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Nakano-Griffiths inequality, holomorphic Morse inequalities, and extension theorems for q-concave domains

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ABSTRACT

We establish a general Nakano–Griffiths inequality with boundary conditions and apply it to derive holomorphic Morse inequalities for domains satisfying analytic convexity assumptions. The proof of these inequalities relies on analyzing the spectral spaces of the Laplace operator with $\overline{\partial}$-Neumann boundary conditions. As an application, we obtain a criterion for Moishezon 1-concave manifolds.

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We further apply the holomorphic Morse inequalities on Levi q-concave domains in conjunction with the Kohn–Rossi extension theorem to obtain the following result. Let X be a compact complex manifold of dimension n equipped with a holomorphic line bundle that is semi-positive everywhere and positive at least at one point, and let $D \subset X$ be a smooth q-concave domain $(1 \leq q \leq n-1)$. We prove that every $\bar{\partial}_b$-closed $(0, \ell)$-form on bD with values in a holomorphic vector bundle, admits a meromorphic extension to D for all $q \leq \ell \leq n-1$.

Contents

1. Introduction

A foundational tool in complex geometry is the Bochner–Kodaira–Nakano identity and its variants, which have far-reaching implications, from vanishing theorems to holomorphic Morse inequalities and extension results. In this paper, we develop a general version of the Nakano–Griffiths inequality with boundary conditions, unifying the pseudoconvex/positive and pseudoconcave/negative settings. This is achieved via a refined Bochner–Kodaira–Nakano formula that incorporates boundary terms in full generality, and extends the classical results of Andreotti–Vesentini [7], Griffiths [20], and their analytic extensions in [30] (see Theorems 1.1 and 1.2).

Our new inequality leads naturally to holomorphic Morse inequalities for domains and manifolds satisfying analytic convexity conditions. Rather than relying on scaling arguments, we adopt a geometric approach based on the spectral theory of the $\overline{\partial}$-Laplacian with $\overline{\partial}$-Neumann boundary conditions. Our method provides new index ranges and accommodates weaker curvature assumptions than those previously required. In particular, we establish holomorphic Morse inequalities for Levi q-concave domains (see Theorem 1.4) and for q-concave complex manifolds (see Theorem 1.5), refining and extending earlier results [8,30,32]. As an application of this analytic framework, we obtain a geometric criterion for Moishezon 1-concave manifolds.

We conclude with an application to the extension problem for $\bar{\partial}_b$-closed forms on the boundary of pseudoconcave domains, where no general vanishing theorem is available. This question has its roots in classical results such as Hartogs' theorem [23, Theorem 2.3.2], which ensures that holomorphic functions on $M \setminus K \subset \mathbb{C}^n$, $n \geq 2$, extend to M when $K \subset M$ is compact and $M \setminus K$ is connected. Bochner [10] refined this by allowing the extension of functions satisfying the tangential Cauchy–Riemann equations on the boundary bM. Further generalizations to complex manifolds were given in [4,27].

A key method, introduced by Ehrenpreis [17], reduces such extension problems to solving the $\bar{\partial}$-equation with compact support, interpreting obstructions via $\bar{\partial}$-cohomology. This has been developed in [2,4,23,26,27].

In our setting, we study the extension of $\bar{\partial}_b$-closed forms with values in a holomorphic vector bundle from the boundary of a pseudoconcave domain. Unlike the case of holomorphic functions, this requires more refined tools. We combine holomorphic Morse inequalities for q-concave domains with the Kohn–Rossi extension theorem to obtain precise results. Specifically, let X be a compact complex manifold of dimension n, equipped with a holomorphic line bundle that is semi-positive everywhere and strictly positive at one point, and let $D \subset X$ be a smooth q-concave domain with $1 \leq q \leq n-1$. We prove that every $\bar{\partial}_b$-closed $(0,\ell)$-form on bD with values in a holomorphic vector bundle, admits a meromorphic extension to D for all $q \leq \ell \leq n-1$ (see Theorem 1.12). Our method also yields new vanishing results, including for domains with Levi-flat boundaries.

1.1. Nakano-Griffiths inequality with boundary terms

Our approach to the holomorphic Morse inequalities is based on a novel Bochner-Kodaira-Nakano formula with boundary terms. Let (X, Θ) be a Hermitian manifold of dimension n, and $M := \{x \in X : \varrho(x) < 0\}$ be a relatively compact domain with a smooth defining function ϱ such that $|d\varrho| = 1$ on bM. Let $\mathscr{L}_\varrho$ denote the Levi form for bM (see (2.1)). For a holomorphic vector bundle $F \to X$ with a smooth Hermitian metric h^F, let ∇^F denote the associated Chern connection and $R^F = (\nabla^F)^2$ the Chern curvature. We refer to Section 3 for the notations used in the following theorem.

Theorem 1.1 *(A Bochner-Kodaira-Nakano formula with boundary terms). Let $M \Subset X$ be given as above. Let $\tau \in \mathscr{C}^\infty(X, [0,1])$ be such that $\tau = 1$ near bM and $\operatorname{supp}\tau$ is contained in an open neighborhood of bM where $d\varrho$ does not vanish. Fix $\varepsilon \in [0,1]$. Then for $s \in B^{0,q}(M,F)$, we have*

$$\begin{aligned}
&\|\overline{\partial}^F s\|^2_{L^2(M)} + \|\overline{\partial}^{F*} s\|^2_{L^2(M)} \\
&= \big\langle [\sqrt{-1}R^F, \Lambda]_\varepsilon s, s\big\rangle_{L^2(M)} + 2(1-\varepsilon)\left\langle \tau R^F(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{L^2(M)} \\
&\quad + \varepsilon \sum_{j=1}^n \|\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s\|^2_{L^2(M)} \\
&\quad + \int_{bM} \langle [\sqrt{-1}\mathscr{L}_\varrho, \Lambda]^{bM}_\varepsilon s, s\rangle_h dv_{bM} + (1-\varepsilon)\int_{bM} e_{\mathfrak{n}}(|d\varrho|)|s|^2_h dv_{bM} \\
&\quad + \Big\langle \sum_{j,k=1}^n R^{K^*_X}(\omega_j, \overline{\omega}_k)\overline{\omega}^k \wedge \iota_{\overline{\omega}_j} s, s\Big\rangle_{L^2(M)} - (1-\varepsilon)\int_M |s|^2_h \sqrt{-1}\partial\overline{\partial}\Omega_\tau \\
&\quad + Q^F_{T,\varepsilon}(s,s) + (1-\varepsilon)\left(\Psi^F_\tau(s,s) + 2\|\sqrt{\tau}\,\widetilde{\nabla}^{TX}_{e^{(0,1)}_{\mathfrak{n}}} s\|^2_{L^2(M)}\right) \\
&\quad + (1-\varepsilon)\left(\sum_{j=1}^n \|\widetilde{\nabla}^{TX}_{\omega_j} s\|^2_{L^2(M)} - 2\|\sqrt{\tau}\,\widetilde{\nabla}^{TX}_{e^{(1,0)}_{\mathfrak{n}}} s\|^2_{L^2(M)}\right).
\end{aligned} \tag{1.1}$$

The proof of (1.1) is based on the Bochner-Kodaira-Nakano formula with boundary terms [30, Theorem 1.4.21] (cf. Theorem 3.2), which uses a Riemannian formalism and the approach of Griffiths [20, Theorem 7.4], based on the existence of the $(n-1,n-1)$-form Ω_τ, specific to Hermitian manifolds. For $\varepsilon = 1$, formula (1.1) reduces to the Bochner-Kodaira-Nakano formula with boundary terms (3.16) by Andreotti and Vesentini [7, p. 113] and by Griffiths [20, (7.14)], which in turn follows from [30, Theorem 1.4.21].

An important feature of (1.1) is the introduction of the modified curvature commutator $[\sqrt{-1}R^F, \Lambda]_\varepsilon := [\sqrt{-1}R^F, \Lambda] + \varepsilon \mathrm{Tr}_\Theta[\sqrt{-1}R^F]$ (cf. (3.38)) and similarly for the Levi form, $[\sqrt{-1}\mathscr{L}_\varrho, \Lambda]^{bM}_\varepsilon$ (cf. (3.39)). The curvature commutator $[\sqrt{-1}R^F, \Lambda]$ plays an important role in the applications of the Bochner-Kodaira-Nakano formula [15,16,30], such as vanishing theorems, holomorphic Morse inequalities, or Bergman kernel asymptotics. In (1.1), the main difficulty is to identify an operator of the same nature on the boundary bM, namely $[\sqrt{-1}\mathscr{L}_\varrho, \Lambda]^{bM}_\varepsilon$, where we need to develop new techniques inspired by the remarkable calculations from [20, VII. §5] used to derive a Nakano-Griffiths inequality [20, (7.8)]; see Subsection 3.2.

If F is a line bundle and R^F is negative, then $[\sqrt{-1}R^F, \Lambda]_\varepsilon$ is a positive operator on $(0,q)$-forms ($q \leq n-1$) for sufficiently small $\varepsilon \geq 0$; if $\varepsilon = 1$ (or ε is sufficiently close to 1), $[\sqrt{-1}R^F, \Lambda]_\varepsilon$ becomes negative; similarly for the term $[\sqrt{-1}\mathscr{L}_\varrho, \Lambda]_\varepsilon^{bM}$, see (4.16). This suggests that a formula like (1.1) is needed to deal with negative bundles on a domain with a pseudoconcave boundary, parallel to a formula such as (3.16) adapted to the situation of positive bundles on a domain with pseudoconvex boundary. Our formula (1.1) actually unifies these two scenarios.

Since we do not assume Θ to be Kähler, the torsion tensor T (see (3.5)) appears in (1.1), and is encoded in the term $Q^F_{T,\varepsilon}(s,s)$ (see Definition 3.7). If (X,Θ) is Kähler, or more generally, if $d\Theta = 0$ in a neighborhood of M, then $T = 0$ in $\overline{M}$, and we have $Q^F_{T,\varepsilon}(s,s) = 0$.

As a consequence of the equality (1.1), we obtain a generalization of [20, (7.8)].

Theorem 1.2 *(Nakano-Griffiths inequality with boundary terms). Let $M = \{\varrho < 0\} \Subset X$ be a relatively compact domain with a smooth boundary bM. Let $\tau \in \mathscr{C}^\infty(X, [0,1])$ be such that $\tau = 1$ near bM and $\operatorname{supp}\tau$ is contained in an open neighborhood of bM where $d\varrho$ does not vanish. Then there exists a constant $C = C(X, g^{TX}, M, \varrho, \tau) > 0$, which is independent of (F, h^F), such that, for any $\varepsilon \in (0,1]$ and any $s \in B^{0,q}(M,F)$, we have*

$$\begin{aligned}
&(1+\varepsilon)\left(\|\overline{\partial}^F s\|^2_{L^2(M)} + \|\overline{\partial}^{F*} s\|^2_{L^2(M)}\right) + C(1+\frac{1}{\varepsilon})\|s\|^2_{L^2(M)} \\
&\geq \left\langle[\sqrt{-1}R^F, \Lambda]_\varepsilon s, s\right\rangle_{L^2(M)} + 2(1-\varepsilon)\left\langle \tau R^F(e_{\mathfrak{n}}^{(1,0)}, e_{\mathfrak{n}}^{(0,1)})s, s\right\rangle_{L^2(M)} \\
&\quad + \int_{bM} \langle[\sqrt{-1}\mathscr{L}_\varrho, \Lambda]^{bM}_\varepsilon s, s\rangle_h dv_{bM} + (1-\varepsilon)\int_{bM} e_{\mathfrak{n}}(|d\varrho|)|s|^2_h dv_{bM} \\
&\quad + (1-\varepsilon)\left(\|(\widetilde{\nabla}^{TX})^{1,0}s\|^2_{L^2(M)} - 2\|\sqrt{\tau}\widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(1,0)}}s\|^2_{L^2(M)}\right).
\end{aligned} \tag{1.2}$$

Formula (1.2) is more involved than [20, (7.8)]. Indeed, in [20] it is assumed that $|d\varrho| = 1$ near bM (not only on bM), so that $e_{\mathfrak{n}}(|d\varrho|) \equiv 0$ on bM and the second boundary term (the integral on bM) in (1.2) does not appear. The last line in (1.2) will always be non-negative. In [20] a modified Hermitian metric on F near bM is used to derive [20, (7.8)] (see [20, the last part of VII. §6]). This is not needed in (1.2).

When $\varepsilon = 1$ in (1.2), we get [30, Corollary 1.4.22] (up to constants), which can be used to prove the holomorphic Morse inequalities for q-convex manifolds or domains (see [30, Section 3.5]). However, for the q-concave case, we need to apply (1.2) for sufficiently small $\varepsilon > 0$, in particular, to obtain the optimal fundamental estimates in Proposition 4.2; see Subsection 4.2. In Subsection 4.5, we combine these two set-ups to obtain the weak holomorphic Morse inequalities for (p,q)-coronas and (p,q)-convex-concave manifolds (see Definition 4.13).

Let $H^{0,\ell}(\overline{M}, F)$ denote the Dolbeault cohomology groups up to the boundary, defined by the Dolbeault complex consisting of differential forms on M that are smooth up to the boundary, see Subsection 2.2. From (1.2), we can also recover the classical Andreotti-Tomassini vanishing theorem for both q-concave and q-convex manifolds; see Subsection

6.1. In fact, we get a more general vanishing theorem as follows. For a Hermitian line bundle $L \to X$ and an integer $k \in \mathbb{N}$, $L^k := L^{\otimes k}$ denotes the k-th tensor power of L.

Theorem 1.3. *Let X be a complex manifold of dimension $n \geq 2$. Let $1 \leq q \leq n-1$, and let $M \Subset X$ be a smooth domain in X. Let E be a holomorphic vector bundle on X.*

(a) Assume that the Levi form $\mathscr{L}_{bM}$ has at least $n-q$ negative eigenvalues on bM. Let (L, h^L) be a Hermitian line bundle on X such that $c_1(L, h^L) < 0$ on $\overline{M}$. Then, there exists an integer $k_0 = k_0(L, E)$ such that

$$H^{0,\ell}(\overline{M}, L^k \otimes E) = 0 \quad \text{for } k \geq k_0 \text{ and } \ell \leq n-q-1. \tag{1.3}$$

(b) Assume that the Levi form $\mathscr{L}_{bM}$ has at least $n-q$ positive eigenvalues on bM. Let (L, h^L) be a Hermitian line bundle on X such that $c_1(L, h^L) > 0$ on $\overline{M}$. Then, there exists an integer $k_0 = k_0(L, E)$ such that

$$H^{0,\ell}(\overline{M}, L^k \otimes E) = 0 \quad \text{for } k \geq k_0 \text{ and } \ell \geq q. \tag{1.4}$$

The cohomology groups $H^{0,\ell}(\overline{M}, L^k \otimes E)$ in the above statements can be replaced by the spaces of harmonic forms $\mathscr{H}^{0,\ell}(M, L^k \otimes E)$ with respect to an arbitrary Hermitian metric on E.

Finally, in Subsection 6.3, employing Theorem 1.2, we derive a vanishing theorem for negative line bundles on relatively compact domains of complex dimension $n \geq 2$ with smooth Levi-flat boundary.

1.2. Holomorphic Morse inequalities for q-concave domains

Let F be a holomorphic vector bundle on complex manifold X. We denote by $H^\ell(X, F)$ the ℓ-th sheaf cohomology group of X with values in F. Let $M \Subset X$ be a relatively compact domain with smooth boundary. Given the smooth metrics on F and on X, we can consider the Kodaira Laplacians on M with $\overline{\partial}$-Neumann boundary conditions. We denote by $\mathscr{H}^{0,\ell}(M, F)$ the corresponding space of harmonic forms. For a Hermitian line bundle (L, h^L), let R^L be the Chern curvature form and let $c_1(L, h^L) := \frac{\sqrt{-1}}{2\pi} R^L$ be the first Chern form associated with h^L. We denote by $(L^k, h^{L^k}) := (L^{\otimes k}, (h^L)^{\otimes k})$ the k-th tensor power.

The formula (1.2), together with suitable geometric conditions, implies the optimal fundamental estimates for the Kodaira Laplacians with $\overline{\partial}$-Neumann boundary conditions on $\overline{M}$; therefore, Theorems 1.4 and 1.8 follow from the abstract holomorphic Morse inequalities for L^2 cohomology; see Subsection 2.3 and [30, Section 3.2]. We have the following holomorphic Morse inequalities for the (Levi) q-concave domains.

Theorem 1.4. *Let $M \Subset X$ be a smooth domain of a Hermitian manifold (X, Θ) of dimension n such that the Levi form $\mathscr{L}_{bM}$ (see (2.1)) has at least $n-q$ negative eigenvalues,*

$1 \leq q \leq n-1$. *Let* (E, h^E) *and* (L, h^L) *be holomorphic Hermitian vector bundles on* X *with* $\mathrm{rk}(L) = 1$.

If $c_1(L, h^L) \leq 0$ *on* $M \setminus K_L$ *for some compact subset* $K_L \subset M$, *then for* $\ell \leq n-q-1$,

$$\limsup_{k\to\infty} k^{-n} \dim \mathscr{H}^{0,\ell}(M, L^k \otimes E) \leq \frac{\mathrm{rk}(E)}{n!} \int_{K_L(\ell, h^L)} (-1)^\ell c_1(L, h^L)^n,$$

$$\limsup_{k\to\infty} \sum_{j=0}^{\ell} (-1)^{\ell-j} k^{-n} \dim \mathscr{H}^{0,j}(M, L^k \otimes E) \leq \frac{\mathrm{rk}(E)}{n!} \int_{K_L(\leq \ell, h^L)} (-1)^\ell c_1(L, h^L)^n, \tag{1.5}$$

where $K_L(\ell, h^L)$ *is the subset of* K_L *consisting of all points* x *such that* $\sqrt{-1}R^L_x$ *is non-degenerate and has exactly* ℓ *negative eigenvalues, and* $K_L(\leq \ell, h^L) = \cup_{0\leq j\leq \ell} K_L(j, h^L)$.

Moreover, if $1 \leq q \leq n-2$, *we also have*

$$\liminf_{k\to\infty} k^{-n} \dim \mathscr{H}^{0,0}(M, L^k \otimes E) \geq \frac{\mathrm{rk}(E)}{n!} \int_{K_L(\leq 1, h^L)} c_1(L, h^L)^n. \tag{1.6}$$

The spaces $\mathscr{H}^{0,\ell}(M, L^k \otimes E)$ *in* (1.5) *and* (1.6) *can be replaced by the cohomology groups* $H^{0,\ell}(\overline{M}, L^k \otimes E)$.

When $E = \mathbb{C}$, the Morse inequalities (1.5) for $0 \leq \ell \leq n-q-1$ were also implicitly given by Berman in [8, Theorems 6.5 and 6.6], via the scaling method for Bergman kernels. Our method develops the Bochner techniques along Griffiths [20] and adapts the abstract holomorphic Morse inequalities for L^2 cohomology as in [30], which also leads to a more geometric approach for the classical Andreotti-Tomassini vanishing theorems [6]; see Theorems 1.3 and 6.1.

The holomorphic Morse inequalities for q-concave manifolds, established in [32,34], are a consequence of Theorem 1.4. In this paper, our method is based on the fundamental estimates for the Kodaira Laplacians with $\overline{\partial}$-Neumann boundary conditions.

Theorem 1.5 *([32]). Let* X *be a* q*-concave complex manifold of dimension* $n \geq 2$, *where* $1 \leq q \leq n-1$. *Let* E, L *be holomorphic vector bundles on* X *with* $\mathrm{rk}(L) = 1$. *Let* h^L *be a smooth Hermitian metric on* L *such that* $c_1(L, h^L) \leq 0$ *on* $X \setminus K_L$ *for some compact subset* K_L. *Then for* $j \leq n-q-1$, *we have the weak Morse inequalities: as* $k \to \infty$,

$$\dim H^j(X, L^k \otimes E) \leq \mathrm{rk}(E) \frac{k^n}{n!} \int_{K_L(j, h^L)} (-1)^j c_1(L, h^L)^n + o(k^n). \tag{1.7}$$

For $j \leq n-q-2$, *we have the strong Morse inequalities: as* $k \to \infty$,

$$\sum_{\ell=0}^{j}(-1)^{j-\ell}\dim H^{\ell}(X,L^k\otimes E)\leq \mathrm{rk}(E)\frac{k^n}{n!}\int\limits_{K_L(\leq j,h^L)}(-1)^j c_1(L,h^L)^n+o(k^n). \quad (1.8)$$

Moreover, if $1\leq q\leq n-2$*, we also have*

$$\dim H^0(X,L^k\otimes E)\geq \mathrm{rk}(E)\frac{k^n}{n!}\int\limits_{K_L(\leq 1\,,h^L)}c_1(L,h^L)^n+o(k^n). \quad (1.9)$$

Remark 1.6. The proof of the above theorem will be given in Subsection 4.3. Note that in [32], the inequality (1.7) was stated only for all $j\leq n-q-2$; here we can state it for all $j\leq n-q-1$ using our approach. Additionally, we would like to make the following remarks:

(i) Let $\varrho\in\mathscr{C}^\infty(X,\mathbb{R})$ be a sublevel exhaustion function for X with exceptional set K (see Definition 2.1) in Theorem 1.5, and set $X_c:=\{x\in X\ :\ \varrho(x)<c\}$ for $c\in\mathbb{R}$. We can modify the assumption from $c_1(L,h^L)\leq 0$ *on* $X\setminus K_L$ to the more relaxed condition $c_1(L,h^L)\leq 0$ *on* $X_c\setminus K_L$, where c is a regular value of ϱ with $K\cup K_L\Subset X_c\Subset X$.

(ii) If X is a compact complex manifold, then we can take $K_L=X$, and all the assumptions in Theorem 1.5 hold trivially, so that the inequalities (1.7) and (1.8) are exactly the holomorphic Morse inequalities proven by Demailly in [15, Théorème 0.1].

(iii) When X is noncompact, we further develop the ideas in [30, Chapter 3] and give a different approach to [32]. Holomorphic Morse inequalities for noncompact manifolds were proved in various contexts; by Nadel-Tsuji [35] for some complete Kähler manifolds with Ricci negative curvature; by Bouche [11] for q-convex manifolds; by Marinescu [31,32,42] for q-concave manifolds, weakly 1-complete manifolds, and covering manifolds; and by Berman [8] for smooth domains satisfying the $Z(q)$ condition. In this paper, parallel to Ma-Marinescu [30, Chapter 3], we present a unified treatment. The Bochner-Kodaira-Nakano-type formulas with boundary terms (Theorems 1.1 and 1.2) play a fundamental role in this treatment. For further details, we refer to Subsection 4.5.

1.3. Characterization of Moishezon manifolds and compactification

We further apply the Bochner-Kodaira-Nakano formula with a boundary term to obtain asymptotic lower bounds for the dimension of the spaces of holomorphic sections of a semi-positive line bundle over a 1-concave manifold. They generalize the well-known lower bounds obtained by Siu [39] and Demailly [15] in order to solve the Grauert-Riemenschneider conjecture, which provides a characterization of Moishezon manifolds in terms of semi-positive line bundles. The corresponding problem for 1-concave manifolds was considered in [8,24,32,34].

Recall that an n-dimensional compact complex manifold M or a q-concave complex manifold M ($1\leq q\leq n-2$) is called *Moishezon* if $\alpha(M)=n$, where $\alpha(M)$ denotes the transcendence degree of the field of meromorphic functions on M (see [30, Definition

2.2.12 and Subsection 3.4] and [32, §5]). This definition also works for a compact irreducible complex space (see [30, Example 3.4.2(ii)]). According to a classical theorem of Moishezon, a compact Moishezon manifold admits a proper modification $\widehat{X} \to X$, given by a finite number of blow-ups with smooth centers, such that $\widehat{X}$ is a projective algebraic manifold (see [30, Theorem 2.2.16]).

The following criterion for Moishezon q-concave manifolds is essentially derived from [32, Theorem 1.1] (or equivalently, (1.9)) in combination with the above definition and [30, Theorem 3.4.7].

Theorem 1.7. *Let X be a q-concave complex manifold of dimension $n \geq 3$, where $1 \leq q \leq n-2$. Let L be a holomorphic line bundle on X with a smooth Hermitian metric h^L such that $c_1(L, h^L) \leq 0$ on $X \setminus K_L$ for some compact subset K_L. If*

$$\int_{K_L(\leq 1, h^L)} c_1(L, h^L)^n > 0, \tag{1.10}$$

then X is Moishezon.

A holomorphic line bundle $L \to X$ on a q-concave manifold X is called big if

$$\limsup_{k\to\infty} k^{-n} \dim H^0(X, L^k) > 0.$$

By (1.9), the criterion (1.10) also gives the bigness of L. In this case, for k sufficiently large, the sections of $H^0(X, L^k)$ provide local coordinates on a Zariski open set of X.

Now let us focus on the 1-concave case, also known as the strictly pseudoconcave case. As we mentioned in the beginning of this part, we would like to study the semi-positive line bundle (L, h^L).

Theorem 1.8. *Let X be a connected complex manifold of dimension $n \geq 3$. Let M be a relatively compact domain in X with a smooth boundary bM, and assume that ϱ is a defining function for bM with $M = \{x \in X : \varrho(x) < 0\}$. Assume that the Levi form $\mathscr{L}_\varrho$ is strictly negative on bM. Let E be a holomorphic vector bundle on X of rank $\mathrm{rk}(E) \geq 1$.*

Let (L, h^L) be a Hermitian line bundle on X such that $c_1(L, h^L) \geq 0$ on an open neighborhood of bM in $\overline{M}$.

(I) Assume that there exists an open neighborhood U of bM and a plurisubharmonic function $\phi \in \mathscr{C}^\infty(U, \mathbb{R})$ such that $c_1(L, h^L) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$ on U. Then we have

$$\begin{aligned} &\liminf_{k\to+\infty} k^{-n} \dim H^0(M, L^k \otimes E) \\ &\geq \frac{\mathrm{rk}(E)}{n!} \int_{M(\leq 1, h^L)} c_1(L, h^L)^n + \frac{\mathrm{rk}(E)}{n!} \int_{bM} c_1(L, h^L)^{n-1} \wedge \frac{\sqrt{-1}}{2\pi} \partial\phi, \end{aligned} \tag{1.11}$$

where bM has the induced orientation as the boundary of M.
(II) Assume $c_1(L,h^L) \geq 0$ on the whole of $\overline{M}$. If we can take $\phi = -\varrho$ in the above situation, that is, $c_1(L,h^L) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varrho$ on U, then

$$\lim_{k\to+\infty} k^{-n}\dim H^0(M, L^k\otimes E) = \frac{\mathrm{rk}(E)}{n!}\int_M c_1(L,h^L)^n - \frac{\mathrm{rk}(E)}{n!}\int_{bM} c_1(L,h^L)^{n-1}\wedge\frac{\sqrt{-1}}{2\pi}\partial\varrho. \tag{1.12}$$

When $E = \mathbb{C}$, (1.12) was proven by Berman [8, Corollary 6.7] and [9]. In his results, it is only assumed that $c_1(L,h^L) = -\frac{\sqrt{-1}}{2\pi}\mathscr{L}_\varrho$ on bM. A version of (1.11) was derived in [34, (2.1)], under the assumption that (L,h_L) is positive on $\overline{M}$ and $\mathscr{L}_\varrho$ has at least two negative eigenvalues on bM.

By Theorem 1.8 and [32, Theorem 5.2] (cf. also [24,34]), we obtain the following result for the compactification of strictly pseudoconcave domains.

Corollary 1.9. *Let X be a connected complex manifold of dimension $n \geq 3$, and let M be a connected, strictly pseudoconcave, relatively compact domain in X with a smooth boundary bM. Let (L,h^L) be a Hermitian line bundle on X such that $c_1(L,h^L) \geq 0$ holds in an open neighborhood of bM in $\overline{M}$. We assume that there exists an open neighborhood U of bM and a plurisubharmonic function $\phi \in \mathscr{C}^\infty(U,\mathbb{R})$ such that $c_1(L,h^L) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi$ on U, and*

$$\int_{M(\leq 1,h^L)} c_1(L,h^L)^n > -\int_{bM} c_1(L,h^L)^{n-1}\wedge\frac{\sqrt{-1}}{2\pi}\partial\phi. \tag{1.13}$$

Then L is big, and M can be compactified to a Moishezon manifold; that is, there exists a compact Moishezon manifold $\widetilde{M}$ and an open set $M_0 \subset \widetilde{M}$ such that M is biholomorphic to M_0.

We illustrate the applications of the above corollary with the following example.

Example 1.10 *(Moishezon 1-concave domains in a compact irreducible complex space with isolated singularities).* Let X be a compact irreducible complex space of dimension $n \geq 3$ with at most isolated singularities, denoted by X_{sing} (see [30, §3.4]). Set $X_{\mathrm{reg}} := X\setminus X_{\mathrm{sing}}$ and let $D \subset X$ be a finite set that contains X_{sing}.

Let $L \to X$ be a holomorphic line bundle equipped with a smooth Hermitian metric h^L on $X \setminus D$ such that, in a local chart near each point $x \in D$, h^L as well as all its derivatives (actually, it is enough to consider all the derivatives up to order 2) are uniformly bounded. This is the case when h^L is smooth on the whole of X in the sense that h^L is smooth on X_{reg} and for each $x_\alpha \in X_{\mathrm{sing}}$, there exists an open chart U_α of X and a holomorphic embedding $\iota_\alpha : U_\alpha \hookrightarrow \mathbb{C}^{N_\alpha}$ such that $x_\alpha \in U_\alpha$, $\iota_\alpha(x_\alpha) = 0 \in \mathbb{C}^{N_\alpha}$, and $(L|_{U_\alpha}, h^L|_{U_\alpha}) = \iota_\alpha^*(\mathbb{C}^{N_\alpha} \times \mathbb{C}, h_\alpha)$ is the pullback of the trivial line bundle on $\mathbb{C}^{N_\alpha}$ together with a smooth Hermitian metric.

We assume $c_1(L, h_L) \geq 0$ on an open deleted neighborhood of D. For each point $x \in D$, we fix a small open domain U_x of x such that

- each U_x is a relatively compact, strictly pseudoconvex domain with a smooth boundary;
- on an open neighborhood V_x of $\overline{U}_x$, the line bundle L can be holomorphically trivialized so that, in this trivialization, h^L has a local weight $\phi_x \in \mathscr{C}^\infty(V_x \setminus \{x\}, \mathbb{R})$. We also take U_x and V_x such that $c_1(L, h_L)|_{V_x \setminus \{x\}} \geq 0$. By our assumption, every derivative of ϕ_x in the deleted neighborhood $V_x \setminus \{x\}$ is uniformly bounded.
- the open subsets V_x, $x \in D$, are disjoint from each other.

Observe that one can always choose a family $\{x \in U_x \Subset V_x \Subset X\}_{x \in D}$ satisfying the above conditions.

Now we set $M := X \setminus \cup_{x \in D} \overline{U}_x$. Then bM is the disjoint union of bU_x, but with the opposite orientation. Then M becomes a strictly pseudoconcave domain with a smooth boundary. Our assumption on (L, h^L) implies that, for $x \in D$, the smooth form $c_1(L, h^L)^{n-1} \wedge \frac{\sqrt{-1}}{2\pi} \partial \phi_x$ is uniformly bounded on a small deleted neighborhood of x, then by Stokes' theorem, we have

$$\int_{U_x \setminus \{x\}} c_1(L, h^L)^n = - \int_{bU_x} c_1(L, h^L)^{n-1} \wedge \frac{\sqrt{-1}}{2\pi} \partial \phi_x \,. \tag{1.14}$$

As a consequence, we have

$$\int_{X_{\mathrm{reg}}(\leq 1, h^L)} c_1(L, h^L)^n = \int_{M(\leq 1, h^L)} c_1(L, h^L)^n + \int_{bM} c_1(L, h^L)^{n-1} \wedge \frac{\sqrt{-1}}{2\pi} \partial \phi, \tag{1.15}$$

where $\phi = \{\phi_x\}_{x \in D}$. By [30, Theorem 3.4.10], if the integral in (1.15) is strictly positive, then X is Moishezon, which implies that M is Moishezon as a consequence of Corollary 1.9.

The subsequent stability result for Moishezon manifolds is established by combining the idea in [34] with Corollary 1.9 and (1.15). The proof is provided at the end of Section 4.4.

Theorem 1.11. *Under the same setup as in the above example, assume in addition that $c_1(L, h^L) > 0$ on $X \setminus D$. Then, for any sufficiently small deformation J' of the complex structure J on M and any sufficiently small deformation of complex structure on L that makes the original smooth line bundle $L|_M \to M$ also holomorphic with respect to the new complex structure, the manifold M is Moishezon 1-concave and it can be compactified to a Moishezon manifold.*

1.4. Kohn-Rossi extension theorem for q-concave domains

Let μ be the projection from the space of the smooth forms with values in holomorphic bundles on the boundary to its subspace consisting of complex tangential forms with values in bundles, and let $\overline{\partial}_b$ be the Kohn-Rossi operator on the space of the smooth forms with values in holomorphic vector bundles on the boundary; see Subsection 2.4.

Under the same conditions on $M \Subset X$ as Theorem 1.4, if L is semi-positive throughout M, then (1.5) indicates that

$$\dim \mathscr{H}^{0,n-\ell-1}(M, L^{-k}\otimes E^*) = o(k^n), \qquad \text{as } k\to\infty \text{ when } q\le \ell\,;$$

see Corollary 4.9. This estimate, along with the Kohn-Rossi criterion (Theorem 2.15), allows for the substitution of cohomology vanishing theorems typically used in the extension results for holomorphic forms, leading to the following theorem.

Theorem 1.12. *Let X be a connected, compact complex manifold of dimension $n\ge 2$. Let (E, h^E) and (L, h^L) be holomorphic Hermitian vector bundles over X and $\mathrm{rk}(L) = 1$. Assume that L is semi-positive on X and positive at least at one point. Let $1\le q\le n-1$ and assume that M is a relatively compact domain in X with a smooth boundary bM on which the Levi form restricted to the analytic tangent space has at least $n-q$ negative eigenvalues. Then, for $q\le \ell\le n-1$, there exists a non-zero holomorphic section $s\in H^0(X, L^{k_0})$ for some $k_0\in\mathbb{N}$, such that for every $\overline{\partial}_b$-closed form $\sigma\in\Omega^{0,\ell}(bM, E)$, there exists a $\overline{\partial}$-closed extension S of $s\otimes\sigma\in\Omega^{0,\ell}(bM, L^{k_0}\otimes E)$, i.e.,*

$$S\in\Omega^{0,\ell}(\overline{M}, L^{k_0}\otimes E)\,, \quad \overline{\partial}S = 0 \text{ on } M\,, \quad \mu(S|_{bM}) = \mu(s\otimes\sigma) \text{ on } bM.$$

Note that the hypothesis on the line bundle L implies that X is a Moishezon manifold by Siu's criterion [30, Theorem 2.2.7]. The hypotheses of the theorem can be weakened as follows. It is enough to assume that the holomorphic line bundle L is big and semipositive in a neighborhood of the domain M. As a corollary, we obtain that under these conditions, if the Levi form on bM has one negative eigenvalue everywhere, then every smooth $(0, n-1)$-form on bM can be extended meromorphically over M; see Theorem 5.3. We will apply vanishing theorems to prove the $\overline{\partial}_b$-extension results in Theorems 6.2 and 6.3. A preliminary version of Theorem 1.12 can be found in [33].

For $k\in\mathbb{N}$, the space $H^0(\mathbb{CP}^n, \mathscr{O}(k))$ is canonically isomorphic to the space of homogeneous polynomials of degree k. By choosing $X = \mathbb{CP}^n$, $L = \mathscr{O}(1)$, E trivial, $\alpha = \sigma$, and $A = S\otimes s^{-1}$ in Theorem 1.12, we obtain the following interesting case.

Corollary 1.13. *Let $M\subset\mathbb{CP}^n$ be an open domain with a smooth boundary bM. Let $1\le q\le n-1$, and assume that the Levi form of bM restricted to the analytic tangent space $T^{(1,0)}bM$ has at least $n-q$ negative eigenvalues. Then, for each $q\le j\le n-1$, there exists a homogeneous polynomial $f_j\in\mathbb{C}[z_0, z_1, \cdots, z_n]$ of degree $k_j\in\mathbb{N}$ such that* *for every $\overline{\partial}_b$-closed form $\alpha \in \Omega^{0,j}(bM)$, there exists a $\overline{\partial}$-closed extension A of α on M outside the zero set $Z(f_j) := \{[z] \in \mathbb{CP}^n : f_j(z) = 0\}$, that is,*

$$A \in \Omega^{0,j}\left(\overline{M} \setminus Z(f_j)\right), \quad \overline{\partial} A = 0 \text{ on } M \setminus Z(f_j), \quad \mu\left(A|_{bM\setminus Z(f_j)}\right) = \mu\left(\alpha|_{bM\setminus Z(f_j)}\right).$$

1.5. Organization of the paper

This paper is structured as follows. Section 2 introduces the notation and reviews known results on cohomology groups for complex manifolds satisfying certain convexity or concavity conditions. In Section 3, we derive our version of the Bochner-Kodaira-Nakano formula and establish the Nakano-Griffiths inequality including boundary contributions. Section 4 is devoted to a systematic investigation of Morse inequalities for q-concave manifolds and domains, based on our Nakano-Griffiths inequality. In Section 5, we prove the Kohn-Rossi extension theorem for q-concave domains; see Theorem 1.12. Finally, in Section 6, we rederive the Andreotti-Tomassini vanishing theorem and apply it to the $\overline{\partial}_b$ extension problem; in addition, we obtain a vanishing theorem for domains with Levi-flat boundaries.

2. Preliminaries

Let (X, Θ) be a connected Hermitian manifold of dimension n, and let (F, h^F) be a holomorphic Hermitian vector bundle over X. Let $p, q \in \mathbb{N}$. We denote by $\Omega^{p,q}(X, F)$ the space of smooth (p, q)-forms with values in F, and by $\langle \cdot, \cdot \rangle_{h^F,\Theta}$ the pointwise Hermitian metric on $\Omega^{p,q}(X, F)$ induced by Θ and h^F. Let $M \Subset X$ be a relatively compact domain with a smooth boundary bM. We denote by $\overline{M} = M \bigcup bM$ the closure of M in X. Throughout this section, we use these notations without further notice.

2.1. Complex convexity

The theory of q-convexity/concavity was introduced by Andreotti and Grauert in [3] and is one of the basic tools in the study of the geometry of noncompact complex spaces. We recall here some important facts of the theory.

Definition 2.1. Let M be a connected complex manifold of dimension n. Let $1 \leqslant q \leqslant n$.

(i) M is called *q-convex* if there exists a smooth function $\varphi : M \longrightarrow [a, b)$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, such that the *sublevel set* $M_c = \{x \in M : \varphi(x) < c\}$ is relatively compact in M for all $c \in [a, b)$ and $\sqrt{-1}\partial\overline{\partial}\varphi$ has at least $n - q + 1$ positive eigenvalues outside a compact set K (called the exceptional set).

(ii) M is called *q-concave* if there exists a smooth function $\varphi : M \longrightarrow (a, b]$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R}$, such that the *superlevel set* $M_c = \{x \in M : \varphi(x) > c\}$ is relatively compact in M for all $c \in (a, b]$ and $\sqrt{-1}\partial\overline{\partial}\varphi$ has at least $n - q + 1$ positive eigenvalues outside a compact set $K \subset M$, called *an exceptional compact set*. In the sequel, we also use $\varrho := -\varphi$ as the sublevel exhaustion function and then $M_c = \{x \in M : \varrho(x) < c\} \Subset M$ for all $c \in [-b, -a)$.

In all these cases, we call φ or ϱ an *exhaustion function*.

Example 2.2. We provide here some examples of q-concave manifolds (cf. [32]).

(1) Let X be a compact Kähler space of pure dimension n and Y an analytic subset of pure dimension q containing the singular locus of X. Then $X \setminus Y$ is a $(q+1)$-concave manifold. In particular, the regular locus of a projective algebraic variety with isolated singularities is 1-concave.

(2) Let X be a compact complex manifold and Y an analytic subset of pure dimension q. Then $X \setminus Y$ is a $(q+1)$-concave manifold.

(3) If $\pi : X \to Y$ is a proper holomorphic map between a complex manifold X and a q-concave manifold Y such that the dimension of its fibers does not exceed r, then X is $(q+r)$-concave.

Example 2.3. A hyperconcave (or hyper 1-concave) manifold is a 1-concave manifold with $a = -\infty$; cf. [30, Definition 3.4.1]. A geometric example due to Siu–Yau is a complete Kähler manifold of finite volume and bounded negative sectional curvature; see more examples in [30, Example 3.4.2].

We also need the following terminology concerning the Levi convexity of domains M with smooth boundaries in a complex manifold X of dimension n, cf. [18]. We assume that there exists a real smooth function ϱ on X such that $M = \{x \in X : \varrho(x) < 0\}$, $bM = \{x \in X : \varrho(x) = 0\}$, and $d\varrho(x) \neq 0$ for any $x \in bM$. In addition, we assume that $|d\varrho| = 1$ on bM. Then, we call ϱ a defining function of M. Let $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$ be the splitting of the complexified tangential bundle. The analytic tangent space to bM at $x \in bM$ is given by

$$T_x^{(1,0)}bM := \{v \in T_x^{(1,0)}X : \partial\varrho(v) = 0\}.$$

The Levi form of ϱ is the 2-form $\mathscr{L}_\varrho \in \mathscr{C}^\infty(bM, T^{(1,0)*}bM \otimes T^{(0,1)*}bM)$ given by

$$\mathscr{L}_\varrho(U, \overline{V}) := (\partial\overline{\partial}\varrho)(U, \overline{V}), \quad \text{for } U, V \in T_x^{(1,0)}bM,\ x \in bM. \tag{2.1}$$

The number of positive and negative eigenvalues of the Levi form is independent of the choice of the defining function. We will denote the Levi form by $\mathscr{L}_{bM}$ when the defining function ϱ is not emphasized.

Definition 2.4. We say that M satisfies the condition $Z(q)$ if the Levi form $\mathscr{L}_\varrho$ has at least $n-q$ positive or at least $q+1$ negative eigenvalues at each point of bM.

Immediately, we have the following result (see [21, Theorem 6.7]).

Proposition 2.5. *Let M be a q-concave manifold with superlevel exhaustion function φ and exceptional compact set K. Then the following claims hold for any $c < \inf_K \varphi$:*

(a) The manifold $M_c = \{x \in M : \varphi > c\}$ is q-concave.

(b) If in addition c is a regular value of φ, then M_c satisfies the $Z(j)$ condition for $j \leq n - q - 1$.

Let $M \Subset X$ be a smooth domain such that its Levi form $\mathscr{L}_{bM}$ has at least $n-q$ negative eigenvalues on bM with $1 \leq q \leq n-1$. Let ϱ be its defining function. For $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ sufficiently small, the Levi form of $M_\varepsilon := \{x \in X : \varrho(x) < \varepsilon\}$ still has at least $n-q$ negative eigenvalues on bM_ε. Following [21, Theorem 6.7], if we set $\varrho_1 = -\exp(-C\varrho)+1$ for a sufficiently large constant $C > 0$, then we can rewrite $M = \{x \in X : \varrho_1 < 0\}$, and $\sqrt{-1}\partial\overline{\partial}\varrho_1$ has at least $n-q+1$ negative eigenvalues in a small neighborhood of bM. So M is a q-concave manifold.

Definition 2.6. A smooth domain $M \Subset X$ is called (Levi) q-concave ($1 \leq q \leq n-1$) if the Levi form $\mathscr{L}_{bM}$ has at least $n-q$ negative eigenvalues on bM.

The following important cohomology finiteness theorem is due to Andreotti-Grauert [3]; see also [7,18,22,36].

Theorem 2.7 *(Andreotti-Grauert). Let M be a q-concave manifold of dimension $n \geq 2$, and let F be a holomorphic vector bundle on M. Then the sheaf cohomology groups of M with values in F satisfy*

$$\dim H^j(M, F) < \infty, \quad j \leq n - q - 1. \tag{2.2}$$

2.2. $\overline{\partial}$-Neumann problem and Hodge theory

Let $p, q \in \mathbb{N}$ and $0 \leq p, q \leq n$. Let $\Omega^{p,q}(M, F)$ be the space of smooth (p,q)-forms with values in F. Let $\Omega^{p,q}(\overline{M}, F) \subset \Omega^{p,q}(M, F)$ be the subspace of those forms that are smooth up to and including the boundary bM. The L^2-scalar product on $\Omega^{p,q}(M, F)$ is given by

$$\langle s_1, s_2 \rangle_{L^2(M,F)} := \int_M \langle s_1(x), s_2(x) \rangle_{h^F, \Theta}\, dv_M(x) \tag{2.3}$$

where $dv_M = \Theta^n/n!$ is the volume form of M. We will write simply $\langle s_1, s_2 \rangle_{L^2}$ or $\langle s_1, s_2 \rangle_{L^2(M)}$ for $\langle s_1, s_2 \rangle_{L^2(M,F)}$ if there is no confusion, and we denote by $\| \cdot \|_{L^2}$ the corresponding L^2-norm and by $L^2_{p,q}(M, F)$ the L^2 completion of $\Omega^{p,q}(\overline{M}, F)$. Let

$$\overline{\partial}^F : \mathrm{Dom}(\overline{\partial}^F) \cap L^2_{p,q}(M, F) \to L^2_{p,q+1}(M, F)$$

be the maximal extension of the Cauchy-Riemann operator. Sometimes we will use $\overline{\partial}$ instead of $\overline{\partial}^F$ to simplify the notation. Let $\overline{\partial}^{F*}$ be the Hilbert space adjoint of $\overline{\partial}^F$.

Let $e_{\mathfrak{n}}$ be the inward pointing unit normal vector at bM. We decompose $e_{\mathfrak{n}} = e_{\mathfrak{n}}^{(1,0)} + e_{\mathfrak{n}}^{(0,1)} \in T^{(1,0)}X \oplus T^{(0,1)}X$. Set

$$B^{p,q}(M,F) = \{s \in \Omega^{p,q}(\overline{M},F) \ : \ \iota_{e_{\mathfrak{n}}^{(0,1)}} s = 0 \text{ on } bM\}. \tag{2.4}$$

Then by [30, Proposition 1.4.19], we have

$$B^{p,q}(M,F) = \mathrm{Dom}(\overline{\partial}^{F*}) \cap \Omega^{p,q}(\overline{M},F). \tag{2.5}$$

Furthermore, we define the Gaffney extension of the Kodaira Laplacian by

$$\begin{aligned} \mathrm{Dom}(\Box^F) &= \{s \in \mathrm{Dom}(\overline{\partial}^F) \cap \mathrm{Dom}(\overline{\partial}^{F*}) : \overline{\partial}^F s \in \mathrm{Dom}(\overline{\partial}^{F*}),\ \overline{\partial}^{F*} s \in \mathrm{Dom}(\overline{\partial}^F)\}, \\ \Box^F &:= \overline{\partial}^F \overline{\partial}^{F*} + \overline{\partial}^{F*} \overline{\partial}^F. \end{aligned} \tag{2.6}$$

We denote the space of harmonic (p,q)-forms with values in F by

$$\mathscr{H}^{p,q}(M,F) := \mathrm{Ker}(\Box^F) \cap L^2_{p,q}(M,F) = \{s \in \mathrm{Dom}(\Box^F) \cap L^2_{p,q}(M,F) : \Box^F s = 0\} \tag{2.7}$$

and the orthogonal projection by

$$H : L^2_{p,q}(M,F) \to \mathscr{H}^{p,q}(M,F). \tag{2.8}$$

We recall the solution of the $\overline{\partial}$-Neumann problem.

Theorem 2.8 *([18, (3.1.11), (3.1.14)]; [26, Theorem 5.11]). If M satisfies the $Z(q)$ condition, then there exists a compact operator (the Neumann operator)*

$$\mathscr{N} : L^2_{p,q}(M,F) \to L^2_{p,q}(M,F)$$

such that

(a) $\mathscr{N} L^2_{p,q}(M,F) \subset \mathrm{Dom}(\Box^F)$ *and* $L^2_{p,q}(M,F) = \Box^F \mathscr{N} L^2_{p,q}(M,F) + \mathscr{H}^{p,q}(M,F)$;
(b) $\mathscr{N}$ *commutes with* $\Box^F$, $\overline{\partial}^F$, $\overline{\partial}^{F*}$, H;
(c) $\mathscr{N}(\Omega^{p,q}(\overline{M},F)) \subset \Omega^{p,q}(\overline{M},F)$ *and* $H(\Omega^{p,q}(\overline{M},F)) \subset \Omega^{p,q}(\overline{M},F)$;
(d) $\mathscr{H}^{p,q}(M,F) \subset \Omega^{p,q}(\overline{M},F)$ *is finite dimensional.*

The reduced L^2-Dolbeault cohomology is defined by

$$\overline{H}^{p,q}_{(2)}(M,F) := \frac{\mathrm{Ker}(\overline{\partial}^F) \cap L^2_{p,q}(M,F)}{[\mathrm{Im}(\overline{\partial}^F) \cap L^2_{p,q}(M,F)]}, \tag{2.9}$$

where $[V]$ denotes the closure of the space V. By the weak Hodge decomposition (see [30, (3.1.21) and (3.1.22)]),

$$\overline{H}^{p,q}_{(2)}(M,F) \cong \mathscr{H}^{p,q}(M,F). \tag{2.10}$$

The (non-reduced) L^2-Dolbeault cohomology is defined by

$$H^{p,q}_{(2)}(M,F) := \frac{\operatorname{Ker}(\overline{\partial}^F) \cap L^2_{p,q}(M,F)}{\operatorname{Im}(\overline{\partial}^F) \cap L^2_{p,q}(M,F)}. \tag{2.11}$$

We say the fundamental estimate holds in bidegree (p,q) for forms with values in F if there exists a compact subset $K \subset M$ and $C > 0$ such that

$$\|s\|^2_{L^2(M)} \le C(\|\overline{\partial}^F s\|^2_{L^2(M)} + \|\overline{\partial}^{F*} s\|^2_{L^2(M)} + \int_K |s|^2 dv_M), \tag{2.12}$$

for $s \in \operatorname{Dom}(\overline{\partial}^F) \cap \operatorname{Dom}(\overline{\partial}^{F,*}) \cap L^2_{p,q}(M,F)$. If the fundamental estimate holds, the reduced and non-reduced L^2-Dolbeault cohomology coincide; see [30, Theorem 3.1.8].

The sheaf cohomology $H^q(M,F)$ for holomorphic sections of F is isomorphic to the Dolbeault cohomology $H^{0,q}(M,F)$. Besides, the Dolbeault cohomology up to the boundary is defined by

$$H^{p,q}(\overline{M},F) = \frac{\operatorname{Ker}(\overline{\partial}^F) \cap \Omega^{p,q}(\overline{M},F)}{\operatorname{Im}(\overline{\partial}^F) \cap \Omega^{p,q}(\overline{M},F)}. \tag{2.13}$$

Regarding the relationships among the aforementioned cohomology groups, we present the following fundamental results.

Theorem 2.9 *([18, (3.1.14), (4.3.1)], [22, Theorem 3.4.9]). If M satisfies $Z(q)$, then*

$$H^{p,q}(\overline{M},F) \simeq H^{p,q}_{(2)}(M,F) \simeq \mathscr{H}^{p,q}(M,F). \tag{2.14}$$

If M satisfies $Z(q)$ and $Z(q+1)$, then

$$H^{p,q}(M,F) \simeq \mathscr{H}^{p,q}(M,F).$$

The spaces $H^{p,q}(\overline{M},F)$ and $H^{p,q}(M,F)$ are independent of the metrics used on the domain M and F, whereas the other spaces mentioned rely on metric data.

Theorem 2.10 *([3, Proposition 18]). If X is a connected q-concave manifold of dimension $n \ge 2$ equipped with an exhaustion function ϱ as in Definition 2.1 and $1 \le q \le n-1$, then for a regular value $c \in \mathbb{R}$ of ϱ such that the sublevel set X_c contains the exceptional set K of ϱ, we have*

$$H^j(X,F) \simeq H^j(X_c,F) \simeq H^{0,j}(X_c,F), \quad \text{for } j \le n-q-1. \tag{2.15}$$

Theorem 2.11 *([22, Theorem 3.4.9 and the subsequent remark]). Under the same hypothesis as Theorem 2.10, we have for* $0 \le j \le n-q-1$,

$$\dim H^j(X,F) \le \dim H^{0,j}_{(2)}(X_c,F) = \dim H^{0,j}(\overline{X}_c,F). \tag{2.16}$$

2.3. Fundamental estimates and abstract holomorphic Morse inequalities

We recall the results presented in [30, Section 3.2] on the abstract holomorphic Morse inequalities for L^2-cohomology. The main idea is to show that the spectral spaces of the Laplacian, associated with the small eigenvalues, inject into the spectral spaces of the Laplacian with Dirichlet boundary conditions on a smooth, relatively compact domain. The asymptotics of the latter operator can be calculated explicitly; for example, see [30, Theorem 3.2.9]. We also refer to [11,15,32] for these results.

Let (X,Θ) be a Hermitian manifold of dimension n. Let L, E be holomorphic vector bundles on X equipped with smooth Hermitian metrics h^L, h^E. Throughout, we assume $\operatorname{rank}(L)=1$. Let $\overline{\partial}^E_k : \Omega^{0,\bullet}_0(X,L^k\otimes E) \to \Omega^{0,\bullet+1}_0(X,L^k\otimes E)$ be the associated $\overline{\partial}$-operator, and we use the same notation for its maximal extension with respect to the L^2-inner product.

We say that *the fundamental estimate* holds in bidegree $(0,q)$ for forms with values in $L^k\otimes E$, if there exists a compact subset $K \subset X$ and $C_0>0$ such that for sufficiently large k, we have for all $s \in \operatorname{Dom}(\overline{\partial}^E_k) \cap \operatorname{Dom}(\overline{\partial}^{E,*}_k) \cap L^2_{(0,q)}(X,L^k\otimes E)$,

$$\|s\|^2_{L^2(X)} \le \frac{C_0}{k}\left(\|\overline{\partial}^E_k s\|^2_{L^2(X)} + \|\overline{\partial}^{E,*}_k s\|^2_{L^2(X)}\right) + C_0\int_K |s|^2 dv_X. \tag{2.17}$$

Note that (2.17) implies the fundamental estimate (2.12) for sufficiently large k.

Theorem 2.12 *([30, Theorem 3.2.13]). Assume that there exists* $m \in \mathbb{N}$ *such that the fundamental estimate* (2.17) *holds for* $(0,j)$*-forms with values in* $L^k\otimes E$ *and any* $j \le m$*. Let* $U \Subset X$ *with a smooth boundary be such that* $K \Subset U$*. Then as* $k\to\infty$*, the following strong and weak Morse inequalities hold for any* $j \le m$,

$$\sum_{\ell=0}^{j}(-1)^{j-\ell}\dim H^{0,\ell}_{(2)}(X,L^k\otimes E) \le \operatorname{rk}(E)\frac{k^n}{n!}\int_{U(\le j,h^L)}(-1)^j c_1(L,h^L)^n + o(k^n), \tag{2.18}$$

and

$$\dim H^{0,j}_{(2)}(X,L^k\otimes E) \le \operatorname{rk}(E)\frac{k^n}{n!}\int_{U(j,h^L)}(-1)^j c_1(L,h^L)^n + o(k^n). \tag{2.19}$$

As a consequence of [30, Theorem 3.2.9, Proposition 3.2.11, and (3.2.63)], we have the following result (implicit in the proof of [30, Theorem 3.2.13]).

Corollary 2.13. *Assume that the fundamental estimate (2.17) holds for $(0,q)$-forms with values in $L^k \otimes E$ for some $q \in \{0,1,\dots,n\}$. Let $U \Subset X$ with a smooth boundary be such that $K \Subset U$. Then as $k \to \infty$, the weak Morse inequalities (2.19) hold at degree $(0,q)$.*

For the space of L^2-holomorphic sections $H^0_{(2)}(X, L^k \otimes E) := H^{0,0}_{(2)}(X, L^k \otimes E)$, there is also a lower bound provided the fundamental estimate holds.

Theorem 2.14 *([30, Theorem 3.2.16]). Assume that the fundamental estimate (2.17) holds for $(0,1)$-forms with values in $L^k \otimes E$. Let $U \Subset X$ with a smooth boundary such that $K \Subset U$. Then as $k \to \infty$, we have*

$$\dim H^0_{(2)}(X, L^k \otimes E) \geq \operatorname{rk}(E) \frac{k^n}{n!} \int_{U(\leq 1, h^L)} c_1(L, h^L)^n + o(k^n). \tag{2.20}$$

2.4. The $\overline{\partial}_b$ operator on the boundary

Let $\Omega^{p,q}(bM, F)$ be the (p,q)-forms with values in F over bM, that is, $\Omega^{p,q}(bM, F) := \Omega^{p,q}(X, F)|_{bM}$. We recall some notions in [26]. A form $\alpha \in \Omega^{p,q}(bM, F)$ is called complex normal, if there exists $\psi \in \Omega^{p,q-1}(bM, F)$ such that $\alpha = \psi \wedge (\overline{\partial}\varrho)|_{bM}$, where ϱ is a defining function of M. We denote by $\mathcal{C}^{p,q}(bM, F)$ the subspace of $\Omega^{p,q}(bM, F)$ consisting of complex normal forms. A form $\beta \in \Omega^{p,q}(bM, F)$ is called complex tangential, if $\langle \alpha(x), \beta(x) \rangle_{h^F, \Theta} = 0$ for every $\alpha \in \mathcal{C}^{p,q}(bM, F)$ and every $x \in bM$. We denote by $\mathcal{D}^{p,q}(bM, F)$ the subspace of $\Omega^{p,q}(bM, F)$ consisting of complex tangential forms. With respect to the pointwise Hermitian product $\langle \cdot, \cdot \rangle_{h^F, \Theta}$, we have

$$\Omega^{p,q}(bM, F) = \mathcal{C}^{p,q}(bM, F) \oplus \mathcal{D}^{p,q}(bM, F). \tag{2.21}$$

Moreover, we denote the projections by $\mu : \Omega^{p,q}(bM, F) \to \mathcal{D}^{p,q}(bM, F)$, $p, q \in \mathbb{N}$.

A form $\Sigma \in \Omega^{p,q}(\overline{M}, E)$ is a $\overline{\partial}$-closed extension of $\sigma \in \Omega^{p,q}(bM, E)$ if $\overline{\partial}\Sigma = 0$ on M and $\mu(\Sigma|_{bM}) = \mu(\sigma)$, i.e., Σ is $\overline{\partial}$-closed and $\Sigma = \sigma$ in the complex tangential direction on bM. We define a map

$$\overline{\partial}_b : \Omega^{p,q}(bM, E) \to \Omega^{p,q+1}(bM, E), \quad \overline{\partial}_b \sigma := \mu((\overline{\partial}\sigma')|_{bM}) \tag{2.22}$$

where $\sigma' \in \Omega^{p,q}(\overline{M}, E)$ and $\sigma'|_{bM} = \sigma$. The definition of $\bar{\partial}_b$ is independent of the choice of σ'. We say σ is $\overline{\partial}_b$-closed if $\overline{\partial}_b \sigma = 0$. Kohn-Rossi provided a general criterion for the extension of $\overline{\partial}_b$-closed forms as follows.

Theorem 2.15 *([26, Proposition 5.13]; [18, Theorem 5.3.1]). Suppose bM has property $Z(n-q-1)$. Let F be a holomorphic vector bundle over X. If $\sigma \in \Omega^{p,q}(bM, F)$, there exists a $\overline{\partial}$-closed extension Σ of σ if and only if*

$$\overline{\partial}_b\sigma = 0 \quad \text{and} \quad \int\limits_{bM} \theta\wedge\sigma = 0 \qquad \text{for } \theta\in \mathscr{H}^{n-p,n-q-1}(M,F^*). \tag{2.23}$$

3. Bochner-Kodaira-Nakano formula with boundary term

Let (X, J, Θ) be a Hermitian manifold of dimension $n \geq 1$ where J denotes the complex structure, and define the associated J-invariant Riemannian metric on TX by

$$g^{TX}(\cdot,\cdot) := \Theta(\cdot, J\cdot). \tag{3.1}$$

The corresponding volume form is $dv_X = \Theta^n/n!$. Let ∇^{TX} denote the Levi-Civita connection of (TX, g^{TX}), and let R^{TX} denote the Riemannian curvature tensor. Let $h^{T^{(1,0)}X}$ (resp. $h^{T^{(0,1)}X}$) denote the Hermitian metrics on $T^{(1,0)}X$ (resp. $T^{(0,1)}X$) induced by g^{TX}.

Let F be a holomorphic vector bundle on X equipped with a smooth Hermitian metric h^F. In this section, we will use the notation $\langle\cdot,\cdot\rangle$ or $\langle\cdot,\cdot\rangle_{g^{TX}}$ to denote the bilinear form on TX or $TX\otimes_{\mathbb{R}}\mathbb{C}$, and we will denote $\langle\cdot,\cdot\rangle_{h^F,\Theta}$ or simply $\langle\cdot,\cdot\rangle_h$ the Hermitian inner products.

Let ∇^F denote the Chern connection of (F, h^F), and let R^F denote the corresponding Chern curvature form. In particular, let $\nabla^{T^{(1,0)}X}$ be the Chern connection of the holomorphic Hermitian vector bundle $(T^{(1,0)}X, h^{T^{(1,0)}X})$. For $s \in \mathscr{C}^\infty(X, T^{(0,1)}X)$, $\overline{s}\in\mathscr{C}^\infty(X,T^{(1,0)}X)$, we have

$$\nabla^{T^{(1,0)}X}\overline{s}\in\Omega^1(X,T^{(1,0)}X) = \mathscr{C}^\infty(X,\mathbb{C}T^*X\otimes T^{(1,0)}X).$$

We define $\nabla^{T^{(0,1)}X}s = \overline{\nabla^{T^{(1,0)}X}\overline{s}}\in\mathscr{C}^\infty(X,\mathbb{C}T^*X\otimes T^{(0,1)}X)$. Set

$$\widetilde{\nabla}^{TX} := \nabla^{T^{(1,0)}X}\oplus\nabla^{T^{(0,1)}X} \tag{3.2}$$

Then $\widetilde{\nabla}^{TX}$ defines a connection on $\mathbb{C}\otimes_{\mathbb{R}} TX$ which preserves the real tangent bundle TX and the Riemannian metric g^{TX}.

We define the covariant derivative

$$\widetilde{\nabla}^{TX}_Y(\alpha\wedge\beta) = \widetilde{\nabla}^{TX}_Y(\alpha)\wedge\beta + \alpha\wedge\widetilde{\nabla}^{TX}_Y(\beta) \tag{3.3}$$

for differential forms α, β. For $\alpha\in T^*X$, $U, V\in TX$,

$$U(\alpha(V)) = (\widetilde{\nabla}^{TX}_U\alpha)V + \alpha(\widetilde{\nabla}^{TX}_U V). \tag{3.4}$$

When we consider the differential forms valued in F, we also use the same notation $\widetilde{\nabla}^{TX}$ for the connection or covariant derivative given by $\widetilde{\nabla}^{TX}\otimes\mathrm{Id}_F + \mathrm{Id}_{\Lambda(T^*X)}\otimes\nabla^F$.

Let $T\in\wedge^2T^*X\otimes TX$ be the torsion of the connection $\widetilde{\nabla}^{TX}$ given by

$$T(U,V) = \widetilde{\nabla}^{TX}_U V - \widetilde{\nabla}^{TX}_V U - [U,V] \tag{3.5}$$

for $U, V \in TX$. By [30, (1.2.37)], T maps $T^{(1,0)}X \otimes T^{(1,0)}X$ (resp. $T^{(0,1)}X \otimes T^{(0,1)}X$) into $T^{(1,0)}X$ (resp. $T^{(0,1)}X$), and vanishes on $T^{(1,0)}X \otimes T^{(0,1)}X$. We will also write $T = T^{(1,0)} + T^{(0,1)}$. Note that Θ is Kähler if and only if ∇^{TX} preserves the splitting $T^{(1,0)}X \oplus T^{(0,1)}X$ and in this case we have $\nabla^{TX} = \widetilde{\nabla}^{TX}$, $T = 0$.

In this section, we consider a smooth, relatively compact domain M in X. As in Subsection 2.1, we will always set $M = \{x \in X \ : \ \varrho(x) < 0\}$, where $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ is a defining function of M such that $|d\varrho| = 1$ on bM. The Hermitian metric $h^{T^{(1,0)}bM}$ on $T^{(1,0)}bM$ is induced by $h^{T^{(1,0)}X}$. In this setup, given any orthonormal basis $\{\omega_j\}_{j=1}^n$ of $(T^{(1,0)}X, h^{T^{(1,0)}X})$ at a point $y \in bM$, we have

$$e_{\mathfrak{n}}^{(1,0)} = -\sum_{j=1}^n \bar{\omega}_j(\varrho)\omega_j, \ e_{\mathfrak{n}}^{(0,1)} = -\sum_{j=1}^n \omega_j(\varrho)\bar{\omega}_j. \tag{3.6}$$

Let dv_{bM} be the volume form on bM defined by

$$dv_X = d\varrho \wedge dv_{bM} \quad \text{on } bM. \tag{3.7}$$

3.1. Bochner-Kodaira-Nakano formula with boundary terms

We recall the Bochner-Kodaira-Nakano formulas with boundary terms given by Andreotti-Vesentini [7, p. 113], Griffiths [20, (7.14)], and Ma-Marinescu [30, Theorem 1.4.21].

First, we introduce the necessary notation. Set the Lefschetz operator $\mathrm{Lef}_\Theta = (\Theta\wedge) \otimes \mathrm{Id}_F$ on $\Lambda^{\bullet,\bullet}(T^*X) \otimes F$, and let $\Lambda = \iota_\Theta$ be its adjoint with respect to the Hermitian product $\langle \cdot, \cdot \rangle_h$.

Definition 3.1. The Hermitian torsion operator $\mathcal{T}$ acting on $\Lambda^{\bullet,\bullet}(T^*X)$ is given as follows

$$\mathcal{T} = \frac{1}{2} \sum_{j,k,m} \langle T(\omega_j, \omega_k), \bar{\omega}_m \rangle \left[2\omega^k \wedge \bar{\omega}^m \wedge \iota_{\omega_j} - 2\delta_{jm}\omega^k - \omega^j \wedge \omega^k \wedge \iota_{\omega_m} \right], \tag{3.8}$$

where $\{\omega_j\}_{j=1}^n$ is an orthonormal basis of $(T^{(1,0)}X, h^{T^{(1,0)}X})$ with the dual basis $\{\omega^j\}_{j=1}^n$ of $T^{(1,0)*}X$. We will use the same notation $\mathcal{T}$ for the operator $\mathcal{T} \otimes \mathrm{Id}_F$ on $\Lambda^{\bullet,\bullet}(T^*X) \otimes F$.

We write $\nabla^F = (\nabla^F)^{1,0} + (\nabla^F)^{0,1}$ as operators acting on differential forms valued in F. Let $(\nabla^F)^{1,0*}$, $(\nabla^F)^{0,1*}$, $\mathcal{T}^*$, $\overline{\mathcal{T}}^*$ denote the (formal) adjoints of $(\nabla^F)^{1,0}$, $(\nabla^F)^{0,1} = \overline{\partial}^F$, $\mathcal{T}$, $\overline{\mathcal{T}}$ with respect to the L^2-metric or the Hermitian metric $\langle \cdot, \cdot \rangle_h$. Analogous to $\square^F$, the holomorphic Kodaira Laplacian is defined as

$$\overline{\square}^F = [(\nabla^F)^{1,0}, (\nabla^F)^{1,0*}]. \tag{3.9}$$

We recall the Bochner-Kodaira-Nakano formula of Demailly [16, Proposition 2.1]:

$$\square^F = \overline{\square}^F + [\sqrt{-1}R^F, \Lambda] + [(\nabla^F)^{1,0}, \mathcal{T}^*] - [\overline{\partial}^F, \overline{\mathcal{T}}^*]. \tag{3.10}$$

Note that the curvature term $[\sqrt{-1}R^F, \Lambda]$ is crucial for various applications of the above formula. By the definition of Λ, in an orthonormal basis $\{\omega_j\}_{j=1}^n$ of $(T^{(1,0)}X, h^{T^{(1,0)}X})$, we have

$$[\sqrt{-1}R^F, \Lambda] = \sum_{j,k} R^F(\omega_j, \overline{\omega}_k)\omega^j \wedge \iota_{\omega_k} - \sum_{j,k} R^F(\omega_j, \overline{\omega}_k)\iota_{\overline{\omega}_j}\overline{\omega}^k \wedge . \tag{3.11}$$

Write

$$\mathrm{Tr}_\Theta[\sqrt{-1}R^F] = \sum_{j=1}^n R^F(\omega_j, \overline{\omega}_j) \in \mathrm{End}(F). \tag{3.12}$$

For $s \in \Omega^{0,q}(X, F)$, $0 \le q \le n$, we have

$$[\sqrt{-1}R^F, \Lambda]s = \sum_{j,k} R^F(\omega_j, \overline{\omega}_k)\overline{\omega}^k \wedge \iota_{\overline{\omega}_j} s - \mathrm{Tr}_\Theta[\sqrt{-1}R^F]s. \tag{3.13}$$

Set $\widetilde{F} = F \otimes K_X^*$ with $K_X^* := \Lambda^n(T^{(1,0)}X) = \det T^{(1,0)}X$. Since $K_X \otimes K_X^* \simeq \mathbb{C}$, there exists a natural isometry

$$\Psi : \Lambda^{0,q}(T^*X) \otimes F \to \Lambda^{n,q}(T^*X) \otimes \widetilde{F}, \quad \Psi s := (\omega^1 \wedge \ldots \wedge \omega^n \wedge s) \otimes (\omega_1 \wedge \ldots \wedge \omega_n). \tag{3.14}$$

Now we consider the smooth, relatively compact domain $M = \{x \in X : \varrho < 0\} \Subset X$ with a defining function $\varrho \in \mathscr{C}^\infty(X)$ satisfying $|d\varrho| = 1$ on bM, and the sections in $B^{0,q}(M, F)$. The Bochner-Kodaira-Nakano formula with a boundary term is as follows:

Theorem 3.2 *(Ma-Marinescu, [30, Theorem 1.4.21]). For $q \in \{0, 1, \ldots, n\}$ and for $s \in B^{0,q}(M, F)$, we have*

$$\begin{aligned}
&\|\overline{\partial}^F s\|^2_{L^2(M)} + \|\overline{\partial}^{F*} s\|^2_{L^2(M)} \\
=&\|(\nabla^{\widetilde{F}})^{1,0*}\Psi s\|^2_{L^2(M)} + \langle \sum_{j,k=1}^n R^{F\otimes K_X^*}(\omega_j, \overline{\omega}_k)\overline{\omega}^k \wedge \iota_{\overline{\omega}_j} s, s\rangle_{L^2(M)} \\
&- \langle \overline{\partial}^F s, \Psi^{-1}\overline{\mathcal{T}}\Psi s\rangle_{L^2(M)} - \langle \Psi^{-1}\overline{\mathcal{T}}^*\Psi s, \overline{\partial}^{F*} s\rangle_{L^2(M)} \\
&+ \langle \overline{\mathcal{T}}^*\Psi s, (\nabla^{\widetilde{F}})^{1,0*}\Psi s\rangle_{L^2(M)} \\
&+ \int_{bM} \langle \sum_{j,k=1}^n \mathscr{L}_\varrho(\omega_j, \overline{\omega}_k)\overline{\omega}^k \wedge \iota_{\overline{\omega}_j} s, s\rangle_h dv_{bM},
\end{aligned} \tag{3.15}$$

where $\{\omega_j\}_{j=1}^n$ is an orthonormal basis of $(T^{(1,0)}X, h^{T^{(1,0)}X})$ and $\mathscr{L}_\varrho$ is the Levi form defined in (2.1). In the boundary term in the last line, we can replace the sum over *the basis* $\{\omega_j\}_{j=1}^n$ *by the sum over an orthonormal basis of* $(T^{(1,0)}bM, h^{T^{(1,0)}bM})$ *since* $\iota_{e_{\mathfrak{n}}^{(0,1)}} s|_{bM} = 0$.

By substituting the explicit form (3.14) for the isometry Ψ into (3.15), we obtain the following form of the Bochner-Kodaira-Nakano formula with boundary term, which is due to Andreotti-Vesentini [7, p. 113] and Griffiths [20, (7.14)].

Theorem 3.3. *For* $s \in B^{0,q}(M,F)$*, we have*

$$\begin{aligned}
&\|\overline{\partial}^F s\|^2_{L^2(M)} + \|\overline{\partial}^{F*} s\|^2_{L^2(M)} \\
&= \sum_{j=1}^n \|\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s\|^2_{L^2(M)} + \langle \sum_{j,k=1}^n R^{F\otimes K_X^*}(\omega_j, \overline{\omega}_k)\overline{\omega}^k \wedge \iota_{\overline{\omega}_j} s, s\rangle_{L^2(M)} \\
&\quad + \int_{bM} \langle \sum_{j,k} \mathscr{L}_\varrho(\omega_j, \overline{\omega}_k)\overline{\omega}^k \wedge \iota_{\overline{\omega}_j} s, s\rangle_h dv_{bM} \\
&\quad + \langle \overline{\partial}^F s, \iota_{T^{(0,1)}} s\rangle_{L^2(M)} + \langle \iota^*_{T^{(0,1)}} s, \overline{\partial}^{F*} s\rangle_{L^2(M)} \\
&\quad + \sum_{j,k,m=1}^n \langle \langle T(\overline{\omega}_k, \overline{\omega}_j), \omega_m\rangle \overline{\omega}^j \wedge \iota_{\overline{\omega}_m} s, \widetilde{\nabla}^{TX}_{\overline{\omega}_k} s\rangle_{L^2(M)},
\end{aligned} \tag{3.16}$$

where $\iota_{T^{(0,1)}} := \frac{1}{2}\sum_{j,k,m=1}^n \langle T(\overline{\omega}_j, \overline{\omega}_k), \omega_m\rangle \overline{\omega}^j \wedge \overline{\omega}^k \wedge \iota_{\overline{\omega}_m}$*, and* $\iota^*_{T^{(0,1)}}$ *denotes the dual of* $\iota_{T^{(0,1)}}$ *with respect to the Hermitian metric.*

3.2. A variant of Bochner-Kodaira-Nakano formula; Proof of Theorem 1.1

Recall that we always assume $|d\varrho|_{bM} = 1$, so that $|d\varrho|$ is a positive smooth function near bM, which might not be constant. By our definition of $e_{\mathfrak{n}}$, it is clear that $\sqrt{-1}(e_{\mathfrak{n}}^{(1,0)} - e_{\mathfrak{n}}^{(0,1)})$ is a real vector field tangent to bM; hence, the derivatives of $|d\varrho|$ in the normal direction on bM satisfy

$$\frac{1}{2} e_{\mathfrak{n}}(|d\varrho|)|_{bM} = e_{\mathfrak{n}}^{(1,0)}(|d\varrho|)|_{bM} = e_{\mathfrak{n}}^{(0,1)}(|d\varrho|)|_{bM} \in \mathscr{C}^\infty(bM,\mathbb{R}). \tag{3.17}$$

Note that the Levi form $\mathscr{L}_\varrho$ is defined on $\Lambda^2 T(bM)$. In a local orthonormal frame $\{\omega_j\}_{j=1}^{n-1}$ of $(T^{1,0}bM, h^{T^{1,0}bM})$, we have

$$\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_\varrho] = \sum_{j=1}^{n-1} \mathscr{L}_\varrho(\omega_j, \overline{\omega}_j) \in \mathscr{C}^\infty(bM,\mathbb{R}). \tag{3.18}$$

Lemma 3.4. *Let* $\tau \in \mathscr{C}^\infty(X,[0,1])$ *such that* $\tau = 1$ *near* bM *and* $\operatorname{supp}\tau$ *is included in an open neighborhood of* bM *where* $d\varrho$ *does not vanish. Let* $\Omega_\tau \in \Omega^{n-1,n-1}(X,\mathbb{R})$ *be given by*

$$\Omega_\tau := \frac{\Theta^{n-1}}{(n-1)!} - 2\sqrt{-1}\tau * (\partial\varrho \wedge \bar{\partial}\varrho),$$

where $*$ denotes the Hodge star operator associated to g^{TX}. Then we have

$$\frac{\sqrt{-1}}{2}(\partial - \bar{\partial})\Omega_\tau|_{bM} = \left(\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_\varrho] - e_{\mathfrak{n}}(|d\varrho|)\right) dv_{bM}, \tag{3.19}$$

and

$$\Omega_\tau|_{bM} = 0. \tag{3.20}$$

Moreover, if $n \geq 2$, we can choose the function τ as above such that

$$\Theta \wedge \Omega_\tau > 0\,, \quad \textit{near } M. \tag{3.21}$$

If $n = 1$, for any $\varepsilon > 0$, we can choose the function τ as above such that

$$\Theta \wedge \Omega_\tau + \varepsilon dv_X > 0\,, \quad \textit{near } M. \tag{3.22}$$

Proof. At first, we consider a special local frame of $T^{(1,0)}X$ near bM. Let $\{\omega_j\}_{j=1}^n$ be an orthonormal frame of $(T^{(1,0)}X, h^{T^{(1,0)}X})$; in particular, near bM, we take

$$\omega_n = \frac{\sqrt{2}}{|d\varrho|} e_{\mathfrak{n}}^{(1,0)}, \tag{3.23}$$

where $1/|d\varrho| > 0$ is a smooth function near bM with $(1/|d\varrho|)|_{bM} \equiv 1$. Let $\{\omega^j\}_{j=1}^n$ be the dual frame of $T^{(1,0)*}X$; in particular, near bM, we have

$$\omega^n = -\frac{\sqrt{2}}{|d\varrho|}\partial\varrho.$$

As a consequence, the following $(1,1)$-form acting on $T^{1,0}bM \otimes T^{0,1}bM$ satisfies

$$\omega^n \wedge \overline{\omega}^n|_{bM} = 0. \tag{3.24}$$

Here the restriction is the pull-backs under the inclusion $i_{bM} : bM \to X$.

Then for $k \in \{1, \ldots, n-1\}$ and near bM, we have

$$\omega_k(\varrho) = \overline{\omega}_k(\varrho) = 0, \tag{3.25}$$

and

$$\omega_n(\varrho) = \overline{\omega}_n(\varrho) = -\frac{|d\varrho|}{\sqrt{2}}.$$

Therefore, at bM, $\{\omega_j\}_{j=1}^{n-1}$ forms a local orthonormal frame of $(T^{(1,0)}bM, h^{T^{(1,0)}bM})$. Moreover, near bM, we have

$$\partial\bar{\partial}\varrho(\omega_k, \overline{\omega}_k) = \begin{cases} -(\bar{\partial}\varrho, [\omega_k, \overline{\omega}_k]) = \langle e_{\mathfrak{n}}^{(1,0)}, \widetilde{\nabla}^{TX}_{\omega_k}\overline{\omega}_k\rangle, \text{ for } k < n; \\ -\dfrac{1}{|d\varrho|} e_{\mathfrak{n}}(|d\varrho|) + \dfrac{2}{|d\varrho|^2}\langle e_{\mathfrak{n}}^{(1,0)}, \widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(1,0)}} e_{\mathfrak{n}}^{(0,1)}\rangle, \text{ for } k = n. \end{cases} \tag{3.26}$$

Note that the Riemannian volume form associated with g^{TX} is given as

$$dv_X = \frac{\Theta^n}{n!} = \Lambda_{j=1}^n(\sqrt{-1}\omega^j \wedge \overline{\omega}^j). \tag{3.27}$$

For $k \in \{1, \dots, n\}$, we denote the local real $(n-1, n-1)$ form

$$\Psi_{k\bar{k}} = \sqrt{-1}\iota_{\omega_k}\iota_{\overline{\omega}_k} dv_X. \tag{3.28}$$

Then we have

$$\Omega_\tau = \sum_{k=1}^{n-1} \Psi_{k\bar{k}} + (1 - \tau|d\varrho|^2)\Psi_{n\bar{n}}. \tag{3.29}$$

It follows directly that $\Omega_\tau|_{bM} = 0$. We have

$$\Theta \wedge \Omega_\tau = (n - \tau|d\varrho|^2)dv_X. \tag{3.30}$$

If $n \geq 2$, we can choose τ such that $n - \tau|d\varrho|^2 > 0$ near the domain M. Therefore we obtain (3.21). When $n = 1$, for any $\varepsilon > 0$, we can choose τ appropriately such that $1 - \tau|d\varrho|^2 > -\varepsilon$ near the domain M, thereby satisfying (3.22).

It remains to prove equation (3.19). Initially, it is clear that

$$\iota_{\omega_n} dv_X|_{bM} = -\frac{1}{\sqrt{2}} dv_{bM}, \; \iota_{\overline{\omega}_n} dv_X|_{bM} = -\frac{1}{\sqrt{2}} dv_{bM}. \tag{3.31}$$

For $1 \leq k \leq n-1$, we compute

$$\begin{aligned} \partial\Psi_{k\bar{k}}|_{bM} &= -\sqrt{-1}\langle\partial\overline{\omega}^n, \omega_k \wedge \overline{\omega}_k\rangle\iota_{\overline{\omega}_n} dv_X|_{bM} \\ &= \frac{\sqrt{-1}}{\sqrt{2}}\langle\partial\overline{\omega}^n, \omega_k \wedge \overline{\omega}_k\rangle dv_{bM} \\ &= -\sqrt{-1}\partial\bar{\partial}\varrho(\omega_k, \overline{\omega}_k) dv_{bM}. \end{aligned} \tag{3.32}$$

Similarly, we have

$$\bar{\partial}\Psi_{k\bar{k}}|_{bM} = \sqrt{-1}\partial\bar{\partial}\varrho(\omega_k, \overline{\omega}_k) dv_{bM}. \tag{3.33}$$

Then we have

$$\frac{\sqrt{-1}}{2}(\partial-\bar{\partial})\Psi_{k\bar{k}}|_{bM}=\partial\bar{\partial}\varrho(\omega_k,\overline{\omega}_k)dv_{bM}. \tag{3.34}$$

Now we deal with the last term $(1-\tau|d\varrho|^2)\Psi_{n\bar{n}}$. We have

$$\begin{aligned}(\partial-\bar{\partial})(1-\tau|d\varrho|^2)\Psi_{n\bar{n}}|_{bM}&=2(-\omega_n(|d\varrho|)\sqrt{-1}\iota_{\overline{\omega}_n}dv_X-\overline{\omega}_n(|d\varrho|)\sqrt{-1}\iota_{\omega_n}dv_X)|_{bM}\\&=2\sqrt{-1}e_{\mathfrak{n}}(|d\varrho|)dv_{bM}.\end{aligned} \tag{3.35}$$

Then (3.19) follows from (3.29), (3.34) and (3.35). This finishes the proof. □

As a consequence, we obtain a slight extension of [20, Theorem 7.3].

Corollary 3.5. *Let $\tau\in\mathscr{C}^\infty(X,[0,1])$ and Ω_τ be as in Lemma 3.4. Then for $g\in\mathscr{C}^\infty(\overline{M})$, we have*

$$\int_M\sqrt{-1}\partial\bar{\partial}g\wedge\Omega_\tau-\int_M g\sqrt{-1}\partial\bar{\partial}\Omega_\tau=\int_{bM}g\left(\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_\varrho]-e_{\mathfrak{n}}(|d\varrho|)\right)dv_{bM}. \tag{3.36}$$

Proof. By Lemma 3.4 and Stokes' formula, we get

$$\begin{aligned}\int_{bM}g\left(\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_\varrho]-e_{\mathfrak{n}}(|d\varrho|)\right)dv_{bM}&=\frac{1}{2}\int_{bM}g\sqrt{-1}(\partial-\bar{\partial})\Omega_\tau\\&=\frac{1}{2}\int_M\sqrt{-1}d\left(g(\partial-\bar{\partial})\Omega_\tau\right)\\&=\int_M\sqrt{-1}\partial\bar{\partial}g\wedge\Omega_\tau-\int_M g\sqrt{-1}\partial\bar{\partial}\Omega_\tau\\&\quad+\frac{\sqrt{-1}}{2}\int_{bM}(\partial-\bar{\partial})g\wedge\Omega_\tau.\end{aligned} \tag{3.37}$$

By (3.20), the last integral on bM vanishes, so that we obtain (3.36). □

Prior to presenting a more complex variant of the Bochner-Kodaira-Nakano formula with boundary terms, we introduce some notation to streamline our expressions.

Definition 3.6 *(Modified curvature commutators).* For $\varepsilon\in[0,1]$, we define the following operator on $\Omega^{p,q}(X,F)$,

$$[\sqrt{-1}R^F,\Lambda]_\varepsilon:=[\sqrt{-1}R^F,\Lambda]+\varepsilon\mathrm{Tr}_\Theta[\sqrt{-1}R^F], \tag{3.38}$$

where $\mathrm{Tr}_\Theta[\sqrt{-1}R^F]$ is defined in (3.12). Similarly, we define the operator acting on $\mathcal{D}^{p,q}(bM,F)$ (see (2.21)) by

$$[\sqrt{-1}\mathscr{L}_\varrho,\Lambda]^{bM}_\varepsilon := [\sqrt{-1}\mathscr{L}_\varrho,\Lambda|_{bM}] + \varepsilon\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_\varrho]. \tag{3.39}$$

Note that, for $s \in B^{0,q}(M,F)$, we have $s|_{bM} \in \mathcal{D}^{0,q}(bM,F)$.

We will also need the following notation which appeared in Theorem 1.1. Recall that for a 1-form β on X, $\mathrm{Tr}[\nabla\beta](x)$ is defined with respect to (TX, g^{TX}). If $\{e_j\}_{j=1}^{2n}$ is an orthonormal basis of the real tangent space T_xX, then

$$\mathrm{Tr}[\nabla\beta](x) = \sum_j e_j\beta(e_j)(x) - \beta(\sum_j \nabla^{TX}_{e_j}e_j)(x).$$

Let $\widetilde{R}^{\Lambda^{(0,\bullet)}T^*X}$ denote the curvature tensor operator associated with the Hermitian connection $\widetilde{\nabla}^{TX}$ on $\Lambda^{(0,\bullet)}T^*X$.

Definition 3.7. Given $\tau \in \mathscr{C}^\infty(X,[0,1])$, Ω_τ as in Lemma 3.4 and $\varepsilon \in [0,1]$; a local orthonormal frame of $(T^{1,0}X, h^{T^{1,0}X})$ denoted by $\{\omega_j\}_{j=1}^n$. For $s \in B^{0,q}(M,F)$, we define the following bilinear forms

$$\begin{aligned} Q^F_{T,\varepsilon}(s,s) :=& \langle \overline{\partial}^F s, \iota_{T^{(0,1)}}s\rangle_{L^2(M)} + \langle \iota^*_{T^{(0,1)}}s, \overline{\partial}^{F*}s\rangle_{L^2(M)} \\ &+ \sum_{j,k,m=1}^n \Big\langle \langle T(\overline{\omega}_k,\overline{\omega}_j),\omega_m\rangle \overline{\omega}^j \wedge \iota_{\overline{\omega}_m}s, \widetilde{\nabla}^{TX}_{\overline{\omega}_k}s\Big\rangle_{L^2(M)} \\ &- 2(1-\varepsilon)\mathrm{Re}\int_M \sum_{j,k=1}^n \langle T(\omega_j,\omega_k),\overline{\omega}_k\rangle\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s,s\rangle_h dv_X, \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \Psi^F_\tau(s,s) :=& -\Big\langle \mathrm{Tr}_\Theta[\sqrt{-1}\widetilde{R}^{\Lambda^{(0,\bullet)}T^*X}]s,s\Big\rangle_{L^2(M)} \\ &+ \Big\langle \tau|d\varrho|^2\left(\widetilde{R}^{\Lambda^{(0,\bullet)}T^*X}\otimes \mathrm{Id}_F\right)(\omega_n,\overline{\omega}_n)s,s\Big\rangle_{L^2(M)} \\ &+ 2\mathrm{Re}\int_M \left(\omega_n(\tau|d\varrho|^2) + \tau|d\varrho|^2\mathrm{Tr}[\nabla\overline{\omega}^n]\right)\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_n}s,s\rangle_h dv_X \\ &+ 2\mathrm{Re}\int_M \tau|d\varrho|^2\langle\widetilde{\nabla}^{TX}_{\widetilde{\nabla}^{TX}_{\omega_n}\overline{\omega}_n}s,s\rangle_h dv_X, \end{aligned} \tag{3.41}$$

where ω_n near bM (on the support of τ) is taken as in (3.23), that is, $\omega_n = \frac{\sqrt{2}}{|d\varrho|}e^{(1,0)}_{\mathfrak{n}}$.

The following proposition, whose proof will be given in Subsection 3.4, is the key ingredient for the proof of Theorem 1.1.

Proposition 3.8. *Given $\tau \in \mathscr{C}^\infty(X,[0,1])$ and Ω_τ as in Lemma 3.4, then for $s \in \Omega^{0,q}(\overline{M},F)$, we have*

$$\begin{aligned}\int\limits_M \sqrt{-1}\partial\bar\partial|s|_h^2\wedge\Omega_\tau = &\sum_{j=1}^n \|\widetilde\nabla^{TX}_{\omega_j}s\|^2_{L^2(M)} - \sum_{j=1}^n \|\widetilde\nabla^{TX}_{\overline\omega_j}s\|^2_{L^2(M)} - \left\langle \mathrm{Tr}_\Theta[\sqrt{-1}R^F]s,s\right\rangle_{L^2(M)}\\ &- 2\|\sqrt{\tau}\widetilde\nabla^{TX}_{e_{\mathfrak n}^{(1,0)}}s\|^2_{L^2(M)} + 2\|\sqrt{\tau}\widetilde\nabla^{TX}_{e_{\mathfrak n}^{(0,1)}}s\|^2_{L^2(M)}\\ &- 2\mathrm{Re}\int\limits_M \sum_{j,k=1}^n \langle T(\omega_j,\omega_k),\overline\omega_k\rangle\langle\widetilde\nabla^{TX}_{\overline\omega_j}s,s\rangle_h dv_X\\ &+ 2\langle\tau R^F(e_{\mathfrak n}^{(1,0)},e_{\mathfrak n}^{(0,1)})s,s\rangle_{L^2(M)} + \Psi^F_\tau(s,s).\end{aligned} \tag{3.42}$$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. In this proof, we always use the local orthonormal frame $\{\omega_j\}_{j=1}^n$ of $(T^{1,0}X,h^{T^{1,0}X})$ constructed at the beginning of the proof of Lemma 3.4, where we have $\omega_n = \frac{\sqrt2}{|d\varrho|}e_{\mathfrak n}^{(1,0)}$ near bM (on the support of τ). Applying (3.36) with $g=|s|_h^2$ and then combining it with Proposition 3.8, we get

$$\begin{aligned}\sum_{j=1}^n \|\widetilde\nabla^{TX}_{\overline\omega_j}s\|^2_{L^2(M)} = &-\int\limits_M |s|_h^2\sqrt{-1}\partial\bar\partial\Omega_\tau + \sum_{j=1}^n \|\widetilde\nabla^{TX}_{\omega_j}s\|^2_{L^2(M)}\\ &- \|\sqrt\tau|d\varrho|\widetilde\nabla^{TX}_{\omega_n}s\|^2_{L^2(M)} + \|\sqrt\tau|d\varrho|\widetilde\nabla^{TX}_{\overline\omega_n}s\|^2_{L^2(M)}\\ &- 2\mathrm{Re}\int\limits_M \sum_{j,k=1}^n \langle T(\omega_j,\omega_k),\overline\omega_k\rangle\langle\widetilde\nabla^{TX}_{\overline\omega_j}s,s\rangle_h dv_X\\ &- \left\langle \sum_{j=1}^n R^F(\omega_j,\overline\omega_j)s,s\right\rangle_{L^2(M)} + \left\langle\tau|d\varrho|^2R^F(\omega_n,\overline\omega_n)s,s\right\rangle_{L^2(M)}\\ &+ \Psi^F_\tau(s,s) - \int\limits_{bM}|s|_h^2\left(\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_\varrho] - e_{\mathfrak n}(|d\varrho|)\right)dv_{bM}.\end{aligned} \tag{3.43}$$

Now we take $s\in B^{0,q}(M,F)$. We split

$$\sum_{j=1}^n \|\widetilde\nabla^{TX}_{\overline\omega_j}s\|^2_{L^2(M)} = \varepsilon\sum_{j=1}^n \|\widetilde\nabla^{TX}_{\overline\omega_j}s\|^2_{L^2(M)} + (1-\varepsilon)\sum_{j=1}^n \|\widetilde\nabla^{TX}_{\overline\omega_j}s\|^2_{L^2(M)},$$

and apply (3.43) to the term $(1-\varepsilon)\sum_{j=1}^n \|\widetilde\nabla^{TX}_{\overline\omega_j}s\|^2_{L^2(M)}$ and insert it back to (3.16) to obtain (1.1). $\square$

3.3. Nakano-Griffiths inequalities; Proof of Theorem 1.2

In this subsection, we prove Theorem 1.2. We will use the following elementary inequality: for any $a, b \in \mathbb{C}$, $\varepsilon > 0$,

$$2|ab| \leq \varepsilon |a|^2 + \frac{1}{\varepsilon}|b|^2. \tag{3.44}$$

Proof of Theorem 1.2. First, fix the function τ as in the theorem. Then there exists a constant $C_1 = C_1(X, g^{TX}, \varrho, \tau) > 0$ such that for all $\varepsilon \in [0,1]$ and $s \in \Omega^{0,q}(\overline{M}, F)$,

$$\left| \Big\langle \sum_{j,k} R^{K_X^*}(\omega_j, \overline{\omega}_k)\overline{\omega}^k \wedge \iota_{\overline{\omega}_j} s, s \Big\rangle_{L^2(M)} - (1-\varepsilon)\int_M |s|_h^2 \sqrt{-1}\partial\overline{\partial}\Omega_\tau \right| \leq C_1 \|s\|^2_{L^2(M)}. \tag{3.45}$$

Estimating separately the terms in $Q^F_{T,\varepsilon}(s,s)$ involving $\overline{\partial}^F s$, $\overline{\partial}^{F*} s$, and the covariant derivatives $\{\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s\}_{j=1}^n$, we obtain that there exists a constant $C_2 = C_2(X, g^{TX}, \varrho, \tau) > 0$ such that for all $\varepsilon \in (0,1]$ and $s \in \Omega^{0,q}(\overline{M}, F)$,

$$\left|Q^F_{T,\varepsilon}(s,s)\right| \leq \frac{C_2}{\varepsilon}\|s\|^2_{L^2(M)} + \varepsilon\left(\|\overline{\partial}^F s\|^2_{L^2(M)} + \|\overline{\partial}^{F*} s\|^2_{L^2(M)}\right) + \frac{\varepsilon}{4}\sum_{j=1}^n \|\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s\|^2_{L^2(M)}. \tag{3.46}$$

Note that C_2 is determined only by the torsion tensor T.

Finally, note that inside $\Psi^F_\tau(s,s)$, we can write

$$\widetilde{\nabla}^{TX}_{\widetilde{\nabla}^{TX}_{\omega_n}\overline{\omega}_n} s = \sum_{j=1}^n \langle \widetilde{\nabla}^{TX}_{\omega_n}\overline{\omega}_n, \omega_j\rangle \widetilde{\nabla}^{TX}_{\overline{\omega}_j} s. \tag{3.47}$$

Since our choice of τ ensures that ω_n is well defined via $e_{\mathfrak{n}}$, we conclude that there exists a constant $C_3 = C_3(X, g^{TX}, \varrho, \tau) > 0$ such that for all $\varepsilon \in (0,1]$ and $s \in \Omega^{0,q}(\overline{M}, F)$,

$$\left|\Psi^F_\tau(s,s)\right| \leq \frac{C_3}{\varepsilon}\|s\|^2_{L^2(M)} + \frac{\varepsilon}{4}\sum_{j=1}^n \|\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s\|^2_{L^2(M)}. \tag{3.48}$$

It is clear that the constants C_1, C_2, C_3 depend only on the geometry of (X, g^{TX}) and the functions τ and ϱ (hence also depend on M). Finally, by combining (3.45)–(3.48), we obtain (1.2) from (1.1). $\square$

Remark 3.9. From the above proof, (1.2) also holds if we add two more positive terms to the right-hand side

$$\frac{\varepsilon}{2}\sum_{j=1}^n \|\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s\|^2_{L^2(M)} + 2(1-\varepsilon)\|\sqrt{\tau}\widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(0,1)}} s\|^2_{L^2(M)}.$$

3.4. Proof of Proposition 3.8

Now we focus on the proof of Proposition 3.8. Let $\{\omega_j\}_{j=1}^n$ denote a local orthonormal frame of $(T^{1,0}X, h^{T^{1,0}X})$. Given $s \in \Omega^{0,q}(\overline{M}, F)$, we have

$$\sqrt{-1}\partial\bar{\partial}|s|_h^2 \wedge \Omega_\tau = \sum_{j=1}^n (\partial\bar{\partial}|s|_h^2)(\omega_j, \overline{\omega}_j) dv_X - 2\tau(\partial\bar{\partial}|s|_h^2)(e_{\mathfrak{n}}^{(1,0)}, e_{\mathfrak{n}}^{(0,1)}) dv_X. \tag{3.49}$$

Note that by $T(\omega_j, \overline{\omega}_j) = 0$, we have

$$[\omega_j, \overline{\omega}_j] = \widetilde{\nabla}^{TX}_{\omega_j}\overline{\omega}_j - \widetilde{\nabla}^{TX}_{\overline{\omega}_j}\omega_j.$$

For $j = 1, \ldots, n$, we have

$$\begin{aligned}(\partial\bar{\partial}|s|_h^2)(\omega_j, \overline{\omega}_j) =& \overline{\omega}_j(\omega_j(|s|_h^2)) - (\partial|s|_h^2, \widetilde{\nabla}^{TX}_{\overline{\omega}_j}\omega_j) \\ =& |\widetilde{\nabla}^{TX}_{\omega_j} s|_h^2 + |\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s|_h^2 \\ &- \left\langle \left(\widetilde{R}^{\Lambda^{(0,\bullet)}T^*X} \otimes \mathrm{Id}_F + \mathrm{Id}_{\Lambda^{(0,\bullet)}T^*X} \otimes R^F\right)(\omega_j, \overline{\omega}_j) s, s\right\rangle_h \\ &+ 2\mathrm{Re}\left\langle \left(\widetilde{\nabla}^{TX}_{\omega_j}\widetilde{\nabla}^{TX}_{\overline{\omega}_j} - \widetilde{\nabla}^{TX}_{\widetilde{\nabla}^{TX}_{\omega_j}\overline{\omega}_j}\right) s, s\right\rangle_h.\end{aligned} \tag{3.50}$$

We need the following technical lemmas.

Lemma 3.10. *In the above setup, we have*

$$\begin{aligned}&\int_M \sum_{j=1}^n \left\langle \left(\widetilde{\nabla}^{TX}_{\omega_j}\widetilde{\nabla}^{TX}_{\overline{\omega}_j} - \widetilde{\nabla}^{TX}_{\widetilde{\nabla}^{TX}_{\omega_j}\overline{\omega}_j}\right) s, s\right\rangle_h dv_X \\ &= -\sum_{j=1}^n \|\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s\|^2_{L^2(M)} - \int_{bM} \langle \widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(0,1)}} s, s\rangle_h dv_{bM} - \int_M \sum_{j,k=1}^n \langle T(\omega_j, \omega_k), \overline{\omega}_k\rangle \langle \widetilde{\nabla}^{TX}_{\overline{\omega}_j} s, s\rangle_h dv_X.\end{aligned} \tag{3.51}$$

Proof. Note that we cannot choose the local orthonormal frame $\{\omega_j\}_{j=1}^n$ globally on $\overline{M}$, but the integrands in (3.51) are clearly well-defined global functions on $\overline{M}$.

First, we have

$$L_{\omega_j}(\langle \widetilde{\nabla}^{TX}_{\overline{\omega}_j} s, s\rangle_h dv_X) = \left\langle \widetilde{\nabla}^{TX}_{\omega_j}\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s, s\right\rangle_h dv_X + |\widetilde{\nabla}^{TX}_{\overline{\omega}_j} s|_h^2 dv_X + \langle \widetilde{\nabla}^{TX}_{\overline{\omega}_j} s, s\rangle_h L_{\omega_j} dv_X, \tag{3.52}$$

where L_{ω_j} denotes the Lie derivative on differential forms.

Using the formula (3.27) for dv_X, we obtain

$$L_{\omega_j} dv_X = \sum_{k=1}^n (\langle T(\omega_j, \omega_k), \overline{\omega}_k\rangle - \langle \omega_j, \widetilde{\nabla}^{TX}_{\omega_k}\overline{\omega}_k\rangle) dv_X. \tag{3.53}$$

Then we have

$$\sum_{j=1}^{n}\left\langle\left(\widetilde{\nabla}^{TX}_{\omega_j}\widetilde{\nabla}^{TX}_{\overline{\omega}_j}-\widetilde{\nabla}^{TX}_{\widetilde{\nabla}^{TX}_{\omega_j}\overline{\omega}_j}\right)s,s\right\rangle_h dv_X=\sum_{j=1}^{n}L_{\omega_j}(\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s,s\rangle_h dv_X)-\sum_{j=1}^{n}|\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s|_h^2 dv_X$$
$$-\sum_{j,k=1}^{n}\langle T(\omega_j,\omega_k),\overline{\omega}_k\rangle\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s,s\rangle_h dv_X. \tag{3.54}$$

It remains to prove that

$$\int_M\sum_{j=1}^{n}L_{\omega_j}(\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s,s\rangle_h dv_X)=-\int_{bM}\langle\widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(0,1)}}s,s\rangle_h dv_{bM} \tag{3.55}$$

We now use the local orthonormal frame $\{\omega_j\}_{j=1}^n$ of $(T^{1,0}X,h^{T^{1,0}X})$ constructed at the beginning of the proof of Lemma 3.4, where $\omega_n=\frac{\sqrt{2}}{|d\varrho|}e_{\mathfrak{n}}^{(1,0)}$ near bM (on the support of τ). Using the fact that ω_j, $j=1,\dots,n-1$, are tangential to bM and Stokes' theorem, we get

$$\int_M\sum_{j=1}^{n}L_{\omega_j}(\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s,s\rangle_h dv_X)=\int_M d\iota_{\omega_n}(\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_n}s,s\rangle_h dv_X)=\int_{bM}\iota_{\omega_n}(\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_n}s,s\rangle_h dv_X), \tag{3.56}$$

then (3.55) follows from (3.23) and (3.31), and this finishes the proof. □

Combining the above lemma with (3.50), we obtain the following result.

Lemma 3.11. *For $s\in\Omega^{0,q}(\overline{M},F)$, we have*

$$\int_M\left(\sum_{j=1}^{n}(\partial\bar\partial|s|_h^2)(\omega_j,\overline{\omega}_j)\right)dv_X$$
$$=\sum_{j=1}^{n}\|\widetilde{\nabla}^{TX}_{\omega_j}s\|^2_{L^2(M)}-\sum_{j=1}^{n}\|\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s\|^2_{L^2(M)}-2\mathrm{Re}\int_{bM}\langle\widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(0,1)}}s,s\rangle_h dv_{bM}$$
$$-2\mathrm{Re}\int_M\sum_{j,k=1}^{n}\langle T(\omega_j,\omega_k),\overline{\omega}_k\rangle\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_j}s,s\rangle_h dv_X$$
$$-\left\langle\sum_{j=1}^{n}\left(\widetilde{R}^{\Lambda^{(0,\bullet)}T^*X}\otimes\mathrm{Id}_F+\mathrm{Id}_{\Lambda^{(0,\bullet)}T^*X}\otimes R^F\right)(\omega_j,\overline{\omega}_j)s,s\right\rangle_{L^2(M)}. \tag{3.57}$$

Lemma 3.12. *Let $\tau\in\mathscr{C}^\infty(X,[0,1])$ and Ω_τ be as in Lemma 3.4 and let ω_n near bM (on the support of τ) be as in (3.23), that is, $\omega_n=\frac{\sqrt{2}}{|d\varrho|}e_{\mathfrak{n}}^{(1,0)}$. Then*

$$\begin{aligned}
&\int_M \tau|d\varrho|^2(\partial\bar{\partial}|s|_h^2)(\omega_n,\overline{\omega}_n)dv_X \\
=&\|\sqrt{\tau}|d\varrho|\widetilde{\nabla}^{TX}_{\omega_n}s\|^2_{L^2(M)} - \|\sqrt{\tau}|d\varrho|\widetilde{\nabla}^{TX}_{\overline{\omega}_n}s\|^2_{L^2(M)} - 2\mathrm{Re}\int_{bM}\langle\widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(0,1)}}s,s\rangle_h dv_{bM} \\
&-\left\langle\tau|d\varrho|^2\left(\widetilde{R}^{\Lambda^{(0,\bullet)}T^*X}\otimes\mathrm{Id}_F+\mathrm{Id}_{\Lambda^{(0,\bullet)}T^*X}\otimes R^F\right)(\omega_n,\overline{\omega}_n)s,s\right\rangle_{L^2(M)} \\
&-2\mathrm{Re}\int_M\left(\omega_n(\tau|d\varrho|^2)+\tau|d\varrho|^2\mathrm{Tr}[\nabla\overline{\omega}^n]\right)\langle\widetilde{\nabla}^{TX}_{\overline{\omega}_n}s,s\rangle_h dv_X \\
&-2\mathrm{Re}\int_M\tau|d\varrho|^2\langle\widetilde{\nabla}^{TX}_{\widetilde{\nabla}^{TX}_{\omega_n}\overline{\omega}_n}s,s\rangle_h dv_X.
\end{aligned} \tag{3.58}$$

Proof. The formula follows from computations analogous to those in the proof of Lemma 3.10. Note that near bM, we have

$$L_{\omega_n}dv_X = \mathrm{Tr}[\nabla\overline{\omega}^n]dv_X. \tag{3.59}$$

By replacing dv_X by $\tau|d\varrho|^2dv_X$ and proceeding as in the calculations of Lemma 3.10 we get (3.58). $\square$

Proof of Proposition 3.8. This proposition follows directly from (3.49), Lemmas 3.11 and 3.12. In particular, the two boundary terms $-2\mathrm{Re}\int_{bM}\langle\widetilde{\nabla}^{TX}_{e_{\mathfrak{n}}^{(0,1)}}s,s\rangle_h dv_{bM}$ cancel out. $\square$

4. Holomorphic Morse inequalities for q-concave domains

In this section, we establish the holomorphic Morse inequalities for Levi q-concave domains stated in Theorem 1.4. Our method is based on the Nakano-Griffiths inequality given in Theorem 1.2. Moreover, the same strategy also applies to Theorem 1.8. In parallel, Theorem 1.4 yields a new proof of holomorphic Morse inequalities for q-concave manifolds, as stated in Theorem 1.5. Finally, in Subsection 4.5, we combine our method with the one used in [30, Section 3.5] to study the domains or manifolds with mixed convexity/concavity such as (p,q)-coronas and (p,q)-convex-concave complex manifolds.

4.1. Holomorphic Morse inequalities with modified metrics

Let (E,h^E) and (L,h^L) be holomorphic Hermitian vector bundles on a connected Hermitian manifold (X,Θ) with $\mathrm{rk}(L)=1$. Let $\overline{\partial}^E_k:\Omega_0^{0,\bullet}(X,L^k\otimes E)\to\Omega_0^{0,\bullet+1}(X,L^k\otimes E)$ be the associated $\overline{\partial}$-operator. After taking into account the L^2 inner product associated with Θ and h^L, h^E, we can define the maximal extension of $\overline{\partial}^E_k$, its Hilbert-space adjoint $\overline{\partial}^{E*}_k$, and then the Gaffney extension $\square^E_k$ on $L^2_{0,\bullet}(X,L^k\otimes E)$.

Recall that R^L denotes the Chern curvature form of (L,h^L). For any subset $M\subset X$, we set

$$M(j,h^L)=\{x\in M\ :\ \sqrt{-1}R^L_x \text{ is non-degenerate}\\ \text{and has exactly } j \text{ negative eigenvalues}\},\tag{4.1}$$

and $M(\leq j,h^L)=\cup_{0\leq \ell\leq j}M(\ell,h^L)$.

First, we prove the following result as an intermediate step towards Theorem 1.4.

Theorem 4.1. *Let $M\Subset X$ be a smooth domain of a Hermitian manifold (X,Θ) of dimension n such that the Levi form $\mathscr{L}_{bM}$ has at least $n-q$ negative eigenvalues ($1\leq q\leq n-1$), that is, M is a q-concave domain of X. Let (E,h^E) and (L,h^L) be holomorphic Hermitian vector bundles on X with $\operatorname{rk}(L)=1$. Then there exists a smooth Hermitian metric h^L_χ on L (which is obtained by modifying h^L near bM) such that, given a sufficiently small neighborhood U_1 of bM with $M'=M\setminus\overline{U}_1$ having a smooth boundary, we have the strong Morse inequalities for $j\leq n-q-1$ as $k\to\infty$,*

$$\sum_{\ell=0}^{j}(-1)^{j-\ell}\dim H^{0,\ell}_{(2)}(M,L^k\otimes E)\leq \frac{\operatorname{rk}(E)k^n}{n!}\int\limits_{M'(\leq j,h^L_\chi)}(-1)^jc_1(L,h^L_\chi)^n+o(k^n).\tag{4.2}$$

For $j\leq n-q-1$, we have the weak Morse inequalities,

$$\dim H^{0,j}_{(2)}(M,L^k\otimes E)\leq \operatorname{rk}(E)\frac{k^n}{n!}\int\limits_{M'(j,h^L_\chi)}(-1)^jc_1(L,h^L_\chi)^n+o(k^n).\tag{4.3}$$

Moreover, if $q\leq n-2$, we also have

$$\dim H^{0}_{(2)}(M,L^k\otimes E)\geq \operatorname{rk}(E)\frac{k^n}{n!}\int\limits_{M'(\leq 1\,,h^L_\chi)}c_1(L,h^L)^n+o(k^n).\tag{4.4}$$

A key step in the proof of Theorem 4.1 is the following proposition, whose proof is deferred to Subsection 4.2.

Proposition 4.2. *Let $M\Subset X$ be a q-concave domain in a complex manifold X of dimension n, and let (E,h^E) and (L,h^L) be holomorphic Hermitian vector bundles on X with $\operatorname{rk}(L)=1$. Then there exists a defining function $\varrho\in\mathscr{C}^\infty(X,\mathbb{R})$ of M with $M=X_0=\{x\in X\ :\ \varrho(x)<0\}$, a Hermitian metric $\widetilde{\Theta}$ on X and a modified Hermitian metric h^L_χ on L such that the following property holds: there exist $C>0$, $C_0>0, C'>0$, and δ, $\varepsilon_1\in(0,1)$ such that given any $c\in[-\delta,\delta]$ and any $\varepsilon\in(0,\varepsilon_1)$, for any $s\in B^{0,j}(X_c,L^k\otimes E)$ with $\operatorname{supp}(s)\subset X_\delta\setminus\overline{X}_{-\delta}$ and $j\leq n-q-1$, we have*

$$(1+\varepsilon)\left(\|\overline{\partial}^E_k s\|^2_{L^2(X_c)}+\|\overline{\partial}^{E*}_k s\|^2_{L^2(X_c)}\right)+C(1+\frac{1}{\varepsilon})\|s\|^2_{L^2(X_c)}\geq (C_0k-C')\|s\|^2_{L^2(X_c)},\tag{4.5}$$

where the L^2 inner product is induced by $\widetilde{\Theta}$ and h^L_χ, h^E, and $X_c:=\{x\in X\ :\ \varrho(x)<c\}$.

Using a cut-off function near the boundary bM and the fact that $B^{0,j}(M, L^k \otimes E)$ is dense in $\mathrm{Dom}(\overline{\partial}_k^E) \cap \mathrm{Dom}(\overline{\partial}_k^{E*}) \cap L^2_{0,j}(M, L^k \otimes E)$ with respect to the graph norm of $\overline{\partial}_k^E + \overline{\partial}_k^{E*}$, we obtain the following *optimal* fundamental estimates for M (see also [30, (3.5.19)] and [28, Proposition 3.4]).

Proposition 4.3. *Let $M \Subset X$ be a q-concave domain in a complex manifold X of dimension n, and let (E, h^E) and (L, h^L) be holomorphic Hermitian vector bundles on X with $\mathrm{rk}(L) = 1$. Let ϱ, $\widetilde{\Theta}$, h^L_χ, δ be as in Proposition 4.2.*

Then there exists $\widetilde{C} > 0$ such that for any given $\delta_1 \in (0, \delta]$ and for $k \geq 1$, we have

$$\left(1 - \frac{\widetilde{C}}{k}\right) \|s\|^2_{L^2(M)} \leq \frac{\widetilde{C}}{k} \left(\|\overline{\partial}_k^E s\|^2_{L^2(M)} + \|\overline{\partial}_k^{E*} s\|^2_{L^2(M)}\right) + \int\limits_{\overline{X}_{-\delta_1}} |s|^2 dv_X \tag{4.6}$$

for $s \in \mathrm{Dom}(\overline{\partial}_k^E) \cap \mathrm{Dom}(\overline{\partial}_k^{E*}) \cap L^2_{0,j}(M, L^k \otimes E)$ with $j \leq n - q - 1$, where the L^2 norm is induced by the metrics $\widetilde{\Theta}$, h^L_χ, h^E.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. The main step is to apply Theorem 2.12 (see [30, Theorem 3.2.13]). The optimal fundamental estimate (4.6) holds for $(0, j)$-forms with any $j \leq n - q - 1$. This allows us to directly apply Theorem 2.12 to the triplet $\overline{X}_{-\delta_1} \subset M' \Subset M$ in our setting, where M' is a smooth domain. Explicitly, after working with the modified Hermitian metrics $\widetilde{\Theta}$ on X and h^L_χ on L given in Proposition 4.2, we have, for $j \leq n - q - 1$, as $k \to \infty$,

$$\begin{aligned} &\sum_{\ell=0}^{j} (-1)^{j-\ell} \dim H^{0,\ell}_{(2)}(M, L^k \otimes E) \leq \mathrm{rk}(E) \frac{k^n}{n!} \int\limits_{M'(\leq j, h^L_\chi)} (-1)^j c_1(L, h^L_\chi)^n + o(k^n), \\ &\dim H^{0,j}_{(2)}(M, L^k \otimes E) \leq \mathrm{rk}(E) \frac{k^n}{n!} \int\limits_{M'(j, h^L_\chi)} (-1)^j c_1(L, h^L_\chi)^n + o(k^n), \end{aligned} \tag{4.7}$$

where each space $H^{0,\ell}_{(2)}(M, L^k \otimes E)$ is defined with respect to $\widetilde{\Theta}$ and h^L_χ, as in Lemmas 4.5 and 4.6, with a function χ satisfying condition (4.41).

By Theorem 2.9 and the fact that M satisfies $Z(j)$ condition for $j \leq n - q - 1$, we conclude that (4.7) holds for the groups $H^{0,\bullet}_{(2)}(M, L^k \otimes E)$ that are defined with respect to the original metrics Θ and h^L. Similarly, (4.4) follows from Theorem 2.14. In this way, we complete the proof. $\square$

Remark 4.4. In Lemma 4.6 and Subsection 4.3 we will explicitly construct a metric h^L_χ by using a well-chosen function χ, in order to prove Theorem 4.1.

4.2. Two technical lemmas and proof of Proposition 4.2

We need two technical lemmas to prove Proposition 4.2. In this subsection, we fix $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ such that

- There exists an interval $[u, v]$ $(u < v)$ of $\mathbb{R}$ which consists of regular values of ϱ such that for $c \in [u, v]$, $\varnothing \neq X_c := \{x \in X \ : \ \varrho(x) < c\} \Subset X$.
- For $c \in [u, v]$, at $x \in bX_c$, the Levi form $\mathscr{L}_\varrho := \partial\overline{\partial}\varrho|_{bX_c}$ has at least $n - q$ negative eigenvalues.

Lemma 4.5. *For any $C_1 > 0$, there exists a Hermitian metric Θ on X, with associated Riemannian metric g^{TX}, and $\varepsilon_1 \in (0, 1)$ such that, for every holomorphic Hermitian vector bundle (F, h^F) over X and every $c \in (u, v)$, the following statements hold.*

(i) At each point of $X_v \setminus \overline{X}_u$,

$$|d\varrho|_{g^{TX}} = 1. \tag{4.8}$$

Thus the function $\varrho_c := \varrho - c$ is a defining function for X_c; let $e_\mathfrak{n}$ denote the metric dual of $-d\varrho$.

(ii) For $j \in \{0, 1, \ldots, n - q - 1\}$ and $0 < \varepsilon \leq \varepsilon_1$, if $s \in \Omega_0^{0,j}(X_v \setminus \overline{X}_u, F)$ with

$$\iota_{e_\mathfrak{n}} s|_{bX_c} \equiv 0,$$

then at each point $x \in bX_c$,

$$\langle [\sqrt{-1}\mathscr{L}_{\varrho_c}, \Lambda]_\varepsilon^{bX_c} s, s\rangle_h(x) \geq C_1 |s(x)|_h^2. \tag{4.9}$$

Proof. Since any $c \in [u, v]$ is a regular value of ϱ, the smooth 1-form $d\varrho$ does not vanish on a neighborhood of $X_v \setminus \overline{X}_u$. Let $HX := \ker d\varrho$ be a (real) subbundle of TX in the neighborhood of $X_v \setminus \overline{X}_u$. Put

$$H^{(1,0)}X := T^{(1,0)}X \cap \mathbb{C}HX. \tag{4.10}$$

Note that we have $H^{(1,0)}X|_{bX_c} = T^{(1,0)}bX_c$, the analytic tangent space to bX_c defined in Subsection 2.1. Set $H_bX := \{w + \overline{w} \in HX \ : \ w \in H^{(1,0)}X\}$; this is a real subbundle of HX with corank 1.

Let $e_\mathfrak{n}$ be a smooth vector field in the neighborhood of $X_v \setminus \overline{X}_u$ such that $e_\mathfrak{n}(\varrho) \equiv -1$. Then we have the splitting of the tangent bundle in the neighborhood of $X_v \setminus \overline{X}_u$

$$TX = HX \oplus \mathbb{R}e_\mathfrak{n} = (H_bX \oplus \mathbb{R}Je_\mathfrak{n}) \oplus \mathbb{R}e_\mathfrak{n}. \tag{4.11}$$

Note that the complex structure J preserves H_bX.

Now we fix any Hermitian metric Θ_0 (and the corresponding J-invariant Riemannian metric g_0^{TX}) such that the splitting (4.11) is orthogonal and $e_{\mathfrak{n}}$ has norm one on the neighborhood of $X_v \setminus \overline{X}_u$. Moreover, we can write

$$g_0^{TX} = g_0^{HX} \oplus (d\varrho \otimes d\varrho), \tag{4.12}$$

where g_0^{HX} is a smooth Euclidean metric on HX. Hence $d\varrho_c = d\varrho$ has norm 1 with respect to g_0^{TX}. In particular, let Θ_0^b be the induced Hermitian metric on H_bX.

With respect to the above g_0^{TX} and proceeding as in the proof of Lemma 3.4, we fix a local orthonormal frame $\{\omega'_j\}_{j=1}^n$ of $(T^{(1,0)}X, h^{T^{(1,0)}X})$ (and the dual frame $\{\omega'^{,j}\}_{j=1}^n$) on any small domain U inside the neighborhood of $X_v \setminus \overline{X}_u$, such that

$$\omega'_n = \sqrt{2}e_{\mathfrak{n}}^{(1,0)}, \quad \omega'^{,n} = -\sqrt{2}\partial\varrho, \tag{4.13}$$

and moreover $\{\omega'_j\}_{j=1}^{n-1}$ forms an orthonormal frame of $(H^{(1,0)}X, g_0^{T^{(1,0)}X})$. Note that for $c \in (u, v)$, $H^{(1,0)}X|_{bX_c}$ is exactly the analytic tangent space to bX_c.

In the sequel, we will work on $U \cap M$ with $M = X_c$ for some $c \in (u, v)$, and set

$$\varrho_c := \varrho - c.$$

The Levi form on bM is given by

$$\mathscr{L}_{\varrho_c} = \partial\overline{\partial}\varrho|_{bM} = \sum_{k=1}^{n-1} \partial\overline{\partial}\varrho(\omega'_k, \overline{\omega}'_k)\omega'^{,k} \wedge \overline{\omega}'^{,k},$$

and the trace with respect to g_0^{TX} is given by

$$\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_{\varrho_c}] = \sum_{k=1}^{n-1} \partial\overline{\partial}\varrho(\omega'_k, \overline{\omega}'_k).$$

Note that $\Lambda|_{bM} = -\sqrt{-1}\sum_{k=1}^{n-1} \iota_{\overline{\omega}'_k}\iota_{\omega'_k}$.

By Proposition 2.5, $\mathscr{L}_{\varrho_c}$ has at least $n - q$ negative eigenvalues on bM, that is, after a unitary change on the frame $\{\omega'_j\}_{j=1}^{n-1}$ of $T^{(1,0)}X \cap \mathbb{C}HX$, we can write

$$\mathscr{L}_{\varrho_c} = \sum_{j=1}^{n-1} \lambda'_j\omega'^{,j} \wedge \overline{\omega}'^{,j}, \quad \text{where } \lambda'_1 \leq \cdots \leq \lambda'_{n-q} < 0, \quad \lambda'_1 \leq \cdots \leq \lambda'_{n-1}. \tag{4.14}$$

Fix an integer $j \in \{0, 1, \ldots, n-q-1\}$. Take $s \in \Omega_0^{0,j}(X_v \setminus \overline{X}_u, F)$ with $\iota_{e_{\mathfrak{n}}} s \equiv 0$. Then there is no $\overline{\omega}'^{,n}$-component in s with respect to the local frame $\{\overline{\omega}'^{,i}\}_{i=1}^n$. For a subset of indices $I \subset \{1, \ldots, n-1\}$ with $|I| = j$, we set $\complement I := \{1, \cdots, n-1\} \setminus I$, and clearly we have $|\complement I| = n - j - 1 \geq q$.

Now we write

$$s=\sum_{|I|=j,\,1\leq\ell\leq\mathrm{rk}(F)} s_I^{\prime,\ell}\overline{\omega}^{\prime,I}\otimes f_\ell \tag{4.15}$$

on the small domain U, where $s_I^{\prime,\ell}=s_I^{\prime,\ell}(x)\in\mathscr{C}^\infty(U)$ and $\{f_\ell:1\leq\ell\leq\mathrm{rk}(F)\}$ is a local orthonormal frame of (F,h^F) on U. For $\varepsilon>0$,

$$\langle[\sqrt{-1}\mathscr{L}_{\varrho_c},\Lambda]^{bM}s,s\rangle_{h_0}+\varepsilon\,\mathrm{Tr}_{bM}[\sqrt{-1}\mathscr{L}_{\varrho_c}]|s|_{h_0}^2=\sum_{I,\,\ell}\left(\sum_{k\in I}\lambda_k'+(\varepsilon-1)\sum_{k=1}^{n-1}\lambda_k'\right)|s_I^{\prime,\ell}|^2. \tag{4.16}$$

Note that

$$\begin{aligned}\sum_{k\in I}\lambda_k'+(\varepsilon-1)\sum_{k=1}^{n-1}\lambda_k'&=\sum_{k\in\complement I}(-\lambda_k')+\varepsilon\sum_{k=1}^{n-1}\lambda_k'\\&\geq(-\lambda_{n-q}')+(|\complement I|-1)(-\lambda_{n-1}')+\varepsilon(n-1)\lambda_1'.\end{aligned} \tag{4.17}$$

Let $C>0$ be sufficiently large such that $C-\frac{n-1}{C}>C_1$. Note that we have the Hermitian metric on H_bX induced from Θ_0 (see (4.11))

$$\Theta_0^b=\sqrt{-1}\sum_{k=1}^{n-1}\omega^{\prime,k}\wedge\overline{\omega}^{\prime,k}. \tag{4.18}$$

Then we modify the metric g_0^{HX} to be a new metric g^{HX} on U; if Θ^b is the Hermitian metric on H_bX induced by g^{HX}, then

$$\Theta^b=\sum_{k=1}^{n-1}\kappa_k\sqrt{-1}\omega^{\prime,k}\wedge\overline{\omega}^{\prime,k}, \tag{4.19}$$

where the frame $\{\omega^{\prime,k}\}_{k=1}^{n-1}$ is orthonormal with respect to the original Θ_0^b and satisfies (4.14), and each $\kappa_k>0$ is some constant to be fixed.

Now we define the new metric g^{TX} associated to the Hermitian metric

$$\Theta=\sum_{k=1}^{n-1}\kappa_k\sqrt{-1}\omega^{\prime,k}\wedge\overline{\omega}^{\prime,k}+2\sqrt{-1}\partial\varrho\wedge\overline{\partial}\varrho. \tag{4.20}$$

With respect to the metric g^{TX}, the Levi form $\mathscr{L}_{\varrho_c}$ on bM has the eigenvalues $\{\lambda_k\}_{k=1}^{n-1}$ such that

$$\lambda_k=\frac{\lambda_k'}{\kappa_k}. \tag{4.21}$$

Subsequently, we select each $\kappa_k > 0$ appropriately to obtain

$$2C > |\lambda_1| \geq \cdots \geq |\lambda_{n-q}| > C, \qquad |\lambda_k| < 1/C, \ \text{ for } n-q+1 \leq k \leq n-1. \tag{4.22}$$

Next we choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \min\{1, \frac{1}{2(n-1)C^2}\}$. By (4.16) and (4.17), with respect to g^{TX} and for $\varepsilon \in (0, \varepsilon_1)$, we obtain at any point $x \in U \cap bM$,

$$\begin{aligned} &\langle [\sqrt{-1}\mathscr{L}_{\varrho_c}, \Lambda]^{bM} s, s\rangle_h(x) + \varepsilon \operatorname{Tr}_{bM}[\sqrt{-1}\mathscr{L}_{\varrho_c}]|s(x)|_h^2 \\ &\quad \geq ((-\lambda_{n-q}) + (n-j-2)(-\lambda_{n-1}) + \varepsilon(n-1)\lambda_1)\, |s(x)|_h^2 \\ &\quad \geq (C - \frac{n-j-1}{C})|s(x)|_h^2 \geq C_1|s(x)|_h^2. \end{aligned} \tag{4.23}$$

Note that $e_{\mathfrak{n}}$ is still the dual vector of $-d\varrho$ with respect to this new metric g^{TX}.

Since $\mathscr{L}_{\varrho_c}$ depends smoothly on $c \in [u, v]$, the ordered eigenvalues of $\mathscr{L}_{\varrho_c}$ vary continuously with c. Then we can always obtain a local Hermitian metric Θ by modifying Θ_0 locally on U such that (4.23) holds for any $c \in (u, v)$ with $bM = bX_c$. Patching the local metrics as constructed above with the help of a partition of unity, we conclude our lemma. □

From now on, we consider (X, Θ) as a Hermitian manifold, where Θ will be taken as constructed in Lemma 4.5. Let $dv_X = \Theta^n/n!$ be the volume form on X with the induced metric dv_{bX_c} on the boundary bX_c.

Lemma 4.6. *Let L be a holomorphic line bundle on X. For any given constant $C_0 > 0$, there exists a Hermitian metric Θ (as well as the associated Riemannian metric g^{TX}), a Hermitian metric h^L_χ on L, and $\varepsilon_1 \in (0, 1)$ such that Θ and ε_1 satisfy all the conditions of Lemma 4.5 and, for any $\varepsilon \in (0, \varepsilon_1)$, $\tau \in [0, 1]$, $s \in \Omega_0^{0,j}(X_v \setminus \overline{X}_u, L^k)$ with $j \leq n-q-1$ and any $k \in \mathbb{N}_{\geq 1}$, we have, at any point of $X_v \setminus \overline{X}_u$,*

$$\left\langle [\sqrt{-1}R^L_\chi, \Lambda]_\varepsilon s, s\right\rangle_h + 2(1-\varepsilon)\tau \left\langle R^L_\chi(e_{\mathfrak{n}}^{(1,0)}, e_{\mathfrak{n}}^{(0,1)})s, s\right\rangle_h \geq C_0|s|_h^2 \tag{4.24}$$

holds, where $e_{\mathfrak{n}}$ is the metric dual of $-d\varrho$ that has norm 1 on $X_v \setminus \overline{X}_u$, and $R^L_\chi := R^{(L, h^L_\chi)}$ is the Chern curvature form of (L, h^L_χ).

Proof. We take the geometric data $(X, J, \Theta, g^{TX}, \varepsilon_1)$ as constructed in Lemma 4.5 with respect to the given constant $C_1 > 0$ satisfying (4.9). In particular, we have $|d\varrho|_{g^{T^*X}} \equiv 1$ on a neighborhood of $X_v \setminus \overline{X}_u$. Recall that $e_{\mathfrak{n}}$ is the dual vector field to $-d\varrho$ with respect to g^{TX}.

Regarding the decomposition $T^{(1,0)}X = H^{(1,0)}X \oplus \mathbb{C}\omega_n$, following (4.19) and (4.20) after appropriately adjusting the constants κ_k, we obtain

$$\Theta = \Theta^b + \sqrt{-1}\omega^n \wedge \overline{\omega}^n, \tag{4.25}$$

where $\omega^n = -\sqrt{2}\partial\varrho$. Let $\{\omega_j\}_{j=1}^{n-1}$ be a local orthonormal frame of $H^{(1,0)}X$ with respect to the Hermitian metric Θ^b such that we can write

$$\begin{aligned} \Theta^b &= \sqrt{-1}\sum_{j=1}^{n-1}\omega^j \wedge \overline{\omega}^j, \\ \mathscr{L}_\varrho|_{H^{(1,0)}X\times H^{(0,1)}X} &= \sum_{j=1}^{n-1}\lambda_j\omega^j \wedge \overline{\omega}^j, \end{aligned} \tag{4.26}$$

where $\lambda_1 \leq \ldots \leq \lambda_{n-q} \leq \ldots \leq \lambda_{n-1}$ satisfy (4.22) and $\lambda_{n-q} < 0$.

Let h^L be an arbitrary smooth Hermitian metric on L. For $T > 0$, we introduce a new family of Hermitian metrics on $X_v \setminus \overline{X}_u$

$$\Theta_T = \frac{1}{T^2}\Theta^b + \sqrt{-1}\omega^n \wedge \overline{\omega}^n. \tag{4.27}$$

Let g_T^{TX} be the Riemannian metric associated with Θ_T, and let $\langle\cdot,\cdot\rangle_{h,T}$, $|\cdot|_{h,T}$ denote the (pointwise) Hermitian inner product or norm with respect to Θ_T and h^L. Note that for any $T > 0$, we always have $|e_{\mathfrak{n}}|_{g_T^{TX}} \equiv 1$ on $X_v \setminus \overline{X}_u$. For simplicity, we set $\mathscr{L}_\varrho := \partial\overline{\partial}\varrho|_{\Lambda^2\mathbb{C}H_bX}$.

Then we have by our proof of Lemma 4.5 that if $s \in \Omega_0^{0,j}(X_v \setminus \overline{X}_u, L^k)$ with

$$\iota_{e_{\mathfrak{n}}}s \equiv 0, \tag{4.28}$$

then for $T > 0$, $c \in (u, v)$, $0 < \varepsilon < \varepsilon_1$, and at each point $x \in bX_c$,

$$\langle[\sqrt{-1}\mathscr{L}_\varrho, \Lambda_T]_\varepsilon^{bX_c}s, s\rangle_{h,T}(x) \geq T^2C_1|s(x)|_{h,T}^2, \tag{4.29}$$

where $\Lambda_T = \iota_{\Theta_T}$ is the adjoint of the Lefschetz operator Lef_{Θ_T} with respect to $\langle\cdot,\cdot\rangle_{h,T}$.

First we need to prove the following claim: For any given $C_2 > 0$, there exists a constant $T_0 > 0$ such that for $s \in \Omega_0^{0,j}(X_v \setminus \overline{X}_u, L^k)$ (with $0 \leq j \leq n-q-1$ and $k \in \mathbb{N}$), $0 < \varepsilon < \varepsilon_1$, $\tau \in [0, 1]$, and at $x \in X_v \setminus \overline{X}_u$, we have

$$\left\langle[\sqrt{-1}\partial\overline{\partial}\varrho, \Lambda_{T_0}]_\varepsilon s, s\right\rangle_{h,T_0} + 2(1-\varepsilon)\tau\left\langle\partial\overline{\partial}\varrho(e_{\mathfrak{n}}^{(1,0)}, e_{\mathfrak{n}}^{(0,1)})s, s\right\rangle_{h,T_0} \geq C_2|s|_{h,T_0}^2. \tag{4.30}$$

Note that the power k for the line bundle L^k does not affect the estimate since all the involved operators are zeroth order linear operators acting only on the differential forms part.

Indeed, after comparing (4.29) and (4.30), we have to deal with the $\overline{\omega}^n$-component of the section s. For the above orthonormal frame $\{\omega_j\}_{j=1}^n$, set $\omega_{j,T} = T\omega_j$ for $j = 1, \ldots, n-1$ and $\omega_{n,T} = \omega_n$. Then $\{\omega_{j,T}\}_{j=1}^n$ is a local orthonormal frame of $(T^{(1,0)}X, h_T^{T^{(1,0)}X})$ (where the Hermitian inner product is induced by Θ_T); denote its dual frame by $\{\omega_T^j\}_{j=1}^n$. By (4.26), we have

$$\begin{aligned}\partial\overline{\partial}\varrho =&T^2\sum_{j=1}^{n-1}\lambda_j\omega_T^j\wedge\overline{\omega}_T^j+\nu_n\omega^n\wedge\overline{\omega}^n\\&+T\sum_{j=1}^{n-1}\nu_j\omega_T^j\wedge\overline{\omega}^n+T\sum_{j=1}^{n-1}\overline{\nu}_j\omega^n\wedge\overline{\omega}_T^j,\end{aligned}\tag{4.31}$$

where $\nu_j=\partial\overline{\partial}\varrho(\omega_j,\overline{\omega}_n)$ for $1\leq j\leq n$.

For any $s\in\Omega_0^{0,j}(X_v\backslash\overline{X}_u,L^k)$ $(0\leq j\leq n-q-1)$, we have an orthogonal decomposition

$$s=s_0+\overline{\omega}^n\wedge s_1\tag{4.32}$$

such that

$$\begin{aligned}&s_0\in\Omega_0^{0,j}(X_v\setminus\overline{X}_u,L^k),\ \iota_{e_{\mathfrak{n}}}s_0\equiv 0,\\&s_1\in\Omega_0^{0,j-1}(X_v\setminus\overline{X}_u,L^k),\ \iota_{e_{\mathfrak{n}}}s_1\equiv 0.\end{aligned}\tag{4.33}$$

Then an elementary computation shows that

$$\begin{aligned}&[\sqrt{-1}\partial\overline{\partial}\varrho,\Lambda_T]_\varepsilon s+(1-\varepsilon)\partial\overline{\partial}\varrho(\omega_n,\overline{\omega}_n)s\\&=[\sqrt{-1}\mathscr{L}_\varrho,\Lambda_T]_\varepsilon^{bM}s_0+T\sum_{k=1}^{n-1}\overline{\nu}_k\overline{\omega}_T^k\wedge s_1\\&\quad+\overline{\omega}^n\wedge\left([\sqrt{-1}\mathscr{L}_\varrho,\Lambda_T]_\varepsilon^{bM}s_1+T\sum_{k=1}^{n-1}\nu_k\iota_{\overline{\omega}_{k,T}}s_0+\partial\overline{\partial}\varrho(\omega_n,\overline{\omega}_n)s_1\right).\end{aligned}\tag{4.34}$$

Note that (4.34) holds for any s as above and for any $T>0$ and $0<\varepsilon<\varepsilon_1$. Then by (4.29), (4.33) and the Cauchy-Schwarz inequality, we obtain that there exists $C_3>0$ such that for $s\in\Omega_0^{0,j}(X_v\setminus\overline{X}_u,L^k)$ $(0\leq j\leq n-q-1$ and $k\in\mathbb{N})$, $T>0$, $0<\varepsilon<\varepsilon_1$, $\tau\in[0,1]$, and at $x\in X_v\setminus\overline{X}_u$

$$\begin{aligned}&\left\langle[\sqrt{-1}\partial\overline{\partial}\varrho,\Lambda_T]_\varepsilon s,s\right\rangle_{h,T}+2(1-\varepsilon)\tau\left\langle\partial\overline{\partial}\varrho(e_{\mathfrak{n}}^{(1,0)},e_{\mathfrak{n}}^{(0,1)})s,s\right\rangle_{h,T}\\&\quad\geq(T^2C_1-TC_3-C_3)|s|_{h,T}^2.\end{aligned}\tag{4.35}$$

Therefore, after fixing a sufficiently large $T_0>0$ such that $T_0^2C_1-T_0C_3-C_3\geq C_2$, we conclude (4.30).

The next step is to find a new Hermitian metric h_χ^L as required in our lemma with the help of (4.30). From now on, we fix $C_2>0$ and $T_0\gg 1$ as in (4.30).

Let $\chi=\chi(t)$ be a real smooth function on $\mathbb{R}$, to be specified later. Let $h_\chi^L:=h^Le^{-\chi(\varrho)}$ be the modified Hermitian metric on L. The Chern curvature of (L,h_χ^L) is given by

$$R_\chi^L=R^L+\chi'(\rho)\partial\overline{\partial}\varrho+\chi''(\varrho)\partial\varrho\wedge\overline{\partial}\varrho.\tag{4.36}$$

Then the term $\langle[\sqrt{-1}R^L_\chi, \Lambda_{T_0}]_\varepsilon s, s\rangle_{h,T_0}$ equals

$$\begin{aligned}\langle[\sqrt{-1}R^L, \Lambda_{T_0}]_\varepsilon s, s\rangle_{h,T_0} + \chi'(\varrho)\langle[\sqrt{-1}\partial\overline{\partial}\varrho, \Lambda_{T_0}]_\varepsilon s, s\rangle_{h,T_0} \\ + \chi''(\varrho)\langle[\sqrt{-1}\partial\varrho\wedge\overline{\partial}\varrho, \Lambda_{T_0}]_\varepsilon s, s\rangle_{h,T_0}.\end{aligned} \tag{4.37}$$

Similarly, note that $\partial\varrho\wedge\overline{\partial}\varrho(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}}) = \frac{1}{4}$, we have

$$\begin{aligned}&\left\langle R^L_\chi(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{h,T_0} \\ &= \left\langle R^L(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{h,T_0} + \chi'(\varrho)\left\langle \partial\overline{\partial}\varrho(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{h,T_0} + \frac{1}{4}\chi''(\varrho)|s|^2_{h,T_0}.\end{aligned} \tag{4.38}$$

There exists a constant $C_4 > 0$ (depending on T_0, Θ and h^L) such that for any $s \in \Omega^{0,j}_0(X_v \setminus \overline{X}_u, L^k)$ (with any $0 \le j \le n-q-1$ and $k \in \mathbb{N}$), $0 < \varepsilon < \varepsilon_1$, $\tau \in [0,1]$, and at $x \in X_v \setminus \overline{X}_u$, we have

$$\begin{aligned}&\left|\left\langle[\sqrt{-1}R^L, \Lambda_{T_0}]_\varepsilon s, s\right\rangle_{h,T_0} + 2(1-\varepsilon)\tau\left\langle R^L(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{h,T_0}\right| \le C_4|s|^2_{h,T_0}, \\ &\left|\left\langle[\sqrt{-1}\partial\varrho\wedge\overline{\partial}\varrho, \Lambda_{T_0}]_\varepsilon s, s\right\rangle_{h,T_0} + \frac{(1-\varepsilon)\tau}{2}|s|^2_{h,T_0}\right| \le C_4|s|^2_{h,T_0}.\end{aligned} \tag{4.39}$$

Combining (4.30), (4.37)– (4.39), we obtain

$$\begin{aligned}&\left\langle[\sqrt{-1}R^L_\chi, \Lambda]_\varepsilon s, s\right\rangle_{h,T_0} + 2(1-\varepsilon)\tau\left\langle R^L_\chi(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{h,T_0} \\ &\ge \left(C_2\chi'(\varrho) - C_4\left(1 + |\chi''(\varrho)|\right)\right)|s|^2_{h,T_0}.\end{aligned} \tag{4.40}$$

Then we choose the function χ such that for $t \in [u,v]$, we have

$$C_2\chi'(t) - C_4(1 + |\chi''(t)|) \ge C_0. \tag{4.41}$$

Such a function always exists. Finally, since $h^L_\chi := h^L e^{-\chi(\varrho)}$ and $h^{L^k}_\chi := h^{L^k} e^{-k\chi(\varrho)}$, we obtain from (4.40) and (4.41) that at each point $x \in X_v \setminus \overline{X}_u$

$$\left\langle[\sqrt{-1}R^L_\chi, \Lambda]_\varepsilon s, s\right\rangle_{h^L_\chi,T_0} + 2(1-\varepsilon)\tau\left\langle R^L_\chi(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{h^L_\chi,T_0} \ge C_0|s|^2_{h^L_\chi,T_0}. \tag{4.42}$$

Consequently, the Hermitian metrics Θ_{T_0} and h^L_χ satisfy the requirements of the lemma, thus completing the proof. □

Proof of Proposition 4.2. As explained in the text before Definition 2.6, since the domain M is assumed to be Levi q-concave, we can take a defining function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ for M with $M = \{x \in X : \varrho(x) < 0\}$, and there exists a sufficiently small $\delta > 0$ such that for all $c \in [-\delta, \delta]$, c is a regular value of ϱ, and $\mathscr{L}_{bX_c}$ has at least $n-q$ negative eigenvalues on bX_c. Moreover, $\sqrt{-1}\partial\overline{\partial}\varrho$ has at least $n-q+1$ negative eigenvalues at each point $x \in X_\delta \setminus X_{-\delta}$.

Apply Lemmas 4.5 and 4.6 to the function ϱ with $[u,v]=[-\delta,\delta]$. Fix the metrics $\widetilde{\Theta}$, h^L_χ and constants ε_1, C_1, C_0 so that (4.8), (4.9) and (4.24) hold.

For any $c \in [-\delta,\delta]$, apply the Nakano-Griffiths formula from Theorem 1.2 to X_c with the defining function $\varrho_c := \varrho - c$. For this purpose, let $\tau \in \mathscr{C}^\infty(X,[0,1])$ be such that $\tau \equiv 1$ near $X_\delta \setminus \overline{X}_{-\delta}$ and $\operatorname{supp}\tau$ is included in a small open neighborhood of $X_\delta \setminus \overline{X}_{-\delta}$ where $d\varrho$ does not vanish.

By the proof of (1.2), when applying it to ϱ_c and X_c, all the involved constants can be chosen uniformly in $c \in [-\delta,\delta]$. Therefore, we conclude that there exists a constant $C = C(X,\widetilde{\Theta},\delta,\varrho,\tau) > 0$ such that for $c \in [-\delta,\delta]$, and for any $\varepsilon \in (0,\varepsilon_1]$, $s \in \Omega^{0,j}_0(X_\delta \setminus \overline{X}_{-\delta}, L^k \otimes E)$ with $\iota_{e_{\mathfrak{n}}} s = 0$ on bX_c for $0 \le j \le n-q-1$, we have

$$
\begin{aligned}
&(1+\varepsilon)\left(\|\overline{\partial}^E_k s\|^2_{L^2(X_c)} + \|\overline{\partial}^{E*}_k s\|^2_{L^2(X_c)}\right) + C(1+\frac{1}{\varepsilon})\|s\|^2_{L^2(X_c)} \\
&\quad \ge k\left(\left\langle[\sqrt{-1}R^L_\chi,\Lambda]_\varepsilon s, s\right\rangle_{L^2(X_c)} + 2(1-\varepsilon)\left\langle \tau R^L_\chi(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{L^2(X_c)}\right) \\
&\qquad + \left\langle[\sqrt{-1}R^E,\Lambda]_\varepsilon s, s\right\rangle_{L^2(X_c)} + 2(1-\varepsilon)\left\langle \tau R^E(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_{L^2(X_c)} \\
&\qquad + \int_{bX_c} \langle[\sqrt{-1}\mathscr{L}_\varrho,\Lambda]^{bX_c}_\varepsilon s, s\rangle_h dv_{bX_c} + (1-\varepsilon)\int_{bX_c} e_{\mathfrak{n}}(|d\varrho|)|s|^2_h dv_{bX_c} \\
&\qquad + (1-\varepsilon)\left(\|(\widetilde{\nabla}^{TX})^{1,0}s\|^2_{L^2(X_c)} - 2\|\sqrt{\tau}\widetilde{\nabla}^{TX}_{e^{(1,0)}_{\mathfrak{n}}} s\|^2_{L^2(X_c)}\right).
\end{aligned}
\tag{4.43}
$$

Since $\operatorname{supp} s \subset X_\delta \setminus \overline{X}_{-\delta}$, $\sqrt{\tau} \in [0,1]$, and $\sqrt{2}e^{(1,0)}_{\mathfrak{n}}$ has norm 1 on the support of τ,

$$
\|(\widetilde{\nabla}^{TX})^{1,0}s\|^2_{L^2(X_c)} - 2\|\sqrt{\tau}\widetilde{\nabla}^{TX}_{e^{(1,0)}_{\mathfrak{n}}} s\|^2_{L^2(X_c)} \ge 0 \tag{4.44}
$$

follows. Then, by Lemmas 4.5 and 4.6, we have

$$
\begin{aligned}
&e_{\mathfrak{n}}(|d\varrho|) \equiv 0 \ \text{ on } \ X_\delta \setminus \overline{X}_{-\delta}\,, \\
&\langle[\sqrt{-1}\mathscr{L}_\varrho,\Lambda]^{bX_c}_\varepsilon s, s\rangle_h \ge C_1|s|^2_h \ \text{ on } \ bX_c\,, \\
&\left\langle[\sqrt{-1}R^L_\chi,\Lambda]_\varepsilon s, s\right\rangle_h + 2(1-\varepsilon)\tau\left\langle R^L_\chi(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_h \ge C_0|s|^2_h \ \text{ on } \ X_\delta \setminus \overline{X}_{-\delta}\,.
\end{aligned}
\tag{4.45}
$$

Combining (4.43)–(4.45), there exists $C' > 0$ such that for any $\varepsilon \in (0,\varepsilon_1]$, and any $s \in \Omega^{0,j}_0(X_\delta \setminus \overline{X}_{-\delta}, L^k \otimes E)$ with $\iota_{e_{\mathfrak{n}}} s = 0$ on bX_c, we have

$$
\begin{aligned}
(1+\varepsilon)&\left(\|\overline{\partial}^E_k s\|^2_{L^2(X_c)} + \|\overline{\partial}^{E*}_k s\|^2_{L^2(X_c)}\right) + C(1+\frac{1}{\varepsilon})\|s\|^2_{L^2(X_c)} \\
&\ge (C_0 k - C')\,\|s\|^2_{L^2(X_c)} + C_1\|s\|^2_{L^2(bX_c)}.
\end{aligned}
\tag{4.46}
$$

The conclusion now follows from the inequality (4.46). □

Remark 4.7. Note that the inequality (4.46) is stronger than the one in Proposition 4.2 because of the boundary term $C_1\|s\|^2_{L^2(bX_c)}$. Actually, the inequality (4.46) implies the basic estimate in the sense of [18, Chapter II] for sufficiently large k.

4.3. Holomorphic Morse inequalities; Proof of Theorems 1.4 and 1.5

Now we prove Theorem 1.4 based on Theorem 4.1 and Lemmas 4.5 and 4.6.

Proof of Theorem 1.4. By the assumption that the domain M is q-concave, as in the first part of the proof of Theorem 4.1, we get a defining function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ for M with the properties mentioned there. In particular, set $K := \overline{X}_{-\delta}$, so that $\sqrt{-1}\partial\overline{\partial}\varrho$ has at least $n - q + 1$ negative eigenvalues at each point $x \in M \setminus K$. Let $K_L \subset M$ be the compact subset as given in Theorem 1.4 such that $\sqrt{-1}R^L \leq 0$ on $M \setminus K_L$. Then we can fix $u_0 \in [-\delta, 0)$ such that

$$K \cup K_L \subset X_{u_0}. \tag{4.47}$$

Then we fix a small interval $[u', v]$ of regular values of ϱ with $u_0 < u' < 0 < v \leq \delta$. Fix another number u such that $u' < u < 0 < v$. Now, we give a proof of Theorem 1.4 by refining the construction of h^L_χ in the proof of Lemma 4.6.

Note that we can apply Lemmas 4.5 and 4.6 to ϱ and the interval $[u', v]$. First, we fix a modified Hermitian metric $\widetilde{\Theta}$ from Lemma 4.5 such that the results hold for all $c \in (u', v)$. In particular, the unit vector field $e_{\mathfrak{n}}$ on $X_v \setminus X_{u'}$ is defined as the metric dual of $-d\varrho$. Recall that $M = X_0$.

Let h^L be the smooth Hermitian metric given in the assumption of Theorem 1.4. Then $\sqrt{-1}R^L \leq 0$ on $M \setminus K_L$. As in the proof of Lemma 4.6, we consider a smooth real function χ and set a new Hermitian metric $h^L_\chi := h^L e^{-\chi(\varrho)}$. Then on $M \setminus X_{u_0}$, we have the comparison of real Hermitian forms

$$\sqrt{-1}R^L_\chi \leq \sqrt{-1}\chi'(\rho)\partial\overline{\partial}\varrho + \sqrt{-1}\chi''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho. \tag{4.48}$$

The first condition for the function χ is that for $t \in [u, v]$, we have

$$C_2\chi'(t) - C_4(1 + |\chi''(t)|) \geq C_0, \tag{4.49}$$

where the constants C_0, C_2, C_4 are given in the proof of Lemma 4.6.

In addition to (4.49), we also require the function χ to satisfy the following conditions,

$$\chi(t) = 0 \ \text{ for } \ t \leq u', \quad \chi'(t) > 0 \ \text{ for } \ t > u'. \tag{4.50}$$

Note that a smooth function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying (4.49) and (4.50) always exists, for example, near u', $\chi(t)$ could have a formula like $e^{-(t-u')^{-2}}$ for $t > u'$.

As in the proof of Lemma 4.5, on $X_v \setminus X_{u'}$, we have the orthogonal splitting of $(1,0)$ tangent bundle

$$T^{(1,0)}X = H^{(1,0)}X \oplus \mathbb{C}\omega_n, \tag{4.51}$$

where $\omega_n = \sqrt{2}e_{\mathfrak{n}}^{(1,0)}$. Since $\mathscr{L}_\varrho$ on $H^{(1,0)}X$ has at least $n-q$ negative eigenvalues, we can find a local orthonormal family $\{\omega_\ell\}_{\ell=1}^{n-q}$ in $H^{(1,0)}X$ such that, the restriction of $\sqrt{-1}\partial\overline{\partial}\varrho$ to the subspace $V_{n-q} := \mathrm{span}_{\mathbb{C}}\{\omega_\ell,\ \ell = 1,\dots,n-q\}$ is strictly negative definite.

By our choice of $V_{n-q} \subset H^{(1,0)}X$, for $v \in V_{n-q}$, we always have

$$(\partial\varrho \wedge \overline{\partial}\varrho)(v,\overline{v}) = 0. \tag{4.52}$$

Then by (4.48), we have at all points in $M \setminus \overline{X}_{u'}$, for any nonzero $v \in V_{n-q}$,

$$R^L_\chi(v,\overline{v}) = R^L(v,\overline{v}) + \chi'(\varrho)\partial\overline{\partial}\varrho(v,\overline{v}) \le \chi'(\varrho)\partial\overline{\partial}\varrho(v,\overline{v}) < 0, \tag{4.53}$$

which is equivalent to $\sqrt{-1}R^L_\chi < 0$ therein. As a consequence (and by Sylvester's law of inertia for Hermitian matrices), for $j = 0,\dots,n-q-1$, we have

$$M(j,h^L_\chi) = \overline{X}_{u'}(j,h^L_\chi) = \overline{X}_{u'}(j,h^L). \tag{4.54}$$

Since $\sqrt{-1}R^L \le 0$ on $\overline{X}_{u'} \setminus K_L$, we have for $j = 0,\dots,n-q-1$,

$$M(j,h^L_\chi) = \overline{X}_{u'}(j,h^L) = K_L(j,h^L). \tag{4.55}$$

Moreover, on a neighborhood of K_L, we have

$$c_1(L,h^L_\chi) = c_1(L,h^L). \tag{4.56}$$

Finally, combining the inequalities in Theorem 4.1 and (4.55), (4.56), we conclude exactly (1.5) and (1.6) for $H^{0,\ell}_{(2)}(M,L^k\otimes E)$ defined with respect to the original metrics Θ, h^E, h^L, and the integrals on the right-hand side are exactly on $K_L(\ell,h^L)$ or $K_L(\le \ell, h^L)$. Moreover, since M satisfies the $Z(j)$ condition for $0 \le j \le n-q-1$, we apply Theorem 2.9 to identify $H^{0,\ell}_{(2)}(M,L^k\otimes E)$ with $H^{0,\ell}(\overline{M},L^k\otimes E)$ and with $\mathscr{H}^{0,\ell}(M,L^k\otimes E)$. This completes the proof. $\square$

Remark 4.8. Note that the equation (4.52) plays a crucial role in achieving (4.53), which is a consequence of constructing Θ adapted to the splitting (4.51). Generally, it is difficult to control R^L_χ in (4.48), since $\sqrt{-1}\partial\varrho\wedge\overline{\partial}\varrho \ge 0$, while the term $\sqrt{-1}\partial\overline{\partial}\varrho$ could be negative along certain directions in $T^{(1,0)}X$.

Another important observation from (4.53) is that, instead of $\sqrt{-1}R^L \le 0$ we only need $\sqrt{-1}R^L|_{H^{(1,0)}X\otimes H^{(0,1)}X} \le 0$ on $X_v \setminus \overline{X}_{u'}$.

When $K_L = \varnothing$ in Theorem 1.4, we obtain:

Corollary 4.9. *If $c_1(L, h^L) \leq 0$ on a smooth domain $M \Subset X$ of a complex manifold X of dimension $n \geq 2$ such that its Levi form $\mathscr{L}_{bM}$ has at least $n-q$ negative eigenvalues on bM with $1 \leq q \leq n-1$, then for $0 \leq \ell \leq n-q-1$, as $k \to \infty$,*

$$\dim \mathscr{H}^{0,\ell}(M, L^k \otimes E) = o(k^n).$$

Note that the spaces $\mathscr{H}^{0,\ell}(M, L^k \otimes E)$ above can be replaced by the cohomology groups $H^{0,\ell}(\overline{M}, L^k \otimes E)$.

Now we turn to Theorem 1.5. Let X be a q-concave manifold equipped with a sublevel exhaustion function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ and the associated exceptional subset $K \Subset X$, then the $(1,1)$-form $\partial\overline{\partial}\varrho$ has at least $n-q+1$ negative eigenvalues on $X \setminus K$. Now we fix $u_0 \in \mathbb{R}$ such that $K \Subset X_{u_0}$. Then for a regular value $c > u_0$, by Proposition 2.5, $Z(n-q-1)$ holds on X_c and then we have for $j \leq n-q-2$ (see Theorem 2.9 and Theorem 2.10),

$$H^j(X, L^k \otimes E) \simeq H^{0,j}(\overline{X}_c, L^k \otimes E) \simeq \mathscr{H}^{0,j}(X_c, L^k \otimes E), \tag{4.57}$$

where $\mathscr{H}^{0,j}(X_c, \cdot)$ is the space of the harmonic forms smooth up to the boundary of X_c (for fixed smooth metrics Θ, h^L and h^E). By Theorem 2.11, we have

$$\dim H^{n-q-1}(X, L^k \otimes E) \leq \dim H^{0,n-q-1}(\overline{X}_c, L^k \otimes E). \tag{4.58}$$

Proof of Theorem 1.5. By the assumption, we take a sublevel exhaustion function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ and the associated exceptional subset $K \Subset X$. Let $u_0 \in \mathbb{R}$ be such that $K \cup K_L \subset X_{u_0} \Subset X$. Then we fix a regular value $c \geq u_0$ of ϱ such that $M := X_c \Subset X$. Then M is a Levi q-concave domain with a smooth boundary.

By our assumption, $\sqrt{-1}R^L \leq 0$ on $M \setminus K_L$, we apply Theorem 1.4 to this domain M and line bundle (L, h^L), combining with (4.57) and (4.58), we obtain exactly (1.7), (1.8) and (1.9). $\square$

Combining Theorem 1.5 with [14, §2.1], we can also obtain a singular version of Theorem 1.5. See [14, §2.1] for the detailed definition of singular metrics with algebraic singularities and the Nadel multiplier ideal sheaf.

Corollary 4.10 *(Singular holomorphic Morse inequalities for q-concave manifolds). Let X be a connected q-concave manifold of dimension n. Let L, E be holomorphic vector bundles on X with* $\mathrm{rk}(L) = 1$. *Let h^L be a Hermitian metric on L with algebraic singularities such that $S(h^L) \subset K$ and $c_1(L, h^L) \leq 0$ on $X \setminus K$ for some compact subset K. Then for $\ell \leq n-q-2$,*

$$\limsup_{k\to\infty} k^{-n}\dim H^{\ell}(X, L^k\otimes E\otimes \mathscr{I}(h^{L^k})) \le \frac{\mathrm{rk}(E)}{n!}\int\limits_{K(\ell,h^L)} (-1)^{\ell} c_1(L,h^L)^n,$$

$$\limsup_{k\to\infty} \sum_{j=0}^{\ell} \frac{(-1)^{\ell-j}}{k^n}\dim H^{j}(X, L^k\otimes E\otimes \mathscr{I}(h^{L^k})) \le \frac{\mathrm{rk}(E)}{n!}\int\limits_{K(\le\ell,h^L)} (-1)^{\ell} c_1(L,h^L)^n, \tag{4.59}$$

where $\mathscr{I}(h^{L^k})$ is the Nadel multiplier ideal sheaf of h^{L^k}, and $K(\ell,h^L) := K\cap(X\setminus S(h^L))(\ell,h^L)$. Moreover, we have

$$\begin{aligned}&\limsup_{k\to\infty} k^{-n}\dim H^{n-q-1}(X, L^k\otimes E\otimes \mathscr{I}(h^{L^k}))\\ &\quad\le \frac{\mathrm{rk}(E)}{n!}\int\limits_{K(n-q-1,h^L)} (-1)^{n-q-1} c_1(L,h^L)^n.\end{aligned} \tag{4.60}$$

The inequalities (4.59) for $\ell \le n-q-2$ already appeared in [14, Theorem 3.12] but the weak Morse inequality (4.60) for degree $n-q-1$ is new.

4.4. Volume of semi-positive line bundles; Proofs of Theorems 1.8 and 1.11

In this subsection, we first present a proof of Theorem 1.8. Our approach is based on the established proofs of Theorems 1.4 and 4.1 as well as the subsequent corollary derived from Theorem 1.4, where we identify $H^{0,0}(\overline{M}, L^k\otimes E)$ with $H^0(M, L^k\otimes E)$ by Theorem 2.9.

Corollary 4.11. *Let X be a connected complex manifold of dimension $n\ge 3$. Let M be a relatively compact domain in X with a smooth boundary bM, and assume that ϱ is a defining function for bM with $M := \{x\in X \,:\, \varrho(x)<0\}$ (and $d\varrho$ does not vanish near bM). Assume that $\mathscr{L}_\varrho$ is strictly negative on bM. Let E be a holomorphic vector bundle on X. Let (L,h^L) be a Hermitian holomorphic line bundle on X such that $c_1(L,h^L)\le 0$ on $M\setminus K$ for some compact subset $K\Subset M$. Then we have*

$$\begin{aligned}&\limsup_{k\to\infty} \frac{n!}{k^n}\dim H^0(M, L^k\otimes E) \le \mathrm{rk}(E)\int\limits_{K(0,h^L)} c_1(L,h^L)^n,\\ &\liminf_{k\to\infty} \frac{n!}{k^n}\dim H^0(M, L^k\otimes E) \ge \mathrm{rk}(E)\int\limits_{K(\le 1,h^L)} c_1(L,h^L)^n.\end{aligned} \tag{4.61}$$

Proof of (1.11). For any sufficiently small $\delta>0$, we first construct a Hermitian metric h^L_δ such that $c_1(L,h^L_\delta)\le 0$ outside $X_{-\delta}$; we then apply (4.61) and let $\delta\to 0$.

Let $\delta_0>0$ be small enough such that the assumptions for (1.11) hold on $\overline{M}_{\delta_0}\setminus M_{-\delta_0}\subset U$, where $M_c := \{x\in X \,:\, \varrho(x)<c\}$. We also assume that $\phi|_{\overline{U}}>0$, and $c_1(L,h^L)\ge 0$ on $\overline{M}\cap\overline{U}$.

For $\delta \in (0, \delta_0)$, let $\chi_\delta : \mathbb{R} \to [0,1]$ be an increasing smooth function such that $\chi_\delta(t) = 0$ for $t \leq -\delta$, and $\chi_\delta(t) = 1$ for $t \geq -\delta/2$. We may also assume that $\chi'_\delta(t) > 0$ for $-\delta/2 > t > -\delta$. Set a new Hermitian metric on L by

$$h^L_\delta := h^L \exp(\chi_\delta(\varrho)\phi). \tag{4.62}$$

Then the corresponding Chern curvature form

$$R^L_\delta = R^L - \chi_\delta(\varrho)\partial\overline{\partial}\phi - \chi'_\delta(\varrho)(\partial\phi \wedge \overline{\partial}\varrho + \partial\varrho \wedge \overline{\partial}\phi) - \chi''_\delta(\varrho)\phi\partial\varrho \wedge \overline{\partial}\varrho - \chi'_\delta(\varrho)\phi\partial\overline{\partial}\varrho. \tag{4.63}$$

Then it is clear that

$$R^L_\delta = R^L \text{ on } M_{-\delta}, \quad \text{and } R^L_\delta \equiv 0 \text{ on } M \setminus M_{-\delta/2}. \tag{4.64}$$

Hence Corollary 4.11 applies.

For any $c \in (-\delta, -\frac{\delta}{2})$ and restricting to $T^{(1,0)}bM_c \otimes T^{(0,1)}bM_c$, we have

$$R^L_\delta|_{bM_c} = (1 - \chi_\delta(c))R^L|_{bM_c} - \chi'_\delta(c)\phi\mathscr{L}_\varrho|_{bM_c} > 0. \tag{4.65}$$

As a consequence, on $M_{-\delta/2} \backslash M_{-\delta}$, the Hermitian form $\sqrt{-1}R^L_\delta$ has at most one negative eigenvalue. Then

$$\int_{(M_{-\delta/2}\backslash M_{-\delta})(\leq 1, h^L_\delta)} c_1(L, h^L_\delta)^n = \int_{M_{-\delta/2}\backslash M_{-\delta}} c_1(L, h^L_\delta)^n. \tag{4.66}$$

Write

$$(R^L_\delta)^n = \left(\partial\overline{\partial}(\phi - \chi_\delta(\varrho)\phi)\right)^n = -d\left\{\left(\partial\overline{\partial}(\phi - \chi_\delta(\varrho)\phi)\right)^{n-1}\partial(\phi - \chi_\delta(\varrho)\phi)\right\}$$

By Stokes' theorem and our condition on χ_δ, we get

$$\int_{M_{-\delta/2}\backslash M_{-\delta}} c_1(L, h^L_\delta)^n = \frac{\sqrt{-1}}{2\pi}\int_{bM_{-\delta}} c_1(L, h^L)^{n-1} \wedge \partial\phi. \tag{4.67}$$

Then by the second line of (4.61), we get

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{n!}{k^n} \dim H^0(M, L^k \otimes E) \\
&\geq \mathrm{rk}(E) \int_{M_{-\delta}(\leq 1, h^L)} c_1(L, h^L)^n + \mathrm{rk}(E)\frac{\sqrt{-1}}{2\pi}\int_{bM_{-\delta}} c_1(L, h^L)^{n-1} \wedge \partial\phi.
\end{aligned} \tag{4.68}$$

Finally (1.11) follows from making $\delta \to 0$ in (4.68). $\square$

Remark 4.12. From the proof of (1.11), we also obtain that

$$\begin{aligned}\limsup_{k\to\infty}\frac{n!}{k^n}&\dim H^0(M,L^k\otimes E)\\ &\le \operatorname{rk}(E)\int\limits_{M_{-\delta/2}(0,h^L_\delta)} c_1(L,h^L_\delta)^n\\ &=\operatorname{rk}(E)\int\limits_{M_{-\delta}(0,h^L)} c_1(L,h^L)^n+\operatorname{rk}(E)\int\limits_{(M_{-\delta/2}\setminus M_{-\delta})(0,h^L_\delta)} c_1(L,h^L_\delta)^n.\end{aligned}\tag{4.69}$$

The integral over $(M_{-\delta/2}\setminus M_{-\delta})(0,h^L_\delta)$ appears to be difficult to compute explicitly, making it challenging to determine the limit as $\delta\to 0$.

The proof of (1.12) is more intricate compared to the preceding one, requiring a thorough review of the arguments presented in the proofs of Theorems 1.5 and 4.1.

Proof of (1.12). We use the same notations as in the proof of (1.11). Let $\delta'>0$ be a small constant such that on $M_{\delta'}\setminus M_{-\delta'}$, we have, for $|\delta|\le\delta'$,

$$R^L=-\partial\overline{\partial}\varrho\,.$$

Note that this implies that $\sqrt{-1}\partial\overline{\partial}\varrho\le 0$.

Fix any $\delta\in(0,\delta')$, and take a smooth real function η on $\mathbb{R}$ such that

- $\eta(t)=0$ when $t\le-\delta$;
- $\eta'(-\delta/2)=1$;
- $\eta'(t)>0$ and $\eta''(t)>0$, for $t>-\delta$.

We then consider the following Hermitian metric $h^L_\eta:=h^Le^{-\eta(\varrho)}$ on L. Its curvature is

$$R^L_\eta=R^L+\eta'(\varrho)\partial\overline{\partial}\varrho+\eta''(\varrho)\partial\varrho\wedge\overline{\partial}\varrho.\tag{4.70}$$

We have the following properties:

- $R^L_\eta=R^L$ on $M_{-\delta}$.
- $R^L_\eta\ge 0$ on $M_{-\delta/2}$.
- R^L_η has at least 2 negative eigenvalues on $M_\delta\setminus\overline{M}_{-\delta/2}$ (since $n\ge 3$).

Following Remark 4.8 and repeating the arguments in the proofs of Proposition 4.2 and Theorem 1.4, we first obtain the optimal fundamental estimates for $(0,0)$ and $(0,1)$-forms on $M_0=M$, with the core subset $K=\overline{M}_{-\delta/2}$ and with respect to a modified Hermitian metric on L that coincides with the above h^L_η on $\overline{M}_{-\delta/2}$; then we obtain for any $\delta\in(0,\delta')$ that

$$\limsup_{k\to\infty} \frac{n!}{k^n} \dim H^0(M, L^k \otimes E) \leq \operatorname{rk}(E) \int\limits_{M_{-\delta/2}(0,h_\eta^L)} c_1(L, h_\eta^L)^n,$$
$$\liminf_{k\to\infty} \frac{n!}{k^n} \dim H^0(M, L^k \otimes E) \geq \operatorname{rk}(E) \int\limits_{M_{-\delta/2}(\leq 1\,,h_\eta^L)} c_1(L, h_\eta^L)^n. \tag{4.71}$$

By our construction, we have $M_{-\delta/2}(0, h_\eta^L) = M_{-\delta/2}(\leq 1\,, h_\eta^L)$, so that

$$\int\limits_{M_{-\delta/2}(0,h_\eta^L)} c_1(L, h_\eta^L)^n = \int\limits_{M_{-\delta/2}} c_1(L, h_\eta^L)^n = \int\limits_{M_{-\delta}} c_1(L, h^L)^n + \int\limits_{M_{-\delta/2}\setminus M_{-\delta}} c_1(L, h_\eta^L)^n. \tag{4.72}$$

By (4.70), we have

$$\begin{aligned} &\int\limits_{M_{-\delta/2}\setminus M_{-\delta}} c_1(L, h_\eta^L)^n \\ &= \int\limits_{M_{-\delta/2}\setminus M_{-\delta}} (1-\eta'(\varrho))^n c_1(L, h^L)^n \\ &\quad + \frac{\sqrt{-1}}{2\pi} \int\limits_{M_{-\delta/2}\setminus M_{-\delta}} n(1-\eta'(\varrho))^{n-1}\eta''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho \wedge c_1(L, h^L)^{n-1}. \end{aligned} \tag{4.73}$$

It is clear that there exists a constant C such that for all $\delta \in (0, \delta')$, we have

$$\left| \int\limits_{M_{-\delta/2}\setminus M_{-\delta}} (1-\eta'(\varrho))^n c_1(L, h^L)^n \right| \leq C\delta. \tag{4.74}$$

Another observation, by calculus in local coordinates, is that there exists a constant $C' > 0$ independent of δ and η such that

$$\begin{aligned} &\Big| \frac{\sqrt{-1}}{2\pi} \int\limits_{bM_{-\delta}} c_1(L, h^L)^{n-1} \wedge \partial\varrho \\ &\quad + \frac{\sqrt{-1}}{2\pi} \int\limits_{M_{-\delta/2}\setminus M_{-\delta}} n(1-\eta'(\varrho))^{n-1}\eta''(\varrho)\partial\varrho \wedge \overline{\partial}\varrho \wedge c_1(L, h^L)^{n-1} \Big| \leq C'\delta. \end{aligned} \tag{4.75}$$

In fact, since all points in $[-\delta, 0]$ are regular values of ϱ, after using the gradient flow of ϱ starting from $bM_{-\delta}$, we can identify

$$M_{-\delta/2} \setminus M_{-\delta} \simeq bM_{-\delta} \times [-\delta, -\delta/2)$$

so that for $t \in [-\delta, -\delta/2)$,

$$M_t \simeq bM_{-\delta} \times \{t\}.$$

Equivalently, for $(y,t) \in bM_{-\delta} \times [-\delta, -\delta/2) \simeq M_{-\delta/2} \setminus M_{-\delta}$, we have $\varrho(y,t) = t$. We fix a smooth volume form $d\sigma$ on $bM_{-\delta}$, and we write in the coordinate (y,t)

$$\frac{\sqrt{-1}}{2\pi} \partial\varrho \wedge \overline{\partial}\varrho \wedge c_1(L, h^L)^{n-1} = a(y,t) dt \wedge d\sigma(y).$$

Set

$$A(t) := \int_{bM_{-\delta}} a(y,t) d\sigma(y).$$

In particular,

$$\frac{\sqrt{-1}}{2\pi} \int_{bM_{-\delta}} c_1(L, h^L)^{n-1} \wedge \partial\varrho = -A(-\delta). \tag{4.76}$$

Therefore, using the fact

$$\int_{-\delta}^{-\delta/2} n(1 - \eta'(t))^{n-1} \eta''(t) dt = 1, \tag{4.77}$$

we deduce

$$\begin{aligned} &\frac{\sqrt{-1}}{2\pi} \int_{bM_{-\delta}} c_1(L, h^L)^{n-1} \wedge \partial\varrho \\ &\quad + \frac{\sqrt{-1}}{2\pi} \int_{M_{-\delta/2} \setminus M_{-\delta}} n(1 - \eta'(\varrho))^{n-1} \eta''(\varrho) \partial\varrho \wedge \overline{\partial}\varrho \wedge c_1(L, h^L)^{n-1} \\ &= \int_{-\delta}^{-\delta/2} n(1 - \eta'(t))^{n-1} \eta''(t) \left(A(t) - A(-\delta)\right) dt \end{aligned} \tag{4.78}$$

Then (4.75) follows from (4.77), (4.78) and the facts that there exists a constant $C' > 0$ independent of the function η and sufficiently small $\delta > 0$ such that for $t \in [-\delta, -\delta/2]$,

$$(1 - \eta'(t))^{n-1} \eta''(t) \geq 0, \quad |A(t) - A(-\delta)| \leq C'\delta.$$

Finally, combining (4.71)–(4.75) and letting $\delta \to 0$, we conclude exactly (1.12). $\square$

Now we provide the proof of Theorem 1.11.

Proof of Theorem 1.11. The proof follows directly from [34, §5], relying on the fact that, after a small deformation of the complex structures on L and M, the Chern curvature $c_1(L, h^L)$, the defining function of M, and the local potential ϕ remain close in the $\mathscr{C}^\infty$-topology. Since $c_1(L, h^L)$ is strictly positive and bM strictly pseudoconcave, these properties are preserved under such deformations.

Moreover, since $c_1(L, h^L) > 0$ on X_{reg}, the integral in (1.15) is strictly positive with respect to the original complex structures. Thus, after sufficiently small deformation of complex structures, the strict inequality (1.13) remains valid. Then we can apply Corollary 1.9 to (M, J'). □

4.5. Weak holomorphic Morse inequalities for (p, q)*-coronas*

By Definition 2.4, when M satisfies the $Z(q)$ condition, the boundary points look like either $(n - q - 1)$-concave or q-convex. It could be an interesting question to combine the estimate in Proposition 4.2 for $(n - q - 1)$-concave points and the estimate in [30, (3.5.19)] for q-convex points to obtain an optimal fundamental estimate for $(0, q)$-forms on M. In this subsection, we aim to apply our methods from the previous subsections to study the domains with mixed convexity/concavity, which shall be simpler than the case of the general $Z(q)$ condition.

Following [5] and the Definition 2.1 of q-convex/concave functions, we introduce the following notions:

Definition 4.13. Let X be a connected complex manifold of dimension $n \geq 1$ and let $p, q \in \{1, \ldots, n\}$.

(i) The complex space X is called (p, q)-*convex-concave* if there exists a smooth real function $\varphi : X \to (a, b)$ with $a \in \mathbb{R} \cup \{-\infty\}$ and $a < b \in \mathbb{R} \cup \{+\infty\}$ such that

- For any c, d with $a < c < d < b$, we have $X_c^d := \{x \in X \ : \ c < \varphi(x) < d\} \Subset X$.
- There exist a_0, b_0 with $a < a_0 < b_0 < b$ such that $\sqrt{-1}\partial\overline{\partial}\varphi$ has at least $n - p + 1$ positive eigenvalues on $\{\varphi > b_0\}$, and has at least $n - q + 1$ positive eigenvalues on $\{\varphi < a_0\}$.

In this case, we call φ an *exhaustion function* for X with exceptional interval $[a_0, b_0]$.

(ii) If $M \Subset X$ is a relatively compact domain with a smooth boundary, and we write $bM = b_+M \cup b_-M$ where b_+M and b_-M are disjoint from each other and consist of connected components of bM. We call M a (p, q)-*corona* with respect to the splitting $bM = b_+M \cup b_-M$ if, near b_+M, there is a local smooth definition function r_+ for M such that r_+ is p-convex; and near b_-M, there is a local smooth definition function r_- for M such that r_- is q-concave.

Note that when $p+q \leq n-1$, then a (p,q)-corona M always satisfies the $Z(j)$ condition for $p \leq j \leq n-q-1$. When $b_+M = \varnothing$ or $b_-M = \varnothing$, we get M to be q-concave or p-convex, respectively. If $p=1$, then b_+M (if nonempty) is a strongly pseudoconvex boundary component; if $q=1$, b_-M (if nonempty) is a strongly pseudoconcave boundary component. The following result is clear by definition and by Proposition 2.5.

Lemma 4.14. *If X is a (p,q)-convex-concave complex manifold with an exhaustion function φ and an exceptional interval $[a_0, b_0] \subset (a,b)$, then for any regular values a' and b' of φ with $a < a' < a_0$ and $b_0 < b' < b$, the domain $X_{a'}^{b'} = \{x \in X \,:\, a' < \varphi(x) < b'\}$ is a (p,q)-corona in X with $b_+X_{a'}^{b'} = \varphi^{-1}(b')$ and $b_-X_{a'}^{b'} = \varphi^{-1}(a')$.*

Analogous to Theorems 2.9, 2.10, and 2.11, we have the following result (see also [3, Propositions 21, 22 and Théorème 15], [22, Theorem 3.4.9 and the subsequent remark], and [5, Proposition 1.2]).

Proposition 4.15. *With the same notation and hypotheses as in Lemma 4.14, let F be a holomorphic vector bundle on X. If $p+q \leq n-1$, then we have*

- *For $p \leq j \leq n-q-1$, the isomorphism among cohomology groups*

$$H^j(X,F) \simeq H^j(X_{a'}^{b'}, F) \simeq H^{0,j}(X_{a'}^{b'}, F). \tag{4.79}$$

- *For $p \leq j \leq n-q-2$, the isomorphism among cohomology groups*

$$H^j(X,F) \simeq H^{0,j}_{(2)}(X_{a'}^{b'}, F) \simeq H^{0,j}(\overline{X}_{a'}^{b'}, F). \tag{4.80}$$

- *For $j = n-q-1$,*

$$\dim H^{n-q-1}(X,F) \leq \dim H^{0,n-q-1}_{(2)}(X_{a'}^{b'}, F) = \dim H^{0,n-q-1}(\overline{X}_{a'}^{b'}, F). \tag{4.81}$$

We can now extend the weak holomorphic Morse inequalities presented in Theorem 1.4 to the setting of the (p,q)-corona.

Theorem 4.16. *Let (X,Θ) be a Hermitian complex manifold of dimension $n \geq 2$. Take $p,q \in \{1,\dots,n-1\}$ such that $p+q \leq n-1$. Let $M \Subset X$ be a relatively compact (p,q)-corona with a smooth boundary in X with respect to the splitting $bM = b_+M \cup b_-M$. Let (E,h^E), (L,h^L) be two Hermitian holomorphic vector bundles on X with $\mathrm{rk}(L) = 1$.*

- *There exists a Hermitian metric h^L_χ on L that coincides with h^L away from bM, such that for $p \leq j \leq n-q-1$,*

$$\limsup_{k\to\infty} k^{-n} \dim H^{0,j}(\overline{M}, L^k \otimes E) \leq \frac{\mathrm{rk}(E)}{n!} \int_{M(j,h^L_\chi)} (-1)^j c_1(L,h^L_\chi)^n. \tag{4.82}$$

- *If $c_1(L,h^L) \le 0$ on an open neighborhood of b_-M in $\overline{M}$, and $c_1(L,h^L) \ge 0$ on an open neighborhood of b_+M in $\overline{M}$, then for $p \le j \le n-q-1$,*

$$\limsup_{k\to\infty} k^{-n} \dim H^{0,j}(\overline{M}, L^k \otimes E) \le \frac{\mathrm{rk}(E)}{n!} \int\limits_{M(j,h^L)} (-1)^j c_1(L,h^L)^n. \tag{4.83}$$

Moreover, if $p = 1$, we also have

$$\liminf_{k\to\infty} k^{-n} \dim H^{0,0}(\overline{M}, L^k \otimes E) \ge \frac{\mathrm{rk}(E)}{n!} \int\limits_{M(\le 1,h^L)} c_1(L,h^L)^n. \tag{4.84}$$

Note that the spaces $H^{0,j}(\overline{M}, L^k \otimes E)$ above can be replaced by the spaces of $\overline{\partial}$-Neumann harmonic forms $\mathscr{H}^{0,j}(M, L^k \otimes E)$ induced by Θ, h^E, h^L.

Proof. Since b_+M and b_-M are disjoint from each other, we could take any small open neighborhoods U_+ and U_- of them in X respectively such that $\overline{U}_+ \cap \overline{U}_- = \varnothing$. Then on U_+, by p-convex condition, we proceed as in [30, Lemmas 3.5.3, 3.5.4 and Theorem 3.5.8] to modify Θ and h^L locally on U_+, so that if $s \in B^{0,j}(M, L^k \otimes E)$ for $j \ge p$ and $\mathrm{supp}\,(s) \subset U_+$, we have

$$\|\overline{\partial}_k^E s\|_{L^2(M)}^2 + \|\overline{\partial}_k^{E*} s\|_{L^2(M)}^2 \ge (C_0 k - C')\,\|s\|_{L^2(M)}^2. \tag{4.85}$$

Then on U_-, by q-concave condition, we proceed as in the proof of Proposition 4.2 to modify Θ and h^L locally on U_-, so that if $s \in B^{0,j}(M, L^k \otimes E)$ for $j \le n-q-1$ and $\mathrm{supp}\,(s) \subset U_-$, we have

$$\|\overline{\partial}_k^E s\|_{L^2(M)}^2 + \|\overline{\partial}_k^{E*} s\|_{L^2(M)}^2 \ge (C_0 k - C')\,\|s\|_{L^2(M)}^2. \tag{4.86}$$

Where $C_0 > 0$ and $C' > 0$ are constants independent of k. Let $V \Subset M$ contain $M \setminus (\overline{U}_+ \cup \overline{U}_-)$. Combining (4.85) and (4.86), we obtain that there exists a constant $\widetilde{C} > 0$ such that for $p \le j \le n-q-1$, $s \in B^{0,j}(M, L^k \otimes E)$, $k \gg 0$,

$$(1 - \frac{\widetilde{C}}{k})\|s\|_{L^2(M)}^2 \le \frac{\widetilde{C}}{k}\left(\|\overline{\partial}_k^E s\|_{L^2(M)}^2 + \|\overline{\partial}_k^{E*} s\|_{L^2(M)}^2\right) + \int\limits_V |s|^2 dv_X, \tag{4.87}$$

where we use the modified metrics $\widetilde{\Theta}$ and h_χ^L. Then (4.82) follows from Corollary 2.13 and (4.87). Moreover, if $p = 1$, we also obtain by Theorem 2.14 that

$$\liminf_{k\to\infty} k^{-n} \dim H^{0,0}(\overline{M}, L^k \otimes E) \ge \frac{\mathrm{rk}(E)}{n!} \int\limits_{V(\le 1,h_\chi^L)} c_1(L,h_\chi^L)^n. \tag{4.88}$$

Using the same method as in the proof of Theorem 1.4 and [30, Theorem 3.5.8], in particular, we construct the function χ (according to the choice of $V \Subset M$) so that results analogous to (4.53)–(4.56) and [30, (3.5.20)] hold. More precisely, for $p \leq j \leq n-q-1$, we have $c_1(L, h^L_\chi) = c_1(L, h^L)$ on

$$V(j\,, h^L_\chi) = V(j\,, h^L). \tag{4.89}$$

Therefore, we obtain (4.83) from (4.82), (4.89) and the fact that $(-1)^j c_1(L, h^L)^n$ is non-negative on $M(j, h^L)$.

Finally, we need to prove (4.84). By (4.88) and (4.89), we obtain for a given $V \Subset M$ as chosen in the beginning

$$\liminf_{k\to\infty} k^{-n} \dim H^{0,0}(\overline{M}, L^k \otimes E) \geq \frac{\mathrm{rk}(E)}{n!} \int\limits_{V(\leq 1\,, h^L)} c_1(L, h^L)^n. \tag{4.90}$$

Since we can choose the open subset $U_\pm$ to be sufficiently small (due to the smoothness of the boundary), then by letting V exhaust M the integral in the right-hand side of (4.90) converges to the integral in (4.84) and this finishes the proof. □

As a direct consequence of Proposition 4.15 and Theorem 4.16, we obtain an extension of the weak holomorphic Morse inequalities in Theorem 1.5 as follows.

Theorem 4.17. *Consider integers $n \geq 2$, $p, q \geq 1$ with $1 \leq p+q \leq n-1$, and let X be a n-dimensional (p, q)-convex-concave connected complex manifold. Using the same notation as in Definition 4.13, let $\varphi : X \to (a, b)$ be an exhaustion function with the exceptional interval $[a_0, b_0]$. Let E, L be holomorphic vector bundles on X with $\mathrm{rk}(L) = 1$. Let h^L be a smooth Hermitian metric on L such that there are $a' < b'$ satisfying $a < a' \leq a_0$, $b_0 \leq b' < b$ such that $c_1(L, h^L) \leq 0$ on $\{\varphi < a'\}$ and $c_1(L, h^L) \geq 0$ on $\{\varphi > b'\}$. Then for $p \leq j \leq n-q-1$, we have the weak Morse inequalities:*

$$\begin{aligned}\limsup_{k\to\infty} k^{-n} \dim H^j(X, L^k \otimes E) &\leq \frac{\mathrm{rk}(E)}{n!} \int\limits_{X^{b'}_{a'}(j, h^L)} (-1)^j c_1(L, h^L)^n \\ &\leq \frac{\mathrm{rk}(E)}{n!} \int\limits_{X(j, h^L)} (-1)^j c_1(L, h^L)^n.\end{aligned} \tag{4.91}$$

Moreover, if $p = 1$, we also have

$$\liminf_{k\to\infty} k^{-n} \dim H^0(X, L^k \otimes E) \geq \frac{\mathrm{rk}(E)}{n!} \int\limits_{X(\leq 1\,, h^L)} c_1(L, h^L)^n, \tag{4.92}$$

where the right or left side might be $+\infty$.

Remark 4.18. Note that analogous versions of Theorems 1.8, 1.3 and 6.1 can also be established for (p, q)-coronas or (p, q)-convex-concave manifolds. Moreover, by combining the above strategy with the techniques in [36] and [32], the results can be extended to compact complex spaces with smooth boundary and isolated singularities, by constructing a complete Hermitian metric in a punctured neighborhood of each singularity.

5. Proof of Theorem 1.12

Let (X, Θ) be an n-dimensional Hermitian complex manifold ($n \geq 2$). Let (E, h^E) and (L, h^L) be holomorphic Hermitian vector bundles over X with $\operatorname{rk} L = 1$. Let M be a relatively compact domain in X with smooth boundary bM. We denote by $L^{-1} = L^*$ and E^* the dual bundle of L and E, respectively. We also recall our assumptions in Theorem 1.12 as follows:

(A) L is semi-positive on X and positive at one point;
(B) The Levi form $\mathscr{L}_\varrho$ of a defining function for $M \Subset X$ has at least $n - q$ negative eigenvalues on bM $(1 \leq q \leq n - 1)$.

Note that (B) implies that M satisfies $Z(j)$ for $0 \leq j \leq n - q - 1$.

We need the following two lemmas to prove Theorem 1.12.

Lemma 5.1. *Assume X is compact. Under the assumptions (A) and (B), for the harmonic projection with $q \leq \ell \leq n - 1$,*

$$H : H^0(X, L^k) \times \mathscr{H}^{0,n-\ell-1}(M, L^{-k} \otimes E^*) \to \mathscr{H}^{0,n-\ell-1}(M, E^*), \tag{5.1}$$

there exists a non-zero holomorphic section $s \in H^0(X, L^{k_0})$ for some $k_0 \in \mathbb{N}$ such that $H(s\theta) = 0$ for every $\theta \in \mathscr{H}^{0,n-\ell-1}(M, L^{-k_0} \otimes E^*)$.

Proof. To simplify notation, we set $V := H^0(X, L^k)$, $U := \mathscr{H}^{0,n-\ell-1}(M, E^*)$ and $W := \mathscr{H}^{0,n-\ell-1}(M, L^{-k} \otimes E^*)$. Moreover, we define a bilinear map $F(s, \alpha) := H(s\alpha)$ for $s \in V$ and $\alpha \in W$. Then we obtain a linear map $G : V \to W^* \otimes U$ by $G(s) := F(s, \cdot)$. Suppose the assertion were false, that is, for any $k \in \mathbb{N}$ and any non-zero $s \in V$, there exists $\alpha_0 \in W$ such that $F(s, \alpha_0) \neq 0$. If $G(s_0) = 0$ for some $s_0 \in V$, then $F(s_0, \alpha) = 0$ for any $\alpha \in W$. Thus s_0 is zero, that is, G is injective. And it follows that $\dim V \leq \dim(W^* \otimes U) = \dim W \times \dim U$, that is, for any $k \in \mathbb{N}$,

$$\dim H^0(X, L^k) \leq \dim \mathscr{H}^{0,n-\ell-1}(M, L^{-k} \otimes E^*) \times \dim \mathscr{H}^{0,n-\ell-1}(M, E^*). \tag{5.2}$$

From Corollary 4.9, the assumptions (B), $L \geq 0$ on X and $q \leq \ell$ imply that as $k \to \infty$,

$$\dim \mathscr{H}^{0,n-\ell-1}(M, L^{-k} \otimes E^*) = o(k^n). \tag{5.3}$$

By the bigness of L, we have

$$\dim H^0(X, L^k) \geq Ck^n, \quad k \to \infty. \tag{5.4}$$

Under the assumption (B), it follows that M satisfies $Z(j)$ for all $0 \leq j \leq n-q-1$, thus Theorem 2.8(d) implies that, for $q \leq \ell$,

$$\dim \mathscr{H}^{0,n-\ell-1}(M, E^*) < \infty. \tag{5.5}$$

We obtain $Ck^n \leq o(k^n)$ for k large which cannot hold. Thus, the assertion is true. $\square$

Lemma 5.2. *Assume X is compact and the assumptions (A) and (B) hold. For $q \leq \ell \leq n-1$, let $s \in H^0(X, L^{k_0})$ be the non-zero holomorphic section in Lemma 5.1 and $\sigma \in \Omega^{n,\ell}(bM, E)$ be $\overline{\partial}_b$-closed. Then, there exists a $\overline{\partial}$-closed extension S of $s \otimes \sigma \in \Omega^{n,\ell}(bM, L^{k_0} \otimes E)$, i.e., $S \in \Omega^{n,\ell}(\overline{M}, L^{k_0} \otimes E)$ such that $\overline{\partial} S = 0$ on M and $\mu(S|_{bM}) = \mu(s \otimes \sigma)$.*

Proof. Let $\sigma' \in \Omega^{n,\ell}(\overline{M}, E)$ such that $\sigma'|_{bM} = \sigma$. Let $\theta \in \mathscr{H}^{0,n-\ell-1}(M, L^{-k_0} \otimes E^*)$. Thus

$$\theta \wedge (s\sigma') \in \Omega^{n,n-1}(\overline{M}) \tag{5.6}$$

and $s\theta \in \Omega^{0,n-\ell-1}(\overline{M}, E^*)$ is $\overline{\partial}$-closed. By Theorem 2.8 (a)(b) and $H(s\theta) = 0$, there exists $\xi \in \Omega^{0,n-\ell-2}(\overline{M}, E^*)$ such that

$$s\theta = \overline{\partial}\xi + H(s\theta) = \overline{\partial}\xi. \tag{5.7}$$

Then, by the Stokes formula,

$$\int_{bM} \theta \wedge (s\sigma) = \int_{bM} (s\theta) \wedge \sigma = \int_{bM} (\overline{\partial}\xi) \wedge \sigma = (-1)^{n-\ell-1} \int_{bM} \xi \wedge (\overline{\partial}\sigma'). \tag{5.8}$$

By (2.21), (2.22) and $\overline{\partial}_b \sigma = 0$, we have

$$(\overline{\partial}\sigma')|_{bM} = \overline{\partial}_b \sigma + (\psi \wedge \overline{\partial}\varrho)|_{bM} = (\psi \wedge \overline{\partial}\varrho)|_{bM} \tag{5.9}$$

where $\psi \in \Omega^{n,\ell}(\overline{M}, E)$, and ϱ is the defining function of M. By the Stokes formula and $bM = \{x \in X : \varrho(x) = 0\}$, we have

$$\int_{bM} \xi \wedge (\overline{\partial}\sigma') = \int_{bM} \xi \wedge \psi \wedge \overline{\partial}\varrho = -\int_{bM} \overline{\partial}(\xi \wedge \psi) \wedge \varrho = 0. \tag{5.10}$$

Then, by (5.8) and (5.10), it follows that

$$\int_{bM} \theta \wedge (s\sigma) = 0 \tag{5.11}$$

for the $\overline{\partial}_b$-closed form $s\sigma \in \Omega^{n,\ell}(bM, L^{k_0} \otimes E)$ and any $\theta \in \mathscr{H}^{0,n-\ell-1}(M, L^{-k_0} \otimes E^*)$. Finally, there exists a $\overline{\partial}$-closed extension S of $s\sigma$ by Theorem 2.15 for $Z(n-\ell-1)$. □

Proof of Theorem 1.12. The result follows by replacing E by $E \otimes \Lambda^n T^{1,0}X$ in Lemmas 5.1 and 5.2. □

Theorem 1.12 can be slightly generalized to the following case without strict positivity.

Theorem 5.3. *Let X be a Moishezon manifold of $\dim X = n \geq 2$. Let $1 \leq q \leq n-1$ and let $M \Subset X$ be a domain with smooth boundary bM on which the Levi form restricted to the analytic tangent space has at least $n-q$ negative eigenvalues. Let L be a big line bundle on X with a singular Hermitian metric h^L. Assume that h^L is smooth on M and $c_1(L, h^L) \geq 0$ on M. Then the conclusion of Theorem 1.12 holds.*

Proof. Since X is compact and L is big, we have $\dim H^0(X, L^k) \geq Ck^n$. Since the assumption (B) holds, Theorem 2.8 (d) entails $\dim \mathscr{H}^{0,n-\ell-1}(M, F) < \infty$. Meanwhile, Corollary 4.9 implies $\dim \mathscr{H}^{0,n-\ell-1}(M, L^{-k} \otimes E^*) = o(k^n)$. As in Lemma 5.1, for $q \leq \ell \leq n-1$ and the harmonic projection $H : H^0(X, L^k) \times \mathscr{H}^{0,n-\ell-1}(M, L^{-k} \otimes E^*) \to \mathscr{H}^{0,n-\ell-1}(M, E^*)$, there exist $k_0 \in \mathbb{N}$ and $s \in H^0(X, L^{k_0}) \setminus \{0\}$ such that $H(s\theta) = 0$ for every $\theta \in \mathscr{H}^{0,n-\ell-1}(M, L^{-k_0} \otimes E^*)$. Finally, as in Lemma 5.2, we obtain $\overline{\partial}$-closed extension $S \in \Omega^{n,\ell}(\overline{M}, L^{k_0} \otimes E)$, and then replace E by $E \otimes \Lambda^n T^{1,0}X$. □

6. Vanishing theorems and applications

In this section, we present some vanishing theorems as consequences of the Bochner-Kodaira-Nakano formula (1.1) with boundary term. These include the classical Andreotti-Tomassini vanishing theorem and a specific case for negative line bundles on domains with Levi-flat boundaries.

6.1. Andreotti-Tomassini vanishing theorem; Proof of Theorem 1.3

In this subsection we revisit the Andreotti-Tomassini vanishing theorem [6] (especially for the q-concave case), and give a proof based on Theorem 1.2 and inspired by Griffiths [20, Theorem 7.1].

Proof of Theorem 1.3. The proof of Theorem 1.3 follows from a modification of the proof of Proposition 4.2. We solely prove (a) via Theorem 1.2, as an analogous proof of (b) can be achieved by applying (3.15) (refer to [30, Theorem 3.5.9]).

Let ϱ be a defining function for M such that $M := \{x \in X \ : \ \varrho(x) < 0\}$ and $d\varrho$ doesn't vanish near bM. First, by Lemma 4.5, since $\mathscr{L}_\varrho$ has at least $n-q$ negative eigenvalues along bM, there exists a smooth Hermitian metric Θ in a neighborhood of $\overline{M}$ such that $d\varrho$ has norm 1 in a neighborhood of bM and for a sufficiently small $\varepsilon > 0$

$$[\sqrt{-1}\mathscr{L}_\varrho, \Lambda]^{bM}_\varepsilon \geq 1 \tag{6.1}$$

as a positive operator acting on $\oplus_{j\leq n-q-1}\Lambda^j T^{(0,1)*}bM$.

Since $c_1(L, h^L) < 0$ on $\overline{M}$, we need not modify further the metric h^L for there exists $C' > 0$ such that for all $s \in \Omega^{0,j}(\overline{M}, L)$ $(j \leq n-1)$ and $x \in \overline{M}$,

$$\langle[\sqrt{-1}R^L, \Lambda]s, s\rangle_h(x) \geq C'|s|^2_h(x). \tag{6.2}$$

Then following the first part of the proof of Lemma 4.6, by replacing $\sqrt{-1}\partial\overline{\partial}\varrho$ with $\sqrt{-1}R^L$, we can conclude that, after modifying Θ again with the parameter $T \geq 1$ as in (4.27), we may always assume that for the Hermitian metric Θ, (6.1) still holds as in (4.29) and moreover, there exists a constant $C_0 > 0$ such that for any sufficiently small $\varepsilon > 0$, and for any $\tau \in [0, 1]$, $s \in \Omega^{0,j}_0(\overline{M}, L^k)$ with $\operatorname{supp}(s)$ included in a small neighborhood of bM, with $j \leq n-q-1$ and any $k \in \mathbb{N}_{\geq 1}$, we have

$$\left\langle[\sqrt{-1}R^L, \Lambda]_\varepsilon s, s\right\rangle_h + 2(1-\varepsilon)\tau\left\langle R^L(e^{(1,0)}_{\mathfrak{n}}, e^{(0,1)}_{\mathfrak{n}})s, s\right\rangle_h \geq C_0|s|^2_h. \tag{6.3}$$

When we work for the points in M that are away from bM, (6.2) still holds no matter how we modify the metric Θ (with possibly a different constant $C' > 0$). So if $\varepsilon > 0$ is sufficiently small, we also have

$$\langle[\sqrt{-1}R^L, \Lambda]_\varepsilon s, s\rangle_h \geq C'|s|^2_h. \tag{6.4}$$

By Theorem 1.2, combining (6.1), (6.2), (6.3), and (6.4), we conclude that there exist $C > 0$, $C_0 > 0$, $C'' > 0$ and a sufficiently small $\varepsilon \in (0, 1)$ such that for all $k \in \mathbb{N}_{\geq 1}$, $s \in \Omega^{0,j}(\overline{M}, L^k \otimes E)$ with $j \leq n-q-1$ and $\iota_{e_{\mathfrak{n}}} s = 0$ on bM, we have

$$(1+\varepsilon)\left(\|\overline{\partial}^E_k s\|^2_{L^2(M)} + \|\overline{\partial}^{E*}_k s\|^2_{L^2(M)}\right) + C(1+\frac{1}{\varepsilon})\|s\|^2_{L^2(M)} \geq (C_0 k - C'')\,\|s\|^2_{L^2(M)}. \tag{6.5}$$

The above inequality still holds for $s \in \mathrm{Dom}(\overline{\partial}^E_k) \cap \mathrm{Dom}(\overline{\partial}^{E*}_k) \cap L^2_{0,j}(M, L^k \otimes E)$ for $j \leq n-q-1$, by the density of $B^{0,j}(M, L^k \otimes E)$ with respect to the graph norm of $\overline{\partial}^E_k + \overline{\partial}^{E*}_k$. Therefore, we conclude that if $k > \frac{1}{C_0}(C'' + C + C/\varepsilon)$, for all $j \leq n-q-1$

$$\mathscr{H}^{0,j}(M, L^k \otimes E) = 0. \tag{6.6}$$

Using Theorem 2.9 to identify $\mathscr{H}^{0,j}(M, L^k \otimes E)$ with $H^{0,j}(\overline{M}, L^k \otimes E)$ for $j \leq n-q-1$ completes the proof. $\square$

As a consequence of the above theorem, we recover the Andreotti-Tomassini vanishing theorem [6].

Theorem 6.1 *([6, Theorems 1 and 2]). Let $n \geq 2$ and $1 \leq q \leq n-1$.*

(a) Let X be a connected q-concave complex manifold of dimension n, and let ϱ be an exhaustion function with an exceptional set $K \Subset X$. Let E and L be holomorphic vector bundles on X and $\mathrm{rk}(L) = 1$. Suppose L is negative in the neighborhood of $\overline{X}_c$ for some regular value c of ϱ such that $K \subset X_c$. Then, there exists an integer $k_0 = k_0(L, E)$ such that

$$H^{\ell}(X, L^k \otimes E) = 0 \qquad \text{for } k \geq k_0 \text{ and } \ell \leq n-q-1. \tag{6.7}$$

(b) Let X be a connected q-convex complex manifold of dimension n, and let ϱ be an exhaustion function with an exceptional set $K \Subset X$. Let E and L be holomorphic vector bundles on X and $\mathrm{rk}(L) = 1$. Suppose L is positive in the neighborhood of $\overline{X}_c$ for some regular value c of ϱ such that $K \subset X_c$. Then there exists an integer $k_0 = k_0(L, E)$ such that

$$H^{\ell}(X, L^k \otimes E) = 0 \qquad \text{for } k \geq k_0 \text{ and } \ell \geq q. \tag{6.8}$$

Proof. For (a): by Theorem 1.3(a), for all sufficiently large k, we have for $j \leq n-q-1$,

$$H^{0,j}(\overline{X}_c, L^k \otimes E) = 0. \tag{6.9}$$

By Theorems 2.9, 2.10 and 2.11, for $j \leq n-q-2$, we have

$$H^{j}(X, L^k \otimes E) \simeq H^{0,j}(\overline{X}_c, L^k \otimes E), \tag{6.10}$$

and

$$\dim H^{n-q-1}(X, L^k \otimes E) \leq \dim H^{0,n-q-1}(\overline{X}_c, L^k \otimes E). \tag{6.11}$$

So we obtain (6.7).

Part (b) follows from Theorem 1.3(b), Theorem 2.9, and by theorems of Hörmander and Andreotti-Grauert [30, Theorem 3.5.6 and 3.5.7]. □

6.2. The $\overline{\partial}_b$ extension via vanishing theorems

If we replace the assumption of Theorem 1.12 that L is semi-positive on X and positive at least at one point by the assumption $L > 0$ on M, the $\overline{\partial}_b$ extension result follows via our proof of Theorem 1.3(a). In this case, the compactness assumption on X can be dropped.

Theorem 6.2. *Let X be a complex manifold of dimension n. Let (E, h^E) and (L, h^L) be holomorphic Hermitian vector bundles over X and $\mathrm{rk}(L) = 1$. Let $1 \le q \le n-1$ and assume that M is a relatively compact domain in X with smooth boundary bM on which the Levi form restricted to the analytic tangent space has at least $n-q$ negative eigenvalues. Assume that $c_1(L, h^L) > 0$ on $\overline{M}$.*

Then there exists $k_0 \in \mathbb{N}$ such that for each $k \ge k_0$, every $q \le \ell \le n-1$, and every $\overline{\partial}_b$-closed form $\sigma \in \Omega^{0,\ell}(bM, L^k \otimes E)$, there exists a $\overline{\partial}$-closed extension Σ of σ, i.e.,

$$\Sigma \in \Omega^{0,\ell}(\overline{M}, L^k \otimes E) \tag{6.12}$$

such that $\overline{\partial}\Sigma = 0$ on M and $\mu(\Sigma|_{bM}) = \mu(\sigma)$ on bM.

Proof. Firstly, M satisfies $Z(i)$ for all $0 \le i \le n-q-1$. By (6.6) for $i \le n-q-1$ and k large, $\mathscr{H}^{n,i}(M, L^{-k} \otimes E^*) = 0$ after identifying (n,i)-forms with $(0,i)$-forms valued in K_X. Finally, we apply Theorem 2.15 for $Z(n-\ell-1)$ with $\ell \ge q$. $\square$

Similarly, if $L < 0$ on M, the $\overline{\partial}_b$ extension result follows via Theorem 1.3(b).

Theorem 6.3. *Let X be a complex manifold of dimension n. Let (E, h^E) and (L, h^L) be holomorphic Hermitian bundles over X and $\mathrm{rk}(L) = 1$. Let $1 \le q \le n-1$ and assume that M is a relatively compact domain in X with smooth boundary bM on which the Levi form restricted to the analytic tangent space has at least q positive eigenvalues. Assume that $c_1(L, h^L) < 0$ on $\overline{M}$.*

Then there exists $k_0 \in \mathbb{N}$ such that for each $k \ge k_0$, every $0 \le \ell \le q-1$, and every $\overline{\partial}_b$-closed form $\sigma \in \Omega^{0,\ell}(bM, L^k \otimes E)$, there exists a $\overline{\partial}$-closed extension Σ of σ, i.e.,

$$\Sigma \in \Omega^{0,\ell}(\overline{M}, L^k \otimes E) \tag{6.13}$$

such that $\overline{\partial}\Sigma = 0$ on M and $\mu(\Sigma|_{bM}) = \mu(\sigma)$ on bM.

Proof. Firstly, M satisfies $Z(j)$ for $n-q \le j \le n-1$. Since $M = \{\varrho < 0\}$ and $M_\varepsilon = \{\varrho < \varepsilon\}$ for sufficiently small $\varepsilon > 0$ are j-convex manifolds for $j \ge n-q$, then Theorem 1.3(b) with the replacement of the degree q by $n-q$ gives $\mathscr{H}^{n,j}(M, L^{-k} \otimes E^*) = 0$ for k large after identifying (n,j)-forms with $(0,j)$-forms valued in K_X. Finally, we apply Theorem 2.15 for $Z(n-\ell-1)$ with $0 \le \ell \le q-1$. $\square$

Theorem 6.3 with $q = 1$ means that, under the assumption $L < 0$ on $\overline{M}$, we can generalize the classical result [26, Theorem 7.5] to forms with values in $L^k \otimes E$.

As a final remark, we proceed in the same vein by using Theorem 2.15 for $q = p = 0$ and [43, Proposition 4.17] that $\mathscr{H}^{n,n-1}(M, L^*) = H^{0,n-1}_{(2)}(M, L^* \otimes K_X) = 0$. It follows that: Let (X, ω) be a Kähler manifold and $M \Subset X$ be a domain with smooth boundary bM such that the Levi form on bM have one positive eigenvalue and semi-positive everywhere. If (L, h^L) is negative on bM and semi-negative on M, then every $\overline{\partial}_b$-closed smooth section of L on bM has a holomorphic extension to all of M.

6.3. Negative line bundles on domains with Levi-flat boundary

A domain $M = \{x \in X : \varrho(x) < 0\} \Subset X$ with defining function $\varrho \in \mathscr{C}^\infty(X, \mathbb{R})$ is said to have a Levi-flat boundary if the Levi form $\mathscr{L}_\varrho$ vanishes identically along bM. In the Levi-flat case, the boundary term on the right-hand side of (1.2) vanishes, allowing us to obtain the following result for the negative line bundles.

Proposition 6.4. *Let X be a complex manifold of dimension $n \geq 2$, and let $M \Subset X$ be a domain with a smooth Levi-flat boundary bM. Let ϱ be a defining function of M, and let Θ_0 be a smooth Hermitian metric on X such that $|d\varrho| = 1$ near bM. Let (L, h^L) be a negative line bundle in a neighborhood of $\overline{M}$ and let (E, h^E) be any holomorphic Hermitian vector bundle on X. Then there exists a Hermitian metric Θ on $\overline{M}$, which is given by modifying Θ_0 locally near bM, and an integer $k_0 > 0$ such that for $q = 0, \ldots, n-1$ and $k \geq k_0$, we have*

$$\mathscr{H}^{0,q}_{(2)}(M, L^k \otimes E; \Theta) := \ker \square_k^{L^k \otimes E}|_{L^2_{0,q}(M, L^k \otimes E)} = 0, \tag{6.14}$$

where the L^2-cohomology and Kodaira Laplacians are given with respect to the metric Θ.

Proof. Using the fact that L is negative near $\overline{M}$, we can proceed as in the proof of Lemma 4.6 to modify Θ_0 near bM such that $d\varrho$ still has norm 1 near bM. Moreover, there exists $C_0 > 0$ such that for a small $0 < \varepsilon \ll 1$, any $\tau \in [0, 1]$, and for any $s \in \Omega^{0,j}(\overline{M}, L^k \otimes E)$ (with $j \leq n-1$ and $k \in \mathbb{N}$), we have near bM,

$$\left\langle [\sqrt{-1} R^L, \Lambda]_\varepsilon s, s \right\rangle_h + 2(1-\varepsilon)\tau \left\langle R^L(e_{\mathfrak{n}}^{(1,0)}, e_{\mathfrak{n}}^{(0,1)}) s, s \right\rangle_h \geq C_0 |s|_h^2, \tag{6.15}$$

where the metric is taken with respect to Θ and $(h^L)^{\otimes k}$. Since bM is Levi-flat and $d\varrho$ has the norm 1 near bM, the boundary terms vanish identically in (1.2). Finally, we repeat the proof of Proposition 4.2. In particular, by applying Theorem 1.2, we obtain, with some constants $C > 0$ and $C' > 0$ independent of k,

$$(1+\varepsilon)\left(\|\overline{\partial}_k^E s\|^2_{L^2(M)} + \|\overline{\partial}_k^{E*} s\|^2_{L^2(M)} \right) + C\left(1 + \frac{1}{\varepsilon}\right) \|s\|^2_{L^2(M)} \geq (C_0 k - C') \|s\|^2_{L^2(M)}. \tag{6.16}$$

The proposition follows. □

As a consequence, for any domain M with a smooth Levi-flat boundary ($\dim_{\mathbb{C}} M \geq 2$), there exists no strictly plurisubharmonic bounded smooth function in any open neighborhood of $\overline{M}$. This also follows from the maximum principle directly applied on the boundary of M. When we specialize to projective spaces, we conclude from Proposition 6.4 that if $M \Subset \mathbb{CP}^n$ is a domain with a smooth Levi-flat boundary, then each hyperplane in $\mathbb{CP}^n$ has a nonempty intersection with $\overline{M}$. This also follows from the solution to the Levi problem over $\mathbb{CP}^n$ by Fujita [19] and Takeuchi [41]. Namely, any proper pseudoconvex domain in $\mathbb{CP}^n$ is Stein and neither M nor the complement of $\overline{M}$ can contain a divisor when the boundary of M is Levi-flat. Furthermore, for $n \geq 3$, this is evidently due to the nonexistence of smooth Levi-flat real hypersurfaces in $\mathbb{CP}^n$ [40].

The conjecture of the nonexistence of smooth real Levi-flat hypersurfaces in projective spaces first appeared in [12] and [13] for the study of minimal sets of holomorphic foliations on $\mathbb{CP}^n$. Then it was proven for $n \geq 3$ by Lins Neto [29] in the real analytic case and by Siu [40] in the smooth case. Despite many attempts, the nonexistence of smooth Levi-flat real hypersurfaces in $\mathbb{CP}^2$ is still considered an open problem; see [25,37,38] and references therein. In [1], Adachi and Brinkschulte deduced a curvature restriction of Levi-flat real hypersurfaces in $\mathbb{CP}^2$. Their method was inspired by the integral formula of Griffiths [20, VII.§5]; see also Lemma 3.4 and Corollary 3.5.

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Data availability

No data was used for the research described in the article.

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