The Journal of Geometric Analysis (2022) 32:266
https://doi.org/10.1007/s12220-022-00990-4

Szegő Kernel Asymptotics on Complete Strictly Pseudoconvex CR Manifolds

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Received: 3 October 2021 / Accepted: 9 June 2022 / Published online: 18 August 2022

Abstract

We prove a Bochner–Kodaira–Nakano formula and establish Szegő kernel expansions on complete strictly pseudoconvex CR manifolds with transversal CR $\mathbb{R}$-action under certain natural geometric conditions. As a consequence we show that such manifolds are locally CR embeddable.

Keywords CR manifolds · Kohn Laplacian · Szegő kernel · Bochner–Kodaira–Nakano formula

Mathematics Subject Classification Primary: 32V20 · 32V30 · Secondary: 32W10 , 32L20

Chin-Yu Hsiao was partially supported by Taiwan Ministry of Science and Technology projects 108-2115-M-001-012-MY5, 109-2923-M-001-010-MY4 and Academia Sinica Career Development Award.

George Marinescu partially supported by the DFG funded project SFB TRR 191 'Symplectic Structures in Geometry, Algebra and Dynamics' (Project-ID 281071066–TRR 191).
Huan Wang thanks his friends and former colleagues for their help during the Coronavirus pandemic.

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1 Introduction

The first goal of this paper is to develop a differential geometric formalism on strictly pseudoconvex CR manifolds with $\mathbb{R}$-action, analogous to the Kähler identities and Bochner–Kodaira–Nakano formula for Hermitian manifolds. We refine in this way Tanaka's formulas in the spirit of Demailly's general version of the latter formulas. This formalism leads to vanishing theorems and L^2-estimates for the $\overline{\partial}_b$-operator for complete CR manifolds.

The second goal is to generalize the result of Boutet de Monvel–Sjöstrand about the singularities of the Szegő kernel for complete strictly pseudoconvex CR manifolds with $\mathbb{R}$-action. This entails global and local embeddability theorems for CR manifolds with $\mathbb{R}$-action, including Sasakian manifolds. Moreover, by applying our result for the Grauert tube of a positive line bundle we obtain a new result about the expansion of the Bergman kernel on complete Kähler manifolds.

Let $(X, T^{(1,0)}X)$ be a CR manifold of dimension $2n+1$, $n \geq 1$. The orthogonal projection $S^{(q)} : L^2_{0,q}(X) \to \ker \Box_b^{(q)}$ onto $\ker \Box_b^{(q)}$ is called the Szegő projection, while its distribution kernel $S^{(q)}(x, y)$ is called the Szegő kernel, where $\Box_b^{(q)}$ denotes the Kohn Laplacian acting on $(0, q)$-forms. The study of the Szegő kernel is a classical subject in several complex variables and CR geometry. If X is compact strictly pseudoconvex and $\Box_b^{(0)}$ has closed range, Boutet de Monvel–Sjöstrand [5] showed that $S^{(0)}(x, y)$ is a complex Fourier integral operator. The Boutet de Monvel–Sjöstrand description of the Szegő kernel had a profound impact in several complex variables, symplectic and contact geometry, geometric quantization, and Kähler geometry. These ideas also partly motivated the introduction of the recent direct approaches and their various extensions, see [23, 24].

However, almost all the results on Szegő kernel assumed that X is compact, while for non-compact complex manifolds the Bergman kernel asymptotics was comprehensively studied [18, 19, 23–25], and used in the several applications mentioned above. Note that for CR manifolds, besides the global embeddability question [4, 26], there is an important delicate specific issue, namely the local embeddability [1, 20, 22, 27], which will be treated here by the analysis of the Szegő kernel.

The Szegő kernel was used by Boutet de Monvel–Guillemin [6] to introduce the Toeplitz quantization on compact contact manifolds. In the same vein, the question of "quantization commutes with reduction" was studied on CR manifolds in the recent paper [17]. It is natural to extend these results to complete Sasakian manifolds.

Let us see some simple examples. Consider the hypersurface $Y := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n;\ \mathrm{Im}\, z_n = f(z_1, \ldots, z_{n-1})\}$, where $f \in \mathscr{C}^\infty(\mathbb{C}^{n-1}, \mathbb{R})$. Then Y is a non-compact CR manifold carrying many smooth CR functions, but even in this simple example we do not know the behavior of the associated Szegő kernel. Another example is the Heisenberg manifold $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ with CR structure $T^{(1,0)}\mathbb{H} := \mathrm{span}\left\{\frac{\partial}{\partial z_j} + i\frac{\partial \phi}{\partial z_j}(z)\frac{\partial}{\partial x_{2n+1}} : 1 \leq j \leq n\right\}$, where $\phi \in \mathscr{C}^\infty(\mathbb{C}^n, \mathbb{R})$. Then, $\mathbb{H}$ is also a non-compact CR manifold and the Szegő kernel has been studied when ϕ is quadratic (see [14]). However, for general ϕ there are fewer results. Both Y and $\mathbb{H}$ are non-compact CR manifolds with transversal CR $\mathbb{R}$-action. Therefore, we think that the study of the Szegő kernels on non-compact CR manifolds with transversal CR $\mathbb{R}$-action is a very natural and interesting question.

In [15], the first author obtained the Szegő kernel asymptotic expansion on the non-degenerate part of the Levi form under the assumption that Kohn Laplacian has closed range in L^2. The method in [15] works well for non-compact setting, but for general non-compact CR manifolds, the closed range property is not a natural assumption. In the Heisenberg case mentioned above, even for ϕ quadratic, $\Box_b^{(0)}$ does not have closed range; however, the Szegő kernel still has an asymptotic expansion.

In this paper, we show that $\Box_b^{(0)}$ has local closed range with respect to a spectral projection Q_λ (see Definition 4.12) under certain geometric conditions. Furthermore, combining this local closed range property with a detailed analysis, we establish Szegő kernel asymptotic expansions on non-compact strictly pseudoconvex complete CR manifolds with transversal CR $\mathbb{R}$-action under certain natural geometric conditions. To study the local closed range property, we establish a CR Bochner–Kodaira–Nakano formula analog to [9], see Theorem 3.3, which has its own interest. This is also a refinement of Tanaka's basic identities [28, Theorems 5.1, 5.2] in our context. We remark that the results in this paper hold both for transversal CR $\mathbb{R}$-action and S^1-action.

We will work in the following setting. Let X be a connected smooth paracompact manifold of dimension $2n+1$, HX be a smooth sub-bundle of TX of rank $2n$, and J be a smooth complex structure on the fibers of HX. Let $T^{(1,0)}X$ be the complex sub-bundle of the complexification $\mathbb{C}HX$ of HX, which corresponds to the i eigenspace of J, that is, $T^{(1,0)}X = \{v - iJv : v \in HX\}$. We say that X is a CR manifold (of hypersurface type) if the formal integrability condition

$$\left[\mathscr{C}^\infty(X, T^{(1,0)}X), \mathscr{C}^\infty(X, T^{(1,0)}X)\right] \subset \mathscr{C}^\infty(X, T^{(1,0)}X). \tag{1.1}$$

holds. The sub-bundle HX is called Levi distribution and the annihilator $(HX)^0 \subset T^*X$ of HX is called the characteristic bundle of the CR manifold X.

We will assume in the sequel that X is orientable. Since HX is oriented by its complex structure, it follows that $(HX)^0$ is a real orientable line bundle, thus trivial. A global frame of $(HX)^0$, that is, a real non-vanishing 1-form $\omega_0 \in \mathscr{C}^\infty(X, T^*X)$ such that $(HX)^0 = \mathbb{R}\omega_0$, is called characteristic 1-form.

Given a characteristic 1-form ω_0 on X the Levi form $\mathscr{L}^{\omega_0}$ is defined by

$$\mathscr{L} = \mathscr{L}_x^{\omega_0}(u, v) = \frac{1}{2}d\omega_0(u, Jv), \quad \text{for } u, v \in H_xX. \tag{1.2}$$

We say that (X, HX, J) is strictly pseudoconvex if there exists a characteristic 1-form ω_0 the Levi form $\mathscr{L}_x^{\omega_0}$ is positive definite at every point $x \in X$. If $\mathscr{L}^{\omega_0}$ is positive definite, then $d\omega_0$ is symplectic on HX, thus ω_0 is a contact form and HX is a contact structure. Associated with a contact form ω_0 one has the Reeb vector field $T = T^{\omega_0}$, uniquely defined by the equations

$$\omega_0(T) = 1, \quad d\omega_0(T, \cdot) = 0 \quad \text{on } X. \tag{1.3}$$

Assumption 1.1 (X, HX, J, ω_0) is an orientable strictly pseudoconvex CR manifold of dimension $2n+1$, $n \geq 1$, where HX is the Levi distribution, J is the complex structure, and ω_0 is a contact form. We assume that X is endowed with a smooth locally free $\mathbb{R}$-action preserving ω_0 and J such that the infinitesimal generator of the $\mathbb{R}$-action is a Reeb vector field, denoted T.

We denote by $T^{(1,0)}X$ and $T^{(0,1)}X$ the bundles of tangent vectors of type $(1,0)$ and $(0,1)$, respectively. By Assumption 1.1 the $\mathbb{R}$-action is CR and transversal (see (2.12) and (2.13)), hence we have a decomposition $\mathbb{C}TX = T^{(1,0)}X \oplus T^{(0,1)}X \oplus \mathbb{C}T$. The Levi form (1.2) induces a Hermitian metric, called the Levi (or Webster) metric,

$$\langle \cdot | \cdot \rangle_{\mathscr{L}} = g_{\mathscr{L}} := \frac{1}{2} d\omega_0(\cdot, J\cdot) + \omega_0(\cdot)\omega_0(\cdot) \tag{1.4}$$

on TX and by extension on $\mathbb{C}TX$, with the following properties:

$$T^{(1,0)}X \perp T^{(0,1)}X, \;\; T \perp (T^{(1,0)}X \oplus T^{(0,1)}X), \;\; \langle T \,|\, T \rangle_{\mathscr{L}} = 1. \tag{1.5}$$

In Sect. 2.2, we observe that Assumption 1.1 implies that the contact metric manifold $(X, \omega_0, T, J, g_{\mathscr{L}})$ is a Sasakian manifold. Conversely, every compact Sasakian manifold admits an $\mathbb{R}$-action as in Assumption 1.1.

Let $K_X^* := \det(T^{(1,0)}X)$ and let $R_{\mathscr{L}}^{K_X^*}$ be the curvature of K_X^* induced by $\langle \cdot | \cdot \rangle_{\mathscr{L}}$ (see (2.26) and (4.31)).

More generally, we consider an arbitrary $\mathbb{R}$-invariant Hermitian metric $g = g_X = \langle \cdot | \cdot \rangle_g = \langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that (1.5) holds. Given such a metric we will denote by Θ_X its fundamental $(1,1)$-form given by $\Theta_X(a, \overline{b}) = \sqrt{-1}\langle a \,|\, b \rangle_g$ for $a, b \in T^{(1,0)}X$. Let dv_X be the volume form induced by the $\mathbb{R}$-invariant metric g_X as in (1.5). Let $(\cdot | \cdot)$ be the L^2 inner product on the space of smooth compactly supported functions $\mathscr{C}_c^\infty(X)$ with respect to dv_X. We denote by $L^2(X, dv_X)$ the completion of $\mathscr{C}_c^\infty(X)$ with respect to $(\cdot | \cdot)$.

We denote by $\overline{\partial}_b$ the tangential Cauchy–Riemann operator (see Definition 2.3). The Szegő projection is the orthogonal projection with respect to $(\cdot | \cdot)$,

$$S^{(0)} : L^2(X, dv_X) \to \ker \overline{\partial}_b \cap L^2(X, dv_X), \tag{1.6}$$

on the space of square-integrable CR functions on X. The distribution kernel $S^{(0)}(x, y) \in \mathscr{D}'(X \times X)$ of the Szegő projection is called the Szegő kernel. The main result of this article is as follows:

Theorem 1.2 *Let (X, HX, J, ω_0) be an orientable strictly pseudoconvex CR manifold of dimension $2n+1$, $n \geq 1$, with an $\mathbb{R}$-action on X as in Assumption 1.1. Let g_X be an $\mathbb{R}$-invariant metric as in (1.5) and let Θ_X be its fundamental form. Assume that the Levi metric $g_{\mathscr{L}}$ is complete and there is $C > 0$ such that*

$$\sqrt{-1} R_{\mathscr{L}}^{K_X^*} \geq -2C\sqrt{-1}\mathscr{L}, \quad (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 \geq C\Theta_X^n \wedge \omega_0. \tag{1.7}$$

Then the Szegő projection is a Fourier integral operator with complex phase, that is, for any local coordinate patch $(D, x = (x_1, \ldots, x_{2n+1}))$ *with* $D \Subset X$*, we have*

$$S^{(0)}(x,y) - \int_0^\infty e^{i\varphi(x,y)t} s(x,y,t)dt \in \mathscr{C}^\infty(D \times D), \tag{1.8}$$

where the phase function $\varphi \in \mathscr{C}^\infty(D \times D)$ *satisfies*

$$\begin{aligned} &\varphi \in \mathscr{C}^\infty(D \times D), \ \ \operatorname{Im}\varphi(x,y) \geq 0, \\ &\varphi(x,x) = 0, \ \ \varphi(x,y) \neq 0 \ \ \textit{if} \ \ x \neq y, \\ &d_x\varphi(x,y)\big|_{x=y} = \omega_0(x), \ \ d_y\varphi(x,y)\big|_{x=y} = -\omega_0(x), \\ &\varphi(x,y) = -\overline{\varphi}(y,x), \end{aligned} \tag{1.9}$$

and $s(x,y,t) \in S^n_{\mathrm{cl}}\big(D \times D \times \mathbb{R}_+\big)$ *is a symbol of order* n *with asymptotic expansion* $s(x,y,t) = \sum_{j=0}^\infty s_j(x,y)t^{n-j}$ *whose leading term* $s_0(x,y)$ *satisfies*

$$s_0(x,x) = \frac{1}{2}\pi^{-n-1}|\det \mathscr{L}_x|, \ \ \textit{for all}\ x \in D, \tag{1.10}$$

where $\det \mathscr{L}_x$ *is the determinant of* $\mathscr{L}_x$ *with respect to* g_X*, cf. (4.73).*

We will show in Lemma 2.7 that $R^{K_X^*}_{\mathscr{L}} = \operatorname{Ric}_{\mathscr{L}}$ where $\operatorname{Ric}_{\mathscr{L}} \in \Omega^{1,1}(X)$ is the pseudohermitian Ricci form with respect to the pseudohermitian structure ω_0 (see (2.10)). We refer to Definition 2.1 for the definition of the symbol space $S^n_{\mathrm{cl}}\big(D \times D \times \mathbb{R}_+\big)$ and to [19, Theorems 3.3, 4.4] for more properties for the phase φ in (1.8).

Examples for the situation described in Theorem 1.2 are given by Galois coverings of compact strictly pseudoconvex CR manifolds (Examples 4.5, 5.3), circle bundles of positive line bundles over complete Kähler manifolds (Example 4.6), and, as mentioned before, the Heisenberg group (Sect. 5).

If we work with $(n, 0)$-forms we can drop some of the hypotheses of Theorem 1.2.

Theorem 1.3 *Let* (X, HX, J, ω_0) *be an orientable strictly pseudoconvex CR manifold of dimension* $2n+1$*,* $n \geq 1$*, with an* $\mathbb{R}$*-action on* X *as in Assumption 1.1. Assume that the Levi metric* $g_{\mathscr{L}}$ *is complete. Then the Szegő projection* $S^{(n,0)} : L^2_{n,0}(X) \to \ker \overline{\partial}_b \subset L^2_{n,0}(X)$ *is a Fourier integral operator with complex phase, that is, for any local coordinate patch* $(D, x = (x_1, \ldots, x_{2n+1}))$ *with* $D \Subset X$*, the Szegő kernel has the form (1.8) with respect to the trivialization of* K_X *given by* $dz_1 \wedge \ldots \wedge dz_n$*.*

The equivariant Kodaira embedding theorems for Sasakian manifolds were obtained in [13, 16]. From Theorem 1.2, we obtain a Boutet de Monvel type embedding theorem [4] for complete Sasakian manifolds as follows, which is a generalization of the embedding theorem for compact Sasakian manifolds [26].

Corollary 1.4 *In the situation of Theorem 1.2 the space of* L^2 *CR functions separate points and give local coordinates on* X*. In particular, for any compact set of* $K \subset X$ *there exists a positive integer* N *and CR functions* $f_1, \ldots, f_N \in L^2(X) \cap \mathscr{C}^\infty(X)$ *such that* $(f_1, \ldots, f_N)$ *is an embedding of* K *in* $\mathbb{C}^N$*.*

As a consequence of Theorem 1.3 we obtain the following:

Corollary 1.5 *In the situation of Theorem 1.3 the space of L^2 CR $(n,0)$-forms separate points and give local coordinates on X. Thus, X is locally CR embeddable in an Euclidean space. In particular, every Sasakian manifold with complete Levi metric $g_{\mathscr{L}}$ is locally CR embeddable by global CR $(n,0)$-forms.*

The question arises if one can extend these results for general strictly pseudoconvex CR manifolds (without Assumption 1.1 about the existence of an $\mathbb{R}$-action). An analytic property that we use is that the spectral projections Q_λ of the operator $\sqrt{-1}T$ (see 4.7) commute to $\overline{\partial}_b$. Beyond that it is not clear what would be general geometric or analytic conditions that would imply that the Szegő projector is a Fourier integral operator.

We now apply our main result to complex manifolds. Let (L,h^L) be a holomorphic line bundle over a Hermitian manifold (M,Θ_M), where h^L denotes a Hermitian metric on L and Θ_M is a positive $(1,1)$ form on M. For every $k\in\mathbb{N}$, let (L^k,h^{L^k}) be the k-th power of (L,h^L). The positive $(1,1)$ form Θ_M and h^{L^k} induces a L^2 inner product $(\cdot\,|\,\cdot)_{\Theta_M}$ on $\Omega^{0,q}_c(M,L^k))$. Let $L^2_{0,q}(M,L^k)$ be the completion of $\Omega^{0,q}_c(M,L^k)$ with respect to $(\cdot\,|\,\cdot)_{\Theta_M}$. We write $L^2(M,L^k):=L^2_{0,0}(M,L^k)$. Let

$$H^0_{(2)}(M,L^k)=\ker\overline{\partial}_k:=\{u\in L^2(M,L^k);\ \overline{\partial}u=0\},$$

be the space of holomorphic square-integrable sections of L^k. Let $\{f^k_j\}_{j=1}^{d_k}$ be an orthonormal basis for $H^0_{(2)}(M,L^k)$ with respect to $(\cdot\,|\,\cdot)_{\Theta_M}$, where $d_k\in\mathbb{N}\cup\{\infty\}$. The Bergman kernel of L^k is

$$P_k(x,y):=\sum_{j=1}^{d_k} f^k_j(x)\otimes f^k_j(y)^*\in\mathscr{C}^\infty(X\times X,L^k\boxtimes(L^k)^*).\tag{1.11}$$

Let s be a local holomorphic frame of L defined on an open set $D\Subset M$, $|s|^2_{h^L}=e^{-2\phi}$, $\phi\in\mathscr{C}^\infty(D,\mathbb{R})$. On D, we write $f^k_j=\tilde{f}^k_j s^{\otimes k}$, $\tilde{f}^k_j\in\mathscr{C}^\infty(D)$, $j=1,\ldots,d_k$. The localized Bergman kernel on D is given by

$$P_{k,s}(x,y):=\sum_{j=1}^{d_k} e^{-k\phi(x)}\tilde{f}^k_j(x)\overline{\tilde{f}^k_j(y)}e^{-k\phi(y)}\in\mathscr{C}^\infty(D\times D).\tag{1.12}$$

Let R^L be the Chern curvature of L induced by h^L. Assume that $\omega=\sqrt{-1}R^L$ is positive. Let $K^*_M:=\det(T^{(1,0)}M)$ and let $R^{K^*_M}_\omega$ be the curvature of K^*_M induced by ω. Applying Theorem 1.2 to the circle bundle of (L,h^L), we get the following:

Theorem 1.6 *Let (L,h^L) be a Hermitian holomorphic line bundle over a Hermitian manifold (M,Θ_M) of dimension n. We assume that $\omega=\sqrt{-1}R^L$ defines a complete*

Kähler metric on M. We assume moreover that there is $C > 0$ such that

$$\sqrt{-1}R^{K^*_M}_{\omega} \geq -C\omega, \quad \omega^n \geq C\Theta^n_M \quad on\, M. \tag{1.13}$$

Let s be a local holomorphic frame of L defined on an open set $D \Subset M$. Then,

$$P_{k,s}(x,y) \equiv e^{ik\Phi(x,y)}b(x,y,k) \mod O(k^{-\infty}) \ \ on\, D, \tag{1.14}$$

where $\Phi \in \mathscr{C}^\infty(D \times D)$, $\operatorname{Im}\Phi(x,y) \geq C|x-y|^2$, $C > 0$, $\Phi(x,x) = 0$, for every $x \in D$,

$$b(x,y,k) \in S^n_{\mathrm{loc}}(1; D \times D), \ \ b(x,y,k) \sim \sum_{j=0}^{\infty} k^{n-j} b_j(x,y) \ \ in\, S^n_{\mathrm{loc}}(1; D \times D), \tag{1.15}$$

$b_j(x,y) \in \mathscr{C}^\infty(D \times D)$, $j = 0, 1, \ldots$, and

$$b_0(x,x) = (2\pi)^{-n}\frac{\omega^n(x)}{\Theta^n_M(x)}, \ \ for\ every\ x \in D.$$

In particular, there exist coefficients $\boldsymbol{b}_r \in \mathscr{C}^\infty(X)$, $r \in \mathbb{N}_0$, such that for any open set U of X with $\overline{U}$ compact, every $\ell \in \mathbb{N}_0$ and every $m \in \mathbb{N}$, there is a $C_{U,\ell,m} > 0$ independent of k such that

$$\|P_k(x,x) - \sum_{r=0}^{m} \boldsymbol{b}_r(x)k^{n-r}\|_{\mathscr{C}^\ell(U)} \leq C_{U,\ell,m}k^{n-m-1}. \tag{1.16}$$

We refer the reader to Sect. 2 for the precise meaning of the notation $A_k \equiv B_k \mod O(k^{-\infty})$ on D in (1.14), $S^n_{\mathrm{loc}}(1; D \times D)$ and the asymptotic sums in (1.15) and (1.16).

For compact or certain complete Kähler–Einstein manifolds, the expansion (1.16) was obtained by Tian [29] for $m = 0$ and $\ell = 4$. For general m, ℓ, and compact manifolds, the existence of the expansion was first obtained in [8, 31]. In [23, Theorem 6.1.1] the expansion was generalized for complete Hermitian manifolds such that $R^{K^*_M}$ and $\partial\Theta_M$ are bounded below. Our conditions (1.13) are different from [23, Theorem 6.1.1], we replace the condition on $\partial\Theta_M$ by a condition on the volume form. The reason is that we use a local closed range condition instead of standard closed range or spectral gap condition.

This paper is organized as follows. In Sect. 2, we recall necessary notions of microlocal analysis, pseudohermitian geometry, and strictly pseudoconvex CR manifolds with transversal CR $\mathbb{R}$-actions. In Sect. 3, we prove the Bochner–Kodaira formula on CR manifolds with $\mathbb{R}$-action. Section 4 is devoted to the proof of the asymptotics of the Szegő kernel. In Sect. 5, we examine the Heisenberg group.

2 Preliminaries

We use the following notations through this article: $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}}_+ = \{x \in \mathbb{R};\ x \geq 0\}$. For $m \in \mathbb{N}$, let $x = (x_1, \ldots, x_m)$ be coordinates of $\mathbb{R}^m$. For $n \in \mathbb{N}$, let $z = (z_1, \ldots, z_n)$, $z_j = x_{2j-1} + \sqrt{-1}x_{2j}$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^n$. We write

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - \sqrt{-1}\frac{\partial}{\partial x_{2j}}\right), \quad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} + \sqrt{-1}\frac{\partial}{\partial x_{2j}}\right), \tag{2.1}$$

$$dz_j = dx_{2j-1} + \sqrt{-1}dx_{2j}, \quad d\overline{z}_j = dx_{2j-1} - \sqrt{-1}dx_{2j}. \tag{2.2}$$

2.1 Notions of Microlocal Analysis

Let X be a $\mathscr{C}^\infty$ paracompact manifold. We let TX and T^*X denote the tangent bundle of X and the cotangent bundle of X, respectively. The complexified tangent bundle of X and the complexified cotangent bundle of X are denoted by $\mathbb{C}TX$ and $\mathbb{C}T^*X$, respectively. Write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TX and T^*X. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TX \times \mathbb{C}T^*X$.

Let $D \subset X$ be an open set . The spaces of distributions of D and smooth functions of D will be denoted by $\mathscr{D}'(D)$ and $\mathscr{C}^\infty(D)$, respectively. Let $\mathscr{E}'(D)$ be the subspace of $\mathscr{D}'(D)$ whose elements have compact support in D. Let $\mathscr{C}^\infty_c(D)$ be the subspace of $\mathscr{C}^\infty(D)$ whose elements have compact support in D. Let $A : \mathscr{C}^\infty_c(D) \to \mathscr{D}'(D)$ be a continuous map. We write $A(x, y)$ to denote the distribution kernel of A. In this work, we will identify A with $A(x, y)$. The following two statements are equivalent:

(I) A is continuous: $\mathscr{E}'(D) \to \mathscr{C}^\infty(D)$,
(II) $A(x, y) \in \mathscr{C}^\infty(D \times D)$.

If A satisfies (I) or (II), we say that A is smoothing on D. Let $A, B : \mathscr{C}^\infty_c(D) \to \mathscr{D}'(D)$ be continuous operators. We write

$$A \equiv B (\text{on } D) \tag{2.3}$$

if $A - B$ is a smoothing operator. We say that A is properly supported if the restrictions of the two projections $(x, y) \to x$, $(x, y) \to y$ to Supp $(A(x, y))$ are proper.

For $m \in \mathbb{R}$, let $H^m(D)$ denote the Sobolev space of order m on D. Put

$$H^m_{\text{loc}}(D) = \left\{u \in \mathscr{D}'(D);\ \varphi u \in H^m(D),\ \forall \varphi \in \mathscr{C}^\infty_c(D)\right\},$$
$$H^m_{\text{comp}}(D) = H^m_{\text{loc}}(D) \cap \mathscr{E}'(D).$$

Let D be an open coordinate patch of X with local coordinates x. We recall the following Hörmander symbol space.

Definition 2.1 For $m \in \mathbb{R}$, $S^m_{1,0}(D \times D \times \mathbb{R}_+)$ is the space of all $a(x, y, t) \in \mathscr{C}^\infty(D \times D \times \mathbb{R}_+)$ such that for all compact $K \Subset D \times D$ and all $\alpha, \beta \in \mathbb{N}_0^{2n+1}$, $\gamma \in \mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta \partial_t^\gamma a(x, y, t)| \leq C_{\alpha,\beta,\gamma}(1 + |t|)^{m-|\gamma|}, \quad \text{for all } (x, y, t) \in K \times \mathbb{R}_+, t \geq 1.$$

Put

$$S^{-\infty}(D \times D \times \mathbb{R}_+) := \bigcap_{m \in \mathbb{R}} S^m_{1,0}(D \times D \times \mathbb{R}_+).$$

Let $a_j \in S^{m_j}_{1,0}(D \times D \times \mathbb{R}_+)$, $j \in \mathbb{N}_0$, with $m_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{m_0}_{1,0}(D \times D \times \mathbb{R}_+)$, unique modulo $S^{-\infty}$, such that $a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{1,0}(D \times D \times \mathbb{R}_+)$ for $k \in \mathbb{N}$. If a and a_j have the properties above, we write

$$a \sim \sum_{j=0}^{\infty} a_j \ \text{ in } S^{m_0}_{1,0}(D \times D \times \mathbb{R}_+).$$

The space $S^m_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$ of classical symbols of order m is defined as the space of symbols $s(x, y, t) \in S^m(D \times D \times \mathbb{R}_+)$ satisfying

$$\begin{aligned} & s(x, y, t) \sim \sum_{j=0}^{\infty} s_j(x, y) t^{m-j} \ \text{ in } \ S^m_{1,0}(D \times D \times \mathbb{R}_+), \\ & s_j(x, y) \in \mathscr{C}^\infty(D \times D), \ \ j \in \mathbb{N}_0. \end{aligned} \tag{2.4}$$

We explain now for the precise meaning of $A_k \equiv B_k \mod O(k^{-\infty})$ on D in (1.14), $S^n_{\mathrm{loc}}(1; D \times D)$ and the asymptotic sum in (1.15) (see also [18, Sect. 3.3]). A k-dependent smoothing operator $A_k : \Omega^{0,q}_0(D) \to \Omega^{0,q}(D)$ is called k-negligible if the kernel $A_k(x, y)$ of A_k satisfies $|\partial_x^\alpha \partial_y^\beta A_k(x, y)| = O(k^{-N})$ uniformly on every compact set in $D \times D$, for all multi-indices α, β, and all $N \in \mathbb{N}$. Let $C_k : \Omega^{0,q}_0(D) \to \Omega^{0,q}(D)$ be another k-dependent smoothing operator. We write $A_k \equiv C_k \mod O(k^{-\infty})$ or $A_k(x, y) \equiv C_k(x, y) \mod O(k^{-\infty})$ if $A_k - C_k$ is k-negligible.

We recall the definition of semi-classical Hörmander symbol spaces:

Definition 2.2 Let U be an open set in $\mathbb{R}^N$. Let $S(1; U) = S(1)$ be the set of $a \in \mathscr{C}^\infty(U)$ such that for every $\alpha \in \mathbb{N}_0^N$, there exists $C_\alpha > 0$, such that $|\partial_x^\alpha a(x)| \leq C_\alpha$ on U. If $a = a(x, k)$ depends on $k \in (1, \infty)$, we say that $a(x, k) \in S_{\mathrm{loc}}(1)$ if $\chi(x) a(x, k)$ is uniformly bounded in $S(1)$ when k varies in $(1, \infty)$, for any $\chi \in \mathscr{C}^\infty_0(U)$. For $m \in \mathbb{R}$, we put $S^m_{\mathrm{loc}}(1) = k^m S_{\mathrm{loc}}(1)$. If $a_j \in S^{m_j}_{\mathrm{loc}}(1)$, $m_j \searrow -\infty$, we say that $a \sim \sum_{j=0}^{\infty} a_j$ in $S^{m_0}_{\mathrm{loc}}(1)$ if $a - \sum_{j=0}^{N_0} a_j \in S^{m_{N_0+1}}_{\mathrm{loc}}(1)$ for every N_0. From this, we form $S^m_{\mathrm{loc}}(1; Y, E)$ in the natural way, where Y is a smooth paracompact manifold and E is a vector bundle over Y.

Let X be an orientable paracompact smooth manifold of dimension $2n+1$ with $n \geq 1$. The Levi form (1.2) of X at $x \in X$ induces a Hermitian quadratic form on $T_x^{(1,0)}X$ by

$$\mathscr{L}_x(u, \overline{v}) = \frac{1}{2i} d\omega_0(u, \overline{v}), \quad \text{for } u, v \in T_x^{(1,0)}X. \tag{2.5}$$

Let $g^{\mathbb{C}TX}$ be a Hermitian metric on $\mathbb{C}TX$ such that the decomposition $\mathbb{C}TX = T^{(1,0)}X \oplus T^{(0,1)}X \oplus \mathbb{C}T$ is orthogonal. For $u, v \in \mathbb{C}TX$ we denote by $\langle u|v\rangle = \langle u|v\rangle_g$ the inner product given by $g^{\mathbb{C}TX}$ and for $u \in \mathbb{C}TX$, we write $|u|_g^2 := \langle u|u\rangle_g$. Given such a metric we will denote by Θ_X its fundamental $(1,1)$-form given by $\Theta_X(a, \overline{b}) = \sqrt{-1}\langle a \,|\, b\rangle_g$ for $a, b \in T^{(1,0)}X$.

For $p, q \in \mathbb{N}_0$, define $T^{(p,q)}X := (\Lambda^p T^{(1,0)}X) \wedge (\Lambda^q T^{(0,1)}X)$ and let $T^{\bullet,\bullet}X = \bigoplus_{p,q\in\mathbb{N}_0} T^{(p,q)}X$. For $u \in \mathbb{C}TX$ and $\phi \in \mathbb{C}T^*X$, the pointwise duality is defined by $\langle u, \phi\rangle := \phi(u)$. Let $T^{*(1,0)}X \subset \mathbb{C}T^*X$ be the dual bundle of $T^{(1,0)}X$ and $T^{*(0,1)}X \subset \mathbb{C}T^*X$ be the dual bundle of $T^{(0,1)}X$. For $p, q \in \mathbb{N}_0$, the bundle of (p,q) forms is denoted by $T^{*(p,q)}X := (\Lambda^p T^{*(1,0)}X) \wedge (\Lambda^q T^{*(0,1)}X)$ and let $T^{*\bullet,\bullet}X := \oplus_{p,q\in\mathbb{N}_0} T^{*(p,q)}X$. The induced Hermitian inner product on $T^{\bullet,\bullet}X$ and $T^{*\bullet,\bullet}X$ by $\langle\cdot\,|\,\cdot\rangle$ are still denoted by $\langle\cdot|\cdot\rangle$. The Hermitian norms are still denoted by $|\cdot|$. Let $\Omega^{p,q}(X) := \mathscr{C}^\infty(X, T^{*(p,q)}X)$ be the space of smooth (p,q)-forms on X and $\Omega^{\bullet,\bullet}(X) := \bigoplus_{p,q\in\mathbb{N}_0} \Omega^{p,q}(X)$. Let $\mathscr{C}^\infty(X) := \Omega^{0,0}(X)$.

Definition 2.3 Let $\pi^{p,q} : \Lambda^{p+q}\mathbb{C}T^*X \longrightarrow T^{*(p,q)}X$ be the natural projection for $p, q \in \mathbb{N}_0, p+q \geq 1$. The tangential (resp. anti-tangential) Cauchy–Riemann operator is given by

$$\begin{aligned} \overline{\partial}_b &:= \pi^{p,q+1} \circ d : \Omega^{p,q}(X) \longrightarrow \Omega^{p,q+1}(X), \\ \partial_b &:= \pi^{p+1,q} \circ d : \Omega^{p,q}(X) \longrightarrow \Omega^{p+1,q}(X). \end{aligned} \tag{2.6}$$

Let $D \subset X$ be an open set. Let $\Omega_c^{p,q}(D)$ be the space of smooth (p,q)-forms on D with compact support in D. Let $\Omega_c^{\bullet,\bullet}(D) := \bigoplus_{p,q\in\mathbb{N}_0} \Omega_c^{p,q}(D)$. We write $\mathscr{C}_c^\infty(D) := \Omega_c^{0,0}(D)$. Let $(\cdot\,|\,\cdot)$ be the L^2 inner product on $\Omega_c^{\bullet,\bullet}(X)$ induced by $\langle\cdot\,|\,\cdot\rangle$. Note that

$$(u|v) := \int_X \langle u(x)|v(x)\rangle dv_X(x), \quad u, v \in \Omega_c^{\bullet,\bullet}(X), \tag{2.7}$$

where $dv_X := (\Theta_X^n/n!) \wedge \omega_0$ is the volume form induced by the Hermitian metric Θ_X on X. Let $L^2_{p,q}(X)$ be the completion of $\Omega_c^{p,q}(X)$ with respect to $(\cdot\,|\,\cdot)$. Let $L^2_{\bullet,\bullet}(X) := \bigoplus_{p,q\in\mathbb{N}_0} L^2_{p,q}(X)$. We write $L^2(X) := L^2_{0,0}(X)$. We denote by $\|u\|^2 := (u|u)$ the L^2-norm on X. Let $\overline{\partial}_b^*$ and ∂_b^* be the formal adjoints of $\overline{\partial}_b$ and ∂_b with respect to $(\cdot\,|\,\cdot)$, respectively. Let $\square_b := \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$ be the Kohn Laplacian on $\Omega^{\bullet,\bullet}(X)$. Let $\overline{\square}_b := \partial_b\partial_b^* + \partial_b^*\partial_b$ be the anti-Kohn Laplacian on $\Omega^{\bullet,\bullet}(X)$. We still denoted by $\overline{\partial}_b$ the maximal extension and by $\overline{\partial}_b^*$ the Hilbert space adjoint with respect to the L^2-inner product on X. We also denote by

$$\square_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b : \mathrm{Dom}\,\square_b \subset L^2_{\bullet,\bullet}(X) \to L^2_{\bullet,\bullet}(X) \tag{2.8}$$

the Gaffney extension of the Kohn Laplacian with the domain

$$\mathrm{Dom}\,\Box_b = \left\{u \in L^2_{\bullet,\bullet}(X) : u \in \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}^*_b,\ \overline{\partial}_b u \in \mathrm{Dom}\,\overline{\partial}^*_b,\ \overline{\partial}^*_b u \in \mathrm{Dom}\,\overline{\partial}_b\right\}. \quad (2.9)$$

By a result of Gaffney, $\Box_b$ is a self-adjoint operator (see e.g., [23, Proposition 3.1.2]).

2.2 Pseudohermitian Geometry

The following is well known:

Proposition 2.4 [28, Proposition 3.1] *Let (X, HX, J, ω_0) be an orientable strictly pseudoconvex CR manifold. Then there exists a unique affine connection, called Tanaka–Webster connection,*

$$\nabla := \nabla^{\omega_0} : \mathscr{C}^\infty(X, TX) \to \mathscr{C}^\infty(X, T^*X \otimes TX)$$

such that

(I) $\nabla_U \mathscr{C}^\infty(X, HX) \subset \mathscr{C}^\infty(X, HX)$ *for* $U \in \mathscr{C}^\infty(X, TX)$.
(II) $\nabla T = \nabla J = \nabla d\omega_0 = 0$.
(III) *The torsion T_∇ of ∇ satisfies: $T_\nabla(U, V) = d\omega_0(U, V)T$, $T_\nabla(T, JU) = -JT_\nabla(T, U)$, $U, V \in \mathscr{C}^\infty(X, HX)$.*

Recall that $\nabla J \in \mathscr{C}^\infty(X, T^*X \otimes \mathscr{L}(HX, HX))$, $\nabla d\omega_0 \in \mathscr{C}^\infty(T^*X \otimes \Lambda^2(\mathbb{C}T^*X))$ are defined by $(\nabla_U J)W = \nabla_U(JW) - J\nabla_U W$ and $\nabla_U d\omega_0(W, V) = U d\omega_0(W, V) - d\omega_0(\nabla_U W, V) - d\omega_0(W, \nabla_U V)$ for $U \in \mathscr{C}^\infty(X, TX)$, $W, V \in \mathscr{C}^\infty(X, HX)$. Moreover, $\nabla J = 0$ and $\nabla d\omega_0 = 0$ imply that the Tanaka–Webster connection is compatible with the Levi metric. By definition, the torsion of ∇ is given by $T_\nabla(W, U) = \nabla_W U - \nabla_U W - [W, U]$ for $U, V \in \mathscr{C}^\infty(X, TX)$ and $\tau(T, U)$ for $U \in \mathscr{C}^\infty(X, HX)$ is called pseudohermitian torsion.

In the following, we will use the Einstein summation convention. Let $\{Z_\alpha\}_{\alpha=1}^n$ be a local frame of $T^{(1,0)}X$ and $\{\theta^\alpha\}_{\alpha=1}^n$ be the dual frame of $\{Z_\alpha\}_{\alpha=1}^n$. We use the notations $Z_{\overline{\alpha}} := \overline{Z_\alpha}$ and $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. Write

$$\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta,\ \nabla Z_{\overline{\alpha}} = \omega_{\overline{\alpha}}^{\overline{\beta}} \otimes Z_{\overline{\beta}},\ \text{and recall that } \nabla T = 0.$$

We call ω_α^β the connection 1-form of Tanaka–Webster connection with respect to the frame $\{Z_\alpha\}_{\alpha=1}^n$. We denote by Θ_α^β the Tanaka–Webster curvature 2-form. Then,

$$\Theta_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta.$$

By direct computation, we also have

$$\Theta_\alpha^\beta = R^\beta_{\alpha j\overline{k}}\theta^j \wedge \theta^{\overline{k}} + A^\beta_{\alpha jk}\theta^j \wedge \theta^k + B^\beta_{\alpha jk}\theta^{\overline{j}} \wedge \theta^{\overline{k}} + C_0 \wedge \omega_0,$$

where C_0 is a 1-form. The term $R^{\beta}_{\alpha j\overline{k}}$ is called the pseudohermitian curvature tensor and the form

$$\mathrm{Ric}_{\mathscr{L}} := R_{\alpha\overline{k}}\theta^{\alpha} \wedge \theta^{\overline{k}}, \quad R_{\alpha\overline{k}} := \sum_{j=1}^{n} R^{j}_{\alpha j\overline{k}} \tag{2.10}$$

is called pseudohermitian Ricci form.

2.3 Strictly Pseudoconvex CR Manifolds with $\mathbb{R}$-Action

Let $(X, T^{(1,0)}X)$ be a CR manifold of $\dim X = 2n + 1$. Let $r : \mathbb{R} \times X \to X$, $r(x) = r \circ x$ for $r \in \mathbb{R}$, be an $\mathbb{R}$-action on X, see [13]. Let $\widehat{T}$ be the infinitesimal generator of the $\mathbb{R}$-action:

$$(\widehat{T}u)(x) := \frac{\partial}{\partial r}(u(r \circ x))\Big|_{r=0}, \quad u \in \mathscr{C}^{\infty}(X). \tag{2.11}$$

Definition 2.5 The $\mathbb{R}$-action is called locally free if $\widehat{T}(x) \neq 0$ at every $x \in X$.

By Assumption 1.1 we have

$$\text{The } \mathbb{R}\text{-action is Cauchy–Riemann (CR)} : \big[\widehat{T}, \mathscr{C}^{\infty}(X, T^{(1,0)}X)\big] \subset \mathscr{C}^{\infty}(X, T^{(1,0)}X). \tag{2.12}$$

$$\text{The } \mathbb{R}\text{-action is transversal} : \mathbb{C}T_xX = T^{1,0}_xX \oplus T^{0,1}_xX \oplus \mathbb{C}\widehat{T}(x) \text{ at every } x \in X. \tag{2.13}$$

Note that (2.12) implies that $L_{\widehat{T}}$ preserves HX and $[L_{\widehat{T}}, J] = 0$. Since $HX = \ker \omega_0$ we have for $U \in \mathscr{C}^{\infty}(X, HX)$,

$$(L_{\widehat{T}}\omega_0)(U) = \widehat{T}(\omega_0(U)) - \omega_0(L_{\widehat{T}}U) = 0$$
$$(L_{\widehat{T}}\omega_0)(\widehat{T}) = \widehat{T}(\omega_0(\widehat{T})) - \omega_0(L_{\widehat{T}}\widehat{T}) = \widehat{T}(\omega_0(\widehat{T})).$$

We pose $f = \omega_0(\widehat{T})$ and $\omega_1 = f^{-1}\omega_0$. Then $L_{\widehat{T}}\omega_1 = 0$ and $\omega_1(\widehat{T}) = 1$ since $(L_{\widehat{T}}\omega_1)(U) = 0$ and $(L_{\widehat{T}}\omega_1)(\widehat{T}) = \widehat{T}(\omega_1(\widehat{T})) = \widehat{T}(1) = 0$. This also implies $\iota_{\widehat{T}}d\omega_1 = 0$. We have thus

$$\iota_{\widehat{T}}\omega_1 = 1, \ \ \iota_{\widehat{T}}d\omega_1 = 0, \ \ L_{\widehat{T}}J = 0.$$

We can therefore assume up to rescaling ω_0 by a smooth function that the infinitesimal generator of the $\mathbb{R}$-action is a Reeb vector field $T = \widehat{T}$. This motivates the equality of the infinitesimal generator to the Reeb field in Assumption 1.1.

By [28, Lemma 3.2 (3)]) we have $2J\tau U = (L_T J)U$ for any $U \in HX$, hence the pseudohermitian torsion τ vanishes, which means that the contact metric manifold $(X, \omega_0, T, J, g_{\mathscr{L}})$ is a Sasakian manifold. Conversely, there exists a natural transversal CR $\mathbb{R}$-action on any compact Sasakian manifold. Recall that compact Sasakian manifolds can be classified in three categories based on the properties of the Reeb foliation consisting of the orbits of the Reeb field (see [7, Definition 6.1.25]). If the orbits of the Reeb field are all closed, then the Reeb field T generates a locally free, isometric S^1-action thus also an $\mathbb{R}$-action on $(X, g_{\mathscr{L}})$. In this case the Reeb foliation is called quasi-regular (and regular if the action is free). If the Reeb foliation is not quasi-regular, it is said to be irregular. In this case, T generates a transversal CR $\mathbb{R}$-action on X.

We use the local coordinates of Baouendi–Rothschild–Trevès (BRT charts) [3, Sect. 1], [19, Theorem 6.5] extensively as follows:

Theorem 2.6 (BRT charts) *For each point $x \in X$, there exists a coordinate neighborhood $D = U \times \mathcal{I}$ with coordinates $x = (x_1, \ldots, x_{2n+1})$ centered at 0, where $U = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z| < \epsilon\}$ and $\mathcal{I} = \{x_{2n+1} \in \mathbb{R} : |x_{2n+1}| < \epsilon_0\}$, $\epsilon, \epsilon_0 > 0$, $z = (z_1, \ldots, z_n)$ and $z_j = x_{2j-1} + \sqrt{-1}x_{2j}$, $j = 1, \ldots, n$, such that*

$$T = \frac{\partial}{\partial x_{2n+1}} \quad on \quad D, \tag{2.14}$$

and there exists $\phi \in \mathscr{C}^\infty(U, \mathbb{R})$ independent of x_{2n+1} satisfying that

$$\left\{ Z_j := \frac{\partial}{\partial z_j} + i \frac{\partial \phi}{\partial z_j}(z) \frac{\partial}{\partial x_{2n+1}} \right\}_{j=1}^{n} \tag{2.15}$$

is a frame of $T^{(1,0)}D$, and $\{dz_j\}_{j=1}^n \subset T^{*(1,0)}D$ is the dual frame.

Let $D = U \times \mathcal{I}$ be a BRT chart. Let $f \in \mathscr{C}^\infty(D)$ and $u \in \Omega^{p,q}(D)$ with $u = \sum_{I,J} u_{IJ} dz_I \wedge d\overline{z}_J$ with ordered sets I, J and $u_{IJ} \in \mathscr{C}^\infty(D)$, for all I, J. We have

$$df = \sum_{j=1}^n Z_j(f) dz_j + \sum_{j=1}^n \overline{Z}_j(f) d\overline{z}_j + T(f)\omega_0, \tag{2.16}$$

$$\partial_b f = \sum_{j=1}^n Z_j(f) dz_j, \quad \overline{\partial}_b f = \sum_{j=1}^n \overline{Z}_j(f) d\overline{z}_j, \tag{2.17}$$

$$\partial_b u = \sum_{I,J} (\partial_b u_{IJ}) \wedge dz_I \wedge d\overline{z}_J, \quad \overline{\partial}_b u = \sum_{I,J} (\overline{\partial}_b u_{IJ}) \wedge dz_I \wedge d\overline{z}_J. \tag{2.18}$$

For $u \in \Omega^{p,q}(X)$, let $\mathcal{L}_T u$ be the Lie derivative of u in the direction of T. For simplicity, we write Tu to denote $\mathcal{L}_T u$. Since the $\mathbb{R}$-action is CR, $Tu \in \Omega^{p,q}(X)$. On a BRT chart D, for $u \in \Omega^{p,q}(D)$, $u = \sum_{I,J} u_{IJ} dz_I \wedge d\overline{z}_J$, we have $Tu = \sum_{I,J} (Tu_{IJ}) \wedge dz_I \wedge d\overline{z}_J$ on D.

The Levi form $\mathscr{L}$ in a BRT chart $D \subset X$ has the form

$$\mathscr{L} = \partial\overline{\partial}\phi|_{T^{(1,0)}X}. \tag{2.19}$$

Indeed, the characteristic 1-form ω_0 and $d\omega_0$ on D are given by

$$\begin{aligned} \omega_0(x) &= dx_{2n+1} - i\sum_{j=1}^{n}\left(\frac{\partial\phi}{\partial z_j}dz_j - \frac{\partial\phi}{\partial\overline{z}_j}d\overline{z}_j\right), \\ d\omega_0(x) &= 2i\sum_{j,k=1}^{n}\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_k}dz_j\wedge d\overline{z}_k. \end{aligned} \tag{2.20}$$

From now on, we assume that Θ_X is $\mathbb{R}$-invariant. Let $D = U\times\mathcal{I}$ be a BRT chart. The $(1,1)$ form $\Theta = \Theta_U$ on U is defined by, for $x = (z, x_{2n+1}) \in D$,

$$\Theta(z) := \Theta_X(x). \tag{2.21}$$

Note that it is independent of x_{2n+1}. More precisely,

$$\Theta(z) = \sqrt{-1}\sum_{j,k=1}^{n}\left\langle Z_j|Z_k\right\rangle(x)dz_j\wedge d\overline{z}_k. \tag{2.22}$$

Note that for another BRT coordinates $D = \widetilde{U}\times\widetilde{\mathcal{I}}$, $y = (w, y_{2n+1})$, there exist biholomorphic map $H \in \mathscr{C}^\infty(U,\tilde{U})$ and $G \in \mathscr{C}^\infty(U,\mathbb{R})$ such that $H(z) = w$, for all $z \in U$, $y_{2n+1} = x_{2n+1} + G(z)$, for all $(z, x_{2n+1}) \in U\times\mathcal{I}$ and $\tilde{U} = H(U)$, $\tilde{\mathcal{I}} = \mathcal{I} + G(U)$. We deduce that Θ is independent of the choice of BRT coordinates, i.e., $\Theta = \Theta_U = \Theta_{\widetilde{U}}$.

Until further notice, we work on a BRT chart $D = U\times\mathcal{I}$. For $p, q \in \mathbb{N}_0$, let $T^{*(p,q)}U$ be the bundle of (p,q) forms on U and let $T^{*\bullet,\bullet}U := \oplus_{p,q\in\mathbb{N}_0}T^{*(p,q)}U$. For $p, q \in \mathbb{N}_0$, let $T^{(p,q)}U$ be the bundle of (p,q) vector fields on U and let $T^{\bullet,\bullet}U := \oplus_{p,q\in\mathbb{N}_0}T^{(p,q)}U$. The $(1,1)$ form Θ induces Hermitian metrics on $T^{\bullet,\bullet}U$ and $T^{*\bullet,\bullet}U$. We shall use $\langle\cdot\,,\,\cdot\rangle_h$ to denote all the induced Hermitian metrics. The volume form on U induced by Θ is given by $d\lambda(z) := \Theta^n/n!$. Thus, the volume form dv_X can be represented by

$$dv_X(x) = d\lambda(z)\wedge dx_{2n+1} \text{ on } D. \tag{2.23}$$

The L^2-inner product on $\Omega_c^{\bullet,\bullet}(U)$ with respect to Θ is given by

$$\langle s_1, s_2\rangle_{L^2(U)} := \int_U\langle s_1(z), s_2(z)\rangle_h d\lambda(z), \quad s_1, s_2 \in \Omega_c^{\bullet,\bullet}(U). \tag{2.24}$$

Let $t \in \mathbb{R}$ be fixed. The L^2-inner product on $\Omega_c^{\bullet,\bullet}(U)$ with respect to Θ and $e^{-2t\phi(z)}$ is given by

$$\langle s_1, s_2\rangle_{L^2(U,e^{-2t\phi})} := \int_U\langle s_1(z), s_2(z)\rangle_h e^{-2t\phi(z)}d\lambda(z), \quad s_1, s_2 \in \Omega_c^{\bullet,\bullet}(U). \tag{2.25}$$

The Chern curvature of $K_U^* := \det(T^{(1,0)}U)$ with respect to Θ is given by

$$R^{K_U^*} := \overline{\partial}\partial \log\det\left(\left\langle\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right\rangle_h\right)_{j,k=1}^n, \quad R^{K_U^*} \in \Omega^{1,1}(U).$$

On X, define $K_X^* := \det(T^{(1,0)}X)$. Then, K_X^* is a CR line bundle over X. The Chern curvature $R^{K_X^*}$ of K_X^* with respect to Θ_X is defined as follows: On a BRT chart D, let

$$R^{K_X^*} := \overline{\partial}_b\partial_b \log\det\left(\langle Z_j|Z_k\rangle\right)_{j,k=1}^n. \tag{2.26}$$

It is easy to see that $R^{K_X^*}$ is independent of the choice of BRT coordinates and hence $R^{K_X^*}$ is globally defined, i.e., $R^{K_X^*} \in \Omega^{1,1}(X)$.

Let $\{L_j\}_{j=1}^n$ be an $\mathbb{R}$-invariant orthonormal frame of $T^{(0,1)}D$ with the dual (orthonormal) frame $\{e_j\}_{j=1}^n$. Then $\{\overline{L}_j\}_{j=1}^n$ is an $\mathbb{R}$-invariant orthonormal frame of $T^{(1,0)}D$ with the dual (orthonormal) frame $\{\overline{e}_j\}_{j=1}^n$. Since Θ_X is $\mathbb{R}$-invariant, there exist $c_j^k = c_j^k(z), w_j^k = w_j^k(z) \in \mathscr{C}^\infty(U)$, $j,k = 1,\ldots,n$, satisfying $\sum_{k=1}^n c_j^k w_k^l = \delta_j^l$, for all $j,l = 1,\ldots,n$, such that for $j = 1,\ldots,n$,

$$\overline{L}_j = \sum_{k=1}^n \overline{c}_j^k Z_k, \qquad \overline{e}_j = \overline{w}_k^j dz_k, \tag{2.27}$$

$$L_j = \sum_{k=1}^n c_j^k \overline{Z}_k, \qquad e_j = w_k^j d\overline{z}_k. \tag{2.28}$$

We can check that $\{w_j := \sum_{k=1}^n \overline{c}_j^k \frac{\partial}{\partial z_k};\ j = 1,\ldots,n\}$ and $\{\overline{w}_j := \sum_{k=1}^n c_j^k \frac{\partial}{\partial \overline{z}_k};\ j = 1,\ldots,n\}$ are orthonormal frames for $T^{(1,0)}U$ and $T^{(0,1)}U$ with respect to Θ, respectively, and $\{\overline{e}_j;\ j = 1,\ldots,n\}$, $\{e_j;\ j = 1,\ldots,n\}$ are dual frames for $\{w_j;\ j = 1,\ldots,n\}$ and $\{\overline{w}_j;\ j = 1,\ldots,n\}$, respectively. We also write w^j and $\overline{w}^j$ to denote $\overline{e}_j$ and e_j, respectively, $j = 1,\ldots,n$.

Lemma 2.7 *We have*

$$\mathrm{Ric}_{\mathscr{L}} = R_{\mathscr{L}}^{K_X^*}$$

on X.

Proof Fix $p \in D$ and let $x = (x_1,\ldots,x_{2n+1})$ be BRT local coordinates defined on an open set D of p with $x(p) = 0$. We take $x = (x_1,\ldots,x_{2n+1}) = (z_1,\ldots,z_n,x_{2n+1})$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1,\ldots,n$, so that

$$\phi(z) = \frac{1}{2}\sum_{j=1}^n |z_j|^2 + O(|z|^4), \tag{2.29}$$

where $\phi \in \mathscr{C}^\infty(D)$ is as in (2.15).

In the following, we will use Einstein summation convention. Write $\nabla_{Z_i} Z_j = \Gamma^l_{ij} Z_l$, where ∇ denotes the Tanaka–Webster connection (see Proposition 2.4). From [28, Lemma 3.2],

$$d\omega_0\left(\nabla_{Z_i} Z_j, \overline{Z}_k\right) = Z_i(d\omega_0(Z_j, \overline{Z}_k)) - d\omega_0\left(Z_j, [Z_i, \overline{Z}_k]_{T^{(0,1)}X}\right). \tag{2.30}$$

Directly,

$$d\omega_0\left(\nabla_{Z_i} Z_j, \overline{Z}_k\right) = d\omega_0\left(\Gamma^l_{ij} Z_l, \overline{Z}_k\right) = 2i\Gamma^l_{ij}\frac{\partial^2\phi}{\partial z_l \partial \overline{z}_k}, \tag{2.31}$$

$$Z_i\left(d\omega_0\left(Z_j, \overline{Z}_k\right)\right) = 2i\frac{\partial^3\phi}{\partial z_i \partial z_j \partial \overline{z}_k}, \tag{2.32}$$

$$[Z_i, \overline{Z}_k]|_{T^{(0,1)}X} = 0$$

and hence

$$2i\Gamma^l_{ij}\frac{\partial^2\phi}{\partial z_l \partial \overline{z}_k} = 2i\frac{\partial^3\phi}{\partial z_i \partial z_j \partial \overline{z}_k}, \tag{2.33}$$

for all $i, j, l, k = 1, \ldots, n$. Accordingly, by (2.29) and (2.33), we get that for all $i, j, k = 1, \cdots, n$,

$$\Gamma^k_{ij}(0) = 0. \tag{2.34}$$

Moreover, by taking $\frac{\partial}{\partial \overline{z}_h}$ both sides in (2.33), from (2.29) and (2.33), it is not difficult to check that

$$\frac{\partial \Gamma^k_{ij}}{\partial \overline{z}_h}(0) = 2\frac{\partial^4\phi}{\partial z_i \partial z_j \partial \overline{z}_k \partial \overline{z}_h}(0). \tag{2.35}$$

It is clear that $\{dz_j\}_{j=1}^n$ and $\{d\overline{z}_j\}_{j=1}^n$ are the dual frames of $\{Z_j\}_{j=1}^n$ and $\{\overline{Z}_j\}_{j=1}^n$, respectively. Denote

$$\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta,$$

and we can check that the $(1, 1)$ part of $d\omega_\alpha^\beta$ is

$$-\sum_{k,l=1}^n \left(\overline{Z}_l \Gamma^\beta_{k\alpha})\right) dz_k \wedge d\overline{z}_l$$

and the $(1, 1)$ part of $\Theta_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$ denoted by

$$\sum_{k,l=1}^n R^\beta_{\alpha k \bar{l}} \theta^k \wedge \theta^{\bar{l}}$$

equals the (1, 1) part of $d\omega_\alpha^\beta$. Hence, the pseudohermitian Ricci curvature tensor at origin is

$$R_{\alpha\bar{l}}(0)=\sum_{k=\beta=1}^{n}R^{\beta}_{\alpha k\bar{l}}(0)=-\sum_{k=\beta=1}^{n}\frac{\partial\Gamma^{\beta}_{k\alpha}}{\partial\bar{z}_l}(0)=-2\sum_{k=1}^{n}\frac{\partial^4\phi}{\partial z_k\partial\bar{z}_k\partial z_\alpha\partial\bar{z}_l}(0).$$

We get that

$$\mathrm{Ric}_{\mathscr{L}}(0)=-2\sum_{k=1}^{n}\frac{\partial^4\phi}{\partial z_k\partial\bar{z}_k\partial z_\alpha\partial\bar{z}_l}(0)dz_\alpha\wedge d\bar{z}_l. \tag{2.36}$$

On the other hand, by directed computation, we can check that

$$R^{K^*_X}(0)=\overline{\partial}_b\partial_b\log\det\big(\langle Z_j|Z_k\rangle\big)_{j,k=1}^{n}(0)=-2\sum_{k=1}^{n}\frac{\partial^4\phi}{\partial z_k\partial\bar{z}_k\partial z_\alpha\partial\bar{z}_l}(0)dz_\alpha\wedge d\bar{z}_l. \tag{2.37}$$

From (2.36) and (2.37), the lemma follows. □

3 Bochner–Kodaira Formula on CR Manifolds with $\mathbb{R}$-Action

In this section, we will prove the Bochner–Kodaira–Nakano for CR manifolds with transversal CR $\mathbb{R}$-action. They are refinements of Tanaka's basic identities [28, Theorems 5.1, 5.2] in our context. Namely, Tanaka's formulas hold for any strictly pseudoconvex manifold endowed with the Levi metric, while our formulas are specific to CR manifolds with $\mathbb{R}$-action endowed with arbitrary Hermitian metric Θ_X.

3.1 The Fourier Transform on BRT Charts

Let $D=U\times\mathcal{I}$ be a BRT chart. Let $f\in\mathscr{C}^\infty_c(D)$. We write $f=f(x)=f(z,x_{2n+1})$. For each fixed $x_{2n+1}\in I$, $f(\cdot,x_{2n+1})\in\mathscr{C}^\infty_c(U)$. For each fixed $z\in U$, $f(z,\cdot)\in\mathscr{C}^\infty_c(\mathcal{I})$. Let $p,q\in\mathbb{N}_0$, $u\in\Omega^{p,q}_c(D)$. We write $u=\sum_{I,J}u_{IJ}dz_I\wedge d\bar{z}_J\in\Omega^{p,q}_c(D)$ and we always assume that the summation is performed only over increasingly ordered indices $I=i_1<i_2<\ldots<i_p$, $J=j_1<j_2<\ldots<j_q$, and $u_{IJ}\in\mathscr{C}^\infty_c(D)$, for all $\{I,J\}$. For each fixed $z\in U$, $u_{IJ}(z,\cdot)\in\mathscr{C}^\infty_c(\mathcal{I})$.

Definition 3.1 The Fourier transform of the function $f\in\mathscr{C}^\infty_c(D)$ with respect to x_{2n+1}, denoted by $\widehat{f}$, is defined by

$$\widehat{f}(z,t):=\int_{-\infty}^{\infty}e^{-itx_{2n+1}}f(z,x_{2n+1})dx_{2n+1}\in\mathscr{C}^\infty(U\times\mathbb{R}). \tag{3.1}$$

The Fourier transform of the form $u = \sum_{I,J} u_{IJ} dz_I \wedge d\overline{z}_J \in \Omega_c^{p,q}(D)$ with respect to x_{2n+1}, denoted by $\widehat{u}$, is defined by

$$\widehat{u}(z,t) = \sum_{I,J} \widehat{u}_{IJ}(z,t) dz_I \wedge d\overline{z}_J \in \Omega^{p,q}(U \times \mathbb{R}) := \mathscr{C}^\infty(U \times \mathbb{R}, T^{*(p,q)}U). \quad (3.2)$$

Note that $\widehat{f} \in \mathscr{C}^\infty(U \times \mathbb{R})$ and $\widehat{f}(\cdot,t) \in \mathscr{C}_c^\infty(U)$ for every $t \in \mathbb{R}$. Similarly, $\widehat{u} \in \Omega^{p,q}(U \times \mathbb{R})$ and $\widehat{u}(\cdot,t) \in \Omega_c^{p,q}(U)$ for every $t \in \mathbb{R}$. From Parseval's formula, we have for $u, v \in \Omega_c^{p,q}(D)$,

$$\int_{-\infty}^{\infty} \langle u(z,x_{2n+1}) | v(z,x_{2n+1}) \rangle dx_{2n+1} = (1/2\pi) \int_{-\infty}^{\infty} \langle \widehat{u}(z,t), \widehat{v}(z,t) \rangle_h dt, \quad (3.3)$$

for every $z \in U$. By using integration by parts, we have for $u \in \Omega_c^{p,q}(D)$,

$$-\sqrt{-1}\widehat{Tu} = t\widehat{u}, \quad ,i.e., \quad -\sqrt{-1}\widehat{\frac{\partial u}{\partial x_{2n+1}}}(z,t) = t\widehat{u}(z,t). \quad (3.4)$$

Let $t \in \mathbb{R}$ be fixed. Let $|(z,1)|_h^2 := e^{-2t\phi(z)}$ be the Hermitian metric on the trivial line bundle $U \times \mathbb{C}$ over U. The Chern connection of $(U \times \mathbb{C}, e^{-2t\phi})$ is given by

$$\nabla^{(U \times \mathbb{C}, e^{-2t\phi})} = \nabla^{1,0} + \nabla^{0,1}, \quad \nabla^{1,0} = \partial - 2t\partial\phi, \quad \nabla^{0,1} = \overline{\partial}. \quad (3.5)$$

Indeed, $\nabla^{(U \times \mathbb{C}, e^{-2t\phi})} = d + h^{-1}\partial h = d + e^{2t\phi}\partial(e^{-2t\phi})$. The curvature of $(U \times \mathbb{C}, e^{-2t\phi})$ is

$$R^{(U \times \mathbb{C}, e^{-2t\phi})} = \left(\nabla^{(U \times \mathbb{C}, e^{-2t\phi})}\right)^2 = 2t\partial\overline{\partial}\phi. \quad (3.6)$$

We can identify $\partial\overline{\partial}\phi$ with Levi form $\mathscr{L}$ and write $R^{(U \times \mathbb{C}, e^{-2t\phi})} = 2t\mathscr{L}$. Moreover, we will identify $\Omega^{\bullet,\bullet}(U)$ and $\Omega_c^{\bullet,\bullet}(U)$ with $\Omega^{\bullet,\bullet}(U, U \times \mathbb{C})$ and $\Omega_c^{\bullet,\bullet}(U, U \times \mathbb{C})$, respectively.

Proposition 3.2 *Let $u, v \in \Omega_c^{\bullet,\bullet}(D)$. We have*

$$\widehat{\overline{\partial}_b u} = e^{-t\phi}\overline{\partial}(e^{t\phi}\widehat{u}) \quad on\ U \times \mathbb{R}, \quad (3.7)$$

$$\widehat{\overline{\partial}_b^* v} = e^{-t\phi}\overline{\partial}^*(e^{t\phi}\widehat{v}) \quad on\ U \times \mathbb{R}, \quad (3.8)$$

$$\widehat{\partial_b u} = e^{-t\phi}\nabla^{1,0}(e^{t\phi}\widehat{u}) \quad on\ U \times \mathbb{R}, \quad (3.9)$$

$$\widehat{\partial_b^* u} = e^{-t\phi}\nabla^{1,0*}(e^{t\phi}\widehat{u}) \quad on\ U \times \mathbb{R}, \quad (3.10)$$

where $\overline{\partial}^*$, $\nabla^{1,0*}$ are the formal adjoints of $\overline{\partial}$, $\nabla^{1,0}$ with respect to $\langle \cdot, \cdot \rangle_{L^2(U, e^{-2t\phi})}$, respectively, and $\overline{\partial}_b^*$, ∂_b^* are the formal adjoints of $\overline{\partial}_b$, ∂_b with respect to $(\cdot | \cdot)$, respectively.

Proof Let $u=\sum_{I,J}u_{IJ}dz_I\wedge d\overline{z}_J$. By $\overline{\partial}_b u=\sum_{I,J}\sum_{j=1}^n\left(\frac{\partial u_{IJ}}{\partial\overline{z}_j}-i\frac{\partial\phi}{\partial\overline{z}_j}\frac{\partial u_{IJ}}{\partial x_{2n+1}}\right)d\overline{z}_j\wedge dz_I\wedge d\overline{z}_J$,

$$\begin{aligned}\left(\widehat{\overline{\partial}_b u}\right)(z,t)&=\sum_{I,J}\sum_{j=1}^n\left(\frac{\partial\widehat{u}_{IJ}}{\partial\overline{z}_j}(z,t)+t\frac{\partial\phi}{\partial\overline{z}_j}(z)\widehat{u}_{IJ}(z,t)\right)d\overline{z}_j\wedge dz_I\wedge d\overline{z}_J\\&=e^{-t\phi(z)}\overline{\partial}(e^{t\phi}\sum_{I,J}\widehat{u}_{IJ}dz_I\wedge d\overline{z}_J)(z,t)\\&=\left(e^{-t\phi}\overline{\partial}(e^{t\phi}\widehat{u})\right)(z,t).\end{aligned}\tag{3.11}$$

Thus the first equality holds. From Parseval's formula,

$$\begin{aligned}(\overline{\partial}_b u|v)&=\int_D\langle\overline{\partial}_b u|v\rangle d\lambda(z)dx_{2n+1}\\&=\int_U\left((2\pi)^{-1}\int_{-\infty}^{\infty}\langle\widehat{\overline{\partial}_b u}(z,t),\widehat{v}(z,t)\rangle_h dt\right)d\lambda(z)\\&=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_U\langle e^{-t\phi}\overline{\partial}(e^{t\phi}\widehat{u}),\widehat{v}\rangle_h d\lambda(z)dt\\&=(2\pi)^{-1}\int_{-\infty}^{\infty}\langle\overline{\partial}(e^{t\phi}\widehat{u}),e^{t\phi}\widehat{v}\rangle_{L^2(U,e^{-2t\phi})}dt\\&=(2\pi)^{-1}\int_{-\infty}^{\infty}\langle e^{t\phi}\widehat{u},\overline{\partial}^*(e^{t\phi}\widehat{v})\rangle_{L^2(U,e^{-2t\phi})}dt\\&=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_U\langle\widehat{u},e^{-t\phi}\overline{\partial}^*(e^{t\phi}\widehat{v})\rangle_h d\lambda(z)dt.\end{aligned}\tag{3.12}$$

Meanwhile, we have

$$(\overline{\partial}_b u|v)=(u|\overline{\partial}_b^*v)=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_U\langle\widehat{u},\widehat{\overline{\partial}_b^*v}\rangle_h d\lambda(z)dt.\tag{3.13}$$

Thus the second equality holds. The proofs of the third and the fourth equalities are similar. □

3.2 CR Bochner–Kodaira–Nakano Formula I

Analog to [23, (1.4.32)], we define the Lefschetz operator $\Theta_X\wedge\cdot$ on $\bigwedge^{\bullet,\bullet}(T^*X)$ and its adjoint $\Lambda=i(\Theta_X)$ with respect to the Hermitian inner product $\langle\cdot|\cdot\rangle$ associated with Θ_X. The Hermitian torsion of Θ_X is defined by

$$\mathcal{T}:=[\Lambda,\partial_b\Theta_X].\tag{3.14}$$

Let $D = U \times \mathcal{I}$ be a BRT chart and let $\{\overline{L}_j\}_{j=1}^n \subset T^{(1,0)}D$, $\{\overline{e}_j\}_{j=1}^n \subset T^{*(1,0)}D$, $\{w_j\}_{j=1}^n \subset T^{(1,0)}U$ be as in the discussion after (2.26). We can check that

$$\Theta_X \wedge \cdot = \sqrt{-1}\overline{e}_j \wedge e_j \wedge \cdot, \quad \Lambda = -\sqrt{-1} i_{L_j} i_{\overline{L}_j} \quad \text{on } D. \tag{3.15}$$

Note that i_{L_j} and $i_{\overline{L}_j}$ are the adjoints of $e_j\wedge$ and $\overline{e}_j\wedge$, respectively.

Since $\partial\Theta(z) = \partial_b\Theta_X(x)$ on D, and $\Theta \wedge \cdot = \sqrt{-1}\overline{e}_j \wedge e_j \wedge \cdot$, $\Lambda = -\sqrt{-1} i_{\overline{w}_j} i_{w_j}$ on U, see [23, 1.4.32], we have $\mathcal{T} = [\Lambda, \partial_b\Theta] = [\Lambda, \partial\Theta]$ on $\Omega^{\bullet,\bullet}(D)$, which is independent of x_{2n+1}. We remark that $\mathcal{T}$ is a differential operator of order zero. With respect to the Hermitian inner product $\langle\cdot|\cdot\rangle$ associated with Θ_X, we have the adjoint operator $\mathcal{T}^*$, the conjugate operator $\overline{\mathcal{T}}$ and the adjoint of the conjugate operator $\overline{\mathcal{T}}^*$ for $\mathcal{T}$.

Theorem 3.3 *With the notations used above, we have on* $\Omega^{\bullet,\bullet}(X)$,

$$\Box_b = \overline{\Box}_b + [2\sqrt{-1}\mathscr{L}, \Lambda](-\sqrt{-1}T) + (\partial_b \mathcal{T}^* + \mathcal{T}^* \partial_b) - (\overline{\partial}_b \overline{\mathcal{T}}^* + \overline{\mathcal{T}}^* \overline{\partial}_b). \tag{3.16}$$

Proof Since the both side of (3.16) are globally defined, we can check (3.16) on a BRT chart. Now, we work on a BRT chart $D = U \times \mathcal{I}$. We will use the same notations as before. Let

$$\Box^{(U\times\mathbb{C},e^{-2t\phi})} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} : \Omega_c^{\bullet,\bullet}(U) \to \Omega_c^{\bullet,\bullet}(U),$$
$$\overline{\Box}^{(U\times\mathbb{C},e^{-2t\phi})} := \nabla^{1,0*}\nabla^{1,0} + \nabla^{1,0}\nabla^{1,0*} : \Omega_c^{\bullet,\bullet}(U) \to \Omega_c^{\bullet,\bullet}(U),$$

where $\nabla^{1,0}$ is given by (3.5), $\overline{\partial}^*$, $\nabla^{1,0*}$ are the formal adjoints of $\overline{\partial}$, $\nabla^{1,0}$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(U,e^{-2t\phi})}$, respectively. From [23, (1.4.44)],

$$\Box^{(U\times\mathbb{C},e^{-2t\phi})} = \overline{\Box}^{(U\times\mathbb{C},e^{-2t\phi})} + [2\sqrt{-1}t\mathscr{L}, \Lambda] + (\nabla^{1,0}\mathcal{T}^* + \mathcal{T}^*\nabla^{1,0}) - (\overline{\partial}\,\overline{\mathcal{T}}^* + \overline{\mathcal{T}}^*\,\overline{\partial}).$$

Let $u, v \in \Omega_c^{\bullet,\bullet}(D)$. Let $s_1(z) := e^{t\phi(z)}\widehat{u}(z,t) \in \Omega^{\bullet,\bullet}(U\times\mathbb{R})$, $s_2(z) := e^{t\phi(z)}\widehat{v}(z,t) \in \Omega^{\bullet,\bullet}(U \times \mathbb{R})$. Firstly, we have

$$(1/2\pi)\int_{-\infty}^{\infty} \langle\Box^{(U\times\mathbb{C},e^{-2t\phi})}s_1, s_2\rangle_{L^2(U,e^{-2t\phi})}dt = (\,\overline{\partial}_b u\,|\,\overline{\partial}_b v\,) + (\,\overline{\partial}_b^* u\,|\,\overline{\partial}_b^* v\,). \tag{3.17}$$

In fact, from Proposition 3.2,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \langle\Box^{(U\times\mathbb{C},e^{-2t\phi})}s_1, s_2\rangle_{L^2(U,e^{-2t\phi})}dt \\
&= \int_{-\infty}^{\infty}\Big(\langle\overline{\partial}s_1, \overline{\partial}s_2\rangle_{L^2(U,e^{-2t\phi})} + \langle\overline{\partial}^* s_1, \overline{\partial}^* s_2\rangle_{L^2(U,e^{-2t\phi})}\Big)dt \\
&= \int_{-\infty}^{\infty}\Big(\langle\widehat{\overline{\partial}_b u}, \widehat{\overline{\partial}_b v}\rangle_{L^2(U)} + \langle\widehat{\overline{\partial}_b^* u}, \widehat{\overline{\partial}_b^* v}\rangle_{L^2(U)}\Big)dt \\
&= 2\pi(\overline{\partial}_b u|\overline{\partial}_b v) + 2\pi(\overline{\partial}_b^* u|\overline{\partial}_b^* v).
\end{aligned}$$

Similarly, we have

$$(1/2\pi)\int_{-\infty}^{\infty}\langle\overline{\Box}^{(U\times\mathbb{C},e^{-2t\phi})}s_1,s_2\rangle_{L^2(U,e^{-2t\phi})}dt=(\partial_b u|\partial_b v)+(\partial_b^* u|\partial_b^* v). \tag{3.18}$$

Thirdly, we have

$$(1/2\pi)\int_{-\infty}^{\infty}t\langle[2\sqrt{-1}\mathscr{L},\Lambda]s_1,s_2\rangle_{L^2(U,e^{-2t\phi})}dt=([2\sqrt{-1}\mathscr{L},\Lambda](-\sqrt{-1}T)u|v). \tag{3.19}$$

In fact, it follows from

$$\begin{aligned}&\int_{-\infty}^{\infty}t\langle[2\sqrt{-1}\mathscr{L},\Lambda]s_1,s_2\rangle_{L^2(U,e^{-2t\phi})}dt\\&=\int_{-\infty}^{\infty}\langle t[2\sqrt{-1}\mathscr{L},\Lambda]\widehat{u},\widehat{v}\rangle_{L^2(U)}dt\\&=\int_{-\infty}^{\infty}\langle[2\sqrt{-1}\mathscr{L},\Lambda](-\widehat{\sqrt{-1}Tu}),\widehat{v}\rangle_{L^2(U)}dt\\&=2\pi([2\sqrt{-1}\mathscr{L},\Lambda](-\sqrt{-1}Tu)|v).\end{aligned}$$

Fourthly, we consider the rest terms

$$\langle(\nabla^{1,0}\mathcal{T}^*+\mathcal{T}^*\nabla^{1,0})s_1,s_2\rangle_{L^2(U,e^{-2t\phi})},\quad\langle(\nabla^{0,1}\overline{\mathcal{T}}^*+\overline{\mathcal{T}}^*\nabla^{0,1})s_1,s_2\rangle_{L^2(U,e^{-2t\phi})}. \tag{3.20}$$

By Proposition 3.2, we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\langle(\nabla^{1,0}\mathcal{T}^*+\mathcal{T}^*\nabla^{1,0})s_1,s_2\rangle_{L^2(U,e^{-2t\phi})}dt\\&=\int_{-\infty}^{\infty}\langle\Big(\nabla^{1,0}\mathcal{T}^*s_1,s_2\rangle_{L^2(U,-2t\phi)}+\langle\mathcal{T}^*\nabla^{1,0}s_1,s_2\rangle_{L^2(U,-2t\phi)}\Big)dt\\&=\int_{-\infty}^{\infty}\langle\Big(\nabla^{1,0}\mathcal{T}^*e^{t\phi}\widehat{u},e^{t\phi}\widehat{v}\rangle_{L^2(U,-2t\phi)}+\langle\mathcal{T}^*\nabla^{1,0}e^{t\phi}\widehat{u},e^{t\phi}\widehat{v}\rangle_{L^2(U,-2t\phi)}\Big)dt\\&=\int_{-\infty}^{\infty}\langle\Big(\mathcal{T}^*e^{t\phi}\widehat{u},\nabla^{1,0*}(e^{t\phi}\widehat{v})\rangle_{L^2(U,-2t\phi)}+\langle\nabla^{1,0}(e^{t\phi}\widehat{u}),\mathcal{T}e^{t\phi}\widehat{v}\rangle_{L^2(U,-2t\phi)}\Big)dt\\&=\int_{-\infty}^{\infty}\langle\Big(\mathcal{T}^*\widehat{u},e^{-t\phi}\nabla^{1,0*}(e^{t\phi}\widehat{v})\rangle_{L^2(U)}+\langle e^{-t\phi}\nabla^{1,0}e^{t\phi}\widehat{u},\mathcal{T}\widehat{v}\rangle_{L^2(U)}\Big)dt\\&=\int_{-\infty}^{\infty}\langle\Big(\mathcal{T}^*\widehat{u},\widehat{\partial_b^*v}\rangle_{L^2(U)}+\langle\widehat{\partial_b u},\mathcal{T}\widehat{v}\rangle_{L^2(U)}\Big)dt\\&=2\pi(\mathcal{T}^*u|\partial_b^*v)+2\pi(\partial_b u|\mathcal{T}v).\end{aligned} \tag{3.21}$$

Thus we obtain

$$\begin{aligned}&(1/2\pi)\int_{-\infty}^{\infty}\langle(\nabla^{1,0}\mathcal{T}^*+\mathcal{T}^*\nabla^{1,0})s_1,s_2\rangle_{L^2(U,e^{-2t\phi})}dt\\&=(\mathcal{T}^*u|\partial_b^*v)+(\partial_b u|\mathcal{T}v)\\&=((\partial_b\mathcal{T}^*+\mathcal{T}^*\partial_b)u|v).\end{aligned}\tag{3.22}$$

Similarly, we obtain

$$(1/2\pi)\int_{-\infty}^{\infty}\langle(\overline{\partial}\,\overline{\mathcal{T}}^*+\overline{\mathcal{T}}^*\overline{\partial})s_1,s_2\rangle_{L^2(U,e^{-2t\phi})}dt=((\overline{\partial}_b\overline{\mathcal{T}}^*+\overline{\partial}_b\overline{\mathcal{T}}^*)u|v).\tag{3.23}$$

From (3.17), (3.18), (3.19), (3.22), and (3.23), we get that for $u,v\in\Omega_c^{\bullet,\bullet}(D)$,

$$(\Box_b u|v)=((\overline{\Box}_b+[2\sqrt{-1}\mathscr{L},\Lambda](-\sqrt{-1}T)+(\partial_b\mathcal{T}^*+\mathcal{T}^*\partial_b)-(\overline{\partial}_b\overline{\mathcal{T}}^*+\overline{\mathcal{T}}^*\overline{\partial}_b))u|v).$$

The theorem follows. □

Corollary 3.4 (CR Nakano's inequality I) *With the notations used above, for any $u\in\Omega_c^{\bullet,\bullet}(X)$,*

$$\begin{aligned}\frac{3}{2}(\Box_b u|u)\geq{}&([2\sqrt{-1}\mathscr{L},\Lambda](-\sqrt{-1}Tu)|u)\\&-\frac{1}{2}(\|\mathcal{T}u\|^2+\|\mathcal{T}^*u\|^2+\|\overline{\mathcal{T}}u\|^2+\|\overline{\mathcal{T}}^*u\|^2).\end{aligned}\tag{3.24}$$

If $(X,T^{(1,0)}X)$ is Kähler, i.e., $d\Theta_X=0$, then

$$(\Box_b u|u)\geq([2\sqrt{-1}\mathscr{L},\Lambda](-\sqrt{-1}Tu)|u).\tag{3.25}$$

Proof By the Cauchy–Schwarz inequality, Theorem 3.3 and since $\mathcal{T}=0$, $\mathcal{T}^*=0$ if $d\Theta_X=0$, we get the corollary. □

The following follows from straightforward calculation, we omit the proof.

Proposition 3.5 *For a real $(1,1)$-form $\sqrt{-1}\alpha\in\Omega^{1,1}(D)$, if we choose local orthonormal frame $\{\overline{L}_j\}_{j=1}^n$ of $T^{(1,0)}D$ with the dual frame $\{\overline{e}_j\}_{j=1}^n$ of $T^{*(1,0)}D$ such that $\sqrt{-1}\alpha=\sqrt{-1}\lambda_j(x)\overline{e}_j\wedge e_j$ at a given point $x\in D$, then for any $f=\sum_{I,J}f_{IJ}(x)\overline{e}^I\wedge e^J\in\Omega^{\bullet,\bullet}(D)$, we have*

$$[\sqrt{-1}\alpha,\Lambda]f(x)=\sum_{I,J}\left(\sum_{j\in I}\lambda_j(x)+\sum_{j\in J}\lambda_j(x)-\sum_{j=1}^n\lambda_j(x)\right)f_{IJ}(x)\overline{e}^I\wedge e^J.\tag{3.26}$$

Corollary 3.6 *With the notations used above, let Θ_X be a Hermitian metric on X such that*

$$2\sqrt{-1}\mathscr{L}=\Theta_X.\tag{3.27}$$

Then for any $u \in \Omega_c^{n,q}(X)$ with $1 \leq q \leq n$,

$$\left(-\sqrt{-1}Tu|u\right) \leq \frac{1}{q}\left(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}_b^* u\|^2\right). \tag{3.28}$$

Proof By applying (3.26) for $\sqrt{-1}\alpha := 2\sqrt{-1}\mathscr{L}$, $\lambda_j = 1$ for all j, we have

$$[2\sqrt{-1}\mathscr{L}, \Lambda](-\sqrt{-1}Tu) = q(-\sqrt{-1}Tu), \quad \text{for all } u \in \Omega_c^{n,q}(X). \tag{3.29}$$

By $d\Theta_X = d(2\sqrt{-1}\mathscr{L}) = 0$ and Corollary 3.4, we obtain

$$(\Box_b u|u) \geq ([2\sqrt{-1}\mathscr{L}, \Lambda](-\sqrt{-1}Tu)|u) = q(-\sqrt{-1}Tu|u). \tag{3.30}$$

□

Let E be a CR line bundle over X (see Definition 2.4 in [13]. We say that E is a $\mathbb{R}$-equivariant CR line bundle over X if the $\mathbb{R}$-action on X can be CR lifted to E and for every point $x \in X$, we can find a T-invariant local CR frame of E defined near x (see [16, Definitions 2.6, 2.9]). Here, we also use T to denote the vector field acting on sections of E induced by the $\mathbb{R}$-action on E. From now on, we assume that E is a $\mathbb{R}$-equivariant CR line bundle over X with a $\mathbb{R}$-invariant Hermitian metric h^E on E. For $p, q \in \mathbb{N}_0$, let $\Omega^{p,q}(X, E)$ be the space of smooth (p, q)-forms of X with values in E and let $\Omega^{\bullet,\bullet}(X, E) := \oplus_{p,q\in\mathbb{N}_0}\Omega^{p,q}(X, E)$. Let $\Omega_c^{p,q}(X, E)$ be the subspace of $\Omega^{p,q}(X, E)$ whose elements have compact support in X and let $\Omega_c^{\bullet,\bullet}(X, E) := \oplus_{p,q\in\mathbb{N}_0}\Omega_c^{p,q}(X, E)$. For $p, q \in \mathbb{N}_0$, let

$$\overline{\partial}_{b,E} : \Omega^{p,q}(X, E) \to \Omega^{p,q+1}(X, E)$$

be the tangential Cauchy–Riemann operator with values in E. Let $(\cdot\,|\,\cdot)_E$ be the L^2 inner product on $\Omega_c^{\bullet,\bullet}(X, E)$ induced by $\langle\cdot\,|\,\cdot\rangle$ and h^E. Let

$$\overline{\partial}_{b,E}^* : \Omega^{p,q+1}(X, E) \to \Omega^{p,q}(X, E)$$

be the formal adjoint of $\overline{\partial}_{b,E}$ with respect to $(\cdot\,|\,\cdot)_E$. Put

$$\Box_{b,E} := \overline{\partial}_{b,E}\,\overline{\partial}_{b,E}^* + \overline{\partial}_{b,E}^*\,\overline{\partial}_{b,E} : \Omega^{\bullet,\bullet}(X, E) \to \Omega^{\bullet,\bullet}(X, E).$$

Let

$$\nabla^E : \Omega^{\bullet,\bullet}(X, E) \to \Omega^{\bullet,\bullet}(X, E \otimes \mathbb{C}T^*X) \tag{3.31}$$

be the connection on E induced by h^E given as follows: Let s be a T-invariant local CR frame of E on an open set D of X,

$$|s|^2_{h^E} = e^{-2\Phi}, \quad \Phi \in \mathscr{C}^\infty(D, \mathbb{R}). \tag{3.32}$$

Then,

$$\nabla^E(u \otimes s) := (\overline{\partial}_b u + \partial_b u - 2(\partial_b \Phi) \wedge u + \omega_0 \wedge (Tu)) \otimes s, \quad u \in \Omega^{\bullet,\bullet}(D). \tag{3.33}$$

It is straightforward to check that (3.33) is independent of the choices of T-invariant local CR trivializing sections s and hence is globally defined. Put

$$(\nabla^E)^{0,1} := \overline{\partial}_b, \quad (\nabla^E)^{1,0} := \partial_b - 2\partial_b \Phi. \tag{3.34}$$

Let

$$\overline{\square}_{b,E} := (\nabla^E)^{1,0}((\nabla^E)^{1,0})^* + ((\nabla^E)^{1,0})^*(\nabla^E)^{1,0} : \Omega^{\bullet,\bullet}(X, E) \to \Omega^{\bullet,\bullet}(X, E), \tag{3.35}$$

where $((\nabla^E)^{1,0})^*$ is the adjoint of $(\nabla^E)^{1,0}$ with respect to $(\,\cdot\,|\,\cdot\,)_E$. Let $R^E \in \Omega^{1,1}(X)$ be the curvature of E induced by h^E given by $R^E := -2\overline{\partial}_b \partial_b \Phi$ on D, where Φ is as in (3.32). It is easy to check that R^E is globally defined. Let $D = U \times \mathcal{I}$ be a BRT chart. Since E is $\mathbb{R}$-equivariant, on D, E is a holomorphic line bundle over U. We can repeat the proof of Theorem 3.3 with minor changes and conclude the following:

Theorem 3.7 *Let E be a $\mathbb{R}$-equivariant CR line bundle over X with a $\mathbb{R}$-invariant Hermitian metric h^E. With the notations used above, we have on $\Omega^{\bullet,\bullet}(X, E)$,*

$$\begin{aligned} \square_{b,E} &= \overline{\square}_{b,E} + [2\sqrt{-1}\mathscr{L}, \Lambda](-\sqrt{-1}T) + [\sqrt{-1}R^E, \Lambda] \\ &\quad + \Big((\nabla^E)^{1,0}\mathcal{T}^* + \mathcal{T}^*(\nabla^E)^{1,0}\Big) - \Big(\overline{\partial}_{b,E}\overline{\mathcal{T}}^* + \overline{\mathcal{T}}^*\overline{\partial}_{b,E}\Big), \end{aligned} \tag{3.36}$$

where $R^E \in \Omega^{1,1}(X)$ is the curvature of E induced by h^E.

3.3 CR Bochner–Kodaira–Nakano Formula II

The bundle $K_X^* := \det(T^{(1,0)}X)$ is a $\mathbb{R}$-equivariant CR line bundle over X. The $(1, 1)$ form Θ_X induces a $\mathbb{R}$-invariant Hermitian metric $h^{K_X^*}$ on K_X^*. Let $R^{K_X^*}$ be the curvature of K_X^* induced by $h^{K_X^*}$. Let

$$\Psi : T^{*0,q}X \to T^{*n,q}X \otimes K_X^*$$

be the natural isometry defined as follows: Let $D = U \times \mathcal{I}$ be a BRT chart. Let $\{\overline{L}_j\}_{j=1}^n \subset T^{(1,0)}D$, $\{\overline{e}_j\}_{j=1}^n \subset T^{*(1,0)}D$ be as in the discussion after (2.26). Then,

$$\Psi u := \overline{e}_1 \wedge \ldots \wedge \overline{e}_n \wedge u \otimes (\overline{L}_1 \wedge \ldots \wedge \overline{L}_n) \in T^{*n,q}X \otimes K_X^*, \quad u \in T^{*0,q}X.$$

It is easy to see that the definition above is independent of the choices of $\mathbb{R}$-invariant orthonormal frame $\{\overline{L}_j\}_{j=1}^n \subset T^{(1,0)}D$ and hence is globally defined. We have the isometry:

$$\Psi : \Omega^{0,q}(X) \to \Omega^{n,q}(X, K_X^*).$$

Moreover, it is straightforward to see that

$$\begin{aligned}&\overline{\partial}_b u=\Psi^{-1}\overline{\partial}_{b,K_X^*}\Psi u,\quad \overline{\partial}^*_b u\\&\quad=\Psi^{-1}\overline{\partial}^*_{b,K_X^*}\Psi u,\quad \Box_b u=\Psi^{-1}\Box_{b,K_X^*}\Psi u,\quad \text{for every } u\in\Omega^{0,q}(X).\end{aligned}\tag{3.37}$$

We can now prove

Theorem 3.8 *With the notations used above, we have on $\Omega^{0,\bullet}(X)$,*

$$\begin{aligned}\Box_b=&\Psi^{-1}\Box_{b,K_X^*}\Psi+2\mathscr{L}(\overline{L}_j,L_k)e_k\wedge i_{L_j}(-\sqrt{-1}T)+R^{K_X^*}(\overline{L}_j,L_k)e_k\wedge i_{L_j}\\&+\Psi^{-1}(\nabla^{K_X^*})^{1,0}\mathcal{T}^*\Psi-\Big(\overline{\partial}_b\Psi^{-1}\overline{\mathcal{T}}^*\Psi+\Psi^{-1}\overline{\mathcal{T}}^*\Psi\overline{\partial}_b\Big),\end{aligned}\tag{3.38}$$

where $\{\overline{L}_j\}_{j=1}^n$ is a local $\mathbb{R}$-invariant orthonormal frame of $T^{(1,0)}X$ with dual frame $\{\overline{e}_j\}_{j=1}^n\subset T^{*(1,0)}X$.

Proof Let $u\in\Omega^{0,q}(X)$. From (3.37) and (3.36), we have

$$\begin{aligned}\Box_b u=&\Psi^{-1}\Box_{b,K_X^*}\Psi u\\=&\Psi^{-1}\overline{\Box}_{b,K_X^*}\Psi u+\Psi^{-1}[2\sqrt{-1}\mathscr{L},\Lambda](-\sqrt{-1}T)(\Psi u)+\Psi^{-1}[\sqrt{-1}R^{K_X^*},\Lambda]\Psi u\\&+\Psi^{-1}\Big((\nabla^{K_X^*})^{1,0}\mathcal{T}^*+\mathcal{T}^*(\nabla^{K_X^*})^{1,0}\Big)\Psi u-\Big(\overline{\partial}_b\Psi^{-1}\overline{\mathcal{T}}^*\Psi u+\Psi^{-1}\overline{\mathcal{T}}^*\Psi\overline{\partial}_b u\Big).\end{aligned}\tag{3.39}$$

It is straightforward to check that

$$\begin{aligned}&[2\sqrt{-1}\mathscr{L},\Lambda]=2\mathscr{L}(\overline{L}_j,L_k)(\overline{e}_j\wedge i_{\overline{L}_k}-i_{L_j}e_k\wedge),\\&[2\sqrt{-1}R^{K_X^*},\Lambda]=R^{K_X^*}(\overline{L}_j,L_k)(\overline{e}_j\wedge i_{\overline{L}_k}-i_{L_j}e_k\wedge).\end{aligned}\tag{3.40}$$

From (3.39), (3.40) and noting that $(\overline{e}_j\wedge i_{\overline{L}_k}-i_{L_j}e_k\wedge)v=e_k\wedge i_{L_j}v$, $\mathcal{T}^*(\nabla^{K_X^*})^{1,0}v=0$, for every $v\in\Omega^{n,q}(X,K_X^*)$ and T commutes with Ψ, we get (3.38). ◻

Corollary 3.9 *With the notations used above, assume that $2\sqrt{-1}\mathscr{L}=\Theta_X$ and there is $C>0$ such that*

$$\sqrt{-1}R^{K_X^*}\geq -C\Theta_X\ \ \text{on } X.$$

Then, for any $u\in\Omega^{0,q}_c(X)$ with $1\leq q\leq n$, we have

$$\Big(-\sqrt{-1}Tu\,|\,u\Big)\leq\frac{1}{q}\Big(\|\overline{\partial}_b u\|^2+\|\overline{\partial}^*_b u\|^2\Big)+C\|u\|^2.\tag{3.41}$$

Proof Since $2\sqrt{-1}\mathscr{L}=\Theta_X$, we can choose $\mathbb{R}$-invariant orthonormal frame $\{\overline{L}_j\}_{j=1}^n$ such that $2\mathscr{L}(\overline{L}_j,L_k)=\delta_{jk}$, for every j,k. We write $u=\sum_J u_Je_J$ on D with

$u_J \in \mathscr{C}^\infty(D)$ and $e_J = e_{j_1} \wedge \ldots \wedge e_{j_q}$, $j_1 < \ldots < j_q$. We have

$$\begin{aligned}\langle 2\mathscr{L}(\overline{L}_j, L_k)e_k \wedge i_{L_j}(-\sqrt{-1}T)u \,|\, u \rangle &= \langle \sum_J q(-\sqrt{-1}Tu_J)e_J \,|\, u \rangle \\ &= q\langle -\sqrt{-1}Tu \,|\, u \rangle. \end{aligned} \tag{3.42}$$

Since $\sqrt{-1}R^{K_X^*} \geq -C\Theta_X$, as (3.42), we can check that

$$\langle R^{K_X^*}(\overline{L}_j, L_k)e_k \wedge i_{L_j}u|u\rangle \geq -Cq|u|^2. \tag{3.43}$$

Since $d\Theta_X = d(2\sqrt{-1}\mathscr{L}) = 0$, we have $\mathcal{T} = [\Lambda, \partial_b \Theta_X] = 0$. From this observation, (3.38), (3.42), and (3.43), we obtain

$$\left(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}_b^* u\|^2\right) \geq q\left(-\sqrt{-1}Tu|u\right) - qC\|u\|^2 \tag{3.44}$$

holds for every $u \in \Omega_c^{0,q}(X)$ with $1 \leq q \leq n$. □

4 Szegő Kernel Asymptotics

In this section, we will establish Szegő kernel asymptotic expansions on X under certain curvature assumptions.

4.1 Complete CR Manifolds

Let X be a CR manifold as in Assumption 1.1. Let g_X be the $\mathbb{R}$-invariant Hermitian metric as in (1.5). We will assume in the following that the Riemannian metric induced by g_X on TX is complete and study the extension $\overline{\partial}_b$, $\overline{\partial}_b^*$, and T. We denote by the same symbols the maximal weak extensions in L^2 of these differential operators.

Since g_X is complete we know by [10, Lemma 2.4, p. 366] that there exists a sequence $\{\chi_k\}_{k=1}^\infty \subset \mathscr{C}_c^\infty(X)$ such that $0 \leq \chi_k \leq 1$, $\chi_{k+1} = 1$ on $\operatorname{supp}\chi_k$, $|d\chi_k|_g \leq \frac{1}{2^k}$, for every $k \in \mathbb{N}$, and $\bigcup_{k=1}^\infty \operatorname{supp}\chi_k = X$. By using this sequence as in the Andreotti–Vesentini lemma on complex Hermitian manifolds (cf. [10, Theorem 3.2, p. 368], [23, Lemma 3.3.1]) and the classical Friedrichs lemma, we obtain the following.

Lemma 4.1 *Assume that (X, g_X) is complete. Then $\Omega_c^{p,q}(X)$ is dense in* $\operatorname{Dom}\overline{\partial}_b$, $\operatorname{Dom}\overline{\partial}_b^*$, $\operatorname{Dom} T$, $\operatorname{Dom}\overline{\partial}_b \cap \operatorname{Dom}\overline{\partial}_b^*$, *and* $\operatorname{Dom} T \cap \operatorname{Dom}\overline{\partial}_b \cap \operatorname{Dom}\overline{\partial}_b^*$ *with respect to the graph norms of $\overline{\partial}_b$, $\overline{\partial}_b^*$, T, $\overline{\partial}_b + \overline{\partial}_b^*$, and $\overline{\partial}_b + \overline{\partial}_b^* + T$. Here the graph norm of a linear operator R is defined by $\|u\| + \|Ru\|$ for $u \in$* $\operatorname{Dom} R$.

As a consequence, analog to [23, Corollary 3.3.3], we obtain the following:

Corollary 4.2 *If (X, g_X) is complete, then the maximal extension of the formal adjoint of $\overline{\partial}_b$ and T coincide with their Hilbert space adjoint, respectively.*

Corollary 4.3 *If (X, g_X) is complete, then $\sqrt{-1}T : \mathrm{Dom}\sqrt{-1}T \subset L^2_{\bullet,\bullet}(X) \to L^2_{\bullet,\bullet}(X)$ is self-adjoint, that is, $(\sqrt{-1}T)^* = \sqrt{-1}T$.*

Using these results, we extend the estimates from Corollary 3.9 as follows:

Theorem 4.4 *Let X be a CR manifold as in Assumption 1.1. Assume that $2\sqrt{-1}\mathscr{L} = \Theta_X$, g_X is complete and there is $C > 0$ such that*

$$\sqrt{-1}R^{K_X^*} \geq -C\Theta_X.$$

Then, for any $u \in L^2_{0,q}(X)$, $1 \leq q \leq n$, $u \in \mathrm{Dom}\,\overline{\partial}_b \bigcap \mathrm{Dom}\overline{\partial}^*_b \bigcap \mathrm{Dom}(\sqrt{-1}T)$, we have

$$(-\sqrt{-1}Tu \,|\, u) \leq \frac{1}{q}\Big(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}^*_b u\|^2\Big) + C\|u\|^2. \tag{4.1}$$

Proof Let $u \in L^2_{0,q}(X)$, $1 \leq q \leq n$, $u \in \mathrm{Dom}\,\overline{\partial}_b \bigcap \mathrm{Dom}\,\overline{\partial}^*_b \bigcap \mathrm{Dom}(\sqrt{-1}T)$. From Lemma 4.1, we can find $\{u_j\}_{j=1}^{\infty} \subset \Omega^{\bullet,\bullet}_c(X)$ such that

$$\lim_{j\to\infty}\Big(\|u_j - u\|^2 + \|\overline{\partial}_b u_j - \overline{\partial}_b u\|^2 + \|\sqrt{-1}Tu_j - \sqrt{-1}Tu\|^2\Big) = 0. \tag{4.2}$$

From (3.41), we have for every $j = 1, 2, \ldots$,

$$(-\sqrt{-1}Tu_j \,|\, u_j) \leq \frac{1}{q}\Big(\|\overline{\partial}_b u_j\|^2 + \|\overline{\partial}^*_b u_j\|^2\Big) + C\|u_j\|^2. \tag{4.3}$$

Taking $j \to \infty$ in (4.3) and using (4.2), we get (4.1). □

Let us describe two examples of CR manifolds with complete $\mathbb{R}$-invariant metric g_X.

Example 4.5 Let (X, HX, J, ω_0) be a compact strictly pseudoconvex CR manifold as in Assumption 1.1 and let g_X be a $\mathbb{R}$-invariant metric as in (1.5). Let $\pi : \widetilde{X} \to X$ be a Galois covering of X, that is, there exists a discrete, proper action Γ such that $X = \widetilde{X}/\Gamma$. By pulling back the objects from X by the projection π we obtain a strictly pseudoconvex CR manifold $(\widetilde{X}, H\widetilde{X}, \widetilde{J}, \widetilde{\omega}_0)$ satisfying Assumption 1.1. Moreover, the metric $\widetilde{g}_X = \pi^* g_X$ is a complete $\mathbb{R}$-invariant metric satisfying (1.5).

Example 4.6 Let us consider now the case of a circle bundle associated to a Hermitian holomorphic line bundle. Let (M, J, Θ_M) be a complete Hermitian manifold. Let $(L, h^L) \to M$ be a Hermitian holomorphic line bundle over M. Let h^{L^*} be the Hermitian metric on L^* induced by h^L. Let

$$X := \{v \in L^*;\ |v|^2_{h^{L^*}} = 1\} \tag{4.4}$$

be the circle bundle of L^*; it is isomorphic to the S^1 principal bundle associated to L. Since X is a hypersurface in the complex manifold L^*, it has a CR structure (X, HX, J) inherited from the complex structure of L^* by setting $T^{(1,0)}X = TX \cap T^{(1,0)}L^*$.

In this situation, S^1 acts on X by fiberwise multiplication, denoted $(x, e^{i\theta}) \mapsto xe^{i\theta}$. A point $x \in X$ is a pair $x = (p, \lambda)$, where λ is a linear functional on L_p, the S^1 action is $xe^{i\theta} = (p, \lambda)e^{i\theta} = (p, e^{i\theta}\lambda)$.

Let ω_0 be the connection 1-form on X associated to the Chern connection ∇^L. Then

$$d\omega_0 = \pi^*(iR^L), \tag{4.5}$$

where R^L is the curvature of ∇^L. Assume R^L is positive, hence X is a strictly pseudoconvex CR manifold. Hence, (X, HX, J, ω_0) fulfills Assumption 1.1. We denote by ∂_θ the infinitesimal generator of the S^1 action on X. The span of ∂_θ defines a rank one sub-bundle $T^V X \cong TS^1 \subset TX$, the vertical sub-bundle of the fibration $\pi : X \to M$. Moreover (1.3) holds for $T = \partial_\theta$.

We construct now a Riemannian metric on X. Let g_M be a J-invariant metric on TM associated to Θ_M. The Chern connection ∇^L on L induces a connection on the S^1-principal bundle $\pi : X \to M$, and let $T^H X \subset TX$ be the corresponding horizontal bundle. Let $g_X = \pi^* g_M \oplus d\theta^2/4\pi^2$ be the metric on $TX = T^H X \oplus TS^1$, with $d\theta^2$ the standard metric on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then g_X is a $\mathbb{R}$-invariant Hermitian metric on X satisfying (1.5). Since g_M is complete it is easy to see that g_X is also complete.

4.2 The Operators Q_λ, $Q_{[\lambda_1,\lambda]}$, Q_τ

From now on, we assume that X is a CR manifold satisfying Assumption 1.1 and (X, g_X) is complete. Let $\mathbb{S}$ denote the spectrum of $\sqrt{-1}T$. By the spectral theorem, there exists a finite measure μ on $\mathbb{S} \times \mathbb{N}$ and a unitary operator

$$U : L^2_{\bullet,\bullet}(X) \to L^2(\mathbb{S} \times \mathbb{N}, d\mu)$$

with the following properties: If $h : \mathbb{S} \times \mathbb{N} \to \mathbb{R}$ is the function $h(s, n) = s$, then the element ξ of $L^2_{\bullet,\bullet}(X)$ lies in $\mathrm{Dom}(\sqrt{-1}T)$ if and only if $hU(\xi) \in L^2(\mathbb{S} \times \mathbb{N}, d\mu)$. We have

$$U\sqrt{-1}TU^{-1}\varphi = h\varphi, \text{ for all } \varphi \in U(\mathrm{Dom}(\sqrt{-1}T)).$$

Let $\lambda_1, \lambda \in \mathbb{R}$, $\lambda_1 < \lambda$, and let $\tau(t) \in \mathscr{C}^\infty(\mathbb{R}, [0, 1])$. Put

$$\begin{aligned}
\mathscr{E}(\lambda, \sqrt{-1}T) &:= U^{-1}\Big(\text{Image } U \cap \{\mathbb{1}_{(-\infty,\lambda]}(s)h(s, n);\ h(s, n) \in L^2(\mathbb{S} \times \mathbb{N}, d\mu)\}\Big),\\
\mathscr{E}([\lambda_1, \lambda], \sqrt{-1}T) &:= U^{-1}\Big(\text{Image } U \cap \{\mathbb{1}_{[\lambda_1,\lambda]}(s)h(s, n);\ h(s, n) \in L^2(\mathbb{S} \times \mathbb{N}, d\mu)\}\Big),\\
\mathscr{E}(\tau, \sqrt{-1}T) &:= U^{-1}\Big(\text{Image } U \cap \{\tau(s)h(s, n);\ h(s, n) \in L^2(\mathbb{S} \times \mathbb{N}, d\mu)\}\Big),
\end{aligned} \tag{4.6}$$

where $\mathbb{1}_A$ denotes the characteristic function of the set A. Let

$$\begin{aligned}
&Q_\lambda : L^2_{\bullet,\bullet}(X) \to \mathscr{E}(\lambda, \sqrt{-1}T),\\
&Q_{[\lambda_1,\lambda]} : L^2_{\bullet,\bullet}(X) \to \mathscr{E}([\lambda_1, \lambda], \sqrt{-1}T),\\
&Q_\tau : L^2_{\bullet,\bullet}(X) \to \mathscr{E}(\tau, \sqrt{-1}T)
\end{aligned} \tag{4.7}$$

be the orthogonal projections with respect to $(\cdot\,|\,\cdot)$.

Since X is strictly pseudoconvex, from [21, Lemma 3.4 (3), p. 239], [13, Theorem 3.5], we have one of the following two cases:

$$\begin{aligned}&\text{(a) The } \mathbb{R}\text{-action is free,}\\ &\text{(b) The } \mathbb{R}\text{-action comes from a CR torus action } \mathbb{T}^d \text{ on } X \text{ and } \omega_0 \text{ and } \Theta_X \text{ are } \mathbb{T}^d \text{ invariant.}\end{aligned} \tag{4.8}$$

When X is non-compact, the $\mathbb{R}$-action does not always come from a CR torus action. For example, when X is the Heisenberg group (see Sect. 5), the standard $\mathbb{R}$-action on X is free and does not come from any CR torus action.

Assume that the $\mathbb{R}$-action is free. Let $D = U \times \mathcal{I}$ be a BRT chart with BRT coordinates $x = (x_1, \ldots, x_{2n+1})$. Since the $\mathbb{R}$-action is free, we can extend $x = (x_1, \ldots, x_{2n+1})$ to $\hat{D} := U \times \mathbb{R}$. We identify $\hat{D}$ with an open set in X.

Lemma 4.7 *Assume that the $\mathbb{R}$-action is free. Let $D = U \times \mathcal{I}$ be a BRT chart with BRT coordinates $x = (x_1, \ldots, x_{2n+1})$. Let $\lambda_1, \lambda \in \mathbb{R}$, $\lambda_1 < \lambda$. For $u \in \Omega_c^{\bullet,\bullet}(D)$, we have*

$$(Q_\lambda u)(x) = \frac{1}{(2\pi)^{2n+1}} \int e^{i<x-y,\eta>} \mathbb{1}(-\infty, \lambda](-\eta_{2n+1}) u(y) dy d\eta \in \Omega^{\bullet,\bullet}(\hat{D}), \tag{4.9}$$

$$(Q_{[\lambda_1,\lambda]} u)(x) = \frac{1}{(2\pi)^{2n+1}} \int e^{i<x-y,\eta>} \mathbb{1}[\lambda_1, \lambda](-\eta_{2n+1}) u(y) dy d\eta \in \Omega^{\bullet,\bullet}(\hat{D}), \tag{4.10}$$

$$(Q_\tau u)(x) = \frac{1}{(2\pi)^{2n+1}} \int e^{i<x-y,\eta>} \tau(-\eta_{2n+1}) u(y) dy d\eta \in \Omega^{\bullet,\bullet}(\hat{D}), \tag{4.11}$$

and $\operatorname{supp} Q_\lambda u \subset \hat{D}$, $\operatorname{supp} Q_{[\lambda_1,\lambda]} u \subset \hat{D}$, $\operatorname{supp} Q_\tau u \subset \hat{D}$, *where $\hat{D}$ is as in the discussion after* (4.8).

Proof Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$, $\chi = 1$ on $[-1, 1]$, $\chi = 0$ outside $[-2, 2]$. For every $M > 0$, put $\tau_M(t) := \chi(\frac{t}{M})\tau(t)$. Then,

$$Q_\tau u = \lim_{M\to\infty} Q_{\tau_M} u \quad \text{in } L^2_{\bullet,\bullet}(X), \text{ for every } u \in L^2_{\bullet,\bullet}(X). \tag{4.12}$$

From the Helffer–Sjöstrand formula [12, Proposition 7.2], we see that

$$Q_{\tau_M} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \tilde{\tau}_M}{\partial \bar{z}} (z - \sqrt{-1}T)^{-1} dz \wedge d\bar{z} \;\text{ on } L^2_{\bullet,\bullet}(X), \tag{4.13}$$

where $\tilde{\tau}_M \in \mathscr{C}_c^\infty(\mathbb{C})$ is an extension of τ_M with $\frac{\partial \tilde{\tau}_M}{\partial \bar{z}} = 0$ on $\mathbb{R}$. It is not difficult to see that for $u \in \Omega_c^{\bullet,\bullet}(D)$,

$$(z - \sqrt{-1}T)^{-1} u = \frac{1}{(2\pi)^{2n+1}} \int e^{i<x-y,\eta>} \frac{1}{z + \eta_{2n+1}} u(y) dy d\eta \in \Omega^{\bullet,\bullet}(\hat{D}) \tag{4.14}$$

and supp $(z-\sqrt{-1}T)^{-1}u\subset\hat{D}$. From (4.13) and (4.14), we have

$$(Q_{\tau_M}u)(x)=\frac{1}{2\pi i}\frac{1}{(2\pi)^{2n+1}}\int_{\mathbb{C}}\int e^{i<x-y,\eta>}\frac{\frac{\partial\tilde{\tau}_M}{\partial\overline{z}}}{z+\eta_{2n+1}}u(y)dyd\eta dz\wedge d\overline{z}\in\Omega^{\bullet,\bullet}(\hat{D}) \tag{4.15}$$

and supp $Q_{\tau_M}u\subset\hat{D}$, for every $u\in\Omega_c^{\bullet,\bullet}(D)$. By Cauchy integral formula, we see that

$$\frac{1}{2\pi i}\int_{\mathbb{C}}\frac{1}{z+\eta_{2n+1}}\frac{\partial\tilde{\tau}_M}{\partial\overline{z}}dz\wedge d\overline{z}=\tau_M(-\eta_{2n+1}).$$

From this observation and (4.15), we deduce that

$$(Q_{\tau_M}u)(x)=\frac{1}{(2\pi)^{2n+1}}\int e^{i\langle x-y,\eta\rangle}\tau_M(-\eta_{2n+1})u(y)dyd\eta\in\Omega^{\bullet,\bullet}(\hat{D}) \tag{4.16}$$

and supp $Q_{\tau_M}u\subset\hat{D}$, for every $u\in\Omega_c^{\bullet,\bullet}(D)$. From (4.12) and (4.16), we get (4.11).

Let $\gamma_\varepsilon\in\mathscr{C}_c^\infty(\mathbb{R})$, $\lim_{\varepsilon\to0}\gamma_\varepsilon(t)=\mathbb{1}(-\infty,\lambda](t)$, for every $t\in\mathbb{R}$. We can repeat the proof above and get that

$$Q_\lambda u=\lim_{\varepsilon\to0}Q_{\gamma_\varepsilon}u=\frac{1}{(2\pi)^{2n+1}}\int e^{i\langle x-y,\eta\rangle}\mathbb{1}(-\infty,\lambda](-\eta_{2n+1})u(y)dyd\eta\in\Omega^{\bullet,\bullet}(\hat{D})$$

and supp $Q_\lambda u\subset\hat{D}$, for every $u\in\Omega_c^{\bullet,\bullet}(D)$. We obtain (4.9). The proof of (4.10) is similar. □

We now assume that the $\mathbb{R}$-action is not free. From (4.8), we know that the $\mathbb{R}$-action comes from a CR torus action $\mathbb{T}^d=(e^{i\theta_1},\ldots,e^{i\theta_d})$ on X and ω_0,Θ_X are $\mathbb{T}^d$ invariant. Since the $\mathbb{R}$-action comes from the $\mathbb{T}^d$-action, there exist $\beta_1,\ldots,\beta_d\in\mathbb{R}$, such that

$$T=\beta_1T_1+\ldots+\beta_dT_d, \tag{4.17}$$

where T_j is the vector field on X given by $T_ju:=\frac{\partial}{\partial\theta_j}((1,\ldots,1,e^{i\theta_j},1,\ldots,1)^*u)|_{\theta_j=0}$, $u\in\Omega^{\bullet,\bullet}(X)$, $j=1,\ldots,d$. For $(m_1,\ldots,m_d)\in\mathbb{Z}^d$, put

$$\begin{aligned}&L^{2,m_1,\ldots,m_d}_{\bullet,\bullet}(X)\\&:=\{u\in L^2_{\bullet,\bullet}(X);\ (e^{i\theta_1},\ldots,e^{i\theta_d})^*u=e^{im_1\theta_1+\ldots+im_d\theta_d}u,\ \text{for all }(\theta_1,\ldots,\theta_d)\in\mathbb{R}^d\}\end{aligned}$$

and let

$$Q_{m_1,\ldots,m_d}:L^2_{\bullet,\bullet}(X)\to L^{2,m_1,\ldots,m_d}_{\bullet,\bullet}(X) \tag{4.18}$$

be the orthogonal projection. It is not difficult to see that for every $u \in L^2_{\bullet,\bullet}(X)$, we have

$$
\begin{aligned}
Q_\lambda u &= \sum_{\substack{(m_1,\ldots,m_d)\in\mathbb{Z}^d,\\ -m_1\beta_1-\ldots-m_d\beta_d\leq\lambda}} Q_{m_1,\ldots,m_d}u,\\
Q_{[\lambda_1,\lambda]}u &= \sum_{\substack{(m_1,\ldots,m_d)\in\mathbb{Z}^d,\\ \lambda_1\leq-m_1\beta_1-\ldots-m_d\beta_d\leq\lambda}} Q_{m_1,\ldots,m_d}u,\\
Q_\tau u &= \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \tau(-m_1\beta_1-\ldots-m_d\beta_d)Q_{m_1,\ldots,m_d}u.
\end{aligned}
\tag{4.19}
$$

From Lemma 4.7 and (4.19), we conclude that

Proposition 4.8 *Let $\lambda_1,\lambda\in\mathbb{R}$, $\lambda_1<\lambda$. For $u\in\operatorname{Dom}\overline{\partial}_b$, we have $Q_\lambda u$, $Q_{[\lambda_1,\lambda]}u$, $Q_\tau u\in\operatorname{Dom}\overline{\partial}_b$ and $\overline{\partial}_bQ_\lambda u=Q_\lambda\overline{\partial}_bu$, $\overline{\partial}_bQ_{[\lambda_1,\lambda]}u=Q_{[\lambda_1,\lambda]}\overline{\partial}_bu$, $\overline{\partial}_bQ_\tau u=Q_\tau\overline{\partial}_bu$. Similarly, for $u\in\operatorname{Dom}\overline{\partial}^*_b$, we have $Q_\lambda u$, $Q_{[\lambda_1,\lambda]}u$, $Q_\tau u\in\operatorname{Dom}\overline{\partial}^*_b$ and $\overline{\partial}^*_bQ_\lambda u=Q_\lambda\overline{\partial}^*_bu$, $\overline{\partial}^*_bQ_{[\lambda_1,\lambda]}u=Q_{[\lambda_1,\lambda]}\overline{\partial}^*_bu$, $\overline{\partial}^*_bQ_\tau u=Q_\tau\overline{\partial}^*_bu$.*

For $\lambda\in\mathbb{R}$, define

$$
\begin{aligned}
&\Box_{b,\lambda}:\operatorname{Dom}\Box_{b,\lambda}\subset\mathscr{E}(\lambda,\sqrt{-1}T)\to\mathscr{E}(\lambda,\sqrt{-1}T),\\
&\operatorname{Dom}\Box_{b,\lambda}:=\operatorname{Dom}\Box_b\bigcap\mathscr{E}(\lambda,\sqrt{-1}T),\quad \Box_{b,\lambda}u=\Box_bu,\quad\text{for } u\in\operatorname{Dom}\Box_{b,\lambda},
\end{aligned}
\tag{4.20}
$$

where $\Box_b$ is defined in (2.8), (2.9). From Proposition 4.8, we see that

$$
\begin{aligned}
&\operatorname{Dom}\Box_{b,\lambda}=Q_\lambda(\operatorname{Dom}\Box_b),\\
&Q_\lambda\Box_b=\Box_bQ_\lambda=\Box_{b,\lambda}Q_\lambda\ \text{ on }\operatorname{Dom}\Box_b.
\end{aligned}
\tag{4.21}
$$

From now on, we write $\Box^{(q)}_b$ and $\Box^{(q)}_{b,\lambda}$ to denote $\Box_b$ and $\Box_{b,\lambda}$ acting on $(0,q)$ forms, respectively.

4.3 Local Closed Range for $\Box^{(0)}_b$

In this section, we will establish the local closed range property for $\Box^{(0)}_b$ under appropriate curvature assumptions. We first need the following.

Lemma 4.9 *Assume that $2\sqrt{-1}\mathscr{L}=\Theta_X$, g_X is complete and there is $C>0$ such that*

$$\sqrt{-1}R^{K^*_X}\geq-C\Theta_X.$$

Then, for any $u \in L^2_{0,q}(X)$, $1 \leq q \leq n$, $u \in \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}^*_b \cap \mathscr{E}(\lambda, \sqrt{-1}T)$, $\lambda \in \mathbb{R}$, $\lambda < -C$, *we have*

$$\|u\|^2 \leq \frac{1}{q(-\lambda - C)}\Big(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}^*_b u\|^2\Big). \tag{4.22}$$

Proof Let $\lambda < -C$ and let $u \in \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}^*_b \cap \mathscr{E}(\lambda, \sqrt{-1}T)$, $u \in L^2_{0,q}(X)$, $1 \leq q \leq n$. Let $M \gg 1$ and let $u_M := Q_{[-M,\lambda]}u$. By Proposition 4.8, we see that

$$u_M \in \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}^*_b \cap \mathscr{E}(\lambda, \sqrt{-1}T) \bigcap \mathrm{Dom}(\sqrt{-1}T).$$

From this observation and (4.1), we have

$$\begin{aligned} -\lambda\|u_M\|^2 &\leq (\,-\sqrt{-1}Tu_M \,|\, u_M\,) \leq \frac{1}{q}\Big(\|Q_{[-M,\lambda]}\overline{\partial}_b u\|^2 + \|Q_{[-M,\lambda]}\overline{\partial}^*_b u\|^2\Big) + C\|u_M\|^2 \\ &\leq \frac{1}{q}\Big(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}^*_b u\|^2\Big) + C\|u_M\|^2. \end{aligned} \tag{4.23}$$

Since $\lambda < -C$ we deduce from (4.23) that

$$\|u_M\|^2 \leq \frac{1}{q(-\lambda - C)}\Big(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}^*_b u\|^2\Big). \tag{4.24}$$

Letting $M \to \infty$ in (4.24) we get (4.22). □

For every $q = 0, 1, \ldots, n$, put $\mathscr{E}^{(q)}(\lambda, \sqrt{-1}T) := \mathscr{E}(\lambda, \sqrt{-1}T) \bigcap L^2_{0,q}(X)$. We prove now a vanishing theorem for harmonic forms which are eigenforms of $\sqrt{-1}T$.

Theorem 4.10 *Assume that* $2\sqrt{-1}\mathscr{L} = \Theta_X$, g_X *is complete and there is* $C > 0$ *such that*

$$\sqrt{-1}R^{K^*_X} \geq -C\Theta_X.$$

Let $q \in \{1, \ldots, n\}$. *Let* $\lambda \in \mathbb{R}$, $\lambda < -C$. *The operator*

$$\Box^{(q)}_{b,\lambda} : \mathrm{Dom}\,\Box^{(q)}_{b,\lambda} \subset \mathscr{E}^{(q)}(\lambda, \sqrt{-1}T) \to \mathscr{E}^{(q)}(\lambda, \sqrt{-1}T)$$

has closed range and $\ker \Box^{(q)}_{b,\lambda} = \{0\}$. *Hence, there is a bounded operator*

$$G^{(q)}_\lambda : \mathscr{E}^{(q)}(\lambda, \sqrt{-1}T) \to \mathrm{Dom}\,\Box^{(q)}_{b,\lambda}$$

such that

$$\Box^{(q)}_{b,\lambda}G^{(q)}_\lambda = I \ \ \text{on}\ \mathscr{E}^{(q)}(\lambda, \sqrt{-1}T). \tag{4.25}$$

Proof Let $u \in \operatorname{Dom} \Box^{(q)}_{b,\lambda}$. From (4.22), we have

$$\|u\|^2 \le \frac{1}{q(-\lambda - C)}\Big(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}^*_b u\|^2\Big) = \frac{1}{q(-\lambda - C)}(\,\Box^{(q)}_{b,\lambda} u \,|\, u\,).$$

Hence,

$$\|u\| \le \frac{1}{q(-\lambda - C)}\|\Box^{(q)}_{b,\lambda} u\|. \tag{4.26}$$

From (4.26), the theorem follows. □

We now consider (0, 1)-forms. Let $G^{(1)}_\lambda$ be as in (4.25). Since $G^{(1)}_\lambda$ is L^2 bounded, there is $C_0 > 0$ such that

$$\|G^{(1)}_\lambda v\| \le C_0\|v\|, \quad \text{for every } v \in \mathscr{E}^{(1)}(\lambda, \sqrt{-1}T). \tag{4.27}$$

We use the previous result to solve the $\overline{\partial}_b$-equation for eigenforms of $\sqrt{-1}T$.

Theorem 4.11 *Assume that $2\sqrt{-1}\mathscr{L} = \Theta_X$, g_X is complete and there is $C > 0$ such that*

$$\sqrt{-1}R^{K^*_X} \ge -C\Theta_X.$$

Let $\lambda \in \mathbb{R}$, $\lambda < -C$. For every $v \in \mathscr{E}^{(1)}(\lambda, \sqrt{-1}T)$ with $\overline{\partial}_b v = 0$, we can find $u \in \operatorname{Dom} \overline{\partial}_b \bigcap \mathscr{E}^{(0)}(\lambda, \sqrt{-1}T)$ such that

$$\begin{aligned} &\overline{\partial}_b u = v,\\ &\|u\|^2 \le C_0\|v\|^2, \end{aligned} \tag{4.28}$$

where $C_0 > 0$ is a constant as in (4.27).

Proof Let $v \in \mathscr{E}^{(1)}(\lambda, \sqrt{-1}T)$ with $\overline{\partial}_b v = 0$. From (4.25), we have

$$v = \overline{\partial}_b\, \overline{\partial}^*_b G^{(1)}_\lambda v + \overline{\partial}^*_b \overline{\partial}_b G^{(1)}_\lambda v. \tag{4.29}$$

Since $\overline{\partial}_b\Big(\overline{\partial}^*_b\overline{\partial}_b G^{(1)}_\lambda v\Big) = \overline{\partial}_b v - \overline{\partial}^2_b\, \overline{\partial}^*_b G^{(1)}_\lambda v = 0$, $\overline{\partial}^*_b\Big(\overline{\partial}^*_b\overline{\partial}_b G^{(1)}_\lambda v\Big) = 0$, $\overline{\partial}^*_b\overline{\partial}_b G^{(1)}_\lambda v \in \ker \Box^{(1)}_{b,\lambda}$. From Theorem 4.10, we see that $\overline{\partial}^*_b\overline{\partial}_b G^{(1)}_\lambda v = 0$. From this observation and (4.29), we get $v = \overline{\partial}_b u$, $u = \overline{\partial}^*_b G^{(1)}_\lambda v$. Now,

$$\begin{aligned} \|u\|^2 &= \|\overline{\partial}^*_b G^{(1)}_\lambda v\|^2 \le \|\overline{\partial}_b G^{(1)}_\lambda v\|^2 + \|\overline{\partial}^*_b G^{(1)}_\lambda v\|^2\\ &= (\,\Box^{(1)}_{b,\lambda} G^{(1)}_\lambda v \,|\, G^{(1)}_\lambda v\,) = (\,v \,|\, G^{(1)}_\lambda v\,) \le C_0\|v\|^2, \end{aligned}$$

where $C_0 > 0$ is as in (4.27). The theorem follows. □

Fix $q \in \{0, 1, \ldots, n\}$. Let $S^{(q)} : L^2_{0,q}(X) \to \operatorname{Ker} \Box_b^{(q)}$ be the orthogonal projection with respect to $(\cdot\,|\,\cdot)$. From Proposition 4.8, we can check that

$$\begin{aligned} Q_\lambda S^{(q)} &= S^{(q)} Q_\lambda \text{on } L^2_{0,q}(X), \\ Q_{[\lambda_1,\lambda]} S^{(q)} &= S^{(q)} Q_{[\lambda_1,\lambda]} \text{on } L^2_{0,q}(X), \\ Q_\tau S^{(q)} &= S^{(q)} Q_\tau \text{on } L^2_{0,q}(X). \end{aligned} \tag{4.30}$$

We recall the following notion introduced in [19, Definition 1.8].

Definition 4.12 Fix $q \in \{0, 1, 2, \ldots, n\}$. Let $Q : L^2_{0,q}(X) \to L^2_{0,q}(X)$ be a continuous operator. We say that $\Box_b^{(q)}$ has local L^2 closed range on an open set $D \subset X$ with respect to Q if for every $D' \Subset D$, there exist constants $C_{D'} > 0$ and $p \in \mathbb{N}$, such that

$$\|Q(I - S^{(q)})u\|^2 \leq C_{D'}\big((\Box_b^{(q)})^p u \,|\, u\big), \quad \text{for all } u \in \Omega_c^{0,q}(D').$$

We remind that we do not assume that $\Theta_X = 2\sqrt{-1}\mathscr{L}$. The Levi form $2\sqrt{-1}\mathscr{L}$ induces a Hermitian metric $\langle \cdot\,|\,\cdot \rangle_{\mathscr{L}}$ on $\mathbb{C}TX$ and $\langle \cdot\,|\,\cdot \rangle_{\mathscr{L}}$ induces a Hermitian metric $\langle \cdot\,|\,\cdot \rangle_{\mathscr{L}}$ on $T^{*\bullet,\bullet}X$. More precisely, if X is strictly pseudoconvex, i.e., $2\sqrt{-1}\mathscr{L} \in \Omega^{1,1}(X)$ is positive definite, then we can construct a Hermitian metric $\langle \cdot\,|\,\cdot \rangle_{\mathscr{L}}$ on $\mathbb{C}TX = T^{(1,0)}X \oplus T^{(0,1)}X \oplus \mathbb{C}\{T\}$ in the following way: For arbitrary $a, b \in T^{(1,0)}X$, $\langle a|b \rangle_{\mathscr{L}} := 2\mathscr{L}(a, \overline{b})$, $\langle \overline{a}|\overline{b} \rangle_{\mathscr{L}} := \langle b|a \rangle_{\mathscr{L}}$, $\langle a|\overline{b} \rangle_{\mathscr{L}} := 0$ and $\langle T|T \rangle_{\mathscr{L}} := 1$. We simply use $2\sqrt{-1}\mathscr{L}$ to represent $\langle \cdot\,|\,\cdot \rangle_{\mathscr{L}}$. Let $(\cdot\,|\,\cdot)_{\mathscr{L}}$ be the L^2 inner product on $\Omega_c^{\bullet,\bullet}(X)$ induced by $\langle \cdot\,|\,\cdot \rangle_{\mathscr{L}}$ and let $L^2_{\bullet,\bullet}(X, \mathscr{L})$ be the completion of $\Omega_c^{\bullet,\bullet}(X)$ with respect to $(\cdot\,|\,\cdot)_{\mathscr{L}}$. We write $L^2(X, \mathscr{L}) := L^2_{0,0}(X, \mathscr{L})$. For $f \in L^2_{\bullet,\bullet}(X, \mathscr{L})$, we write $\|f\|^2_{\mathscr{L}} := (\, f \,|\, f\,)_{\mathscr{L}}$.

Let $R^{K_X^*}_{\mathscr{L}}$ be the Chern curvature of K_X^* with respect to the Hermitian metric $\langle, \rangle_{\mathscr{L}}$ on X, see (2.26). Locally it can be represented by

$$R^{K_X^*}_{\mathscr{L}} = \overline{\partial}_b \partial_b \log \det \big(\langle Z_j | Z_k \rangle_{\mathscr{L}}\big)^n_{j,k=1}, \tag{4.31}$$

where $\{Z_j\}_{j=1}^n \subset T^{(1,0)}X$ is as in (2.15).

For $u \in \Omega_c^{\bullet,\bullet}(X)$, from Lemma 4.7 and (4.19), we see that $Q_\lambda u$, $Q_{[\lambda_1,\lambda]}u$, $Q_\tau u$ are independent of the choices of $\mathbb{R}$-invariant Hermitian metrics on X.

Theorem 4.13 *Assume that $g_{\mathscr{L}}$ is complete and there is $C > 0$ such that*

$$\sqrt{-1}R^{K_X^*}_{\mathscr{L}} \geq -2C\sqrt{-1}\mathscr{L}, \quad (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 \geq C\Theta_X^n \wedge \omega_0. \tag{4.32}$$

Let $D \Subset X$ be an open set. Let $\lambda \in \mathbb{R}$, $\lambda < -C$. Then, $\Box_b^{(0)}$ has local closed range on D with respect to Q_λ.

Proof Let $u \in \mathscr{C}_c^\infty(D)$. Let $v := \overline{\partial}_b Q_\lambda u = Q_\lambda \overline{\partial}_b u$. Since $\overline{\partial}_b u \in \Omega_c^{0,1}(D)$,

$$Q_\lambda \overline{\partial}_b u \in L^2_{\bullet,\bullet}(X, \mathscr{L}) \bigcap L^2_{\bullet,\bullet}(X).$$

From Theorem 4.11, there exists $g \in L^2(X, \mathscr{L})$ with

$$\|g\|^2_{\mathscr{L}} \leq C_0\|\overline{\partial}_b Q_\lambda u\|^2_{\mathscr{L}} \leq C_0\|\overline{\partial}_b u\|^2_{\mathscr{L}} \tag{4.33}$$

such that

$$\overline{\partial}_b g = \overline{\partial}_b Q_\lambda u, \tag{4.34}$$

where $C_0 > 0$ is a constant as in (4.28). Since $\overline{\partial}_b(I - S^{(0)})Q_\lambda u = \overline{\partial}_b Q_\lambda u$ and $(I - S^{(0)})Q_\lambda u \perp \ker \overline{\partial}_b$, we have

$$\|(I - S^{(0)})Q_\lambda u\|^2 \leq \|g\|^2 \leq \frac{1}{C}\|g\|^2_{\mathscr{L}}, \tag{4.35}$$

where $C > 0$ is a constant as in (4.32). From (4.32), (4.33), and (4.35), we have

$$\|Q_\lambda(I - S^{(0)})u\|^2 = \|(I - S^{(0)})Q_\lambda u\|^2 \leq \frac{1}{C}\|g\|^2_{\mathscr{L}} \leq \frac{C_0}{C}\|\overline{\partial}_b u\|^2_{\mathscr{L}}. \tag{4.36}$$

Since $\overline{\partial}_b u$ has compact support in D, there exists $C_1 > 0$ independent of u such that

$$\|\overline{\partial}_b u\|^2_{\mathscr{L}} \leq C_1\|\overline{\partial}_b u\|^2. \tag{4.37}$$

(4.36) and (4.37), the theorem follows. □

For $\lambda \in \mathbb{R}$, $\lambda \leq 0$, let $\tau_\lambda \in \mathscr{C}^\infty(\mathbb{R}, [0, 1])$, $\tau_\lambda = 1$ on $]-\infty, 2\lambda]$, $\tau_\lambda = 0$ outside $(-\infty, \lambda]$. It is clear that $\|Q_{\tau_\lambda}(I - S^{(0)})u\| \leq \|Q_\lambda(I - S^{(0)})u\|$, for every $u \in L^2(X)$. From this observation and Theorem 4.13, we deduce that

Theorem 4.14 *Assume that $g_{\mathscr{L}}$ is complete and there is $C > 0$ such that*

$$\sqrt{-1}R^{K^*_X}_{\mathscr{L}} \geq -2C\sqrt{-1}\mathscr{L}, \ \ (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 \geq C\Theta^n_X \wedge \omega_0. \tag{4.38}$$

Let $D \Subset X$ be an open set. Let $\lambda \in \mathbb{R}$, $\lambda < -C$. Then, $\Box^{(0)}_b$ has local closed range on D with respect to Q_{τ_λ}.

4.4 Local Closed Range for $\Box^{(n,0)}_b$

In this section, we will establish the local closed range property for $\Box^{(n,0)}_b$ under appropriate curvature assumptions. We observe that the condition (4.32) can be removed, if we consider $(n, 0)$-forms instead of smooth function. We will adopt the same notation as before.

Let $\Box_b$ be the Gaffney extension of the usual Kohn Laplacian. Let $\Box^{(n,q)}_b$ be the restriction of $\Box_b$ acting on (n, q)-forms. Set

$$\mathscr{E}^{(n,q)}(\lambda, \sqrt{-1}T) := \mathscr{E}(\lambda, \sqrt{-1}T) \cap L^2_{n,q}(X), \quad \Box^{(n,q)}_{b,\lambda} := \Box^{(n,q)}_b|_{\mathscr{E}(\lambda,\sqrt{-1}T)\cap \mathrm{Dom}\,\Box^{(n,q)}_b}. \tag{4.39}$$

Let $S^{(n,q)} : L^2_{n,q}(X) \to \mathrm{Ker}\,\Box_b^{(n,q)}$ be orthogonal projection. It is known that Q_τ, Q_λ, $Q_{[\lambda_1,\lambda]}$ commutes with $S^{(n,q)}$ on $L^2_{n,q}(X)$. Now, we present the main result of this section as follows:

Theorem 4.15 *Let X be a CR manifold with a transversal CR $\mathbb{R}$-action. Let Θ_X be an $\mathbb{R}$-invariant Hermitian metric on X. Assume that $g_{\mathscr{L}}$ is complete. Let $D \Subset X$ be an open set. Let $\lambda \in \mathbb{R}$, $\lambda < 0$. Then, $\Box_b^{(n,0)}$ has local closed range on D with respect to Q_λ, i.e., there exists $C > 0$ such that for all $u \in \Omega_c^{n,0}(D)$,*

$$\|Q_\lambda(I - S^{(n,0)})u\|^2 \leq C\|\overline{\partial}_b u\|^2. \tag{4.40}$$

This result is very natural in view of the Kodaira vanishing theorem, in the same way as Theorem 4.13 is parallel to the Kodaira–Serre type vanishing theorem. The proof is analog to the proof of Theorem 4.13.

Firstly, from Corollary 3.6 and the density Lemma 4.1, we obtain the following:

Lemma 4.16 *With the notations used above, let Θ_X be a Hermitian metric on X such that*

$$2\sqrt{-1}\mathscr{L} = \Theta_X. \tag{4.41}$$

Then for any $u \in L^2_{n,q}(X) \cap \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}_b^* \cap \mathrm{Dom}(\sqrt{-1}T)$, $1 \leq q \leq n$, we have

$$\left(-\sqrt{-1}Tu|u\right) \leq \frac{1}{q}\left(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}_b^* u\|^2\right). \tag{4.42}$$

Lemma 4.17 *Assume that $2\sqrt{-1}\mathscr{L} = \Theta_X$ and g_X is complete. Then, for any $u \in L^2_{n,q}(X) \cap \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}_b^* \cap \mathscr{E}(\lambda, \sqrt{-1}T)$, $\lambda < 0$, and $1 \leq q \leq n$, we have*

$$\|u\|^2 \leq \frac{1}{q(-\lambda)}\left(\|\overline{\partial}_b u\|^2 + \|\overline{\partial}_b^* u\|^2\right). \tag{4.43}$$

Proof Let $u \in L^2_{n,q}(X) \cap \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}_b^* \cap \mathscr{E}(\lambda, \sqrt{-1}T)$, $\lambda < 0$, and $1 \leq q \leq n$. Let $M \gg 1$ be a sufficiently large positive real number such that $-M < \lambda$ and $u_M := Q_{[-M,\lambda]}u$. By Proposition 4.8, we see that

$$u_M \in L^2_{n,q}(X) \cap \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}_b^* \cap \mathscr{E}(\lambda, \sqrt{-1}T) \cap \mathrm{Dom}(\sqrt{-1}T). \tag{4.44}$$

Then

$$\begin{aligned} -\lambda\|u_M\|^2 \leq \left(-\sqrt{-1}Tu_M|u_M\right) &\leq \frac{1}{q}\left(\|\overline{\partial}_b u_M\|^2 + \|\overline{\partial}_b^* u_M\|^2\right) \\ &\leq \frac{1}{q}\left(\|Q_{[-M,\lambda]}\overline{\partial}_b u\|^2 + \|Q_{[-M,\lambda]}\overline{\partial}_b^* u\|^2\right). \end{aligned} \tag{4.45}$$

By letting $M \to +\infty$ we complete the proof. □

Moreover, we have the following analog of Theorem 4.10.

Theorem 4.18 *Let X be a CR manifold with a transversal CR $\mathbb{R}$-action. Let Θ_X be an $\mathbb{R}$-invariant Hermitian metric on X. Assume that $2\sqrt{-1}\mathscr{L} = \Theta_X$ and g_X is complete. Let $1 \leq q \leq n$ and $\lambda < 0$. Then the operator*

$$\Box_{b,\lambda}^{(n,q)} : \mathrm{Dom}\,\Box_{b,\lambda}^{(n,q)} \subset \mathscr{E}^{(n,q)}(\lambda, \sqrt{-1}T) \to \mathscr{E}^{(n,q)}(\lambda, \sqrt{-1}T) \tag{4.46}$$

has closed range, and

$$\mathrm{Ker}\,\Box_{b,\lambda}^{(n,q)} = \{0\}. \tag{4.47}$$

Hence, there exists a bounded operator

$$G_\lambda^{(n,q)} : \mathscr{E}^{(n,q)}(\lambda, \sqrt{-1}T) \to \mathrm{Dom}\,\Box_{b,\lambda}^{(n,q)} \tag{4.48}$$

such that

$$\Box_{b,\lambda}^{(n,q)} G_\lambda^{(n,q)} = I \quad on\ \mathscr{E}^{(n,q)}(\lambda, \sqrt{-1}T). \tag{4.49}$$

Therefore, we have $C_0 > 0$ such that, for all $v \in \mathscr{E}^{(n,1)}(\lambda, \sqrt{-1}T)$,

$$\|G_\lambda^{(n,1)} v\| \leq C_0 \|v\|. \tag{4.50}$$

Secondly, we solve the $\overline{\partial}_b$-equation as follows.

Theorem 4.19 *Let X be a CR manifold with a transversal CR $\mathbb{R}$-action. Let Θ_X be an $\mathbb{R}$-invariant Hermitian metric on X. Assume that $2\sqrt{-1}\mathscr{L} = \Theta_X$ and g_X is complete. Then for every $\lambda < 0$ and every $v \in \mathscr{E}^{(n,1)}(\lambda, \sqrt{-1}T)$ with $\overline{\partial}_b v = 0$, there exists $u \in \mathrm{Dom}\,\overline{\partial}_b \cap \mathscr{E}^{(n,0)}(\lambda, \sqrt{-1}T)$ such that*

$$\overline{\partial}_b u = v, \quad \|u\|^2 \leq C_0 \|v\|^2. \tag{4.51}$$

Proof Let $\lambda < 0$. Let $v \in \mathscr{E}^{(n,1)}(\lambda, \sqrt{-1}T)$ with $\overline{\partial}_b v = 0$. We have

$$v = \Box_{b,\lambda}^{(n,1)} G_\lambda^{(n,1)} v = \overline{\partial}_b\, \overline{\partial}_b^* G_\lambda^{(n,1)} v + \overline{\partial}_b^* \overline{\partial}_b G_\lambda^{(n,1)} v. \tag{4.52}$$

Since $\overline{\partial}_b \Big(\overline{\partial}_b^* \overline{\partial}_b G_\lambda^{(n,1)} v \Big) = \overline{\partial}_b v - \overline{\partial}_b^2\, \overline{\partial}_b^* G_\lambda^{(n,1)} v = 0$, $\overline{\partial}_b^* \Big(\overline{\partial}_b^* \overline{\partial}_b G_\lambda^{(n,1)} v \Big) = 0$,

$$\overline{\partial}_b^* \overline{\partial}_b G_\lambda^{(n,1)} v \in \mathrm{Ker}\,\Box_{b,\lambda}^{(n,1)}. \tag{4.53}$$

Thus, we see that $\overline{\partial}_b^* \overline{\partial}_b G_\lambda^{(n,1)} v = 0$ and $v = \overline{\partial}_b u$ with $u = \overline{\partial}_b^* G_\lambda^{(n,1)} v$. Thus obtain

$$\begin{aligned} \|u\|^2 &= \|\overline{\partial}_b^* G_\lambda^{(n,1)} v\|^2 \leq \|\overline{\partial}_b G_\lambda^{(n,1)} v\|^2 + \|\overline{\partial}_b^* G_\lambda^{(n,1)} v\|^2 \\ &= (\, \Box_{b,\lambda}^{(n,1)} G_\lambda^{(n,1)} v \,|\, G_\lambda^{(n,1)} v \,) = (\, v \,|\, G_\lambda^{(n,1)} v \,) \\ &\leq C_0 \|v\|^2. \end{aligned} \tag{4.54}$$

The proof is complete. □

Proof of Theorem 4.15 Note that Θ_X is not necessarily equal to $2\sqrt{-1}\mathscr{L}$ so we have to deduce the general case to the particular case considered in Theorems 4.16–4.19. Let $\lambda < 0$. Let $v := \overline{\partial}_b Q_\lambda u = Q_\lambda \overline{\partial}_b u$. Note that Q_λ is independent of the choice of Θ_X. Then $v \in L^2_{n,1}(X, \mathscr{L}) \cap L^2_{n,1}(X)$. From the above theorem, we can find $g \in L^2_{n,0}(X, \mathscr{L})$ such that $\overline{\partial}_b g = v$ and

$$\|g\|^2_{\mathscr{L}} \le C_0 \|v\|^2_{\mathscr{L}} = C_0 \|Q_\lambda \overline{\partial}_b u\|^2_{\mathscr{L}} \le C_0 \|\overline{\partial}_b u\|^2_{\mathscr{L}}. \tag{4.55}$$

where the first inequality in (4.55) follows from Theorem 4.19. We claim that

$$\|g\| = \|g\|_{\mathscr{L}}$$

and thus $g \in L^2_{n,0}(X, \mathscr{L}) \cap L^2_{n,0}(X)$. In fact, we write locally

$$g = \alpha dz_1 \wedge \cdots \wedge dz_n. \tag{4.56}$$

With respect to $\Theta_X = \sqrt{-1}\langle Z_i | Z_j \rangle dz_i \wedge dz_j$ and $2\sqrt{-1}\mathscr{L} = \sqrt{-1}\langle Z_i | Z_j \rangle_{\mathscr{L}} dz_i \wedge dz_j$,

$$\begin{aligned} |g|^2 &= |\alpha|^2 \det(\langle Z_i | Z_j \rangle)^{-1}, \\ |g|^2_{\mathscr{L}} &= |\alpha|^2 \det(\langle Z_i | Z_j \rangle_{\mathscr{L}})^{-1}, \end{aligned} \tag{4.57}$$

and, respectively, the volume forms are given by

$$\begin{aligned} \Theta_X^n \wedge \omega_0 &= n!(\sqrt{-1})^n \det(\langle Z_i | Z_j \rangle) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n \wedge \omega_0, \\ (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 &= n!(\sqrt{-1})^n \det(\langle Z_i | Z_j \rangle_{\mathscr{L}}) dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n \wedge \omega_0. \end{aligned} \tag{4.58}$$

Thus, the claim follows from

$$\|g\|^2 = \int_X |g|^2 \Theta_X^n \wedge \omega_0 = \int_X |g|^2_{\mathscr{L}} (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 = \|g\|^2_{\mathscr{L}}. \tag{4.59}$$

Since $\overline{\partial}_b (I - S^{(n,0)}) Q_\lambda u = \overline{\partial}_b Q_\lambda u = v$ and $(I - S^{(n,0)}) Q_\lambda u \perp \operatorname{Ker} \overline{\partial}_b$, we have $\overline{\partial}_b (I - S^{(n,0)}) Q_\lambda u$ is the solution of minimal norm with respect to Θ_X, i.e.,

$$\|\overline{\partial}_b (I - S^{(n,0)}) Q_\lambda u\|^2 \le \|g\|^2 = \|g\|^2_{\mathscr{L}} \le C_0 \|\overline{\partial}_b u\|^2_{\mathscr{L}} \le C_0 C_1 \|\overline{\partial}_b u\|^2 \tag{4.60}$$

by $\operatorname{supp}(u) \Subset D$. □

4.5 L^2 Estimates for $\overline{\partial}_{b,E}$

In this section, we prove an analog for the $\overline{\partial}_{b,E}$-operator of the L^2-estimates of the Hörmander–Andreotti–Vesentini estimates for $\overline{\partial}$. As in the case of complex manifolds, we use the Bochner–Kodaira–Nakano formula in the present form (3.36). In order to eliminate the first-order error term $[(\nabla^E)^{1,0}, \mathcal{T}^*] - [\overline{\partial}_{b,E}, \overline{\mathcal{T}}^*]$ in (3.36), we reformulate (3.36) can be reformulated as in [10, VII.1]. Under the hypothesis of Theorem 3.7, we have on $\Omega^{\bullet,\bullet}(X, E)$,

$$\Box_b^E = \overline{\Box}_{b,\mathcal{T}}^E + [2\mathscr{L}T + \sqrt{-1}R^E, \Lambda] + \mathcal{T}_\Theta, \tag{4.61}$$

where $\overline{\Box}_{b,\mathcal{T}}^E := [(\nabla^E)^{1,0} + \mathcal{T}, (\nabla^E)^{1,0*} + \mathcal{T}^*]$ is a positive formally self-adjoint operator, and

$$\mathcal{T}_\Theta := [\Lambda, [\Lambda, \frac{\sqrt{-1}}{2}\partial_b\overline{\partial}_b\Theta_X]] - [\overline{\partial}_b\Theta_X, (\overline{\partial}_b\Theta_X)^*] \tag{4.62}$$

is an operator of order zero depending only on the torsion of Hermitian metric Θ_X.

Theorem 4.20 (L^2-estimates for $\overline{\partial}_b$) *Let X be a CR manifold with a smooth locally free CR $\mathbb{R}$-action. Let Θ_X be an $\mathbb{R}$-invariant Hermitian metric on X. Assume g_X is complete. Let E be a $\mathbb{R}$-equivariant CR line bundle over X with a $\mathbb{R}$-invariant Hermitian metric h^E. Assume that for some (r, q), $q \geq 1$, there exists a function $\psi : X \to [0, \infty)$ such that, for all $s \in \Omega_c^{r,q}(X, E)$, pointwisely*

$$\left\langle [2\mathscr{L}T + \sqrt{-1}R^E, \Lambda] + \mathcal{T}_\Theta s | s \right\rangle \geq \psi |s|^2. \tag{4.63}$$

Then, for any $f \in L^2_{r,q}(X, E)$ satisfying $\overline{\partial}_{b,E} f = 0$ and $\int_X \psi^{-1}|f|^2 dv_X < \infty$, there exists $g \in L^2_{r,q-1}(X, E)$ such that $\overline{\partial}_{b,E} g = f$ and $\|g\|^2 \leq \int_X \psi^{-1}|f|^2 dv_X$.

Proof Consider the complex of closed densely defined operators

$$L^2_{r,q-1}(X, E) \xrightarrow{T=\overline{\partial}_{b,E}} L^2_{r,q}(X, E) \xrightarrow{S=\overline{\partial}_{b,E}} L^2_{r,q+1}(X, E), \tag{4.64}$$

where T and S are maximal extensions of $\overline{\partial}_{b,E}$. We apply (4.61) and obtain for all $s \in \Omega_c^{r,q}(X, E)$, it follows that

$$\|\overline{\partial}_{b,E} s\|^2 + \|\overline{\partial}^*_{b,E} s\|^2 \geq \left([2\mathscr{L}T + \sqrt{-1}R^E, \Lambda] + \mathcal{T}_\Theta s | s \right) \geq \int_X \psi |s|^2 dv_X. \tag{4.65}$$

By Cauchy–Schwarz inequality,

$$|(f|s)|^2 = |(\psi^{-1/2} f | \psi^{1/2} s)|^2 \leq \int_X \psi^{-1}|f|^2 dv_X \cdot \left(\|\overline{\partial}_{b,E} s\|^2 + \|\overline{\partial}^*_{b,E}\|^2 \right) \tag{4.66}$$

Since g_X is complete, the above inequality still holds for all $s \in \mathrm{Dom}(S) \cap \mathrm{Dom}(T^*)$ by the density Lemma 4.1. Consider now $s \in \mathrm{Dom}(T^*)$ and write the orthogonal decomposition $s = s_1 + s_2$ with $s_1 \in \mathrm{Ker}(S)$ and $s_2 \in \mathrm{Ker}(S)^\perp \subset [\mathrm{Im}(S^*)] \subset \mathrm{Ker}\, T^*$.

So $s_1 = s - s_2 \in \mathrm{Ker}(S) \cap \mathrm{Dom}(T^*)$. Recall $f \in \mathrm{Ker}(S)$,

$$|(f|s)|^2 = |(f|s_1)|^2 \le \int_X \psi^{-1}|f|^2 dv_X \cdot \|T^*s_1\|^2 = \int_X \psi^{-1}|f|^2 dv_X \cdot \|T^*s\|^2. \tag{4.67}$$

We consider $\lambda : \mathrm{Im}(T^*) \to \mathbb{C}$ defined by $\lambda(T^*s) = (f|s)$ for $s \in \mathrm{Dom}(T^*)$. We see that λ is $\mathbb{C}$-antilinear map and $|\lambda(T^*s)| \le (\int_X \psi^{-1}|f|^2 dv_X)^{1/2}\|T^*s\|$, i.e., λ is bounded with norm $\|\lambda\| \le (\int_X \psi^{-1}|f|^2 dv_X)^{1/2}$. By the complex Hahn–Banach theorem we can extend λ to $L^2_{r,q-1}(X, E)$ with the same norm $\|\lambda\| \le (\int_X \psi^{-1}|f|^2 dv_X)^{1/2}$.

By the Riesz representation theorem, there exists $g \in L^2_{r,q-1}(X, E)$ such that $\lambda(\alpha) = (g|\alpha)$ for $\alpha \in L^2_{r,q-1}(X, E)$ with $\|g\|^2 = \|\lambda\|^2 \le \int_X \psi^{-1}|f|^2 dv_X$. We set $\alpha = T^*s$, $s \in \mathrm{Dom}(T^*)$. Thus $(g|T^*s) = \lambda(T^*s) = (f|s)$ and $g \in \mathrm{Dom}(T^{**}) = \mathrm{Dom}(T)$ satisfying $Tg = f$. □

For certain complete CR manifold endowed with a Nakano q-semipositive line bundle, the L^2 method applies to solve the $\overline{\partial}_{b,E}$-equation for (n, q)-forms as follows. For the cohomology aspect of Nakano q-semipositive line bundles on complex manifolds, see [30].

Corollary 4.21 *Let X be a CR manifold with a smooth locally free CR $\mathbb{R}$-action. Let Θ_X be an $\mathbb{R}$-invariant Hermitian metric on X. Assume g_X is complete. Assume $\mathscr{L} = 0$ and $d\Theta_X = 0$. Let E be a $\mathbb{R}$-equivariant CR line bundle over X with a $\mathbb{R}$-invariant Hermitian metric h^E. Let $\lambda_1 \le \cdots \le \lambda_n$ be eigenvalues of R^E with respect to Θ_X. Assume (E, h^E) is Nakano q-semipositive with respect to Θ_X, i.e., $\lambda_1 + \cdots + \lambda_q \ge 0$. Then, for any $f \in L^2_{n,q}(X, E)$ satisfying $\overline{\partial}_{b,E} f = 0$ and $\int_X (\lambda_1 + \cdots + \lambda_q)^{-1}|f|^2 dv_X < \infty$, there exists $g \in L^2_{n,q-1}(X, E)$ such that $\overline{\partial}_{b,E} g = f$ and $\|g\|^2 \le \int_X (\lambda_1 + \cdots + \lambda_q)^{-1}|f|^2 dv_X$.*

4.6 Vanishing Theorems

In this section, we present some vanishing theorems that follow from the previous L^2 estimates. We obtain first a CR counterpart of the Kodaira vanishing theorem [23, Theorem 1.5.4.(a)] as follows:

Corollary 4.22 *Assume that $2\sqrt{-1}\mathscr{L} = \Theta_X$, g_X is complete and let $\lambda < 0$ and $1 \le q \le n$. Then, we have*

$$\mathrm{Ker}\,\Box_b \cap \mathscr{E}(\lambda, \sqrt{-1}T) \cap L^2_{n,q}(X) = 0. \tag{4.68}$$

This follows immediately from Theorem 4.18.

We obtain a CR counterpart of the Kodaira–Serre vanishing theorem [23, Theorem 1.5.6] as follows:

Corollary 4.23 *Assume that $2\sqrt{-1}\mathscr{L} = \Theta_X$, g_X is complete and let $C > 0$ such that*

$$\sqrt{-1}R^{K^*_X} \ge -C\Theta_X. \tag{4.69}$$

Let $\lambda < -C$ and $1 \leq q \leq n$. Then, we have

$$\operatorname{Ker} \Box_b \cap \mathscr{E}(\lambda, \sqrt{-1}T) \cap L^2_{0,q}(X) = 0. \tag{4.70}$$

This follows immediately from Theorem 4.10. We note that the previous vanishing theorems on CR manifolds imply the following generalizations due to Andreotti–Vesentini [2] of the Kodaira–Serre and Kodaira vanishing theorems for complete Kähler manifolds.

Corollary 4.24 (Andreotti–Vesentini) *Let (M, ω) be a complete Kähler manifold of dimension n and let $(L, h^L) \to M$ be a Hermitian holomorphic line bundle such that $\sqrt{-1}R^L = \omega$ and there is $C > 0$ such that $\sqrt{-1}R^{K^*_M}_\omega \geq -C\omega$ on M. Then there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ we have $H^q_{(2)}(M, L^m) = 0$ for $q \geq 1$, where $H^q_{(2)}(M, L^m)$ denotes the L^2 q-th Dolbeault cohomology group with respect to the metric $(h^L)^m$ and volume form $\omega^n/n!$ with values in L^m.*

Proof We apply the previous results for the CR manifold X constructed in Example 4.6. In this case $T = \partial_\theta$. For $m \in \mathbb{Z}$, the space $L^2_{0,q}(M, L^m)$ is isometric to the space of m-equivariant L^2 forms on X, $L^2_{0,q}(X)_m = \{u \in L^2_{0,q}(X) : (e^{i\theta})^* u = e^{im\theta}u, \text{for any } e^{i\theta} \in S^1\}$. Note that $L^2_{0,q}(X)_m = \mathscr{E}^{(q)}(-m, \sqrt{-1}\partial_\theta)$ and the L^2-Dolbeault complex $(L^2_{0,\bullet}(M, L^m), \overline{\partial})$ is isomorphic to the $\overline{\partial}_b$-complex $(L^2_{0,\bullet}(X)_m, \overline{\partial}_b)$. Hence, the assertion follows from Theorem 4.10. □

In the same vein recover from Theorem 4.18 the following vanishing theorem for the L^2-cohomology of positive bundles twisted with the canonical bundle on complete Kähler manifolds.

Corollary 4.25 (Andreotti–Vesentini) *Let (M, ω) be a complete Kähler manifold of dimension n and let $(L, h^L) \to M$ be a Hermitian holomorphic line bundle such that $\sqrt{-1}R^L = \omega$. Then $H^{n,q}_{(2)}(M, L) = 0$ for $q \geq 1$, where $H^{n,q}_{(2)}(M, L)$ denotes the L^2 q-th Dolbeault cohomology group with respect to the metric h^L and volume form $\omega^n/n!$ with values in $K_X \otimes L$.*

4.7 Szegö Kernel Asymptotic Expansions

In this section, we prove Theorem 1.2 and Corollary 1.4. We first introduced some notations. Let $D \subset X$ be an open coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. Let $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$. Let $S^m_{\rho,\delta}(T^*D)$ denote the Hörmander symbol space on T^*D of order m type (ρ, δ) and let $S^m_{\mathrm{cl}}(T^*D)$ denote the space of classical symbols on T^*D of order m, see Grigis–Sjöstrand [11, Definition 1.1 and p. 35] and Definition 2.1. Let $L^m_{\rho,\delta}(D)$ and $L^m_{\mathrm{cl}}(D)$ denote the space of pseudodifferential operators on D of order m type (ρ, δ) and the space of classical pseudodifferential operators on D of order m, respectively.

Let Σ be the characteristic manifold of $\Box_b$. We have

$$\begin{aligned}\Sigma &= \Sigma^- \cup \Sigma^+,\\ \Sigma^- &= \{(x, -c\omega_0(x)) \in T^*X;\ c < 0\},\\ \Sigma^+ &= \{(x, -c\omega_0(x)) \in T^*X;\ c > 0\}.\end{aligned} \tag{4.71}$$

We recall the following definition introduced in [19, Definition 2.4].

Definition 4.26 Let $Q : L^2(X) \to L^2(X)$ be a continuous operator. Let $D \Subset X$ be an open local coordinate patch of X with local coordinates $x = (x_1, \ldots, x_{2n+1})$ and let $\eta = (\eta_1, \ldots, \eta_{2n+1})$ be the dual variables of x. We write

$$Q \equiv 0 \text{at}\, \Sigma^- \cap T^*D\,,$$

if for every $D' \Subset D$,

$$Q(x, y) \equiv \int e^{i\langle x-y,\eta\rangle} q(x, \eta) d\eta \ \text{ on } D',$$

where $q(x, \eta) \in S^0_{1,0}(T^*D')$ and there exist $M > 0$ and a conic open neighborhood Λ_- of Σ^- such that for every $(x, \eta) \in T^*D' \cap \Lambda_-$ with $|\eta| \geq M$, we have $q(x, \eta) = 0$.

For a given point $x_0 \in D$, let $\{W_j\}_{j=1}^n$ be an orthonormal frame of $T^{(1,0)}X$ with respect to $\langle \cdot | \cdot \rangle$ near x_0, for which the Levi form is diagonal at x_0. Put

$$\mathscr{L}_{x_0}(W_j, \overline{W}_\ell) = \mu_j(x_0)\delta_{j\ell}\,, \quad j, \ell = 1, \ldots, n\,. \tag{4.72}$$

We will denote by

$$\det \mathscr{L}_{x_0} = \prod_{j=1}^{n} \mu_j(x_0)\,. \tag{4.73}$$

We recall the following results in [19, Theorems 1.9, 5.1].

Theorem 4.27 *Let $D \Subset X$ be an open coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. Let $Q : L^2(X) \to L^2(X)$ be a continuous operator and let Q^* be the L^2 adjoint of Q with respect to $(\cdot | \cdot)$. Suppose that $\Box_b^{(0)}$ has local L^2 closed range on D with respect to Q and $QS^{(0)} = S^{(0)}Q$ on $L^2(X)$ and*

$$Q - Q_0 \equiv 0 \quad \text{at}\, \Sigma^- \cap T^*D,$$

where $Q_0 \in L^0_{\mathrm{cl}}(D)$. Then,

$$(Q^*S^{(0)}Q)(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)t} a(x, y, t) dt \ \ \text{on}\ D, \tag{4.74}$$

where

$$\begin{aligned}&\varphi\in\mathscr{C}^\infty(D\times D),\ \ \operatorname{Im}\varphi(x,y)\geq 0,\\&\varphi(x,x)=0,\ \ \varphi(x,y)\neq 0\ \ \textit{if}\ \ x\neq y,\\&d_x\varphi(x,y)\big|_{x=y}=\omega_0(x),\ \ d_y\varphi(x,y)\big|_{x=y}=-\omega_0(x),\\&\varphi(x,y)=-\overline{\varphi}(y,x),\end{aligned}\tag{4.75}$$

$a(x,y,t)\in S^n_{\rm cl}\left(D\times D\times\mathbb{R}_+\right)$ *and the leading term* $a_0(x,y)$ *of the expansion* (2.4) *of* $a(x,y,t)$ *satisfies*

$$a_0(x,x)=\frac{1}{2}\pi^{-n-1}|\det\mathcal{L}_x|\overline{q(x,\omega_0(x))}q(x,\omega_0(x)),\ \ \textit{for all}\ x\in D,\tag{4.76}$$

where $\det\mathcal{L}_x$ *is the determinant of the Levi form defined in* (4.73)*,* $q(x,\eta)\in\mathscr{C}^\infty(T^*D)$ *is the principal symbol of* Q*.*

We refer the reader to [19, Theorems 3.3, 4.4] for more properties for the phase φ in (4.75). Let $D=U\times\mathcal{I}$ be a BRT chart with BRT coordinates $x=(x_1,\ldots,x_{2n+1})$. For $\lambda\in\mathbb{R}$, put

$$\hat{Q}_{\tau_\lambda}:=(2\pi)^{-(2n+1)}\int e^{i\langle x-y,\eta\rangle}\tau_\lambda(-\eta_{2n+1})d\eta\in L^0_{1,0}(D),\tag{4.77}$$

where $\tau_\lambda\in\mathscr{C}^\infty(\mathbb{R})$ is as in the discussion before Theorem 4.14. It is not difficult to see that

$$\hat{Q}_{\tau_\lambda}-I\equiv 0\ \text{ at }\Sigma^-\bigcap T^*D.\tag{4.78}$$

Assume that the $\mathbb{R}$-action is free. From (4.11), we see that $Q_{\tau_\lambda}=\hat{Q}_{\tau_\lambda}$ on D. From this observation, Theorems 4.14, 4.27, (4.78), and noticing that $Q^*_{\tau_\lambda}S^{(0)}Q_{\tau_\lambda}=Q_{\tau^2_\lambda}S^{(0)}$, where $Q^*_{\tau_\lambda}$ is the L^2 adjoint of Q_{τ_λ} with respect to $(\cdot\,|\,\cdot)$, we get

Theorem 4.28 *Suppose that the* $\mathbb{R}$*-action is free. Assume that* $g_{\mathscr{L}}$ *is complete and there is* $C>0$ *such that*

$$\sqrt{-1}R^{K^*_X}_{\mathscr{L}}\geq -2C\sqrt{-1}\mathscr{L},\ \ (2\sqrt{-1}\mathscr{L})^n\wedge\omega_0\geq C\Theta^n_X\wedge\omega_0.$$

Let $D=U\times\mathcal{I}\Subset X$ *be a BRT chart with BRT coordinates* $x=(x_1,\ldots,x_{2n+1})$*. Let* $\lambda\in\mathbb{R}$*,* $\lambda<-C$*. Then,*

$$(Q_{\tau^2_\lambda}S^{(0)})(x,y)\equiv\int_0^\infty e^{i\varphi(x,y)t}s(x,y,t)dt\ \ \textit{on}\ D,\tag{4.79}$$

where $\varphi\in\mathscr{C}^\infty(D\times D)$ *is as in* (4.74)*,* $s(x,y,t)\in S^n_{\rm cl}\left(D\times D\times\mathbb{R}_+\right)$ *and the leading term* $s_0(x,y)$ *of the expansion* (2.4) *of* $s(x,y,t)$ *satisfies*

$$s_0(x,x)=\frac{1}{2}\pi^{-n-1}|\det\mathcal{L}_x|,\ \ \textit{for all}\ x\in D.\tag{4.80}$$

We now assume that the $\mathbb{R}$-action is not free. From (4.8), we know that the $\mathbb{R}$-action comes from a CR torus action $\mathbb{T}^d = (e^{i\theta_1}, \ldots, e^{i\theta_d})$ on X and ω_0, Θ_X are $\mathbb{T}^d$ invariant. We will use the same notations as in the discussion before Proposition 4.8. We need

Lemma 4.29 *Suppose that the $\mathbb{R}$-action is not free. With the notations and assumptions used above, let $D = U \times \mathcal{I} \Subset X$ be a BRT chart with BRT coordinates $x = (x', x_{2n+1})$, $x' = (x_1, \ldots, x_{2n})$. Fix $D_0 \Subset D$ and $\lambda \in \mathbb{R}$. For $u \in \mathscr{C}_c^\infty(D_0)$, we have*

$$\begin{aligned}
&Q_{\tau_\lambda} u = \hat{Q}_{\tau_\lambda} u + \hat{R}_{\tau_\lambda} u \quad \text{on } D_0,\\
&(\hat{R}_{\tau_\lambda} u)(x) = \frac{1}{2\pi} \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i\langle x_{2n+1}-y_{2n+1},\eta_{2n+1}\rangle + i(\sum_{j=1}^d m_j\beta_j)y_{2n+1} - im_1\theta_1 - \ldots - im_d\theta_d}\\
&\times \tau_\lambda(-\eta_{2n+1})(1-\chi(y_{2n+1}))u((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x')d\mathbb{T}_d d\eta_{2n+1} dy_{2n+1} \quad \text{on } D_0,
\end{aligned} \tag{4.81}$$

where $\chi \in \mathscr{C}_c^\infty(I)$, $\chi(x_{2n+1}) = 1$ for every $(x', x_{2n+1}) \in D_0$ and $\beta_1 \in \mathbb{R}, \ldots, \beta_d \in \mathbb{R}$ are as in (4.17).

Proof We also write $y = (y', y_{2n+1}) = (y_1, \ldots, y_{2n+1})$, $y' = (y_1, \ldots, y_{2n})$, to denote the BRT coordinates x. Let $u \in \mathscr{C}_c^\infty(D_0)$. From (4.19), it is easy to see that on D,

$$\begin{aligned}
Q_{\tau_\lambda} u(y) = &\sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \tau_\lambda\Big(-\sum_{j=1}^d m_j\beta_j\Big) e^{i(\sum_{j=1}^d m_j\beta_j)y_{2n+1}} \times\\
&\int_{\mathbb{T}^d} e^{-(im_1\theta_1+\ldots+im_d\theta_d)} u((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ y')dT_d.
\end{aligned} \tag{4.82}$$

Now, we claim that

$$\hat{Q}_{\tau_\lambda} + \hat{R}_{\tau_\lambda} = Q_{\tau_\lambda} \quad \text{on } \mathscr{C}_0^\infty(D_0). \tag{4.83}$$

Let $u \in \mathscr{C}_c^\infty(D_0)$. From Fourier inversion formula, it is straightforward to see that

$$\begin{aligned}
\hat{Q}_{\tau_\lambda} u(x) = &\frac{1}{2\pi} \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int e^{i\langle x_{2n+1}-y_{2n+1},\eta_{2n+1}\rangle} \tau_\lambda(-\eta_{2n+1})\chi(y_{2n+1})\\
&\times e^{i(\sum_{j=1}^d m_j\beta_j)y_{2n+1} - im_1\theta_1 - \ldots - im_d\theta_d} u((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x')d\mathbb{T}_d dy_{2n+1} d\eta_{2n+1}.
\end{aligned} \tag{4.84}$$

From (4.84) and the definition of $\hat{R}_{\tau_\lambda}$, we have

$$\begin{aligned}
&(\hat{Q}_{\tau_\lambda} + \hat{R}_{\tau_\lambda})u(x)\\
&= \frac{1}{2\pi} \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int e^{i\langle x_{2n+1}-y_{2n+1},\eta_{2n+1}\rangle} \tau_\lambda(-\eta_{2n+1})\\
&\times e^{i(\sum_{j=1}^d m_j\beta_j)y_{2n+1} - im_1\theta_1 - \ldots - im_d\theta_d} u((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ x')d\mathbb{T}_d dy_{2n+1} d\eta_{2n+1}.
\end{aligned} \tag{4.85}$$

Note that the following formula holds for every $\alpha \in \mathbb{R}$,

$$\int e^{i\alpha y_{2n+1}} e^{-iy_{2n+1}\eta_{2n+1}} dy_{2n+1} = 2\pi\delta_\alpha(\eta_{2n+1}), \tag{4.86}$$

where the integral is defined as an oscillatory integral and δ_α is the Dirac measure at α. Using (4.82), (4.86), and the Fourier inversion formula, (4.85) becomes

$$
\begin{aligned}
(\hat{Q}_{\tau_\lambda} + \hat{R}_{\tau_\lambda})u(x) = & \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \tau_\lambda(-\sum_{j=1}^{d} m_j\beta_j)e^{i(\sum_{j=1}^{d} m_j\beta_j)x_{2n+1}} \times \\
& \int_{\mathbb{T}_d} e^{-im_1\theta_1-\ldots-im_d\theta_d} u((e^{i\theta_1},\ldots,e^{i\theta_d})\circ x')d\mathbb{T}_d \\
= & Q_{\tau_\lambda} u(x).
\end{aligned} \tag{4.87}
$$

From (4.87), the claim (4.83) follows. □

To study $Q_{\tau_\lambda^2} S^{(0)}$ when the $\mathbb{R}$ is not free, we also need the following two known results [19, Theorems 3.2, 5.2].

Theorem 4.30 *We assume that the $\mathbb{R}$-action is arbitrary. Let $D \Subset X$ be a coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. Then there exist properly supported continuous operators $A \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(D)$, $\tilde{S} \in L^{0}_{\frac{1}{2},\frac{1}{2}}(D)$, such that*

$$
\begin{aligned}
& \Box_b^{(0)} A + \tilde{S} = I \, on \, D, \\
& A^* \Box_b^{(0)} + \tilde{S} = I \, on \, D, \\
& \Box_b^{(0)} \tilde{S} \equiv 0 \ \ on \ D, \\
& A \equiv A^* \ \ on \ D, \ \ \tilde{S} A \equiv 0 \ \ on \ D, \\
& \tilde{S} \equiv \tilde{S}^* \equiv \tilde{S}^2 \ \ on \ D,
\end{aligned} \tag{4.88}
$$

where A^*, $\tilde{S}^*$ are the formal adjoints of A, $\tilde{S}$ with respect to $(\cdot\,|\,\cdot)$, respectively, and $\tilde{S}(x, y)$ satisfies

$$
\tilde{S}(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)t} s(x, y, t)dt \ \ on \ D, \tag{4.89}
$$

where $\varphi(x, y) \in \mathscr{C}^\infty(D \times D)$ and $s(x, y, t) \in S^n_{\mathrm{cl}}(D \times D \times \mathbb{R}_+)$ are as in (4.79).

Theorem 4.31 *Let us consider an arbitrary $\mathbb{R}$-action and let $Q : L^2(X) \to L^2(X)$ be a continuous operator and let Q^* be the L^2 adjoint of Q with respect to $(\cdot\,|\,\cdot)$. Suppose that $\Box_b^{(0)}$ has local L^2 closed range on D with respect to Q and $QS^{(0)} = S^{(0)}Q$ on $L^2(X)$. Let D be a coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. We have*

$$
Q^* S^{(0)} Q \equiv \tilde{S}^* Q^* Q \tilde{S} on \, D, \tag{4.90}
$$

where $\tilde{S}$ is as in Theorem 4.30.

For the proof we need the following.

Lemma 4.32 *Suppose that the* $\mathbb{R}$*-action is not free. Fix* $p \in X$*. Let* $D = U \times \mathcal{I}$ *be a BRT chart defined near* p *with BRT coordinates* $x = (x', x_{2n+1})$*,* $x' = (x_1, \ldots, x_{2n})$*,* $x(p) = 0$*. Fix* $D_0 \Subset D$*,* $p \in D_0$*, and* $\lambda \in \mathbb{R}$*. Then,*

$$\tilde{S}\hat{R}_{\tau_\lambda} \equiv 0 \ \ on\ D_0,$$

where $\hat{R}_{\tau_\lambda}$ *and* $\tilde{S}$ *are as in Lemma 4.29 and Theorem 4.30, respectively.*

Proof From (4.75), we may assume D_0 is small so that

$$|\partial_{y_{2n+1}}\varphi(x, y)| \geq C, \text{ for every } (x, y) \in D_0, \tag{4.91}$$

where $C > 0$ is a constant. Let $g \in \mathscr{C}_c^\infty(D_0)$. From (4.81) and (4.89), we have

$$\begin{aligned}
&(\tilde{S}\hat{R}_{\tau_\lambda} g)(x)\\
&= \frac{1}{2\pi} \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int e^{it\varphi(x,u)} a(x, u, t) e^{i\langle u_{2n+1}-y_{2n+1},\eta_{2n+1}\rangle + i(\sum_{j=1}^d m_j\beta_j)y_{2n+1} - i\sum_{j=1}^d m_j\theta_j}\\
&\times \tau_\lambda(-\eta_{2n+1})(1 - \chi(y_{2n+1}))g((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ u')d\mathbb{T}_d d\eta_{2n+1} dy_{2n+1} dv_X(u) \ \ \text{on } D_0,
\end{aligned} \tag{4.92}$$

where we also write $u = (u', u_{2n+1})$, $u' = (u_1, \ldots, u_{2n})$, to denote the BRT coordinates x. Since $u_{2n+1} \neq y_{2n+1}$, for every $(u', u_{2n+1}) \in D_0$, $y_{2n+1} \in \text{Supp}\,(1 - \chi(y_{2n+1}))$, we can integrate by parts in η_{2n+1} and rewrite (4.92):

$$\begin{aligned}
&(\tilde{S}\hat{R}_{\tau_\lambda} g)(x)\\
&= \frac{1}{2\pi} \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int e^{it\varphi(x,u)} a(x, u, t) \frac{1}{i(u_{2n+1} - y_{2n+1})}\\
&\times e^{i\langle u_{2n+1}-y_{2n+1},\eta_{2n+1}\rangle + i(\sum_{j=1}^d m_j\beta_j)y_{2n+1} - i\sum_{j=1}^d m_j\theta_j}\\
&\times \tau'_\lambda(-\eta_{2n+1})(1 - \chi(y_{2n+1}))g((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ u')d\mathbb{T}_d d\eta_{2n+1} dy_{2n+1} dv_X(u) \ \ \text{on } D_0.
\end{aligned} \tag{4.93}$$

Let

$$\begin{aligned}
&A(x, u', y_{2n+1})\\
&:= \int e^{it\varphi(x,u) + i\langle u_{2n+1}-y_{2n+1},\eta_{2n+1}\rangle} a(x, u, t)(1 - \chi(y_{2n+1})) \frac{1}{i(u_{2n+1} - y_{2n+1})}\\
&\quad \times \tau'_\lambda(-\eta_{2n+1}) d\eta_{2n+1} du_{2n+1} dt.
\end{aligned}$$

By (4.91) there exists $c > 0$ such that

$$\begin{aligned}
&|\partial_{u_{2n+1}}(it\varphi(x, u) + i\langle u_{2n+1} - y_{2n+1}, \eta_{2n+1}\rangle)| \geq ct,\\
&\quad \text{for } t \gg |\lambda|, \eta_{2n+1} \in \text{Supp}\,\tau'_\lambda(-\eta_{2n+1})
\end{aligned}$$

for every $(x, u) \in D_0 \times D_0$. Hence, we can integrate by parts in u_{2n+1} and η_{2n+1} and deduce that

$$A(x, u', y_{2n+1}) \in \mathscr{C}^\infty(D_0 \times D_0 \times \mathbb{R}) \text{and } A(x, u', y_{2n+1}) \text{is a Schwartz function in } y_{2n+1}. \tag{4.94}$$

We have

$$\begin{aligned}&(\tilde{S}\hat{R}_{\tau_\lambda} g)(x)\\&= \frac{1}{2\pi} \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int A(x, u', y_{2n+1}) e^{i(\sum_{j=1}^d m_j\beta_j) y_{2n+1} - im_1\theta_1 - \ldots - im_d\theta_d}\\&\times g((e^{i\theta_1}, \ldots, e^{i\theta_d}) \circ u') dv(u') dy_{2n+1} d\mathbb{T}_d,\end{aligned} \tag{4.95}$$

where $dv_X(u) = dv(u')du_{2n+1}$. From (4.94) and (4.95), we have

$$\begin{aligned}\|\tilde{S}\hat{R}_{\tau_\lambda} g\|_{D_0,s} &\leq C \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int \Big| \int_{\mathbb{T}^d} e^{-im_1\theta_1-\ldots-im_d\theta_d} g((e^{i\theta_1}, \ldots, e^{i\theta_d} \circ u') \Big|^2 \tau(u) dv_X(u)\\&= C \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int \Big| \int_{\mathbb{T}^d} e^{-im_1\theta_1-\ldots-im_d\theta_d} g((e^{i\theta_1}, \ldots, e^{i\theta_d} \circ u) \Big|^2 \tau(u) dv_X(u)\\&\leq \hat{C} \sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d} \int \Big| \int_{\mathbb{T}^d} e^{-im_1\theta_1-\ldots-im_d\theta_d} g((e^{i\theta_1}, \ldots, e^{i\theta_d} \circ u) \Big|^2 dv_X(u)\\&\leq \hat{C}_0 \|g\|^2,\end{aligned} \tag{4.96}$$

where $\|\cdot\|_{D_0,s}$ denotes the standard Sobolev norm of order s on D_0, $C, \hat{C}, \hat{C}_0 > 0$ are constants, $\tau \in \mathscr{C}_0^\infty(D)$, $\tau = 1$ near D_0. From (4.96), we deduce that

$$\tilde{S}\hat{R}_{\tau_\lambda} : L^2_c(D_0) \to H^s_{\mathrm{loc}}(D_0) \text{is continuous, for every } s \in \mathbb{N}.$$

Let $\triangle_X : \mathscr{C}^\infty(X) \to \mathscr{C}^\infty(X)$ be the standard Laplacian on X induced by $\langle\cdot|\cdot\rangle$. Since $\langle\cdot|\cdot\rangle$ is $\mathbb{T}^d$ invariant, $\triangle_X$ is $\mathbb{T}^d$ invariant. Fix $s \in \mathbb{N}$. Let

$$G_s : \mathscr{C}^\infty_c(D_0) \to \mathscr{C}^\infty_c(D_0)$$

be a parametrix of $\triangle^s_X$ on D_0 and G_s is properly supported on D_0. Hence, there is a properly supported smoothing operator

$$F_s : \mathscr{E}'(D_0) \to \mathscr{C}^\infty_c(D_0)$$

such that

$$g = (\triangle^s_X G_s + F_s)g \text{ on } D_0, \tag{4.97}$$

for all $g \in \mathscr{C}^\infty_c(D_0)$. Now, on D_0,

$$\tilde{S}\hat{R}_{\tau_\lambda} g = \tilde{S}\hat{R}_{\tau_\lambda}(\triangle^s_X G_s g) + \tilde{S}\hat{R}_{\tau_\lambda}(F_s g). \tag{4.98}$$

Since F_s is smoothing, we have

$$\|\tilde{S}\hat{R}_{\tau_\lambda}(F_s g)\|_{D_0,s} \leq C\|g\|_{-s}, \tag{4.99}$$

where $C > 0$ is a constant. Now, we can integrate by parts and repeat the proof of (4.95) and show that

$$\begin{aligned}&(\tilde{S}\hat{R}_{\tau_\lambda}\triangle_X^s G_s g)(x)\\&= \frac{1}{2\pi}\sum_{(m_1,\ldots,m_d)\in\mathbb{Z}^d}\int A_s(x,u',y_{2n+1})e^{i(\sum_{j=1}^d m_j\beta_j)y_{2n+1}-im_1\theta_1-\ldots-im_d\theta_d}\\&\times(G_s g)((e^{i\theta_1},\ldots,e^{i\theta_d})\circ u')dv(u')dy_{2n+1}d\mathbb{T}_d,\end{aligned} \tag{4.100}$$

where

$$A_s(x,u',y_{2n+1}) \in \mathscr{C}^\infty(D_0\times D_0\times\mathbb{R})\text{is a Schwartz function in } y_{2n+1}. \tag{4.101}$$

From (4.100), (4.101), and noticing that $G_s : H^{-2s}_{\text{comp}}(D) \to H^0_{\text{comp}}(D)$ is continuous, we can repeat the proof of (4.96) and conclude that

$$\|\tilde{S}\hat{R}_{\tau_\lambda}(\triangle_X^s G_s g)\|_{D_0,s} \leq C\|G_s g\| \leq C_1\|g\|_{-s}, \tag{4.102}$$

where $C, C_1 > 0$ are constants. From (4.97), (4.98), (4.99), and (4.102), we get that

$$\tilde{S}\hat{R}_{\tau_\lambda} : H^{-s}_{\text{comp}}(D_0) \to H^s_{\text{loc}}(D_0)\text{is continuous for every } s \in \mathbb{N}.$$

Hence, $\tilde{S}\hat{R}_{\tau_\lambda}$ is smoothing on D_0. □

Theorem 4.33 *Suppose that the $\mathbb{R}$-action is not free. Assume that $g_{\mathscr{L}}$ is complete and there is $C > 0$ such that*

$$\sqrt{-1}R_{\mathscr{L}}^{K_X^*} \geq -2C\sqrt{-1}\mathscr{L}, \ \ (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 \geq C\Theta_X^n \wedge \omega_0.$$

Let $D = U \times \mathcal{I} \Subset X$ be a BRT chart with BRT coordinates $x = (x_1, \ldots, x_{2n+1})$. Let $\lambda \in \mathbb{R}$, $\lambda < -C$. Then,

$$(Q_{\tau_\lambda^2}S^{(0)})(x,y) \equiv \int_0^\infty e^{i\varphi(x,y)t}s(x,y,t)dt \ \ \text{on } D, \tag{4.103}$$

where $\varphi \in \mathscr{C}^\infty(D\times D)$ and $s(x,y,t) \in S^n_{\text{cl}}\left(D\times D\times\mathbb{R}_+\right)$ are as in (4.79).

Proof From (4.81), (4.90), and Lemma 4.32, we see that on D,

$$Q_{\tau_\lambda^2}S^{(0)} \equiv \tilde{S}^*\hat{Q}^*_{\tau_\lambda}\hat{Q}_{\tau_\lambda}\tilde{S}.$$

Using this observation, we can repeat the proof of [19, Theorem 5.8] and obtain the conclusion. □

Theorem 4.34 *Let us consider an arbitrary* $\mathbb{R}$*-action. Assume that* $g_{\mathscr{L}}$ *is complete and there is* $C > 0$ *such that*

$$\sqrt{-1}R^{K^*_X}_{\mathscr{L}} \geq -2C\sqrt{-1}\mathscr{L}, \ \ (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 \geq C\Theta^n_X \wedge \omega_0.$$

Let $D = U \times I$ *be a BRT chart with BRT coordinates* $x = (x_1, \dots, x_{2n+1})$*. Let* $\lambda \in \mathbb{R}$*,* $\lambda < -C$*. Then,*

$$(I - Q_{\tau^2_\lambda})^2 S^{(0)} \equiv 0 \ \ on\ D. \tag{4.104}$$

Proof From (4.88), we have

$$(I - Q_{\tau^2_\lambda})S^{(0)} = \tilde{S}(I - Q_{\tau^2_\lambda})S^{(0)}. \tag{4.105}$$

From Lemma 4.32, we have

$$\tilde{S}(I - Q_{\tau^2_\lambda}) \equiv \tilde{S}(I - \hat{Q}_{\tau^2_\lambda}) \ \text{ on } D_0.$$

Since WF $(I - \hat{Q}_{\tau^2_\lambda}) \bigcap \Sigma^- = \emptyset$ and WF$'$ $(\tilde{S}) = \text{diag}\,(\Sigma^- \times \Sigma^-)$, we have

$$\tilde{S}(I - \hat{Q}_{\tau^2_\lambda}) \equiv 0 \ \text{ on } D_0, \tag{4.106}$$

where WF $(I - \hat{Q}_{\tau^2_\lambda})$ denotes the wave front set of $I - \hat{Q}_{\tau^2_\lambda}$ and

$$\text{WF}'\,(\tilde{S}) = \{(x, \xi, y, \eta) \in T^*D \times T^*D;\ (x, \xi, y, -\eta) \in \text{WF}\,(\tilde{S}).$$

From (4.105) and (4.106), we get

$$(I - Q_{\tau^2_\lambda})S^{(0)} : L^2(X) \to \mathscr{C}^\infty(D) \text{is continuous} \tag{4.107}$$

and hence

$$S^{(0)}(I - Q_{\tau^2_\lambda}) : \mathscr{E}'(D) \to L^2(X) \text{is continuous.} \tag{4.108}$$

From (4.107) and (4.108), we get

$$(I - Q_{\tau^2_\lambda})S^{(0)}(I - Q_{\tau^2_\lambda}) : \mathscr{E}'(D) \to \mathscr{C}^\infty(D) \text{is continuous.}$$

The theorem follows. □

We can now prove the main result of this work.

Theorem 4.35 (=Theorem 1.2) *Let the* $\mathbb{R}$*-action be arbitrary. Assume that* $g_{\mathscr{L}}$ *is complete and there is* $C > 0$ *such that*

$$\sqrt{-1}R^{K^*_X}_{\mathscr{L}} \geq -2C\sqrt{-1}\mathscr{L}, \ \ (2\sqrt{-1}\mathscr{L})^n \wedge \omega_0 \geq C\Theta^n_X \wedge \omega_0.$$

Let $D \Subset X$ be a local coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. Then,

$$S^{(0)}(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)t} s(x, y, t) dt \ \ on\ D, \tag{4.109}$$

where $\varphi \in \mathscr{C}^\infty(D \times D)$ satisfies (4.75), $s(x, y, t) \in S^n_{\mathrm{cl}}\left(D \times D \times \mathbb{R}_+\right)$ and the leading term $s_0(x, y)$ of the expansion (2.4) of $s(x, y, t)$ satisfies

$$s_0(x, x) = \frac{1}{2}\pi^{-n-1}|\det \mathcal{L}_x|, \ \ for\ all\ x \in D. \tag{4.110}$$

Proof Let $\lambda \in \mathbb{R}$, $\lambda < -C$. From Theorem 4.34, we have

$$(I - 2Q_{\tau_\lambda^2} + Q_{\tau_\lambda^4})S^{(0)} \equiv 0 \ \text{ on } D. \tag{4.111}$$

We can repeat the proofs of Theorems 4.28 and 4.33 and get

$$\begin{aligned} Q_{\tau_\lambda^2} S^{(0)} &\equiv \int_0^\infty e^{i\varphi(x,y)t} \hat{s}(x, y, t) \ \text{ on } D, \\ Q_{\tau_\lambda^4} S^{(0)} &\equiv \int_0^\infty e^{i\varphi(x,y)t} \tilde{s}(x, y, t) \ \text{ on } D, \end{aligned} \tag{4.112}$$

where $\varphi \in \mathscr{C}^\infty(D \times D)$, $\hat{s}(x, y, t), \tilde{s}(x, y, t) \in S^n_{\mathrm{cl}}\left(D \times D \times \mathbb{R}_+\right)$ are as in (4.79). From (4.111) and (4.112), the theorem follows. □

Proof of Theorem 1.3 The proof is analogous to the proof of Theorem 1.2 by using Theorem 4.15 instead of Theorem 4.13. □

Proof of Corollary 1.4 Let K be a compact set of X. Fix $x \in K$. From Theorem 1.2 and the fact that the Szegő kernel is smoothing away the diagonal, we can repeat the proof of [19, Theorem 1.10] and deduce that there are open neighborhoods $V_x \subset U_x$ of x and global smooth L^2 CR functions $(f_{0,x}, f_{1,x}, \cdots, f_{N_x,x}) = F_x$ such that $F_x : U_x \to \mathbb{C}^{N_x+1}$ is an embedding and $\sup_{K \setminus U_x} |f_{0,x}| \leq \frac{1}{2}$, $\inf_{V_x} |f_{0,x}| \geq 1$. There exists $x_1, x_2, \ldots, x_m \in K$ such that $K \subset V_{x_1} \cup V_{x_2} \cup V_{x_m} \subset U_{x_1} \cup U_{x_1} \cup U_{x_2} \cup U_{x_m}$. Then $K \ni x \mapsto (F_{x_1}, \cdots, F_{x_m})$ is an embedding. □

Proof of Corollary 1.5 We proceed as in the proof of Corollary 1.5 by working on a compact coordinate patch K with coordinates $(x_1, \ldots, x_{2n+1})$ and observing that in these coordinates a CR $(n, 0)$-form equals $f dz_1 \wedge \ldots \wedge dz_n$ with f a CR function on K. □

5 Examples

We now consider Heisenberg group $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ with CR structure

$$T^{(1,0)}\mathbb{H} := \text{span}\left\{\frac{\partial}{\partial z_j} + i\frac{\partial \phi}{\partial z_j}(z)\frac{\partial}{\partial x_{2n+1}}\right\}_{j=1}^n, \tag{5.1}$$

where $\phi \in \mathscr{C}^\infty(\mathbb{C}^n, \mathbb{R})$. Let $(\,\cdot\,|\,\cdot\,)_{\mathbb{H}}$ be the L^2 inner product on $\mathbb{H}$ induced by the Euclidean measure dx on $\mathbb{R}^{2n+1}$. Let

$$S_{\mathbb{H}} : L^2(\mathbb{H}) \to \Big\{u \in L^2(\mathbb{H});\ \Big(\frac{\partial}{\partial \overline{z}_j} - i\frac{\partial \phi}{\partial \overline{z}_j}(z)\frac{\partial}{\partial x_{2n+1}}\Big)u = 0\Big\}$$

be the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)_{\mathbb{H}}$ and let $S_{\mathbb{H}}(x, y) \in \mathscr{D}'(\mathbb{H}\times\mathbb{H})$ be the distribution kernel of $S_{\mathbb{H}}$. From Theorem 1.2, we deduce

Corollary 5.1 *With the notations used above, assume that* $\left(\frac{\partial^2\phi(z)}{\partial z_j \partial \overline{z}_k}\right)_{j,k=1}^n$ *is positive definite, at every* $z \in \mathbb{C}^n$. *Let* $0 < \lambda_1(z) \le \ldots \le \lambda_n(z)$ *be the eigenvalues of* $\left(\frac{\partial^2\phi(z)}{\partial z_j \partial \overline{z}_k}\right)_{j,k=1}^n$, *for every* $z \in \mathbb{C}^n$. *Suppose that there is* $C > 0$ *such that*

$$\begin{aligned} &\sqrt{-1}\partial\overline{\partial}\left(-\log\det\left(\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_k}\right)_{j,k=1}^n\right) \ge -C\sqrt{-1}\partial\overline{\partial}\phi\ , \\ &\frac{1}{\lambda_1(z)} \le C,\ \ \textit{for every}\, z \in \mathbb{C}^n. \end{aligned} \tag{5.2}$$

Let $D \Subset \mathbb{H}$ *be any open set. Then,*

$$S_{\mathbb{H}}(x, y) \equiv \int_0^\infty e^{i\varphi(x,y)t} s(x, y, t)dt\ \ \textit{on}\ D, \tag{5.3}$$

where $\varphi \in \mathscr{C}^\infty(D\times D)$ *and* $s(x, y, t) \in S^n_{\mathrm{cl}}\left(D\times D\times\mathbb{R}_+\right)$ *are as in Theorem* 1.2.

Example 5.2 With the notations used in Corollary 5.1, assume that

$$\phi(z) = |z|^2 + r(z), \tag{5.4}$$

with $r(z) \in \mathscr{C}^\infty_c(\mathbb{C}^n)$ and $\sqrt{-1}\partial\overline{\partial}(|z|^2 + r(z)) > 0$ on $\mathbb{C}^n$. With this ϕ, we can check the conditions of Corollary 5.1 fulfilled as follows. In fact, in this case, we have

$$\det\left(\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_k}\right)_{j,k=1}^n = \det\left(\frac{\partial^2(|z|^2 + r(z))}{\partial z_j\partial\overline{z}_k}\right)_{j,k=1}^n = 1 + F(z) > 0$$

with some $F(z) \in \mathscr{C}^\infty_c(\mathbb{C}^n)$. And we have

$$\sqrt{-1}\partial\overline{\partial}\left(-\log\det\left(\frac{\partial^2\phi}{\partial z_j\partial\overline{z}_k}\right)_{j,k=1}^n\right) = \sqrt{-1}\partial\overline{\partial}\,(-\log(1 + F(z))) \in \Omega^{1,1}_c(\mathbb{C}^n).$$

Since $r(z) \in \mathscr{C}^\infty_c(\mathbb{C}^n)$ and $\sqrt{-1}\partial\overline{\partial}\phi = \sqrt{-1}\partial\overline{\partial}(|z|^2 + r(z)) > 0$, we have a uniform lower bound for the smallest eigenvalue, i.e., $\lambda_1(z) > 1/C_1$ for some $C_1 > 0$. Moreover, we can choose $C_2 > 0$ sufficiently large such that

$$\sqrt{-1}\partial\overline{\partial}\,(-\log(1 + F(z))) + C_2\sqrt{-1}\partial\overline{\partial}\phi \ge 0,$$

since the first term $\sqrt{-1}\partial\overline{\partial}\,(-\log(1+F(z)))$ is a real $(1,1)$-form with compact support in $\mathbb{C}^n$ and the second term $\sqrt{-1}\partial\overline{\partial}\phi$ is a real positive $(1,1)$-form with a uniformly positive lower bound for the smallest eigenvalue $\lambda_1(z) > 1/C_1$ on $\mathbb{C}^n$. Finally, we obtain $C := \max\{C_1, C_2\} > 0$ as desired in (5.2).

With this ϕ, it is easy to see that (5.2) hold. This example shows that, after small perturbation of the Levi form of Heisenberg group, we still can obtain the Szegő kernel expansion via Corollary 5.1.

Example 5.3 Let $(X, T^{(1,0)}X)$ be a strictly pseudoconvex, CR manifold of dimension $2n+1$, $n \geq 1$, with a discrete, proper, CR action Γ such that the quotient X/Γ is compact. Assume X admits a transversal CR $\mathbb{R}$-action on X and let Θ_X be a Γ-invariant, $\mathbb{R}$-invariant, Hermitian metric on X. Then the conclusion of Theorem 1.2 holds. In fact, the Γ-covering manifold is complete and we can find the desired constant C depending on the fundamental domain $U \Subset X$ given by the Γ-action such that (1.7) is fulfilled. As a consequence, if we consider the circle bundle case in which $R^L = 2\mathscr{L}$, we could obtain the Bergman kernel expansion for covering manifold [23, 6.1.2].

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest The authors declare that there are no conflict of interest regarding the publication of this paper.

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