Semi-classical Spectral Asymptotics of Toeplitz Operators on Strictly Pseudodonvex Domains

Chin-Yu Hsiao and George Marinescu

Abstract On a relatively compact strictly pseudoconvex domain with smooth boundary in a complex manifold of dimension n we consider a Toeplitz operator T_R with symbol a Reeb-like vector field R near the boundary. We show that the kernel of a weighted spectral projection $\chi(k^{-1}T_R)$, where χ is a cut-off function with compact support in the positive real line, is a semi-classical Fourier integral operator with complex phase, hence admits a full asymptotic expansion as $k \to +\infty$. More precisely, the restriction to the diagonal $\chi(k^{-1}T_R)(x,x)$ decays at the rate $O(k^{-\infty})$ in the interior and has an asymptotic expansion on the boundary with leading term of order k^{n+1} expressed in terms of the Levi form and the pairing of the contact form with the vector field R.

Keywords Bergman projector · Szegö projector · Toeplitz operator · Semi-classical Fourier intergral operator

1 Introduction

Since the introduction of the Bergman kernel in [1], and the subsequent groundbreaking work by Hörmander [9], Fefferman [7], and Boutet de Monvel and Sjöstrand [4], the study of the Bergman kernel has been a central subject in several complex variables and complex geometry.

Let M be a relatively compact strictly pseudoconvex domain with smooth boundary in a complex manifold M' and let $B : L^2(M) \to H^0_{(2)}(M)$ be the Bergman projection, that is, the orthogonal projection from the space of square-integrable func-

C.-Y. Hsiao (✉)
Institute of Mathematics, Academia Sinica, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road, Taipei 10617, Taiwan
e-mail: chsiao@math.sinica.edu.tw; chinyu.hsiao@gmail.com

G. Marinescu
Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany
e-mail: gmarines@math.uni-koeln.de

Institute of Mathematics ‘Simion Stoilow’, Romanian Academy, Bucharest, Romania

K. Hirachi et al. (eds.), *The Bergman Kernel and Related Topics*, Springer Proceedings in Mathematics & Statistics 447, https://doi.org/10.1007/978-981-99-9506-6_8

tions $L^2(M)$ onto the space of L^2 holomorphic functions on M. The Bergman kernel $B(x, y)$ is the Schwartz kernel of B. Fefferman [7] obtained the complete asymptotic expansion of the diagonal Bergman kernel $B(x, x)$ at the boundary. Subsequently, Boutet de Monvel-Sjöstrand [4] described the singularity of the full Bergman kernel $B(x, y)$ by showing that it is a Fourier integral operator with complex phase. They also obtained in [4] a full asymptotic expansion for the Szegő projection on a strictly pseudoconvex CR manifold. All the results mentioned above are about microlocal behavior of the Szegő and Bergman kernels. Some of these results were recently extended to weakly pseudoconvex domains of finite type in $\mathbb{C}^2$ in [16]. The structure of the Szegő projector also plays an important role in the quantization of CR manifolds [15].

On the other hand, semi-classical analysis plays an important role in modern complex geometry. For example, we can study many important problems in complex geometry by using semi-classical Bergman kernel asymptotics [5, 11, 12, 17]. Therefore, we believe that it is important to study classical several complex variables from semi-classical viewpoint. To this end, it is important to have semiclassical versions of the Boutet de Monvel-Sjöstrand's and Fefferman's results on strictly pseudoconvex CR manifolds and on complex manifolds with strictly pseudoconvex boundary. Recently, we obtained jointly with Herrmann and Shen [8] a semi-classical version of the Boutet de Monvel and Sjöstrand's result on a strictly peudoconvex CR manifold and as applications, we established Kodaira type embedding theorem and Tian type theorem on a strictly pseudoconvex CR manifold.

It is natural to establish similar results as in [8] for complex manifolds with boundary. In this paper, we consider the operator $\chi_k(T_R)$ constructed by functional calculus, where $\chi_k(\lambda) = \chi(k^{-1}\lambda)$ is a rescaled cut-off function χ with compact support in the positive real line, $k \in \mathbb{R}_+$ is a semi-classical parameter, and T_R is the Toeplitz operator on the domain M associated with a first-order differential operator given by a Reeb-like vector field from in the neighborhood of X. We show that $\chi_k(T_R)$ admits a full asymptotic expansion as $k \to +\infty$. This result can be seen as a semi-classical version of the Boutet de Monvel-Sjöstrand's and Fefferman's results on complex manifolds with boundary.

We now formulate our main result. We refer the reader to Section 2 for the notations used here. Let (M', J) be a complex manifold of dimension n with complex structure J. We fix a Hermitian metric Θ on M' and let $g^{TM'} = \Theta(\cdot, J\cdot)$ be the Riemannian metric on TM' associated to Θ and let $dv_{M'}$ be its volume form. We denote by $\langle \cdot | \cdot \rangle$ the pointwise Hermitian product induced by $g^{TM'}$ on the fibers of $\mathbb{C}TM'$ and by duality on $\mathbb{C}T^*M'$.

Let M be a relatively compact open subset in M' with smooth boundary. We set $X = \partial M$. We assume throughout the paper that M is strictly pseudoconvex. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of M (cf. (20)), let $\mathscr{L}_x = \mathscr{L}_x(\rho)$ be the the Levi form associated to ρ at $x \in X$ (cf. (23)) and let $\det(\mathscr{L}_x)$ be the determinant of the Levi form (cf. (24)). We consider the 1-form $\omega_0 = -d\rho \circ J = i(\overline{\partial}\rho - \partial\rho)$ and we fix the contact form $\omega_0|_{TX} = 2i\overline{\partial}\rho|_{TX} = -2i\partial\rho|_{TX}$ on X (cf. (21)–(23)).

Let $(\cdot | \cdot)_M, (\cdot | \cdot)_{M'}$ be the L^2 inner products on $\mathscr{C}^\infty(\overline{M}), \mathscr{C}^\infty_c(M')$ induced by the given Hermitian metric $\langle \cdot | \cdot \rangle$ respectively (see (28)). Let $L^2(M)$ be the completion of $\mathscr{C}^\infty(\overline{M})$ with respect to $(\cdot\,|\,\cdot)_M$. Let $H^0(\overline{M}) := \big\{u \in \mathscr{C}^\infty(\overline{M});\ \overline{\partial}u = 0\big\}$, where $\overline{\partial} : \mathscr{C}^\infty(M') \to \Omega^{0,1}(M')$ denotes the standard Cauchy-Riemann operator on M'. Let $H^0_{(2)}(M)$ be the completion of $H^0(\overline{M})$ with respect to $(\cdot\,|\,\cdot,)_M$.

We denote by $\nabla\rho$ the gradient of ρ with respect to the Riemannian metric $g^{TM'}$. We consider the vector field $T = \alpha J\big(\nabla\rho\big) + Z$ on M', where $\alpha \in \mathscr{C}^\infty(M')$, $a|_X > 0$, and $Z \in \mathscr{C}^\infty(M', TM')$, $Z|_X \in \mathscr{C}^\infty(X, HX)$, cf. (25)–(26). Let R be a formally selfadjoint first order partial differential operator on M', given near X by $R = \frac{1}{2}((-iT) + (-iT)^*)$.

Since M is strictly pseudoconvex we have by [4, 7] that the Bergman projection maps the space $\mathscr{C}^\infty(\overline{M})$ of smooth functions up to the boundary into itself, $B : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M})$. Let

$$T_R := BRB : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(\overline{M}).$$

We extend T_R to $L^2(M)$:

$$\begin{aligned} &T_R : \mathrm{Dom}(T_R) \subset L^2(M) \to L^2(M), \\ &\mathrm{Dom}(T_R) = \big\{u \in L^2(M);\ BRBu \in L^2(M)\big\}, \end{aligned} \tag{1}$$

where $BRBu$ is defined in the sense of distributions on M. In Theorem 2, we will show that T_R is self-adjoint. We consider a function

$$\chi \in \mathscr{C}^\infty_c(\mathbb{R}_+, \mathbb{R}), \tag{2}$$

and set for $k > 0$,

$$\chi_k \in \mathscr{C}^\infty_c(\mathbb{R}_+, \mathbb{R}), \quad \chi_k(\lambda) := \chi(k^{-1}\lambda). \tag{3}$$

We let

$$\chi_k(T_R) : L^2(M) \to L^2(M), \tag{4}$$

be obtained by functional calculus of T_R and let $\chi_k(T_R)(\cdot\,,\cdot) \in \mathscr{D}'(M \times M)$ be the distribution kernel of $\chi_k(T_R)$. We will show that $\chi_k(T_R)(\cdot\,,\cdot) \in \mathscr{C}^\infty(\overline{M} \times \overline{M})$ (cf. Corollary 2). We consider a function χ with support in $(0, +\infty)$ in order to avoid that the spectral operator $\chi_k(T_R)$ takes into account the zero eigenvalue of T_R, whose eigenspace contains the kernel of B. With this choice the image of $\chi_k(T_R)$ is contained in $H^0_{(2)}(M)$.

The main result of this paper is the following.

Theorem 1 *Let M be a relatively compact strictly pseudoconvex domain with smooth boundary X of a complex manifold M' of dimension n. Let $T_R : \mathrm{Dom}(T_R) \subset L^2(M) \to L^2(M)$ be the Toeplitz operator (1) and let $\chi_k(T_R)$ be as in (4). Then the following assertion hold:*

(i) For any $\tau, \hat{\tau} \in \mathscr{C}^\infty(\overline{M})$ with $\operatorname{supp}\tau \cap \operatorname{supp}\hat{\tau} = \emptyset$ we have

$$\tau\chi_k(T_R)\hat{\tau} = O(k^{-\infty}) \quad \text{on } \overline{M} \times \overline{M}. \tag{5}$$

(ii) For any $\tau \in \mathscr{C}^\infty_c(M)$ *we have*

$$\tau\chi_k(T_R) = O(k^{-\infty}) \quad \text{on } \overline{M} \times \overline{M}. \tag{6}$$

(iii) For any $p \in X$ *and any open local coordinate patch* U *around* p *in* M' *we have*

$$\chi_k(T_R)(x,y) = \int_0^{+\infty} e^{ikt\Psi(x,y)} b(x,y,t,k)dt + O(k^{-\infty}) \quad \text{on } (U \times U) \cap (\overline{M} \times \overline{M}), \tag{7}$$

where $b(x,y,t,k) \in S^{n+1}_{\mathrm{loc}}(1;((U \times U) \cap (\overline{M} \times \overline{M})) \times \mathbb{R}_+)$*,*

$$\begin{aligned}
&b(x,y,t,k) \sim \sum_{j=0}^{\infty} b_j(x,y,t)k^{n+1-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1;((U \times U) \cap (\overline{M} \times \overline{M})) \times \mathbb{R}_+),\\
&b_j(x,y,t) \in \mathscr{C}^\infty(((U \times U) \cap (\overline{M} \times \overline{M})) \times \mathbb{R}_+), \ \ j = 0,1,2,\ldots,\\
&b_0(x,x,t) = \frac{1}{\pi^n} \det(\mathscr{L}_x)\, \chi(t\omega_0(T(x)))t^n \not\equiv 0, \ \ x \in U \cap X,
\end{aligned} \tag{8}$$

and for some compact interval $I \Subset \mathbb{R}_+$*,*

$$\mathrm{supp}_t\, b(x,y,t,k), \ \ \mathrm{supp}_t\, b_j(x,y,t) \subset I, \quad j = 0,1,2,\ldots, \tag{9}$$

and

$$\begin{aligned}
&\Psi(z,w) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \ \ \mathrm{Im}\,\Psi \geq 0,\\
&\Psi(z,z) = 0, \ \ z \in U \cap X,\\
&\mathrm{Im}\,\Psi(z,w) > 0 \text{ if } (z,w) \notin (U \times U) \cap (X \times X),\\
&d_x\Psi(x,x) = -d_y\Psi(x,x) = -2i\partial\rho(x), \ \ x \in U \cap X,\\
&\Psi|_{(U \times U) \cap (X \times X)} = \varphi_-,
\end{aligned} \tag{10}$$

where φ_- *is a phase function as in* (39)*, cf. [13, Theorem 4.1]. Moreover, using the local coordinates* $z = (x_1,\ldots,x_{2n-1},\rho)$ *on* M' *near* p*, where* $x = (x_1,\ldots,x_{2n-1})$ *are local coordinates on* X *near* p *with* $x(p) = 0$*, we have*

$$\Psi(z,w) = \Psi(x,y) - i\rho(z)(1+f(z)) - i\rho(w)(1+\overline{f(w)}) + O(|(z,w)|^3) \ \text{ near } (p,p), \tag{11}$$

where f *is smooth near* p *and* $f = O(|z|)$*.*

The representation (7) shows that near the boundary $\chi_k(T_R)$ is a semi-classical Fourier integral operator with complex phase and canonical relation generated by the phase $\Psi(x,y)t$. The integral in (7) is a smooth kernel, since t runs in the bounded interval I. The term $O(k^{-\infty})$ denotes a k-negligible smooth kernel (cf. (17)–(18)).

The idea of the proof of Theorem 1 follows the strategy of [4, 10]. We express the Bergman projection in terms of the Poisson operator and a projector $\mathcal{S}$ on a subspace of functions annihilated by a system of pseudo-differential operators simulating $\overline{\partial}_b$.

We can express in the same way a Toeplitz operator T_R in terms of a Toeplitz operator on the boundary X, given by $\mathcal{T}_{\mathcal{R}} = \mathcal{S}\mathcal{R}\mathcal{S}$. The operator $\mathcal{S}$ is a Fourier integral operator having a structure similar to the Szegő projector cf. [3, 4, 8, 10] and we can apply the results obtained in [8] for the asymptotics of $\chi_k(\mathcal{T}_{\mathcal{R}})$.

As a consequence we have the following asymptotics of the kernel of $\chi_k(T_R)$ on the diagonal.

Corollary 1 *In the situation of Theorem 1 we have:*

$$\chi_k(T_R)(z,z) = O(k^{-\infty}), \quad \text{as } k \to \infty \text{ on } M. \tag{12}$$

$$\chi_k(T_R)(x,x) = \sum_{j=0}^{\infty} b_j(x) k^{n+1-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1;X) \text{ on } X, \tag{13}$$

where for $x \in X$ and with $b_j(x,x,t)$ as in (8),

$$b_j(x) = \int_0^{+\infty} b_j(x,x,t)dt, \quad j \in \mathbb{N}_0, \tag{14}$$

with

$$b_0(x) = \frac{1}{\pi^n}\det(\mathscr{L}_x)\int_0^{+\infty} \chi(t\omega_0(T(x)))t^n dt, \tag{15}$$

Moreover, there exist $C_1, C_2 > 0$ such that for k large enough $C_1 k^n \le \operatorname{Tr}\chi_k(T_R) \le C_2 k^n$.

2 Preliminaries

2.1 Notions from Microlocal and Semi-classical Analysis

We shall use the following notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R};\ x > 0\}$.

Let W be a smooth paracompact manifold. We let TW and T^*W denote the tangent bundle of W and the cotangent bundle of W respectively. The complexified tangent bundle of W and the complexified cotangent bundle of W are be denoted by $\mathbb{C}TW$ and $\mathbb{C}T^*W$, respectively. Write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TW and T^*W. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TW \times \mathbb{C}T^*W$. Let G be a smooth vector bundle over W. The fiber of G at $x \in W$ will be denoted by G_x. Let $Y \subset W$ be an open set. From now on, the spaces of distributions of Y and smooth functions of Y will be denoted by $\mathscr{D}'(Y)$ and $\mathscr{C}^\infty(Y)$ respectively. Let $\mathscr{E}'(Y)$ be the subspace of $\mathscr{D}'(Y)$ whose elements have compact support in Y and let $\mathscr{C}^\infty_c(Y)$ be the subspace of $\mathscr{C}^\infty(Y)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, let $H^m(Y)$ denote the Sobolev space of order m of Y. Put

$$H^m_{\mathrm{loc}}(Y) = \left\{u \in \mathscr{D}'(Y);\ \varphi u \in H^m(Y),\ \text{for every } \varphi \in \mathscr{C}^\infty_c(Y)\right\},$$
$$H^m_{\mathrm{comp}}(Y) = H^m_{\mathrm{loc}}(Y) \cap \mathscr{E}'(Y)\,.$$

If $A : \mathscr{C}^\infty_c(W) \to \mathscr{D}'(W)$ is continuous, we write $A(x, y)$ to denote the distribution kernel of A. The following two statements are equivalent

1. A is continuous: $\mathscr{E}'(W) \to \mathscr{C}^\infty(W)$,
2. $A(x, y) \in \mathscr{C}^\infty(W \times W)$.

If A satisfies (1) or (2), we say that A is smoothing on W. Let $A, B : \mathscr{C}^\infty_c(W) \to \mathscr{D}'(W)$ be continuous operators. We write

$$A \equiv B \text{ (on } W \times W) \tag{16}$$

if $A - B$ is a smoothing operator on W.

Let $H(x, y) \in \mathscr{D}'(W \times W)$. We write H to denote the unique continuous operator $\mathscr{C}^\infty_c(W) \to \mathscr{D}'(W)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

Let D be an open set of a smooth manifold X. For $0 \leq \rho, \delta \leq 1$, $m \in \mathbb{R}$, let

$$L^m_{\rho,\delta}(D)\,, \quad L^m_{\mathrm{cl}}(D),$$

denote the space of pseudodifferential operators on D of order m type (ρ, δ) and the space of classical pseudodifferential operators on D of order m respectively. Let $W \subset \mathbb{R}^N$ be an open set. For $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$, let $S^m_{\rho,\delta}(W \times \mathbb{R}^{N_1})$ be the Hörmander symbol space on $W \times \mathbb{R}^{N_1}$ of order m and type (ρ, δ). Let $S^m_{\mathrm{cl}}(W \times \mathbb{R}^{N_1})$ be the classical symbol space on $W \times \mathbb{R}^{N_1}$ of order m.

Let W_1 be an open set in $\mathbb{R}^{N_1}$ and let W_2 be an open set in $\mathbb{R}^{N_2}$. A k-dependent continuous operator $F_k : \mathscr{C}^\infty_c(W_2) \to \mathscr{D}'(W_1)$ is called k-negligible on $W_1 \times W_2$ if, for k large enough, F_k is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices α, β and any $N \in \mathbb{N}$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial^\alpha_x \partial^\beta_y F_k(x, y)\right| \leq C_{K,\alpha,\beta,N} k^{-N} \text{ on } K, \quad \text{for } k \gg 1. \tag{17}$$

In that case we write

$$F_k(x, y) = O(k^{-\infty}) \text{ or } F_k = O(k^{-\infty}) \text{ on } W_1 \times W_2. \tag{18}$$

If $F_k, G_k : \mathscr{C}^\infty_c(W_2) \to \mathscr{D}'(W_1)$ are k-dependent continuous operators, we write $F_k = G_k + O(k^{-\infty})$ on $W_1 \times W_2$ or $F_k(x, y) = G_k(x, y) + O(k^{-\infty})$ on $W_1 \times W_2$ if $F_k - G_k = O(k^{-\infty})$ on $W_1 \times W_2$.

Let Ω_1 and Ω_2 be smooth manifolds. Let $F_k, G_k : \mathscr{C}^\infty(\Omega_2) \to \mathscr{C}^\infty(\Omega_1)$ be k-dependent smoothing operators. We write $F_k = G_k + O(k^{-\infty})$ on $\Omega_1 \times \Omega_2$ if on every local coordinate patch D of Ω_1 and local coordinate patch D_1 of Ω_2, $F_k = G_k + O(k^{-\infty})$ on $D \times D_1$.

We recall the definition of the semi-classical symbol spaces.

Definition 1 Let W be an open set in $\mathbb{R}^N$. Let

$$S(1; W) := \left\{a \in \mathscr{C}^\infty(W);\ \text{for every } \alpha \in \mathbb{N}_0^N : \sup_{x \in W} \left|\partial^\alpha a(x)\right| < \infty\right\},$$

$$S^0_{\mathrm{loc}}(1; W) := \left\{(a(\cdot, k))_{k\in\mathbb{R}};\ \text{for all } \alpha \in \mathbb{N}_0^N,\ \chi \in \mathscr{C}_c^\infty(W) : \sup_{k\geq 1}\sup_{x\in W} \left|\partial^\alpha(\chi a(x, k))\right| < \infty\right\}.$$

For $m \in \mathbb{R}$, let

$$S^m_{\mathrm{loc}}(1) := S^m_{\mathrm{loc}}(1; W) = \left\{(a(\cdot, k))_{k\in\mathbb{R}};\ (k^{-m}a(\cdot, k)) \in S^0_{\mathrm{loc}}(1; W)\right\}.$$

Hence $a(\cdot, k) \in S^m_{\mathrm{loc}}(1; W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in \mathscr{C}_c^\infty(W)$, there exists $C_\alpha > 0$ independent of k, such that $|\partial^\alpha(\chi a(\cdot, k))| \leq C_\alpha k^m$ holds on W.

Consider a sequence $a_j \in S^{m_j}_{\mathrm{loc}}(1)$, $j \in \mathbb{N}_0$, where $m_j \searrow -\infty$, and let $a \in S^{m_0}_{\mathrm{loc}}(1)$. We say that

$$a(\cdot, k) \sim \sum_{j=0}^{\infty} a_j(\cdot, k) \ \text{ in } S^{k_0}_{\mathrm{loc}}(1),$$

if for every $\ell \in \mathbb{N}_0$, we have $a - \sum_{j=0}^{\ell} a_j \in S^{m_{\ell+1}}_{\mathrm{loc}}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\mathrm{loc}}(1) = S^{-\infty}_{\mathrm{loc}}(1; W) := \cap_m S^m_{\mathrm{loc}}(1)$.

Similarly, we can define $S^m_{\mathrm{loc}}(1; Y)$ in the standard way, where Y is a smooth manifold.

2.2 Set Up of Complex Manifolds with Smooth Boundary

Let (M', J) be a complex manifold of dimension n, where $J : TM' \to TM'$ is the complex structure of M'. We fix a Hermitian metric Θ on M' and let $g^{TM'} = \Theta(\cdot, J\cdot)$ be the Riemannian metric on TM' associated to Θ and let $dv_{M'}$ be its volume form. We denote by $\langle \cdot \,|\, \cdot \rangle$ the pointwise Hermitian product induced by $g^{TM'}$ on the fibers of the bundle $\Lambda^q(T^{*(0,1)}M')$ of $(0, q)$-forms for every $q \in \{0, \ldots, n\}$. Let $\Omega^{0,q}(M')$ be the space of smooth $(0, q)$-forms on M' and let $\Omega^{0,q}_c(M')$ be the subspace of $\Omega^{0,q}(M')$ whose elements have compact support in M'. The L^2 inner product on $\Omega^{0,q}_c(M')$ is given by

$$(\alpha \,|\, \beta)_{M'} = \int_{M'} \langle \alpha \,|\, \beta \rangle dv_{M'}. \tag{19}$$

The corresponding L^2 space is denoted by $L^2_{0,q}(M')$, and we set $L^2(M') = L^2_{0,0}(M')$.

Let M be a relatively compact open subset of M' with smooth boundary. Hence $X := \partial M$ is a submanifold of M' of real dimension $2n - 1$. We denote by $HX =$

$TX \cap J(TX)$ the complex tangent bundle of X. The triple (X, HX, J) forms a CR structure on X and we set $T^{1,0}X := T^{1,0}M' \cap \mathbb{C}TX$, $T^{0,1}X := \overline{T^{1,0}X}$. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of X, that is,

$$M = \{x \in M' : \rho(x) < 0\}, \quad X = \partial M = \{x \in M' : \rho(x) = 0\}, \quad \text{and } d\rho \neq 0 \text{ on } X. \tag{20}$$

From now on, we fix a defining function ρ so that $|d\rho| = 1$ on X. Define a real 1-form ω_0 on M' by

$$\omega_0 = -d\rho \circ J. \tag{21}$$

Hence

$$\omega_0 = i(\overline{\partial}\rho - \partial\rho), \quad d\omega_0 = 2i\partial\overline{\partial}\rho. \tag{22}$$

The Levi form of ρ is Hermitian symmetric map $\mathscr{L}_x = \mathscr{L}_x(\rho)$ given by

$$\mathscr{L}_x : T_x^{1,0}X \times T_x^{1,0}X \to \mathbb{C}, \quad \mathscr{L}_x(U, \overline{V}) = \frac{1}{2i} d\omega_0(U, \overline{V}) = \partial\overline{\partial}\rho(U, \overline{V}), \quad U, V \in T_x^{1,0}X. \tag{23}$$

We assume that M is a strictly pseudoconvex domain, that is, the Levi form $\mathscr{L}_x$ is positive definite for every $x \in X$. In this case the hyperplane field HX is a contact structure on X. Indeed, $HX = \ker(\omega_0|_{TX})$ and for every $u \in HX \setminus \{0\}$ we have $d\omega_0(u, Ju) = 4\mathscr{L}(U, \overline{U}) > 0$, where $U = \frac{1}{2}(u - iJu) \in T^{1,0}X$. So $d\omega_0|_{HX}$ is symplectic, and hence HX is a contact structure, with $\omega_0|_{TX} = 2i\overline{\partial}\rho|_{TX} = -2i\partial\rho|_{TX}$ a contact form.

We denote by $\lambda_j(x)$, $j = 1, \ldots, n-1$, the eigenvalues of $\mathscr{L}_x$ with respect to $\langle \cdot | \cdot \rangle$ (note that $T^{0,1}X$ has rank $n-1$). The determinant of the Levi form is defined by

$$\det(\mathscr{L})_x := \lambda_1(x) \ldots \lambda_{n-1}(x). \tag{24}$$

Let $\nabla\rho$ be the gradient of ρ with respect to the Riemannian metric $g^{TM'}$. We define the vector field T on M' by

$$T = \alpha J\big(\nabla\rho\big) + Z \in \mathscr{C}^\infty(M', TM'), \tag{25}$$

where

$$\alpha \in \mathscr{C}^\infty(M'), \;\; \alpha|_X > 0, \;\; Z \in \mathscr{C}^\infty(M', TM'), \;\; Z|_X \in \mathscr{C}^\infty(X, HX). \tag{26}$$

The vector field T does not vanish on a neighborhood of X. Indeed, we have on X that $\langle J(\nabla\rho), Z\rangle = -\langle \nabla\rho, JZ\rangle = 0$ hence $|T|^2 = a^2 + |Z|^2 > 0$. Note also that

$$\omega_0(T) = -(d\rho \circ J)(\alpha J(\nabla\rho) + Z) = \alpha d\rho(\nabla\rho) = \alpha|\nabla\rho|^2 = \alpha \quad \text{on } X, \tag{27}$$

since $|\nabla\rho| = |d\rho| = 1$ on X.

Let U be an open set in M'. Let

$$\mathscr{C}^\infty(U\cap\overline{M}),\ \ \mathscr{D}'(U\cap\overline{M}),\ \ \mathscr{C}^\infty_c(U\cap\overline{M}),\ \ \mathscr{E}'(U\cap\overline{M}),$$
$$H^s(U\cap\overline{M}),\ \ H^s_{\rm comp}(U\cap\overline{M}),\ \ H^s_{\rm loc}(U\cap\overline{M}),$$

(where $s\in\mathbb{R}$) denote the spaces of restrictions to $U\cap\overline{M}$ of elements in

$$\mathscr{C}^\infty(U\cap M'),\ \ \mathscr{D}'(U\cap M'),\ \ \mathscr{C}^\infty(U\cap M'),\ \ \mathscr{E}'(U\cap M'),$$
$$H^s(M'),\ \ H^s_{\rm comp}(M'),\ \ H^s_{\rm loc}(M'),$$

respectively. Write

$$L^2(U\cap\overline{M}):=H^0(U\cap\overline{M}),\ \ L^2_{\rm comp}(U\cap\overline{M}):=H^0_{\rm comp}(U\cap\overline{M}),$$
$$L^2_{\rm loc}(U\cap\overline{M}):=H^0_{\rm loc}(U\cap\overline{M}).$$

Let $dv_{M'}$ be the volume form on M' induced by the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TM'$ and and let $(\cdot\,|\,\cdot)_M$ and $(\cdot\,|\,\cdot)_{M'}$ be the inner products on $\mathscr{C}^\infty(\overline{M})$ and $\mathscr{C}^\infty_c(M')$ defined by

$$\begin{aligned}(\,f\,|\,h\,)_M&=\int_M f\overline{h}dv_{M'},\quad f,h\in\mathscr{C}^\infty(\overline{M}),\\ (\,f\,|\,h\,)_{M'}&=\int_{M'} f\overline{h}dv_{M'},\quad f,h\in\mathscr{C}^\infty_c(M').\end{aligned}\tag{28}$$

Let $\|\cdot\|_M$ and $\|\cdot\|_{M'}$ be the corresponding norms with respect to $(\cdot\,|\,\cdot)_M$ and $(\cdot\,|\,\cdot)_{M'}$ respectively. Let $L^2(M)$ be the completion of $\mathscr{C}^\infty(\overline{M})$ with respect to $(\cdot\,|\,\cdot)_M$. We extend $(\cdot\,|\,\cdot)_M$ to $L^2(M)$ in the standard way. For $q=1,2,\ldots,n$, let $\Omega^{0,q}(M')$ be the space of smooth $(0,q)$ forms on M' and let $\Omega^{0,q}_c(M')$ be the subspace of $\Omega^{0,q}(M')$ whose elements have compact support in M'. As in (28), let $(\cdot\,|\,\cdot)_{M'}$ be the L^2 inner product on $\Omega^{0,q}_c(M')$ induced by $dv_{M'}$ and $\langle\cdot\,|\,\cdot\rangle$.

The boundary $X=\partial M$ is a compact CR manifold of dimension $2n-1$ with natural CR structure $T^{1,0}X:=T^{1,0}M'\cap\mathbb{C}TX$. Let $T^{0,1}X:=\overline{T^{1,0}X}$. The Hermitian metric on $\mathbb{C}TM'$ induces a Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TX$ and let $(\cdot\,|\,\cdot)_X$ be the L^2 inner product on $\mathscr{C}^\infty(X)$ induced by $\langle\cdot\,|\,\cdot\rangle$.

Let U be an open set in M'. Let

$$F_1,F_2:\mathscr{C}^\infty_c(U\cap M)\to\mathscr{D}'(U\cap M)$$

be continuous operators. Let $F_1(x,y),F_2(x,y)\in\mathscr{D}'((U\times U)\cap(M\times M))$ be the distribution kernels of F_1 and F_2 respectively. We write

$$F_1\equiv F_2\mod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$$

or $F_1(x,y) \equiv F_2(x,y) \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ if $F_1(x,y) = F_2(x,y) + r(x,y)$, where $r(x,y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Let $F_k, G_k : \mathscr{C}^\infty_c(U \cap M) \to \mathscr{D}'(U \cap M)$ be k-dependent continuous operators. Let $F_k(x,y), G_k(x,y) \in \mathscr{D}'((U \times U) \cap (M \times M))$ be the distribution kernels of F_k and G_k respectively. We write

$$F_k(x,y) \equiv G_k(x,y) \mod O(k^{-\infty}) \quad \text{on } (U \times U) \cap (\overline{M} \times \overline{M}) \tag{29}$$

or $F_k \equiv G_k \mod O(k^{-\infty})$ on $(U \times U) \cap (\overline{M} \times \overline{M})$ if there is a $r_k(x,y) \in \mathscr{C}^\infty(U \times U)$ with $r_k(x,y) = O(k^{-\infty})$ on $U \times U$ such that

$$r_k(x,y)|_{(U \times U) \cap (\overline{M} \times \overline{M})} = F_k(x,y) - G_k(x,y), \text{ for } k \gg 1.$$

Let $m \in \mathbb{R}$. Let U be an open set in M'. Let

$$S^m_{\mathrm{loc}}(1, (U \times U) \cap (\overline{M} \times \overline{M})) \tag{30}$$

denote the space of restrictions to $(U \times U) \cap (\overline{M} \times \overline{M})$ of elements in $S^m_{\mathrm{loc}}(1, U \times U)$. Let

$$a_j \in S^{m_j}_{\mathrm{loc}}(1, (U \times U) \cap (\overline{M} \times \overline{M})), \quad j = 0, 1, 2, \ldots,$$

with $m_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{m_0}_{\mathrm{loc}}(1, (U \times U) \cap (\overline{M} \times \overline{M}))$ such that for every $\ell \in \mathbb{N}$,

$$a - \sum_{j=0}^{\ell-1} a_j \in S^{m_\ell}_{\mathrm{loc}}(1, (U \times U) \cap (\overline{M} \times \overline{M})).$$

If a and a_j have the properties above, we write

$$a \sim \sum_{j=0}^{\infty} a_j \text{ in } S^{m_0}_{\mathrm{loc}}(1, (U \times U) \cap (\overline{M} \times \overline{M})).$$

3 The Toeplitz Operator T_R

Let R be a first order partial differential operator on M' such that R is formally self-adjoint with respect to $(\cdot \,|\, \cdot)_{M'}$ and near X,

$$R = \frac{1}{2}((-iT) + (-iT)^*), \tag{31}$$

where T is given by (25) and $(-iT)^*$ is the formal adjoint of $-iT$ with respect to $(\cdot\,|\,\cdot)_{M'}$. Let T_R the Toeplitz operator introduced in (1). The goal of this section is to prove the following.

Theorem 2 *The operator* $T_R : \mathrm{Dom}(T_R) \subset L^2(M) \to L^2(M)$ *is self-adjoint.*

For the proof of the theorem 2 we need some preparation. Let

$$\overline{\partial}^*_f : \Omega^{0,1}(M') \to \mathscr{C}^\infty(M')$$

be the formal adjoint of $\overline{\partial}$ with respect to $(\cdot\,|\,\cdot)_{M'}$, that is, $(\,\overline{\partial} f\,|\,h\,)_{M'} = (\,f\,|\,\overline{\partial}^*_f h\,)_{M'}$, for any $f \in \mathscr{C}^\infty_c(M')$, $h \in \Omega^{0,1}(M')$. Let

$$\Box_f = \overline{\partial}^*_f\,\overline{\partial} : \mathscr{C}^\infty(M') \to \mathscr{C}^\infty(M')$$

denote the complex Laplace-Beltrami operator on functions. Let

$$P : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(\overline{M}) \tag{32}$$

be the Poisson operator associated to $\Box_f$. The Poisson operator P satisfies

$$\begin{aligned} &\Box_f Pu = 0, \quad u \in \mathscr{C}^\infty(X), \\ &\gamma Pu = u, \quad u \in \mathscr{C}^\infty(X), \end{aligned} \tag{33}$$

where γ denotes the operator of restriction to the boundary X. It is known that P extends continuously

$$P : H^s(X) \to H^{s+\frac{1}{2}}(\overline{M}), \quad \forall s \in \mathbb{R}$$

(see [2, Page 29]). Let

$$P^* : \hat{\mathscr{D}}'(\overline{M}) \to \mathscr{D}'(X)$$

be the operator defined by

$$(\,P^* u\,|\,v\,)_X = (\,u\,|\,Pv\,)_M, \quad u \in \hat{\mathscr{D}}'(\overline{M}), \quad v \in \mathscr{C}^\infty(X),$$

where $\hat{\mathscr{D}}'(\overline{M})$ denotes the space of continuous linear maps from $\mathscr{C}^\infty(\overline{M})$ to $\mathbb{C}$ with respect to $(\cdot\,|\,\cdot)_M$. It is well-known (see [2, page 30]) that P^* is continuous $P^* : H^s(\overline{M}) \to H^{s+\frac{1}{2}}(X)$ for every $s \in \mathbb{R}$ and

$$P^* : \mathscr{C}^\infty(\overline{M}) \to \mathscr{C}^\infty(X).$$

It is well-known that the operator

$$P^*P : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$$

is a classical elliptic pseudodifferential operator of order -1 and invertible since P is injective (see [2]). Moreover, the operator

$$(P^*P)^{-1} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$$

is a classical elliptic pseudodifferential operator of order one. We define a new inner product on $H^{-\frac{1}{2}}(X)$ as follows:

$$[\, u \mid v\,]_X := (\, Pu \mid Pv)_M, \quad u, v \in H^{-\frac{1}{2}}(X). \tag{34}$$

For an operator A on $H^{-\frac{1}{2}}(X)$ we denote by $A^\dagger$ the formal adjoint of A with respect to the inner product $[\,\cdot \mid \cdot\,]_X$.

The next result shows that we can link the Bergman projection with a certain approximate projector on the boundary X, which is a Fourier integral operator.

Theorem 3 ([10]) *There exists a continuous operator*

$$\mathcal{S} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X), \quad \mathcal{S} \in L^0_{\frac{1}{2},\frac{1}{2}}(X) \tag{35}$$

such that

$$B = P\mathcal{S}(P^*P)^{-1}P^*, \quad \text{on } \mathscr{C}^\infty(\overline{M}), \tag{36}$$

with the following properties,

$$\mathcal{S}^\dagger = \mathcal{S}, \quad \mathcal{S}^2 = \mathcal{S} \ \text{ on } \mathscr{D}'(X), \tag{37}$$

and for any local coordinate patch (D, x), we have

$$\mathcal{S}(x, y) \equiv \int_0^\infty e^{it\varphi(x,y)} s(x, y, t)dt \quad \text{on } D \times D, \tag{38}$$

where $\varphi = \varphi_- \in \mathscr{C}^\infty(D \times D)$ is the phase function φ_- as in [13, Theorem 4.1] satisfying

$$\begin{aligned}
&\varphi \in \mathscr{C}^\infty(D \times D), \ \ \operatorname{Im}\varphi(x, y) \geq 0,\\
&\varphi(x, x) = 0, \ \ \varphi(x, y) \neq 0 \ \text{ if } \ x \neq y,\\
&d_x\varphi(x, y)\big|_{x=y} = -d_y\varphi(x, y)\big|_{x=y} = \omega_0(x),\\
&\varphi(x, y) = -\overline{\varphi}(y, x).
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
&s(x,y,t) \sim \sum_{j=0}^{+\infty} s_j(x,y)t^{n-1-j} \;\; \textit{in } S^{n-1}_{1,0}(D\times D\times \mathbb{R}_+),\\
&s_j(x,y) \in \mathscr{C}^\infty(D\times D),\ j=0,1,\ldots,\\
&s_0(x,x) = \frac{1}{2\pi^n}\det(\mathscr{L}_x),\ \ \textit{for all } x\in D_0.
\end{aligned} \tag{40}$$

Proof We recall here the construction from [10] for the convenience of the reader. In order to link the Bergman projection to a boundary operator we consider a version of the tangential Cauchy-Riemann operator $\overline{\partial}_b$ on X, denoted $\overline{\partial}_\beta$, expressed in terms of the Poisson extension operator, the $\overline{\partial}$ operator and the restriction to the boundary, namely

$$\overline{\partial}_\beta : \Omega^{0,\star}(X) \to \Omega^{0,\star+1}(X), \quad \overline{\partial}_\beta = Q\gamma\overline{\partial}P,$$

where P is the Poisson operator, $\gamma : \Omega^{0,\star}(\overline{M}) \to \Omega^{0,\star}(X)$ is the restriction operator and $Q : H^{-1/2}(X, \Lambda^{0,\star}TM') \to \ker(\overline{\partial}\rho\wedge\cdot)^* \subset H^{-1/2}(X,\Lambda^{0,\star}TM')$ is the orthogonal projection, cf. [10, (5.1), p. 103]. Note that Q is the operator T in [10, (3.6), p. 96] and Q is the identity in degree zero. The operator $\overline{\partial}_\beta$ is a classical pseudo-differential operator of order one on X, such that $\overline{\partial}_\beta = \overline{\partial}_b + \text{l.o.t.}$ and $\overline{\partial}_\beta^2 = 0$. The corresponding Laplace operator (cf. [10, (5.6), p. 104]),

$$\Box^{(\star)}_\beta = \overline{\partial}_\beta\,\overline{\partial}_\beta^\dagger + \overline{\partial}_\beta^\dagger\,\overline{\partial}_\beta : \Omega^{0,\star}(X) \to \Omega^{0,\star}(X)$$

is a classical pseudo-differential operator of order two on X, with the same principal symbol and the same characteristic manifold as the Kohn Laplacian $\Box_b = \overline{\partial}_b\,\overline{\partial}_b^* + \overline{\partial}_b^*\,\overline{\partial}_b$,

$$\Sigma = \left\{(x, t\omega_0(x)) \in T^*X : x\in X,\ t\in\mathbb{R}\setminus\{0\}\right\}. \tag{41}$$

We have

$$\Sigma = \Sigma^+ \cup \Sigma^-, \ \ \Sigma^+ := \{(x,t\omega_0(x)) \in T^*X : x\in X,\ t>0\}, \ \ \Sigma^- := \Sigma\setminus\Sigma^+. \tag{42}$$

Note that we use here a different sign convention than in [10], where ω_0 equals $d\rho\circ J$ (compare [10, (1.9), p. 84], (21)), thus we swap here the roles of Σ^+ and Σ^- compared to [10].

By Theorem [10, Theorem 6.15, p. 114] the operator $\Box^{(0)}_\beta$ acting in degree $q=0$ (that is on functions) has a parametrix A and an approximate projector S (denoted B_- in [10]) such that

$$\begin{gathered}
A\in L^{-1}_{\frac12,\frac12}(X), \quad S\in L^0_{\frac12,\frac12}(X),\\
A\Box^{(0)}_\beta + S \equiv I, \quad \Box^{(0)}_\beta A + S \equiv I,\\
S^2\equiv S, \quad S^\dagger \equiv S,\\
\overline{\partial}_\beta\, S \equiv 0, \quad \overline{\partial}_\beta^\dagger\, S\equiv 0.
\end{gathered} \tag{43}$$

Morevover the wavefront set of the distribution kernel $S(\cdot,\cdot)$ of S is given by

$$\mathrm{WF}(S(\cdot,\cdot)) = \{(x,\xi,x,-\xi) : (x,\xi) \in \Sigma^+\}\,. \tag{44}$$

In [10, (7.4), p. 120] the operator $P\mathcal{S}Q(P^*P)^{-1}P^*$ is defined on $\Omega^{0,q}(\overline{M})$ and it is shown in [10, Proposition 7.5] that its kernel equals the Bergman kernel on $(0,q)$-forms up to a smooth form on $\overline{M}\times\overline{M}$. For $q=0$ the operator Q is the identity so we have $B = PS(P^*P)^{-1}P^* + F$, where F is smoothing. We set

$$\mathcal{S} = S + (P^*P)^{-1}P^*FP. \tag{45}$$

Then $\mathcal{S}^2 = \mathcal{S}$, $\mathcal{S}^\dagger = \mathcal{S}$, and $B = P\mathcal{S}(P^*P)^{-1}P^*$. We have thus obtained (36) and (37). The prpperties (38), (39) and (38) follow from the corresponding properties of S. $\square$

Remark 1 The operator $\mathcal{S}$ is a Toeplitz structure on Σ^+ in the sense of [3, Definition 2.10].

Lemma 1 *For any* $u\in\mathrm{Dom}(T_R)$ *there exists* $u_j\in\mathscr{C}^\infty(\overline{M})$*,* $j=1,2,\ldots$*, such that* $\lim_{j\to+\infty}u_j = Bu$ *in* $L^2(M)$ *and* $\lim_{j\to+\infty}BRBu_j = BRBu$ *in* $L^2(M)$*.*

Proof Let $u\in\mathrm{Dom}(T_R)$. We may assume that $u=Bu$. Then,

$$u = P(P^*P)^{-1}P^*u = P\mathcal{S}(P^*P)^{-1}P^*u.$$

From (36), we have

$$BRBu = P\mathcal{S}(P^*P)^{-1}P^*RP\mathcal{S}(P^*P)^{-1}P^*u = P\mathcal{S}L\mathcal{S}(P^*P)^{-1}P^*u, \tag{46}$$

where $L=(P^*P)^{-1}P^*RP$. It is straightforward to check that $L\in L^1_{\mathrm{cl}}(X)$ and

$$\sigma^0_L(x,\omega_0(x))\neq 0$$

at every $x\in X$, where σ^0_L denotes the principal symbol of L. Since $\mathcal{S}L\mathcal{S}(P^*P)^{-1}P^*u \in H^{-\frac{1}{2}}(X)$, we can repeat the proof of [8, Theorem 3.3] and deduce that $(P^*P)^{-1}P^*u\in H^{\frac{1}{2}}(X)$. Let $v_j\in\mathscr{C}^\infty(X)$, $j=1,2,\ldots$, $v_j\to(P^*P)^{-1}P^*u$ in $H^{\frac{1}{2}}(X)$ as $j\to+\infty$. Then, $u_j:=Pv_j\to P(P^*P)^{-1}P^*u = u$ in $H^1(\overline{M})$ as $j\to+\infty$ and $BRBu_j\to BRBu$ in $L^2(\overline{M})$ as $j\to+\infty$. $\square$

Proof (*Proof of Theorem* 2) Let $T^*_R:\mathrm{Dom}(T^*_R)\subset L^2(X)\to L^2(X)$ be the L^2 adjoint of T_R. Let $u\in\mathrm{Dom}(T_R)$. From Lemma 1, for every $v\in\mathrm{Dom}(T_R)$, we have

$$(\,u\,|\,Av\,)_M = (\,Bu\,|\,Av\,)_M = \lim_{j\to+\infty}(\,Bu_j\,|\,BRBv_j\,)_M, \tag{47}$$

where $u_j, v_j\in\mathscr{C}^\infty(\overline{M})$, $j=1,2,\ldots$, such that $\lim_{j\to+\infty}u_j = Bu$ in $L^2(M)$, $\lim_{j\to+\infty}v_j = Bv$ in $L^2(M)$, $\lim_{j\to+\infty}BRBu_j = BRBu$ in $L^2(M)$ and $\lim_{j\to+\infty}$

$BRBv_j = BRBv$ in $L^2(M)$. From (47) and since $R\rho = 0$ on X, we can integrate by parts and deduce that

$$(\, u \,|\, T_R v \,)_M = (\, T_R u \,|\, v \,)_M, \quad \text{for every } v \in \mathrm{Dom}(T_R).$$

Thus, $u \in \mathrm{Dom}(T_R^*)$ and $T_R^* u = T_R u$. Let $u \in \mathrm{Dom}(T_R^*)$. Since $\mathscr{C}^\infty(\overline{M}) \subset \mathrm{Dom}(T_R)$, we deduce that there is a constant $C > 0$ such that

$$|(\, u \,|\, BRBv \,)_M| \le C \,\|v\|_M\,, \quad \text{for every } v \in \mathscr{C}^\infty(\overline{M}).$$

Thus, $BRBu \in L^2(M)$ and hence $u \in \mathrm{Dom}(T_R)$. $\square$

4 Asymptotic Expansion of $\chi_k(T_R)$

In this Section we will reduce the study of the Toeplitz operator T_R to the study of a Toeplitz operator $\mathcal{T}_{\mathcal{R}}$ on the boundary X and apply results from [8] in order to prove Theorem 1. The Toeplitz operator on the boundary is defined by

$$\mathcal{T}_{\mathcal{R}} := \mathcal{S}\mathcal{R}\mathcal{S} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X), \tag{48}$$

where $\mathcal{S}$ is as in Theorem 3 and

$$\mathcal{R} := (P^*P)^{-1}P^*RP \in L^1_{\mathrm{cl}}(X). \tag{49}$$

Note that by (46) we have

$$T_R = P(\mathcal{S}\mathcal{R}\mathcal{S})(P^*P)^{-1}P^* = P\mathcal{T}_{\mathcal{R}}(P^*P)^{-1}P^*. \tag{50}$$

We extend $\mathcal{T}_{\mathcal{R}}$ to $H^{-\frac{1}{2}}(X)$:

$$\begin{aligned} &\mathcal{T}_{\mathcal{R}} : \mathrm{Dom}(\mathcal{T}_{\mathcal{R}}) \subset H^{-\frac{1}{2}}(X) \to H^{-\frac{1}{2}}(X), \\ &\mathrm{Dom}(\mathcal{T}_{\mathcal{R}}) = \left\{ u \in H^{-\frac{1}{2}}(X);\ \mathcal{S}\mathcal{R}\mathcal{S}u \in H^{-\frac{1}{2}}(X) \right\}. \end{aligned} \tag{51}$$

The operator $\mathcal{S}$ is a Toeplitz structure (generalized Szegő projector) in the sense of [3, Definition 2.10]. Let $\mathrm{Im}(\mathcal{S})$ be the image of $\mathcal{S}$ in $L^2(X)$. By [3, Proposition 2.14] the spectrum of the operator $\mathcal{T}_{\mathcal{R}}|_{\mathrm{Im}(\mathcal{S})} : \mathrm{Im}(\mathcal{S}) \to \mathrm{Im}(\mathcal{S})$ consists only of isolated eigenvalues of finite multiplicity, is bounded from below and has only $+\infty$ as a point of accumulation. We have $\mathrm{Spec}(\mathcal{T}_{\mathcal{R}}) \setminus \{0\} = \mathrm{Spec}(\mathcal{T}_{\mathcal{R}}|_{\mathrm{Im}(\mathcal{S})}) \setminus \{0\}$ and the restrictions to $\mathbb{R} \setminus \{0\}$ of spectral measures of these operators coincide. We conclude that the operator $\mathcal{T}_{\mathcal{R}}$ in (51) is self-adjoint with respect to $[\,\cdot\,|\,\cdot\,]_X$ and its spectrum consists only of isolated eigenvalues, is bounded from below and has only $+\infty$ as a point of accumulation. Moreover, for every $\lambda \in \mathrm{Spec}(\mathcal{T}_{\mathcal{R}})$, $\lambda \neq 0$, the eigenspace

$$E_\lambda = \{u \in \mathrm{Dom}(\mathcal{T}_{\mathcal{R}}) : \mathcal{T}_{\mathcal{R}} u = \lambda u\}$$

is a finite dimensional subspace of $\mathscr{C}^\infty(X)$.

Remark 2 The kernel of $\mathcal{T}_{\mathcal{R}}$ contains the kernel of $\mathcal{S}$, so in order to avoid the zero eigenvalue we consider the operator $\chi_k(\mathcal{T}_{\mathcal{R}})$ associated to a function χ with support in $(0, +\infty)$. In this way the image of $\chi_k(\mathcal{T}_{\mathcal{R}})$ is contained in $\mathrm{Im}(\mathcal{S})$.

Lemma 2 *For every $z \in \mathbb{C}$, $z \notin \mathbb{R}$, we have*

$$(z - T_R)^{-1} B = P(z - \mathcal{T}_{\mathcal{R}})^{-1} \mathcal{S}(P^* P)^{-1} P^*. \tag{52}$$

Proof From (36) and (46) we have

$$(z - T_R)B = P(z - \mathcal{S}\mathcal{R}\mathcal{S})(P^* P)^{-1} P^* B = P(z - \mathcal{T}_{\mathcal{R}})\mathcal{S}(P^* P)^{-1} P^*. \tag{53}$$

From (53), we have

$$P(z - \mathcal{T}_{\mathcal{R}})^{-1}(P^* P)^{-1} P^*(z - T_R)B = B.$$

Thus,

$$\begin{aligned}(z - T_R)^{-1} B &= P(z - \mathcal{T}_{\mathcal{R}})^{-1}(P^* P)^{-1} P^* B \\ &= P(z - \mathcal{T}_{\mathcal{R}})^{-1}(P^* P)^{-1} P^* P \mathcal{S}(P^* P)^{-1} P^* \\ &= P(z - \mathcal{T}_{\mathcal{R}})^{-1} \mathcal{S}(P^* P)^{-1} P^*.\end{aligned}$$

The lemma follows. □

Lemma 3 *We have*

$$\chi_k(T_R) = P \chi_k(\mathcal{T}_{\mathcal{R}})(P^* P)^{-1} P^*. \tag{54}$$

Proof From the Helffer-Sjöstrand formula [6, §8] and (52), we have

$$\begin{aligned}\chi_k(T_R) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{\chi}_k}{\partial \overline{z}}(z)(z - T_R)^{-1} dz d\overline{z} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{\chi}_k}{\partial \overline{z}}(z) P(z - \mathcal{T}_{\mathcal{R}})^{-1} \mathcal{S}(P^* P)^{-1} P^* dz d\overline{z} \\ &= P \chi_k(\mathcal{T}_{\mathcal{R}})(P^* P)^{-1} P^*,\end{aligned}$$

where $\tilde{\chi}_k$ denotes an almost analytic extension of χ_k. The lemma follows. □

Corollary 2 *We have*

$$\chi_k(T_R)(x, y) \in \mathscr{C}^\infty(\overline{M} \times \overline{M}). \tag{55}$$

Proof This follows from (54) and from the fact that $\chi_k(\mathcal{T}_{\mathcal{R}}) \in \mathscr{C}^\infty(X \times X)$. □

We need the following variant of [8, Theorem 1.1].

Theorem 4 *Let (X, HX, J) be an orientable compact strictly pseudoconvex Cauchy-Riemann manifold of dimension $2n-1$, $n \geq 2$. We consider:*

(a) A Riemannian metric g^{TX} compatible with J, with volume form dv_X and the associated L^2-space $L^2(X) = L^2(X, dv_X)$.

(b) A contact form ω_0 on X such that the Levi form $\mathcal{L} = \frac{1}{2}d\omega_0(\cdot, J\cdot)$ is positive definite. We denote by $dv_{\omega_0} = \omega_0 \wedge (d\omega_0)^{n-1}$.

(c) An operator $\mathcal{S} : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$, satisfying (35) and (37)- (40).

(d) For a formally self-adjoint first order pseudodifferential operator $Q \in L^1_{\mathrm{cl}}(X)$ we consider the Toeplitz operator $\mathcal{T}_Q = \mathcal{S}Q\mathcal{S} : L^2(X) \to L^2(X)$.

Let (D, x) be any coordinates patch and let $\varphi : D \times D \to \mathbb{C}$ be the phase function satisfying (38) and (39). Then for any formally self-adjoint first order pseudodifferential operator $Q \in L^1_{\mathrm{cl}}(X)$ whose symbol σ_Q satisfies $\sigma_Q(\omega_0) > 0$ on X, and for any $\chi \in \mathscr{C}^\infty_c((0, +\infty))$, $\chi \not\equiv 0$, the Schwartz kernel of $\chi_k(T_Q)$, $\chi_k(\lambda) := \chi\left(k^{-1}\lambda\right)$, can be represented for k large by

$$\chi_k(\mathcal{T}_Q)(x, y) = \int_0^{+\infty} e^{ikt\varphi(x,y)} A(x, y, t, k)dt + O\left(k^{-\infty}\right) \text{ on } D \times D, \tag{56}$$

where $A(x, y, t, k) \in S^n_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+)$,

$$\begin{aligned} &A(x, y, t, k) \sim \sum_{j=0}^{+\infty} A_j(x, y, t)k^{n-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1; D \times D \times \mathbb{R}_+), \\ &A_j(x, y, t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+),\ \ j = 0, 1, 2, \ldots, \\ &A_0(x, x, t) = \frac{1}{2\pi^n}\frac{dv_{\omega_0}}{dv_X}(x)\,\chi\left(t\sigma_Q(\omega_0(x))\right) t^{n-1} \not\equiv 0, \end{aligned} \tag{57}$$

and for some compact interval $I \Subset \mathbb{R}_+$,

$$\operatorname{supp}_t A(x, y, t, k),\ \ \operatorname{supp}_t A_j(x, y, t) \subset I, \quad j = 0, 1, 2, \ldots. \tag{58}$$

Moreover, for any $\tau_1, \tau_2 \in \mathscr{C}^\infty(X)$ such that $\operatorname{supp}(\tau_1) \cap \operatorname{supp}(\tau_2) = \emptyset$, *we have*

$$\tau_1\chi_k(T_P)\tau_2 = O\left(k^{-\infty}\right). \tag{59}$$

The proof of Theorem 4 is completely analogous to the proof of [8, Theorem 1.1] on account of the structure of $\mathcal{S}$ as a Fourier integral operator given in Theorem 3.

Proof (*Proof of Theorem* 1) We will apply Theorem 4 for $X = \partial M$ as in Theorem 1. The metric g^{TX} in (a) is induced by the metric $g^{TM'}$ and the contact form ω_0 in (b) is given by (21)–(23). The operator $\mathcal{S}$ in (c) is the operator constructed in Theorem 3, which in particular fulfills (36). Moreover, we apply Theorem 4 for $Q = \mathcal{R}$ given by (49). In this situation, we have

$$\frac{dv_{\omega_0}}{dv_X}(x) = \det(\mathscr{L}_x), \quad \sigma_{\mathcal{R}}(\omega_0) = \omega_0(T). \tag{60}$$

By (26) and (27) we have $\sigma_{\mathcal{R}}(\omega_0) = \omega_0(T) > 0$ on X.

We first prove (i). Let $\tau, \hat{\tau} \in \mathscr{C}^\infty(\overline{M})$, $\operatorname{supp}\tau \cap \operatorname{supp}\hat{\tau} = \emptyset$. We have

$$\begin{aligned}
&\tau\chi_k(T_R)\hat{\tau} \\
&= \tau P\chi_k(\mathcal{T}_{\mathcal{R}})(P^*P)^{-1}P^*\hat{\tau} \\
&= \tau P\tau_1\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau} + \tau P(1-\tau_1)\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau} \\
&\quad + \tau P\chi_k(\mathcal{T}_{\mathcal{R}})(1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau},
\end{aligned} \tag{61}$$

where $\tau_1, \hat{\tau}_1 \in \mathscr{C}^\infty(X)$, $\operatorname{supp}\tau_1 \cap \operatorname{supp}\hat{\tau}_1 = \emptyset$, $\operatorname{supp}\tau \cap \operatorname{supp}(1-\tau_1) = \emptyset$,

$$\operatorname{supp}(1-\hat{\tau}_1) \cap \operatorname{supp}\hat{\tau} = \emptyset.$$

We apply now Theorem 4 for the operator $Q = \mathcal{R}$ and we see that $\tau_1\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1 = O(k^{-\infty})$ and hence

$$\tau P\tau_1\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau} = O(k^{-\infty}) \text{ on } \overline{M}\times\overline{M}. \tag{62}$$

From [14, Lemma 4.1], we see that

$$(\tau P(1-\tau_1))(x,y) \in \mathscr{C}^\infty(\overline{M}\times X), \tag{63}$$

where $(\tau P(1-\tau_1))(x,y)$ denotes the distribution kernel of $\tau P(1-\tau_1)$. From (63), we can repeat the proof of [8, Theorem 4.6] with minor changes and deduce that

$$\tau P(1-\tau_1)\chi_k(\mathcal{T}_{\mathcal{R}})\hat{\tau}_1(P^*P)^{-1}P^*\hat{\tau} = O(k^{-\infty}) \text{ on } \overline{M}\times\overline{M}. \tag{64}$$

Similarly, from [14, Lemma 4.2], we see that

$$((1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau})(x,y) \in \mathscr{C}^\infty(\overline{X}\times\overline{M}), \tag{65}$$

where $((1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau})(x,y)$ denotes the distribution kernel of $(1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau}$. From (65), we can repeat the proof of [8, Theorem 4.6] with minor changes and deduce that

$$\tau P\chi_k(\mathcal{T}_{\mathcal{R}})(1-\hat{\tau}_1)(P^*P)^{-1}P^*\hat{\tau} = O(k^{-\infty}) \text{ on } \overline{M}\times\overline{M}. \tag{66}$$

From (61), (62), (64) and (66), we get (5).

We prove now (ii) and (iii). Fix $p \in \overline{M}$. We first assume that $p \notin X$ and let U be an open set of p with $U \cap X = \emptyset$. Let $\tau \in \mathscr{C}^\infty_c(U)$. Since $(\tau P)(x,y) \in \mathscr{C}^\infty(\overline{M}\times X)$, we can repeat the proof of [8, Theorem 4.6] with minor changes and get

$$\tau P\chi_k(\mathcal{T}_{\mathcal{R}})(P^*P)^{-1}P^* = O(k^{-\infty}) \text{ on } \overline{M}\times\overline{M}. \tag{67}$$

From (54) and (67), we get (6).

Now, assume that $p\in X$ and let U be an open local coordinate patch of p in M'. Let $D := U\cap X$. We can repeat the proof of [8, Theorem 1.1] (in the situation of Theorem 4) and deduce

$$\chi_k(\mathcal{T}_{\mathcal{R}})(x,y) = \int_0^{+\infty} e^{ikt\varphi(x,y)}a(x,y,t,k)dt + O\left(k^{-\infty}\right) \quad \text{on } D\times D, \tag{68}$$

where $a(x,y,t,k)\in S^n_{\mathrm{loc}}(1;D\times D\times\mathbb{R}_+)$,

$$\begin{aligned}
&a(x,y,t,k)\sim\sum_{j=0}^{\infty}a_j(x,y,t)k^{n-j} \text{ in } S^n_{\mathrm{loc}}(1;D\times D\times\mathbb{R}_+),\\
&a_j(x,y,t)\in\mathscr{C}^\infty(D\times D\times\mathbb{R}_+),\ \ j=0,1,2,\ldots,\\
&a_0(x,x,t)=\frac{1}{2\pi^n}\det(\mathscr{L}_x)\,\chi(t\omega_0(T(x)))t^{n-1}\not\equiv 0,
\end{aligned}$$

and for some compact interval $I\Subset\mathbb{R}_+$,

$$\mathrm{supp}_t\, a(x,y,t,k),\ \ \mathrm{supp}_t\, a_j(x,y,t)\subset I,\quad j=0,1,2,\ldots.$$

From (68), we can repeat the WKB procedure in [10, Part II,Proposition 7.8, Theorem 7.9] and get (7). □

Proof (*Proof of Corollary* 1) The asymptotics $\chi_k(T_R)(x,x)=O(k^{-\infty})$, $k\to\infty$, on M from (12) follow immediately from (6). Let $p\in X$ be fixed and consider local coordinates near p on M' of the form $z=(x_1,\ldots,x_{2n-1},\rho)$, where $x=(x_1,\ldots,x_{2n-1})$ are local coordinates on X near p with $x(p)=0$ and the phase function ψ in (7) has the form (11). In this local chart we have near (p,p),

$$i\Psi(z,z)=2\rho(z)\big(1+O(|z|\big)+O(|z|^3). \tag{69}$$

By (7) we have

$$\chi_k(T_P)(z,z)=\sum_{j=0}^{\infty}k^{n+1-j}\int_0^{+\infty}e^{ikt\psi(z,z)}b_j(z,z,t)\,dt+O(k^{-\infty}) \tag{70}$$

Since $\Psi(x,x)=0$ for $x\in X$ this yields the asymptotic expansion (13) with the coefficients (14). The expression (15) of $b_0(x)$ follows from (8). We have $b_0(x)>0$ for every $x\in X$. Note also the exponential decay of the integrands in (70) for $z\in M$ near p due to (69) and on account of $\rho(z)<0$.

The trace of the operator $\chi_k(T_P)$ is given by

$$\begin{aligned}\operatorname{Tr}\chi_k(T_P) &= \int_M \chi_k(T_P)(z,z)\,dv_{M'}(z)\\ &= \int_{\{\rho<\varepsilon\}} \chi_k(T_P)(z,z)\,dv_{M'}(z) + \int_{\{\varepsilon\leq\rho<0\}} \chi_k(T_P)(z,z)\,dv_{M'}(z)\\ &=: I_1(k)+I_2(k).\end{aligned} \tag{71}$$

where $\varepsilon < 0$ is chosen small enough. We have $I_1(k) = O(k^{-\infty})$ by (12). By using (69) and (70) and the fact that $2k\int_\varepsilon^0 e^{2k\rho}d\rho \to 1$ as $k\to\infty$, we obtain that there exist $C_1, C_2 > 0$ such that $C_1k^n \leq I_2(k) \leq C_2k^n$ for k large enough. □

Remark 3 It is interesting to compare the result of Corollary 1 to the corresponding result regarding Toeplitz operators on the boundary X (see also [8, Corollary 1.2]). By Theorem 4 we have for the operator $\mathcal{T}_{\mathcal{R}}$ from (48),

$$\chi_k(\mathcal{T}_{\mathcal{R}})(x,x) = \sum_{j=0}^{\infty} A_j(x)k^{n-j} \text{ in } S^{n+1}_{\mathrm{loc}}(1;X) \text{ on } X, \tag{72}$$

where

$$A_j(x) = \int_0^{+\infty} A_j(x,x,t)dt, \quad j\in\mathbb{N}_0, \tag{73}$$

with $A_j(x,x,t)$ as in (57), and

$$A_0(x) = \frac{1}{2\pi^n}\det(\mathscr{L}_x)\int_0^{+\infty}\chi(t\omega_0(T(x)))t^{n-1}dt. \tag{74}$$

Moreover,

$$\operatorname{Tr}\chi_k(\mathcal{T}_{\mathcal{R}}) = \frac{k^n}{2\pi^n}\int_X\int_0^{+\infty}\det(\mathscr{L}_x)\chi(t\omega_0(x))\,t^{n-1}dt + O(k^{n-1})\,. \tag{75}$$

We see that $\chi_k(T_R)(x,x)$ and $\chi_k(\mathcal{T}_{\mathcal{R}})(x,x)$ have an asymptotic expansion on the boundary X, the former with leading term of order k^{n+1}, the latter of order k^n. On the other hand both traces $\operatorname{Tr}\chi_k(T_R)$ and $\operatorname{Tr}\chi_k(\mathcal{T}_{\mathcal{R}})$ have growth of order k^n as $k\to+\infty$.

Remark 4 If we do not normalize the definition function ρ such that $|d\rho| = 1$, Theorem 1 holds with the same proof, but we need to take $\omega_0 = -J\circ d(\rho/|d\rho|)$. With this ω_0 the leading term $b_0(x,x,t)$ has the same formula as in (8).

Acknowledgements Chin-Yu Hsiao would like to thank the Department of Mathematics and Computer Science of the University of Cologne for the hospitality during his visits in September-November 2022 and May 2023. George Marinescu is partially supported by the DFG funded project SFB TRR 191 'Symplectic Structures in Geometry, Algebra and Dynamics' (Project-ID 281071066-TRR 191) and the ANR-DFG project 'QuasiDy-Quantization, Singularities, and Holomorphic Dynamics' (Project-ID 490843120).

References

1. Bergman, S.: Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande. I. J. Reine Angew. Math. **169**, 1–42 (1932)
2. Boundary problems for pseudo-differential operators: Boutet de Monvel, L. Acta Math. **126**, 11–51 (1971)
3. Boutet de Monvel, L., Guillemin, V.: The Spectral Theory of Toeplitz Operators. Annals of Mathematics Studies, vol. 99, p. v+161. Princeton University Press, Princeton (1981)
4. Boutet de Monvel, L., Sjöstrand, J., Sur la singularité des noyaux de Bergman et de Szegő, Journées Équations aux Dérivées Partielles,: 123–164, pp. 34–35. Astérisque, No (1975)
5. Dai, X., Liu, K., Ma, X.: On the asymptotic expansion of Bergman kernel. J. Differential Geom. **72**, 1–41 (2006)
6. Dimassi, M., Sjöstrand, J.: Spectral Asymptotics in the Semi-classical Limit. London Mathematical Society Lecture Note Series, vol. 268, p. xii+227. Cambridge University Press, Cambridge (1999)
7. Fefferman, C.: The Bergman kernel and biholomorphic mappings of pseudoconvex domains. Invent. Math. **26**, 1–65 (1974)
8. Herrmann, H., Hsiao, C.-Y., Marinescu, G., Shen, W.-C.: Semi-classical spectral asymptotics of Toeplitz operators on CR manifolds. arXiv.org:2303.17319
9. Hörmander, L.: L^2 estimates and existence theorems for the $\overline{\partial}$ operator. Acta Math. **113**, 89–152 (1965)
10. Hsiao, C.-Y.: Projections in several complex variables. Mémoires de la Société Mathématique de France **123**, 131 (2010)
11. Hsiao, C.-Y., Marinescu, G.: Asymptotics of spectral function of lower energy forms and Bergman kernel of semi-positive and big line bundles. Comm. Anal. Geom. **22**(1), 1–108 (2014)
12. Hsiao, C.-Y., Marinescu, G.: Berezin-Toeplitz quantization for lower energy forms. Comm. Partial Differential Equ. **42**(6), 895–942 (2017)
13. Hsiao, C.-Y., Marinescu, G.: On the singularities of the Szegő projections on lower energy forms. J. Differential Geom. **107**(1), 83–155 (2017)
14. Hsiao, C.-Y., Marinescu, G.: On the singularities of the Bergman projections for lower energy forms on complex manifolds with boundary. Anal. PDE (to appear). arXiv:1911.10928
15. Hsiao, C.-Y., Ma, X., Marinescu, G.: Geometric quantization on CR manifolds. Commun. Contemp. Math. **25**(10), Paper No. 2250074 (2023)
16. Hsiao, C.-Y., Savale, N.: Bergman-Szegő kernel asymptotics in weakly pseudoconvex finite type cases. J. Reine Angew. Math. **791**, 173–223 (2022)
17. Ma, X., Marinescu, G.: Holomorphic Morse Inequalities and Bergman Kernels. Progress in Mathematics, vol. 254, p. 422. Birkhäuser, Basel (2007)