since $\beta(x,x)=x$ for every $x\in Y$, we conclude that $\widetilde{e}_0(x,\beta(x,y))$ and $\widetilde{f}_0(\beta(x,y),y)$ have vanishing order at least N_1 and N_2 on $\operatorname{diag}(Y\times Y)$, respectively. This observation and (4.38) yield (4.37). ☐

Theorem 4.3. *Consider the Fourier integral operators*

$$\mathcal{A}(x,\underline{y}'') = \int_0^\infty e^{i\Phi(x,\underline{y}'')t}\alpha(x,\underline{y}'',t)dt, \quad \mathcal{B}(\underline{x}'',y) = \int_0^\infty e^{i\Phi(\underline{x}'',y)t}\beta(\underline{x}'',y,t)dt,$$

with symbols $\alpha(x,\underline{y}'',t)\in S^k_{\mathrm{cl}}(\mathcal{U}\times\Omega_3\times\mathbb{R}_+)$ *and* $\beta(\underline{x}'',y,t)\in S^\ell_{\mathrm{cl}}(\Omega_3\times\mathcal{U}\times\mathbb{R}_+)$. *Let* $\chi_1(\underline{x}'')\in\mathscr{C}^\infty_0(\Omega_3)$. *Then, we have*

$$\int\mathcal{A}(x,\underline{u}'')\chi_1(\underline{u}'')\mathcal{B}(\underline{u}'',y)dv(\underline{u}'') \equiv \int_0^\infty e^{i\Phi(x,y)t}\gamma(x,y,t)dt \quad \text{on } \mathcal{U}\times\mathcal{U},$$

with $\gamma(x,y,t)\in S^{k+\ell-(n-d)}_{\mathrm{cl}}(\mathcal{U}\times\mathcal{U}\times\mathbb{R}_+)$ *where*

$$\gamma_0(x,x) = 2^{-n+1}\pi^{n-d+1}|\det\mathscr{L}_x|^{-1}|\det\mathcal{R}_x|\alpha_0(x,\underline{x}'')\beta_0(\underline{x}'',x)\chi_1(\underline{x}''), \quad x\in Y\cap U. \tag{4.39}$$

Moreover, if there are $N_1,N_2\in\mathbb{N}^*$, $C>0$, *such that for all* $x_0\in Y\cap\mathcal{U}$ *we have*

$$|\alpha_0(x,\underline{y}'')|\leq C|(x,\underline{y}'')-(x_0,x_0)|^{N_1}, \quad |\beta_0(x,\underline{y}'')|\leq C|(x,\underline{y}'')-(x_0,x_0)|^{N_2},$$

then there exists $\widehat{C}>0$ *such that for all* $x_0\in Y\cap\mathcal{U}$,

$$|\gamma_0(x,y)|\leq\widehat{C}|(x,y)-(x_0,x_0)|^{N_1+N_2}. \tag{4.40}$$

The proof of Theorem 4.3 is similar to the proof of Theorem 4.2. We omit the details.

Remark 4.4. (i) The reason why in Theorem 4.3 we integrate only with respect to $\underline{y}''$ is that in the proof of our main results, we need to integrate over the reduced space and this corresponds to the integration only with respect to $\underline{y}''$ variables.

(ii) Theorems 4.2 and 4.3 are about compositions of Fourier integral operators. In general, the composition of Fourier integral operators correspond to composition of canonical relation but we do not use this point of view since in the proof of our main results we need to know the precise form of the phase function of the composition of our Fourier integral operators.

We introduce next the following notion.

Definition 4.5. Let $H : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)^G$ be a continuous operator with distribution kernel $H(x, y) \in \mathscr{D}'(X \times X)$ and $k \in \mathbb{R}$, $\ell \in \mathbb{N}$.

(i) We say that H is a *complex Fourier integral operator of G-Szegő type of leading order* (k, ℓ), if for every open set D of X with $D \cap Y = \emptyset$,

$$\chi H \text{ and } H\chi \text{ are smoothing operators on } X, \text{ for every } \chi \in \mathscr{C}_0^\infty(D) \tag{4.41}$$

and for every $p \in Y$ and any open coordinate neighborhood $(U, x = (x_1, \ldots, x_{2n+1}))$ of p, we have

$$H(x, y) \equiv \int_0^\infty e^{i\Phi(x,y)t} a(x, y, t) dt \quad \text{on } U \times U \tag{4.42}$$

with $\Phi(x, y) \in \mathscr{C}^\infty(U \times U)$ as in Theorem 3.25, and

$$a(x, y, t) \in S^k_{\mathrm{cl}}(U \times U \times \mathbb{R}_+) \tag{4.43}$$

and under the notation (3.76),

$$\left.\frac{\partial^{|\alpha|+|\beta|} a_0(x, y)}{\partial x^\alpha \partial y^\beta}\right|_{x=y\in Y} = 0, \quad \text{for } \alpha, \beta \in \mathbb{N}^{2n+1}, \quad |\alpha| + |\beta| \le \ell - 1. \tag{4.44}$$

(ii) We say that H is a *complex Fourier integral operator of G-Szegő type of order* (k, ℓ), $k \in \mathbb{R}$, $\ell \in \mathbb{N}$, if (4.41), (4.42), (4.43) hold and there is a $r(x, y, t) \in S^{-\infty}_{\mathrm{cl}}(U \times U \times \mathbb{R}_+)$ such that

$$\left.\frac{\partial^{|\alpha|+|\beta|}(a(x, y, t) - r(x, y, t))}{\partial x^\alpha \partial y^\beta}\right|_{x=y\in Y} = 0, \quad \text{for } \alpha, \beta \in \mathbb{N}^{2n+1},$$
$$|\alpha| + |\beta| \le \ell - 1. \tag{4.45}$$

Let $G_{k,\ell}(X)$ denote the space of all complex Fourier integral operators of G-Szegő type of leading order (k, ℓ) and let $\widehat{G}_{k,\ell}(X)$ denote the space of all complex Fourier integral operators of G-Szegő type of order (k, ℓ).

Note that (4.45) means that each coefficient a_j has vanishing order at least ℓ at $\mathrm{diag}\,(Y \times Y)$. In Definition 4.5 we use the terminology G-Szegő type in order to stress the dependence on the set Y and thus on the group G.

Let us explain briefly the role of the spaces $G_{k,\ell}(X)$ and $\widehat{G}_{k,\ell}(X)$. Our goal is to study distribution kernel of $\sigma^*\sigma$. In Theorem 4.13, we will show that $C_0\sigma^*\sigma$ is of the same type as S_G and with the same leading term, where C_0 is a constant. In the terminology introduced in Definition 4.5, $C_0\,\sigma^*\sigma - S_G \in G_{n-(d/2),1}(X)$. To prove our main result, we need to show that $C_0\,\sigma^*\sigma - S_G$ is "microlocally small", and it suffices to prove that elements $H \in G_{n-(d/2),1}(X)$ have good regularity properties (see (4.58)).

Let $H = H_0 + H_1$, where $H_0 \in G_{n-(d/2),1}(X)$ is the leading term of H and $H_1 \in G_{n-(d/2)-1,0}(X)$ is the lower order term of H. By using calculus of complex Fourier integral operators, we can show that when we compose H_1 with itself, the order of the composition will decrease. More precisely, $H_1^N \in G_{n-(d/2)-N,0}(X)$, for every $N \in \mathbb{N}^*$. Hence, for large N, H_1^N has good regularity properties and hence H_1 itself has good regularity properties.

In order to handle H_0 we observe that when we compose H_0 with itself the order of the composition will not decrease, that is, H_0^N is still in $G_{n-(d/2),1}(X)$ for every $N \in \mathbb{N}^*$. To get good regularity properties, we need the space $\widehat{G}_{k,\ell}(X)$. Note that the space $\widehat{G}_{k,\ell}(X)$ is the subspace of $G_{k,\ell}(X)$ whose elements have *full symbols* have vanishing order at least ℓ on $\mathrm{diag}(Y \times Y)$. The key observation is that the leading symbol of H_0^N has vanishing order at least N on $\mathrm{diag}(Y \times Y)$. We write $H_0^N = R_{0,N} + R_{1,N}$, where $R_{1,N} \in G_{n-(d/2)-1,0}(X)$ is the lower order term of H_0^N and $R_{0,N} \in \widehat{G}_{n-(d/2),N}(X)$ is the the leading term of H_0^N. Since the *full symbol* of $R_{0,N}$ has vanishing order at least N at $\mathrm{diag}(Y \times Y)$, even the order of $R_{0,N}$ is high, $R_{0,N}$ still has good regularity properties if N is large. Note that for an element $A \in G_{k,\ell}(X)$, only the leading symbol of A has vanishing order at least ℓ on $\mathrm{diag}(Y \times Y)$. Hence even for large ℓ we still do not have good regularity property for A in general. That is why we need the space $\widehat{G}_{k,\ell}(X)$.

Remark 4.6. In the following, we will establish L^2 continuity and regularity properties for $H \in G_{k,\ell}(X)$ under certain assumptions on k, ℓ (see Theorem 4.10). Hörmander [25] already established the L^2 continuity for complex Fourier integral operators but we cannot apply Hörmander's results to our situation. More precisely, from the main results in [25], we have that for $H \in G_{n-(d/2)-m,0}(X)$ with $m \in \mathbb{N}$, the map $H : H^s(X) \to H^{s+m}(X)$ is continuous for every $s \in \mathbb{R}$. In this work, we need the regularity property for $H \in G_{n-(d/2),\ell}(X)$ for ℓ large (see Lemma 4.9 and (4.58)) but Hörmander's result can only be applied to $\ell = 0$.

From Theorem 4.2, we deduce the following.

Theorem 4.7. *Let $H_1 \in G_{k_1,\ell_1}(X)$, $H_2 \in G_{k_2,\ell_2}(X)$, where $k_1, \ell_1, k_2, \ell_2 \in \mathbb{R}$. Then,*

$$H_1 \circ H_2 \in G_{k_1+k_2-(n-\frac{d}{2}),\ell_1+\ell_2}(X).$$

Recall that $\|\cdot\|_s$ denotes the standard Sobolev norm on X of order s.

Theorem 4.8. *Let $H \in G_{k,0}(X)$ with $k \le n - \frac{d}{2} - 1$. Then, there exists $C > 0$ such that for any $u \in \mathscr{C}^\infty(X)$,*

$$\|Hu\| \le C\|u\|. \tag{4.46}$$

Moreover, for every $s \in \mathbb{N}^*$, there exist $N_s \in \mathbb{N}^*$ and $C_s > 0$ such that for any $u \in \mathscr{C}^\infty(X)$,

$$\|H^{N_s}u\|_s \le C_s\|u\|. \tag{4.47}$$

Proof. Fix $s \in \mathbb{N}^*$. By Theorem 4.7, for any $N_s \in \mathbb{N}^*$ we have $H^{N_s} \in G_{N_s k-(N_s-1)(n-\frac{d}{2}),0}(X)$. Taking now $N_s \in \mathbb{N}^*$ such that $N_s k - (N_s-1)(n-\frac{d}{2}) < -s-2$. We claim that $H^{N_s}(x,y) \in \mathscr{C}^s(X \times X)$. Since H^{N_s} is smoothing outside

Y, we only need to check this property near Y. Let U be an open set of $p \in Y$. Fix $\alpha, \beta \in \mathbb{N}^{2n+1}$ with $|\alpha| + |\beta| \leq s$. Since $H^{N_s} \in G_{N_s k-(N_s-1)(n-\frac{d}{2}),0}(X)$, we have

$$\partial_x^\alpha \partial_y^\beta H^{N_s}(x,y) \equiv \int_0^\infty e^{it\Phi(x,y)} a(x,y,t),$$

$a(x,y,t) \in S_{\rm cl}^{N_s k-(N_s-1)(n-\frac{d}{2})+|\alpha|+|\beta|}(U \times U \times \mathbb{R}_+)$. Since there exist $C, \hat{C} > 0$ such that

$$\int_1^\infty e^{-t{\rm Im}\,\Phi(x,y)}|a(x,y,t)|dt \leq C \int_1^\infty t^{N_s k-(N_s-1)(n-\frac{d}{2})+|\alpha|+|\beta|} dt \\ \leq \hat{C} \int_1^\infty t^{-2} dt < \infty,$$

we conclude that $H^{N_s}(x,y) \in \mathscr{C}^s(X \times X)$ and (4.47) follows.

We now prove (4.46). We claim that for every $\ell \in \mathbb{N}^*$, we have for any $u \in \mathscr{C}^\infty(X)$,

$$\|Hu\|^2 \leq \|(H^*H)^{2^\ell} u\|^{2^{-\ell}} \|u\|^{2-2^{-\ell}}, \tag{4.48}$$

where H^* is the adjoint of H. We have for any $u \in \mathscr{C}^\infty(X)$,

$$\|Hu\|^2 = (\,Hu\,,\,Hu\,) = (\,H^*Hu\,,\,u) \leq \|H^*Hu\|\|u\| \tag{4.49}$$

and

$$\|H^*Hu\|^2 = (\,H^*Hu\,,\,H^*Hu\,) = (\,(H^*H)^2 u\,,\,u) \leq \|(H^*H)^2 u\|\|u\|. \tag{4.50}$$

We prove (4.48) by induction on ℓ. From (4.49) and (4.50), we get (4.48) for $\ell = 1$. Suppose that (4.48) holds for $\ell \in \mathbb{N}^*$. We have for every $u \in \mathscr{C}^\infty(X)$,

$$\|(H^*H)^{2^\ell} u\|^2 = (\,(H^*H)^{2^\ell} u, (H^*H)^{2^\ell} u\,) \\ = (\,(H^*H)^{2^{\ell+1}} u\,,\,u) \leq \|(H^*H)^{2^{\ell+1}} u\|\|u\|, \tag{4.51}$$

From the induction hypothesis and (4.51), we get (4.48) for $\ell + 1$.

It is obvious that $H^* \in G_{k,0}$ and hence $H^*H \in G_{2k-(n-\frac{d}{2}),0}$. From this observation and (4.47), we deduce that for ℓ large, there exists $C > 0$ such that for any $u \in \mathscr{C}^\infty(X)$,

$$\|(H^*H)^{2^\ell} u\| \leq C\|u\|. \tag{4.52}$$

From (4.48) and (4.52), we get (4.46). ☐

Lemma 4.9. *Let $H \in \widehat{G}_{k,2\ell}(X)$. If $k - \ell \leq -s - 2$, for some $s \in \mathbb{N}$, then $H(x,y) \in \mathscr{C}^s(X \times X)$.*

Proof. Since H is smoothing away Y, we only need to prove that $H(x,y)$ is in $\mathscr{C}^s$ near Y. Let $p \in Y$ and $x = (x_1, \ldots, x_{2n+1})$ be local coordinates as in the discussion before Theorem 3.28 defined in an open set U of p. We will use the same notations as in the beginning of Sec. 4.1. On U, write

$$H(x,y) = \int_0^\infty e^{i\Phi(x,y)t} a(x,y,t)dt + F(x,y),$$

$F(x,y) \in \mathscr{C}^\infty(U \times U)$, $a \in S^k_{\mathrm{cl}}(U \times U \times \mathbb{R}_+)$. Let $u \in \mathscr{C}^\infty_c(U)$. Since Hu is G-invariant, Hu is independent of x' and hence on U,

$$(Hu)(x) = (Hu)(x'') = \int_0^\infty e^{i\Phi(x,y)t} a(x'',y,t)u(y)dv(y)dt + \int F(x'',y)u(y)dv(y).$$

Hence,

$$H(x,y) \equiv \int_0^\infty e^{i\Phi(x,y)t} a(x'',y,t)dt \text{ on } U \times U.$$

We are going to prove the lemma by induction over $\ell \in \mathbb{N}$. Let $\ell = 0$. Let P and Q be differential operators on U with $\mathrm{ord}\,(P) + \mathrm{ord}\,(Q) \le s$. Then the symbol of PHQ is of order $\le k + \mathrm{ord}\,(P) + \mathrm{ord}\,(Q) \le k + s \le -2$ and hence the integral over t converges. We get that the lemma holds for $\ell = 0$. We assume that the lemma holds for some $\ell = \ell_0 \in \mathbb{N}$. We are going to prove that the lemma holds for $\ell = \ell_0 + 1$. Since all the asymptotics of a have vanishing order at least $2\ell_0 + 2$ on $\mathrm{diag}\,(Y \times Y)$, we may assume for simplicity that $a(x'',y,t) = t^k b(x'',y)$ and b has vanishing order at least $2\ell_0 + 2$ on $\mathrm{diag}\,(Y \times Y)$. The Malgrange preparation theorem entails that there exist a neighborhood $U_0 \subset U$ of p and functions $f, g, \hat{\Phi}, \hat{b} \in \mathscr{C}^\infty(U_0 \times U_0)$, such that $\hat{\Phi}$ and $\hat{b}$ are independent of y_{2n+1} and we have on $U_0 \times U_0$,

$$\begin{aligned} \Phi(x,y) &= f(x,y)(y_{2n+1} + \hat{\Phi}(x'', \mathring{y}'')), \\ b(x'',y) &= g(x'',y)(y_{2n+1} + \hat{\Phi}(x'', \mathring{y}'')) + \hat{b}(x'', \mathring{y}''). \end{aligned} \tag{4.53}$$

We claim that $g(x,y)$ and $\hat{b}$ have vanishing order at least $2\ell_0 + 1$ and $2\ell_0 + 2$ on $\mathrm{diag}\,(Y \times Y)$, respectively. For every $j = 1, \ldots, 2\ell_0 + 1$, we have

$$\frac{\partial^j b}{\partial y^j_{2n+1}} = j \frac{\partial^{j-1} g}{\partial y^{j-1}_{2n+1}} + \frac{\partial^j g}{\partial y^j_{2n+1}} (y_{2n+1} + \hat{\Phi}(x'', \mathring{y}'')). \tag{4.54}$$

Taking $j = 2\ell_0 + 1$ in (4.54) and using that $(y_{2n+1} + \hat{\Phi}(x'', \mathring{y}''))|_{\mathrm{diag}\,(Y \times Y)} = 0$, we conclude that $\partial^{2\ell_0}_{y_{2n+1}} g$ vanishes on $\mathrm{diag}\,(Y \times Y)$. Taking $j = 2\ell_0$ in (4.54) we conclude that $\partial^{2\ell_0 - 1}_{y_{2n+1}} g$ has vanishing order at least 2 on $\mathrm{diag}\,(Y \times Y)$. Continuing in a similar way, we can show that for every $j = 1, \ldots, 2\ell_0 + 1$, $\partial^{j-1}_{y_{2n+1}} g$ has vanishing order at least $2\ell_0 + 2 - j$ on $\mathrm{diag}\,(Y \times Y)$. Now we have from (4.53),

$$\frac{\partial b}{\partial y_{2n+1}} = g + \frac{\partial g}{\partial y_{2n+1}} (y_{2n+1} + \hat{\Phi}(x'', \mathring{y}'')). \tag{4.55}$$

Since $\partial_{y_{2n+1}} g$ has vanishing order at least $2\ell_0$ on $\mathrm{diag}\,(Y \times Y)$, we deduce that $g(x,y)$ has vanishing order at least $2\ell_0 + 1$ on $\mathrm{diag}\,(Y \times Y)$. Thus, $\hat{b} = b - g(y_{2n+1} + \hat{\Phi}(x'', \mathring{y}''))$ has vanishing order at least $2\ell_0 + 2$ at $\mathrm{diag}\,(Y \times Y)$. The claim follows.

By using integration by parts, we have

$$\begin{aligned}\int_0^\infty e^{it\Phi(x,y)}t^k b(x'',y)dt &\equiv \int_0^\infty e^{it\Phi(x,y)}t^k(g(x,y)(y_{2n+1}+\hat{\Phi}(x'',\mathring{y}''))+\hat{b}(x'',\mathring{y}))dt\\ &\equiv I_1+I_2, \quad \text{where}\\ I_1(x,y) &\equiv \int_0^\infty e^{it\Phi(x,y)}kit^{k-1}\frac{g}{f}(x,y)dt,\\ I_2(x,y) &\equiv \int_0^\infty e^{it\Phi(x,y)}t^k\hat{b}(x'',\mathring{y})dt.\end{aligned}$$

Since g has vanishing order at least $2\ell_0+1$ and $k-1-\ell_0\le -s-2$, we deduce by the induction assumption that $I_1(x,y)\in\mathscr{C}^s(U_0\times U_0)$. Since $\hat{b}(x'',\mathring{y})$ has vanishing order at least $2\ell_0+2$ on $\operatorname{diag}(Y\times Y)$, there exist $C,\hat{C}>0$ such that we have

$$\begin{aligned}|\hat{b}(x'',\mathring{y})| &\le C|(x'',y)-((0,\underline{x}''),(0,\underline{\mathring{x}}''))|^{2\ell_0+2}\\ &\le \hat{C}(|\hat{x}''|^2+|\mathring{x}''-\mathring{y}''|^2+|\hat{y}''|^2)^{\ell_0+1},\end{aligned} \tag{4.56}$$

where $\underline{\mathring{x}}''=(x_{2d+1},\ldots,x_{2n})$. From (3.93) and (4.56), there exist $c,\hat{C}>0,C_1,C_2>0$ such that

$$\begin{aligned}\left|\int_1^\infty e^{it\Phi(x,y)}t^k\hat{b}(x'',\mathring{y})dt\right| &\le \int_1^\infty e^{-t\operatorname{Im}\Phi}t^k|\hat{b}(x'',\mathring{y})|dt\\ &\le \int_1^\infty e^{-tc(|\hat{x}''|^2+|\mathring{x}''-\mathring{y}''|^2+|\hat{y}''|)^2}t^k\\ &\quad\times\hat{C}(|\hat{x}''|^2+|\mathring{x}''-\mathring{y}''|^2+|\hat{y}''|^2)^{\ell_0+1}dt\\ &\le C_1\int_1^\infty t^{k-\ell_0-1}dt\le C_2\int_1^\infty t^{-s-2}dt<\infty.\end{aligned}$$

Let $\alpha,\beta\in\mathbb{N}^{2n+1}$ with $|\alpha|+|\beta|\le s$. We can repeat the procedure above with minor changes and deduce that

$$\int_1^\infty|\partial_x^\alpha\partial_y^\beta(e^{it\Phi(x,y)}t^k\hat{b}(x'',\mathring{y}))|dt\le C_3\int_1^\infty t^{k-\ell_0-1+|\alpha|+|\beta|}\le C_4\int_1^\infty t^{-2}dt<\infty,$$

where $C_3,C_4>0$ are constants. Hence $I_2(x,y)\in\mathscr{C}^s(U_0\times U_0)$. The assertion holds thus for $\ell=\ell_0+1$ and the lemma follows. □

In the proof of our main result, we need the following.

Theorem 4.10. *Let $H\in G_{n-\frac{d}{2},1}(X)$. Then there exists $C>0$ such that for any $u\in\mathscr{C}^\infty(X)$,*

$$\|Hu\|\le C\|u\|. \tag{4.57}$$

Moreover, for every $s \in \mathbb{N}^*$*, there exist* $N_s \in \mathbb{N}^*$ *and* $C_s > 0$ *such that for any* $u \in \mathscr{C}^\infty(X)$,

$$\|H^{N_s}u\|_s \leq C_s\|u\|. \tag{4.58}$$

Proof. From Theorem 4.2, we see that for every $N \in \mathbb{N}^*$, we have

$$\begin{aligned} &H^N = H_{1,N} + H_{2,N}, \\ &H_{1,N} \in \widehat{G}_{n-\frac{d}{2},N}(X), \quad H_{2,N} \in G_{n-\frac{d}{2}-1,0}(X). \end{aligned} \tag{4.59}$$

Fix $s \in \mathbb{N}^*$. Due to Lemma 4.9 there exists $N \gg 1$ such that $H_{1,N}(x,y) \in \mathscr{C}^s(X \times X)$ and for every $j \in \mathbb{N}^*$, there exists $C_j > 0$ such that for every $u \in \mathscr{C}^\infty(X)$ we have

$$\|H^j_{1,N}u\|_s \leq C_j\|u\|. \tag{4.60}$$

Since $H_{2,N} \in G_{n-\frac{d}{2}-1,0}(X)$, Theorem 4.8 shows that there exist $K_s \in \mathbb{N}^*$ and $\widehat{C}_s > 0$ so that

$$\|H^{K_s}_{2,N}u\|_s \leq \widehat{C}_s\|u\|, \quad \text{for } u \in \mathscr{C}^\infty(X). \tag{4.61}$$

We have

$$H^{NK_s} = (H_{1,N} + H_{2,N})^{K_s}. \tag{4.62}$$

Let $A := H^{a_1}_{1,N}H^{b_1}_{2,N} \cdots H^{a_p}_{1,N}H^{b_p}_{2,N}$, where $a_j, b_j \in \mathbb{N}$, $j = 1, \ldots, p$, $\sum_{j=1}^p (a_j + b_j) = K_s$. We claim that

$$A : L^2(X) \to H^s(X) \text{ is continuous.} \tag{4.63}$$

If $a_1 = a_2 = \ldots = a_p = 0$, then, $A = H^{K_s}_{2,N}$. From (4.61), we get (4.63). Assume that $a_{j_0} \neq 0$, for some $j_0 \in \{1, 2, \ldots, p\}$. Let Λ_s be a classical elliptic pseudodifferential operator on X of order s with inverse Λ_{-s}. By the complex stationary phase formula of Melin-Sjöstrand, we have $\Lambda_s \circ H_{2,N} \circ \Lambda_{-s} \in G_{n-\frac{d}{2}-1,0}(X)$. From this observation and (4.46), we see that

$$H_{2,N} : H^s(X) \to H^s(X) \text{ is continuous.} \tag{4.64}$$

By (4.60) and (4.64) we deduce that

$$H_{1,N} \circ H_{2,N} \text{ and } H_{2,N} \circ H_{1,N} \; : \; L^2(X) \to H^s(X) \text{ are continuous.} \tag{4.65}$$

From (4.65), we get the claim (4.63). From (4.62) and (4.63), we get (4.58) with $N_s = NK_s$. Using (4.58) we can repeat the proof of (4.46) and obtain (4.57). ☐

Let $H \in G_{n-\frac{d}{2},1}(X)$. From (4.57), we can extend $I - H$ to a bounded operator in

$$I - H : L^2(X) \to L^2(X).$$

Theorem 4.11. *Let $H \in G_{n-\frac{d}{2},1}(X)$ and extend $I - H$ to a bounded operator in*

$$I - H : L^2(X) \to L^2(X)$$

by density. Then Ker$\,(I-H)$ *is a finite-dimensional subspace of* $\mathscr{C}^\infty(X)$ *and there exists* $C > 0$ *such that*

$$\|(I-H)u\| \geq C\|u\|, \quad \text{for any } u \in L^2(X), \quad u \perp \operatorname{Ker}(I-H). \tag{4.66}$$

Proof. Fix $s \in \mathbb{N}^*$. Theorem 4.10 shows that we can extend H^{N_s} to a bounded operator

$$H^{N_s} : L^2(X) \to H^s(X), \tag{4.67}$$

by density. Now, let $u \in \operatorname{Ker}(I-H)$. Then,

$$(I - H^{N_s})u = (I + H + \cdots + H^{N_s-1})(I-H)u = 0 \tag{4.68}$$

and hence $u = H^{N_s}u \in H^s(X)$. Since s is arbitrary, we deduce that $u \in \mathscr{C}^\infty(X)$. Moreover, from (4.68), we can apply Rellich's theorem and conclude that Ker$\,(I-H)$ is a finite-dimensional subspace of $\mathscr{C}^\infty(X)$. Since the argument is standard, we omit the details.

We now prove (4.66). Assume that (4.66) is not true. For every $j \in \mathbb{N}^*$ we can find $u_j \in L^2(X)$ with $u_j \perp \operatorname{Ker}(I-H)$ and $\|u_j\| = 1$ such that

$$\|(I-H)u_j\| \leq \frac{1}{j}\cdot \tag{4.69}$$

Put $v_j := (I-H)u_j$. We have for any $j \in \mathbb{N}^*$,

$$(I - H^{N_s})u_j = (I + H + \ldots + H^{N_s-1})v_j. \tag{4.70}$$

From (4.57) and (4.69) we see that there exists $C > 0$ such that

$$\|(I + H + \ldots + H^{N_s-1})v_j\| \leq \frac{C}{j}, \quad j \in \mathbb{N}^*. \tag{4.71}$$

By (4.67) and since $\|u_j\| = 1$ we conclude that there exists $\widehat{C} > 0$ such that for any $j \in \mathbb{N}^*$,

$$\|H^{N_s}u_j\|_s \leq \widehat{C}. \tag{4.72}$$

By Rellich's theorem, we can find a subsequence $H^{N_s}u_{j_k}$, $1 \leq j_1 < j_2 < \ldots$, such that $H^{N_s}u_{j_k}$ converges to some u in $L^2(X)$ as $k \to \infty$. From this observation, (4.70) and (4.71), we deduce that u_{j_k} converges to u in $L^2(X)$ with $\|u\| = 1$ as $k \to \infty$. By (4.57) and (4.69), we get $u \in \operatorname{Ker}(I-H)$. Since $u_{j_k} \perp \operatorname{Ker}(I-H)$ for every k, we have $u \perp \operatorname{Ker}(I-H)$. We get a contradiction and (4.66) follows. ◻

4.2. *The distribution kernels of σ and $\sigma^*\sigma$; proof of Theorem* 1.5

We are now ready to study the distribution kernels of σ and $\sigma\sigma^*$ in (4.1). We will use the same notations as before. Let $\mathscr{L}_{X_G,q}$ be the Levi form on X_G at $q \in X_G$ induced naturally from $\mathscr{L}$. The Hermitian metric $g^{\mathbb{C}TX}$ on $\mathbb{C}TX$ restricts to a metric on $T^{1,0}X$ which in turn induces a Hermitian metric on $T^{1,0}X_G$. Let $\mu_{d+1}, \ldots, \mu_n$ be the eigenvalues of $\mathscr{L}_{X_G,q}$ with respect to this Hermitian metric. We set

$$\det \mathscr{L}_{X_G,q} = \mu_{d+1} \ldots \mu_n. \tag{4.73}$$

Recall that $\pi : Y = \mu^{-1}(0) \to X_G$ is the natural projection. Let

$$S_{X_G} : L^2(X_G) \to \operatorname{Ker} \overline{\partial}_{b,X_G} = H^0_b(X_G), \tag{4.74}$$

be the Szegő projection on X_G (cf. (2.14)). Since X_G is assumed to be strictly pseudoconvex and $\overline{\partial}_{b,X_G}$ has closed range in L^2 on X_G, S_{X_G} is smoothing away the diagonal (see [8], [27, Theorem 1.2], [30, Theorem 4.7]). Hence, for any $x, y \in Y$ with $\pi(x) \neq \pi(y)$, there are open neighborhoods U of $\pi(x)$ in X_G and U_1 of $\pi(y)$ in X_G such that for all $\widehat{\chi} \in \mathscr{C}^\infty_0(U)$, $\widetilde{\chi} \in \mathscr{C}^\infty_0(U_1)$, we have

$$\widehat{\chi} S_{X_G} \widetilde{\chi} \equiv 0 \quad \text{on } X_G \times X_G. \tag{4.75}$$

We will use the same notations as in Sec. 4.1. Fix $p \in Y$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates and $\Omega_3 \subset \mathbb{R}^{2n+1-2d}$ be an open set as in the discussion at the beginning of Sec. 4.1. From now on, we identify $\underline{x}''$ as local coordinates of X_G near $q := \pi(p) \in X_G$ and we identify $W := \Omega_3$ with an neighborhood of $\pi(p)$ in X_G. Note that from (3.90), on $\mu^{-1}(0)/G$,

$$\begin{aligned} \omega_{0,G}(x) = (1 + O(|x|))dx_{2n+1} + \sum_{j=d+1}^{n} 2\mu_j(x_{2j}dx_{2j-1} - x_{2j-1}dx_{2j}) \\ + \sum_{j=2d+1}^{2n} b_j x_{2n+1} dx_j + O(|x|^2). \end{aligned} \tag{4.76}$$

From (3.79), (4.76), we see that μ_j, $j = d+1, \ldots, n$ in (4.76) are the same μ_j in (3.79) and $c_j = b_{2j-1} - ib_{2j}$. From this observation and applying [8] (more precisely Theorems 3.22, 3.24) to $\mu^{-1}(0)/G$, we have

$$S_{X_G}(\underline{x}'', \underline{y}'') \equiv \int_0^\infty e^{i\varphi(\underline{x}'',\underline{y}'')t} \beta(\underline{x}'', \underline{y}'', t)dt \quad \text{on } W \times W, \tag{4.77}$$

where $\beta(\underline{x}'', \underline{y}'', t) \in S^{n-d}_{\mathrm{cl}}(W \times W \times \mathbb{R}_+)$ with

$$\beta_0(\underline{x}'', \underline{x}'') = \frac{1}{2}\pi^{-(n-d)-1} |\det \mathscr{L}_{X_G,\underline{x}''}|, \quad \underline{x}'' \in W, \tag{4.78}$$

and $\varphi(\underline{x}'', \underline{y}'') \in \mathscr{C}^\infty(W \times W)$ with

$$d_{\underline{x}''}\varphi(\underline{x}'', \underline{x}'') = -d_{\underline{y}''}\varphi(\underline{x}'', \underline{x}'') = -\lambda(\underline{x}'')\omega_{0,G}(\underline{x}''), \quad \lambda(\underline{x}'') > 0,$$

$$\operatorname{Im}\varphi(\underline{x}'', \underline{y}'') \geq c \sum_{j=2d+1}^{2n} |x_j - y_j|^2, \quad \text{for some } c > 0,$$

$$\overline{\partial}_{b,\underline{x}''}\varphi(\underline{x}'', \underline{y}'') \text{ vanishes to infinite order at } \underline{x}'' = \underline{y}'',$$

$$\begin{aligned}\varphi(\underline{x}'', \underline{y}'') = &-x_{2n+1} + y_{2n+1} + i \sum_{j=d+1}^{n} |\mu_j||z_j - w_j|^2 \\ &+ \sum_{j=d+1}^{n} i\mu_j(\overline{z}_j w_j - z_j \overline{w}_j) \\ &+ \sum_{j=d+1}^{n} \frac{1}{2}(b_{2j-1} - ib_{2j})(-z_j x_{2n+1} + w_j y_{2n+1}) \\ &+ \sum_{j=d+1}^{n} \frac{1}{2}(b_{2j-1} + ib_{2j})(-\overline{z}_j x_{2n+1} + \overline{w}_j y_{2n+1}) \\ &+ (x_{2n+1} - y_{2n+1})r(\underline{x}'', \underline{y}'') + O(|(\underline{x}'', \underline{y}'')|^3),\end{aligned} \tag{4.79}$$

where $r(\underline{x}'', \underline{y}'') \in \mathscr{C}^\infty(W \times W)$, $r(0, 0) = 0$, $z_j = x_{2j-1} + ix_{2j}$, $j = d+1, \ldots, n$.

From Theorem 3.28, it is not difficult to see that the phase function $\Phi(\underline{x}'', \underline{y}'')$ in Theorem 3.25 satisfies (4.79). Hence, there is a function $h \in \mathscr{C}^\infty(W \times W)$ with $h(\underline{x}'', \underline{x}'') \neq 0$ for any $\underline{x}'' \in W$, such that $\varphi(\underline{x}'', \underline{y}'') - h(\underline{x}'', \underline{y}'')\Phi(\underline{x}'', \underline{y}'')$ vanishes to infinite order at $\underline{x}'' = \underline{y}''$ (see Theorem 3.24). We can replace the phase $\varphi(\underline{x}'', \underline{y}'')$ by $\Phi(\underline{x}'', \underline{y}'')$ and we have

$$S_{X_G}(\underline{x}'', \underline{y}'') \equiv \int_0^\infty e^{i\Phi(\underline{x}'', \underline{y}'')t}\beta(\underline{x}'', \underline{y}'', t)dt \quad \text{on } W \times W. \tag{4.80}$$

Theorem 4.12. *If $y \notin Y$, then for any open neighborhood D of y with $\overline{D} \cap Y = \emptyset$, we have*

$$\sigma \equiv 0 \quad \text{on } X_G \times D. \tag{4.81}$$

Let $x_0, y_0 \in Y$. If $\pi(x_0) \neq \pi(y_0)$, then there are open neighborhoods U_G of $\pi(x_0)$ in X_G and U_1 of y_0 in X such that

$$\sigma \equiv 0 \quad \text{on } U_G \times U_1. \tag{4.82}$$

Let $p \in Y$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates, and U an open neighborhood of p as at the discussion in the beginning of Sec. 4.1. *Then under the* *notations at* (3.76), (4.77), *there exists* $\alpha(\underline{x}'', y'', t) \in S^{n-\frac{3}{4}d}_{\rm cl}(W \times U \times \mathbb{R}_+)$ *such that*

$$\begin{aligned}\sigma(\underline{x}'', y) &\equiv \int_0^\infty e^{i\Phi(\underline{x}'',y'')t}\alpha(\underline{x}'', y'', t)dt \quad \textit{on } W \times U,\\ \alpha_0(\underline{x}'', \underline{x}'') &= 2^{-n+2d-1}\pi^{\frac{d}{2}-n-1}\frac{1}{\sqrt{V_{\rm eff}\,(\underline{x}'')}}|\det \mathscr{L}_{\underline{x}''}| \\ &\quad \times |\det \mathcal{R}_{\underline{x}''}|^{-\frac{3}{4}}, \quad \textit{for } \underline{x}'' \in W.\end{aligned} \tag{4.83}$$

Proof. By Theorem 3.20, S_G is smoothing away Y, which implies (4.81) from Theorem 3.27. Let $x_0, y_0 \in Y$. Assume that $\pi(x_0) \neq \pi(y_0)$. Let V_1 be a G-invariant neighborhood of y_0 and $x_0 \notin V_1$. We have

$$S_G(x, y) = \int_G S_{\leq\lambda_0}(x, g \circ y)d\mu(g),$$

where $\lambda_0 > 0$ is as in Theorem 3.19. Since $S_{\leq\lambda_0}$ is smoothing away the diagonal near Y, for any neighborhood U_1 of x_0 in X with $\overline{U}_1 \cap V_1 = \emptyset$, we have

$$S_G \equiv 0 \quad \text{on } U_1 \times V_1. \tag{4.84}$$

Let $\widehat{V}_G := \{\pi(y) \in X_G : y \in V_1 \cap Y\}$. By (4.75) there is an open set $\widehat{U}_G$ of $\pi(x_0)$ in X_G such that

$$S_{X_G} \equiv 0 \quad \text{on } \widehat{U}_G \times \widehat{V}_G. \tag{4.85}$$

The definition (4.1) of σ and Theorem 3.27, relations (4.84) and (4.85) yield (4.82).

Fix $u = (u_1, \ldots, u_{2n+1}) \in Y \cap U$. In view of (4.81) and (4.82), we only need to show that (4.83) holds near u. We may assume that $u = (0, \ldots, 0, u_{2d+1}, \ldots, u_{2n+1}) = \underline{u}''$. Let U_2 be a small neighborhood of u. Let $\chi(\underline{x}'') \in \mathscr{C}^\infty_0(\Omega_3)$. By (4.5) we can extend $\chi(\underline{x}'')$ to $G{\cdot}\Omega_3$ by setting $\chi(g{\cdot}\underline{x}'') := \chi(\underline{x}'')$ for every $g \in G$. Assume that $\chi = 1$ on some neighborhood of $\overline{U}_2$. Let $\chi_1 \in \mathscr{C}^\infty_0(X_G)$ with $\chi_1 = 1$ on some neighborhood of $\pi(U_2 \cap Y) \subset X_G$ and $\operatorname{supp}\chi_1 \subset \pi(\{x \in Y : \chi(x) = 1\})$. We have by (4.1),

$$\begin{aligned}\chi_1\sigma &= \chi_1 S_{X_G} \circ E \circ \iota_G \circ f_G \circ \iota^* \circ S_G\\ &= \chi_1 S_{X_G} \circ E \circ \iota_G \circ f_G \circ \iota^* \circ \chi S_G\\ &\quad + \chi_1 S_{X_G} \circ E \circ \iota_G \circ f_G \circ \iota^* \circ (1-\chi)S_G.\end{aligned} \tag{4.86}$$

If $u \in Y$ but $u \notin \{x \in X : \chi(x) = 1\}$, then $\pi(u) \notin \operatorname{supp}\chi_1$. From this observation, Theorem 3.27 and (4.75), we get

$$\chi_1 S_{X_G} \circ E \circ \iota_G \circ f_G \circ \iota^* \circ (1-\chi)S_G \equiv 0 \quad \text{on } X_G \times X. \tag{4.87}$$

From (4.86) and (4.87), we get

$$\chi_1\sigma \equiv \chi_1 S_{X_G} \circ E \circ \iota_G \circ f_G \circ \iota^* \circ \chi S_G \quad \text{on } X_G \times X. \tag{4.88}$$

From Theorem 3.25, (4.80) and (4.88), we can check that on $W \times U$,

$$\chi_1(\underline{x}'')\sigma(\underline{x}'', y) \equiv \int_0^\infty \int e^{i\Phi(\underline{x}'', \underline{v}'')t}\beta(\underline{x}'', \underline{v}'', t)E \circ \\ \times \left(\int_0^\infty \chi(\underline{v}'') f_G(\underline{v}'') e^{i\Phi(\underline{v}'', y)s} a(\underline{v}'', y, s) ds \right) dv(\underline{v}'') dt. \quad (4.89)$$

From the asymptotic formula of Melin–Sjöstrand [45, (2.28)], we have

$$E \circ \left(\int_0^\infty \chi(\underline{v}'') f_G(\underline{v}'') e^{i\Phi(\underline{v}'', y)s} a(\underline{v}'', y, s) ds \right) \equiv \int_0^\infty e^{i\Phi(\underline{v}'', y)s} b(\underline{v}'', y, s) ds, \quad (4.90)$$

where $b(\underline{v}'', y, s) \in S^{n-\frac{d}{2}-\frac{d}{4}}_{\mathrm{cl}}(W \times U \times \mathbb{R}_+)$ with

$$b_0(\underline{v}'', \underline{v}'') = \sigma^0_E(\underline{v}'', -\omega_{0,G}(\underline{v}''))\chi(\underline{v}'') f_G(\underline{v}'') a_0(\underline{v}'', \underline{v}''),$$

where b_0 denotes the leading term of b and σ^0_E denotes the principal symbol of E, a_0 is as in (3.85). Since $\sigma^0_E(x, \xi) = |\xi|^{-\frac{d}{4}}$ and $|\omega_0| = 1$, we have

$$b_0(\underline{v}'', \underline{v}'') = \chi(\underline{v}'') f_G(\underline{v}'') a_0(\underline{v}'', \underline{v}''). \quad (4.91)$$

From Theorem 4.3, (4.89) and (4.90), we get that there exists $\alpha(\underline{x}'', y'', t) \in S^{n-\frac{3}{4}d}_{\mathrm{cl}}(W \times U \times \mathbb{R}_+)$ such that

$$\sigma(\underline{x}'', y) \equiv \int_0^\infty e^{i\Phi(\underline{x}'', y'')t} \alpha(\underline{x}'', y'', t) dt \quad \text{on } W \times U. \quad (4.92)$$

Now, we compute α_0 at $(\underline{x}'', \underline{x}'')$. From (1.19), (3.85), (4.39), (4.78) and (4.91), we have that on $\{\underline{x}'' \in W;\ \chi(\underline{x}'') = 1\}$,

$$\begin{aligned}
\alpha_0(\underline{x}'', \underline{x}'') &= 2^{-n+1}\pi^{n-d+1}|\det \mathscr{L}_{\underline{x}''}|^{-1}|\det \mathcal{R}_{\underline{x}''}| b_0(\underline{x}'', \underline{x}'')\beta_0(\underline{x}'', \underline{x}'') \\
&= 2^{-n+1}\pi^{n-d+1}|\det \mathscr{L}_{\underline{x}''}|^{-1}|\det \mathcal{R}_{\underline{x}''}| f_G(\underline{x}'') a_0(\underline{x}'', \underline{x}'')\beta_0(\underline{x}'', \underline{x}'') \\
&= 2^{-n+1}\pi^{n-d+1}|\det \mathscr{L}_{\underline{x}''}|^{-1}|\det \mathcal{R}_{\underline{x}''}||\det \mathcal{R}_{\underline{x}''}|^{-\frac{1}{4}}\sqrt{V_{\mathrm{eff}}(\underline{x}'')} \\
&\quad \times 2^{d-1}\frac{1}{V_{\mathrm{eff}}(\underline{x}'')}\pi^{-n-1+\frac{d}{2}}|\det \mathcal{R}_{\underline{x}''}|^{-\frac{1}{2}}|\det \mathscr{L}_{\underline{x}''}| \\
&\quad \times \frac{1}{2}\pi^{-(n-d)-1}|\det \mathscr{L}_{X_G, \underline{x}''}| \\
&= 2^{-n+2d-1}\pi^{\frac{d}{2}-n-1}\frac{1}{\sqrt{V_{\mathrm{eff}}(\underline{x}'')}}|\det \mathscr{L}_{\underline{x}''}||\det \mathcal{R}_{\underline{x}''}|^{-\frac{3}{4}}. \quad (4.93)
\end{aligned}$$

Here we used the fact that $|\det \mathscr{L}_{X_G, \underline{x}''}| = 2^d|\det \mathscr{L}_{\underline{x}''}||\det \mathcal{R}_{\underline{x}''}|^{-1}$. From (4.92) and (4.93), we get (4.83). □

Let $\sigma^*:\mathscr{C}^\infty(X_G)\to\mathscr{D}'(X)$ be the formal adjoint of σ. From Theorem 4.12 we deduce that

$$\sigma^*:\mathscr{C}^\infty(X_G)\to H^0_b(X)^G\cap\mathscr{C}^\infty(X)^G. \tag{4.94}$$

Let

$$\begin{aligned} A_1 &:= \sigma^*\sigma:\mathscr{C}^\infty(X)\to H^0_b(X)^G,\\ A_2 &:= \sigma\sigma^*:\mathscr{C}^\infty(X_G)\to H^0_b(X_G). \end{aligned} \tag{4.95}$$

Let $A_1(x,y)$ and $A_2(x,y)$ be the distribution kernels of A_1 and A_2, respectively. In view of Theorems 4.2, 4.3 we can repeat the proof of Theorem 4.12 with minor changes and deduce the following two theorems.

Theorem 4.13. *With the notations used above, if $y\notin Y$, then for any neighborhood D of y with $\overline{D}\cap Y=\emptyset$, we have*

$$A_1\equiv 0\quad \text{on } X\times D. \tag{4.96}$$

Let $x,y\in Y$. If $\pi(x)\neq\pi(y)$, then there are neighborhoods D_1 of x in X and D_2 of y in X such that

$$A_1\equiv 0\quad \text{on } D_1\times D_2. \tag{4.97}$$

Let $p\in Y$ and let $x=(x_1,\ldots,x_{2n+1})$ be the local coordinates as in the discussion in the beginning of Sec. 4.1. *Then there exists an open neighborhood U of p and a symbol $a(x'',y'',t)\in S^{n-\frac{d}{2}}_{\mathrm{cl}}(U\times U\times\mathbb{R}_+)$ such that the following holds under the notation* (3.76),

$$A_1(x,y)\equiv\int_0^\infty e^{i\Phi(x'',y'')t}a(x'',y'',t)dt\quad \text{on } U\times U \tag{4.98}$$

with

$$\begin{aligned} a_0(\underline{x}'',\underline{x}'') = {} & 2^{-3n+4d-1}\pi^{-n-1}\frac{1}{V_{\mathrm{eff}}\,(\underline{x}'')}|\det\mathscr{L}_{\underline{x}''}|\\ & \times|\det\mathcal{R}_{\underline{x}''}|^{-\frac{1}{2}},\quad \text{for } \underline{x}''\in U\cap Y. \end{aligned} \tag{4.99}$$

Theorem 4.14. *Let $x,y\in Y$. If $\pi(x)\neq\pi(y)$, then there are neighborhoods D_G of $\pi(x)$ and V_G of $\pi(y)$ in X_G such that*

$$A_2\equiv 0\quad \text{on } D_G\times V_G. \tag{4.100}$$

Let $p\in Y$ and let $x=(x_1,\ldots,x_{2n+1})$ be the local coordinates as in the discussion at the beginning of Sec. 4.1. *Then there exists $\widehat{a}(\underline{x}'',\underline{y}'',t)\in S^{n-d}_{\mathrm{cl}}(W\times W\times\mathbb{R}_+)$ such that by using the notations* (3.76), (4.77), *we have*

$$A_2(\underline{x}'',\underline{y}'')\equiv\int_0^\infty e^{i\Phi(\underline{x}'',\underline{y}'')t}\widehat{a}(\underline{x}'',\underline{y}'',t)dt\quad \text{on } W\times W, \tag{4.101}$$

with

$$\widehat{a}_0(\underline{x}'', \underline{x}'') = 2^{-3n+\frac{5}{2}d-1}\pi^{-n+\frac{d}{2}-1}|\det \mathscr{L}_{X_G,\underline{x}''}|, \quad \textit{for } \underline{x}'' \in W. \tag{4.102}$$

Set

$$Q := -C_0\,\sigma^*\sigma + S_G := -C_0 A_1 + S_G :$$
$$\mathscr{C}^\infty(X) \to H^0_b(X)^G, \quad \text{with } C_0 = 2^{3(n-d)}\pi^{d/2}. \tag{4.103}$$

Since $A_1 = A_1 S_G = S_G A_1$, it is clear that

$$C_0 A_1 = S_G - Q = S_G - Q S_G = (I-Q)S_G = S_G(I-Q) \tag{4.104}$$

and

$$Q^* = Q, \tag{4.105}$$

where Q^* is the formal adjoint of Q. From Theorems 3.25, 4.11 and 4.13, we get:

Theorem 4.15. *The operator Q belongs to the class $G_{n-\frac{d}{2},1}(X)$ and hence $I-Q$ extends by density to a bounded self-adjoint operator $I-Q: L^2(X) \to L^2(X)$.*

By Theorem 4.15 there exists $C > 0$ such that for every $u \in H^0_b(X)^G \cap \mathscr{C}^\infty(X)$, we have

$$(\,\sigma u, \sigma u\,)_{X_G} = (\,\sigma^*\sigma u, u\,) = \frac{1}{C_0}(\,(I-Q)u, u) \leq C\|u\|^2. \tag{4.106}$$

From (4.106), we deduce:

Corollary 4.16. *There exists $C > 0$ such that*

$$(\,\sigma u, \sigma u\,)_{X_G} \leq C\|u\|^2, \quad \textit{for any } u \in H^0_b(X)^G \cap \mathscr{C}^\infty(X). \tag{4.107}$$

Hence we can extend σ by density to a bounded operator

$$\sigma: H^0_b(X)^G \to H^0_b(X_G).$$

From now on, we consider σ as a bounded operator $\sigma: H^0_b(X)^G \to H^0_b(X_G)$.

Theorem 4.17. $\operatorname{Ker}\sigma$ *is a finite-dimensional subspace of $H^0_b(X) \cap \mathscr{C}^\infty(X)$.*

Proof. From Theorem 4.11 we see that $\operatorname{Ker}(I-Q)$ is a finite-dimensional subspace of the space $\mathscr{C}^\infty(X)$. Note that $\operatorname{Ker}\sigma \subset H^0_b(X)^G \cap \operatorname{Ker}(I-Q)$, so the theorem follows. □

From Theorem 4.14 we see that $\sigma\sigma^*$ is a complex Fourier integral operator with the same type as S_{X_G}. It is known that $\sigma\sigma^*$ is a pseudodifferential operator of order zero type $(\frac{1}{2}, \frac{1}{2})$ (see [27, Proposition 5.18]). Set $C_1 = \pi^{\frac{d}{2}}2^{3n-\frac{5}{2}d}$. Then the leading of the symbol of $-C_1\sigma\sigma^* + S_{X_G}$ vanishes on the diagonal $x = y$, and it is known that $-C_1\sigma\sigma^* + S_{X_G}$ is a pseudodifferential operator of order $-\frac{1}{2}$ and type $(\frac{1}{2}, \frac{1}{2})$ (see [27, Proposition 5.18]). By the classical Calderon–Vaillancourt theorem [26, Chap. XVIII], we have for every $s \in \mathbb{R}$,

$$\begin{aligned} \sigma\sigma^* : H^s(X_G) \to H^s(X_G) \quad &\text{is continuous,} \\ -C_1\sigma\sigma^* + S_{X_G} : H^s(X_G) \to H^{s+\frac{1}{2}}(X_G) \quad &\text{is continuous.} \end{aligned} \tag{4.108}$$

From (4.108), we have a result similar to Theorem 4.10. Hence, we can apply the proof of Corollary 4.16 with minor changes and deduce that there exists $\widehat{C} > 0$ such that

$$(\,\sigma^* v, \sigma^* v\,) \leq \widehat{C}\|v\|^2_{X_G}, \quad \text{for any } v \in H^0_b(X_G) \cap \mathscr{C}^\infty(X_G). \tag{4.109}$$

Therefore we can extend σ^* by density to a bounded operator

$$\sigma^* : H^0_b(X_G) \to H^0_b(X)^G.$$

We repeat the proof of Theorem 4.17 with minor changes and deduce:

Theorem 4.18. $\operatorname{Ker}\sigma^*$ *is a finite-dimensional subspace of* $H^0_b(X_G) \cap \mathscr{C}^\infty(X_G)$.

Finally, we obtain:

Theorem 4.19. $\operatorname{Ker}\sigma$ *and* $(\operatorname{Im}\sigma)^\perp$ *are finite-dimensional subspaces of* $H^0_b(X)^G \cap \mathscr{C}^\infty(X)$ *and* $H^0_b(X_G) \cap \mathscr{C}^\infty(X_G)$, *respectively.*

Proof. We only need to prove that $(\operatorname{Im}\sigma)^\perp$ is a finite-dimensional subspace of $\mathscr{C}^\infty(X_G)$. Note that $(\operatorname{Im}\sigma)^\perp \subset \operatorname{Ker}\sigma^*$. From this observation and Theorem 4.18, the theorem follows. □

Theorem 4.19 implies Theorem 1.5.

5. Proof of Theorem 1.2

The main goal of this section is to prove Theorem 1.2. Recall that the Riemannian metrics g^{TX} on X and g^{TX_G} on X_G are given by Convention 2.8. Let Δ^X and Δ^{X_G} be the (positive) Laplacians on (X, g^{TX}) and (X_G, g^{TX_G}), respectively. For $s \in \mathbb{R}$ we consider the classical pseudodifferential operators of order s on X and X_G, respectively,

$$\Lambda_s = (1 + \Delta^X)^{s/2}, \quad \widehat{\Lambda}_s = (1 + \Delta^{X_G})^{s/2}. \tag{5.1}$$

They are self-adjoint and positive with respect to the inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_{X_G}$, respectively. In particular, the maps

$$\Lambda_s : H^\ell(X) \to H^{\ell-s}(X), \quad \widehat{\Lambda}_s : H^\ell(X_G) \to H^{\ell-s}(X_G), \tag{5.2}$$

are injective for any $\ell \in \mathbb{R}$. For $u, v \in H^s(X)$, $u', v' \in H^s(X_G)$, we define the inner products

$$(u,v)_s := (\Lambda_s u, \Lambda_s v)_X, \quad (u',v')_s := (\widehat{\Lambda}_s u, \widehat{\Lambda}_s v)_{X_G}, \tag{5.3}$$

and let $\|\cdot\|_s$, $\|\cdot\|_{X_G,s}$ be the corresponding norms.

Recall that the space $G_{k,\ell}(X)$ is given by Definition 4.5 and $S_G \in G_{n-\frac{d}{2},0}(X)$.

Theorem 5.1. *Let $A \in G_{n-\frac{d}{2}+\gamma,0}(X)$, $\gamma \in \mathbb{R}$. Then A can be extended continuously to $A : H^s(X) \to H^{s-\gamma}(X)$ for every $s \in \mathbb{R}$.*

Proof. For every $s \in \mathbb{R}$, put $B := \Lambda_{s-\gamma} A \Lambda_{-s}$. As Λ_s is G-invariant, we see from complex stationary phase formula of Melin–Sjöstrand that $B \in G_{n-\frac{d}{2},0}(X)$. Near Y, write $B(x,y) \equiv \int_0^\infty e^{i\Phi(x,y)t} b(x,y,t)dt$ as in (4.42) and let $b_0(x,y)$ be the leading term of $b(x,y,t)$. Let P be a classical pseudodifferential operator of order 0 on X with

$$\sigma_P(x, d_x\Phi(x,x)) a_0(x,x) = b_0(x,x), \tag{5.4}$$

for every $x \in Y$, where σ_P denotes the principal symbol of P and a_0 is the leading term of the expansion of S_G (see (3.85)). The complex stationary phase formula yields

$$B = PS_G + R, \quad R \in G_{n-\frac{d}{2},1}(X). \tag{5.5}$$

From Theorem 4.10, (5.5) and the fact that S_G is L^2 bounded, we deduce that $B = \Lambda_{s-\gamma} A \Lambda_{-s}$ is L^2 bounded. This implies that there exists $C_1 > 0$ such that for every $u \in \mathscr{C}^\infty(X)$,

$$\|Au\|_{s-\ell} = \|\Lambda_{s-\gamma} A \Lambda_{-s} \Lambda_s u\| \le C_1 \|\Lambda_s u\| = C_1 \|u\|_s. \tag{5.6}$$

The theorem follows. □

From Theorems 3.27, 5.1 and note that $S_G \in G_{n-\frac{d}{2},0}(X)$, we deduce the following regularity property of the G-invariant Szegő projector.

Corollary 5.2. *For every $s \in \mathbb{R}$, $S_G : H^s(X) \to H^s(X)$ is continuous, and in particular, $H^0_b(X)^G \cap \mathscr{C}^\infty(X)$ is dense in $H^0_b(X)^G_s$ in $H^s(X)$.*

Due to Theorem 3.27 the map σ_G given by (1.5) has a well-defined extension

$$\sigma_G : \mathscr{C}^\infty(X) \to H^0_b(X_G) \cap \mathscr{C}^\infty(X_G), \quad u \mapsto \iota_G \circ \iota^* \circ S_G u.$$

Theorem 5.3. *For every $s \in \mathbb{Z}$ there exists $C_s > 0$ such that*

$$\|\sigma_G u\|^2_{X_G, s-\frac{d}{4}} \leq C_s \|u\|^2_s, \quad \text{for every } u \in \mathscr{C}^\infty(X). \tag{5.7}$$

Moreover, the map (1.5) *can be extended continuously by density to a bounded operator*

$$\sigma_{G,s} := \sigma_G : H^0_b(X)^G_s \to H^0_b(X_G)_{s-\frac{d}{4}}, \tag{5.8}$$

for every $s \in \mathbb{R}$.

Proof. Fix $s \in \mathbb{R}$. Let $(\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G)^* : \mathscr{C}^\infty(X_G) \to \mathscr{D}'(X)$ be the formal adjoint of $\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X_G)$. For $u \in \mathscr{C}^\infty(X)$, we have

$$\begin{aligned} \|\sigma_G u\|^2_{X_G, s-\frac{d}{4}} &= (\, \hat{\Lambda}_{s-\frac{d}{4}} \sigma_G u, \quad \hat{\Lambda}_{s-\frac{d}{4}} \sigma_G u \,)_{X_G} \\ &= (\, (\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G)^* (\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G) u \,, \, u \,). \end{aligned} \tag{5.9}$$

We repeat the proof of Theorem 4.13 and conclude that

$$(\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G)^* (\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G) \in G_{n-\frac{d}{2}+2s,0}(X). \tag{5.10}$$

From Theorem 5.1 and (5.10), we deduce that there exist $C, C_1 > 0$ such that

$$\begin{aligned} |(\, (\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G)^* (\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G) u \,, \, u \,)| &\leq C \|(\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G)^* (\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G) u\|_{-s} \|u\|_s \\ &\leq C_1 \|u\|^2_s, \end{aligned} \tag{5.11}$$

for every $u \in \mathscr{C}^\infty(X)$. From (5.9) and (5.11) we get (5.7).

By Theorem 3.27 and Corollary 5.2 we see that $H^0_b(X)^G \cap \mathscr{C}^\infty(X)$ is dense in $H^0_b(X)^G_s$, for every $s \in \mathbb{R}$. From this observation and (5.7), we deduce that for every $s \in \mathbb{R}$ the map (1.5) can be extended continuously by density to a bounded operator $\sigma_{G,s}$ as in (5.8). ☐

Let $f_G \in \mathscr{C}^\infty(Y)^G$ be as in (1.19). We identify f_G with a smooth function on X_G.

Theorem 5.4. *For every $s \in \mathbb{R}$, $\operatorname{Ker} \sigma_{G,s}$ is a finite-dimensional subspace of $\mathscr{C}^\infty(X)$. Moreover, $\operatorname{Ker} \sigma_{G,s}$ is independent of s.*

Proof. Let E be a classical pseudodifferential operator on X_G with principal symbol $\sigma_E(x, \xi) = |\xi|^{-\frac{d}{4}}$. Let for every $s \in \mathbb{R}$,

$$\hat{\sigma}_G := S_{X_G} \circ E \circ f_G \circ \sigma_G : H^0_b(X)^G_s \to H^0_b(X_G)_s. \tag{5.12}$$

We repeat the proof of Theorem 4.17 and deduce that $\operatorname{Ker} \hat{\sigma}_G$ is a finite-dimensional subspace of $\mathscr{C}^\infty(X)$. Since $\operatorname{Ker} \sigma_G \subset \operatorname{Ker} \hat{\sigma}_G$, the theorem follows. ☐

Theorem 5.5. *With $\sigma_{G,s}$ as in* (5.8), $(\operatorname{Im} \sigma_{G,s})^\perp$ *in* (1.7) *is a finite-dimensional subspace of $\mathscr{C}^\infty(X_G)$ for every $s \in \mathbb{R}$.*

Proof. Fix $s \in \mathbb{R}$. By Corollary 5.2 and Theorem 5.3 we can extend $\sigma_{G,s}$ in (5.8) to $H^s(X)$ by

$$\sigma_G : H^s(X) \to H^{s-\frac{d}{4}}(X_G), \quad u \mapsto \sigma_G S_G u. \tag{5.13}$$

We have $\hat{\Lambda}_{s-\frac{d}{4}}\sigma_G : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X_G)$. Let $(\hat{\Lambda}_{s-\frac{d}{4}}\sigma_G)^* : \mathscr{D}'(X_G) \to \mathscr{D}'(X)$ be the formal adjoint of $\hat{\Lambda}_{s-\frac{d}{4}}\sigma_G$. We repeat the proof of Theorem 4.12 with minor changes and deduce that $(\hat{\Lambda}_{s-\frac{d}{4}}\sigma_G)^* : \mathscr{C}^\infty(X_G) \to \mathscr{C}^\infty(X)$. Let

$$F_s := S_{X_G} f_G^2 \hat{\Lambda}_{-2s}\sigma_G(\hat{\Lambda}_{s-\frac{d}{4}}\sigma_G)^*\hat{\Lambda}_{s-\frac{d}{4}} S_{X_G} : \mathscr{D}'(X_G) \to \mathscr{D}'(X_G). \tag{5.14}$$

For any $u \in (\operatorname{Im}\sigma_{G,s})^\perp$ and $v \in \mathscr{C}^\infty(X)$, by (1.7), we have

$$\begin{aligned}(v, S_G(\hat{\Lambda}_{s-\frac{d}{4}}\sigma_G)^*\hat{\Lambda}_{s-\frac{d}{4}} S_{X_G} u) &= (\hat{\Lambda}_{s-\frac{d}{4}}\sigma_G S_G v, \hat{\Lambda}_{s-\frac{d}{4}} S_{X_G} u)_{X_G} \\ &= (\sigma_G v, u)_{X_G, s-\frac{d}{4}} = 0. \end{aligned}\tag{5.15}$$

In view of (5.15), we see that

$$(\operatorname{Im}\sigma_{G,s})^\perp \subset \operatorname{Ker} F_s \cap H^0_b(X_G)_s. \tag{5.16}$$

We repeat the proof of Theorem 4.14 and conclude as in (4.104) that $F_s = C(I - R)S_{X_G}$, where $C \neq 0$ is a constant and R is a complex Fourier integral operator with the same phase and order as S_{X_G} and vanishing leading term on the diagonal. More precisely, in the local coordinates $\underline{x}''$ of X_G defined in an open set W of X_G, we have

$$R(\underline{x}'', \underline{y}'') \equiv \int_0^\infty e^{i\Phi(\underline{x}'', \underline{y}'')t} r(\underline{x}'', \underline{y}'', t)dt, \tag{5.17}$$

where Φ is as in (4.101), $r \in S^{n-d}_{\rm cl}(W \times W \times \mathbb{R}_+)$, $r_0(\underline{x}'', \underline{x}'') = 0$ for every $\underline{x}'' \in W$, where r_0 is the leading term of r. We repeat now the proof of Theorem 4.11 with minor changes and deduce that $\operatorname{Ker}(I - R)$ is a finite-dimensional subspace of $\mathscr{C}^\infty(X_G)$. From (5.16) we deduce that $(\operatorname{Im}\sigma_{G,s})^\perp \subset \operatorname{Ker}(I - R)$. The theorem follows. ∎

Theorem 5.6. $\dim(\operatorname{Im}\sigma_{G,s})^\perp$ *is independent of* s.

Proof. Fix $s \in \mathbb{R}$ and let $u \in (\operatorname{Im}\sigma_{G,s})^\perp$. By Theorem 5.5 we have $u \in \mathscr{C}^\infty(X_G) \cap H^0_b(X_G)$. We have an orthogonal decomposition

$$u = \sigma_{G,\frac{d}{4}} v + w, \quad v \in H^0_b(X)^G_{\frac{d}{4}}, w \in (\operatorname{Im}\sigma_{G,\frac{d}{4}})^\perp. \tag{5.18}$$

We define a linear map

$$\gamma_s : (\operatorname{Im}\sigma_{G,s})^\perp \to (\operatorname{Im}\sigma_{G,\frac{d}{4}})^\perp, \quad \gamma_s u := w \in (\operatorname{Im}\sigma_{G,\frac{d}{4}})^\perp. \tag{5.19}$$

We claim that γ_s is injective. Assume that $\gamma_s u = 0$. Then there exists $v \in H^0_b(X)^G_{\frac{d}{4}}$ such that $u = \sigma_{G,\frac{d}{4}} v$. Let (v_j) be a sequence in $\mathscr{C}^\infty(X)$ with $v_j \to v$ in $H^{\frac{d}{4}}(X)$ as $j \to \infty$. By Corollary 5.2 we have $S_G v_j \to S_G v = v$ in $H^{\frac{d}{4}}(X)$ as $j \to \infty$.

By Theorem 5.3 we have $\sigma_{G,\frac{d}{4}} S_G v_j \to \sigma_{G,\frac{d}{4}} v = u$ in $L^2(X_G)$ as $j \to \infty$. Since $u \in (\operatorname{Im} \sigma_{G,s})^\perp$, we have as $j \to \infty$

$$0 = (\hat{\Lambda}_{s-\frac{d}{4}} \sigma_G S_G v_j, \quad \hat{\Lambda}_{s-\frac{d}{4}} u)_{X_G} = (\sigma_G S_G v_j \,, (\hat{\Lambda}_{s-\frac{d}{4}})^2 u)_{X_G} \to (u \,, (\hat{\Lambda}_{s-\frac{d}{4}})^2 u)_{X_G}.$$

Hence $(u \,, (\hat{\Lambda}_{s-\frac{d}{4}})^2 u)_{X_G} = 0$ and thus $\hat{\Lambda}_{s-\frac{d}{4}} u = 0$. Since $\hat{\Lambda}_{s-\frac{d}{4}}$ is injective, we get $u = 0$ and hence γ_s is injective. Since γ_s is injective, $\dim (\operatorname{Im} \sigma_{G,s})^\perp \leq \dim (\operatorname{Im} \sigma_{G,\frac{d}{4}})^\perp$.

Similarly, we can repeat the procedure above and conclude that $\dim (\operatorname{Im} \sigma_{G,\frac{d}{4}})^\perp \leq \dim (\operatorname{Im} \sigma_{G,s})^\perp$. Thus, $\dim (\operatorname{Im} \sigma_{G,s})^\perp = \dim (\operatorname{Im} \sigma_{G,\frac{d}{4}})^\perp$. The theorem follows. ☐

Proof of Theorem 1.2. Theorems 5.4, 5.5 and 5.6 yield Theorem 1.2 for the case when $\dim X_G \geq 3$.

Assume now that $\dim X_G = 1$. Then X_G is a union of circles. For simplicity, suppose that $X_G = S^1$. The circle X_G admits a natural S^1 action $e^{i\theta} : S^1 \times X_G \to X_G$, $(e^{i\theta}, z) \to e^{i\theta} z$. For every $m \in \mathbb{Z}$, put

$$L^2_m(X_G) := \left\{ u \in L^2(X_G) : (e^{i\theta})^* u = e^{im\theta} u, \text{ for every } e^{i\theta} \in S^1 \right\}.$$

It is clear that $L^2(X_G) = \oplus_{m \in \mathbb{Z}} L^2_m(X_G)$. By definition $H^0_b(X_G) = \oplus_{m \in \mathbb{N}} L^2_m(X_G)$. The Szegő projection on X_G is the orthogonal projection: $S_{X_G} : L^2(X_G) \to H^0_b(X_G)$. The S^1 action induces a smooth vector field $\frac{\partial}{\partial \theta}$ on X_G. Fix a point $p \in X_G$. Let x be local coordinate of X_G such that $x(p) = 0$ and $\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}$. Then

$$S_{X_G}(x, y) \equiv \frac{1}{2\pi} \int_0^\infty e^{it(x-y)} dt. \tag{5.20}$$

In particular, $S_{X_G}(x, y)$ is a Fourier integral operator with complex phase. Therefore, the above proof of Theorem 1.2 in the case $\dim X_G \geq 3$ works also when $\dim X_G = 1$. ☐

Theorem 5.7. *Let X be a three-dimensional compact orientable CR manifold and let G be a compact Lie group acting on X such that the G-action preserves J and ω_0. We assume that X is pseudoconvex of finite type and that $\overline{\partial}_b$ has closed range in L^2 on X. Then the conclusions of Theorem* 1.2 *hold.*

Proof. It was shown in [12, Proposition 4.1] that there exists a bounded linear operator $\mathcal{G} : \operatorname{Im} \overline{\partial}_b \to L^2(X)$ such that $S = I - \mathcal{G} \bar{\partial}_b$. Furthermore $\mathcal{G}$ is smoothing away the diagonal and maps smooth functions to smooth functions, where S denotes the Szegő projector. Hence, the Szegő projector is smoothing outside the diagonal and preserves the space of smooth functions. Moreover, G is a circle and $\dim X_G = 1$ in this case, thus the arguments above apply again. ☐

Example 5.8. If X be a compact pseudoconvex three-dimensional CR manifold of finite type admitting a transversal CR circle action, then $\overline{\partial}_b$ has closed range in L^2, see [31, §5.2]. Let M be a compact Riemann surface and (L, h^L) be semi-positive line bundle over M whose curvature R^L vanishes to finite order at any point. Then the Grauert tube X of L^* (2.19) is a compact pseudoconvex three-dimensional CR manifold of finite type (cf. [31], [41, Proposition 11]) admitting a transversal CR circle action. If G is a compact Lie group acting holomorphically on M and whose action lifts to (L, h^L), Theorem 5.7 applies to X.

6. An Almost Complex Version of Theorem 1.4

In this section, we will prove a version of Theorem 1.4 in the case of an almost complex manifold. We will mostly follow [52] and adopt the notations therein.

Let (L, h^L) be a Hermitian line bundle with Hermitian connection ∇^L and associated curvature R^L on a compact almost complex manifold (M, J).

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ acting (on left) on M, whose action lifts on L such that h^L, ∇^L are G-equivariant. Then the moment map $\mu : M \to \mathfrak{g}^*$ is defined by the Kostant formula

$$2\pi\sqrt{-1}\langle \mu, \xi \rangle : = \nabla^L_{\xi_M} - L_\xi, \quad \text{for } \xi \in \mathfrak{g}. \tag{6.1}$$

For any $\xi \in \mathfrak{g}$, we have

$$d\langle \mu, \xi \rangle = i_{\xi_M}\omega, \quad \text{with } \omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{6.2}$$

We assume that the almost complex structure J on M is G-invariant and R^L is J-invariant, G acts freely on $\mu^{-1}(0)$ and $\omega(\cdot, J\cdot)$ defines a metric on $TM|_{\mu^{-1}(0)}$.

By choosing any G- and J-invariant Riemannian metric g^{TM} on TM, we can define an associated Dirac operator D^L on $\Lambda(T^{*(0,1)}M) \otimes L$ and $D^L_\pm$ its restriction on $\Omega^{0,\text{even/odd}}(M, L)$ (cf. [52, Definition 1.1]). Its index as a finite-dimensional virtual representation of G,

$$\operatorname{Ind}(D^L_+) = \operatorname{Ker}(D^L_+) - \operatorname{Ker}(D^L_-) \in R(G), \tag{6.3}$$

does not depend on the choice of g^{TM}.

Moreover (L, h^L, ∇^L), J, ω on M induce canonically $(L_G, h^{L_G}, \nabla^{L_G})$, J_G, ω_G on $M_G = \mu^{-1}(0)/G$. In particular, (M_G, ω_G) is a compact symplectic manifold with compatible almost complex structure J_G. Thus $\operatorname{Ind}(D^{L_G}_+)$ is well defined as a virtual vector space.

Theorem 6.1. *Let (M, J) be a compact almost complex manifold and (L, h, ∇^L) be a Hermitian line bundle with connection on M. Let G be a compact Lie group acting (on left) on M, whose action lifts on L such that J, h^L and ∇^L are G-equivariant. We assume that G acts freely on $\mu^{-1}(0)$ and $\omega(\cdot, J\cdot)$ defines a metric on $TM|_{\mu^{-1}(0)}$. Then there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ we have*

$$\operatorname{Ind}(D^{L^m}_+)^G = \operatorname{Ind}(D^{L^m_G}_+). \tag{6.4}$$

Note that by the main result of [44], [52, Theorem 0.1], if $\omega(\cdot, J\cdot) > 0$ on the whole M, then Theorem 6.1 holds for $m_0 = 1$.

Proof. We adapt directly the notation and argument from [52]. We fix a G- and J-invariant metric g^{TM} on TM such that near $\mu^{-1}(0)$, g^{TM} is given by $\omega(\cdot, J\cdot)$. We fix an Ad_G-invariant scalar product on $\mathfrak{g}$ and identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via this product.

Let $h_1, \ldots, h_d$ be an orthonormal basis of $\mathfrak{g}$. Set

$$X^{\mathcal{H}}(x) = 2(\mu(x))_M(x) = 2\sum_i \mu_i(x)V_i(x) \ \text{ with } \mu = \sum_i \mu_i(x)h_i \ \text{ and } \ V_i = h_{i,M}. \tag{6.5}$$

Following [52, Definition 1.2] we set for $T \in \mathbb{R}$,

$$D_T^{L^m} = D^{L^m} + \frac{\sqrt{-1}}{2}Tc(X^{\mathcal{H}}) : \Omega^{0,*}(M, L) \to \Omega^{0,*}(M, L), \tag{6.6}$$

where $c(\cdot)$ is the Clifford action. Then we have (cf. [52, (1.26)])

$$(D_T^{L^m})^2 = (D^{L^m})^2 + \frac{\sqrt{-1}}{2}T\sum_j c(e_j)c(\nabla_{e_j}X^{\mathcal{H}}) - \sqrt{-1}T\nabla_{X^{\mathcal{H}}} + \frac{T^2}{4}|X^{\mathcal{H}}|^2, \tag{6.7}$$

where $\{e_j\}_j$ is an orthonormal frame of (TM, g^{TM}). Now as in [52, (1.27)],

$$\nabla_{X^{\mathcal{H}}} = 2\sum_i \mu_i L_{V_i} + 4m\sqrt{-1}\mathcal{H} + A, \tag{6.8}$$

where A is an endomorphism of $\Lambda(T^{*(0,1)}M)$ and does not depend on m.

We fix a sufficiently small G-invariant open neighborhoods $U' \Subset U$ of $\mu^{-1}(0)$ such that G acts freely on U and $\omega(\cdot, J\cdot) > 0$ on U. Note that on $\Omega^{0,*}(M, L)^G$, $L_{V_i} = 0$. From (6.7) and (6.8), there exists $m_0 > 0$ such that for any $m \geq m_0$, the assertion of [52, Theorem 2.1] holds for U': there exist $C > 0, b > 0$ such that for any $T \geq 1$ and any $s \in \Omega^{0,*}(M, L^m)$ with $\mathrm{supp}\, s \subset M \backslash U'$, we have

$$\|D_T^{L^m} s\|_0^2 \geq C(\|s\|_1^2 + (T - b)\|s\|_0^2). \tag{6.9}$$

Thus we are on U exactly in same situation as considered in [52, §3(b)–3(e)]. Hence Theorem 6.1 follows as in [52, (3.36), (3.37)]. $\square$

It is an interesting question to show that in the holomorphic situation, under the assumption of this section, we have

$$H^j(M, L^m)^G = 0 \quad \text{for any } j > 0, m \gg 1. \tag{6.10}$$

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