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ON THE SINGULARITIES OF THE SPECTRAL AND BERGMAN PROJECTIONS ON COMPLEX MANIFOLDS WITH BOUNDARY

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We show that the spectral kernel of the $\bar\partial$-Neumann Laplacian acting on $(0,q)$-forms on a smooth relatively compact domain admits a full asymptotic expansion near the nondegenerate part of the boundary. We show further that the Bergman projection admits an asymptotic expansion under certain local closed range condition. In particular, if condition $Z(q)$ fails but conditions $Z(q-1)$ and $Z(q+1)$ hold, the Bergman projection on $(0,q)$-forms admits an asymptotic expansion. As applications, we establish Bergman kernel asymptotic expansions near nondegenerate points of some domains with weakly pseudoconvex boundary and S^1-equivariant asymptotic expansions and embedding theorems for domains with holomorphic S^1-action.

1. Introduction

1.1. ***Setting and statement of the main results.*** Let M be a relatively compact open subset with smooth boundary X of a complex manifold M' of complex dimension $n \geq 2$. The study of the $\bar\partial$-Neumann Laplacian on M is a classical subject in several complex variables. For $q \in \{0,1,\dots,n-1\}$, let $\Box^{(q)}$ be the $\bar\partial$-Neumann Laplacian for $(0,q)$-forms on M. The domain M is said to satisfy condition $Z(q)$ $(0 \leq q \leq n-1)$ at $p \in X$ if the Levi form of a (and hence any) defining function of M near p has at least $n-q$ positive or at least $q+1$ negative eigenvalues on the holomorphic tangential space to ∂M at p. When condition $Z(q)$ holds at each point of X, Kohn proved subelliptic estimates with gain of one

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derivative in Sobolev norms for the solutions of $\Box^{(q)}u = f$ (see [Chen and Shaw 2001; Folland and Kohn 1972; Hörmander 1965; Kohn 1963; 1964; Kohn and Nirenberg 1965]). This means that for each $(0,q)$-form f orthogonal to $\operatorname{Ker}\Box^{(q)}$ with derivatives of order $\leq s$ in L^2 the equation $\Box^{(q)}u = f$ has a solution u with derivatives of order $\leq s+1$ in L^2. Moreover, $\operatorname{Ker}\Box^{(q)}$ is a finite-dimensional subspace of $\Omega^{0,q}(\overline{M})$. A closely related notion to the condition $Z(q)$ is the notion of q-convexity (and q-concavity) in the sense of [Andreotti and Grauert 1962] and is one of the basic tools in the study of the geometry of noncompact complex manifolds.

The Bergman projection $B^{(q)}$ is the orthogonal projection onto the kernel of $\Box^{(q)}$ in the L^2 space. The Schwartz kernel $B^{(q)}(\cdot,\cdot)$ of $B^{(q)}$ is called the Bergman kernel. If $Z(q)$ holds, the above results show that the Bergman projection $B^{(q)}$ is a smoothing operator on $\overline{M}$ and $B^{(q)}(\cdot,\cdot)$ is smooth on $\overline{M}\times\overline{M}$. When $Z(q)$ fails at some point of X, the study of the boundary behavior of the Bergman kernel $B^{(q)}(\cdot,\cdot)$ is a very interesting problem.

The case when $q = 0$ and the Levi form is positive definite on X (so $Z(0)$ fails) is especially a classical subject with a rich history. After the seminal paper [Bergmann 1933], Hörmander [1965, Theorem 3.5.1] (see also [Diederich 1970]) determined the limit of $B^{(0)}(x,x)$ when x approaches a strictly pseudoconvex point of the boundary of a domain for which the maximal $\bar\partial$ operator acting on functions has closed range.

More precisely, let $\rho \in \mathscr{C}^\infty(M')$ be a defining function of M, that is, $M = \{\rho < 0\}$, $X = \{\rho = 0\}$, and $d\rho \neq 0$ near X. We can and will assume that $|d\rho| = 1$ on the boundary X. Let $x_0 \in X$ be a point where the Levi form $\mathcal{L}_{x_0}(\rho)$ is positive definite. Then we have[1]

$$(-\rho(x))^{n+1}B^{(0)}(x,x) \to 2^{\frac{-n+1}{2}}\frac{n!}{4\pi^n}\det\mathcal{L}_{x_0}(\rho), \quad x \to x_0. \tag{1-1}$$

There are many extensions and variations of Hörmander's asymptotics for weakly pseudoconvex or hyperconvex domains; see, e.g., [Boas et al. 1995; Catlin 1989; Hsiao and Savale 2022; Nagel et al. 1989; Ohsawa 1984].

The existence of the complete asymptotic expansion $B^{(0)}(x,x)$ at the boundary was obtained by Fefferman [1974] on the diagonal; namely, there are functions $a, b \in \mathscr{C}^\infty(\overline{M})$ such that

$$B^{(0)}(x,x) = a(x)(-\rho(x))^{-(n+1)} + b(x)\log(-\rho(x)) \tag{1-2}$$

in M. Subsequently, Boutet de Monvel and Sjöstrand [1976] described the singularity of the full Bergman kernel $B^{(0)}(x,y)$ by showing that it is a Fourier integral operator with complex phase (see (1-15), (1-19)).

If $q = n-1$ and the Levi form is negative definite (so $Z(n-1)$ fails), Hörmander [2004, Theorem 4.6] obtained the corresponding asymptotics for the Bergman projection for $(0,n-1)$-forms in the distribution sense. For general $q > 0$, the first author showed in [Hsiao 2010, Part II] that if $Z(q)$ fails, the Levi form is nondegenerate on X and $\Box^{(q)}$ has L^2 closed range, the singularities of the Bergman projection on $(0,q)$-forms admits a full asymptotic expansion.

The developments about the Bergman projection mentioned above regard the points of the boundary where the Levi form is nondegenerate. For points where the Levi form is degenerate there are fewer

[1]The constant before the determinant of the Levi form here differs by rescaling from the corresponding constant in [Hörmander 1965, Theorem 3.5.1], since in this reference ρ satisfies $|d\rho| = 1/\sqrt{2}$ on the boundary.

results. For example, in [Hsiao and Savale 2022] a pointwise asymptotic expansion of the Bergman kernel of a weakly pseudoconvex domain of finite type in $\mathbb{C}^2$ was obtained.

Fix a point $p \in X$. Suppose that $Z(q)$ fails at p and the Levi form is nondegenerate near p (the Levi form can be degenerate away from p). In this work, we show that the spectral kernel of $\Box^{(q)}$ admits a full asymptotic expansion near p and the Bergman projection for $(0,q)$-forms admits an asymptotic expansion near p under a certain closed range condition. Our results are natural generalizations of the asymptotics of the Bergman kernel for strictly pseudoconvex domains by Fefferman [1974] and Boutet de Monvel and Sjöstrand [1976].

Another motivation to study the spectral kernel of $\Box^{(q)}$ comes from geometric quantization. An important question in the presence of a Lie group G acting on M' is "quantization commutes with reduction" [Guillemin and Sternberg 1982]; see [Ma 2010] for a survey. The study of G-invariant Bergman projection plays an important role in geometric quantization. If we consider a manifold with boundary as above, the $\bar\partial$-Neumann Laplacian may not have L^2 closed range but the G-invariant $\bar\partial$-Neumann Laplacian has L^2 closed range. In these cases, we can use the asymptotic expansion for the spectral kernel of $\Box^{(q)}$ to study G-invariant Bergman projection. Therefore, our results about spectral kernels for the $\bar\partial$-Neumann Laplacian could have applications in geometric quantization on complex manifolds with boundary. In [Hsiao et al. 2023], we used the asymptotic expansions of the spectral kernels for the Kohn Laplacian to study the geometric quantization on CR manifolds.

We now formulate the main results. We refer to Section 2 for some notation and terminology used here. Let (M', J) be a complex manifold of dimension n with complex structure J. We denote by $T^{1,0}M'$ the holomorphic and antiholomorphic tangent bundles of M', and by $T^{*p,q}M'$ the bundle of (p,q)-forms. We fix a J-invariant Riemannian metric $g^{TM'}$ on TM' and let $dv_{M'}$ be its volume form. We denote by $\langle \cdot \,|\, \cdot \rangle$ the pointwise Hermitian product induced by $g^{TM'}$ on the fibers of $\mathbb{C}TM'$ and by duality on $\mathbb{C}T^*M'$; hence on $T^{*p,q}M'$.

Let M be a relatively compact open subset with $\mathscr{C}^\infty$ boundary of M'. We denote by $X = \partial M$ the boundary of M. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of M with $|d\rho| = 1$ on X. Let $\frac{\partial}{\partial\rho} \in \mathscr{C}^\infty(M', TM')$ be the gradient of ρ with respect to the metric $g^{TM'}$. Then

$$d\rho\Big(\frac{\partial}{\partial\rho}\Big) = 1 \quad \text{on } X, \qquad \Big\langle \frac{\partial}{\partial\rho}(x) \,\Big|\, v \Big\rangle = 0 \quad \text{at every } x \in X, \text{ for every } v \in T_xX. \tag{1-3}$$

Put

$$T = J\Big(\frac{\partial}{\partial\rho}\Big) \in \mathscr{C}^\infty(M', TM'). \tag{1-4}$$

It is easy to see that T is orthogonal to $T^{1,0}X \oplus T^{0,1}X$ and $|T| = 1$ on X. We consider the 1-form on M',

$$\omega_0 = -d\rho \circ J = i(\bar\partial\rho - \partial\rho). \tag{1-5}$$

We have

$$\begin{aligned} \omega_0(x)(u) &= 0 \quad \text{for every } x \in X \text{ and every } u \in T_x^{1,0}X \oplus T_x^{0,1}X, \\ \omega_0(T) &= 1 \quad \text{on } X. \end{aligned} \tag{1-6}$$

For $x \in X$, the Levi form $\mathcal{L}_x$ is the Hermitian quadratic form on $T_x^{1,0}X$ given by

$$\mathcal{L}_x(Z, \overline{W}) = \frac{1}{2i} d\omega_0(x)(Z, \overline{W}) = \partial\bar\partial\rho(x)(Z, \overline{W}), \quad Z, W \in T_x^{1,0}X. \tag{1-7}$$

For a given point $x \in X$ let $\{W_j\}_{j=1}^{n-1}$ be an orthonormal frame of $(T^{1,0}X, \langle\cdot\,|\,\cdot\rangle)$ near x for which the Levi form is diagonal at x. We define the eigenvalues $\mu_j(x)$, $j = 1, \dots, n-1$, of the Levi form $\mathcal{L}_x$ by

$$\mathcal{L}_x(W_j, \overline{W}_\ell) = \mu_j(x)\delta_{j\ell}, \quad j, \ell = 1, \dots, n-1. \tag{1-8}$$

The determinant of the Levi form at x is denoted by

$$\det \mathcal{L}_x = \prod_{j=1}^{n-1} \mu_j(x). \tag{1-9}$$

For every $q = 0, 1, \dots, n-1$, let $T^{*0,q}X$ be the bundle of $(0, q)$-forms on X. We assume that $\mu_j(x) < 0$ if $1 \le j \le n_-$ and $\mu_j(x) > 0$ if $n_- + 1 \le j \le n-1$. Let $\{e_j\}_{j=1}^{n-1}$ denote the basis of $T^{*0,1}X$, dual to $\{\overline{W}_j\}_{j=1}^{n-1}$. Put

$$\mathcal{N}(x, n_-) := \mathbb{C}e_1(x) \wedge \cdots \wedge e_{n_-}(x), \tag{1-10}$$

and let

$$\tau_{x,n_-} : T_x^{*0,n_-}X \to \mathcal{N}(x, n_-) \tag{1-11}$$

be the orthogonal projection onto $\mathcal{N}(x_0, n_-)$ with respect to $\langle\cdot\,|\,\cdot\rangle$.

Fix $x \in M'$. Let $L \in T_x^{*0,1}M'$ and let $L^\wedge : T_x^{*0,q}M' \to T_x^{*0,q+1}M'$ be the operator with wedge multiplication by L and let $L^{\wedge,*} : T_x^{*0,q+1}M' \to T_x^{*0,q}M'$ be its adjoint with respect to $\langle\cdot\,|\,\cdot\rangle$, that is,

$$\langle L \wedge u \,|\, v \rangle = \langle u \,|\, L^{\wedge,*}v \rangle, \quad u \in T_x^{*0,q}M',\ v \in T_x^{*0,q+1}M'. \tag{1-12}$$

Let $(\cdot\,|\,\cdot)_M$ be the L^2 inner product on $\Omega^{0,q}(\overline{M})$ induced by $\langle\cdot\,|\,\cdot\rangle$ (see (2-7)). Let $L^2_{(0,q)}(M)$ be the completion of $\Omega^{0,q}(\overline{M})$ with respect to $(\cdot\,|\,\cdot)_M$. Let

$$\Box^{(q)} : \operatorname{Dom} \Box^{(q)} \subset L^2_{(0,q)}(M) \to L^2_{(0,q)}(M), \quad q \in \{0, 1, \dots, n-1\},$$

be the $\bar\partial$-Neumann Laplacian on $(0, q)$-forms (see (2-8)). The operator $\Box^{(q)}$ is a nonnegative self-adjoint operator. We denote by $\mathcal{E}^{(q)}$ the spectral measure of $\Box^{(q)}$. For a Borel set $B \subset \mathbb{R}$, $\mathcal{E}^{(q)}(B)$ is the spectral projection of $\Box^{(q)}$ corresponding to the set B. For $\lambda \ge 0$ we consider the spectral projectors,

$$B^{(q)}_{\le\lambda} := \mathcal{E}^{(q)}\big((-\infty, \lambda]\big) : L^2_{(0,q)}(M) \to \mathcal{H}^q_{\le\lambda}(\overline{M}) := \operatorname{Ran} B^{(q)}_{\le\lambda}, \tag{1-13}$$

and denote by

$$B^{(q)}_{\le\lambda}(x, y) \in \mathscr{D}'(M \times M, T^{*0,q}M \boxtimes (T^{*0,q}M)^*)$$

their distribution kernels. For $\lambda = 0$ we obtain the *Bergman projection* $B^{(q)} := B^{(q)}_{\le 0}$, the *Bergman kernel* $B^{(q)}(x, y) := B^{(q)}_{\le 0}(x, y)$ and the space of harmonic forms $\mathcal{H}^q(\overline{M}) := \mathcal{H}^q_{\le 0}(\overline{M}) = \operatorname{Ker} \Box^{(q)}$. Let us define

$$\Lambda^{(0,q)|(0,q)}_{M'\times M'} := T^{*0,q}M' \boxtimes (T^{*0,q}M')^*$$

and set, for $W \subset M' \times M'$ open,

$$\Omega^{(0,q)|(0,q)}(W) := \mathscr{C}^\infty(W, \Lambda^{(0,q)|(0,q)}_{M'\times M'}) = \mathscr{C}^\infty(W, T^{*0,q}M' \boxtimes (T^{*0,q}M')^*),$$
$$\Omega^{(0,q)|(0,q)}(W \cap (\overline{M} \times \overline{M})) := \mathscr{C}^\infty(W \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M'\times M'}).$$

Let U be an open set of M' with $U \cap X \neq \emptyset$. We shall consider $B^{(q)}_{\leq\lambda}$ as a continuous operator,

$$B^{(q)}_{\leq\lambda} : \Omega^{0,q}_c(U \cap M) \to \mathscr{D}'(U \cap M, T^{*0,q}M'),$$

and let $B^{(q)}_{\leq\lambda}(x, y) \in \mathscr{D}'((U \times U) \cap (M \times M), \Lambda^{(0,q)\,|\,(0,q)}_{M'\times M'})$ be the distribution kernel of $B^{(q)}_{\leq\lambda}$. We denote in the sequel by $S^n_{1,0}$ the Hörmander symbol space. Our first main result is the following.

Theorem 1.1. *Let $M = \{\rho < 0\}$ be a relatively compact open subset with smooth boundary X of a complex manifold M' of complex dimension n. Let U be an open set of M' with $U \cap X \neq \emptyset$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$, where n_- denotes the number of negative eigenvalues of the Levi form on $U \cap X$. Fix $\lambda > 0$. If $q \neq n_-$, we have*

$$B^{(q)}_{\leq\lambda}(x, y) \in \Omega^{(0,q)\,|\,(0,q)}((U \times U) \cap (\overline{M} \times \overline{M})), \tag{1-14}$$

and for $q = n_-$ the operator $B^{(q)}_{\leq\lambda}$ is a Fourier integral operator with complex phase. More precisely,

$$B^{(q)}_{\leq\lambda}(x, y) - \int_0^\infty e^{i\phi(x,y)t} b(x, y, t)\, dt \in \Omega^{(0,q)\,|\,(0,q)}((U \times U) \cap (\overline{M} \times \overline{M})), \tag{1-15}$$

where $b(x, y, t) \in S^n_{1,0}((U \times U) \cap (\overline{M} \times \overline{M}) \times (0, \infty), \Lambda^{(0,q)\,|\,(0,q)}_{M'\times M'})$ has asymptotic expansion $b(x, y, t) \sim \sum_{j=0}^\infty b_j(x, y)t^{n-j}$ in $S^n_{1,0}$, $b_j(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)\,|\,(0,q)}_{M'\times M'})$, $j = 0, 1, \dots,$ and the leading term is given by

$$b_0(x, x) = 2\pi^{-n}|\det \mathcal{L}_x|\tau_{x,n_-} \circ (\bar\partial\rho(x))^{\wedge,*}(\bar\partial\rho(x))^{\wedge} \quad \textit{for every } x \in U \cap X. \tag{1-16}$$

Moreover,

$$\begin{gathered} \phi(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \quad \operatorname{Im}\phi \geq 0, \\ \phi(x, x) = 0, \quad x \in U \cap X, \quad \phi(x, y) \neq 0 \quad \textit{if } (x, y) \notin \operatorname{diag}((U \times U) \cap (X \times X)), \\ \operatorname{Im}\phi(x, y) > 0 \quad \textit{if } (x, y) \notin (U \times U) \cap (X \times X), \\ \phi(x, y) = -\overline{\phi(y, x)}, \\ d_x\phi(x, y)|_{x=y} = -2i\,\partial\rho(x) \quad \textit{for every } x \in U \cap X. \end{gathered} \tag{1-17}$$

Moreover, we can describe the phase function ϕ from (1-15) in the following complement to Theorem 1.1. Let $\bar\partial^*_f$ denote the formal adjoint of $\bar\partial$, and let $\Box^{(q)}_f := \bar\partial^*_f\,\bar\partial + \bar\partial\,\bar\partial^*_f$ be the $\bar\partial$-Laplacian acting on $\Omega^{0,*}(M')$. We denote by $\sigma(\Box^{(q)}_f)$ its principal symbol.

Zusatz 1.2. Fix $p \in U \cap X$ and choose local holomorphic coordinates $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, vanishing at p such that the metric on $T^{1,0}M'$ is $\sum_{j=1}^n dz_j \otimes d\bar z_j$ at p and $\rho(z) = \sqrt{2}\operatorname{Im} z_n + \sum_{j=1}^{n-1}\mu_j|z_j|^2 + O(|z|^3)$, where μ_j, $j = 1, \dots, n-1$, are the eigenvalues of $\mathcal{L}_p$. We also write $w = (w_1, \dots, w_n)$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$. Then, we can take $\phi(z, w)$ in (1-15) so that in some neighborhood of (p,p) in $M'\times M'$ we have

$$\phi(z,w)=-\sqrt{2}x_{2n-1}+\sqrt{2}y_{2n-1}-i\rho(z)\Big(1+\sum_{j=1}^{2n-1}a_jx_j+\frac{1}{2}a_{2n}x_{2n}\Big)-i\rho(w)\Big(1+\sum_{j=1}^{2n-1}\bar{a}_jy_j+\frac{1}{2}\bar{a}_{2n}y_{2n}\Big)$$
$$+i\sum_{j=1}^{n-1}\lvert\mu_j\rvert\lvert z_j-w_j\rvert^2+\sum_{j=1}^{n-1}i\mu_j(\bar{z}_jw_j-z_j\bar{w}_j)+O\big(\lvert(z,w)\rvert^3\big),\quad (1\text{-}18)$$

where $a_j=\frac{1}{2}\partial_{x_j}\sigma(\Box_f^{(q)})(p,-2i\partial\rho(p)),\ j=1,\dots,2n.$

The essential step in the proof of Theorem 1.1 is the construction of a microlocal Hodge decomposition (Theorems 5.9, 5.23) up to smoothing operators. Namely, there exists an approximate Neumann operator $N^{(q)}$ and an approximate Bergman operator Π^q on $U\cap\overline{M}$ such that $\Box^{(q)}N^{(q)}+\Pi^{(q)}-I$, $N^{(q)}\Box^{(q)}+\Pi^{(q)}-I$, $\Box^{(q)}\Pi^{(q)}$ are smoothing on $U\cap\overline{M}$ (here I denotes the identity) and $\Pi^{(q)}$ differs from the Fourier integral operator $\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt$ by a smoothing operator on $U\cap\overline{M}$. In Theorem 6.7 we prove that, for every $\lambda>0$, $B^{(q)}_{\le\lambda}-\Pi^{(q)}$ is smoothing on $U\cap\overline{M}$. Since $\Pi^{(q)}$ is independent of λ, the complex Fourier integral operator $\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt$ in (1-15) can be taken to be independent of λ. Hence, for every $\lambda_1>\lambda>0$, $B^{(q)}_{\le\lambda_1}(x,y)$ and $B^{(q)}_{\le\lambda}(x,y)$ differ by a smooth section on $(U\times U)\cap(\overline{M}\times\overline{M})$.

By integrating over t in the oscillatory integral $\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt$ in (1-15), we have the following corollary of Theorem 1.1.

Corollary 1.3. *Under the assumptions of Theorem 1.1, let U be an open set of M' with $U\cap X\neq\varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-,n_+) on $U\cap X$. Let $q=n_-$. There exist forms $F,G\in\Omega^{(0,q)\vert(0,q)}((U\times U)\cap(\overline{M}\times\overline{M}))$ such that for every $\lambda>0$ we have*

$$B^{(q)}_{\le\lambda}(x,y)=F(x,y)(-i(\phi(x,y)+i0))^{-n-1}+G(x,y)\log(-i(\phi(x,y)+i0))+R_\lambda(x,y),\quad (1\text{-}19)$$

where $R_\lambda(x,y)\in\Omega^{(0,q)\vert(0,q)}((U\times U)\cap(\overline{M}\times\overline{M}))$ is a λ-dependent smooth form. Moreover, we have

$$\begin{aligned}F(x,y)&=\sum_{j=0}^{n}(n-j)!b_j(x,y)(-i\phi(x,y))^j,\\ G(x,y)&\sim\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}b_{n+j+1}(x,y)(-i\phi(x,y))^j \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\end{aligned}\quad (1\text{-}20)$$

where $b_j(x,y)$, $j\in\mathbb{N}_0$, and $\phi(x,y)$ are as in Theorem 1.1.

We introduce now a condition which allows us to pass from spectral projections $B^{(q)}_{\le\lambda}$ with $\lambda>0$ to the Bergman projector $B^{(q)}=B^{(q)}_{\le 0}$.

Definition 1.4. Let U be an open set in M' with $U\cap X\neq\varnothing$. We say that $\Box^{(q)}$ has local closed range in U if, for every open set $W\subset U$ with $W\cap X\neq\varnothing$, $\overline{W}\subset U$, there is a constant $C_W>0$ such that

$$\lVert(I-B^{(q)})u\rVert_M\le C_W\lVert\Box^{(q)}u\rVert_M,\quad u\in\Omega_c^{0,q}(W\cap\overline{M})\cap\operatorname{Dom}\Box^{(q)}.$$

Note that if $\Box^{(q)}$ has closed range then $\Box^{(q)}$ has local closed range in U for any U.

Our second main result is the following.

Theorem 1.5. *Under the assumptions of Theorem 1.1, let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Let $q = n_-$. Suppose that $\Box^{(q)}$ has local closed range in U. Then*

$$B^{(q)}(x, y) - \int_0^\infty e^{i\phi(x,y)t} b(x, y, t)\, dt \in \Omega^{(0,q)|(0,q)}((U \times U) \cap (\overline{M} \times \overline{M})), \tag{1-21}$$

where $b(x, y, t)$ and $\phi(x, y)$ are as in Theorem 1.1. In particular, $B^{(q)}(x, y)$ has the asymptotics (1-19).

Hörmander [2004, Theorem 4.6] determined the leading asymptotics of $B^{(n-1)}(x, y)$ near a boundary point where the Levi form is negative definite under the condition that $\Box^{(n-1)}$ has closed range. Theorem 1.5 thus generalizes this result and gives the full asymptotics.

Remark 1.6. Let (E, h^E) be a Hermitian holomorphic vector bundle over M'. As in (2-8) below, we can consider the $\bar\partial$-Neumann Laplacian on $(0, q)$-forms with values in E:

$$\Box^{(q)} = \bar\partial\, \bar\partial^* + \bar\partial^*\, \bar\partial : \mathrm{Dom}\, \Box^{(q)} \subset L^2_{(0,q)}(M, E) \to L^2_{(0,q)}(M, E), \tag{1-22}$$

where $L^2_{(0,q)}(M, E)$ denotes the L^2 space of $(0, q)$-forms with values in E. We can define $B^{(q)}_{\leq\lambda}(x, y)$ in the same way as above and by the same proofs, Theorems 1.1 and 1.5 hold also in the presence of a vector bundle E.

In particular, we can consider the trivial line bundle $E = \mathbb{C}$ with the metric $h^E = e^{-\varphi}$, where $\varphi \in \mathscr{C}^\infty(M')$ is a weight function. In this case the space $L^2_{(0,q)}(M, E)$ is the completion of $\Omega^{0,q}(\overline{M})$ with respect to the weighted L^2 inner product $(u|v)_\varphi = \int_M \langle u|v\rangle e^{-\varphi}\, dv_{M'}$ and is denoted by $L^2_{(0,q)}(M, \varphi)$. The Bergman projection is denoted by $B^{(q)}_\varphi$ and the Bergman kernel by $B^{(q)}_\varphi(\cdot, \cdot)$. So all the results above have versions for *weighted Bergman kernels* $B^{(q)}_\varphi(\cdot, \cdot)$.

We now give some applications of the results above.

Corollary 1.7. (i) *Let M be a bounded domain of holomorphy in $\mathbb{C}^n$ with smooth boundary and let φ be any function in $\mathscr{C}^\infty(\overline{M})$. Let U be an open set in $\mathbb{C}^n$ such that $U \cap \partial M$ is strictly pseudoconvex. Then the weighted Bergman kernel $B^{(0)}_\varphi(\cdot, \cdot)$ has the asymptotics (1-21) on $U \cap \overline{M}$.*

(ii) *Let M be an open relatively compact domain with smooth boundary X in a complex manifold M' of dimension n. Assume that X satisfies condition $Z(1)$, i.e., the Levi form of X has everywhere either $n-1$ positive or two negative eigenvalues. Let U be an open set in M' such that $U \cap X$ is strictly pseudoconvex. Then the Bergman kernel $B^{(0)}(\cdot, \cdot)$ has the asymptotics (1-21) on $U \cap \overline{M}$.*

(iii) *Let M be a pseudoconvex domain with smooth boundary in $\mathbb{P}^n$. Let U be an open set in $\mathbb{P}^n$ such that $U \cap \partial M$ is strictly pseudoconvex. Then the Bergman kernel $B^{(0)}(\cdot, \cdot)$ has the asymptotics (1-21) on $U \cap \overline{M}$.*

(iv) *Let M be an open relatively compact domain with smooth boundary X in a complex manifold M' of dimension n. Fix $p \in X$ and assume that the Levi form is nondegenerate of constant signature (n_-, n_+) at every point of $U \cap X$, where U is an open set of p in M'. Let $q = n_-$. Assume that $Z(q-1)$, $Z(q+1)$ hold of every point of X. The Bergman kernel $B^{(q)}(\cdot, \cdot)$ has the asymptotics (1-21) on $U \cap \overline{M}$.*

Note that by the solution of the Levi problem [Range 1986, Theorem V.1.5], for a bounded domain $M \subset \mathbb{C}^n$ with smooth boundary the notions of domain of holomorphy and weak (Levi) pseudoconvexity are equivalent. We can apply the L^2 estimates for $\bar\partial$ of [Hörmander 1965, Theorem 2.2.1′] to obtain that $\Box^{(0)}$ has closed range in L^2, and hence settle case (i). Note that the analogous L^2 estimate for $\bar\partial_b$ along the boundary was done in [Shaw 1985]. Moreover, it follows from [Folland and Kohn 1972, Theorem 3.1.19], [Hörmander 1965, Theorem 3.4.1] in case (ii), and [Henkin and Iordan 2000, Corollary 3.6] in case (iii), that $\Box^{(0)}$ has closed range. Note that these assertions are independent of the choice of the function $\varphi \in \mathscr{C}^\infty(\overline{M})$, since changing φ only means introducing equivalent norms in the Hilbert spaces concerned. Obviously, the items (i) and (ii) hold also if we work with Bergman kernels of holomorphic sections in a Hermitian holomorphic vector bundle (E, h^E) defined in a neighborhood of $\overline{M}$ (see Remark 1.6).

We now explain point (iv). Let M be an open relatively compact domain with smooth boundary X in a complex manifold M' of dimension n. We recall that X satisfies condition $Z(q)$ if the Levi from of X has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at every point of X. It was proved in [Folland and Kohn 1972, Proposition 3.1.18] that if $Z(q-1)$, $Z(q+1)$ hold at every point of X, then $\Box^{(q)}$ has closed range. If the Levi form is nondegenerate of signature (n_-, n_+) then $Z(q)$ holds if and only if $q \neq n_-$. We call n_- the critical degree.

Next we consider Bergman kernels on shell domains. These are domains with two boundary components, one pseudoconvex, the other pseudoconcave They appear for example in Andreotti–Grauert theory, e.g., as $(1,1)$-convex-concave domains (roughly speaking of the form $M = \{c \le \varphi \le d\}$, where $\varphi : M' \to \mathbb{R}$ is a strictly plurisubharmonic exhaustion function on M'). Such domains play an important role in problems of compactification of complex manifolds; see, e.g., [Andreotti and Siu 1970].

Corollary 1.8. *Let $M \Subset \mathbb{C}^n$ be the shell domain $M = M_0 \setminus \overline{M}_1$ between two pseudoconvex domains M_0 and M_1 with smooth boundary and $M_1 \Subset M_0$. Let U an open set such that $U \cap \partial M_1$ is strictly pseudoconvex and $U \cap \partial M_0 = \varnothing$. Then the Bergman kernel $B^{(n-1)}(x, y)$ on $(0, n-1)$-forms has the asymptotics (1-21) and (1-19).*

By [Shaw 2010, Theorem 3.5], the operator $\Box^{(n-1)}$ has closed range in L^2 for a shell domain between two pseudoconvex domains as above. Moreover, the Levi form of ∂M is negative definite on $U \cap \partial M$, so the corollary follows from Theorem 1.5.

We consider further shell domains $M = M_0 \setminus \overline{M}_1$ in a complex manifold M'. For general shell domains, e.g., $(1,1)$-convex-concave domains, the associated $\bar\partial$-Neumann Laplacian may not have closed range. This happens for example for domains which cannot be compactified on the pseudoconcave end [Andreotti and Siu 1970] (the pseudoconcave boundary component is not embeddable in the Euclidean space). To overcome this difficulty, we consider a holomorphic line bundle L over M'. In Theorem 1.9 below, we will see that the associated $\bar\partial$-Neumann Laplacian with values in L^k has closed range if k is large and the curvature of L is positive. We refer to [Ma and Marinescu 2007] for a comprehensive study of Bergman kernel asymptotics for high tensor powers of a line bundles.

Suppose that there is a holomorphic line bundle (L, h^L) over M', where h^L denotes a Hermitian metric of L and R^L is the curvature of L induced by h^L. For every $k \in \mathbb{N}$, let (L^k, h^{L^k}) be the k-th power of (L, h^L). Let $(\cdot\,|\,\cdot)_k$ be the L^2 inner product on $\Omega^{0,q}(M, L^k)$ induced by the given Hermitian metric

$\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TM'$ and h^L and let $L^2_{(0,q)}(M,L^k)$ be the completion of $\Omega^{0,q}(M,L^k)$. Let

$$\Box^{(q)}_k:\bar\partial\,\bar\partial^*+\bar\partial^*\,\bar\partial:\mathrm{Dom}\,\Box^{(q)}\subset L^2_{(0,q)}(M,L^k)\to L^2_{(0,q)}(M,L^k)$$

be the $\bar\partial$-Neumann operator on M with values in L^k and let

$$B^{(q)}_k:L^2_{(0,q)}(M,L^k)\to\mathrm{Ker}\,\Box^{(q)}_k$$

be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_k$ and let $B^{(q)}_k(\cdot\,,\cdot)$ be the distribution kernel of $B^{(q)}_k$.

Theorem 1.9. *Let $M=M_0\setminus\overline{M}_1$ be the shell domain between two pseudoconvex domains M_0 and M_1 with smooth boundary, $M_1\Subset M_0\Subset M'$. Let $X_0=\partial M_0$ and $X_1=\partial M_1$. Assume that (L,h^L) is a positive line bundle in a neighborhood of $\overline{M}_0$. Let U be an open set in M' with of $U\cap X_0\neq\varnothing$ and $U\cap X_1=\varnothing$. There exists $k_0\in\mathbb{N}$ such that, for every $k\in\mathbb{N}$, $k\geq k_0$,*

$$\Box^{(0)}_k \text{ has local closed range in } U. \tag{1-23}$$

Moreover, for every $k\in\mathbb{N}$, $k\geq k_0$, the Bergman kernel of M with values in L^k satisfies

$$B^{(0)}_k(x,y)\equiv\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}),L^k\boxtimes(L^k)^*), \tag{1-24}$$

where $\phi(x,y)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $b(x,y,t)\in S^n_{1,0}(((U\times U)\cap(\overline{M}\times\overline{M}))\times(0,\infty),L^k\boxtimes(L^k)^*)$ are as in Theorem 1.1.

The next applications concerns the asymptotics of the S^1-equivariant Bergman kernel and embedding theorems. We assume that M' admits a holomorphic S^1-action

$$S^1\times M'\to M',\quad (e^{i\theta},x)\mapsto e^{i\theta}\circ x.$$

The S^1-action preserves the complex structure J of M'. Let $T_0\in\mathscr{C}^\infty(M',TM')$ be the infinitesimal generator of the S^1-action on M', that is $(T_0u)(x)=\frac{\partial}{\partial\theta}u(e^{i\theta}\circ x)\big|_{\theta=0}$ for every $u\in\mathscr{C}^\infty(M')$.

We take the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TM'$ to be S^1-invariant and $|T_0|=1$ on X. We take an S^1-invariant defining function ρ so that $|d\rho|=1$ on X. Fix an open connected component X_0 of X. Suppose that

$$\omega_0(T_0)>0\quad\text{on } X_0, \tag{1-25}$$

where J is the complex structure map on T^*M'. From (1-4), (1-25) and noting that $|T_0|=|d\rho|=1$ on X, it is easy to see that

$$T_0=T\quad\text{on } X_0. \tag{1-26}$$

For every $m\in\mathbb{Z}$, put

$$\Omega^{0,q}_m(M')=\{u\in\Omega^{0,q}(M'):Tu=imu\}, \tag{1-27}$$

where Tu is the Lie derivative of u along direction T. Similarly, let $\Omega^{0,q}_m(\overline{M})$ denote the space of restrictions to M of elements in $\Omega^{0,q}_m(M')$. We write $\mathscr{C}^\infty_m(\overline{M}):=\Omega^{0,0}_m(\overline{M})$. Let $L^2_{(0,q),m}(M)$ be the completion of $\Omega^{0,q}_m(\overline{M})$ with respect to $(\cdot\,|\,\cdot)_M$. For $q=0$, we write $L^2_m(M):=L^2_{(0,0),m}(M)$. Fix $\lambda\geq 0$ and $m\in\mathbb{Z}$. Put

$$\mathcal{H}^q_{\leq\lambda,m}(\overline{M}):=\mathcal{H}^q_{\leq\lambda}(\overline{M})\cap L^2_{(0,q),m}(M), \tag{1-28}$$

where $\mathcal{H}^q_{\leq\lambda}(\overline{M})$ is given by (1-13). Let

$$B^{(q)}_{\leq\lambda,m}:L^2_{(0,q)}(M)\to\mathcal{H}^q_{\leq\lambda,m}(\overline{M}) \tag{1-29}$$

be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_M$ and let

$$B^{(q)}_{\leq\lambda,m}(x,y)\in\mathscr{D}'(M\times M,\Lambda^{(0,q)\boxtimes(0,q)}_{M'\times M'})$$

be the distribution kernel of $B^{(q)}_{\leq\lambda,m}$. For $\lambda=0$, we write $\mathcal{H}^q_m(\overline{M}):=\mathcal{H}^q_{\leq 0,m}(\overline{M})$, $B^{(q)}_m:=B^{(q)}_{\leq 0,m}$, $B^{(q)}_m(x,y):=B^{(q)}_{\leq 0,m}(x,y)$. From [Hsiao et al. 2020, Theorem 3.3], we see that $\mathcal{H}^q_{\leq\lambda,m}(\overline{M})$ is a finite-dimensional subspace of $\Omega^{0,q}_m(\overline{M})$ and hence

$$B^{(q)}_{\leq\lambda,m}(x,y)\in\Omega^{(0,q)\boxtimes(0,q)}(\overline{M}\times\overline{M}).$$

Moreover, it is straightforward to see that

$$B^{(q)}_{\leq\lambda,m}(x,y)=\frac{1}{2\pi}\int_{-\pi}^{\pi}B^{(q)}_{\leq\lambda}(x,e^{i\theta}\circ y)e^{im\theta}\,d\theta. \tag{1-30}$$

We have the following asymptotic expansion for the S^1-equivariant Bergman kernel. We use here the symbol spaces S^n_{loc}; see Definition 2.1 and the discussion after (2-6).

Theorem 1.10. *Assume that M' admits a holomorphic S^1-action that is boundary preserving, locally free and transversal to the CR structure on the boundary. Let X_0 be a connected component of X such that* (1-26) *holds, let $p\in X_0$ and let U be an open set of p in M' with $U\cap X_0\neq\varnothing$. Suppose that $Z(1)$ holds on X and that the Levi form is positive $U\cap X_0$. Let $N_p:=\{g\in S^1:g\circ p=p\}=\{g_0:=e,g_1,\dots,g_r\}$, where e denotes the identity element in S^1 and $g_j\neq g_\ell$ if $j\neq\ell$ for every $j,\ell=0,1,\dots,r$. Then*

$$B^{(0)}_m(x,y)\equiv\sum_{\alpha=0}^{r}g^m_\alpha e^{im\phi(x,g_\alpha\circ y)}b_\alpha(x,y,m)\ \mathrm{mod}\ O(m^{-\infty})\quad\text{on } U\cap\overline{M}, \tag{1-31}$$

where, for every $\alpha=0,1,\dots,r$,

$$\begin{gathered}b_\alpha(x,y,m)\in S^n_{\mathrm{loc}}((U\times U)\cap(\overline{M}\times\overline{M})),\\ b_\alpha(x,y,m)\sim\textstyle\sum_{j=0}^{\infty}b_{\alpha,j}(x,y)m^{n-j}\quad\text{in } S^n_{\mathrm{loc}}((U\times U)\cap(\overline{M}\times\overline{M})),\\ b_{\alpha,0}(x,x)=b_0(x,x),\end{gathered} \tag{1-32}$$

where $b_0(x,x)$ is given by (5-124) *and $\phi(x,y)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in* (1-17).

Actually, we have more general results than Theorem 1.10. In Theorem 8.2, we get an asymptotic expansion for $B^{(q)}_{\leq\lambda,m}$ in m for every $\lambda>0$, and in Theorem 8.3, we get an asymptotic expansion for $B^{(q)}_m$ in m under the local closed range condition of $\Box^{(q)}$. Moreover, when $Z(q-1)$ and $Z(q+1)$ hold, then $\Box^{(q)}$ has closed range and an analogous statement to Theorem 1.10 holds for $B^{(q)}_m$.

For every $m \in \mathbb{N}$, let

$$\Phi_m : \overline{M} \to \mathbb{C}^{d_m}, \quad x \mapsto (f_1(x), \dots, f_{d_m}(x)), \tag{1-33}$$

where $\{f_1(x), \dots, f_{d_m}(x)\}$ is an orthonormal basis for $\mathcal{H}^0_m(\overline{M})$ with respect to $(\cdot \mid \cdot)_M$ and $d_m = \dim \mathcal{H}^0_m(\overline{M})$. We have the following S^1-equivariant embedding theorem

Theorem 1.11. *Assume that M' admits a holomorphic S^1-action that is boundary preserving, locally free and transversal to the CR structure on the boundary. Let X_0 be a connected component of X such that (1-26) holds. Assume that the Levi form is positive definite on X_0 and $Z(1)$ holds on X. For every $m_0 \in \mathbb{N}$, there exist $m_1, \dots, m_k \in \mathbb{N}$, with $m_j \geq m_0$, $j = 1, \dots, k$, and an S^1-invariant open neighborhood V of X_0 such that the map*

$$\Phi_{m_1,\dots,m_k} : V \cap \overline{M} \to \mathbb{C}^{\hat{d}_m}, \quad x \mapsto (\Phi_{m_1}(x), \dots, \Phi_{m_k}(x)), \tag{1-34}$$

is a holomorphic embedding, where Φ_{m_j} is given by (1-33) and $\hat{d}_m = d_{m_1} + \dots + d_{m_k}$.

Without the $Z(1)$ condition, we can still formulate the following S^1-equivariant embedding theorem.

Theorem 1.12. *Assume that M' admits a holomorphic S^1-action that is boundary preserving, locally free and transversal to the CR structure on the boundary. Let X_0 be a connected component of X such that (1-26) holds and the Levi form is positive definite on X_0. For every $m_0 \in \mathbb{N}$, there exist an S^1-invariant open neighborhood V of X_0 and $f_j \in \mathscr{C}^\infty(V \cap \overline{M})$ with $\bar\partial f_j = 0$ on $V \cap \overline{M}$, $f_j(e^{i\theta}x) = e^{im_j\theta} f(x)$, $j = 1, \dots, k$, for $e^{i\theta} \in S^1$ and every $x \in V$ and some $m_j \geq m_0$, such that the map*

$$\Phi : V \cap \overline{M} \to \mathbb{C}^k, \quad x \mapsto (f_1(x), \dots, f_k(x)), \tag{1-35}$$

is a holomorphic embedding.

1.2. ***Methods and further previous results.*** In [Hsiao 2010] the first author extended the results of the fundamental paper [Boutet de Monvel and Sjöstrand 1976] on the off-diagonal and boundary asymptotics of the Szegő and Bergman kernels to the case of domains whose Levi form is everywhere nondegenerate on the boundary. Building on [Hsiao 2010] we constructed in [Hsiao and Marinescu 2017] a parametrix for the Szegő kernel on the boundary, and extended the above results in several directions: (i) the global nondegeneracy condition on the Levi form was relaxed to local nondegeneracy near the point where the parametrix is being constructed; (ii) a more general projector onto low-energy eigenspaces of the Kohn Laplacian was considered; (iii) the boundary and domain were allowed to be noncompact. In the present paper we achieve the passage from the Szegő parametrix on the boundary to the Bergman parametrix in the interior.

The main technical part of this paper is the construction of the microlocal parametrices for the $\bar\partial$-Neumann problem done in Sections 4 and 5 (see Theorems 4.7, 5.9, 5.23). More precisely, in Section 4, we construct parametrices for $\Box^{(q)}$ near a point $p \in X$ under the assumptions that $Z(q)$ holds at p and the Levi form is nondegenerate at p. Our result generalizes the global result [Folland and Kohn 1972, Theorem 3.1.14] (see also [Kohn 1963; 1964]) about the solution of the $\bar\partial$-Neumann problem under the hypothesis that the $Z(q)$ condition holds on the whole boundary. In this case $\Box^{(q)}$ has a parametrix $N^{(q)}$, the $\bar\partial$-Neumann operator, which has a local character. Our method uses a reduction to the analysis on the boundary and the use of a boundary pseudodifferential operator $\Box_-^{(q)}$ which is elliptic along the negative component $\Sigma_- \subset T^*X$ of the characteristic cone (see Section 3).

In Section 5, we construct microlocal Hodge decomposition theorems for $\Box^{(q)}$ near a point $p \in X$ under the assumptions that $Z(q)$ fails at p and the Levi form is nondegenerate at p. This is the most technical part of the paper. Again, this is the local counterpart of the global result [Folland and Kohn 1972, Proposition 3.1.17] saying that if $Z(q)$ fails but $Z(q-1)$ and $Z(q+1)$ hold on X, there exists a global Hodge decomposition theorem for $\Box^{(q)}$. Our method is to first construct a parametrix $N^{(q)}$ of the $\bar\partial$-Neumann Laplacian and an approximate Bergman projector $\Pi^{(q)}$, then to link $\Pi^{(q)}$ to an approximate Szegő projector, which turns out to be a Fourier operator with complex phase, on the boundary via the Poisson operator. Note that already in [Boutet de Monvel and Sjöstrand 1976] the analysis of the Bergman projector on a strictly pseudoconvex domain was done by reduction to the Szegő projector.

The localization of the $\bar\partial$-Neumann operator was observed in several papers under global assumptions. It was remarked in [Folland and Kohn 1972, p. 52] that the $\bar\partial$-Neumann operator localizes assuming that $\Box^{(q)}$ has globally closed range (see also Theorem 3.6 and Remark (ii) on page 70 of [Straube 2010]). Near a strictly pseudoconvex point ($n_- = 0$), the existence of the localized $\bar\partial$-Neumann operator in Theorems 4.7 and 5.9 follows from the main results of [Henkin et al. 1996; Henkin and Iordan 1997; Michel and Shaw 1998], under various hypotheses, such as piecewise smooth boundary. The generalizations of these articles for higher q have been considered in [Hefer and Lieb 2000, Theorem 3.16].

As mentioned above, a geometric counterpart of the condition $Z(q)$ is the notion of q-convexity [Andreotti and Grauert 1962]. A manifold M of dimension n is called q-convex ($1 \leq q \leq n$) if there exists an exhaustion function $\varphi : M \to \mathbb{R}$ such that its Levi form $i\partial\bar\partial\varphi$ has $n-q+1$ positive eigenvalues outside a compact set K. If $c \in \mathbb{R}$ is a regular value of φ such that $M_c := \{x \in M : \varphi(x) \leq c\} \Subset M$ contains K, then M_c satisfies condition $Z(\ell)$ for every $\ell \geq q$. By [Andreotti and Grauert 1962], if M is q-convex then the cohomology $H^\ell(M, E)$ with values in any holomorphic bundle E is finite-dimensional for any $\ell \geq q$. This can be also deduced from the fact that the $\dim \operatorname{Ker} \Box^{(\ell)} < \infty$ for $\ell \geq q$ and from Hodge theory of the $\bar\partial$-Neumann Laplacian; see [Hörmander 1965]. If M is a domain such that the Levi form of the boundary is nondegenerate of signature (n_-, n_+), it follows from Andreotti–Grauert theory and [Andreotti and Hill 1972] that $\dim H^\ell(M, E) < \infty$ for $\ell \neq n_-$ and $\dim H^\ell(M, E) = \infty$ for $\ell = n_-$. This reflects the fact that in this case the Bergman projector on $(0, n_-)$-forms has infinite-dimensional range.

1.3. ***Organization of the paper.*** The paper is organized as follows. In Section 2, we collect some standard notation, terminology, definitions and statements we use throughout. To construct parametrices for $\Box^{(q)}$, we introduce in Section 3 the boundary operator $\Box_-^{(q)}$. In Section 4, we construct parametrices for $\Box^{(q)}$ near a point $p \in X$ under the assumption that $Z(q)$ holds at p. Up to the authors' knowledge, the parametrices construction in Section 4 under no global assumptions is also a new result. In Section 5, we obtain microlocal Hodge decomposition theorems for $\Box^{(q)}$ near a point $p \in X$ under the assumption that $Z(q)$ fails at p. By using the results in Sections 4 and 5, we prove Theorems 1.1 and 1.5 in Section 6. In Section 7, we prove Theorem 1.9. In Section 8, we prove Theorems 1.10, 1.11 and 1.12 about the asymptotic expansions of the S^1-equivariant Bergman kernel and embedding theorems for domains with holomorphic S^1-action.

2. Preliminaries

2.1. ***Notions from microlocal and semiclassical analysis.*** We shall use the following notation: $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, and $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R} : x \geq 0\}$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we denote by $|\alpha| = \alpha_1 + \dots + \alpha_n$ its norm and by $l(\alpha) = n$ its length. For $m \in \mathbb{N}$, write $\alpha \in \{1, \dots, m\}^n$ if $\alpha_j \in \{1, \dots, m\}$, $j = 1, \dots, n$. A multi-index α is strictly increasing if $\alpha_1 < \alpha_2 < \dots < \alpha_n$. For $x = (x_1, \dots, x_n)$ we write

$$\begin{gathered} x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \\ \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \\ D_{x_j} = \frac{1}{i}\partial_{x_j}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D_x = \frac{1}{i}\partial_x. \end{gathered}$$

Let $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, be coordinates of $\mathbb{C}^n$. We write

$$\begin{gathered} z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \bar{z}^\alpha = \bar{z}_1^{\alpha_1} \cdots \bar{z}_n^{\alpha_n}, \\ \partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\Big), \quad \partial_{\bar{z}_j} = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\Big), \\ \partial_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, \quad \partial_{\bar{z}}^\alpha = \partial_{\bar{z}_1}^{\alpha_1} \cdots \partial_{\bar{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha}. \end{gathered}$$

For $j, s \in \mathbb{Z}$, set $\delta_{j,s} = 1$ if $j = s$, and $\delta_{j,s} = 0$ if $j \neq s$.

Let $\mathcal{M}$ be a smooth paracompact manifold. We let $T\mathcal{M}$ and $T^*\mathcal{M}$ denote the tangent bundle of $\mathcal{M}$ and the cotangent bundle of $\mathcal{M}$ respectively. The complexified tangent bundle of $\mathcal{M}$ and the complexified cotangent bundle of $\mathcal{M}$ are denoted by $\mathbb{C}T\mathcal{M}$ and $\mathbb{C}T^*\mathcal{M}$, respectively. Write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between $T\mathcal{M}$ and $T^*\mathcal{M}$. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}T\mathcal{M} \times \mathbb{C}T^*\mathcal{M}$. Let E be a $\mathscr{C}^\infty$ vector bundle over $\mathcal{M}$. The fiber of E at $x \in \mathcal{M}$ will be denoted by E_x. Let $\widehat{E}$ be another vector bundle over $\mathcal{M}$. We write $E \boxtimes \widehat{E}^*$ to denote the vector bundle over $\mathcal{M} \times \mathcal{M}$ with fiber over $(x, y) \in \mathcal{M} \times \mathcal{M}$ consisting of the linear maps from $\widehat{E}_y$ to E_x. Let $Y \subset \mathcal{M}$ be an open set. From now on, the spaces of distribution sections of E over Y and smooth sections of E over Y will be denoted by $\mathscr{D}'(Y, E)$ and $\mathscr{C}^\infty(Y, E)$ respectively. Let $\mathscr{E}'(Y, E)$ be the subspace of $\mathscr{D}'(Y, E)$ whose elements have compact support in Y and let $\mathscr{C}_c^\infty(Y, E)$ be the subspace of $\mathscr{C}^\infty(Y, E)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, let $H^m(Y, E)$ denote the Sobolev space of order m of sections of E over Y. Put

$$\begin{aligned} H^m_{\mathrm{loc}}(Y, E) &= \{u \in \mathscr{D}'(Y, E) : \varphi u \in H^m(Y, E) \text{ for every } \varphi \in \mathscr{C}_c^\infty(Y)\}, \\ H^m_c(Y, E) &= H^m_{\mathrm{loc}}(Y, E) \cap \mathscr{E}'(Y, E). \end{aligned}$$

Let E and $\widehat{E}$ be $\mathscr{C}^\infty$ vector bundles over a paracompact orientable $\mathscr{C}^\infty$ manifold $\mathcal{M}$ equipped with a smooth density of integration. If $A : \mathscr{C}_c^\infty(\mathcal{M}, E) \to \mathscr{D}'(\mathcal{M}, \widehat{E})$ is continuous, we write $A(x, y)$ to denote the distribution kernel of A. The following two statements are equivalent:

(a) A is continuous: $\mathscr{E}'(\mathcal{M}, E) \to \mathscr{C}^\infty(\mathcal{M}, \widehat{E})$.

(b) $A(x, y) \in \mathscr{C}^\infty(\mathcal{M} \times \mathcal{M}, \widehat{E} \boxtimes E^*)$.

If A satisfies (a) or (b), we say that A is smoothing on $\mathcal{M}$. Let $A, B : \mathscr{C}^\infty_c(\mathcal{M}, E) \to \mathscr{D}'(\mathcal{M}, \widehat{E})$ be continuous operators. We write

$$A \equiv B \text{ (on } \mathcal{M}) \tag{2-1}$$

if $A - B$ is a smoothing operator.

We say that A is properly supported if the restrictions of the two projections $(x, y) \mapsto x,\ (x, y) \mapsto y$ to $\operatorname{supp} A(x, y)$ are proper.

Let $H(x, y) \in \mathscr{D}'(\mathcal{M} \times \mathcal{M}, \widehat{E} \boxtimes E^*)$. We denote by H the unique continuous operator $\mathscr{C}^\infty_c(\mathcal{M}, E) \to \mathscr{D}'(\mathcal{M}, \widehat{E})$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

Let D be an open set of a smooth manifold X and let E be a vector bundle over X. Let

$$L^m_{\frac{1}{2},\frac{1}{2}}(D, E \boxtimes E^*), \quad L^m_{\mathrm{cl}}(D, E \boxtimes E^*)$$

denote the space of pseudodifferential operators on D of order m and type $\left(\frac{1}{2}, \frac{1}{2}\right)$ from sections of E to sections of E and the space of classical pseudodifferential operators on D of order m from sections of E to sections of E respectively. The classical result of Calderon and Vaillancourt [Hörmander 1985, Chapter 18] tells us that any $A \in L^m_{1/2,1/2}(D, E \boxtimes E^*)$ induces for any $s \in \mathbb{R}$ a continuous operator

$$A : H^s_c(D, E) \to H^{s-m}_{\mathrm{loc}}(D, E). \tag{2-2}$$

Let $A \in L^m_{1/2,1/2}(D, E \boxtimes E^*)$, $B \in L^{m_1}_{1/2,1/2}(D, E \boxtimes E^*)$, where $m, m_1 \in \mathbb{R}$. If A or B is properly supported, then the composition of A and B is well-defined. Moreover, we can repeat the proof of [Boutet de Monvel 1974, Proposition 3.2] and conclude that

$$AB \in L^{m+m_1}_{\frac{1}{2},\frac{1}{2}}(D, E \boxtimes E^*). \tag{2-3}$$

For $m \in \mathbb{R}$, $\rho, \delta \in \mathbb{R}$, $0 \le \rho, \delta \le 1$, let

$$S^m_{\rho,\delta}(T^*D, E \boxtimes E^*)$$

be the Hörmander symbol space on T^*D with values in $E \boxtimes E^*$ of order m and type (ρ, δ); see [Hörmander 1983, Definition 7.8.1]. Let

$$S^{-\infty}_{\rho,\delta}(T^*D, E \boxtimes E^*) := \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}(T^*D, E \boxtimes E^*).$$

Let $a_j \in S^{m_j}_{\rho,\delta}(T^*D, E \boxtimes E^*)$ with $m_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{m_0}_{\rho,\delta}(T^*D, E \boxtimes E^*)$ such that, for any $k \in \mathbb{N}$,

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{1,0}(T^*D, E \boxtimes E^*).$$

In this case we write

$$a \sim \sum_{j=0}^{+\infty} a_j \quad \text{in } S^{m_0}_{\rho,\delta}(T^*D, E \boxtimes E^*).$$

The symbol a is unique modulo $S^{-\infty}_{\rho,\delta}(T^*D, E \boxtimes E^*)$.

Let W_1 be an open set in $\mathbb{R}^{N_1}$ and let W_2 be an open set in $\mathbb{R}^{N_2}$. Let E_1 and E_2 be vector bundles over W_1 and W_2, respectively. An m-dependent continuous operator $F_m : \mathscr{C}_c^\infty(W_2, E_2) \to \mathscr{D}'(W_1, E_1)$ is called m-negligible on $W_1 \times W_2$ if, for m large enough, F_m is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices α, β and any $N \in \mathbb{N}$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta F_m|(x, y) \le C_{K,\alpha,\beta,N} m^{-N} \quad \text{on } K \text{ for } m \gg 1. \tag{2-4}$$

In that case we write

$$F_m(x, y) = O(m^{-\infty}) \quad \text{or} \quad F_m = O(m^{-\infty}) \quad \text{on } W_1 \times W_2.$$

If $F_m, G_m : \mathscr{C}_c^\infty(W_2, E_2) \to \mathscr{D}'(W_1, E_1)$ are m-dependent continuous operators, we write $F_m = G_m + O(m^{-\infty})$ on $W_1 \times W_2$ or $F_m(x, y) = G_m(x, y) + O(m^{-\infty})$ on $W_1 \times W_2$ if $F_m - G_m = O(m^{-\infty})$ on $W_1 \times W_2$. When $W = W_1 = W_2$, we sometimes write "on W".

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be smooth manifolds and let E_1 and E_2 be vector bundles over $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let $F_m, G_m : \mathscr{C}^\infty(\mathcal{M}_2, E_2) \to \mathscr{C}^\infty(\mathcal{M}_1, E_1)$ be m-dependent smoothing operators. We write $F_m = G_m + O(m^{-\infty})$ on $\mathcal{M}_1 \times \mathcal{M}_2$ if, on every local coordinate patch D of $\mathcal{M}_1$ and local coordinate patch D_1 of $\mathcal{M}_2$, $F_m = G_m + O(m^{-\infty})$ on $D \times D_1$. When $\mathcal{M}_1 = \mathcal{M}_2$, we sometimes write "on $\mathcal{M}_2$".

We recall the definition of the semiclassical symbol spaces.

Definition 2.1. Let W be an open set in $\mathbb{R}^N$. Let

$$S(W) := \big\{a \in \mathscr{C}^\infty(W) \;\big|\; \text{for every } \alpha \in \mathbb{N}_0^N : \sup_{x \in W} |\partial^\alpha a(x)| < \infty\big\},$$

$$S^0_{\mathrm{loc}}(W) := \big\{(a(\cdot, m))_{m \in \mathbb{R}} \;\big|\; \text{for all } \alpha \in \mathbb{N}_0^N,\ \chi \in \mathscr{C}_c^\infty(W),\ \sup_{m \ge 1} \sup_{x \in W} |\partial^\alpha(\chi a(x, m))| < \infty\big\}.$$

For $k \in \mathbb{R}$, let

$$S^k_{\mathrm{loc}} = S^k_{\mathrm{loc}}(W) = \{(a(\cdot, m))_{m \in \mathbb{R}} \mid (m^{-k} a(\cdot, m)) \in S^0_{\mathrm{loc}}(W)\}.$$

Hence $a(\cdot, m) \in S^k_{\mathrm{loc}}(W)$ if, for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in \mathscr{C}_c^\infty(W)$, there exists $C_\alpha > 0$ independent of m such that $|\partial^\alpha(\chi a(\cdot, m))| \le C_\alpha m^k$ holds on W.

Consider a sequence $a_j \in S^{k_j}_{\mathrm{loc}}$, $j \in \mathbb{N}_0$, where $k_j \searrow -\infty$, and let $a \in S^{k_0}_{\mathrm{loc}}$. We say that

$$a(\cdot, m) \sim \sum_{j=0}^\infty a_j(\cdot, m) \quad \text{in } S^{k_0}_{\mathrm{loc}}$$

if, for every $\ell \in \mathbb{N}_0$, we have $a - \sum_{j=0}^\ell a_j \in S^{k_{\ell+1}}_{\mathrm{loc}}$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\mathrm{loc}} = S^{-\infty}_{\mathrm{loc}}(W) := \bigcap_k S^k_{\mathrm{loc}}$.

Similarly, we can define $S^k_{\mathrm{loc}}(Y, A)$ in the standard way, where Y is a smooth manifold and A is a vector bundle over Y.

2.2. ***Manifolds with smooth boundary.*** Let M be a relatively compact open subset with smooth boundary X of a smooth manifold M'. Let A be a $\mathscr{C}^\infty$ vector bundle over M'. Let U be an open set in M'. Let

$$\mathscr{C}^\infty(U \cap \overline{M}, A), \quad \mathscr{D}'(U \cap \overline{M}, A), \quad \mathscr{C}_c^\infty(U \cap \overline{M}, A), \quad \mathscr{E}'(U \cap \overline{M}, A),$$
$$H^s(U \cap \overline{M}, A), \quad H^s_c(U \cap \overline{M}, A), \quad H^s_{\mathrm{loc}}(U \cap \overline{M}, A)$$

(where $s \in \mathbb{R}$) denote the spaces of restrictions to $U \cap \overline{M}$ of elements in

$$\mathscr{C}^\infty(U \cap M', A), \quad \mathscr{D}'(U \cap M', A), \quad \mathscr{C}^\infty(U \cap M', A), \quad \mathscr{E}'(U \cap M', A),$$
$$H^s(M', A), \quad H^s_c(M', A), \quad H^s_{\mathrm{loc}}(M', A),$$

respectively. Write

$$L^2(U \cap \overline{M}, A) := H^0(U \cap \overline{M}, A), \quad L^2_c(U \cap \overline{M}, A) := H^0_c(U \cap \overline{M}, A),$$
$$L^2_{\mathrm{loc}}(U \cap \overline{M}, A) := H^0_{\mathrm{loc}}(U \cap \overline{M}, A).$$

Let A and B be $\mathscr{C}^\infty$ vector bundles over M'. Let U be an open set in M'. Let

$$F_1, F_2 : \mathscr{C}^\infty_c(U \cap M, A) \to \mathscr{D}'(U \cap M, B)$$

be continuous operators. Let $F_1(x, y), F_2(x, y) \in \mathscr{D}'((U \times U) \cap (M \times M), A \boxtimes B^*)$ be the distribution kernels of F_1 and F_2 respectively. We write

$$F_1 \equiv F_2 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$$

or $F_1(x, y) \equiv F_2(x, y) \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ if $F_1(x, y) = F_2(x, y) + r(x, y)$, where $r(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), A \boxtimes B^*)$. Similarly, let

$$\widehat{F}_1, \widehat{F}_2 : \mathscr{C}^\infty_c(U \cap M, A) \to \mathscr{D}'(U \cap X, B)$$

be continuous operators. Let

$$\widehat{F}_1(x, y), \widehat{F}_2(x, y) \in \mathscr{D}'((U \times U) \cap (X \times M), A \boxtimes B^*)$$

be the distribution kernels of $\widehat{F}_1$ and $\widehat{F}_2$ respectively. We write $\widehat{F}_1 \equiv \widehat{F}_2 \mod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M}))$ or $\widehat{F}_1(x, y) \equiv \widehat{F}_2(x, y) \mod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M}))$ if $\widehat{F}_1(x, y) = \widehat{F}_2(x, y) + \hat{r}(x, y)$, where $\hat{r}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M}), A \boxtimes B^*)$. Similarly, let $\widetilde{F}_1, \widetilde{F}_2 : \mathscr{C}^\infty_c(U \cap X, A) \to \mathscr{D}'(U \cap M, B)$ be continuous operators. Let

$$\widetilde{F}_1(x, y), \widetilde{F}_2(x, y) \in \mathscr{D}'((U \times U) \cap (M \times X), A \boxtimes B^*)$$

be the distribution kernels of $\widetilde{F}_1$ and $\widetilde{F}_2$ respectively. We write $\widetilde{F}_1 \equiv \widetilde{F}_2 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X))$ or $\widetilde{F}_1(x, y) \equiv \widetilde{F}_2(x, y) \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X))$ if $\widetilde{F}_1(x, y) = \widetilde{F}_2(x, y) + \tilde{r}(x, y)$, where $\tilde{r}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X), A \boxtimes B^*)$.

Let

$$F_m, G_m : \mathscr{C}^\infty_c(U \cap M, A) \to \mathscr{D}'(U \cap M, B)$$

be m-dependent continuous operators. Let

$$F_m(x, y), G_m(x, y) \in \mathscr{D}'((U \times U) \cap (M \times M), A \boxtimes B^*)$$

be the distribution kernels of F_m and G_m respectively. We write

$$F_m \equiv G_m \mod O(m^{-\infty}) \quad \text{on } U \cap \overline{M} \tag{2-5}$$

if there is a $r_m(x, y) \in \mathscr{C}^\infty(U \times U, A \boxtimes B^*)$ with $r_m(x, y) = O(m^{-\infty})$ on $U \times U$ such that

$$r_m(x, y)|_{(U \times U) \cap (\overline{M} \times \overline{M})} = F_m(x, y) - G_m(x, y) \quad \text{for } m \gg 1.$$

Let $k \in \mathbb{R}$. Let U be an open set in M' and let E be a vector bundle over $M' \times M'$. Let

$$S^k_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E) \tag{2-6}$$

denote the space of restrictions to $U \cap \overline{M}$ of elements in $S^k_{\mathrm{loc}}(U \times U, E)$. Let

$$a_j \in S^{k_j}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E), \quad j = 0, 1, 2, \dots,$$

with $k_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E)$ such that, for every $\ell \in \mathbb{N}$,

$$a - \sum_{j=0}^{\ell-1} a_j \in S^{k_\ell}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E).$$

If a and a_j have the properties above, we write

$$a \sim \sum_{j=0}^{\infty} a_j \quad \text{in } S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E).$$

If E is trivial, then we write $S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}))$ to denote $S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E)$.

2.3. ***The $\bar\partial$-Neumann Laplacian.*** Let M be a relatively compact open subset with $\mathscr{C}^\infty$ boundary X of a complex manifold M' of dimension n. Let $T^{1,0}M'$ and $T^{0,1}M'$ be the holomorphic tangent bundle of M' and the antiholomorphic tangent bundle of M'. We fix a Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TM'$ so that $T^{1,0}M' \perp T^{0,1}M'$. For $p, q \in \mathbb{N}$, let $T^{*p,q}M'$ be the vector bundle of (p,q)-forms on M'. The Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TM'$ induces by duality a Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\bigoplus_{p,q=1}^{p,q=n} T^{*p,q}M'$. Let $|\cdot|$ be the corresponding pointwise norm with respect to $\langle \cdot \mid \cdot \rangle$. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of X, that is, $\rho = 0$ on X, $\rho < 0$ on M and $d\rho \neq 0$ near X. From now on, we take a defining function ρ so that $|d\rho| = 1$ on X. Let U be an open set of M'. For every $p, q = 0, \dots, n$, we define

$$\Omega^{p,q}(U \cap \overline{M}) := \mathscr{C}^\infty(U \cap \overline{M}, T^{*p,q}M'), \quad \Omega^{p,q}(M') := \mathscr{C}^\infty(M', T^{*p,q}M'),$$
$$\Omega^{p,q}_c(U \cap \overline{M}) := \mathscr{C}^\infty_c(U \cap \overline{M}, T^{*p,q}M'),$$
$$\Omega^{p,q}_c(M') := \mathscr{C}^\infty_c(M', T^{*p,q}M'), \quad \Omega^{p,q}_c(M) := \mathscr{C}^\infty_c(M, T^{*p,q}M').$$

Let $dv_{M'}$ be the volume form on M' induced by the Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TM'$ and let $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_{M'}$ be the inner products on $\Omega^{0,q}(\overline{M})$ and $\Omega^{0,q}_c(M')$ defined by

$$\begin{aligned} (f \mid h)_M &= \int_M \langle f \mid h \rangle \, dv_{M'}, \quad f, h \in \Omega^{0,q}(\overline{M}), \\ (f \mid h)_{M'} &= \int_{M'} \langle f \mid h \rangle \, dv_{M'}, \quad f, h \in \Omega^{0,q}_c(M'). \end{aligned} \tag{2-7}$$

Let $\|\cdot\|_M$ and $\|\cdot\|_{M'}$ be the corresponding norms with respect to $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_{M'}$ respectively. Let $L^2_{(0,q)}(M)$ be the completion of $\Omega^{0,q}(\overline{M})$ with respect to $(\cdot \mid \cdot)_M$. We extend $(\cdot \mid \cdot)_M$ to $L^2_{(0,q)}(M)$ in the standard way. Let $\bar\partial : \Omega^{0,q}(M') \to \Omega^{0,q+1}(M')$ be the part of the exterior differential operator which maps forms of type $(0,q)$ to forms of type $(0,q+1)$ and we denote by $\bar\partial^*_f : \Omega^{0,q+1}(M') \to \Omega^{0,q}(M')$ the formal adjoint of $\bar\partial$. That is,

$$(\bar\partial f \mid h)_{M'} = (f \mid \bar\partial^*_f h)_{M'},$$

$f\in\Omega^{0,q}_c(M')$, $h\in\Omega^{0,q+1}(M')$. We shall also use the notation $\bar\partial$ for the closure in L^2 of the $\bar\partial$ operator, initially defined on $\Omega^{0,q}(\overline M)$ and $\bar\partial^*$ for the Hilbert space adjoint of $\bar\partial$. Recall that for $u\in L^2_{(0,q)}(M)$, we say that $u\in\operatorname{Dom}\bar\partial$ if we can find a sequence $u_j\in\Omega^{0,q}(\overline M)$, $j=1,2,\dots$, with $\lim_{j\to\infty}\|u_j-u\|_M=0$ such that $\lim_{j\to\infty}\|\bar\partial u_j-v\|_M=0$ for some $v\in L^2_{(0,q+1)}(M)$. We set $\bar\partial u=v$. The $\bar\partial$-Neumann Laplacian on $(0,q)$-forms is then the nonnegative self-adjoint operator in the space $L^2_{(0,q)}(M)$ (see [Folland and Kohn 1972, Chapter 1]):

$$\Box^{(q)}=\bar\partial\,\bar\partial^*+\bar\partial^*\,\bar\partial:\operatorname{Dom}\Box^{(q)}\subset L^2_{(0,q)}(M)\to L^2_{(0,q)}(M), \tag{2-8}$$

where

$$\operatorname{Dom}\Box^{(q)}=\{u\in L^2_{(0,q)}(M),\ u\in\operatorname{Dom}\bar\partial^*\cap\operatorname{Dom}\bar\partial,\ \bar\partial^*u\in\operatorname{Dom}\bar\partial,\ \bar\partial u\in\operatorname{Dom}\bar\partial^*\} \tag{2-9}$$

and $\Omega^{0,q}(\overline M)\cap\operatorname{Dom}\Box^{(q)}$ is dense in $\operatorname{Dom}\Box^{(q)}$ for the norm

$$\operatorname{Dom}\Box^{(q)}\ni u\mapsto\|u\|_M+\|\bar\partial u\|_M+\|\bar\partial^*u\|_M;$$

see [Folland and Kohn 1972, p. 14]. We denote by $\operatorname{Spec}\Box^{(q)}$ the spectrum of $\Box^{(q)}$.

Now, we consider the boundary X of M. The boundary X is a compact CR manifold of dimension $2n-1$ with natural CR structure $T^{1,0}X:=T^{1,0}M'\cap\mathbb{C}TX$. Let $T^{0,1}X:=\overline{T^{1,0}X}$. The Hermitian metric on $\mathbb{C}TM'$ induces Hermitian metrics $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TX$ and also on the bundle $\bigoplus_{j=1}^{2n-1}\Lambda^j(\mathbb{C}T^*X)$. Let dv_X be the volume form on X induced by the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TX$ and let $(\cdot\,|\,\cdot)_X$ be the L^2 inner product on $\mathscr{C}^\infty\big(X,\bigoplus_{j=1}^{2n-1}\Lambda^j(\mathbb{C}T^*X)\big)$ induced by dv_X and the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\bigoplus_{j=0}^{2n-1}\Lambda^j(\mathbb{C}T^*X)$. Put

$$T^{*1,0}X:=(T^{0,1}X\oplus\mathbb{C}T)^\perp\subset\mathbb{C}T^*X,\quad T^{*0,1}X:=(T^{1,0}X\oplus\mathbb{C}T)^\perp\subset\mathbb{C}T^*X.$$

We have the pointwise orthogonal decomposition (see (1-5))

$$\begin{aligned}\mathbb{C}T^*X&=T^{*1,0}X\oplus T^{*0,1}X\oplus\{\lambda\omega_0:\lambda\in\mathbb{C}\},\\ \mathbb{C}TX&=T^{1,0}X\oplus T^{0,1}X\oplus\{\lambda T:\lambda\in\mathbb{C}\}.\end{aligned} \tag{2-10}$$

Define the vector bundle of $(0,q)$-forms by $T^{*0,q}X:=\Lambda^qT^{*0,1}X$. Let $D\subset X$ be an open set. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega^{0,q}_c(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D.

In order to describe the $\bar\partial$-Neumann boundary conditions we introduce the operator of restriction to the boundary X: let γ denote the operator of restriction to the boundary X,

$$\gamma:\Omega^{0,\bullet}(\overline M)\to\mathscr{C}^\infty(X,T^{*0,\bullet}M'|_X),\quad u\mapsto\gamma u:=u|_X. \tag{2-11}$$

We have $\{u\in\Omega^{0,q}(\overline M):(\bar\partial\rho)^{\wedge,*}\gamma u=0\}=\operatorname{Dom}\bar\partial^*\cap\Omega^{0,q}(\overline M)$. We have thus

$$u\in\operatorname{Dom}\Box^{(q)}\cap\Omega^{0,q}(\overline M)\quad\Longleftrightarrow\quad(\bar\partial\rho)^{\wedge,*}\gamma u=0,\,(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial u=0. \tag{2-12}$$

The conditions on the right-hand side are called first and second $\bar\partial$-Neumann boundary conditions.

3. The boundary operator $\Box_{-}^{(q)}$

In this section, we introduce a boundary operator on $X = \partial M$ defined for a form u on X as the complex tangential component of the form $\bar\partial v$, where $v = \widetilde{P}u$ is the extension of u from X to M by the Poisson operator $\widetilde{P}$. This operator will play a central role in Section 4 for the construction of the parametrix of the $\bar\partial$-Neumann problem and Section 5 (Lemma 5.18). We fix $q \in \{0, 1, \dots, n-1\}$. Let

$$\Box_f^{(q)} = \bar\partial\,\bar\partial_f^* + \bar\partial_f^*\,\bar\partial : \Omega^{0,q}(M') \to \Omega^{0,q}(M')$$

denote the complex Laplace–Beltrami operator on $(0,q)$-forms. The subscript f indicates that the operator is not subject to any boundary conditions. The boundary problem $(\Box_f^{(q)}, \gamma)$ on $\overline{M}$ is the Dirichlet boundary problem, which is a regular elliptic boundary problem; see, e.g., [Taylor 2011, Chapter 5, Proposition 11.10]. Let us consider the map

$$F^{(q)} : H^2(\overline{M}, T^{*0,q}M') \to L^2_{(0,q)}(M) \oplus H^{\frac{3}{2}}(X, T^{*0,q}M'), \quad u \mapsto (\Box_f^{(q)}u, \gamma u). \tag{3-1}$$

By the general theory of regular elliptic boundary problems [Boutet de Monvel 1971; Taylor 2011, Chapter 5, Proposition 11.16], we know that $\dim \operatorname{Ker} F^{(q)} < \infty$ and $\operatorname{Ker} F^{(q)} \subset \Omega^{0,q}(\overline{M})$. Let

$$K^{(q)} : H^2(\overline{M}, T^{*0,q}M') \to \operatorname{Ker} F^{(q)} \tag{3-2}$$

be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_M$. Put $\widetilde{\Box}_f^{(q)} = \Box_f^{(q)} + K^{(q)}$ and consider the map

$$\widetilde{F}^{(q)} : H^2(\overline{M}, T^{*0,q}M') \to L^2_{(0,q)}(M) \oplus H^{\frac{3}{2}}(X, T^{*0,q}M'), \quad u \mapsto (\widetilde{\Box}_f^{(q)}u, \gamma u). \tag{3-3}$$

It is easy to see that $\widetilde{F}^{(q)}$ is injective. Let

$$\widetilde{P} : \mathscr{C}^\infty(X, T^{*0,q}M') \to \Omega^{0,q}(\overline{M}) \tag{3-4}$$

be the Poisson operator for $\widetilde{\Box}_f^{(q)}$ which is well-defined since (3-3) is injective. The Poisson operator $\widetilde{P}$ satisfies

$$\widetilde{\Box}_f^{(q)}\widetilde{P}u = 0, \quad \gamma\widetilde{P}u = u \quad \text{for every } u \in \mathscr{C}^\infty(X, T^{*0,q}M'). \tag{3-5}$$

By [Boutet de Monvel 1971, p. 29] the operator $\widetilde{P}$ extends continuously

$$\widetilde{P} : H^s(X, T^{*0,q}M') \to H^{s+\frac{1}{2}}(\overline{M}, T^{*0,q}M') \quad \text{for all } s \in \mathbb{R}, \tag{3-6}$$

and there is a continuous operator

$$D^{(q)} : H^s(\overline{M}, T^{*0,q}M') \to H^{s+2}(\overline{M}, T^{*0,q}M') \quad \text{for all } s \in \mathbb{R} \tag{3-7}$$

such that

$$D^{(q)}\widetilde{\Box}_f^{(q)} + \widetilde{P}\gamma = I \quad \text{on } \Omega^{0,q}(\overline{M}). \tag{3-8}$$

Let $\widehat{\mathscr{E}}'(\overline{M}, T^{*0,q}M')$ denote the space of continuous linear map from $\Omega^{0,q}(\overline{M})$ to $\mathbb{C}$ with respect to $(\cdot\,|\,\cdot)_M$. Let

$$\widetilde{P}^* : \widehat{\mathscr{E}}'(\overline{M}, T^{*0,q}M') \to \mathscr{D}'(X, T^{*0,q}M') \tag{3-9}$$

be the operator defined by

$$(\widetilde{P}^* u \mid v)_X = (u \mid \widetilde{P} v)_M, \quad u \in \widehat{\mathscr{E}}'(\overline{M}, T^{*0,q}M'),\ v \in \mathscr{C}^\infty(X, T^{*0,q}M').$$

By [Boutet de Monvel 1971, p. 30] the operator

$$\widetilde{P}^* : H^s(\overline{M}, T^{*0,q}M') \to H^{s+\frac{1}{2}}(X, T^{*0,q}M') \tag{3-10}$$

is continuous for every $s \in \mathbb{R}$ and

$$\widetilde{P}^* : \Omega^{0,q}(\overline{M}) \to \mathscr{C}^\infty(X, T^{*0,q}M').$$

Let

$$\Box_-^{(q)} := (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widetilde{P} : \Omega^{0,q}(X) \to \Omega^{0,q}(X). \tag{3-11}$$

In this section, we will construct a parametrix for $\Box_-^{(q)}$ under certain Levi curvature assumptions. Put

$$\begin{aligned}\Sigma^- &= \{(x, \lambda\omega_0(x)) \in T^*X : \lambda < 0\},\\ \Sigma^+ &= \{(x, \lambda\omega_0(x)) \in T^*X : \lambda > 0\}.\end{aligned} \tag{3-12}$$

Note that we use here a different sign convention than in [Hsiao 2010], where ω_0 equals $d\rho \circ J$ (compare [loc. cit., (1.9), p. 84], (1-5)), thus we swap here the roles of Σ^+ and Σ^- compared to [loc. cit.].

Definition 3.1. Let $A \in L^m_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, where $m \in \mathbb{R}$. We write

$$A \equiv 0 \quad \text{near } \Sigma^+ \cap T^*D$$

if there exists $A' \in L^m_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ with full symbol

$$a(x,\eta) \in S^m_{1/2,1/2}(T^*D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

such that

$$A \equiv A' \text{ on } D$$

and $a(x,\eta)$ vanishes in an open neighborhood of $\Sigma^+ \cap T^*D$.

For A as in Definition 3.1 we have $\mathrm{WF}(A) \cap \Sigma^+ = \varnothing$, where $\mathrm{WF}(A)$ denotes the wave front set of the pseudodifferential operator A; see [Grigis and Sjöstrand 1994, Chapter 7].

Let us consider the Hodge–de Rham Laplacian

$$\triangle_X := dd^* + d^*d : \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)) \to \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)), \tag{3-13}$$

where $d^* : \mathscr{C}^\infty(X, \Lambda^{q+1}(\mathbb{C}T^*X)) \to \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X))$ is the formal adjoint of the exterior derivative d with respect to $(\cdot \mid \cdot)_X$. Let $\sqrt{\triangle_X}$ be the nonnegative square root of $\triangle_X$.

Theorem 3.2 [Hsiao 2010, Part II, Proposition 4.1]. *The operator $\Box_-^{(q)}$ from (3-11) is a classical pseudodifferential operator of order* 1 *and we have*

$$\Box_-^{(q)} = \tfrac{1}{2}(iT + \sqrt{\triangle_X}) + \Psi^0, \tag{3-14}$$

where $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. In particular, $\Box_-^{(q)}$ is elliptic outside Σ^+.

Let $\bar\partial_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator. It is not difficult to see that

$$\bar\partial_b = 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial\widetilde{P} : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X). \tag{3-15}$$

We notice that, for $u \in \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X))$,

$$u \in \Omega^{0,q}(X) \quad \text{if and only if} \quad u = 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}u \ \text{ on } X \tag{3-16}$$

and

$$2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge} + 2(\bar\partial\rho)^{\wedge}(\bar\partial\rho)^{\wedge,*} = I \quad \text{on } \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)). \tag{3-17}$$

Consider

$$\gamma\bar\partial_f^*\widetilde{P} : \mathscr{C}^\infty(X, \Lambda^{q+1}(\mathbb{C}T^*X)) \to \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)).$$

It is not difficult to check that (see [Hsiao 2010, Part II, Lemma 2.2])

$$\gamma\bar\partial_f^*\widetilde{P} : \Omega^{0,q+1}(X) \to \Omega^{0,q}(X). \tag{3-18}$$

Put

$$\widetilde{\Box}_b^{(q)} := \gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + \bar\partial_b\gamma\bar\partial_f^*\widetilde{P} : \Omega^{0,q}(X) \to \Omega^{0,q}(X). \tag{3-19}$$

Lemma 3.3. *We have*

$$\widetilde{\Box}_b^{(q)} = -4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^{\wedge}\Box_{-}^{(q)} + R^{(q)} \quad \textit{on } \Omega^{0,q}(X),$$

where $R^{(q)} : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ is a smoothing operator.

Proof. From (3-5), (3-15), (3-16), (3-17), (3-18), we have

$$\begin{aligned}
\widetilde{\Box}_b^{(q)} &= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\widetilde{\Box}_b^{(q)} \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\big(\gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + \bar\partial_b\gamma\bar\partial_f^*\widetilde{P}\big) \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\bar\partial_b\gamma\bar\partial_f^*\widetilde{P} \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial\widetilde{P}\gamma\bar\partial_f^*\widetilde{P} \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}\big(\gamma\bar\partial\widetilde{P} - 2(\bar\partial\rho)^{\wedge}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widetilde{P}\big) + 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial\widetilde{P}\gamma\bar\partial_f^*\widetilde{P}.
\end{aligned} \tag{3-20}$$

From (3-8), we have

$$\begin{aligned}
\bar\partial\widetilde{P} &= \widetilde{P}\gamma\bar\partial\widetilde{P} + D^{(q+1)}\widetilde{\Box}_f^{(q+1)}\bar\partial\widetilde{P} \\
&= \widetilde{P}\gamma\bar\partial\widetilde{P} + D^{(q+1)}(\Box_f^{(q+1)} + K^{(q+1)})\bar\partial\widetilde{P} \\
&= \widetilde{P}\gamma\bar\partial\widetilde{P} + D^{(q+1)}\bar\partial(\Box_f^{(q)} + K^{(q)})\widetilde{P} - D^{(q+1)}\bar\partial K^{(q)}\widetilde{P} + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P} \\
&\equiv \widetilde{P}\gamma\bar\partial\widetilde{P} \mod \mathscr{C}^\infty(\overline{M} \times X).
\end{aligned}$$

Similarly, we have

$$\bar\partial_f^*\widetilde{P} \equiv \widetilde{P}\gamma\bar\partial_f^*\widetilde{P} \mod \mathscr{C}^\infty(\overline{M} \times X).$$

Thus,

$$\gamma\bar\partial_f^*\widetilde{P}\gamma\bar\partial\widetilde{P} + \gamma\bar\partial\widetilde{P}\gamma\bar\partial_f^*\widetilde{P} \equiv \gamma(\bar\partial_f^*\,\bar\partial + \bar\partial\,\bar\partial_f^*)\widetilde{P}. \tag{3-21}$$

From (3-11), (3-20) and (3-21), we get

$$\begin{aligned}\widetilde{\Box}_b^{(q)} &\equiv 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\Box_f^{(q)}\widetilde{P}-4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^{\wedge}\Box_-^{(q)}\\ &\equiv -2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma K^{(q)}\widetilde{P}-4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^{\wedge}\Box_-^{(q)},\end{aligned}\tag{3-22}$$

where $K^{(q)}$ is as in (3-2). Note that $K^{(q)}\equiv 0 \bmod \mathscr{C}^\infty(\overline{M}\times\overline{M})$. From this observation and (3-6), we deduce that

$$-2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma K^{(q)}\widetilde{P}: H^s(X,T^{*0,q}M')\to H^{s+N}(X,T^{*0,q}M'),$$

for every $s\in\mathbb{R}$ and every $N\in\mathbb{N}$. Hence, $-2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma K^{(q)}\widetilde{P}$ is smoothing. From this observation and (3-20), the lemma follows. □

Lemma 3.3 gives a relation between $\widetilde{\Box}_b^{(q)}$ and $\Box_-^{(q)}$. Put

$$A^{(q)}:=-4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^{\wedge}:\Omega^{0,q}(X)\to\Omega^{0,q}(X).\tag{3-23}$$

Then, $\widetilde{\Box}_b^{(q)}\equiv A^{(q)}\Box_-^{(q)}$. We are going to show that $A^{(q)}$ is an elliptic classical pseudodifferential operator near Σ^+. We pause and introduce some notation. Near X, put

$$\widetilde{T}_z^{*0,1}M'=\{u\in T_z^{*0,1}M':\bar\partial\rho(u)=0\},\tag{3-24}$$

$$\widetilde{T}_z^{0,1}M'=\Big\{u\in T_z^{0,1}M':\Big(iT+\frac{\partial}{\partial\rho}\Big)(u)=0\Big\}.\tag{3-25}$$

We have the orthogonal decompositions with respect to $\langle\cdot\,|\,\cdot\rangle$ for every $z\in M'$, z is near X:

$$\begin{aligned}T_z^{*0,1}M'&=\widetilde{T}_z^{*0,1}M'\oplus\{\lambda(\bar\partial\rho)(z):\lambda\in\mathbb{C}\},\\ T_z^{0,1}M'&=\widetilde{T}_z^{0,1}M'\oplus\Big\{\lambda\Big(iT+\frac{\partial}{\partial\rho}\Big)(z):\lambda\in\mathbb{C}\Big\}.\end{aligned}\tag{3-26}$$

Note that $\widetilde{T}_z^{*0,1}M'=T_z^{*0,1}X$ and $\widetilde{T}_z^{0,1}M'=T_z^{0,1}X$ for every $z\in X$. Fix $z_0\in X$. We can choose an orthonormal frame $t_1(z),\dots,t_{n-1}(z)$ for $\widetilde{T}_z^{*,0,1}M'$ varying smoothly with z in a neighborhood U of z_0 in M'. Then

$$t_1(z),\dots,t_{n-1}(z),t_n(z):=\frac{\bar\partial\rho(z)}{|\bar\partial\rho(z)|}$$

is an orthonormal frame for $T_z^{*0,1}M'$. Let

$$T_1(z),\ \dots,\ T_{n-1}(z),\ T_n(z)$$

denote the basis of $T_z^{0,1}M'$ which is dual to $t_1(z),\dots,t_n(z)$. We have $T_j(z)\in\widetilde{T}_z^{0,1}M'$, $j=1,\dots,n-1$, and

$$T_n=\frac{iT+\frac{\partial}{\partial\rho}}{\big|iT+\frac{\partial}{\partial\rho}\big|}.$$

From now on, we write Ψ^0 to denote any element in $L^0_{\mathrm{cl}}(X,T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$. By [Hsiao 2010, Part II, (4.11)] we have

$$\gamma\bar\partial_f^*\widetilde{P}=\sum_{j=1}^{n-1}t_j^{\wedge,*}\circ T_j^*+(\bar\partial\rho)^{\wedge,*}\circ(iT-\sqrt{\triangle_X})+\Psi^0,\tag{3-27}$$

where T_j^* is the adjoint of T_j with respect to $(\cdot\mid\cdot)$, i.e., $(T_j f \mid g)_X = (f \mid T_j^* g)_X$ for every $f, g \in \mathscr{C}_c^\infty(U \cap X)$, $j = 1, \dots, n-1$, and $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$.

Theorem 3.4. *The operator $A^{(q)}$ from (3-23) is a classical pseudodifferential operator with*

$$A^{(q)} = -(iT - \sqrt{\triangle_X}) + \Psi^0 \quad \text{on } \Omega^{0,q}(X), \tag{3-28}$$

where $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. Hence $A^{(q)}$ is elliptic near Σ^+.

Proof. From (3-23) and (3-27), we have

$$A^{(q)} = -4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\Bigl(\sum_{j=1}^{n-1} t_j^{\wedge,*}(\bar\partial\rho)^{\wedge} T_j^* + (\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}(iT - \sqrt{\triangle_X}) + \Psi^0\Bigr). \tag{3-29}$$

We notice that

$$\begin{aligned} (\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}(\bar\partial\rho)^{\wedge} T_j^* = 0 \quad &\text{on } \Omega^{0,q}(X) \text{ for every } j = 1, \dots, n-1, \\ 4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge} = I \quad &\text{on } \Omega^{0,q}(X). \end{aligned} \tag{3-30}$$

From (3-29) and (3-30), we get (3-28). □

Let $D \subset X$ be an open coordinate patch with local coordinates $x = (x_1, \dots, x_{2n-1})$. Assume that the Levi form is nondegenerate of constant signature (n_-, n_+) on D. Note that $(\bar\partial\rho)^{\wedge,*}u = 0$, $u \in \Omega^{0,\bullet}(X)$. From this observation and (3-27), we deduce that

$$\gamma\bar\partial_f^*\widetilde{P} = \sum_{j=1}^{n-1} t_j^{\wedge,*} \circ T_j^* + \Psi^0 \quad \text{on } \Omega^{0,\bullet}(X),$$

and hence

$$\gamma\bar\partial_f^*\widetilde{P} = \bar\partial_b^* + \Psi^0 \quad \text{on } \Omega^{0,\bullet}(X),$$

where $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$.

We can apply the method in [Sjöstrand 1974] to construct a parametrix of $\widetilde{\Box}_b^{(q)}$ near Σ^+ (see also [Hsiao 2010, Part I, Proposition 6.3]) and deduce the following.

Theorem 3.5. *Let $D \subset X$ be an open coordinate patch such that the Levi form is nondegenerate of constant signature (n_-, n_+) on D. Then for any $q \neq n_-$ there exists a properly supported operator $E^{(q)} \in L^{-1}_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that*

$$\widetilde{\Box}_b^{(q)} E^{(q)} \equiv I + R \quad \text{on } D, \tag{3-31}$$

where $R \in L^1_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ with $R \equiv 0$ near $\Sigma^+ \cap T^*D$.

If $q \neq n_-, n_+$, then $\widetilde{\Box}_b^{(q)}$ is hypoelliptic with loss of one derivative and from [Sjöstrand 1974], we can find $\widetilde{E}^{(q)}$ so that $\widetilde{\Box}_b^{(q)}\widetilde{E}^{(q)} \equiv I$. In Theorem 3.5, q could be equal to n_+ and $\widetilde{\Box}_b^{(n_+)}$ is not hypoelliptic; therefore we have R in (3-31). In [Hsiao 2010, Part I, Proposition 6.3], we do not have R since $q \neq n_-, n_+$.

We can now prove the main result of this section. We will use it in the proof of Theorem 4.3 for the definition of the operator $N_5^{(q)}$; see (4-15).

Theorem 3.6. *Let $D \subset X$ be an open coordinate patch such that the Levi form is nondegenerate of constant signature (n_-, n_+) on D. Then for any $q \neq n_-$ there exists a properly supported operator $G^{(q)} \in L^0_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that*

$$\Box^{(q)}_{-} G^{(q)} \equiv I \text{ on } D. \tag{3-32}$$

Proof. Let $A^{(q)} \in L^1_{\mathrm{cl}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be as in (3-23). Since $A^{(q)}$ is elliptic near Σ^+ (see Theorem 3.4), there are properly supported elliptic pseudodifferential operators $H^{(q)}, H^{(q)}_1 \in L^{-1}_{\mathrm{cl}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that

$$\begin{aligned} A^{(q)}H^{(q)} - I &\equiv 0 \quad \text{near } \Sigma^+ \cap T^*D, \\ H^{(q)}_1 A^{(q)} - I &\equiv 0 \quad \text{near } \Sigma^+ \cap T^*D. \end{aligned} \tag{3-33}$$

From Lemma 3.3, (3-23) and (3-33), we have $\widetilde{\Box}^{(q)}_b \equiv A^{(q)}\Box^{(q)}_{-}$, $H^{(q)}_1 \widetilde{\Box}^{(q)}_b \equiv H^{(q)}_1 A^{(q)} \Box^{(q)}_{-}$ and hence

$$\Box^{(q)}_{-} \equiv H^{(q)}_1 \widetilde{\Box}^{(q)}_b \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-34}$$

Let $E^{(q)} \in L^{-1}_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be as in Theorem 3.5. From (3-34), we have

$$\Box^{(q)}_{-} E^{(q)} A^{(q)} - I \equiv H^{(q)}_1 \widetilde{\Box}^{(q)}_b E^{(q)} A^{(q)} - I \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-35}$$

From (3-31), we have $H^{(q)}_1 \widetilde{\Box}^{(q)}_b E^{(q)} A^{(q)} - I \equiv H^{(q)}_1 (I + R) A^{(q)} - I$ and hence

$$H^{(q)}_1 \widetilde{\Box}^{(q)}_b E^{(q)} A^{(q)} - I \equiv H^{(q)}_1 A^{(q)} - I \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-36}$$

From (3-36) and (3-33), we get

$$H^{(q)}_1 \widetilde{\Box}^{(q)}_b E^{(q)} A^{(q)} - I \equiv 0 \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-37}$$

From (3-35), (3-36) and (3-37), we conclude that

$$\Box^{(q)}_{-} E^{(q)} A^{(q)} = I + r,$$

where $r \in L^1_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ with $r \equiv 0$ near $\Sigma^+ \cap T^*D$. Since $\Box^{(q)}_{-}$ is elliptic outside Σ^+, we can find a properly supported operator $r_1 \in L^1_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that $\Box^{(q)}_{-} r_1 \equiv -r$ on D. Let $G^{(q)} \in L^0_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be a properly supported operator so that $G^{(q)} \equiv E^{(q)}A^{(q)} + r_1$ on D. Hence $\Box^{(q)}_{-} G^{(q)} \equiv I$ on D. ◻

4. Parametrices for the $\bar\partial$-Neumann Laplacian outside the critical degree

In this section we consider boundary points where the Levi form is nondegenerate of constant signature (n_-, n_+) on D. In the neighborhood of such points we construct a local parametrix of the $\bar\partial$-Neumann Laplacian on $(0,q)$-forms with $q \neq n_-$.

We briefly recall the global situation [Chen and Shaw 2001; Folland and Kohn 1972; Kohn 1963; 1964]. If $Z(q)$ holds at each point of the boundary X, then $\operatorname{Ker}\Box^{(q)}$ is a finite-dimensional subspace of $\Omega^{0,q}(\overline{M})$, $\Box^{(q)}$ has closed range in L^2 and the Bergman projector $B^{(q)}$ on $\operatorname{Ker}\Box^{(q)}$ is a smoothing operator on $\overline{M}$. Moreover, there exists a continuous partial inverse $N^{(q)} : L^2(M, T^{*0,q}M) \to \operatorname{Dom} \Box^{(q)}$ of $\Box^{(q)}$, called the Neumann operator, such that we have the Hodge decomposition at the operator level, $\Box^{(q)}N^{(q)} + B^{(q)} = I$ on $L^2_{(0,q)}(M)$ and $N^{(q)}\Box^{(q)} + B^{(q)} = I$ on $\operatorname{Dom} \Box^{(q)}$. Moreover, the Neumann operator $N^{(q)}$ maps continuously the Sobolev spaces H^s to H^{s+1} for every $s \in \mathbb{Z}$, and maps the space of smooth forms on $\overline{M}$ into itself. If the Levi form is nondegenerate of signature (n_-, n_+) on X, then $Z(q)$ holds if and only if $q \neq n_-$. We will show in this section a local version of these global results, in which case the Neumann operator will be a local parametrix of the $\bar\partial$-Neumann operator.

Let D be a local coordinate patch of X with local coordinates $x = (x_1, \dots, x_{2n-1})$. Then $\hat{x} := (x_1, \dots, x_{2n-1}, \rho)$ are local coordinates of M' defined in an open set U of M' with $U \cap X = D$. Until further notice, we work on U.

Let $\widehat{\mathscr{E}}'(U \cap \overline{M}, T^{*0,q}M')$ be the space of continuous linear forms from $\Omega^{0,q}(U \cap \overline{M})$ to $\mathbb{C}$. Let $F : \Omega^{0,q}_c(U \cap \overline{M}) \to \mathscr{D}'(U \cap \overline{M}, T^{*0,q}M')$ be a continuous operator. We say that F is properly supported on $U \cap \overline{M}$ if, for every $\chi \in \mathscr{C}^\infty_c(U \cap \overline{M})$, there are $\chi_1 \in \mathscr{C}^\infty_c(U \cap \overline{M})$, $\chi_2 \in \mathscr{C}^\infty_c(U \cap \overline{M})$, such that $F\chi u = \chi_2 F u$, $\chi F u = F\chi_1 u$ for every $u \in \Omega^{0,q}_c(U \cap \overline{M})$. We say that F is smoothing away the diagonal on $U \cap \overline{M}$ if, for every $\chi, \chi_1 \in \mathscr{C}^\infty_c(U \cap \overline{M})$ with $\operatorname{supp}\chi \cap \operatorname{supp}\chi_1 = \varnothing$, we have

$$\chi F \chi_1 \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

Lemma 4.1. *Let $\tau_1 \in \mathscr{C}^\infty(X)$, $\tau \in \mathscr{C}^\infty(\overline{M})$ with* $\operatorname{supp}\tau \cap \operatorname{supp}\tau_1 = \varnothing$. *Then,*

$$\tau \widetilde{P} \tau_1 \equiv 0 \mod \mathscr{C}^\infty(\overline{M} \times X).$$

Proof. Since $\gamma\tau\widetilde{P}\tau_1 = \tau|_X\tau_1 = 0$, we have $\widetilde{P}\gamma\tau\widetilde{P}\tau_1 = 0$. From this observation and (3-8), we have

$$\tau\widetilde{P}\tau_1 = (D^{(q)}\widetilde{\Box}^{(q)}_f + \widetilde{P}\gamma)\tau\widetilde{P}\tau_1 = D^{(q)}\widetilde{\Box}^{(q)}_f\tau\widetilde{P}\tau_1 = -D^{(q)}[\tau, \widetilde{\Box}^{(q)}_f]\widetilde{P}\tau_1. \tag{4-1}$$

By (3-7) the operator

$$D^{(q)}[\tau, \widetilde{\Box}^{(q)}_f] : H^s(\overline{M}, T^{*0,q}M') \to H^{s+1}(\overline{M}, T^{*0,q}M')$$

is continuous, for every $s \in \mathbb{Z}$. Using this observation, (3-6) and (4-1), we have

$$\tau\widetilde{P}\tau_1 : H^s(X, T^{*0,q}M') \to H^{s+\frac{3}{2}}(\overline{M}, T^{*0,q}M')$$

is continuous for every $s \in \mathbb{Z}$. We have proved that, for any $\tilde\tau \in \mathscr{C}^\infty(\overline{M})$ with $\operatorname{supp}\tilde\tau \cap \operatorname{supp}\tau_1 = \varnothing$,

$$\tilde\tau\widetilde{P}\tau_1 : H^s(X, T^{*0,q}M') \to H^{s+\frac{3}{2}}(\overline{M}, T^{*0,q}M') \tag{4-2}$$

is continuous for every $s \in \mathbb{Z}$. Let $\tilde\tau \in \mathscr{C}^\infty(\overline{M})$ with $\tilde\tau = 1$ near $\operatorname{supp}\tau$ and $\operatorname{supp}\tilde\tau \cap \operatorname{supp}\tau_1 = \varnothing$. From (4-1), we have

$$\tau\widetilde{P}\tau_1 = D^{(q)}[\tau, \widetilde{\Box}^{(q)}_f]\tilde\tau\widetilde{P}\tau_1. \tag{4-3}$$

From (4-3), (4-2) and (3-8), $\tau\widetilde{P}\tau_1 : H^s(X, T^{*0,q}M') \to H^{s+5/2}(\overline{M}, T^{*0,q}M')$ is continuous for every $s \in \mathbb{Z}$. Continuing in this way, we conclude that

$$\tau\widetilde{P}\tau_1 : H^s(X, T^{*0,q}M') \to H^{s+\frac{2N+1}{2}}(\overline{M}, T^{*0,q}M')$$

is continuous for every $s \in \mathbb{Z}$ and $N > 0$. The lemma follows. $\Box$

From Lemma 4.1 we obtain the following result for the adjoint $\widetilde{P}^*$ given by (3-9)

Lemma 4.2. *Let* $\tau_1 \in \mathscr{C}^\infty(X)$, $\tau \in \mathscr{C}^\infty(\overline{M})$ *with* $\operatorname{supp}\tau \cap \operatorname{supp}\tau_1 = \varnothing$. *Then,*

$$\tau_1 \widetilde{P}^* \tau \equiv 0 \mod \mathscr{C}^\infty(X \times \overline{M}).$$

We come back to our situation. Until further notice, we assume that the Levi form is nondegenerate of constant signature (n_-, n_+) on $D \subset X$. In the following theorem we construct a local parametrix $N^{(q)}$ for the $\bar\partial$-Neumann Laplacian on $(0,q)$-forms for $q \neq n_-$.

Theorem 4.3. *We assume that the Levi form is nondegenerate of constant signature* (n_-, n_+) *on* D *and let* $q \neq n_-$. *Then there exist a properly supported operator* $N^{(q)}$ *on* $U \cap \overline{M}$ *that is continuous for every* $s \in \mathbb{Z}$ *between*

$$N^{(q)} : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \text{for every } s \in \mathbb{Z}, \tag{4-4}$$

and such that $N^{(q)}u$ *satisfies the* $\bar\partial$-*Neumann conditions*

$$(\bar\partial\rho)^{\wedge,*}\gamma N^{(q)}u|_D = 0, \quad u \in \Omega^{0,q}(U \cap \overline{M}), \tag{4-5}$$

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N^{(q)}u|_D = 0, \quad u \in \Omega^{0,q}(U \cap \overline{M}), \tag{4-6}$$

$$\Box^{(q)}_f N^{(q)} = I + F^{(q)} \quad \text{on } \Omega^{0,q}_c(U \cap M), \tag{4-7}$$

where $F^{(q)} : \mathscr{D}'(U \cap M) \to \Omega^{0,q}(U \cap M)$ *is a properly supported smoothing operator on* $U \cap \overline{M}$.

Hence for $u \in \Omega^{0,q}_c(U \cap M)$ we have $N^{(q)}u \in \operatorname{Dom}\Box^{(q)}$ and $\Box^{(q)}N^{(q)} = I + F^{(q)}$, with $F^{(q)}$ a smoothing operator on $U \cap \overline{M}$.

Proof. Since $\Box^{(q)}_f$ is an elliptic operator on M', we can find a properly supported continuous operator

$$N^{(q)}_1 : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^{s+2}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \text{for every } s \in \mathbb{Z}$$

such that $N^{(q)}_1$ is smoothing away the diagonal on $U \cap \overline{M}$ and

$$\Box^{(q)}_f N^{(q)}_1 = I + F_1 \quad \text{on } \Omega^{0,q}_c(U \cap M'), \tag{4-8}$$

where $F_1 \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

For $u \in \Omega^{0,q}(U \cap \overline{M})$ the form $N^{(q)}_1 u$ doesn't necessarily satisfy the $\bar\partial$-Neumann conditions (4-5), (4-6). We will now construct corrections $N^{(q)}_j$, $j = 2, \dots, 7$, and finally $N^{(q)}$, starting with $N^{(q)}_1$, such that at the end the operator $N^{(q)}$ satisfies (4-4)–(4-8). Consider, for every $s \in \mathbb{Z}$,

$$N^{(q)}_2 := N^{(q)}_1 - \widetilde{P}\gamma N^{(q)}_1 : H^s_c(U \cap \overline{M}, T^{*0,q}M') \to H^{s+2}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M').$$

From (3-5) and (4-8), we see that

$$\gamma N^{(q)}_2 u|_D = 0 \quad \text{for every } u \in \Omega^{0,q}_c(U \cap \overline{M}) \tag{4-9}$$

and

$$\Box^{(q)}_f N^{(q)}_2 = I + F_2 \quad \text{on } \Omega^{0,q}_c(U \cap M'), \tag{4-10}$$

where $F_2 \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From Lemma 4.1, it is not difficult to check that $N_2^{(q)}$ is smoothing away the diagonal on $U\cap\overline{M}$. Hence, we can find a properly supported continuous operator

$$N_3^{(q)}: H^s_{\rm loc}(U\cap\overline{M}, T^{*0,q}M')\to H^{s+2}_{\rm loc}(U\cap\overline{M}, T^{*0,q}M')\quad \text{for every } s\in\mathbb{Z}$$

such that

$$N_3^{(q)}\equiv N_2^{(q)} \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{4-11}$$

From (4-9) and (4-11), we conclude that

$$\gamma N_3^{(q)}\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})). \tag{4-12}$$

Let $E^{(q)}\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ be any smoothing properly supported extension of $\gamma N_3^{(q)}$, that is, $\gamma E^{(q)}u|_D=\gamma N_3^{(q)}u|_D$, for every $u\in\Omega^{0,q}(U\cap\overline{M})$ and $E^{(q)}$ is properly supported on $U\cap\overline{M}$. For every $s\in\mathbb{Z}$ let

$$N_4^{(q)}:=N_3^{(q)}-E^{(q)}: H^s_{\rm loc}(U\cap\overline{M}, T^{*0,q}M')\to H^{s+2}_{\rm loc}(U\cap\overline{M}, T^{*0,q}M'). \tag{4-13}$$

Then $N_4^{(q)}$ is properly supported on $U\cap\overline{M}$ and

$$\begin{aligned}\gamma N_4^{(q)}u|_D&=0 &&\text{for every } u\in\Omega^{0,q}(U\cap\overline{M}),\\ \Box_f^{(q)}N_4^{(q)}&=I+F_3 &&\text{on } \Omega_c^{0,q}(U\cap M),\end{aligned} \tag{4-14}$$

where $F_3\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. Let $G^{(q)}\in L^0_{1/2,1/2}(D, T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$ be as in Theorem 3.6. Put, for every $s\in\mathbb{Z}$,

$$\begin{gathered}N_5^{(q)}: H^s_c(U\cap\overline{M}, T^{*0,q}M')\to H^{s+1}_{\rm loc}(U\cap\overline{M}, T^{*0,q}M'),\\ N_5^{(q)}:=N_4^{(q)}-\widetilde{P}G^{(q)}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_4^{(q)}.\end{gathered} \tag{4-15}$$

From Theorem 3.6, (3-11), (3-32) and (4-14), we can check that

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma N_5^{(q)}u|_D&=0 \quad \text{for every } u\in\Omega^{0,q}(U\cap\overline{M}),\\ (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_5^{(q)}&\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})),\\ \Box_f^{(q)}N_5^{(q)}&=I+F_4 \quad \text{on } \Omega_c^{0,q}(U\cap M),\end{aligned} \tag{4-16}$$

where $F_4\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. We explain the first equation in (4-16). From (4-14), we have $(\bar\partial\rho)^{\wedge,*}\gamma N_5^{(q)}u=-(\bar\partial\rho)^{\wedge,*}G^{(q)}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_4^{(q)}u=0$ since $G^{(q)}$ maps $\Omega^{0,q}(X)$ to $\Omega^{0,q}(X)$. It is not difficult to check that $N_5^{(q)}$ is smoothing away the diagonal on $U\cap\overline{M}$. Hence, we can find a properly supported continuous operator

$$N_6^{(q)}: H^s_{\rm loc}(U\cap\overline{M}, T^{*0,q}M')\to H^{s+1}_{\rm loc}(U\cap\overline{M}, T^{*0,q}M')\quad \text{for every } s\in\mathbb{Z}$$

such that

$$N_5^{(q)}\equiv N_6^{(q)} \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{4-17}$$

Let $R^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ be any smoothing properly supported extension of $2(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)}$. For every $s\in\mathbb{Z}$ put

$$N_7^{(q)} := N_6^{(q)} - R^{(q)} : H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M'). \tag{4-18}$$

From (3-17), we have

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma N_7^{(q)} &= (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} - (\bar\partial\rho)^{\wedge,*}\gamma R^{(q)}\\ &= (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} - 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)}\\ &= (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} - (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} = 0.\end{aligned} \tag{4-19}$$

From (4-16) and (4-19), we have

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma N_7^{(q)}u|_D = 0 &\quad \text{for every } u\in\Omega^{0,q}(U\cap\overline{M}),\\ (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)} \equiv 0 &\bmod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})),\\ \Box_f^{(q)}N_7^{(q)} = I + F_5 &\quad \text{on } \Omega_c^{0,q}(U\cap M),\end{aligned} \tag{4-20}$$

where $F_5 \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. Let $J^{(q)}$ be any smoothing properly supported extension of $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}$. Let $\chi\in\mathscr{C}_c^\infty((-\varepsilon,\varepsilon))$ with $\chi\equiv 1$ near 0, where $\varepsilon>0$ is a sufficiently small constant. For every $s\in\mathbb{Z}$ put

$$N^{(q)} := N_7^{(q)} - 2\chi(\rho)\rho J^{(q)} : H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M'). \tag{4-21}$$

It is not difficult to see that $N^{(q)}$ is properly supported on $U\cap\overline{M}$,

$$N^{(q)} \equiv N_7^{(q)} \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$$

and

$$(\bar\partial\rho)^{\wedge,*}\gamma N^{(q)}u|_D = (\bar\partial\rho)^{\wedge,*}\gamma N_7^{(q)}u|_D = 0 \quad \text{for every } u\in\Omega^{0,q}(U\cap\overline{M}).$$

From (3-17), we have, for every $u\in\Omega^{0,q}(U\cap M)$,

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N^{(q)}u|_D &= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D - 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma J^{(q)}u|_D\\ &= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D - 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D\\ &= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D - (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D = 0.\end{aligned} \tag{4-22}$$

We have proved that $N^{(q)}$ satisfies (4-5), (4-6) and (4-7). The theorem follows. □

Let $N^{(q)}$ be as in Theorem 4.3 and let $(N^{(q)})^* : \Omega_c^{0,q}(U\cap M) \to \mathscr{D}'(U\cap M, T^{*0,q}M')$ be the formal adjoint of $N^{(q)}$ given by

$$((N^{(q)})^*u \mid v)_M = (u \mid N^{(q)}v)_M \quad \text{for every } u,v\in\Omega_c^{0,q}(U\cap M).$$

The following result shows that $N^{(q)}$ is formally self-adjoint up to a smoothing operator.

Lemma 4.4. *With the assumptions and notation used above, we have*

$$(N^{(q)})^* u = N^{(q)} u + H^{(q)} u \quad \text{for every } u \in \Omega_c^{0,q}(U \cap M), \tag{4-23}$$

where $H^{(q)} : \mathscr{D}'(U \cap M, T^{*0,q}M') \to \Omega^{0,q}(U \cap M)$ *is a properly supported continuous operator on* $U \cap \overline{M}$ *with* $H^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Proof. Let $u, v \in \Omega_c^{0,q}(U \cap M)$. From (4-7), we have

$$\begin{aligned}((N^{(q)})^* u \mid v)_M &= ((N^{(q)})^* (\Box_f^{(q)} N^{(q)} - F^{(q)}) u \mid v)_M \\ &= (\Box_f^{(q)} N^{(q)} u \mid N^{(q)} v)_M - (F^{(q)} u \mid N^{(q)} v)_M.\end{aligned} \tag{4-24}$$

From (4-5) and (4-6), we can integrate by parts and get

$$(\Box_f^{(q)} N^{(q)} u \mid N^{(q)} v)_M = (N^{(q)} u \mid \Box_f^{(q)} N^{(q)} v)_M = (N^{(q)} u \mid (I + F^{(q)}) v)_M, \tag{4-25}$$

where we used (4-7). From (4-24) and (4-25), we deduce that

$$((N^{(q)})^* u \mid v)_M = ((N^{(q)} + ((F^{(q)})^* N^{(q)})) u \mid v)_M - (u \mid (F^{(q)})^* N^{(q)} v)_M, \tag{4-26}$$

where $(F^{(q)})^* : \Omega_c^{0,q}(U \cap M) \to \mathscr{D}'(U \cap M, T^{*0,q}M')$ is the formal adjoint of $F^{(q)}$ with respect to $(\cdot \mid \cdot)_M$. It is clear that $(F^{(q)})^*$ is a properly supported continuous operator on $U \cap \overline{M}$ with $(F^{(q)})^* \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

It is not difficult to check that $(F^{(q)})^* N^{(q)}$ is a properly supported continuous operator on $U \cap \overline{M}$ with $(F^{(q)})^* N^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$. Let

$$((F^{(q)})^* N^{(q)})^* : \Omega_c^{0,q}(U \cap M) \to \mathscr{D}'(U \cap M, T^{*0,q}M')$$

be the formal adjoint of $(F^{(q)})^* N^{(q)}$ with respect to $(\cdot \mid \cdot)_M$. Then $((F^{(q)})^* N^{(q)})^*$ is a properly supported continuous operator on $U \cap \overline{M}$ with

$$((F^{(q)})^* N^{(q)})^* \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

From this observation and (4-26), we have

$$((N^{(q)})^* u \mid v)_M = \big((N^{(q)} + (F^{(q)})^* N^{(q)} - ((F^{(q)})^* N^{(q)})^*) u \mid v\big)_M.$$

Relation (4-23) follows. □

From (4-23), we can extend $(N^{(q)})^*$ to

$$(N^{(q)})^* : L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \text{for every } s \in \mathbb{Z}$$

as a properly supported continuous operator on $U \cap \overline{M}$ and we have

$$(N^{(q)})^* u = N^{(q)} u + H^{(q)} u \quad \text{for every } u \in L^2_{\mathrm{loc}}(U \cap M, T^{*0,q}M'), \tag{4-27}$$

where $H^{(q)}$ is as in (4-23). Moreover, for every $g \in L^2_c(U \cap M, T^{*0,q}M')$ and $u \in L^2_{\mathrm{loc}}(U \cap M, T^{*0,q}M')$, we have

$$((N^{(q)})^* u \mid g)_M = (u \mid N^{(q)} g)_M, \quad ((N^{(q)})^* g \mid u)_M = (g \mid N^{(q)} u)_M. \tag{4-28}$$

We can now improve Theorem 4.3.

Theorem 4.5. *With the assumptions and notation used above, let $q \neq n_-$. We have*

$$N^{(q)}\Box^{(q)}u = u + F_1^{(q)}u \quad \text{on } U\cap M \text{ for every } u \in \operatorname{Dom}\Box^{(q)}, \tag{4-29}$$

$$\Box_f^{(q)}N^{(q)}u = u + F_2^{(q)}u \quad \text{on } U\cap M \text{ for every } u \in \Omega^{0,q}(U\cap \overline{M}), \tag{4-30}$$

where $F_1^{(q)}, F_2^{(q)} : \mathscr{D}'(U\cap M) \to \Omega^{0,q}(U\cap M)$ are properly supported smoothing operators on $U\cap\overline{M}$.

Remark 4.6. Let $u \in \operatorname{Dom}\Box^{(q)}$. By (4-29) we have, for every $g \in \Omega_c^{0,q}(U\cap M)$,

$$(N^{(q)}\Box^{(q)}u \mid g)_M = (u + F_1^{(q)}u \mid g)_M. \tag{4-31}$$

Since $N^{(q)}$ and $F_1^{(q)}$ are properly supported operators on $U\cap\overline{M}$, (4-31) makes sense. For $u \in \Omega^{0,q}(U\cap M)$, equation (4-30) means that, for every $g \in \Omega_c^{0,q}(U\cap M)$, we have

$$(\Box_f^{(q)}N^{(q)}u \mid g)_M = (u + F_2^{(q)}u \mid g)_M. \tag{4-32}$$

Proof of Theorem 4.5. Let $u \in \operatorname{Dom}\Box^{(q)}$. Then, $\Box^{(q)}u \in L^2_{(0,q)}(M) \subset L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$. Let $g \in \Omega_c^{0,q}(U\cap M)$. From (4-27) and (4-28), we have

$$\begin{aligned}(N^{(q)}\Box^{(q)}u \mid g)_M &= ((N^{(q)})^* - H^{(q)})\Box^{(q)}u \mid g)_M \\ &= (\Box^{(q)}u \mid N^{(q)}g)_M - (H^{(q)}\Box_f^{(q)}u \mid g)_M.\end{aligned} \tag{4-33}$$

Since $u \in \operatorname{Dom}\Box^{(q)}$ and by (4-5), (4-6), $N^{(q)}g \in \operatorname{Dom}\Box^{(q)}$, we can integrate by parts and get

$$(\Box^{(q)}u \mid N^{(q)}g)_M = (u \mid \Box^{(q)}N^{(q)}g)_M = (u \mid (I + F^{(q)})g)_M = (u + (F^{(q)})^*u \mid g)_M, \tag{4-34}$$

where $F^{(q)}$ is as in (4-7) and $(F^{(q)})^*$ is the formal adjoint of $F^{(q)}$. From (4-33) and (4-34), we have

$$(N^{(q)}\Box^{(q)}u \mid g)_M = (u + (F^{(q)})^*u - H^{(q)}\Box_f^{(q)}u \mid g)_M. \tag{4-35}$$

From (4-35), we get (4-29) with $F_1^{(q)} = (F^{(q)})^* - H^{(q)}\Box_f^{(q)}$.

Let $u \in \Omega^{0,q}(U\cap\overline{M})$ and let $g \in \Omega_c^{0,q}(U\cap M)$. From (4-27), (4-28), (4-29), and since $N^{(q)}$ is properly supported on $U\cap\overline{M}$, we have

$$\begin{aligned}(\Box_f^{(q)}N^{(q)}u \mid g)_M &= (N^{(q)}u \mid \Box_f^{(q)}g)_M = (u \mid (N^{(q)})^*\Box_f^{(q)}g)_M \\ &= (u \mid (N^{(q)} + H^{(q)})\Box_f^{(q)}g)_M = (u \mid g + F_1^{(q)}g + H^{(q)}\Box_f^{(q)}g)_M \\ &= (u + (F_1^{(q)})^*u + (H^{(q)}\Box_f^{(q)})^*u \mid g)_M,\end{aligned} \tag{4-36}$$

where $(F_1^{(q)})^*$ and $(H^{(q)}\Box_f^{(q)})^*$ are the formal adjoints of $F_1^{(q)}$ and $H^{(q)}\Box_f^{(q)}$ respectively. From (4-36), we get (4-30) with $F_2^{(q)} = (F_1^{(q)})^* + (H^{(q)}\Box_f^{(q)})^*$. □

From Theorems 4.3 and 4.5, we get the main result of this section about the local parametrix of the $\bar\partial$-Neumann Laplacian.

Theorem 4.7. *Let U be an open set of M' with $U\cap X\neq\varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-,n_+) on $U\cap X$. Let $q\neq n_-$. We can find properly supported continuous operators on $U\cap\overline{M}$:*

$$N^{(q)}:H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\to H^{s+1}_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\quad \text{for every } s\in\mathbb{Z}$$

such that (4-5), (4-6), (4-23), (4-28), (4-29) and (4-30) hold.

5. Microlocal Hodge decomposition in the critical degree

In this section we will construct a local parametrix of $N^{(q)}$ of the $\bar\partial$-Neumann Laplacian acting on $(0,q)$-forms and a local approximate Bergman operator $\Pi^{(q)}$ in the critical degree $q=n_-$.

We briefly recall the global situation [Folland and Kohn 1972, (3.1.7)–(3.1.19)]. Assume that $Z(q-1)$ and $Z(q+1)$ hold everywhere on X (but $Z(q)$ does not necessarily hold). Then $\Box^{(q)}$ is bounded away from zero on $(\operatorname{Ker}\Box^{(q)})^\perp$, so $\Box^{(q)}$ has closed range in L^2 and one can define a bounded operator $N^{(q)}:L^2(M,T^{*0,q}M')\to\operatorname{Dom}\Box^{(q)}$ (the $\bar\partial$-Neumann operator) such that

$$\begin{gathered}u=\bar\partial\bar\partial^*N^{(q)}u+\bar\partial^*\bar\partial N^{(q)}u+B^{(q)}u,\quad u\in L^2(M,T^{*0,q}M'),\\ B^{(q)}N^{(q)}=N^{(q)}B^{(q)}=0,\quad N^{(q)}\Box^{(q)}=\Box^{(q)}N^{(q)}=I-B^{(q)}\quad\text{on }\operatorname{Dom}\Box^{(q)},\\ B^{(q)}=I-\bar\partial N^{(q-1)}\bar\partial^*-\bar\partial^*N^{(q+1)}\bar\partial\quad\text{on }\operatorname{Dom}\bar\partial\cap\operatorname{Dom}\bar\partial^*,\\ B^{(q)}(\operatorname{Dom}\bar\partial^*\cap\Omega^{0,q}(\overline{M}))\subset\operatorname{Dom}\Box^{(q)}\cap\Omega^{0,q}(\overline{M}).\end{gathered}\tag{5-1}$$

If the Levi form is nondegenerate of signature (n_-,n_+) on an open set $D\subset X$, then $Z(q)$ holds on D if and only if $q\neq n_-$. We will give in this section a (micro-)local version of the above global results in the critical degree $q=n_-$, in which case the Neumann operator will be a local parametrix of the $\bar\partial$-Neumann operator and the Bergman projection $B^{(q)}$ will be replaced by an approximate Bergman projection $\Pi^{(q)}$.

5.1. ***The parametrix and the approximate Bergman operator.*** We recall the following lemma about integration by parts.

Lemma 5.1 [Folland and Kohn 1972, p. 13]. *For all $f\in\Omega^{0,q}(\overline{M})$, $g\in\Omega^{0,q+1}(\overline{M})$, we have*

$$(g\mid\bar\partial f)_M=(\bar\partial^*_f g\mid f)_M+((\bar\partial\rho)^{\wedge,*}\gamma g\mid\gamma f)_X.\tag{5-2}$$

Let D be a local coordinate patch of X with local coordinates $x=(x_1,\dots,x_{2n-1})$. Then, $\hat x:=(x_1,\dots,x_{2n-1},\rho)$ are local coordinates of M' defined in an open set U of M' with $U\cap X=D$. Until further notice, we work on U.

Lemma 5.2. *Let $u\in\Omega^{0,q}(U\cap\overline{M})$. Assume that $(\bar\partial\rho)^{\wedge,*}\gamma u|_D=0$. Then,*

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u|_D=0.\tag{5-3}$$

Proof. Let $g\in\Omega^{0,q-2}_c(U\cap\overline{M})$. From (5-2), we have

$$(\bar\partial^*_f u\mid\bar\partial g)_M=((\bar\partial^*_f)^2u\mid g)_M+((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u\mid\gamma g)_X=((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u\mid\gamma g)_X.\tag{5-4}$$

On the other hand, from (5-2) again, we have

$$0=(u\mid\bar\partial^2 g)_M=(\bar\partial^*_f u\mid\bar\partial g)_M+((\bar\partial\rho)^{\wedge,*}\gamma u\mid\gamma\bar\partial g)_X=(\bar\partial^*_f u\mid\bar\partial g)_M \tag{5-5}$$

since $(\bar\partial\rho)^{\wedge,*}\gamma u|_D=0$. From (5-4) and (5-5), we conclude that

$$((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u\mid\gamma g)_X=0.$$

Since g is arbitrary, $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u|_D=0$. □

We now assume that the Levi form is nondegenerate of constant signature (n_-,n_+) on $D=U\cap X$. Let $q=n_-$. Let $N^{(q+1)}$ and $N^{(q-1)}$ be local parametrices of the $\bar\partial$-Neumann Laplacian as in Theorem 4.7. We define, for every $s\in\mathbb{Z}$,

$$\begin{aligned}\widehat N^{(q)}&:H^s_{\mathrm{loc}}(U\cap\overline M,T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline M,T^{*0,q}M'),\\ \widehat N^{(q)}&:=\bar\partial^*_f(N^{(q+1)})^2\bar\partial+\bar\partial(N^{(q-1)})^2\bar\partial^*_f.\end{aligned}\tag{5-6}$$

Put

$$A^{0,q}(U\cap\overline M):=\{u\in\Omega^{0,q}(U\cap\overline M):(\bar\partial\rho)^{\wedge,*}\gamma u|_D=0\}=\operatorname{Dom}\bar\partial^*\cap\Omega^{0,q}(U\cap\overline M).\tag{5-7}$$

We define

$$\widehat\Pi^{(q)}:=I-\bar\partial^*_fN^{(q+1)}\bar\partial-\bar\partial N^{(q-1)}\bar\partial^*_f:A^{0,q}(U\cap\overline M)\to\Omega^{0,q}(U\cap\overline M).\tag{5-8}$$

We show in Theorem 5.3 below that the operators $\widehat N^{(q)}$ and $\widehat\Pi^{(q)}$ provide a rough version of the microlocal Hodge decomposition. By (5-9) the operator $\widehat N^{(q)}$ satisfies the first $\bar\partial$-Neumann condition. However, by (5-10), the second $\bar\partial$-Neumann condition is satisfied only modulo a smoothing operator (analogously for $\widehat\Pi^{(q)}$ by (5-12)). In the sequel we will modify these operators in order to obtain operators $N^{(q)}$ (the parametrix of the $\bar\partial$-Neumann Laplacian) and $\Pi^{(q)}$ (the approximate Bergman projector) which satisfy exactly the $\bar\partial$-Neumann condition (see Theorems 5.9, 5.11, 5.23).

Theorem 5.3. *With the assumptions and notation above, let $q=n_-$. We have*

$$(\bar\partial\rho)^{\wedge,*}\gamma\widehat N^{(q)}u=0\quad\text{for every } u\in\Omega^{0,q}(U\cap\overline M),\tag{5-9}$$

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widehat N^{(q)}u=H^{(q)}_1u\quad\text{for every } u\in\Omega^{0,q}(U\cap\overline M),\tag{5-10}$$

$$\widehat\Pi^{(q)}u\in A^{0,q}(U\cap\overline M)\quad\text{for every } u\in A^{0,q}(U\cap\overline M),\tag{5-11}$$

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widehat\Pi^{(q)}u=H^{(q)}_2u\quad\text{for every } u\in A^{0,q}(U\cap\overline M),\tag{5-12}$$

$$\Box^{(q)}_f\widehat N^{(q)}u+\widehat\Pi^{(q)}u=u+H^{(q)}_3u\quad\text{for every } u\in A^{0,q}(U\cap\overline M),\tag{5-13}$$

$$\bar\partial\widehat\Pi^{(q)}u=H^{(q)}_4u\quad\text{for every } u\in A^{0,q}(U\cap\overline M)\cap\Omega^{0,q}_c(U\cap\overline M),\tag{5-14}$$

$$\bar\partial^*_f\widehat\Pi^{(q)}u=H^{(q)}_5u\quad\text{for every } u\in A^{0,q}(U\cap\overline M)\cap\Omega^{0,q}_c(U\cap\overline M),\tag{5-15}$$

where $H^{(q)}_j$, $j=1,\dots,5$, are properly supported on $U\cap\overline M$ and

$$\begin{aligned}H^{(q)}_j&\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(X\times\overline M)),\quad j=1,2,\\ H^{(q)}_j&\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline M\times\overline M)),\quad j=3,4,5.\end{aligned}$$

Proof. From (4-5), (4-6), Lemma 5.2 and the definitions of $\widehat{N}^{(q)}$, $\widehat{\Pi}^{(q)}$, we get (5-9) and (5-11). Let $u \in \Omega^{0,q}(U \cap \overline{M})$. From (4-30) and (5-6), we have

$$\begin{aligned}
&(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widehat{N}^{(q)}u \\
&\quad= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial^*_f(N^{(q+1)})^2\bar\partial u \\
&\quad= (\bar\partial\rho)^{\wedge,*}\gamma\Box^{(q+1)}_f(N^{(q+1)})^2\bar\partial u - (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u \\
&\quad= (\bar\partial\rho)^{\wedge,*}\gamma(I+F^{(q+1)}_2)N^{(q+1)}\bar\partial u - (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u \\
&\quad= (\bar\partial\rho)^{\wedge,*}\gamma F^{(q+1)}_2N^{(q+1)}\bar\partial u + (\bar\partial\rho)^{\wedge,*}\gamma N^{(q+1)}\bar\partial u - (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u, \qquad (5\text{-}16)
\end{aligned}$$

where $F^{(q+1)}_2 \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (4-30). Again, from (4-5), (4-6) and Lemma 5.2, we see that

$$(\bar\partial\rho)^{\wedge,*}\gamma N^{(q+1)}\bar\partial u|_D = 0,$$
$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u|_D = 0.$$

From this observation, (5-16) and noticing that

$$(\bar\partial\rho)^{\wedge,*}\gamma F^{(q+1)}_2N^{(q+1)}\bar\partial \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})),$$

we get (5-10). We now prove (5-12). From (5-8), we have

$$\begin{aligned}
\bar\partial\widehat{\Pi}^{(q)} &= \bar\partial - \bar\partial\,\bar\partial^*_fN^{(q+1)}\bar\partial \\
&= \bar\partial - \Box^{(q+1)}_fN^{(q+1)}\bar\partial + \bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial \\
&= -F^{(q+1)}_2\bar\partial + \bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial, \qquad (5\text{-}17)
\end{aligned}$$

where $F^{(q+1)}_2$ is as in (4-30). Let $u \in A^{0,q}(U \cap \overline{M})$. From (5-17), we have

$$\bar\partial\widehat{\Pi}^{(q)}u = -F^{(q+1)}_2\bar\partial u + \bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial u. \qquad (5\text{-}18)$$

From (4-6) and (5-3), we see that $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial u|_D = 0$. From this observation and (5-18), we get (5-12).

Let $u \in A^{0,q}(U \cap \overline{M})$. From (4-30), (5-6) and (5-8), we have

$$\begin{aligned}
\Box^{(q)}_f\widehat{N}^{(q)}u &= \Box^{(q)}_f\big(\bar\partial^*_f(N^{(q+1)})^2\bar\partial + \bar\partial(N^{(q-1)})^2\bar\partial^*_f\big)u \\
&= \bar\partial^*_f\Box^{(q+1)}_f(N^{(q+1)})^2\bar\partial u + \bar\partial\Box^{(q-1)}_f(N^{(q-1)})^2\bar\partial^*_fu \\
&= \bar\partial^*_f(I+F^{(q+1)}_2)N^{(q+1)}\bar\partial u + \bar\partial(I+F^{(q-1)}_2)N^{(q-1)}\bar\partial^*_fu \\
&= \bar\partial^*_fN^{(q+1)}\bar\partial u + \bar\partial N^{(q-1)}\bar\partial^*_fu + \bar\partial^*_fF^{(q+1)}_2N^{(q+1)}\bar\partial u + \bar\partial F^{(q-1)}_2N^{(q-1)}\bar\partial^*_fu \\
&= (I-\widehat{\Pi}^{(q)})u + \big(\bar\partial^*_fF^{(q+1)}_2N^{(q+1)}\bar\partial + \bar\partial F^{(q-1)}_2N^{(q-1)}\bar\partial^*_f\big)u, \qquad (5\text{-}19)
\end{aligned}$$

where

$$F_2^{(q+1)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),$$
$$F_2^{(q-1)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$$

are as in (4-30). It is clear that

$$\bar\partial_f^* F_2^{(q+1)} N^{(q+1)}\bar\partial + \bar\partial F_2^{(q-1)} N^{(q-1)}\bar\partial_f^* \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From this observation and (5-19), we get (5-13).

Let $u\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$, from (4-29), (4-30), (5-6) and (5-8), we have

$$\begin{aligned}\bar\partial_f^*\widehat{\Pi}^{(q)}u &= \bar\partial_f^*u - \bar\partial_f^*(\bar\partial_f^* N^{(q+1)}\bar\partial u - \bar\partial N^{(q-1)}\bar\partial_f^* u)\\ &= \bar\partial_f^*u - \bar\partial_f^*\,\bar\partial N^{(q-1)}\bar\partial_f^* u\\ &= \bar\partial_f^*u - (\Box_f^{(q-1)} - \bar\partial\,\bar\partial_f^*)N^{(q-1)}\bar\partial_f^* u\\ &= \bar\partial_f^*u - (I + F_2^{(q-1)})\bar\partial_f^* u + \bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u\\ &= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u.\end{aligned}\tag{5-20}$$

For every $g\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$, from (4-6), (4-30) and (5-3), we have

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* g &= (\bar\partial\rho)^{\wedge,*}\gamma(\Box_f^{(q-1)} - \bar\partial_f^*\,\bar\partial)N^{(q-1)}\bar\partial_f^* g\\ &= (\bar\partial\rho)^{\wedge,*}\gamma(I + F_2^{(q-1)})\bar\partial_f^* g - (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial_f^*\,\bar\partial N^{(q-1)}\bar\partial_f^* g\\ &= (\bar\partial\rho)^{\wedge,*}\gamma F_2^{(q-1)}\bar\partial_f^* g.\end{aligned}\tag{5-21}$$

Thus,

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial_f^* N^{(q-1)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})).$$

Let $\hat\varepsilon^{(q-1)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ be any smoothing properly supported extension of $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial_f^* N^{(q-1)}$. Put

$$\varepsilon^{(q-1)} := 2\chi(\rho)\rho\hat\varepsilon^{(q-1)} : \mathscr{D}'(U\cap M, T^{*0,q-1}M') \to \Omega^{0,q-2}(U\cap M),$$

where $\chi\in\mathscr{C}_c^\infty((-\varepsilon,\varepsilon))$, $\chi\equiv 1$ near $0\in\mathbb{R}$, for a sufficiently small constant $\varepsilon>0$. We have

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* g = 0,\\ (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* g = 0\end{aligned}\tag{5-22}$$

for every $g\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$ and hence

$$(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* g \in \operatorname{Dom}\Box^{(q-2)}\tag{5-23}$$

for every $g\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$. From (4-29), (4-30), (5-20) and (5-23), we have

$$\begin{aligned}\bar\partial_f^*\widehat{\Pi}^{(q)}u &= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u\\ &= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u\end{aligned}\tag{5-24}$$

$$
\begin{aligned}
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,(N^{(q-2)}\Box^{(q-2)} - F_1^{(q-2)})(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u\\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\Box_f^{(q-2)}\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u\\
&\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u\\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\bar\partial_f^*\Box_f^{(q-1)} N^{(q-1)}\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u\\
&\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u\\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\bar\partial_f^*(I + F_2^{(q-1)})\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u\\
&\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u\\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\bar\partial_f^* F_2^{(q-1)}\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u\\
&\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u, \qquad \text{(5-24 cont.)}
\end{aligned}
$$

where $u \in A^{0,q}(U\cap\overline{M}) \cap \Omega_c^{0,q}(U\cap\overline{M})$. It is clear that

$$
\begin{aligned}
&-F_2^{(q-1)}\bar\partial_f^* + \bar\partial N^{(q-2)}\bar\partial_f^* F_2^{(q-1)}\bar\partial_f^* - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^*\\
&\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).
\end{aligned}
$$

From this observation and (5-24), we get (5-15). The proof of (5-14) is similar but simpler and therefore we omit the details. □

From (5-14) and (5-15), we get

$$
\Box_f^{(q)}\hat\Pi^{(q)} u = H_6^{(q)} u \quad \text{for every } u \in A^{0,q}(U\cap\overline{M}) \cap \Omega_c^{0,q}(U\cap\overline{M}), \tag{5-25}
$$

where $H_6^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $H_6^{(q)}$ is properly supported on $U\cap\overline{M}$.

Lemma 5.4. *With the assumptions and notation above, let $q = n_-$. We have*

$$
(\hat N^{(q)} u \mid v)_M = (u \mid \hat N^{(q)} v)_M + (u \mid \hat\Gamma^{(q)} v)_M
$$

for every $u \in L_c^2(U\cap\overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$, where $\hat\Gamma^{(q)}$ is properly supported on $U\cap\overline{M}$ and $\hat\Gamma^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$.

Proof. Let $u \in L_c^2(U\cap\overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$. Let $u_j \in \Omega_c^{0,q}(U\cap\overline{M})$, $v_j \in \Omega_c^{0,q}(U\cap\overline{M})$, $j = 1, 2, \ldots$, such that $u_j \to u$ in $L_c^2(U\cap\overline{M}, T^{*0,q}M')$ as $j\to\infty$ and $v_j \to v$ in $L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$ as $j\to\infty$. From (5-6), we see that

$$
(\hat N^{(q)} u \mid v)_M = \lim_{j\to+\infty} (\hat N^{(q)} u_j \mid v_j)_M. \tag{5-26}
$$

We infer from (4-28) that for every $j \in \mathbb{N}$ we have $(\hat N^{(q)} u_j \mid v_j)_M = (u_j \mid (\hat N^{(q)})^* v_j)_M$. From (4-23), we see that $(\hat N^{(q)})^* = N^{(q)} + \hat\Gamma^{(q)}$ on $\Omega_c^{0,q}(U\cap\overline{M})$, where $\hat\Gamma^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $\hat\Gamma^{(q)}$ is properly supported on $U\cap\overline{M}$. From this observation, we conclude that

$$
(\hat N^{(q)} u_j \mid v_j)_M = (u_j \mid \hat N^{(q)} v_j)_M + (u_j \mid \hat\Gamma^{(q)} v_j)_M \quad \text{for every } j \in \mathbb{N}. \tag{5-27}
$$

From (5-26) and (5-27), the lemma follows. □

Lemma 5.5. *With the assumptions and notation used above, let $q=n_-$. Fix an open set $W\subset U$ with $\overline{W}$ a compact subset of U. There is a constant $C_W>0$ such that, for every $u\in A^{0,q}(U\cap\overline{M})\cap\Omega^{0,q}_c(W\cap\overline{M})$,*

$$\|\widehat{\Pi}^{(q)}u\|_M\leq C_W\|u\|_M. \tag{5-28}$$

Proof. Let $u\in A^{0,q}(U\cap\overline{M})\cap\Omega^{0,q}_c(W\cap\overline{M})$. From (5-13), we have

$$\begin{aligned}(\widehat{\Pi}^{(q)}u\mid\widehat{\Pi}^{(q)}u)_M&=(\widehat{\Pi}^{(q)}u\mid u)_M-(\widehat{\Pi}^{(q)}u\mid(I-\widehat{\Pi}^{(q)})u)_M\\&=(\widehat{\Pi}^{(q)}u\mid u)_M-(\widehat{\Pi}^{(q)}u\mid(\Box^{(q)}_f\widehat{N}^{(q)}-H^{(q)}_3)u)_M.\end{aligned}\tag{5-29}$$

From (5-9) and (5-10), we can repeat the proof of Theorem 4.3 and deduce that there is a properly supported operator $N^{(q)}:\mathscr{D}'(U\cap M,T^{*0,q}M')\to\Omega^{0,q}(U\cap M)$ on $U\cap\overline{M}$ with $N^{(q)}-\widehat{N}^{(q)}\equiv 0\bmod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ such that

$$N^{(q)}g\in\operatorname{Dom}\Box^{(q)}\tag{5-30}$$

for every $g\in A^{0,q}(U\cap\overline{M})\cap\Omega^{0,q}_c(W\cap\overline{M})$. From (5-11), (5-13), (5-14), (5-15), (5-29), (5-30), we have

$$\begin{aligned}(\widehat{\Pi}^{(q)}u\mid\widehat{\Pi}^{(q)}u)_M&=(\widehat{\Pi}^{(q)}u\mid u)_M-(\widehat{\Pi}^{(q)}u\mid(\Box^{(q)}_f\widehat{N}^{(q)}-H^{(q)}_3)u)_M\\&=(\widehat{\Pi}^{(q)}u\mid u)_M-(\widehat{\Pi}^{(q)}u\mid(\Box^{(q)}_fN^{(q)}-H^{(q)}_3)u)_M+(\widehat{\Pi}^{(q)}u\mid\Box^{(q)}_f(N^{(q)}-\widehat{N}^{(q)})u)_M\\&=(\widehat{\Pi}^{(q)}u\mid u)_M-(\bar\partial\widehat{\Pi}^{(q)}u\mid\bar\partial N^{(q)}u)_M-(\bar\partial^*_f\widehat{\Pi}^{(q)}u\mid\bar\partial^*N^{(q)}u)_M\\&\qquad+(\widehat{\Pi}^{(q)}u\mid H^{(q)}_3u)_M+(\widehat{\Pi}^{(q)}u\mid\Box^{(q)}_f(N^{(q)}-\widehat{N}^{(q)})u)_M\\&=(\widehat{\Pi}^{(q)}u\mid u)_M-(H^{(q)}_4u\mid\bar\partial N^{(q)}u)_M-(H^{(q)}_5u\mid\bar\partial^*N^{(q)}u)_M\\&\qquad+(\widehat{\Pi}^{(q)}u\mid H^{(q)}_3u)_M+(\widehat{\Pi}^{(q)}u\mid\Box^{(q)}_f(N^{(q)}-\widehat{N}^{(q)})u)_M\\&=(\widehat{\Pi}^{(q)}u\mid u)_M-(u\mid((H^{(q)}_4)^*\bar\partial N^{(q)}+(H^{(q)}_5)^*\bar\partial^*N^{(q)})u)_M\\&\qquad+(\widehat{\Pi}^{(q)}u\mid H^{(q)}_3u)_M+(\widehat{\Pi}^{(q)}u\mid\Box^{(q)}_f(N^{(q)}-\widehat{N}^{(q)})u)_M,\end{aligned}\tag{5-31}$$

where

$$H^{(q)}_3,H^{(q)}_4,H^{(q)}_5\equiv 0\bmod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$$

are as in (5-13), (5-14), (5-15), and $(H^{(q)}_4)^*$ and $(H^{(q)}_5)^*$ are the formal adjoints of $H^{(q)}_4$ and $H^{(q)}_5$, respectively. Note that the operators

$$(H^{(q)}_4)^*\bar\partial N^{(q)}+(H^{(q)}_5)^*\bar\partial^*N^{(q)},\quad H^{(q)}_3,\quad\Box^{(q)}_f(N^{(q)}-\widehat{N}^{(q)})$$

map $L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')$ into itself continuously. From this observation and (5-31), we deduce that there exists $\widehat{C}>0$ such that

$$\|\widehat{\Pi}^{(q)}u\|^2_M\leq\widehat{C}(\|\widehat{\Pi}^{(q)}u\|_M\|u\|_M+\|u\|^2_M),\quad u\in A^{0,q}(U\cap\overline{M})\cap\Omega^{0,q}_c(W\cap\overline{M}).\tag{5-32}$$

From (5-32), we get (5-28). □

As a comment regarding the proof of Lemma 5.5, one could try to use $\widehat{N}^{(q)}$ directly, since $\bar\partial\widehat{N}^{(q)}$, $\bar\partial^*\widehat{N}^{(q)}$ are also bounded in L^2_{loc}. However, the range of $\widehat{N}^{(q)}$ is not contained in $\operatorname{Dom}\Box^{(q)}$, since $(\bar\partial\rho)^{\wedge,*}\gamma\widehat{N}^{(q)}$ and $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widehat{N}^{(q)}$ do not necessarily vanish on the boundary (we only know that they are smoothing operators). Thus, we use the operator $N^{(q)}$ which satisfies (5-30).

Remark 5.6. Since $N^{(q-1)}$ and $N^{(q+1)}$ are properly supported on $U \cap \overline{M}$, $\hat{\Pi}$ is properly supported on $U \cap \overline{M}$. Hence for every $\chi \in \mathscr{C}_c^\infty(U \cap \overline{M})$, there are $\chi_1 \in \mathscr{C}_c^\infty(U \cap \overline{M})$, $\chi_2 \in \mathscr{C}_c^\infty(U \cap \overline{M})$, such that

$$\hat{\Pi}^{(q)} \chi u = \chi_2 \hat{\Pi}^{(q)} u \quad \text{for every } u \in A^{0,q}(U \cap \overline{M}),$$
$$\chi \hat{\Pi}^{(q)} u = \hat{\Pi}^{(q)} \chi_1 u \quad \text{for every } u \in A^{0,q}(U \cap \overline{M}).$$

By Lemma 5.5 we can extend $\hat{\Pi}^{(q)}$ to $L^2_c(U \cap \overline{M}, T^{*0,q}M')$ by density. More precisely, let $u \in L^2_c(U \cap \overline{M}, T^{*0,q}M')$. Suppose that $\operatorname{supp} u \subset W$, where $W \subset U$ is an open set with $\overline{W} \Subset U$. Take any sequence $(u_j)_j$ in $A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(W \cap \overline{M})$, with $\lim_{j\to+\infty}\|u_j - u\|_M = 0$. Since $\hat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$, we have

$$\hat{\Pi}^{(q)} u := \lim_{j\to+\infty} \hat{\Pi}^{(q)} u_j \quad \text{in } L^2_c(U \cap \overline{M}, T^{*0,q}M'). \tag{5-33}$$

By using that $\hat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$, we extend $\hat{\Pi}^{(q)}$ to $L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$ and the extensions

$$\begin{aligned} \hat{\Pi}^{(q)} &: L^2_c(U \cap \overline{M}, T^{*0,q}M') \to L^2_c(U \cap \overline{M}, T^{*0,q}M'), \\ \hat{\Pi}^{(q)} &: L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \end{aligned} \tag{5-34}$$

are continuous.

Lemma 5.7. *With the assumptions and notation above, let $q = n_-$. We have*

$$(\hat{\Pi}^{(q)} u \mid v)_M = (u \mid \hat{\Pi}^{(q)} v)_M + (u \mid \hat{\Gamma}_1^{(q)} v)_M \tag{5-35}$$

for every $u \in L^2_c(U \cap \overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$, where $\hat{\Gamma}_1^{(q)}$ is a properly supported continuous operator on $U \cap \overline{M}$ and $\hat{\Gamma}_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Proof. From (4-23), (4-28) and (5-8), we get (5-35) for $u, v \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(U \cap \overline{M})$. By using a density argument and noticing that $\hat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$, we get (5-35). ☐

Theorem 5.8. *We have*

$$\chi \hat{\Pi}^{(q)} u \in \operatorname{Dom} \bar{\partial}^* \quad \textit{for every } \chi \in \mathscr{C}_c^\infty(U \cap \overline{M}),\ u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{5-36}$$

$$\bar{\partial} \hat{\Pi}^{(q)} u = H_4^{(q)} u \quad \textit{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{5-37}$$

$$\bar{\partial}_f^* \hat{\Pi}^{(q)} u = H_5^{(q)} u \quad \textit{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{5-38}$$

$$\Box_f^{(q)} \hat{N}^{(q)} u + \hat{\Pi}^{(q)} u = u + H_3^{(q)} u \quad \textit{for every } u \in \Omega^{0,q}(U \cap \overline{M}), \tag{5-39}$$

where $H_j^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$, $j = 3, 4, 5$, are as in Theorem 5.3.

Proof. Let $u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$ and let $\chi \in \mathscr{C}_c^\infty(U \cap \overline{M})$. Since $\hat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$ (see Remark 5.6), there is a $\chi_1 \in \mathscr{C}_c^\infty(U \cap \overline{M})$ such that $\chi \hat{\Pi}^{(q)} = \hat{\Pi}^{(q)} \chi_1$ on $L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$. Let $g \in \operatorname{Dom} \bar{\partial} \cap L^2_{(0,q)}(\overline{M})$. Let $u_j \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(U \cap \overline{M})$, $j = 1, 2, \dots$, with $\lim_{j\to+\infty}\|u_j - \chi_1 u\|_M = 0$. Then,

$$(\chi\widehat{\Pi}^{(q)}u\,|\,\bar\partial g)_M=(\widehat{\Pi}^{(q)}\chi_1u\,|\,\bar\partial g)_M=\lim_{j\to+\infty}(\widehat{\Pi}^{(q)}u_j\,|\,\bar\partial g)_M$$
$$=\lim_{j\to+\infty}(\bar\partial^*\widehat{\Pi}^{(q)}u_j\,|\,g)_M=\lim_{j\to+\infty}(H_5^{(q)}u_j\,|\,g)_M=(H_5^{(q)}u\,|\,g)_M,\tag{5-40}$$

where $H_5^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-15).

From (5-40), we deduce that $\chi\widehat{\Pi}^{(q)}u\in\operatorname{Dom}\bar\partial^*$, we get (5-36) and we also get (5-38). The proof of (5-37) is similar. We now prove (5-39).

Let $u\in\Omega^{0,q}(U\cap\overline{M})$ and let $g\in\Omega_c^{0,q}(U\cap M)$. Since $\widehat{\Pi}^{(q)}$, $\widehat{N}^{(q)}$ and $H_3^{(q)}$ are properly supported on $U\cap\overline{M}$, there is a $\tau\in\mathscr{C}_c^\infty(U\cap\overline{M})$ such that

$$\begin{gathered}((\Box_f^{(q)}\widehat{N}^{(q)}+\widehat{\Pi}^{(q)})u\,|\,g)_M=((\Box_f^{(q)}\widehat{N}^{(q)}+\widehat{\Pi}^{(q)})\tau u\,|\,g)_M,\\ ((I+H_3^{(q)})u\,|\,g)_M=((I+H_3^{(q)})\tau u\,|\,g)_M.\end{gathered}\tag{5-41}$$

Let $u_j\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$, $j=1,2,\dots$, with $\lim_{j\to+\infty}\|u_j-\tau u\|_M=0$. From (5-13) and (5-41), we have

$$\begin{aligned}((\Box_f^{(q)}\widehat{N}^{(q)}+\widehat{\Pi}^{(q)})u\,|\,g)_M&=((\Box_f^{(q)}\widehat{N}^{(q)}+\widehat{\Pi}^{(q)})\tau u\,|\,g)_M\\&=(\widehat{N}^{(q)}\tau u\,|\,\Box_f^{(q)}g)_M+(\widehat{\Pi}^{(q)}\tau u\,|\,g)_M\\&=\lim_{j\to+\infty}\big((\widehat{N}^{(q)}u_j\,|\,\Box_f^{(q)}g)_M+(\widehat{\Pi}^{(q)}u_j\,|\,g)_M\big)\\&=\lim_{j\to+\infty}((\Box_f^{(q)}\widehat{N}^{(q)}+\widehat{\Pi}^{(q)})u_j\,|\,g)_M=\lim_{j\to+\infty}((I+H_3^{(q)})u_j\,|\,g)_M\\&=((I+H_3^{(q)})\tau u\,|\,g)_M=((I+H_3^{(q)})u\,|\,g)_M.\end{aligned}\tag{5-42}$$

Let $h\in\Omega_c^{0,q}(U\cap\overline{M})$. Take $h_j\in\Omega_c^{0,q}(U\cap M)$, $j=1,2,\dots$, so that $\lim_{j\to+\infty}\|h_j-h\|_M=0$. From (5-34) and (5-42), we have

$$\begin{aligned}((\Box_f^{(q)}\widehat{N}^{(q)}+\widehat{\Pi}^{(q)})u\,|\,h)_M&=\lim_{j\to+\infty}((\Box_f^{(q)}\widehat{N}^{(q)}+\widehat{\Pi}^{(q)})u\,|\,h_j)_M\\&=\lim_{j+\infty}((I+H_3^{(q)})u\,|\,h_j)_M=((I+H_3^{(q)})u\,|\,h)_M.\end{aligned}\tag{5-43}$$

From (5-43), we get (5-39). ☐

The following result is the first version of the local approximate Hodge decomposition for the $\bar\partial$-Neumann Laplacian in the critical degree $q=n_-$.

Theorem 5.9. *With the assumptions and notation used above, let $q=n_-$. We can find properly supported continuous operators on $U\cap\overline{M}$,*

$$\begin{gathered}N^{(q)}:H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\quad\textit{for every } s\in\mathbb{Z},\\ \Pi^{(q)}:L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\to L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\end{gathered}\tag{5-44}$$

such that

$$\begin{gathered}N^{(q)}-\widehat{N}^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\\ \Pi^{(q)}-\widehat{\Pi}^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\end{gathered}\tag{5-45}$$

$$\begin{aligned}\Box_f^{(q)} N^{(q)} u + \Pi^{(q)} u &= u + R_0^{(q)} u && \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}),\\ \Box_f^{(q)} \Pi^{(q)} u &= R_1^{(q)} u && \text{for every } u \in L^2_{\mathrm{loc}}(U \cap M),\\ \bar{\partial}\Pi^{(q)} u &= R_2^{(q)} u && \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),\\ \bar{\partial}_f^* \Pi^{(q)} u &= R_3^{(q)} u && \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),\end{aligned} \tag{5-46}$$

$$\begin{aligned}(\bar{\partial}\rho)^{\wedge,*}\gamma N^{(q)} u|_D = 0 &\quad \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}),\\ \chi \Pi^{(q)} u \in \operatorname{Dom} \bar{\partial}^* &\quad \text{for every } \chi \in \mathscr{C}_c^\infty(U \cap \overline{M}),\ u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),\end{aligned} \tag{5-47}$$

$$\begin{aligned}(\bar{\partial}\rho)^{\wedge,*}\gamma\bar{\partial}N^{(q)} u|_D = 0 &\quad \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}),\\ (\bar{\partial}\rho)^{\wedge,*}\gamma\bar{\partial}\Pi^{(q)} u|_D = 0 &\quad \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),\end{aligned} \tag{5-48}$$

where $R_j^{(q)} : \mathscr{D}'(U \cap M) \to \Omega^{0,q}(U \cap M)$ *is a properly supported continuous operator on* $U \cap \overline{M}$ *with* $R_j^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$, $j = 0, 1, 2, 3$.

Proof. We define, following (4-21), $N^{(q)} := \hat{N}_7^{(q)} - 2\chi(\rho)\rho\tilde{H}_1^{(q)}$, $\Pi^{(q)} := \hat{\Pi}_7^{(q)} - 2\chi(\rho)\rho\tilde{H}_2^{(q)}$, where $\tilde{H}_1^{(q)}$ is a smoothing extension of $H_1^{(q)}$ from (5-10), and $\tilde{H}_2^{(q)}$ is a smoothing extension of $H_2^{(q)}$ from (5-12). We show as in the proof of Theorem 4.3 that $N^{(q)}$ and $\Pi^{(q)}$ satisfy the $\bar{\partial}$-Neumann conditions and by using Theorems 5.3 and 5.8 we conclude the result. □

From Lemmas 5.4 and 5.7, we get:

Theorem 5.10. *With the assumptions and notation used above, let* $q = n_-$. *We have*

$$(N^{(q)} u \mid v)_M = (u \mid N^{(q)} v)_M + (u \mid \Gamma^{(q)} v)_M, \tag{5-49}$$

$$(\Pi^{(q)} u \mid v)_M = (u \mid \Pi^{(q)} v)_M + (u \mid \Gamma_1^{(q)} v)_M \tag{5-50}$$

for every $u \in L^2_c(U \cap \overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$, *where* $N^{(q)}$ *and* $\Pi^{(q)}$ *are as in Theorem 5.9*, $\Gamma^{(q)}, \Gamma_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$, $\Gamma^{(q)}$ *and* $\Gamma_1^{(q)}$ *are properly supported on* $U \cap \overline{M}$.

Theorem 5.11. *With the assumptions and notation used above, let* $q = n_-$. *Let* $N^{(q)}$ *and* $\Pi^{(q)}$ *be as in Theorem 5.9. Then we have on* $U \cap \overline{M}$, *for every* $u \in \operatorname{Dom} \Box^{(q)}$,

$$\Pi^{(q)} \Box^{(q)} u = \Lambda_0^{(q)} u, \tag{5-51}$$

$$N^{(q)} \Box^{(q)} u + \Pi^{(q)} u = u + \Lambda^{(q)} u, \tag{5-52}$$

where $\Lambda_0^{(q)}, \Lambda^{(q)}$ *are properly supported on* $U \cap \overline{M}$ *and* $\Lambda_0^{(q)}, \Lambda^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Proof. Let $u \in \operatorname{Dom} \Box^{(q)}$ and let $v \in \Omega_c^{0,q}(U \cap M)$. From (5-46), (5-47), (5-48) and (5-50), we have

$$\begin{aligned}(\Pi^{(q)} \Box^{(q)} u \mid v)_M &= (\Box^{(q)} u \mid \Pi^{(q)} v)_M + (\Box^{(q)} u \mid \Gamma_1^{(q)} v)_M\\ &= (u \mid \Box^{(q)} \Pi^{(q)} v)_M + (\Box^{(q)} u \mid \Gamma_1^{(q)} v)_M\\ &= (u \mid R_1^{(q)} v)_M + (\Box^{(q)} u \mid \Gamma_1^{(q)} v)_M\\ &= (((R_1^{(q)})^* + (\Gamma_1^{(q)})^* \Box_f^{(q)}) u \mid v)_M,\end{aligned} \tag{5-53}$$

where $R_1^{(q)}$, $\Gamma_1^{(q)}$ are as in (5-46) and (5-50) respectively and $(R_1^{(q)})^*$ and $(\Gamma_1^{(q)})^*$ are the formal adjoints of $R_1^{(q)}$ and $\Gamma_1^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$ respectively. It is clear that $(R_1^{(q)})^* + (\Gamma_1^{(q)})^*\Box_f^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From this observation and (5-53), we get (5-51).

Let $u \in \operatorname{Dom}\Box^{(q)}$ and let $v \in \Omega_c^{0,q}(U\cap M)$. From (5-46), (5-47), (5-48), (5-49) and (5-50), we have

$$\begin{aligned}(N^{(q)}\Box^{(q)}u+\Pi^{(q)}u\,|\,v)_M &= (\Box^{(q)}u\,|\,N^{(q)}v)_M+(\Box^{(q)}u\,|\,\Gamma^{(q)}v)_M+(u\,|\,\Pi^{(q)}v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (u\,|\,\Box^{(q)}N^{(q)}v)_M+((\Gamma^{(q)})^*\Box_f^{(q)}u\,|\,v)_M+(u\,|\,\Pi^{(q)}v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (u\,|\,(\Box^{(q)}N^{(q)}+\Pi^{(q)})v)_M+((\Gamma^{(q)})^*\Box_f^{(q)}u\,|\,v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (u\,|\,R_0^{(q)}v)_M+((\Gamma^{(q)})^*\Box_f^{(q)}u\,|\,v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (((R_0^{(q)}+\Gamma_1^{(q)})^*+(\Gamma^{(q)})^*\Box_f^{(q)})u\,|\,v)_M,\end{aligned}\tag{5-54}$$

where $R_0^{(q)}$, $\Gamma^{(q)}$, $\Gamma_1^{(q)}$ are as in (5-46), (5-49) and (5-50) respectively, $(\Gamma^{(q)})^*$ is the formal adjoint of $\Gamma^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$ and $(R_0^{(q)}+\Gamma_1^{(q)})^*$ is the formal adjoint of $R_0^{(q)}+\Gamma_1^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$. It is clear that $(\Gamma^{(q)})^*\Box_f^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $(R_0^{(q)}+\Gamma_1^{(q)})^* \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From this observation and (5-54), we get (5-52). □

5.2. ***The distribution kernel of the approximate Bergman kernel.*** In this section, we will study the distribution kernel of $\Pi^{(q)}$ and regularity properties of the operators $\Pi^{(q)}$ and $N^{(q)}$. We will refine in this way the Hodge decomposition from Theorem 5.9 in Theorem 5.23.

Let $[\cdot\,|\,\cdot]_X$ be the L^2 inner product on $H^{-1/2}(X, T^{*0,q}M')$ given by

$$[u\,|\,v]_X := (\widetilde{P}u\,|\,\widetilde{P}v)_M,\tag{5-55}$$

where $\widetilde{P}$ is the Poisson operator given by (3-4). Let $\widetilde{P}^*:\Omega^{0,q}(\overline{M})\to\mathscr{C}^\infty(X, T^{*0,q}M')$ be the adjoint of $\widetilde{P}$ as defined in (3-9). Then,

$$\widetilde{P}^*\widetilde{P}:\mathscr{C}^\infty(X, T^{*0,q}M')\to\mathscr{C}^\infty(X, T^{*0,q}M')$$

is an injective continuous operator. Let

$$(\widetilde{P}^*\widetilde{P})^{-1}:\mathscr{C}^\infty(X, T^{*0,q}M')\to\mathscr{C}^\infty(X, T^{*0,q}M')$$

be the inverse of $\widetilde{P}^*\widetilde{P}$. It is well known that $(\widetilde{P}^*\widetilde{P})^{-1}$ is a classical pseudodifferential operator of order 1 on X (see [Boutet de Monvel 1971]).

Sections of $T^{*0,q}M'$ over X annihilated by $(\bar\partial\rho)^{\wedge,*}$ can be identified with sections of $T^{*0,q}X$, so they are called tangential. We have

$$\operatorname{Ker}(\bar\partial\rho)^{\wedge,*} := \{u\in H^{-\frac12}(X, T^{*0,q}M') : (\bar\partial\rho)^{\wedge,*}u=0\} = H^{-\frac12}(X, T^{*0,q}X).$$

Let

$$Q^{(q)}: H^{-\frac12}(X, T^{*0,q}M')\to\operatorname{Ker}(\bar\partial\rho)^{\wedge,*}\tag{5-56}$$

be the orthogonal projection with respect to $[\cdot\,|\,\cdot]_X$.

Theorem 5.12 [Hsiao 2010, Part II, Lemma 3.3]. *$Q^{(q)}$ is a classical pseudodifferential operator of order 0 with principal symbol $2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}$. Moreover,*

$$I - Q^{(q)} = (\tilde P^* \tilde P)^{-1}(\bar\partial\rho)^{\wedge} R, \tag{5-57}$$

where $R: \mathscr{C}^\infty(X, T^{*0,q}M') \to \mathscr{C}^\infty(X, T^{*0,q-1}M')$ is a classical pseudodifferential operator of order -1.

Let $u \in \Omega_c^{0,q}(U \cap M)$. From Theorem 4.3, (5-8) and Theorem 5.9, we see that $\Pi^{(q)}u \in \Omega_c^{0,q}(U \cap \overline{M})$ and $\gamma\Pi^{(q)}u \in \mathscr{C}^\infty(X, T^{*0,q}M')$.

Theorem 5.13. *Under the assumptions and notation used before we have, for $q = n_-$,*

$$\Pi^{(q)}u = \tilde P \gamma \Pi^{(q)}u + \varepsilon^{(q)}u \quad \text{for every } u \in \Omega_c^{0,q}(U \cap M), \tag{5-58}$$

where $\varepsilon^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Proof. Let $u \in \Omega_c^{0,q}(U \cap M)$. Since $\Pi^{(q)}$ is properly supported on $U \cap \overline{M}$,

$$\Pi^{(q)}u \in \Omega_c^{0,q}(U \cap \overline{M}) \subset \Omega^{0,q}(\overline{M}).$$

From (3-8), we have

$$D^{(q)}\widetilde{\Box}_f^{(q)}\Pi^{(q)}u + \tilde P \gamma \Pi^{(q)}u = \Pi^{(q)}u. \tag{5-59}$$

From (5-46) and $\widetilde{\Box}_f^{(q)} - \Box_f^{(q)} \equiv 0 \bmod \mathscr{C}^\infty(\overline{M} \times \overline{M})$, we see that

$$D^{(q)}\widetilde{\Box}_f^{(q)}\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

From this observation and (5-59), we get (5-58). □

From (5-59), we have

$$\begin{aligned}(\tilde P^* \tilde P)^{-1}\tilde P^* \Pi^{(q)}u &= (\tilde P^* \tilde P)^{-1}\tilde P^* \tilde P \gamma \Pi^{(q)}u + (\tilde P^* \tilde P)^{-1}\tilde P^* \varepsilon^{(q)}u \\ &= \gamma\Pi^{(q)}u + (\tilde P^* \tilde P)^{-1}\tilde P^* \varepsilon^{(q)}u\end{aligned} \tag{5-60}$$

and

$$\Pi^{(q)}u = \tilde P(\tilde P^* \tilde P)^{-1}\tilde P^* \Pi^{(q)}u + \varepsilon_1^{(q)}u \tag{5-61}$$

for every $u \in \Omega_c^{0,q}(U \cap M)$, where $\varepsilon_1^{(q)} = -\tilde P(\tilde P^* \tilde P)^{-1}\tilde P^* \varepsilon^{(q)}u \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$. From (3-6) and (3-10), we see that $\tilde P(\tilde P^* \tilde P)^{-1}\tilde P^* \Pi^{(q)}$ is well-defined as a continuous operator

$$\tilde P(\tilde P^* \tilde P)^{-1}\tilde P^* \Pi^{(q)} : L_c^2(U \cap \overline{M}, T^{*0,q}M') \to L_{\mathrm{loc}}^2(U \cap \overline{M}, T^{*0,q}M').$$

From this observation, (5-61) and by using a density argument, we conclude that

$$\Pi^{(q)} - \tilde P(\tilde P^* \tilde P)^{-1}\tilde P^* \Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{5-62}$$

Similarly, from (3-6) and (3-10), we see that $\Pi^{(q)}\tilde P(\tilde P^* \tilde P)^{-1}\tilde P^*$ is well-defined as a continuous operator

$$\Pi^{(q)}\tilde P(\tilde P^* \tilde P)^{-1}\tilde P^* : L^2(\overline{M}, T^{*0,q}M') \to L_{\mathrm{loc}}^2(U \cap \overline{M}, T^{*0,q}M').$$

Lemma 5.14. *Under the assumptions and notation used before we have, for $q = n_-$,*

$$\Pi^{(q)}\tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^* - \Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

Proof. Let $u \in L^2_{(0,q)}(\overline{M})$ and let $v \in \Omega_c^{0,q}(U\cap\overline{M})$. From (5-50) and (5-61), we have

$$\begin{aligned}(\Pi^{(q)}u \mid v)_M &= (u \mid \Pi^{(q)}v)_M + (u \mid \Gamma_1^{(q)}v)_M\\ &= (u \mid \tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*\Pi^{(q)}v)_M + (u \mid \varepsilon_1^{(q)}v)_M + (u \mid \Gamma_1^{(q)}v)_M\\ &= (\tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \Pi^{(q)}v)_M + (u \mid \varepsilon_1^{(q)}v)_M + (u \mid \Gamma_1^{(q)}v)_M\\ &= (\Pi^{(q)}\tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid v)_M\\ &\qquad - (\tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \Gamma_1^{(q)}v)_M + (u \mid \varepsilon_1^{(q)}v)_M + (u \mid \Gamma_1^{(q)}v)_M\\ &= (\Pi^{(q)}\tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid v)_M\\ &\qquad - ((\Gamma_1^{(q)})^*\tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid v)_M + ((\varepsilon_1^{(q)} + \Gamma_1^{(q)})^*u \mid v)_M, \end{aligned} \tag{5-63}$$

where $(\Gamma_1^{(q)})^*$ and $(\varepsilon_1^{(q)} + \Gamma_1^{(q)})^*$ are the formal adjoints of $\Gamma_1^{(q)}$ and $\varepsilon_1^{(q)} + \Gamma_1^{(q)}$ respectively. Note that

$$(\Gamma_1^{(q)})^*\tilde{P}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*,\ (\varepsilon_1^{(q)} + \Gamma_1^{(q)})^* \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From this observation and (5-63), the lemma follows. □

Theorem 5.15. *With the assumptions and notation used before, we have*

$$\Pi^{(q)} - \Pi^{(q)}\tilde{P}Q^{(q)}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^* \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})), \tag{5-64}$$

$$\Pi^{(q)} - \tilde{P}Q^{(q)}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*\Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-65}$$

Proof. Let $u \in L^2_{(0,q)}(\overline{M})$ and let $v \in \Omega_c^{0,q}(U\cap M)$. From (5-50) and (5-58), we have

$$\begin{aligned}&(\Pi^{(q)}\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid v)_M\\ &\quad= (\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \Pi^{(q)}v)_M + (\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \Gamma_1^{(q)}v)_M\\ &\quad= (\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \tilde{P}\gamma\Pi^{(q)}v)_M + (\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \varepsilon^{(q)}v)_M\\ &\qquad\qquad + (\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \Gamma_1^{(q)}v)_M\\ &\quad= [(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \gamma\Pi^{(q)}v]_X + ((\varepsilon^{(q)})^*\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid v)_M\\ &\qquad\qquad + ((\Gamma_1^{(q)})^*\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid v)_M, \end{aligned} \tag{5-66}$$

where $(\varepsilon^{(q)})^*$ and $(\Gamma_1^{(q)})^*$ are the formal adjoints of $\varepsilon^{(q)}$ and $\Gamma_1^{(q)}$ respectively. From the second formula of (5-47) and noticing that $\Pi^{(q)}$ is properly supported on $U\cap\overline{M}$, we get $(\bar{\partial}\rho)^{\wedge,*}\gamma\Pi^{(q)}v = 0$; hence $\gamma\Pi^{(q)}v \in \operatorname{Ker}(\bar{\partial}\rho)^{\wedge,*}$. Thus, $[(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid \gamma\Pi^{(q)}v]_X = 0$. From this observation, (5-66) and noticing that

$$(\varepsilon^{(q)})^*\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*,\ (\Gamma_1^{(q)})^*\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^* \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),$$

we get

$$\Pi^{(q)}\tilde{P}(I-Q^{(q)})(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^* \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-67}$$

From (5-67) and Lemma 5.14, we get (5-64).

Let $u \in L^2_c(\overline{M})$ and let $v \in \Omega^{0,q}_c(U \cap M)$. From (5-50), we have

$$\begin{aligned}
&(\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid v)_M \\
&\quad = ((I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid \widetilde{P}^*v)_X \\
&\quad = ((I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid (\widetilde{P}^*\widetilde{P})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_X \\
&\quad = [(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid (\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v]_X \\
&\quad = [(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid (I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v]_X \\
&\quad = (\Pi^{(q)}u \mid \widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_M \\
&\quad = (u \mid \Pi^{(q)}\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_M + (u \mid \Gamma^{(q)}_1\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_M. \qquad (5\text{-}68)
\end{aligned}$$

From (5-68) and (5-67), we deduce that

$$\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From this observation and (5-62), we get (5-65). □

We can now prove the following regularity property for $\Pi^{(q)}$.

Theorem 5.16. *With the assumptions and notation used before, $\Pi^{(q)}$ can be continuously extended to*

$$\begin{aligned}
&\Pi^{(q)}: H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^{s-1}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \textit{for every } s\in\mathbb{Z},\\
&\Pi^{(q)}: H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s-1}_c(U\cap\overline{M}, T^{*0,q}M') \quad \textit{for every } s\in\mathbb{Z}.
\end{aligned} \qquad (5\text{-}69)$$

Proof. Let $u \in \Omega^{0,q}_c(U\cap\overline{M})$. From (5-64), we see that

$$\Pi^{(q)}u = \Pi^{(q)}\widetilde{P}Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u + \gamma^{(q)}u, \qquad (5\text{-}70)$$

where $\gamma^{(q)}: \Omega^{0,q}_c(U\cap M) \to \mathscr{D}'(U\cap M, T^{*0,q}M')$ is a continuous operator with

$$\gamma^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From Theorem 4.3, (5-8), Theorem 5.9 and noticing that

$$\widetilde{P}Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \in A^{0,q}(U\cap\overline{M}),$$

we conclude that

$$\Pi^{(q)}u = (I - \bar\partial^*_f N^{(q+1)}\bar\partial - \bar\partial N^{(q-1)}\bar\partial^*_f)\widetilde{P}Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u + \gamma^{(q)}_1 u, \qquad (5\text{-}71)$$

where $\gamma^{(q)}_1: \Omega^{0,q}_c(U\cap M) \to \mathscr{D}'(U\cap M, T^{*0,q}M')$ is a continuous operator with

$$\gamma^{(q)}_1 \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From (5-71),

$$\begin{aligned}
&N^{(q-1)}: H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_c(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z},\\
&N^{(q+1)}: H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_c(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z}
\end{aligned}$$

are continuous and note that $\Omega_c^{0,q}(U \cap \overline{M})$ is dense in $H_c^s(U \cap \overline{M}, T^{*0,q}M')$ for every $s \in \mathbb{Z}$, and thus we get (5-69). $\square$

The reason why in the proof of Theorem 5.16 we do not use $\widehat{\Pi}^{(q)}$ directly is the following: In (5-8), $\widehat{\Pi}^{(q)}$ is just defined on the space $A^{0,q}(U \cap \overline{M})$. If $u \in \Omega_c^{0,q}(U \cap \overline{M})$, we cannot define $\widehat{\Pi}^{(q)}u$ by using (5-8) since in general

$$(I - \bar{\partial}_f^* N^{(q+1)} \bar{\partial} - \bar{\partial} N^{(q-1)} \bar{\partial}_f^*) u \notin \operatorname{Dom} \square^{(q)}.$$

We extend $\widehat{\Pi}^{(q)}$ to $L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$ by density and we have (5-71) for the relation between $\Pi^{(q)}$ and (5-8).

5.3. ***Reduction to the analysis on the boundary.*** In order to refine the approximate Hodge decomposition of Theorem 5.9 and show that $\Pi^{(q)}$ is a Fourier integral operator we will bring in an approximate Szegő projector on the boundary, which is a Fourier integral operator, and link it to $\Pi^{(q)}$ by means of the Poisson operator. The approximate Szegő projector appears in the microlocal Hodge decomposition of the boundary Laplacian $\square_\beta^{(q)}$, which is a perturbation of the Kohn Laplacian.

We recall the operators $\bar{\partial}_\beta$ and $\square_\beta^{(q)}$ introduced in [Hsiao 2010, Part II, Chapter 5]. Recall that $Q^{(q+1)}$ is given by (5-56). The operator $\bar{\partial}_\beta$ is defined by

$$\bar{\partial}_\beta = Q^{(q+1)} \gamma \bar{\partial} \widetilde{P} : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) \tag{5-72}$$

and it is obtained by taking the $\bar{\partial}$ derivative of the extension of a form to the interior by the Poisson operator and then taking the projection on the space of the tangential forms to the boundary. It is a classical pseudodifferential operator of order 1 which is a perturbation of the $\bar{\partial}_b$ operator by a zeroth-order operator. It has the advantage that it involves directly the Poisson operator. Let

$$\bar{\partial}_\beta^\dagger : \Omega^{0,q+1}(X) \to \Omega^{0,q}(X) \tag{5-73}$$

be the formal adjoint of $\bar{\partial}_\beta$ with respect to $[\cdot \mid \cdot]_X$, that is, $[\bar{\partial}_\beta f \mid h] = [f \mid \bar{\partial}_\beta^\dagger h]_X$, $f \in \Omega^{0,q}(X)$, $h \in \Omega^{0,q+1}(X)$. Then $\bar{\partial}_\beta^\dagger$ is a classical pseudodifferential operator of order 1 and we have

$$\bar{\partial}_\beta^\dagger = \gamma \bar{\partial}_f^* \widetilde{P} \quad \text{on } \Omega^{0,q}(X) \text{ for } q = 1, \dots, n-1; \tag{5-74}$$

see [Hsiao 2010, Part II, Chapter 5]. Set

$$\square_\beta^{(q)} = \bar{\partial}_\beta^\dagger \, \bar{\partial}_\beta + \bar{\partial}_\beta \, \bar{\partial}_\beta^\dagger : \mathscr{D}'(X, T^{*0,q}X) \to \mathscr{D}'(X, T^{*0,q}X). \tag{5-75}$$

It was shown in [Hsiao 2010, Part II, Chapter 5] that $\square_\beta^{(q)}$ is a classical pseudodifferential operator of order 2 and the characteristic manifold of $\square_\beta^{(q)}$ is given by $\Sigma = \Sigma^+ \cup \Sigma^-$, where Σ^+, Σ^- are as in (3-12). Roughly speaking, forms annihilated by $\square_\beta^{(q)}$ on the boundary are microlocally boundary values of harmonic forms. More precisely, if S_β is the orthogonal projection onto the kernel of $\square_\beta^{(q)}$, then $\widetilde{P} S_\beta$ is in the kernel of the $\bar{\partial}$-Neumann Laplacian up to a smoothing operator. If S is the orthogonal projection onto the kernel of $\square_b^{(q)}$ (the Szegő projector), then $\widetilde{P} S$ does not have this property.

Let D be a local coordinate patch of X with local coordinates $x = (x_1, \dots, x_{2n-1})$ and we assume the Levi form is nondegenerate of constant signature (n_-, n_+) on D. Let $H \in L^{-1}_{\mathrm{cl}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$

be a properly supported pseudodifferential operator of order -1 on D such that

$$H - \widetilde{P}^* \widetilde{P} \equiv 0 \quad \text{on } D. \tag{5-76}$$

The following microlocal Hodge decomposition for $\Box^{(q)}_\beta$ was established in [Hsiao 2010, Part II, Theorem 6.15].

Theorem 5.17. *With the assumptions and notation above, let $q = n_-$. Then there exist properly supported operators*

$$A \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),$$
$$S_-, S_+ \in L^{0}_{\frac{1}{2},\frac{1}{2}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

such that

$$\begin{aligned} \mathrm{WF}'(S_-(x,y)) &= \mathrm{diag}((\Sigma^+ \cap T^*D) \times (\Sigma^+ \cap T^*D)), \\ \mathrm{WF}'(S_+(x,y)) &\subset \mathrm{diag}((\Sigma^- \cap T^*D) \times (\Sigma^- \cap T^*D)) \end{aligned} \tag{5-77}$$

and

$$A\Box^{(q)}_\beta + S_- + S_+ = I, \tag{5-78}$$
$$\bar\partial_\beta S_- \equiv 0, \quad \bar\partial^\dagger_\beta S_- \equiv 0, \tag{5-79}$$
$$S_- \equiv S_-^\dagger \equiv S_-^2, \tag{5-80}$$
$$S_+ \equiv 0 \quad \text{if } q \neq n_+, \tag{5-81}$$

where

$$S_-^\dagger := 2Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1} S_-^* (\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge H : \Omega^{0,q}_c(D) \to \Omega^{0,q}(X), \tag{5-82}$$

H is given by (5-76) and S_-^* is the formal adjoint of S_- with respect to $(\cdot\,|\,\cdot)_X$. Moreover, the kernel $S_-(x,y)$ satisfies

$$S_-(x,y) \equiv \int_0^\infty e^{i\varphi_-(x,y)t} a(x,y,t)\,dt,$$

with

$$\begin{aligned} &a(x,y,t) \in S^{n-1}_{1,0}(D \times D \times (0,\infty), T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ &a(x,y,t) \sim \sum_{j=0}^\infty a_j(x,y) t^{n-1-j} \quad \text{in } S^{n-1}_{1,0}(D \times D \times (0,\infty), T^{*0,q}X \boxtimes (T^{*0,q}X)^*) \end{aligned} \tag{5-83}$$

and

$$a_0(x,x) = \frac{1}{2\pi^n} |\det \mathcal{L}_x| \tau_{x,n_-} \quad \text{for every } x \in D, \tag{5-84}$$

where $a_j(x,y) \in \mathscr{C}^\infty(D \times D; T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, $j = 0, 1, \dots,$ and the phase function φ_- is the same as the phase function appearing in the description of the singularities of the Szegő kernels for lower-energy forms in [Hsiao and Marinescu 2017, Theorems 3.3, 3.4]. *In particular, we have*

$$\varphi_-(x,y) \in \mathscr{C}^\infty(X \times X), \quad \mathrm{Im}\,\varphi_-(x,y) \geq 0, \tag{5-85}$$
$$\varphi_-(x,x) = 0, \quad \varphi_-(x,y) \neq 0 \quad \text{if } x \neq y, \tag{5-86}$$
$$d_x\varphi_- \neq 0, \quad d_y\varphi_- \neq 0 \quad \text{where } \mathrm{Im}\,\varphi_- = 0, \tag{5-87}$$

$$d_x\varphi_-(x,y)|_{x=y} = -d_y\varphi_-(x,y)|_{x=y} = \omega_0(x), \tag{5-88}$$

$$\varphi_-(x,y) = -\bar{\varphi}_-(y,x). \tag{5-89}$$

We have denoted by $\mathrm{WF}(S_\pm(x,y))$ the wave front set in the sense of Hörmander of the distributions $S_\pm(x,y)$ and

$$\mathrm{WF}'(S_\pm(x,y)) := \{(x,\xi,y,\eta) \in T^*X \times T^*X : (x,\xi,y,-\eta) \in \mathrm{WF}(S_\pm(x,y))\}.$$

The leading coefficient $a_0(x,x)$ from (5-83) was obtained in [Hsiao 2010, Part II, Proposition 6.17].

We come back to our situation. In view of Lemmas 4.1, 4.2 and Theorem 5.12, we see that $Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*$ is smoothing away the diagonal. Hence, there is a continuous operator $L^{(q)} : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega^{0,q}(D)$ such that

$$L^{(q)} - Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^* \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})) \tag{5-90}$$

and $L^{(q)}$ is properly supported on $U\cap\overline{M}$, that is, for every $\chi \in \mathscr{C}_c^\infty(U\cap\overline{M})$, there is a $\tau \in \mathscr{C}_c^\infty(D)$ such that $L^{(q)}\chi = \tau L^{(q)}$ on $\Omega_c^{0,q}(U\cap\overline{M})$ and, for every $\tau_1 \in \mathscr{C}_c^\infty(D)$, there is a $\chi_1 \in \mathscr{C}_c^\infty(U\cap\overline{M})$ such that $\tau_1 L^{(q)} = L^{(q)}\chi_1$ on $\Omega_c^{0,q}(U\cap\overline{M})$. We can extend $L^{(q)}$ to a continuous operator

$$L^{(q)} : \Omega^{0,q}(U\cap\overline{M}) \to \Omega^{0,q}(D), \quad L^{(q)} : \Omega_c^{0,q}(U\cap\overline{M}) \to \Omega_c^{0,q}(D).$$

From Theorem 5.15, we have

$$\Pi^{(q)} - \widetilde{P}L^{(q)}\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-91}$$

Lemma 5.18. *With the notation and assumptions above, we have*

$$S_+L^{(q)}\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})), \tag{5-92}$$

where S_+ is as in Theorem 5.17.

Proof. Since $\mathrm{WF}'(S_+(x,y)) \subset \mathrm{diag}\big((\Sigma^-\cap T^*D)\times(\Sigma^-\cap T^*D)\big)$ and by Theorem 3.2 the operator $\Box_-^{(q)}$ is elliptic near Σ^-, there is a classical pseudodifferential operator $E^{(q)} \in L_{\mathrm{cl}}^{-1}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that

$$S_+ - S_+E^{(q)}\Box_-^{(q)} \equiv 0. \tag{5-93}$$

From (5-46) and (5-91), we deduce that

$$\Box_-^{(q)}L^{(q)}\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})). \tag{5-94}$$

From (5-93) and (5-94), we get (5-92). ☐

Theorem 5.19. *With the notation and assumptions above, we have*

$$S_-L^{(q)}\Pi^{(q)} - L^{(q)}\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})), \tag{5-95}$$

$$\widetilde{P}S_-L^{(q)}\Pi^{(q)} - \Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})), \tag{5-96}$$

$$\Pi^{(q)}\widetilde{P}S_-L^{(q)} - \Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-97}$$

Proof. From (5-46) and (5-91), we see that

$$\Box_\beta^{(q)} L^{(q)} \Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M})). \tag{5-98}$$

From (5-78), (5-92) and (5-98), we have

$$\begin{aligned} L^{(q)} \Pi^{(q)} &= (A \Box_\beta^{(q)} + S_- + S_+) L^{(q)} \Pi^{(q)} \\ &\equiv S_- L^{(q)} \Pi^{(q)} \bmod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M})) \end{aligned}$$

and we get (5-95). From (5-95) and (5-91), we get (5-96). We now prove (5-97). Put

$$\begin{aligned} &\gamma^{(q)} := \Pi^{(q)} - \tilde{P} L^{(q)} \Pi^{(q)} : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega^{0,q}(\overline{M}), \\ &\gamma_0^{(q)} := \tilde{P}^* \tilde{P} - H : \Omega_c^{0,q}(D) \to \Omega^{0,q}(X), \\ &\gamma_1^{(q)} := S_-^\dagger - S_- : \Omega_c^{0,q}(D) \to \Omega^{0,q}(X), \\ &\gamma_2^{(q)} := L^{(q)} - Q^{(q)} (\tilde{P}^* \tilde{P})^{-1} \tilde{P}^* : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega^{0,q}(X), \\ &\gamma_3^{(q)} := S_- L^{(q)} \Pi^{(q)} - L^{(q)} \Pi^{(q)} : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega_c^{0,q}(D), \end{aligned}$$

where $S_-^\dagger$ is given by (5-82). From (5-80), (5-90), (5-91) and (5-95), we see that

$$\begin{gathered} \gamma^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \\ \gamma_2^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M})), \\ \gamma_3^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M})), \\ \gamma_1^{(q)} \equiv 0, \quad \gamma_0^{(q)} \equiv 0. \end{gathered} \tag{5-99}$$

Let

$$(\gamma^{(q)})^* : \Omega^{0,q}(\overline{M}) \to \Omega^{0,q}(U \cap \overline{M})$$

be the formal adjoint of $\gamma^{(q)}$ with respect to $(\cdot \mid \cdot)_M$ and let

$$(\gamma_2^{(q)})^* : \Omega^{0,q}(X) \to \Omega^{0,q}(U \cap \overline{M})$$

be the formal adjoint of $\gamma_2^{(q)}$ with respect to $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_X$, that is,

$$(\gamma_2^{(q)} u \mid v)_X = (u \mid (\gamma_2^{(q)})^* v)_M \quad \text{for every } u \in \Omega_c^{0,q}(U \cap \overline{M}),\ v \in \Omega^{0,q}(X).$$

It is obvious that

$$(\gamma^{(q)})^* \equiv 0, \ \ (\gamma_2^{(q)})^* \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X)). \tag{5-100}$$

Let $u, v \in \Omega_c^{0,q}(U \cap \overline{M})$. From (5-50), it is straightforward to check that

$$\begin{aligned} &(\Pi^{(q)} \tilde{P} S_- L^{(q)} u \mid v)_M \\ &\quad = (\tilde{P} S_- L^{(q)} u \mid \Pi^{(q)} v)_M + (\tilde{P} S_- L^{(q)} u \mid \Gamma_1^{(q)} v)_M \\ &\quad = (\tilde{P} S_- L^{(q)} u \mid \tilde{P} L^{(q)} \Pi^{(q)} v)_M + (\tilde{P} S_- L^{(q)} u \mid \gamma^{(q)} v)_M + (\tilde{P} S_- L^{(q)} u \mid \Gamma_1^{(q)} v)_M \\ &\quad = (S_- L^{(q)} u \mid H L^{(q)} \Pi^{(q)} v)_X + (S_- L^{(q)} u \mid \gamma_0^{(q)} L^{(q)} \Pi^{(q)} v)_X \\ &\qquad\qquad + (\tilde{P} S_- L^{(q)} u \mid \gamma^{(q)} v)_M + (\tilde{P} S_- L^{(q)} u \mid \Gamma_1^{(q)} v)_M \end{aligned} \tag{5-101}$$

$$
\begin{aligned}
&= [L^{(q)}u \mid S_-^{\dagger}L^{(q)}\Pi^{(q)}v]_X + (S_-L^{(q)}u \mid \gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_X \\
&\qquad + (\tilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\tilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= [L^{(q)}u \mid S_-L^{(q)}\Pi^{(q)}v]_X + [L^{(q)}u \mid \gamma_1^{(q)}L^{(q)}\Pi^{(q)}v]_X + (S_-L^{(q)}u \mid \gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_X \\
&\qquad + (\tilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\tilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= [Q^{(q)}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid S_-L^{(q)}\Pi^{(q)}v]_X + [\gamma_2^{(q)}u \mid S_-L^{(q)}\Pi^{(q)}v]_X + [L^{(q)}u \mid \gamma_1^{(q)}L^{(q)}\Pi^{(q)}v]_X \\
&\qquad + (S_-L^{(q)}u \mid \gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_X + (\tilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\tilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= (u \mid \tilde{P}S_-L^{(q)}\Pi^{(q)}v)_M + (\tilde{P}\gamma_2^{(q)}u \mid \tilde{P}S_-L^{(q)}\Pi^{(q)}v)_M + (\tilde{P}L^{(q)}u \mid \tilde{P}\gamma_1^{(q)}L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (\tilde{P}S_-L^{(q)}u \mid \tilde{P}(\tilde{P}^*\tilde{P})^{-1}\gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_M + (\tilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\tilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= (u \mid \tilde{P}L^{(q)}\Pi^{(q)}v)_M + (u \mid \tilde{P}\gamma_3^{(q)}v)_M + (u \mid (\gamma_2^{(q)})^*\tilde{P}^*\tilde{P}S_-L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (u \mid (L^{(q)})^*\tilde{P}^*\tilde{P}\gamma_1^{(q)}L^{(q)}\Pi^{(q)}v)_M + (u \mid (L^{(q)})^*S_-^*\gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (u \mid (L^{(q)})^*(S_-)^*(\tilde{P})^*\gamma^{(q)}v)_M + (u \mid (L^{(q)})^*(S_-)^*(\tilde{P})^*\Gamma_1^{(q)}v)_M \\
&= (u \mid \Pi^{(q)}v)_M - (u \mid \gamma^{(q)}v)_M + (u \mid \tilde{P}\gamma_3^{(q)}v)_M + (u \mid (\gamma_2^{(q)})^*\tilde{P}^*\tilde{P}S_-L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (u \mid (L^{(q)})^*\tilde{P}^*\tilde{P}\gamma_1^{(q)}L^{(q)}\Pi^{(q)}v)_M + (u \mid (L^{(q)})^*S_-^*\gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (u \mid (L^{(q)})^*(S_-)^*(\tilde{P})^*\gamma^{(q)}v)_M + (u \mid (L^{(q)})^*(S_-)^*(\tilde{P})^*\Gamma_1^{(q)}v)_M, \qquad \text{(5-101 cont.)}
\end{aligned}
$$

where $(L^{(q)})^* : \Omega^{0,q}(D) \to \Omega^{0,q}(U \cap \overline{M})$ is the formal adjoint of $L^{(q)}$ with respect to $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_X$. We explain the third-to-last equality of (5-101). Since $S_-L^{(q)}\Pi^{(q)}v \in \mathrm{Ker}(\bar{\partial}\rho)^{\wedge,*}$, we have

$$
[Q^{(q)}(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid S_-L^{(q)}\Pi^{(q)}v]_X = [(\tilde{P}^*\tilde{P})^{-1}\tilde{P}^*u \mid S_-L^{(q)}\Pi^{(q)}v]_X. \tag{5-102}
$$

From (5-102), we get the third-to-last equality of (5-101).

Note that $(L^{(q)})^*$ is properly supported. From (5-101), we conclude that there is a continuous operator $\varepsilon^{(q)} : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega^{0,q}(U \cap \overline{M})$ with $\varepsilon^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ such that

$$
(\Pi^{(q)}\tilde{P}S_-L^{(q)}u \mid v)_M = (u \mid \Pi^{(q)}v)_M + (u \mid \varepsilon^{(q)}v)_M \tag{5-103}
$$

for every $u, v \in \Omega_c^{0,q}(U \cap \overline{M})$. From (5-50) and (5-103), we get

$$
(\Pi^{(q)}\tilde{P}S_-L^{(q)}u \mid v)_M = (\Pi^{(q)}u \mid v)_M - ((\Gamma_1^{(q)})^*u \mid v)_M + ((\varepsilon^{(q)})^*u \mid v)_M \tag{5-104}
$$

for every $u, v \in \Omega_c^{0,q}(U \cap \overline{M})$, where $(\Gamma_1^{(q)})^*, (\varepsilon^{(q)})^* : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega^{0,q}(U \cap \overline{M})$ are the formal adjoints of $\Gamma_1^{(q)}$ and $\varepsilon^{(q)}$ with respect to $(\cdot \mid \cdot)_M$ respectively. Note that $(\Gamma_1^{(q)})^*, (\varepsilon^{(q)})^* \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$. From this observation and (5-104), we get (5-97). □

Theorem 5.20. *With the notation and assumptions used above, we have*

$$
\bar{\partial}\tilde{P}S_-L^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{5-105}
$$

Proof. From [Hsiao 2010, Part II, Proposition 6.18], we have

$$
\gamma\bar{\partial}\tilde{P}S_- \equiv 0. \tag{5-106}
$$

From (3-8), we have

$$
D^{(q+1)}\widetilde{\Box}_f^{(q+1)}\bar{\partial}\tilde{P}S_- + \tilde{P}\gamma\bar{\partial}\tilde{P}S_- = \bar{\partial}\tilde{P}S_-. \tag{5-107}
$$

Now,

$$\begin{aligned}D^{(q+1)}\widetilde{\Box}^{(q+1)}_f\bar\partial\widetilde{P}S_- &= D^{(q+1)}(\Box^{(q+1)}_f+K^{(q+1)})\bar\partial\widetilde{P}S_-\\ &= D^{(q+1)}\bar\partial\Box^{(q)}_f\widetilde{P}S_- + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P}S_-\\ &= D^{(q+1)}\bar\partial\widetilde{\Box}^{(q)}_f\widetilde{P}S_- - D^{(q+1)}\bar\partial K^{(q)}\widetilde{P}S_- + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P}S_-\\ &= -D^{(q+1)}\bar\partial K^{(q)}\widetilde{P}S_- + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P}S_-\\ &\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times X)).\end{aligned}\tag{5-108}$$

From (5-106), (5-107) and (5-108), we get (5-105). □

Let $\delta^{(q)} := 2\rho\widetilde{P}((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widetilde{P}S_-L^{(q)}) : \Omega^{0,q}_c(U\cap\overline{M})\to\Omega^{0,q}(\overline{M})$. By (5-105) we have

$$\delta^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).\tag{5-109}$$

Moreover, it is easy to check that

$$(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u\in \operatorname{Dom}\Box^{(q)}\cap\Omega^{0,q}(\overline{M})\quad\text{for every } u\in\Omega^{0,q}_c(U\cap\overline{M}).\tag{5-110}$$

We come now to the crucial relation between the approximate Bergman and Szegő kernels via the Poisson operator.

Theorem 5.21. *With the notation and assumptions used above, we have*

$$\Pi^{(q)}-\widetilde{P}S_-L^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).\tag{5-111}$$

Proof. We first claim that

$$\Pi^{(q)}\widetilde{P}S_-L^{(q)}-\widetilde{P}S_-L^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).\tag{5-112}$$

From (5-52) and (5-110), we have

$$\begin{aligned}&N^{(q)}\Box^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u+\Pi^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u\\ &\qquad=(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u+\Lambda^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u\end{aligned}\tag{5-113}$$

for every $u\in\Omega^{0,q}_c(U\cap\overline{M})$, where $\Lambda^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-52). From (5-79), (5-105) and (5-109), we have

$$N^{(q)}\Box^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u=N^{(q)}\Box^{(q)}_f(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u=\Lambda^{(q)}_1u\tag{5-114}$$

for every $u\in\Omega^{0,q}_c(U\cap\overline{M})$, where $\Lambda^{(q)}_1\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From (5-109), (5-113) and (5-114) we get the claim (5-112).

From (5-97) and (5-112), we get (5-111). □

Note that $S_-\in L^0_{1/2,1/2}(D, T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$. From this observation and the classical result of Calderon and Vaillancourt (see (2-2)), (3-6), (3-10) and (5-111), we can improve Theorem 5.16 as follows.

Theorem 5.22. *With the notation used above, $\Pi^{(q)}$ can be continuously extended to*

$$\begin{aligned}\Pi^{(q)}&: H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')\quad\textit{for every } s\in\mathbb{Z},\\ \Pi^{(q)}&: H^s_c(U\cap\overline{M}, T^{*0,q}M')\to H^s_c(U\cap\overline{M}, T^{*0,q}M')\quad\textit{for every } s\in\mathbb{Z}.\end{aligned}\tag{5-115}$$

5.4. *Final version of the microlocal Hodge decomposition.* We can now prove the our final version of the approximate Hodge decomposition by constructing a parametrix $N^{(q)}$ and an approximate Bergman projector $\Pi^{(q)}$, which is a Fourier integral operator with complex phase.

Theorem 5.23. *Let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Let $q = n_-$. There exist properly supported continuous operators on $U \cap \overline{M}$,*

$$\begin{aligned} N^{(q)} &: H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \text{for every } s \in \mathbb{Z}, \\ \Pi^{(q)} &: H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \text{for every } s \in \mathbb{Z}, \end{aligned} \tag{5-116}$$

such that

$$\begin{aligned} N^{(q)}u &\in \operatorname{Dom} \Box^{(q)} \quad \text{for every } u \in \Omega^{0,q}_c(U \cap \overline{M}), \\ \Pi^{(q)}u &\in \operatorname{Dom} \Box^{(q)} \quad \text{for every } u \in \Omega^{0,q}_c(U \cap \overline{M}), \end{aligned} \tag{5-117}$$

and on $U \cap \overline{M}$, we have

$$\begin{aligned} \Box^{(q)}_f N^{(q)}u + \Pi^{(q)}u &= u + r^{(q)}_0 u \quad \text{for every } u \in \Omega^{0,q}(U \cap M), \\ N^{(q)}\Box^{(q)}u + \Pi^{(q)}u &= u + r^{(q)}_1 u \quad \text{for every } u \in \operatorname{Dom} \Box^{(q)}, \\ \bar\partial \Pi^{(q)}u &= r^{(q)}_2 u \quad \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \\ \bar\partial^*_f \Pi^{(q)}u &= r^{(q)}_3 u \quad \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \\ \Pi^{(q)}\Box^{(q)}u &= r^{(q)}_4 u \quad \text{for every } u \in \operatorname{Dom} \Box^{(q)}, \\ \Box^{(q)}_f \Pi^{(q)}u &= r^{(q)}_5 u \quad \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}), \\ (\Pi^{(q)})^2 u - \Pi^{(q)}u &= r^{(q)}_6 u \quad \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}), \end{aligned} \tag{5-118}$$

where $r^{(q)}_j$ is properly supported on $U \cap \overline{M}$ with $r^{(q)}_j \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ for every $j = 0, \dots, 6$, and the distribution kernel of $\Pi^{(q)}$ satisfies

$$\Pi^{(q)}(z,w) \equiv \int_0^\infty e^{i\phi(z,w)t} b(z,w,t)\,dt \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \tag{5-119}$$

with

$$\begin{aligned} &b(z,w,t) \in S^n_{1,0}((U \times U) \cap (\overline{M} \times \overline{M}) \times (0,\infty), \Lambda^{(0,q)|(0,q)}_{M' \times M'}), \\ &b(z,w,t) \sim \sum_{j=0}^\infty b_j(z,w) t^{n-j} \quad \text{in } S^n_{1,0}((U \times U) \cap (\overline{M} \times \overline{M}) \times (0,\infty), \Lambda^{(0,q)|(0,q)}_{M' \times M'}), \end{aligned} \tag{5-120}$$

with $b_0(z,z)$ given by (5-124) below. Moreover,

$$\begin{gathered} \phi(z,w) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \quad \operatorname{Im} \phi \geq 0, \\ \phi(z,z) = 0, \quad z \in U \cap X, \quad \phi(z,w) \neq 0 \quad \text{if } (z,w) \notin \operatorname{diag}((U \times U) \cap (X \times X)), \\ \operatorname{Im} \phi(z,w) > 0 \quad \text{if } (z,w) \notin (U \times U) \cap (X \times X), \\ \phi(z,w) = -\bar\phi(w,z), \\ d_x \phi(x,y)|_{x=y} = -2i\,\partial\rho(x) \quad \text{for every } x \in U \cap X, \end{gathered} \tag{5-121}$$

$\phi(z,w) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in [Hsiao 2010, Part II, Theorem 1.4] and $\phi(z,w) = \varphi_-(z,w)$ if $z, w \in U \cap X$, where $\varphi_- \in \mathscr{C}^\infty((U \times U) \cap (X \times X))$ is as in Theorem 5.17.

Proof. Let $N^{(q)}$ and $\Pi^{(q)}$ be as in Theorem 5.9. From (5-47), (5-48) and noticing that $N^{(q)}$ and $\Pi^{(q)}$ are properly supported on $U\cap\overline{M}$, we get (5-117).

From (5-46), (5-51) and (5-52), we get the first six equations in (5-118). From the second and sixth equations in (5-118), we have

$$\Pi^{(q)}\equiv N^{(q)}\Box^{(q)}\Pi^{(q)}+(\Pi^{(q)})^2\equiv(\Pi^{(q)})^2 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

We get (5-118). We now study distribution kernel of $\Pi^{(q)}$. From Theorem 5.21, we see that

$$\Pi^{(q)}-\widetilde{P}S_-L^{(q)}\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

We just need to study distribution kernel of $\widetilde{P}S_-L^{(q)}$. Let $x=(x_1,\dots,x_{2n-1})$ be local coordinates of X and extend $x_1,\dots,x_{2n-1}$ to real smooth functions in some neighborhood of X. We may assume that $z=(x,\rho)=(x_1,\dots,x_{2n-1},\rho)$ are local coordinates of U. In view of Theorem 5.17, we have

$$S_-(x,y)\equiv\int_0^{+\infty}e^{i\varphi_-(x,y)t}a(x,y,t)\,dt.$$

We can repeat the proof of [Hsiao 2010, Part II, Proposition 7.6] and find a phase

$$\tilde{\phi}(z,y)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times X))$$

such that

$$\tilde{\phi}(x,y)=\varphi_-(x,y),\ (d_z\tilde{\phi})(x,x)=-\omega_0(x)-i\,d\rho(x)\quad\text{for all }(x,y)\in(U\times U)\cap(X\times X),$$

$\operatorname{Im}\tilde{\phi}(z,y)>0$ if $\rho\neq 0$ and $q_0(z,\tilde{\phi}'_z)$ vanishes to infinite order at $\rho=0$, where q_0 denotes the principal symbol of $\Box^{(q)}_f$. We can repeat the procedure in the proof of [Hsiao 2010, Part II, Proposition 7.8] and deduce that the distribution kernel of $\widetilde{P}S_-$ is of the form

$$\begin{gathered}\widetilde{P}S_-(z,y)\equiv\int_0^{\infty}e^{i\tilde{\phi}(z,y)t}\tilde{b}(z,y,t)\,dt \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times X)),\\ \tilde{b}(z,y,t)\in S^{n-1}_{\mathrm{cl}}((U\times U)\cap(\overline{M}\times X))\times(0,+\infty),\Lambda^{(0,q)|(0,q)}_{M'\times M'}),\\ \tilde{b}(x,y,t)=a(x,y,t)\quad\text{for all }(x,y)\in(U\times U)\cap(X\times X).\end{gathered}$$

Similarly, we can repeat the procedure above and deduce that

$$\widetilde{P}S_-L^{(q)}(z,w)\equiv\int_0^{\infty}e^{i\phi(z,w)t}b(z,w,t)\,dt \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\tag{5-122}$$

where $\phi(z,w)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ satisfies (5-121),

$$b(z,w,t)\in S^n_{\mathrm{cl}}(((U\times U)\cap(\overline{M}\times\overline{M}))\times(0,+\infty),\Lambda^{(0,q)|(0,q)}_{M'\times M'}).$$

Since

$$(\widetilde{P}^*\widetilde{P})^{-1}=2\sqrt{\triangle_X}+\Psi^0$$

and

$$Q^{(q)}=2((\bar\partial\rho)(x))^{\wedge,*}((\bar\partial\rho)(x))^{\wedge}+\Psi^0$$

for some elements $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, we deduce as in [Hsiao 2010, Part II, (7.22)],

$$b_0(x,x) = 4a_0(x,x)((\bar\partial\rho)(x))^{\wedge,*}((\bar\partial\rho)(x))^{\wedge}, \quad x \in U \cap X,$$

where $a_0(x,x)$ is as in (5-84).

From Theorems 5.9, 5.10, 5.11, 5.22, (5-111) and (5-122), the theorem follows. □

The following result describes the phase function ϕ (see (5-119)) of the Fourier integral operator $\Pi^{(q)}$.

Theorem 5.24 [Hsiao 2010, Part II, Theorem 1.4]. *Under the assumptions and notation of Theorem 5.23, fix $p \in U \cap X$. We choose local holomorphic coordinates $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, vanishing at p such that the metric on $T^{1,0}M'$ is $\sum_{j=1}^n dz_j \otimes d\bar z_j$ at p and $\rho(z) = \sqrt2 \operatorname{Im} z_n + \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + O(|z|^3)$, where λ_j, $j = 1, \dots, n-1$, are the eigenvalues of $\mathcal{L}_p$. We also write $w = (w_1, \dots, w_n)$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$. Then, we can take $\phi(z,w)$ in (5-119) so that*

$$\begin{aligned}\phi(z,w) = -\sqrt2 x_{2n-1} + \sqrt2 y_{2n-1} - i\rho(z)\Big(1 + \sum_{j=1}^{2n-1} a_j x_j + \frac12 a_{2n}x_{2n}\Big)\\ - i\rho(w)\Big(1 + \sum_{j=1}^{2n-1} \bar a_j y_j + \frac12 \bar a_{2n} y_{2n}\Big) + i\sum_{j=1}^{n-1}|\lambda_j||z_j - w_j|^2\\ + \sum_{j=1}^{n-1} i\lambda_j(\bar z_j w_j - z_j \bar w_j) + O(|(z,w)|^3)\end{aligned} \tag{5-123}$$

in some neighborhood of (p,p) in $M' \times M'$, where $a_j = \frac12 \partial_{x_j}\sigma(\Box_f^{(q)})(p, -2i\partial\rho(p))$ for $j = 1, \dots, 2n$, and $\sigma(\Box_f^{(q)})$ denotes the principal symbol of $\Box_f^{(q)}$.

The following result describes the restriction to the diagonal of the coefficient b_0 from the expansion of the symbol $b(z,w,t)$ of $\Pi^{(q)}$; see (5-119), (5-120).

Theorem 5.25 [Hsiao 2010, Part II, Proposition 1.6]. *Under the assumptions and notation of Theorem 5.23, fix $p \in U \cap X$. The coefficient $b_0(z,w)$ from (5-120) satisfies*

$$b_0(x,x) = 2\pi^{-n}|\det\mathcal{L}_x|\tau_{x,n_-} \circ (\bar\partial\rho(x))^{\wedge,*}(\bar\partial\rho(x))^{\wedge} \quad \text{for every } x \in U \cap X, \tag{5-124}$$

where $\det\mathcal{L}_x$, τ_{x,n_-} are given by (1-9), (1-11) respectively and $(\bar\partial\rho(x))^{\wedge,*}$ is given by (1-12).

6. Microlocal spectral theory for the $\bar\partial$-Neumann Laplacian

In this section, we will apply the approximate Hodge decomposition theorems for the $\bar\partial$-Neumann Laplacian $\Box^{(q)}$ from Sections 4 and 5 to study the singularities for the kernel $B^{(q)}_{\le\lambda}(x,y)$ near the nondegenerate part of the Levi form. In particular, we give the proof of Theorem 1.1.

Until further notice, we fix $\lambda > 0$. Since $\Box^{(q)}$ is bounded below by $\lambda > 0$ on $\operatorname{Ker} B^{(q)}_{\le\lambda}$ there exists a continuous operator

$$A^{(q)}_\lambda : L^2_{(0,q)}(M) \to \operatorname{Dom}\Box^{(q)}$$

such that

$$\begin{aligned}\Box^{(q)}A^{(q)}_\lambda + B^{(q)}_{\le\lambda} = I \quad \text{on } L^2_{(0,q)}(M),\\ A^{(q)}_\lambda\Box^{(q)} + B^{(q)}_{\le\lambda} = I \quad \text{on } \operatorname{Dom}\Box^{(q)}.\end{aligned} \tag{6-1}$$

Let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Until further notice, we let $q = n_-$.

Theorem 6.1. *Let $q = n_-$. The operators*

$$\bar\partial B^{(q)}_{\le\lambda} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M'), \tag{6-2}$$

$$\bar\partial^* B^{(q)}_{\le\lambda} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{6-3}$$

$$\Box^{(q)} B^{(q)}_{\le\lambda} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \tag{6-4}$$

are continuous for every $s \in \mathbb{N}$.

Proof. Let $u \in L^2(M, T^{*0,q}M')$. Since $B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\Box^{(q)}$, $\bar\partial B^{(q)}_{\le\lambda}u \in L^2_{(0,q+1)}(M)$. We claim that

$$\bar\partial B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\Box^{(q+1)}. \tag{6-5}$$

It is clear that $\bar\partial B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\bar\partial \cap \mathrm{Dom}\,\bar\partial^*$ and $\bar\partial^2 B^{(q)}_{\le\lambda}u = 0$. Hence, $\bar\partial^2 B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\bar\partial^*$. We only need to show that $\bar\partial^*\,\bar\partial B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\bar\partial$. We have

$$\bar\partial^*\,\bar\partial B^{(q)}_{\le\lambda}u = \Box^{(q)} B^{(q)}_{\le\lambda}u - \bar\partial\,\bar\partial^* B^{(q)}_{\le\lambda}u. \tag{6-6}$$

By spectral theory [Ma and Marinescu 2007, Theorem C.2.1], we see that $\Box^{(q)} B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\Box^{(q)}$ and hence $\Box^{(q)} B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\bar\partial$. Note that $\bar\partial^2\bar\partial^* B^{(q)}_{\le\lambda}u = 0$, $\bar\partial\,\bar\partial^* B^{(q)}_{\le\lambda}u \in \mathrm{Dom}\,\bar\partial$. From this observation and (6-6), we get (6-5). From (4-29), we have

$$N^{(q+1)}\Box^{(q+1)}\bar\partial B^{(q)}_{\le\lambda}u = \bar\partial B^{(q)}_{\le\lambda}u + F^{(q+1)}_1\bar\partial B^{(q)}_{\le\lambda}u, \tag{6-7}$$

where $F^{(q+1)}_1 \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (4-29). It is clear that

$$F^{(q+1)}_1\bar\partial : L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M') \tag{6-8}$$

is continuous for every $s \in \mathbb{Z}$. We have

$$N^{(q+1)}\Box^{(q+1)}\bar\partial B^{(q)}_{\le\lambda} = N^{(q+1)}\bar\partial\Box^{(q)} B^{(q)}_{\le\lambda} \quad \text{on } L^2_{(0,q)}(M). \tag{6-9}$$

By spectral theory,

$$\Box^{(q)} B^{(q)}_{\le\lambda} : L^2_{(0,q)}(M) \to L^2_{(0,q)}(M) \tag{6-10}$$

is continuous. In view of Theorem 4.3, we see that

$$N^{(q+1)}\bar\partial : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M') \tag{6-11}$$

is continuous for every $s \in \mathbb{Z}$. From (6-7), (6-8), (6-9), (6-10) and (6-11), we deduce that

$$\bar\partial B^{(q)}_{\le\lambda} : L^2_{(0,q)}(M) \to L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M') \tag{6-12}$$

is continuous. We have

$$N^{(q+1)}\Box^{(q+1)}\bar\partial B^{(q)}_{\le\lambda}u = N^{(q+1)}\bar\partial\Box^{(q)} B^{(q)}_{\le\lambda}u = N^{(q+1)}\bar\partial B^{(q)}_{\le\lambda}\Box^{(q)} B^{(q)}_{\le\lambda}u.$$

From this observation and (6-7), we have

$$N^{(q+1)}\bar{\partial}B^{(q)}_{\leq\lambda}\Box^{(q)}B^{(q)}_{\leq\lambda}u=\bar{\partial}B^{(q)}_{\leq\lambda}u+F^{(q)}_1\bar{\partial}B^{(q)}_{\leq\lambda}u. \tag{6-13}$$

From (6-8), (6-10), (6-12), (6-13) and since that

$$N^{(q+1)}:H^s_{\rm loc}(U\cap\overline{M},T^{*0,q+1}M')\to H^{s+1}_{\rm loc}(U\cap\overline{M},T^{*0,q+1}M') \tag{6-14}$$

is continuous for every $s\in\mathbb{Z}$, we deduce that

$$\bar{\partial}B^{(q)}_{\leq\lambda}:L^2_{(0,q)}(M)\to H^1_{\rm loc}(U\cap\overline{M},T^{*0,q+1}M') \tag{6-15}$$

is continuous. The continuity of (6-2) follows by induction. The proof of the continuity of (6-3) is analogous, and that of (6-4) follows then immediately. $\square$

Lemma 6.2. *Let $q=n_-$. For every $m\in\mathbb{N}$, the operator $B^{(q)}_{\leq\lambda}\bar{\partial}(\Box^{(q-1)}_f)^m:\Omega^{0,q-1}_c(M)\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\bar{\partial}(\Box^{(q-1)}_f)^m:L^2_{(0,q-1)}(M)\to L^2_{(0,q)}(M). \tag{6-16}$$

Proof. Let $u\in\Omega^{0,q-1}_c(M)$, $v\in L^2_{(0,q)}(M)$. We have

$$(B^{(q)}_{\leq\lambda}\bar{\partial}(\Box^{(q-1)}_f)^m u\mid v)_M=(B^{(q)}_{\leq\lambda}(\Box^{(q)}_f)^m\bar{\partial}u\mid v)_M=(u\mid\bar{\partial}^*(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v)_M. \tag{6-17}$$

We have

$$\begin{aligned}\|\bar{\partial}^*(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\|^2_M&\leq\|\bar{\partial}^*(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\|^2_M+\|\bar{\partial}(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\|^2_M\\&=\big((\Box^{(q)})^{m+1}B^{(q)}_{\leq\lambda}v\mid(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\big)_M\leq\lambda^{2m+1}\|v\|^2_M.\end{aligned} \tag{6-18}$$

From (6-17), (6-18) and taking $v=B^{(q)}_{\leq\lambda}\bar{\partial}(\Box^{(q-1)}_f)^m u$, it is straightforward to see that

$$\|B^{(q)}_{\leq\lambda}\bar{\partial}(\Box^{(q-1)}_f)^m u\|_M\leq\lambda^{m+\frac{1}{2}}\|u\|_M. \tag{6-19}$$

From (6-19) and noticing that $\Omega^{0,q-1}_c(M)$ is dense in $L^2_{(0,q-1)}(M)$, the lemma follows. $\square$

Theorem 6.3. (i) *The operator $B^{(q)}_{\leq\lambda}\bar{\partial}:\Omega^{0,q-1}_c(U\cap\overline{M})\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\bar{\partial}:H^{-s}_c(U\cap\overline{M},T^{*0,q-1}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-20}$$

(ii) *The operator $B^{(q)}_{\leq\lambda}\bar{\partial}^*_f:\Omega^{0,q+1}_c(U\cap\overline{M})\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\bar{\partial}^*_f:H^{-s}_c(U\cap\overline{M},T^{*0,q+1}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-21}$$

(iii) *The operator $B^{(q)}_{\leq\lambda}\Box^{(q)}_f:\Omega^{0,q}_c(U\cap\overline{M})\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\Box^{(q)}_f:H^{-s}_c(U\cap\overline{M},T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-22}$$

Proof. Let $u\in\Omega_c^{0,q-1}(U\cap\overline{M})$. From (4-30), we have

$$B^{(q)}_{\le\lambda}\bar\partial\Box_f^{(q-1)}N^{(q-1)}u=B^{(q)}_{\le\lambda}\bar\partial u+B^{(q)}_{\le\lambda}\bar\partial F_2^{(q-1)}u,\tag{6-23}$$

where $F_2^{(q-1)}\equiv 0$ on $U\cap\overline{M}$. From Theorem 4.3, (6-16), (6-23) and since

$$N^{(q-1)}:H_c^s(U\cap\overline{M},T^{*0,q-1}M')\to H_c^{s+1}(U\cap\overline{M},T^{*0,q-1}M')\tag{6-24}$$

is continuous for every $s\in\mathbb{Z}$, we deduce that $B^{(q)}_{\le\lambda}\bar\partial$ can be continuously extended to

$$B^{(q)}_{\le\lambda}\bar\partial:H_c^{-1}(U\cap\overline{M},T^{*0,q-1}M')\to L^2_{(0,q)}(M).\tag{6-25}$$

From Lemma 6.2, we can repeat the proof of (6-25) and deduce that $B^{(q)}_{\le\lambda}\bar\partial\Box_f^{(q-1)}$ can be continuously extended to

$$B^{(q)}_{\le\lambda}\bar\partial\Box_f^{(q-1)}:H_c^{-1}(U\cap\overline{M},T^{*0,q-1}M')\to L^2_{(0,q)}(M).\tag{6-26}$$

From (6-23), (6-24) and (6-26), we deduce that $B^{(q)}_{\le\lambda}\bar\partial$ can be continuously extended to

$$B^{(q)}_{\le\lambda}\bar\partial:H_c^{-2}(U\cap\overline{M},T^{*0,q-1}M')\to L^2_{(0,q)}(M).$$

Continuing by induction we get (i). Item (ii) follows analogously and (iii) follows from (i) and (ii). □

We consider

$$\Box^{(q)}B^{(q)}_{\le\lambda}\Box_f^{(q)}:\Omega_c^{0,q}(U\cap\overline{M})\to L^2_{(0,q)}(M)\subset L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M'),$$
$$(\Box^{(q)})^2B^{(q)}_{\le\lambda}:\Omega_c^{0,q}(U\cap\overline{M})\to L^2_{(0,q)}(M)\subset L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M').$$

Theorem 6.4. *We have*

$$\Box^{(q)}B^{(q)}_{\le\lambda}\Box_f^{(q)}\equiv 0\ \mathrm{mod}\ \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\tag{6-27}$$
$$(\Box^{(q)})^2B^{(q)}_{\le\lambda}\equiv 0\ \mathrm{mod}\ \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).\tag{6-28}$$

Proof. From (6-4) and (6-22), we have

$$\Box^{(q)}B^{(q)}_{\le\lambda}\Box_f^{(q)}:H_c^{-s}(U\cap\overline{M},T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')$$

for every $s\in\mathbb{N}$. This proves (6-27). Let $u\in L^2_{(0,q)}(M)$. Take $u_j\in\Omega_c^{0,q}(M)$, $j=1,2,\dots$, so that $\lim_{j\to+\infty}\|u_j-u\|_M=0$. Since $(\Box^{(q)})^2B^{(q)}_{\le\lambda}$ is L^2 continuous, we have

$$(\Box^{(q)})^2B^{(q)}_{\le\lambda}u=\lim_{j\to+\infty}(\Box^{(q)})^2B^{(q)}_{\le\lambda}u_j\quad\text{in }L^2_{(0,q)}(M).\tag{6-29}$$

From the fact that $u_j\in\mathrm{Dom}\,\Box^{(q)}$ for every $j=1,2,\dots$, we can check that

$$(\Box^{(q)})^2B^{(q)}_{\le\lambda}u_j=\Box^{(q)}B^{(q)}_{\le\lambda}\Box^{(q)}u_j=\Box^{(q)}B^{(q)}_{\le\lambda}\Box_f^{(q)}u_j\quad\text{for every }j=1,2,\dots.\tag{6-30}$$

From (6-29) and (6-30), we conclude that

$$(\Box^{(q)})^2B^{(q)}_{\le\lambda}=\Box^{(q)}B^{(q)}_{\le\lambda}\Box_f^{(q)}\quad\text{on }L^2_{(0,q)}(M).\tag{6-31}$$

From (6-27) and (6-31), we get (6-28). □

Lemma 6.5. *The operator $B^{(q)}_{\leq\lambda}$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda} : H^{-s}_c(U\cap\overline{M}, T^{*0,q}M') \to H^{-s}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \tag{6-32}$$

for every $s\in\mathbb{N}$.

Proof. Let $u\in\Omega^{0,q}_c(U\cap\overline{M})$. From (5-118), we have

$$B^{(q)}_{\leq\lambda}\Box^{(q)}_f N^{(q)}u + B^{(q)}_{\leq\lambda}\Pi^{(q)}u = B^{(q)}_{\leq\lambda}u + B^{(q)}_{\leq\lambda}r^{(q)}_0 u, \tag{6-33}$$

where $r^{(q)}_0 \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-118). From (5-116), (6-22) and (6-33) and noting that $\Omega^{0,q}_c(U\cap\overline{M})$ is dense in $H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')$ for every $s\in\mathbb{N}$, we deduce that $B^{(q)}_{\leq\lambda} - B^{(q)}_{\leq\lambda}\Pi^{(q)}$ can be continuously extended to

$$B^{(q)}_{\leq\lambda} - B^{(q)}_{\leq\lambda}\Pi^{(q)} : H^{-s}_c(U\cap\overline{M}, T^{*0,q}M') \to L^2_{(0,q)}(M) \quad \text{for every } s\in\mathbb{N}. \tag{6-34}$$

On the other hand, from (6-1) and (5-118), we have

$$\begin{aligned}\Pi^{(q)}u &= (A^{(q)}_\lambda\Box^{(q)} + B^{(q)}_{\leq\lambda})\Pi^{(q)}u\\ &= A^{(q)}_\lambda\Box^{(q)}_f\Pi^{(q)}u + B^{(q)}_{\leq\lambda}\Pi^{(q)}u\\ &= A^{(q)}_\lambda r^{(q)}_5 u + B^{(q)}_{\leq\lambda}\Pi^{(q)}u\end{aligned} \tag{6-35}$$

for every $u\in\Omega^{0,q}_c(U\cap\overline{M})$, where $r^{(q)}_5 \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-118). From (6-35), we conclude that $\Pi^{(q)} - B^{(q)}_{\leq\lambda}\Pi^{(q)}$ can be continuously extended to

$$\Pi^{(q)} - B^{(q)}_{\leq\lambda}\Pi^{(q)} : H^{-s}_c(U\cap\overline{M}, T^{*0,q}M') \to L^2_{(0,q)}(M) \quad \text{for every } s\in\mathbb{N}. \tag{6-36}$$

From (6-34) and (6-36), we deduce that $\Pi^{(q)} - B^{(q)}_{\leq\lambda}$ can be continuously extended to

$$\Pi^{(q)} - B^{(q)}_{\leq\lambda} : H^{-s}_c(U\cap\overline{M}, T^{*0,q}M') \to L^2_{(0,q)}(M) \quad \text{for every } s\in\mathbb{N}. \tag{6-37}$$

From (5-116) and (6-37), we get (6-32). □

Theorem 6.6. *We have*

$$\Box^{(q)}B^{(q)}_{\leq\lambda} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{6-38}$$

Proof. By (6-28), $\varepsilon^{(q)} := (\Box^{(q)})^2 B^{(q)}_{\leq\lambda}$ is smoothing on $U\cap\overline{M}$. Let $u\in\Omega^{0,q}_c(U\cap\overline{M})$. From the second equation in (5-118), we have

$$\begin{aligned}\Box^{(q)}B^{(q)}_{\leq\lambda}u &= N^{(q)}(\Box^{(q)})^2 B^{(q)}_{\leq\lambda}u + \Pi^{(q)}\Box^{(q)}B^{(q)}_{\leq\lambda}u - r^{(q)}_1\Box^{(q)}B^{(q)}_{\leq\lambda}u\\ &= N^{(q)}\varepsilon^{(q)}u + r^{(q)}_4 B^{(q)}_{\leq\lambda}u - r^{(q)}_1\Box^{(q)}B^{(q)}_{\leq\lambda}u,\end{aligned} \tag{6-39}$$

where $r^{(q)}_1$, $r^{(q)}_4$ are the smoothing operators from (5-118). From (6-32), we see that

$$r^{(q)}_1\Box^{(q)}B^{(q)}_{\leq\lambda},\ r^{(q)}_4 B^{(q)}_{\leq\lambda} : H^{-s}_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$$

are continuous for every $s \in \mathbb{N}$, and hence they are smoothing on $U \cap \overline{M}$. From this observation and (6-39), we get (6-38). $\square$

We can now prove one of the main results of this work.

Theorem 6.7. *Let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Let $q = n_-$ and fix $\lambda > 0$. We have*

$$B^{(q)}_{\leq\lambda} - \Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})),$$

where $\Pi^{(q)}$ is as in Theorem 5.23.

Proof. From the second equation in (5-118), we have

$$N^{(q)} \square^{(q)} B^{(q)}_{\leq\lambda} u + \Pi^{(q)} B^{(q)}_{\leq\lambda} u = r_1^{(q)} B^{(q)}_{\leq\lambda} u + B^{(q)}_{\leq\lambda} u \tag{6-40}$$

for every $u \in \Omega_c^{0,q}(U \cap \overline{M})$, where $r_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (5-118). From (5-116), (6-32), (6-38) and (6-40), we deduce that

$$B^{(q)}_{\leq\lambda} - \Pi^{(q)} B^{(q)}_{\leq\lambda} =: \varepsilon^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-41}$$

Similarly, from the first equation in (5-118), we have

$$B^{(q)}_{\leq\lambda} \square^{(q)}_f N^{(q)} u + B^{(q)}_{\leq\lambda} \Pi^{(q)} u = B^{(q)}_{\leq\lambda} u + B^{(q)}_{\leq\lambda} r_0^{(q)} u \tag{6-42}$$

for every $u \in \Omega_c^{0,q}(U \cap \overline{M})$, where $r_0^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (5-118). Since $N^{(q)} u \in \operatorname{Dom} \square^{(q)}$, we have

$$B^{(q)}_{\leq\lambda} \square^{(q)}_f N^{(q)} u = B^{(q)}_{\leq\lambda} \square^{(q)} N^{(q)} u = \square^{(q)} B^{(q)}_{\leq\lambda} N^{(q)} u$$

for every $u \in \Omega_c^{0,q}(U \cap \overline{M})$. From this observation and (6-42), we deduce that

$$\square^{(q)} B^{(q)}_{\leq\lambda} N^{(q)} u + B^{(q)}_{\leq\lambda} \Pi^{(q)} u = B^{(q)}_{\leq\lambda} u + B^{(q)}_{\leq\lambda} r_0^{(q)} u \tag{6-43}$$

for every $u \in \Omega_c^{0,q}(U \cap \overline{M})$. From (6-32), (6-38) and (6-43), we deduce that

$$B^{(q)}_{\leq\lambda} - B^{(q)}_{\leq\lambda} \Pi^{(q)} =: \varepsilon_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-44}$$

Let $u \in \Omega_c^{0,q}(U \cap \overline{M})$. From (6-1), we have

$$\Pi^{(q)} \square^{(q)} A^{(q)}_\lambda u + \Pi^{(q)} B^{(q)}_{\leq\lambda} u = \Pi^{(q)} u \quad \text{on } U \cap \overline{M}, \tag{6-45}$$

$$A^{(q)}_\lambda \square^{(q)} \Pi^{(q)} u + B^{(q)}_{\leq\lambda} \Pi^{(q)} u = \Pi^{(q)} u \quad \text{on } U \cap \overline{M}. \tag{6-46}$$

From (5-118), we have

$$\Pi^{(q)} \square^{(q)} A^{(q)}_\lambda u = r_4^{(q)} A^{(q)}_\lambda u \quad \text{on } U \cap X, \tag{6-47}$$

$$A^{(q)}_\lambda \square^{(q)} \Pi^{(q)} u = A^{(q)}_\lambda r_5^{(q)} u \quad \text{on } U \cap X, \tag{6-48}$$

where $r_4^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $r_5^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ are as in (5-118). From (6-47), (6-48) and (6-46), we deduce that

$$\begin{aligned}\Pi^{(q)}-\Pi^{(q)}B^{(q)}_{\leq\lambda}&=r_4^{(q)}A_\lambda^{(q)},\\ \Pi^{(q)}-B^{(q)}_{\leq\lambda}\Pi^{(q)}&=A_\lambda^{(q)}r_5^{(q)}.\end{aligned}\tag{6-49}$$

From (6-41), (6-44) and (6-49), we get

$$\begin{aligned}\Pi^{(q)}-B^{(q)}_{\leq\lambda}&=r_4^{(q)}A_\lambda^{(q)}-\varepsilon^{(q)},\\ \Pi^{(q)}-B^{(q)}_{\leq\lambda}&=A_\lambda^{(q)}r_5^{(q)}-\varepsilon_1^{(q)}.\end{aligned}\tag{6-50}$$

From (6-50), we have

$$\begin{aligned}(\Pi^{(q)}&-B^{(q)}_{\leq\lambda})(\Pi^{(q)}-B^{(q)}_{\leq\lambda})\\ &=(r_4^{(q)}A_\lambda^{(q)}-\varepsilon^{(q)})(A_\lambda^{(q)}r_5^{(q)}-\varepsilon_1^{(q)})\\ &=r_4^{(q)}(A_\lambda^{(q)})^2r_5^{(q)}-r_4^{(q)}A_\lambda^{(q)}\varepsilon_1^{(q)}-\varepsilon^{(q)}A_\lambda^{(q)}r_5^{(q)}+\varepsilon^{(q)}\varepsilon_1^{(q)}\quad\text{on } \Omega_c^{0,q}(U\cap\overline{M}).\end{aligned}\tag{6-51}$$

Note that $r_5^{(q)}$ and $r_4^{(q)}$ are properly supported on $U\cap\overline{M}$ and $r_4^{(q)}(A_\lambda^{(q)})^2r_5^{(q)}$, $r_4^{(q)}A_\lambda^{(q)}\varepsilon_1^{(q)}$, $\varepsilon^{(q)}A_\lambda^{(q)}r_5^{(q)}$, $\varepsilon^{(q)}\varepsilon_1^{(q)}$ are well-defined as continuous operators: $\Omega_c^{0,q}(U\cap\overline{M})\to\Omega^{0,q}(U\cap\overline{M})$. Now,

$$\begin{aligned}r_4^{(q)}(A_\lambda^{(q)})^2r_5^{(q)}:H_c^{-s}(U\cap\overline{M},T^{*0,q}M')\to H_c^s(U\cap\overline{M},T^{*0,q}M')\subset L^2_{(0,q)}(M)\\ \to L^2_{(0,q)}(M)\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\end{aligned}$$

is continuous for every $s\in\mathbb{N}$. Hence, $r_4^{(q)}(A_\lambda^{(q)})^2r_5^{(q)}\equiv 0\mod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. Similarly, $r_4^{(q)}A_\lambda^{(q)}\varepsilon_1^{(q)}$, $\varepsilon^{(q)}A_\lambda^{(q)}r_5^{(q)}$, $\varepsilon^{(q)}\varepsilon_1^{(q)}\equiv 0\mod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From this observation and (6-51), we get

$$(\Pi^{(q)}-B^{(q)}_{\leq\lambda})(\Pi^{(q)}-B^{(q)}_{\leq\lambda})\equiv 0\mod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).\tag{6-52}$$

Now,

$$\begin{aligned}(\Pi^{(q)}-B^{(q)}_{\leq\lambda})(\Pi^{(q)}-B^{(q)}_{\leq\lambda})&=(\Pi^{(q)})^2-\Pi^{(q)}B^{(q)}_{\leq\lambda}-B^{(q)}_{\leq\lambda}\Pi^{(q)}+(B^{(q)}_{\leq\lambda})^2\\ &=\Pi^{(q)}+r_6^{(q)}-B^{(q)}_{\leq\lambda}+\varepsilon^{(q)}-B^{(q)}_{\leq\lambda}+\varepsilon_1^{(q)}+B^{(q)}_{\leq\lambda}\\ &=\Pi^{(q)}-B^{(q)}_{\leq\lambda}+r_6^{(q)}+\varepsilon^{(q)}+\varepsilon_1^{(q)},\end{aligned}\tag{6-53}$$

where $r_6^{(q)},\varepsilon^{(q)},\varepsilon_1^{(q)}\equiv 0\mod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ are as in (5-118), (6-41) and (6-44) respectively. From (6-52) and (6-53), the theorem follows. □

By using Theorem 4.7, we can repeat the proof of Theorem 6.7 with minor changes and deduce:

Theorem 6.8. *Let U be an open set of M' with $U\cap X\neq\varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-,n_+) on $U\cap X$. Let $q\neq n_-$. Fix $\lambda>0$. We have*

$$B^{(q)}_{\leq\lambda}\equiv 0\mod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

Proof of Theorem 1.1. This follows immediately from Theorems 5.23, 6.7 and 6.8. □

We remind the reader that the local closed range condition is given by Definition 1.4. The following is our second main result.

Theorem 6.9. *Let U be an open set of M' with $U \cap X \neq \varnothing$. Assume that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Let $q = n_-$. Suppose that $\Box^{(q)}$ has local closed range in U. Then the Bergman projection $B^{(q)}$ satisfies*

$$B^{(q)} - \Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})),$$

where $\Pi^{(q)}$ is as in Theorem 5.23.

Proof. Let W be any open set of U with $W \cap U \neq \varnothing$, $\overline{W} \Subset U$. Since $\Pi^{(q)}$ is properly supported on $U \cap \overline{M}$, there is an open set $W' \subset U$ with $W' \cap X \neq \varnothing$, $\overline{W}' \Subset U$, such that $\Pi^{(q)} u \in \Omega_c^{0,q}(W' \cap \overline{M}) \cap \operatorname{Dom} \Box^{(q)}$ for every $u \in \Omega_c^{0,q}(W \cap \overline{M})$. Since $\Box^{(q)}$ has local closed range on U, there is a constant $C_{W'} > 0$ such that, for every $u \in \Omega_c^{0,q}(W \cap \overline{M})$,

$$\|(I - B^{(q)})\Pi^{(q)} u\|_M \leq C_{W'} \|\Box^{(q)} \Pi^{(q)} u\|_M = \|r_5^{(q)} u\|_M \,, \tag{6-54}$$

where $r_5^{(q)}$ is as in (5-118). Let $u \in H_c^{-s}(W \cap \overline{M}, T^{*0,q} M')$. Let $u_j \in \Omega_c^{0,q}(W \cap \overline{M})$, $\lim_{j \to \infty} u_j = u$ in $H_c^{-s}(W \cap \overline{M}, T^{*0,q} M')$. Since $r_5^{(q)}$ is smoothing on $U \cap \overline{M}$ the sequence $r_5^{(q)} u_j$ is Cauchy, so by (6-54) $(\Pi^{(q)} - B^{(q)} \Pi^{(q)}) u_j$ converges in $L^2_{(0,q)}(M)$, as $j \to \infty$. Thus, u is in the domain of $\Pi^{(q)} - B^{(q)} \Pi^{(q)}$ and $(\Pi^{(q)} - B^{(q)} \Pi^{(q)}) u \in L^2_{(0,q)}(M)$. We conclude that $\Pi^{(q)} - B^{(q)} \Pi^{(q)}$ can be extended continuously to

$$\Pi^{(q)} - B^{(q)} \Pi^{(q)} : H_c^{-s}(U \cap \overline{M}, T^{*0,q} M') \to L^2_{(0,q)}(M) \quad \text{for every } s \in \mathbb{N}. \tag{6-55}$$

From the first two equations in (5-118) we have, on $U \cap X$,

$$\begin{aligned} \Pi^{(q)} B^{(q)} u &= N^{(q)} \Box^{(q)} B^{(q)} u + \Pi^{(q)} B^{(q)} u = B^{(q)} u + r_1^{(q)} B^{(q)} u, \quad u \in L^2_{(0,q)}(M), \\ B^{(q)} \Pi^{(q)} u &= B^{(q)} \Box^{(q)} N^{(q)} u + B^{(q)} \Pi^{(q)} u = B^{(q)} u + B^{(q)} r_0^{(q)} u, \quad u \in \Omega_c^{0,q}(U \cap \overline{M}), \end{aligned} \tag{6-56}$$

where $r_0^{(q)}, r_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ are as in (5-118). From (6-56), we conclude that $B^{(q)} - \Pi^{(q)} B^{(q)}$ and $B^{(q)} - B^{(q)} \Pi^{(q)}$ can be extended continuously to

$$B^{(q)} - \Pi^{(q)} B^{(q)} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q} M') \quad \text{for every } s \in \mathbb{N}, \tag{6-57}$$

$$B^{(q)} - B^{(q)} \Pi^{(q)} : H_c^{-s}(U \cap \overline{M}, T^{*0,q} M') \to L^2_{(0,q)}(M) \quad \text{for every } s \in \mathbb{N}. \tag{6-58}$$

Form (6-55) and (6-58), we deduce that $\Pi^{(q)} - B^{(q)}$ can be extended continuously to

$$\Pi^{(q)} - B^{(q)} : H_c^{-s}(U \cap \overline{M}, T^{*0,q} M') \to L^2_{(0,q)}(M) \quad \text{for every } s \in \mathbb{N}. \tag{6-59}$$

Since $\Pi^{(q)} : H_c^s(U \cap \overline{M}, T^{*0,q} M') \to H_c^s(U \cap \overline{M}, T^{*0,q} M')$ is continuous for every $s \in \mathbb{Z}$, we deduce that $B^{(q)}$ can be extended continuously to

$$B^{(q)} : H_c^{-s}(U \cap \overline{M}, T^{*0,q} M') \to H^{-s}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q} M') \quad \text{for every } s \in \mathbb{N}. \tag{6-60}$$

From (6-60), we deduce that

$$r_1^{(q)} B^{(q)}, (r_0^{(q)})^* B^{(q)} : H_c^{-s}(U \cap \overline{M}, T^{*0,q} M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q} M') \quad \text{for every } s \in \mathbb{N},$$

where $r_1^{(q)}, r_0^{(q)}$ are as in (5-118) and $(r_0^{(q)})^*$ is the formal adjoint of $r_0^{(q)}$ with respect to $(\cdot \mid \cdot)_M$. Hence,

$$r_1^{(q)} B^{(q)}, (r_0^{(q)})^* B^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-61}$$

By taking adjoint of $(r_0^{(q)})^* B^{(q)}$, we get

$$B^{(q)} r_0^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{6-62}$$

From (6-56), (6-61) and (6-62), we get

$$\begin{aligned} &\Pi^{(q)}B^{(q)}u - B^{(q)}u = f_1^{(q)}u \quad \text{on } U\cap X \text{ for every } u\in L^2_{(0,q)}(M),\\ &B^{(q)}\Pi^{(q)}u - B^{(q)}u = f_2^{(q)}u \quad \text{on } U\cap X \text{ for every } u\in \Omega^{0,q}_c(U\cap\overline{M}), \end{aligned} \tag{6-63}$$

where

$$\begin{aligned} &f_1^{(q)} : L^2_{(0,q)}(M)\to \Omega^{0,q}(U\cap\overline{M}), \quad f_1^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\\ &f_2^{(q)} : \Omega^{0,q}_c(U\cap\overline{M})\to L^2_{(0,q)}(M), \quad f_2^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \end{aligned}$$

Taking adjoint in (6-59), we conclude that $(\Pi^{(q)})^* - B^{(q)}$ can be extended continuously to

$$(\Pi^{(q)})^* - B^{(q)} : L^2_{(0,q)}(M)\to H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{N}, \tag{6-64}$$

where $(\Pi^{(q)})^*$ is the formal adjoint of $\Pi^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$. From (5-50), we see that

$$(\Pi^{(q)})^* = \Pi^{(q)} + \Gamma_1^{(q)},$$

where $\Gamma_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From this observation and (6-64), we deduce that $\Pi^{(q)} - B^{(q)}$ can be extended continuously to

$$\Pi^{(q)} - B^{(q)} : L^2_{(0,q)}(M)\to H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{N}. \tag{6-65}$$

From (6-59) and (6-65), we get

$$(\Pi^{(q)} - B^{(q)})(\Pi^{(q)} - B^{(q)}) : H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$$

is continuous for every $s\in\mathbb{N}$. Hence,

$$(\Pi^{(q)} - B^{(q)})(\Pi^{(q)} - B^{(q)}) \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{6-66}$$

On the other hand, we have

$$\begin{aligned} &(\Pi^{(q)}-B^{(q)})(\Pi^{(q)}-B^{(q)})u\\ &\quad = (\Pi^{(q)})^2u - \Pi^{(q)}B^{(q)}u - B^{(q)}\Pi^{(q)}u + (B^{(q)})^2u\\ &\quad = \Pi^{(q)}u - B^{(q)}u - B^{(q)}u + B^{(q)}u + ((\Pi^{(q)})^2 - \Pi^{(q)})u + (B^{(q)} - \Pi^{(q)}B^{(q)})u + (B^{(q)} - B^{(q)}\Pi^{(q)})u\\ &\quad = \Pi^{(q)}u - B^{(q)}u + r_6^{(q)}u - f_1^{(q)}u - f_2^{(q)}u \quad \text{for every } u\in\Omega^{0,q}_c(U\cap\overline{M}), \end{aligned} \tag{6-67}$$

where $r_6^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-118), $f_1^{(q)}, f_2^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ are as in (6-63). From (6-66) and (6-67), the theorem follows. □

7. Proof of Theorem 1.9

To prove Theorem 1.9, we need a result of [Takegoshi 1983], which is a generalization of [Kohn 1973]. Consider an open relatively compact subset $M_0 := \{z\in M' : \rho(z) < 0\}$ with smooth boundary X_0 of M'. We have the following (see [Takegoshi 1983, Section 3, Theorem N]).

Theorem 7.1. *Let M_0 be a pseudoconvex domain with smooth boundary X_0 in a complex manifold M' and let L be a holomorphic line bundle on M' which is positive on a neighborhood of M_0. Then there exists $k_0 \in \mathbb{N}$ such that the following statement holds for every $k \in \mathbb{N}$, $k \geq k_0$: For every $f \in L^2_{(0,1)}(M_0, L^k)$ with $\bar{\partial} f = 0$ on M_0 there exists $g \in L^2(M_0, L^k)$ such that $\bar{\partial} g = f$ on M_0 and*

$$\int_{M_0} |g|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_{M_0} |f|^2_{h^{L^k}}\, dv_{M'}, \tag{7-1}$$

where $C_k > 0$ is a constant independent of f and g and $|\cdot|_{h^{L^k}}$ denotes the pointwise norm on $\bigoplus_{q=0}^{n} T^{*0,q}M' \otimes L^k$ induced by the given Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TM'$ and h^L.

Proof of (1-23). Let $k_0 \in \mathbb{N}$ be as in Theorem 7.1. Let $k \geq k_0$, $k \in \mathbb{N}$, and let U be any open set of X_0 with $U \cap X_1 = \varnothing$. Let $u \in \mathscr{C}^\infty_c(U \cap \overline{M}, L^k) \cap \operatorname{Dom} \Box^{(0)}_k$ and let $f := \bar{\partial} u \in \Omega^{0,1}_c(U \cap M, L^k) \subset L^2_{(0,1)}(M_0, L^k)$. From Theorem 7.1, we see that there is a $g \in L^2(M_0, L^k) \subset L^2(M, L^k)$ such that $\bar{\partial} g = \bar{\partial} u$ on M_0 (hence on M) and

$$\int_{M_0} |g|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_{M_0} |\bar{\partial} u|^2_{h^{L^k}}\, dv_{M'}, \tag{7-2}$$

where $C_k > 0$ is a constant independent of u and g. Since $(I - B^{(0)}_k)u$ is the solution of $\bar{\partial} g = \bar{\partial} u$ on M of minimal L^2 norm, we have

$$\int_{M} |(I - B^{(0)}_k)u|^2_{h^{L^k}}\, dv_{M'} \leq \int_{M} |g|^2_{h^{L^k}}\, dv_{M'}. \tag{7-3}$$

From (7-2) and (7-3), we get

$$\int_{M} |(I - B^{(0)}_k)u|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_{M_0} |\bar{\partial} u|^2_{h^{L^k}}\, dv_{M'}. \tag{7-4}$$

Since $\bar{\partial} u$ has compact support in $U \cap \overline{M}$, we have

$$\int_{M_0} |\bar{\partial} u|^2_{h^{L^k}}\, dv_{M'} = \int_{M} |\bar{\partial} u|^2_{h^{L^k}}\, dv_{M'}. \tag{7-5}$$

From (7-4) and (7-5), we get

$$\int_{M} |(I - B^{(0)}_k)u|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_{M} |\bar{\partial} u|^2_{h^{L^k}}\, dv_{M'}. \tag{7-6}$$

Since $u \in \operatorname{Dom} \Box^{(0)}_k$, we can check that

$$\begin{aligned}\int_{M} |\bar{\partial} u|^2_{h^{L^k}}\, dv_{M'} &= (\bar{\partial} u \,|\, \bar{\partial} u\,)_k = (\bar{\partial} u \,|\, \bar{\partial}(I - B^{(0)}_k)u\,)_k \\ &= (\Box^{(0)}_k u \,|\, (I - B^{(0)}_k)u\,)_k \leq \|\Box^{(0)}_k u\|_k \|(I - B^{(0)}_k)u\|_k.\end{aligned} \tag{7-7}$$

Since (1-23) follows from (7-6) and (7-7), we are done. □

From Theorem 1.5, Remark 1.6 and Theorem 1.9, we immediately get (1-24).

8. S^1-equivariant Bergman kernel asymptotics and embedding theorems

In this section, we assume that M' admits a holomorphic S^1-action $e^{i\theta}$, $\theta \in [0, 2\pi)$, $e^{i\theta} : M' \to M'$, $x \in M' \to e^{i\theta} \circ x \in M'$. Recall that X_0 is an open connected component of X such that (1-26) holds and we work with the following assumption.

Assumption 8.1. For every $x \in X$ we have $\mathbb{C}T_0(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_xX$, and the S^1-action preserves the boundary X, that is, there exists a defining function $\rho \in \mathscr{C}^\infty(M',\mathbb{R})$ of X such that $\rho(e^{i\theta}\circ x) = \rho(x)$ for every $x \in M'$ and every $\theta \in [0, 2\pi]$.

Theorem 8.2. *Assume that M' admits a holomorphic S^1-action and Assumption 8.1 holds. Let X_0 be a connected component of X such that (1-26) holds, let $p \in X_0$ and let U be an open set of p in M' with $U \cap X_0 \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X_0$, where n_- denotes the number of the negative eigenvalues of the Levi form on $U \cap X_0$. Fix $\lambda > 0$. If $q \neq n_-$, then as $m \to +\infty$,*

$$B^{(q)}_{\leq\lambda,m} \equiv 0 \mod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}. \tag{8-1}$$

Let $q = n_-$. Let $N_p := \{g \in S^1 : g \circ p = p\} = \{g_0 := e, g_1, \dots, g_r\}$, where e denotes the identify element in S^1 and $g_j \neq g_\ell$ if $j \neq \ell$ for every $j, \ell \in \{0, 1, \dots, r\}$. We have

$$B^{(q)}_{\leq\lambda,m}(x, y) \equiv \sum_{\alpha=0}^{r} g_\alpha^m e^{im\phi(x,g_\alpha y)} b_\alpha(x, y, m) \mod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}, \tag{8-2}$$

where, for every $\alpha = 0, 1, \dots, r$,

$$\begin{gathered} b_\alpha(x, y, m) \in S^n_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M'\times M'}), \\ b_\alpha(x, y, m) \sim \textstyle\sum_{j=0}^\infty b_{\alpha,j}(x, y) m^{n-j} \quad \text{in } S^n_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M'\times M'}), \\ b_{\alpha,j}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M'\times M'}), \quad j = 0, 1, \dots, \\ b_{\alpha,0}(x, x) = b_0(x, x), \quad b_0(x, x) \text{ is given by (5-124)}, \end{gathered} \tag{8-3}$$

and $\phi(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (1-17).

Proof. From (1-14) and (1-30), we can integrate by parts in θ and get (8-1). We now prove (8-2). From Theorem 6.7 and (5-111), it is straightforward to see that

$$B^{(q)}_{\leq\lambda} \equiv \widetilde{P} S_{-,m} L^{(q)} \mod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}, \tag{8-4}$$

where $S_{-,m}(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_-(x, e^{i\theta}y) e^{im\theta}\, d\theta$ and $S_-(x, y)$ is as in Theorem 5.17. From Theorem 5.17, we can repeat the proof of [Hsiao and Marinescu 2014, Theorem 3.12] with minor changes and deduce that

$$S_{-,m}(x, y) \equiv \sum_{\alpha=0}^{r} g_\alpha^m e^{im\varphi_-(x,g_\alpha y)} a_\alpha(x, y, m) \mod O(m^{-\infty}) \quad \text{on } U \cap X, \tag{8-5}$$

where, for every $\alpha = 0, 1, \dots, r$,

$$\begin{gathered} a_\alpha(x, y, m) \in S^{n-1}_{\mathrm{loc}}((U \times U) \cap (X \times X), \Lambda^{(0,q)|(0,q)}_{M'\times M'}), \\ a_\alpha(x, y, m) \sim \textstyle\sum_{j=0}^\infty a_{\alpha,j}(x, y) m^{n-1-j} \quad \text{in } S^{n-1}_{\mathrm{loc}}((U \times U) \cap (X \times X), \Lambda^{(0,q)|(0,q)}_{M'\times M'}), \\ a_{\alpha,j}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M'\times M'}), \quad j = 0, 1, \dots, \\ a_{\alpha,0}(x, x) = a_0(x, x), \end{gathered} \tag{8-6}$$

where $a_0(x, x)$ is given by (5-84) and $\varphi_-(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in Theorem 5.17. From (8-5), we can repeat the procedure in the proof of [Hsiao 2010, Part II, Proposition 7.8] and deduce that the distribution kernel of $\widetilde{P} S_{-,m} L^{(q)}$ is of the form (8-2). ☐

By Theorem 1.5, we can repeat the proof of Theorem 8.2 and deduce:

Theorem 8.3. *Assume that M' admits a holomorphic S^1-action and Assumption 8.1 holds. Let X_0 be a connected component of X such that (1-26) holds, let $p \in X_0$ and let U be an open set of p in M' with $U \cap X_0 \neq \emptyset$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X_0$, where n_- denotes the number of the negative eigenvalues of the Levi form on $U \cap X_0$. Suppose that $\Box^{(q)}$ has local closed range in U. If $q \neq n_-$, then*

$$B_m^{(q)} \equiv 0 \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}. \tag{8-7}$$

Let $q = n_-$. Let $N_p := \{g \in S^1 : g \circ p = p\} = \{g_0 := e, g_1, \dots, g_r\}$, where e denotes the identify element in S^1 and $g_j \neq g_\ell$ if $j \neq \ell$ for every $j, \ell = 0, 1, \dots, r$. We have

$$B_m^{(q)}(x, y) \equiv \sum_{\alpha=0}^{r} g_\alpha^m e^{im\phi(x, g_\alpha y)} b_\alpha(x, y, m) \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}, \tag{8-8}$$

where $\phi(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ and $b_\alpha(x, y, m) \in S^n_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M' \times M'})$, $\alpha = 0, 1, \dots, r$, are as in Theorem 8.2.

Proof of Theorem 1.10. We now consider the case $q = 0$. When $Z(1)$ holds on X, it is well known (see [Folland and Kohn 1972]) that $\Box^{(0)}$ has L^2 closed range. From this observation and Theorem 8.3, we deduce Theorem 1.10. $\Box$

We will now prove Theorem 1.11 about the S^1-equivariant embedding theorem.

Proof of Theorem 1.11. Fix $m_0 \in \mathbb{N}$. By using Theorem 1.10, we can repeat the proof of [Herrmann et al. 2018, Theorem 1.2] with minor changes and conclude that we can find $m_1 \in \mathbb{N}, \dots, m_k \in \mathbb{N}$, with $m_j \geq m_0,\ j = 1, \dots, k$, such that

$$\Phi_{m_1,\dots,m_k} : X_0 \to \mathbb{C}^{\hat{d}_m} \text{ is an embedding} \tag{8-9}$$

and there is an S^1-invariant open set U of X_0 such that

$$\Phi_{m_1,\dots,m_k} : U \cap \overline{M} \to \mathbb{C}^{\hat{d}_m} \text{ is an immersion.} \tag{8-10}$$

Fix $x_0 \in X_0$. From (8-10), it is straightforward to see that there are S^1-invariant open sets $\Omega_{x_0} \Subset W_{x_0} \Subset U_{x_0}$ of x_0 in M' such that

$$\Phi_{m_1,\dots,m_k} : U_{x_0} \cap \overline{M} \to \mathbb{C}^{\hat{d}_m} \text{ is injective.} \tag{8-11}$$

Let

$$\delta_{x_0} := \inf\{|\Phi_{m_1,\dots,m_k}(x) - \Phi_{m_1,\dots,m_k}(y)| : x \in \Omega_{x_0} \cap X_0,\ y \in X_0,\ y \notin W_{x_0} \cap X_0\}. \tag{8-12}$$

From (8-9), we see that $\delta_{x_0} > 0$. Let V^{x_0} be a small S^1-invariant open set of X_0 in M' such that, for every $x \in V^{x_0} \cap \overline{M},\ x \notin U_{x_0}$, there is a $y \in X_0,\ y \notin W_{X_0} \cap X_0$, such that

$$|\Phi_{m_1,\dots,m_k}(x) - \Phi_{m_1,\dots,m_k}(y)| \leq \tfrac{1}{2}\delta_{x_0}. \tag{8-13}$$

Assume that $X_0=\bigcup_{j=1}^N(\Omega_{x_j}\cap X_0)$, $N\in\mathbb{N}$, and let

$$V:=U\cap\Bigl(\bigcap_{j=1}^N V^{x_j}\Bigr)\cap\Bigl(\bigcup_{j=1}^N\Omega_{x_j}\Bigr),$$

where Ω_{x_j}, V^{x_j}, $j=1,\dots,N$, are as above, and U is as in (8-10). From (8-10), we see that $\Phi_{m_1,\dots,m_k}:V\cap\overline{M}\to\mathbb{C}^{\hat{d}_m}$ is an immersion. We claim that $\Phi_{m_1,\dots,m_k}:V\cap\overline{M}\to\mathbb{C}^{\hat{d}_m}$ is injective. Let $p,q\in V\cap\overline{M}$, $p\neq q$. We may assume that $p\in\Omega_{x_1}\cap\overline{M}$. If $q\in U_{x_1}$ we see from (8-11) that $\Phi_{m_1,\dots,m_k}(p)\neq\Phi_{m_1,\dots,m_k}(q)$. Assume that $q\notin U_{x_1}$. From the discussion before (8-13), we see that there is $y_0\in X_0$, $y_0\notin W_{x_1}\cap X_0$ such that

$$|\Phi_{m_1,\dots,m_k}(q)-\Phi_{m_1,\dots,m_k}(y_0)|\leq\tfrac{1}{2}\delta_{x_1}. \tag{8-14}$$

From (8-14) and (8-12), we have

$$\begin{aligned}|\Phi_{m_1,\dots,m_k}(p)-\Phi_{m_1,\dots,m_k}(q)|&\geq|\Phi_{m_1,\dots,m_k}(p)-\Phi_{m_1,\dots,m_k}(y_0)|-|\Phi_{m_1,\dots,m_k}(y_0)-\Phi_{m_1,\dots,m_k}(q)|\\&\geq\delta_{x_1}-\tfrac{1}{2}\delta_{x_1}>0.\end{aligned}$$

Hence $\Phi_{m_1,\dots,m_k}(p)\neq\Phi_{m_1,\dots,m_k}(q)$, so $\Phi_{m_1,\dots,m_k}$ is injective and the theorem follows. □

Proof of Theorem 1.12. We may assume that $X_0=\{x\in M':\rho(x)=0\}$. Consider the shell domain

$$\widehat{M}:=\{x\in M':-\varepsilon<\rho(x)<0\},$$

where $\varepsilon>0$ is a small constant. Then $\widehat{M}$ is a complex manifold with smooth boundary $\widehat{X}$. Moreover, it is easy to see that X_0 is an open connected component of $\widehat{X}$ and $Z(1)$ holds on $\widehat{X}$. Hence, we can apply Theorem 1.11 to get Theorem 1.12. □

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