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Optimal convergence speed of Bergman metrics on symplectic manifolds

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It is known that a compact symplectic manifold endowed with a prequantum line bundle can be embedded in the projective space generated by the eigensections of low energy of the Bochner Laplacian acting on high p-tensor powers of the prequantum line bundle. We show that the Fubini-Study forms induced by these embeddings converge at speed rate $1/p^2$ to the symplectic form. This result implies the generalization to the almost-Kähler case of the lower bounds on the Calabi functional given by Donaldson for Kähler manifolds, as shown by Lejmi and Keller.

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0. Introduction

A very useful tool in the study of canonical Kähler metrics is the use of Bergman metrics to approximate arbitrary Kähler metrics in a given integral cohomology class, see e. g., [7, 11, 17].

Let (X, ω) be a compact Kähler manifold endowed with a Hermitian holomorphic line bundle (L, h^L) such that $\frac{\sqrt{-1}}{2\pi} R^L = \omega$. Since the bundle L is positive, Kodaira's theorem shows that high powers L^p give rise to holomorphic embeddings $\Phi_p : X \to \mathbb{P}(H^0(X, L^p)^*)$. The *Bergman form* ω_p at level p is defined as the rescaled induced Fubini-Study form $\frac{1}{p}\Phi_p^*\omega_{\rm FS}$, where $\omega_{\rm FS}$ is the natural Fubini-Study form on $\mathbb{P}(H^0(X, L^p)^*)$. Tian [17] showed that ω_p converges to ω in the $\mathscr{C}^2$ topology with speed rate $p^{-1/2}$, as $p \to \infty$, that is, there exists $C > 0$ such that for any $p \in \mathbb{N}^*$ we have

$$\left| \frac{1}{p}\Phi_p^*(\omega_{\rm FS}) - \omega \right|_{\mathscr{C}^2} \leqslant \frac{C}{p^{1/2}} \cdot \tag{0.1}$$

This was improved by Ruan [15] to convergence in $\mathscr{C}^\infty$ with speed rate p^{-1} (see also [13, Theorem 5.1.4]). Tian's result was motivated by a problem of Yau [18].

The process described above can be seen in the general framework of quantization. The Bergman forms ω_p can be thought as quantization at level p of the original Kähler form ω. The number $1/p$ is to be thought of as analogous to Planck's constant and in the semiclassical limit $p \to \infty$ the quantized objects ω_p converge to the original Kähler one.

The proof of the convergence in [15, 17] is based on the diagonal expansion of the Bergman kernel up to second order. A full diagonal asymptotic expansion of the Bergman kernel in powers of p in the $\mathscr{C}^\infty$ topology was obtained by Catlin [5] and Zelditch [19] as an application of Boutet de Monvel and Sjöstrand's work [4], see also [6, 14] for different approaches and generalizations. We refer to [13] for a comprehensive study of several analytic and geometric aspects of Bergman kernel. One advantage of the expansion in the $\mathscr{C}^\infty$ topology is that it easily implies the convergence of the Bergman forms ω_p to ω with speed rate p^{-2}, see [13, (5.1.23)]. This convergence speed is optimal. Note that the scalar curvature is up to a multiplicative constant the coefficient of the second term of the Bergman kernel expansion. The purpose of this paper is to extend this optimal result to the case of symplectic manifolds.

The Bergman kernel of a holomorphic line bundle L on a complex manifold is the smooth kernel of the orthogonal projection from the space of square integrable sections on the space of holomorphic sections, or, equivalently, on the kernel of the Kodaira Laplacian $\square^L = \overline{\partial}^L \overline{\partial}^{L*} + \overline{\partial}^{L*} \overline{\partial}^L$ on L. In order to find a suitable notion of "holomorphic section" of a prequantum line bundle on a compact *symplectic manifold*, Guillemin and Uribe [9] introduced a renormalized Bochner Laplacian $\Delta_{p,0}$ (cf. (0.5)) which reduces to $2\square^L$ in the Kähler case.

We describe this construction in detail. Let (X,ω) be a compact symplectic manifold of real dimension $2n$. Let (L,h^L) be a Hermitian line bundle on X, and let ∇^L be a Hermitian connection on (L,h^L) with the curvature $R^L = (\nabla^L)^2$. We will assume throughout the paper that (L,h^L,∇^L) is a prequantum line bundle of (X,ω), i.e.,

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{0.2}$$

We choose an almost complex structure J such that ω is J-invariant and $\omega(\cdot, J\cdot) > 0$. The almost complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Let $g^{TX}(\cdot,\cdot) := w(\cdot, J\cdot)$ be the Riemannian metric on TX induced by ω and J. The Riemannian volume form dv_X of (X, g^{TX}) has the form $dv_X = \omega^n/n!$. The L^2-Hermitian product on the space $\mathscr{C}^\infty(X, L^p)$ of smooth sections of L^p on X, with $L^p := L^{\otimes p}$, is given by

$$\langle s_1, s_2\rangle = \int_X \langle s_1, s_2\rangle(x) dv_X(x). \tag{0.3}$$

Let ∇^{TX} be the Levi-Civita connection on (X, g^{TX}) with curvature R^{TX}, and let ∇^{L^p} be the connection on L^p induced by ∇^L. Let $\{e_k\}$ be a local orthonormal frame of (TX, g^{TX}). The Bochner Laplacian acting on $\mathscr{C}^\infty(X, L^p)$ is given by

$$\Delta^{L^p} = -\sum_k \Big[\big(\nabla^{L^p}_{e_k}\big)^2 - \nabla^{L^p}_{\nabla^{TX}_{e_k} e_k} \Big]. \tag{0.4}$$

Given $\Phi \in \mathscr{C}^\infty(X, \mathbb{R})$, the renormalized Bochner Laplacian is defined by

$$\Delta_{p,\Phi} = \Delta^{L^p} - 2\pi n p + \Phi. \tag{0.5}$$

By [9], [12, Corollary 1.2], there exists $C_L > 0$ independent of p such that

$$\operatorname{Spec}(\Delta_{p,\Phi}) \subset [-C_L, C_L] \cup [4\pi p - C_L, +\infty), \tag{0.6}$$

where Spec(A) denotes the spectrum of the operator A. Since $\Delta_{p,\Phi}$ is an elliptic operator on a compact manifold, it has discrete spectrum and its eigensections are smooth. Let $\mathcal{H}_p$ be the direct sum of eigenspaces of $\Delta_{p,\Phi}$ corresponding to the eigenvalues lying in $[-C_L, C_L]$. In mathematical physics terms, the operator $\Delta_{p,\Phi}$ is a semiclassical Schrödinger operator and the space $\mathcal{H}_p$ is the space of its bound states as $p \to \infty$. The space $\mathcal{H}_p$ proves to be an appropriate replacement for the space of holomorphic sections $H^0(X, L^p)$ from the Kähler case. In particular, we have for p large enough (cf. [13, (8.3.3)]),

$$\dim \mathcal{H}_p = \int_X \mathrm{Td}(T^{(1,0)}X) e^{p\omega}, \tag{0.7}$$

where $\mathrm{Td}(T^{(1,0)}X)$ is the Todd class of $T^{(1,0)}X$, which corresponds to the Riemann-Roch-Hirzebruch formula from complex geometry.

Let $\mathbb{P}(\mathcal{H}_p^*)$ be the projective space associated to the dual space of $\mathcal{H}_p$; we identify $\mathbb{P}(\mathcal{H}_p^*)$ with the Grassmannian of hyperplanes in $\mathcal{H}_p$. The base locus of $\mathcal{H}_p$ is the set $\mathrm{Bl}(\mathcal{H}_p) = \big\{x \in X : s(x) = 0 \text{ for all } s \in \mathcal{H}_p\big\}$. We define the Kodaira map

$$\Phi_p : X \backslash \mathrm{Bl}(\mathcal{H}_p) \to \mathbb{P}(\mathcal{H}_p^*), \quad \Phi_p(x) = \big\{s \in \mathcal{H}_p : s(x) = 0\big\}, \tag{0.8}$$

which sends $x \in X \backslash \mathrm{Bl}(\mathcal{H})$ to the hyperplane of sections vanishing at x. Note that $\mathcal{H}_p$ is endowed with the induced L^2 Hermitian product (0.3) so there is a well-defined Fubini-Study metric g_{FS} on $\mathbb{P}(\mathcal{H}_p^*)$ with the associated form ω_{FS}.

The symplectic Kodaira embedding theorem [14, Theorem 3.6], [13, Theorem 8.3.12], states that for large p the Kodaira maps $\Phi_p : X \to \mathbb{P}(\mathcal{H}_p^*)$ are embeddings and the Bergman forms converge to the symplectic form with speed rate p^{-1}. We note that in this case the near-diagonal expansion of the Bergman kernel is essential for the proof, in contrast to the the Kähler case, where the diagonal expansion already implies the result. Let us also observe that [14, Theorem 3.6] and [13, Theorem 8.3.12] are valid in a more general context, namely when g^{TX} is an arbitrary J-invariant Riemannian metric.

There exists in the literature another replacement of the notion of holomorphic section, see e. g., [2, 16]. It is based on a construction by Boutet de Monvel and Guillemin [3] of a first-order pseudodifferential operator D_b on the circle bundle of L^*. The associated Szegő kernels are well defined modulo smooth operators on the associated circle bundle, even though D_b is neither canonically defined nor unique. Indeed, Boutet de Monvel–Guillemin define the Szegő kernels first, and construct the operator D_b from the Szegő kernels. For these spaces the Bergman forms converge to the symplectic form with speed rate p^{-1}, too.

The main result of this paper is as follows.

Theorem 0.1. *Let (X,ω) be a compact symplectic manifold and (L,h^L) be a Hermitian line bundle endowed with a Hermitian connection ∇^L such that $\frac{\sqrt{-1}}{2\pi}R^L=\omega$ holds. Let J be an almost complex structure on TX such that $g^{TX}(\cdot,\cdot):=\omega(\cdot,J\cdot)$ is a J-invariant Riemannian metric on TX. Then for any $\ell\in\mathbb{N}$, there exists $C_\ell>0$ such that*

$$\left|\frac{1}{p}\Phi_p^*(\omega_{\text{FS}})-\omega\right|_{\mathscr{C}^\ell}\leqslant\frac{C_\ell}{p^2}\,, \tag{0.9}$$

where Φ_p is the Kodaira map (0.8) defined by the space $\mathcal{H}_p$ of bound states of the renormalized Bochner Laplacian $\Delta_{p,\Phi}$ associated with g^{TX},∇^L,Φ in (0.5).

The proof is based on the near diagonal expansion of the Bergman kernel of $\mathcal{H}_p$ from [13, 14]. The sharp bound of $\mathscr{O}(p^{-2})$ is due to some remarkable cancellations of the coefficients in this expansions, reminiscent of the local properties of the curvature of Kähler metrics.

The main motivation for approximating Kähler metrics by Fubini-Study metrics arises from questions about the existence and uniqueness of Kähler metrics of constant scalar curvature, or more generally, Kähler-Einstein metrics, see [7, 8, 17, 18]. It is natural to study such questions also in the symplectic framework, for example, it is interesting to generalize to the almost-Kähler case the lower bounds on the Calabi functional given by Donaldson [8]. This is done by Lejmi and Keller [10]. Theorem 0.1 plays a crucial role in their proof in the symplectic case.

The organization of this paper is as follows. In Section 1, we recall the formal calculus on $\mathbb{C}^n$ for the model operator $\mathscr{L}$ (cf. (1.2)), which is the main ingredient of our approach. In Section 2, we review the asymptotic expansion of the generalized Bergman kernel. In Section 3, we reduce the proof of Theorem 0.1 to Theorem 3.3. In Section 4, we prove Theorem 3.3 and thus finish the proof of Theorem 0.1.

We shall use the following notations. For $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, $z\in\mathbb{C}^n$, we set $|\alpha|=\sum_{j=1}^n\alpha_j$, $\alpha!=\prod_j(\alpha_j!)$ and $z^\alpha:=z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. Moreover, when an index variable appears twice in a single term, it means that we are summing over all its possible values.

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1. Kernel calculus on $\mathbb{C}^n$

In this section, we recall the formal calculus on $\mathbb{C}^n$ for the model operator $\mathscr{L}$ introduced in [14, § 1.4], [13, § 4.1.6] (with $a_j = 2\pi$ therein). This calculus is the main ingredient of our approach.

Let us consider the canonical coordinates $(Z_1, \dots, Z_{2n})$ on the real vector space $\mathbb{R}^{2n}$. On the complex vector space $\mathbb{C}^n$ we consider the complex coordinates $(z_1, \dots, z_n)$. The two sets of coordinates are linked by the relation $z_j = Z_{2j-1} + \sqrt{-1} Z_{2j}$, $j = 1, \dots, n$.

We consider the L^2-norm

$$\|\cdot\|_{L^2} = \left(\int_{\mathbb{R}^{2n}} |\cdot|^2 dZ \right)^{1/2} \text{ on } \mathbb{R}^{2n}, \tag{1.1}$$

where $dZ = dZ_1 \dots dZ_{2n}$ is the Lebesgue measure. We define the differential operators:

$$\begin{aligned} b_j &= -2\frac{\partial}{\partial z_j} + \pi \overline{z}_j, \quad b_j^+ = 2\frac{\partial}{\partial \overline{z}_j} + \pi z_j, \\ b &= (b_1, \dots, b_n), \quad \mathscr{L} = \sum_{j=1}^n b_j b_j^+, \end{aligned} \tag{1.2}$$

which extend to closed densely defined operators on $\big(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2}\big)$. As such, b_j^+ is the adjoint of b_j and $\mathscr{L}$ defines as a densely defined self-adjoint operator on $\big(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2}\big)$. The following result was established in [14, Theorem 1.15] (cf. also [13, Theorem 4.1.20]).

Theorem 1.1. *The spectrum of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\operatorname{Spec}(\mathscr{L}) = \big\{ 4\pi|\alpha| : \alpha \in \mathbb{N}^n \big\}, \tag{1.3}$$

and an orthogonal basis of the eigenspace of $4\pi|\alpha|$ is given by

$$b^\alpha \left(z^\beta \exp\left(-\pi \sum_j |z_j|^2/2 \right) \right), \quad \text{with } \beta \in \mathbb{N}^n. \tag{1.4}$$

In particular, an orthonormal basis of $\mathrm{Ker}(\mathscr{L})$ *is*

$$\left\{\phi_\beta(z) = \left(\frac{\pi^{|\beta|}}{\beta!}\right)^{1/2} z^\beta e^{-\pi\sum_j |z_j|^2/2} : \beta\in\mathbb{N}^n\right\}. \tag{1.5}$$

Let $\mathscr{P}(Z,Z')$ denote the kernel of the orthogonal projection $\mathscr{P}: L^2(\mathbb{R}^{2n})\to\mathrm{Ker}(\mathscr{L})$ with respect to dZ'. Set $\mathscr{P}^\perp = \mathrm{Id}-\mathscr{P}$.

Obviously $\mathscr{P}(Z,Z') = \sum_\beta \phi_\beta(z)\overline{\phi_\beta(z')}$, so we infer from (1.5) that

$$\mathscr{P}(Z,Z') = \exp\left(-\frac{\pi}{2}\sum_{j=1}^n \big(|z_j|^2 + |z_j'|^2 - 2z_j\overline{z}_j'\big)\right). \tag{1.6}$$

By (1.2) and (1.6), we obtain

$$\big(b_j^+\mathscr{P}\big)(Z,Z') = 0,\quad \big(b_j\mathscr{P}\big)(Z,Z') = 2\pi(\overline{z}_j - \overline{z}_j')\mathscr{P}(Z,Z'). \tag{1.7}$$

The following commutation relations are very useful in the computations. Namely, for any polynomial $g(z,\overline{z})$ in z and $\overline{z}$, we have

$$\begin{aligned}
[b_j, b_k^+] &= b_j b_k^+ - b_k^+ b_j = -4\pi\delta_{jk},\\
[b_j, b_k] &= [b_j^+, b_k^+] = 0,\\
[g(z,\overline{z}), b_j] &= 2\frac{\partial}{\partial z_j} g(z,\overline{z}),\\
[g(z,\overline{z}), b_j^+] &= -2\frac{\partial}{\partial \overline{z}_j} g(z,\overline{z}).
\end{aligned} \tag{1.8}$$

For a polynomial F in Z, Z', we denote by $F\mathscr{P}$ the operator on $L^2(\mathbb{R}^{2n})$ defined by the kernel $F(Z,Z')\mathscr{P}(Z,Z')$ and the volume form dZ.

In the calculations involving the kernel $\mathscr{P}(\cdot,\cdot)$, we prefer however to use the orthogonal decomposition of $L^2(\mathbb{R}^{2n})$ given in Theorem 1.1 and the fact that $\mathscr{P}$ is an orthogonal projection, rather than integrating against the expression (1.6) of $\mathscr{P}(\cdot,\cdot)$. This point of view leads to streamlined computations and to a better understanding of the operators involved. As an example, Theorem 1.1 implies that

$$\big(\mathscr{P} b^\alpha z^\beta \mathscr{P}\big)(Z,Z') = \begin{cases} \big(z^\beta\mathscr{P}\big)(Z,Z'), & \text{if } |\alpha| = 0,\\ 0, & \text{if } |\alpha| > 0.\end{cases} \tag{1.9}$$

We will also identify z to $\sum_j z_j \frac{\partial}{\partial z_j}$ and $\overline{z}$ to $\sum_j \overline{z}_j \frac{\partial}{\partial \overline{z}_j}$ when we consider z and $\overline{z}$ as vector fields, and

$$\mathcal{R} = \sum_j Z_j \frac{\partial}{\partial Z_j} = z + \overline{z} = Z. \tag{1.10}$$

2. Asymptotic expansion of the generalized Bergman kernel

Let a^X be the injectivity radius of (X, g^{TX}). We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in X and T_xX with center x and radius ε, respectively. Then the exponential map $T_xX \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\varepsilon)$ onto $B^X(x,\varepsilon)$ for $\varepsilon \leqslant a^X$. From now on, we identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ via the exponential map for $\varepsilon \leqslant a^X$.

We fix $x_0 \in X$. For $Z \in B^{T_{x_0}X}$ we identify (L_Z, h^L_Z) to $(L_{x_0}, h^L_{x_0})$ by parallel transport with respect to the connection ∇^L along the curve $\gamma_Z : [0,1] \ni u \to \exp^X_{x_0}(uZ)$.

In general, for functions in normal coordinates, we will add a subscript x_0 to indicate the base point $x_0 \in X$. Similarly, $P_{\mathcal{H}_p}(x,y)$ induces in terms of the above trivialization (note that $\mathrm{End}(L^p_{x_0}) = \mathbb{C}$) a smooth function

$$\big\{(Z,Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\big\} \ni (Z,Z') \longmapsto P_{\mathcal{H}_p,x_0}(Z,Z') \in \mathbb{C}\,,$$

which also depends smoothly on the parameter x_0.

Let us choose an orthonormal basis $\{w_j\}_{j=1}^n$ of $T^{(1,0)}_{x_0}X$. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \dots, n$, forms an orthonormal basis of $T_{x_0}X$. We use coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ given by the identification

$$\mathbb{R}^{2n} \ni (Z_1, \dots, Z_{2n}) \longmapsto \sum_{j=1}^{2n} Z_j e_j \in T_{x_0}X. \tag{2.1}$$

In the sequel we also use complex coordinates $z = (z_1, \dots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa_{x_0} : T_{x_0}X \to \mathbb{R}$, $Z \mapsto \kappa_{x_0}(Z)$ be a smooth positive function defined by

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1, \tag{2.2}$$

where the subscript x_0 of $\kappa_{x_0}(Z)$ indicates the base point $x_0 \in X$.

Rescaling $\Delta_{p,\Phi}$ and Taylor expansion. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \mathbb{C})$, $Z \in \mathbb{R}^{2n}$, $|Z| \leqslant \varepsilon$, and for $t = \frac{1}{\sqrt{p}}$, set

$$(S_t s)(Z) := s(Z/t), \quad \mathscr{L}_t := S_t^{-1} \kappa^{1/2} t^2 \Delta_{p,\Phi} \kappa^{-1/2} S_t. \tag{2.3}$$

For $U \in T_{x_0}X$, we denote ∇_U the ordinary differential in direction U. Set

$$\begin{aligned} \nabla_{0,\bullet} &= \nabla_\bullet + \frac{1}{2} R^L_{x_0}(Z, \cdot), \\ \mathscr{L}_0 &= -\sum_{j=1}^{2n} (\nabla_{0,e_j})^2 - 2n\pi = \sum_{j=1}^{n} b_j b_j^+ = \mathscr{L}. \end{aligned} \tag{2.4}$$

By [14, Theorem 1.4], there exist second order differential operators $\mathcal{O}_r$ such that we have an asymptotic expansion in t when $t \to 0$,

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathscr{O}(t^{m+1}). \tag{2.5}$$

Moreover,

$$\mathcal{O}_1(Z) = -\frac{2}{3} \big(\partial_j R^L\big)_{x_0}(\mathcal{R}, e_i) Z_j \nabla_{0,e_i} - \frac{1}{3} \big(\partial_i R^L\big)_{x_0}(\mathcal{R}, e_i), \tag{2.6}$$

and

(2.7)

$$\begin{aligned} \mathcal{O}_2(Z) = \; & \frac{1}{3} \Big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j \Big\rangle_{x_0} \nabla_{0,e_i} \nabla_{0,e_j} \\ & + \left[\frac{2}{3} \Big\langle R^{TX}_{x_0}(\mathcal{R}, e_j) e_j, e_i \Big\rangle_{x_0} - \frac{1}{2} \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} (\mathcal{R}, e_i) \right] \nabla_{0,e_i} \\ & - \frac{1}{4} \nabla_{e_i} \left(\sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} (\mathcal{R}, e_i) \frac{Z^\alpha}{\alpha!} \right) - \frac{1}{9} \sum_i \left[\sum_j (\partial_j R^L)_{x_0} (\mathcal{R}, e_i) Z_j \right]^2 \\ & - \frac{1}{12} \Big[\mathscr{L}_0, \Big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_i \Big\rangle_{x_0} \Big] + \Phi_{x_0}. \end{aligned}$$

From (2.1) and (2.3), as in [13, Remark 4.1.8], $\mathscr{L}_t$ is a formally self-adjoint elliptic operator with respect to $\|\cdot\|_{L^2}$ on $\mathbb{R}^{2n}$ and is a smooth family of operators with respect to the parameter $x_0 \in X$. Thus $\mathscr{L}, \mathscr{L}_0$ and $\mathcal{O}_r$ in (2.5) are formally self-adjoint with respect to $\|\cdot\|_{L^2}$.

By [13, Theorem 8.3.8], the following asymptotic expansion of the generalized Bergman kernel holds.

Theorem 2.1. *There exist polynomials $J_r(Z,Z')$ in Z,Z' with the same parity as r and of degree $\deg J_r(Z,Z') \leqslant 3r$, such that if we define*

$$\mathscr{F}_r(Z,Z') = J_r(Z,Z')\mathscr{P}(Z,Z'), \quad J_0 = 1, \tag{2.8}$$

then for any $k,\ell,m \in \mathbb{N}$, $q>0$, there exists $C>0$ such that if $p \geqslant 1$, $Z,Z' \in T_{x_0}X$ and $|Z|,|Z'| \leqslant \frac{q}{\sqrt{p}}$, we have

$$\begin{aligned}\sup_{|\alpha|+|\alpha'|\leqslant m} \Bigg| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}} \bigg(\frac{1}{p^n} P_{\mathcal{H}_p}(Z,Z') \\ - \sum_{r=0}^{k} \mathscr{F}_r(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}} \bigg) \Bigg|_{\mathscr{C}^{\ell}(X)} \leqslant Cp^{-\frac{k-m+1}{2}},\end{aligned} \tag{2.9}$$

where $\mathscr{C}^{\ell}(X)$ is $\mathscr{C}^{\ell}$-norm for the parameter $x_0 \in X$.

Moreover, by [13, (4.1.93), (8.3.45)], $\mathscr{F}_1$ and $\mathscr{F}_2$ are given by (cf. [13, (8.3.65)], [14, (1.111)])

$$\begin{aligned}
\mathscr{F}_1 &= -\mathscr{P}^{\perp}\mathscr{L}^{-1}\mathcal{O}_1\mathscr{P} - \mathscr{P}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp},\\
\mathscr{F}_2 &= \mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{P} - \mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{P}\\
&\quad + \mathscr{P}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp} - \mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp}\\
&\quad + \mathscr{P}^{\perp}\mathscr{L}^{-1}\mathcal{O}_1\mathscr{P}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp} - \mathscr{P}\mathcal{O}_1\mathscr{L}^{-2}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{P}.
\end{aligned} \tag{2.10}$$

From Theorem 2.1, we get in particular [13, Theorem 8.3.3]: there exist $\boldsymbol{b}_r \in \mathscr{C}^{\infty}(X,\mathbb{R})$ such that for any $k,\ell \in \mathbb{N}$, there exists $C_{k,\ell}>0$ such that

$$\left| \frac{1}{p^n} P_{\mathcal{H}_p}(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x)p^{-r} \right|_{\mathscr{C}^{\ell}} \leqslant C_{k,\ell}\, p^{-k-1}, \tag{2.11}$$

and

$$\boldsymbol{b}_0(x_0) = \mathscr{F}_0(0,0) = 1, \quad \boldsymbol{b}_r(x_0) = \mathscr{F}_{2r}(0,0), \quad \mathscr{F}_{2r+1}(0,0) = 0. \tag{2.12}$$

3. Proof of Theorem 0.1

In this section we reduce Theorem 0.1 to Theorem 3.3. Let us fix $x_0 \in X$. As in section 2, we identify a small geodesic ball $B^X(x_0,\varepsilon)$ to $B^{T_{x_0}X}$ by means of the exponential map and we trivialize L by using a unit frame $e_L(Z)$ which is parallel with respect to ∇^L along the curve $[0,1] \ni u \to uZ$ for $Z \in B^{T_{x_0}X}(0,\varepsilon)$.

Set $d_p := \dim \mathcal{H}_p$ and for $v = (v_1, \ldots, v_{d_p}) \in \mathbb{C}^{d_p}$, set $\|v\|^2 = \sum_{j=1}^{d_p} |v_j|^2$. We can now express the Fubini-Study form in the homogeneous coordinate $[v] = [v_1, \ldots, v_{d_p}] \in \mathbb{P}(\mathcal{H}_p^*)$ as

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\|v\|^2\right) = \frac{\sqrt{-1}}{2\pi}\left[\frac{1}{\|v\|^2}\sum_{j=1}^{d_p} dv_j \wedge d\overline{v}_j - \frac{1}{\|v\|^4}\sum_{j,k=1}^{d_p} \overline{v}_j v_k dv_j \wedge d\overline{v}_k\right]. \tag{3.1}$$

Let $\{s_j\}$ be an orthonormal basis of $\mathcal{H}_p$, and let $\{s^j\}$ be its dual basis. We write locally $s_j = f_j e_L^{\otimes p}$, then by (0.8), as in [13, (5.1.17)], we have

$$\Phi_p(x) = \left[\sum_{j=1}^{d_p} f_j(x) s^j\right] \in \mathbb{P}(\mathcal{H}_p^*). \tag{3.2}$$

Set

$$f^p(x,y) = \sum_{i=1}^{d_p} f_i(x)\overline{f_i}(y) \quad \text{and} \quad \left|f^p(x)\right|^2 = f^p(x,x). \tag{3.3}$$

Then

$$P_{\mathcal{H}_p}(x,y) = f^p(x,y) e_L^{\otimes p}(x) \otimes e_L^{\otimes p}(y)^*, \quad \left|f^p(x)\right|^2 = P_{\mathcal{H}_p}(x,x). \tag{3.4}$$

By (3.1), (3.2) and (3.3), we get

$$\begin{aligned}\Phi_p^*(\omega_{FS})(x_0) &= \frac{\sqrt{-1}}{2\pi}\left[\frac{1}{|f^p|^2}\sum_{j=1}^{d_p} df_j \wedge d\overline{f_j} - \frac{1}{|f^p|^4}\sum_{j,k=1}^{d_p} \overline{f_j} f_k df_j \wedge d\overline{f_k}\right](x_0) \\ &= \frac{\sqrt{-1}}{2\pi}\Big[\left|f^p(x_0)\right|^{-2} d_x d_y f^p(x,y) \\ &\qquad - \left|f^p(x_0)\right|^{-4} d_x f^p(x,y) \wedge d_y f^p(x,y)\Big]\Big|_{x=y=x_0},\end{aligned} \tag{3.5}$$

where $\big|_{x=y=x_0}$ means the pull-back by the diagonal map $\jmath : X \to X \times X$, $x_0 \mapsto (x_0, x_0)$.

By (3.4), $P_{\mathcal{H}_p}(x,y)$ is represented by $f^p(x,y)$ under our trivialization of L. Since we work with normal coordinates, we get from (2.2) (cf. [13, (4.1.101)])

$$\kappa(Z) = 1 + \mathscr{O}(|Z|^2). \tag{3.6}$$

By (2.9), (2.12), (3.5) and (3.6), we get

$$\begin{aligned}
&\frac{1}{p}\Phi_p^*(\omega_{FS})(x_0)\\
&= \frac{\sqrt{-1}}{2\pi}\Bigg\{ \left[\frac{1}{\mathscr{F}_0}d_xd_y\mathscr{F}_0 - \frac{1}{\mathscr{F}_0^2}d_x\mathscr{F}_0\wedge d_y\mathscr{F}_0\right](0,0)\\
&\quad + p^{-1/2}\left[\frac{1}{\mathscr{F}_0}d_xd_y\mathscr{F}_1 - \frac{1}{\mathscr{F}_0^2}\big(d_x\mathscr{F}_1\wedge d_y\mathscr{F}_0 + d_x\mathscr{F}_0\wedge d_y\mathscr{F}_1\big)\right](0,0)\\
&\quad + p^{-1}\Bigg[\frac{1}{\mathscr{F}_0}d_xd_y\mathscr{F}_2 - \frac{\mathscr{F}_2}{\mathscr{F}_0^2}d_xd_y\mathscr{F}_0 + \frac{2\mathscr{F}_2}{\mathscr{F}_0^3}d_x\mathscr{F}_0\wedge d_y\mathscr{F}_0\\
&\qquad - \frac{1}{\mathscr{F}_0^2}\big(d_x\mathscr{F}_0\wedge d_y\mathscr{F}_2 + d_x\mathscr{F}_1\wedge d_y\mathscr{F}_1 + d_x\mathscr{F}_2\wedge d_y\mathscr{F}_0\big)\Bigg](0,0)\\
&\quad + p^{-3/2}\Bigg[\frac{1}{\mathscr{F}_0}d_xd_y\mathscr{F}_3 - \frac{\mathscr{F}_2}{\mathscr{F}_0^2}d_xd_y\mathscr{F}_1 - \frac{1}{\mathscr{F}_0^2}\big(d_x\mathscr{F}_0\wedge d_y\mathscr{F}_3\\
&\qquad + d_x\mathscr{F}_1\wedge d_y\mathscr{F}_2 + d_x\mathscr{F}_2\wedge d_y\mathscr{F}_1 + d_x\mathscr{F}_3\wedge d_y\mathscr{F}_0\big)\\
&\qquad + \frac{2\mathscr{F}_2}{\mathscr{F}_0^3}\big(d_x\mathscr{F}_0\wedge d_y\mathscr{F}_1 + d_x\mathscr{F}_1\wedge d_y\mathscr{F}_0\big)\Bigg](0,0)\Bigg\} + \mathscr{O}(p^{-2}).
\end{aligned} \tag{3.7}$$

From (1.6) and (2.8), we obtain

$$d_x\mathscr{F}_0(0,0) = d_y\mathscr{F}_0(0,0) = 0. \tag{3.8}$$

As J_r is a polynomial in Z, Z' with the same parity as r, we know from (1.6) and (2.8) that for $\alpha,\alpha'\in\mathbb{N}^{2n}$, there exists a polynomial $J_{r,\alpha,\alpha'}$ in Z, Z' with the same parity as $r-|\alpha|-|\alpha'|$ such that

$$\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}}\mathscr{F}_r(Z,Z') = \big(J_{r,\alpha,\alpha'}\mathscr{P}\big)(Z,Z'). \tag{3.9}$$

In particular, (3.9) yields

$$d_xd_y\mathscr{F}_1(0,0) = d_xd_y\mathscr{F}_3(0,0) = 0, \quad d_y\mathscr{F}_2(0,0) = d_x\mathscr{F}_2(0,0) = 0. \tag{3.10}$$

By (1.6) and (2.8), we get

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}(d_x d_y \mathscr{F}_0)(0,0) &= \frac{\sqrt{-1}}{2\pi}(d_x d_y \mathscr{P})(0,0)\\ &= \frac{\sqrt{-1}}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \omega(x_0).\end{aligned} \tag{3.11}$$

Substituting (2.12), (3.8), (3.10) and (3.11) into (3.7) yields

$$\begin{aligned}\frac{1}{p}\Phi_p^*(\omega_{FS})(x_0) = \omega(x_0) &+ \frac{\sqrt{-1}}{2\pi}\big(d_x d_y \mathscr{F}_2 - d_x\mathscr{F}_1 \wedge d_y \mathscr{F}_1\big)(0,0)\, p^{-1}\\ &- \boldsymbol{b}_1(x_0)\omega(x_0)\, p^{-1} + \mathscr{O}(p^{-2}).\end{aligned} \tag{3.12}$$

Recall that for a tensor ψ, $\nabla^X\psi$ is the covariant derivative of ψ induced by the Levi-Civita connection ∇^{TX}. We will denote by $\langle\,\cdot,\,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $TX\otimes_{\mathbb{R}}\mathbb{C}$ induced by g^{TX}.

The following observation [13, (8.3.54)] is very useful.

Lemma 3.1. *For $U\in T_{x_0}X$, $\nabla^X_U J$ is skew-adjoint and the tensor $\big\langle(\nabla^X_{\cdot} J)\cdot,\cdot\big\rangle$ is of type $\big(T^{*(1,0)}X\big)^{\otimes 3}\oplus\big(T^{*(0,1)}X\big)^{\otimes 3}$.*

Lemma 3.2. *We have*

$$(d_x\mathscr{F}_1)(0,0) = (d_y\mathscr{F}_1)(0,0) = 0. \tag{3.13}$$

Proof. By (1.2) and (2.6), we have (cf. [13, (8.3.51)])

$$\begin{aligned}\mathcal{O}_1 &= -\frac{2}{3}\left[\left\langle(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}, \frac{\partial}{\partial z_i}\right\rangle b_i^+ - b_i\left\langle(\nabla^X_{\mathcal{R}}\mathcal{J})\mathcal{R}, \frac{\partial}{\partial\overline{z}_i}\right\rangle\right],\\ \mathcal{J} &= -2\pi\sqrt{-1}J.\end{aligned} \tag{3.14}$$

From Theorem 1.1, (1.7), (3.14) and Lemma 3.1, we get (cf. [13, (8.3.67)])

(3.15)

$$\begin{aligned}&\Big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{P}\Big)(Z,Z')\\ &= -\frac{\sqrt{-1}}{3}\left\{\left(\frac{b_i b_j}{4\pi}\left\langle\left(\nabla^X_{\frac{\partial}{\partial\overline{z}_j}}J\right)\overline{z}', \frac{\partial}{\partial\overline{z}_i}\right\rangle + b_i\left\langle\left(\nabla^X_{\overline{z}'}J\right)\overline{z}', \frac{\partial}{\partial\overline{z}_i}\right\rangle\right)\mathscr{P}\right\}(Z,Z')\\ &= -\frac{\sqrt{-1}\pi}{3}\left[\left\langle(\nabla^X_{\overline{z}}J)\overline{z}', \overline{z}\right\rangle + \left\langle(\nabla^X_{\overline{z}'}J)\overline{z}', \overline{z}\right\rangle\right]\mathscr{P}(Z,Z').\end{aligned}$$

Note that if K is an operator on $\big(\mathbb{R}^{2n}, \|\cdot\|_{L^2}\big)$ with smooth kernel $K(Z,Z')$ with respect to dZ', then the kernel $K^*(Z,Z')$ of the adjoint K^* of K, with respect to dZ', is given by

$$K^*(Z,Z') = \overline{K(Z',Z)}. \tag{3.16}$$

As $\mathscr{L}$, $\mathcal{O}_1$ are formally self-adjoint with respect to $\|\cdot\|_{L^2}$, thus $\mathscr{P}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}$ is the adjoint of $\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{P}$. From Lemma 3.1, (3.15) and (3.16), we get

$$\begin{aligned}
&\Big(\mathscr{P}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}\Big)(Z,Z')\\
&= \frac{\sqrt{-1}\pi}{3}\Big[\Big\langle(\nabla^X_{z'}J)z,z'\Big\rangle + \Big\langle(\nabla^X_{z}J)z,z'\Big\rangle\Big]\mathscr{P}(Z,Z')\\
&= \frac{\sqrt{-1}\pi}{3}\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_j}}J\right)\frac{\partial}{\partial z_k},\frac{\partial}{\partial z_l}\right\rangle\big(z'_j z_k z'_l + z_j z_k z'_l\big)\mathscr{P}(Z,Z').
\end{aligned} \tag{3.17}$$

As the coefficients of $\mathscr{P}(Z,Z')$ in (3.15) and (3.17) are polynomials of degree 3, from (2.10), (3.15) and (3.17), we get (3.13). The proof of Lemma 3.2 is completed. □

Theorem 3.3. *The following identity holds,*

$$\frac{\sqrt{-1}}{2\pi}(d_x d_y \mathscr{F}_2)(0,0) = \boldsymbol{b}_1(x_0)\omega(x_0). \tag{3.18}$$

Lemma 3.2, Theorem 3.3 and (3.12) yield Theorem 0.1.

4. Proof of Theorem 3.3

This section is devoted to the proof of Theorem 3.3. We will compute the contribution of each term in (2.10) to $\mathscr{F}_2$. Set

$$\begin{aligned}
I_1 &= \mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{P}, & I_2 &= -\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_2\mathscr{P},\\
I_3 &= \mathscr{P}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}, & I_4 &= -\mathscr{P}\mathcal{O}_2\mathscr{L}^{-1}\mathscr{P}^{\perp},\\
I_5 &= \mathscr{P}^{\perp}\mathscr{L}^{-1}\mathcal{O}_1\mathscr{P}\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}, & I_6 &= -\mathscr{P}\mathcal{O}_1\mathscr{L}^{-2}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{P}.
\end{aligned} \tag{4.1}$$

For $j \in \{1,\dots,6\}$, let $I_j(Z,Z')$ be the smooth kernel of the operator I_j with respect to dZ'. By (2.10),

$$(d_x d_y \mathscr{F}_2)(0,0) = \sum_{j=1}^{6}(d_x d_y I_j)(0,0). \tag{4.2}$$

In the context of (3.16), by denoting $b_{jk} = \dfrac{\partial^2 K}{\partial Z_j \partial Z'_k}(Z, Z')\Big|_{Z=Z'=0}$ we have

$$
\begin{aligned}
(4.3) \qquad \big(d_Z d_{Z'} K^*\big)(0,0) &= \sum_{j,k} dZ_j \wedge dZ_k \, \frac{\partial^2 K^*}{\partial Z_j \partial Z'_k}(Z, Z')\big|_{Z=Z'=0} \\
&= \sum_{j<k} \big(\overline{b}_{kj} - \overline{b}_{jk}\big) dZ_j \wedge dZ_k \\
&= -\overline{\big(d_Z d_{Z'} K\big)(0,0)}.
\end{aligned}
$$

Since the operators $\mathscr{O}_r$ from (2.5) are formally self-adjoint with respect to $\|\cdot\|_{L^2}$, (4.1) implies that I_1 and I_2 are the adjoints of I_3 and I_4, respectively, as operators acting on $\big(\mathbb{R}^{2n}, \|\cdot\|_{L^2}\big)$. Hence by (4.3),

$$
(4.4) \quad (d_x d_y I_3)(0,0) = -\overline{(d_x d_y I_1)(0,0)}, \quad (d_x d_y I_4)(0,0) = -\overline{(d_x d_y I_2)(0,0)}.
$$

4.1. Evaluation of $(d_x d_y I_j)(0,0)$ for $j = 1, 3, 5, 6$

To simplify the notation, for polynomials Q_1, Q_2 in Z, Z', we will denote

$$
(4.5) \qquad (Q_1 \mathscr{P})(Z, Z') \sim (Q_2 \mathscr{P})(Z, Z'),
$$

if the constant coefficient and the coefficient of Z'_j for all j in $Q_1 - Q_2$ as a polynomial in Z are zero; we denote

$$
(4.6) \qquad (Q_1 \mathscr{P})(Z, Z') \approx (Q_2 \mathscr{P})(Z, Z'),
$$

if the constant coefficient and the coefficients of $Z_j, Z'_k, Z_j Z'_k$ for all j, k in $Q_1 - Q_2$ are zero.

Set

$$
(4.7) \qquad
\begin{aligned}
\mathscr{J}_{jir} &:= \left\langle \left(\nabla^X_{\frac{\partial}{\partial z_j}} J \right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_r} \right\rangle, \\
\mathscr{J}_{\overline{j}\,\overline{i}\overline{r}} &:= \left\langle \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}} J \right) \frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial \overline{z}_r} \right\rangle = \overline{\mathscr{J}_{jir}}.
\end{aligned}
$$

From Lemma 3.1, (1.7), (3.14), and (3.15), we get

$$
\begin{aligned}
&\Big(\mathcal{O}_1\mathscr{L}^{-1}\mathscr{P}^{\perp}\mathcal{O}_1\mathscr{P}\Big)(Z,Z') \\
&= \frac{4\pi}{9}\Bigg\{\Bigg(\Big\langle(\nabla^X_z J)z, \frac{\partial}{\partial z_i}\Big\rangle b_i^+ - b_i\Big\langle(\nabla^X_{\overline{z}} J)\overline{z}, \frac{\partial}{\partial \overline{z}_i}\Big\rangle\Bigg) \\
&\qquad \times\Bigg(\frac{b_jb_k}{4\pi}\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial\overline{z}_k}} J\Big)\overline{z}', \frac{\partial}{\partial \overline{z}_j}\Big\rangle + b_j\Big\langle(\nabla^X_{\overline{z}'} J)\overline{z}', \frac{\partial}{\partial \overline{z}_j}\Big\rangle\Bigg)\mathscr{P}\Bigg\}(Z,Z') \\
&\sim \frac{1}{9}\Bigg\{\Bigg[\Big\langle(\nabla^X_z J)z, \frac{\partial}{\partial z_i}\Big\rangle b_i^+ - b_i\Big\langle(\nabla^X_{\overline{z}} J)\overline{z}, \frac{\partial}{\partial \overline{z}_i}\Big\rangle\Bigg] \\
&\qquad \times b_jb_k\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial\overline{z}_k}} J\Big)\overline{z}', \frac{\partial}{\partial \overline{z}_j}\Big\rangle\mathscr{P}\Bigg\}(Z,Z') \\
&\sim \frac{1}{9}\Bigg\{\Big\langle(\nabla^X_z J)z, \frac{\partial}{\partial z_i}\Big\rangle\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial\overline{z}_k}} J\Big)\overline{z}', \frac{\partial}{\partial \overline{z}_j}\Big\rangle b_i^+b_jb_k\mathscr{P}\Bigg\}(Z,Z'),
\end{aligned}
\tag{4.8}
$$

where in the last relation $\sim$ of (4.8) we used

$$
b_i\Big\langle(\nabla^X_{\overline{z}} J)\overline{z}, \frac{\partial}{\partial \overline{z}_i}\Big\rangle\mathscr{P} = \big\langle(\nabla^X_{\overline{z}} J)\overline{z}, -2\pi\overline{z}'\big\rangle\mathscr{P}.
$$

By Theorem 1.1, (1.7) and (1.8), we get $b_i^+b_jb_k = b_jb_kb_i^+ + 4\pi\big(\delta_{ij}b_k + \delta_{ik}b_j\big)$ and

$$
\begin{aligned}
\mathscr{L}^{-1}\mathscr{P}^{\perp}\big(z_sz_tb_i^+b_jb_k\mathscr{P}\big) &= 4\pi\mathscr{L}^{-1}\mathscr{P}^{\perp}\big(z_sz_t(\delta_{ij}b_k + \delta_{ik}b_j)\mathscr{P}\big) \\
&= 4\pi\mathscr{L}^{-1}\mathscr{P}^{\perp}\big((\delta_{ij}b_k + \delta_{ik}b_j)z_sz_t\mathscr{P}\big) \\
&= (\delta_{ij}b_k + \delta_{ik}b_j)z_sz_t\mathscr{P}.
\end{aligned}
\tag{4.9}
$$

By (1.7), (1.8), (4.1), (4.8) and (4.9) we obtain

$$
\begin{aligned}
I_1(Z,Z') &\sim \frac{1}{9}\mathscr{J}_{sti}\mathscr{J}_{\overline{k}\,\overline{l}\overline{j}}\big((\delta_{ij}b_kz_sz_t + \delta_{ik}b_jz_sz_t)\overline{z}'_l\mathscr{P}\big)(Z,Z') \\
&= \frac{1}{9}\mathscr{J}_{sti}\mathscr{J}_{\overline{k}\,\overline{l}\overline{j}}\Big[-2\delta_{ij}\delta_{ks}z_t - 2\delta_{ij}\delta_{kt}z_s + 2\pi\delta_{ij}z_sz_t(\overline{z}_k - \overline{z}'_k) \\
&\qquad - 2\delta_{ik}\delta_{js}z_t - 2\delta_{ik}\delta_{jt}z_s + 2\pi\delta_{ik}z_sz_t(\overline{z}_j - \overline{z}'_j)\Big]\overline{z}'_l\mathscr{P}(Z,Z').
\end{aligned}
\tag{4.10}
$$

Recall that $\mathscr{J}_{sti}$ is anti-symmetric on t and i, thus the contribution of $-2\delta_{ij}\delta_{kt}z_s - 2\delta_{ik}\delta_{jt}z_s$ in (4.10) is zero. Thus (4.10) yields

$$I_1(Z,Z') \approx -\frac{2}{9}\mathscr{J}_{sri}\mathscr{J}_{\overline{k}\,\overline{q}\overline{j}}\big(\delta_{ik}\delta_{js} + \delta_{ij}\delta_{ks}\big)z_r\overline{z}'_q\mathscr{P}(Z,Z'). \tag{4.11}$$

By Lemma 3.1, (1.6), and (4.11), we get

$$(d_xd_yI_1)(0,0) = -\frac{2}{9}\mathscr{J}_{jir}\big(\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}} + \mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big)dz_r \wedge d\overline{z}_q. \tag{4.12}$$

From (4.4), (4.7) and (4.12), we get

$$(d_xd_yI_3)(0,0) = (d_xd_yI_1)(0,0). \tag{4.13}$$

By (1.6), (3.15), (3.17) and (4.1), we get

$$\begin{aligned} I_5(Z,Z') &\sim \frac{\pi^2}{9}\Bigg\{\bigg(\Big\langle(\nabla^X_{\overline{z}}J)\overline{z}'' + \big(\nabla^X_{\overline{z}''}J\big)\overline{z}'', \overline{z}\Big\rangle\mathscr{P}\bigg) \\ &\qquad\circ\bigg(\bigg\langle\Big(\nabla^X_{\frac{\partial}{\partial z_j}}J\Big)\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_l}\bigg\rangle z''_j z''_k z'_l\mathscr{P}\bigg)\Bigg\}(Z,Z') \\ &\approx \frac{\pi^2}{9}\bigg\langle\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_s}}J\Big)\frac{\partial}{\partial \overline{z}_t}, \overline{z}\bigg\rangle\bigg\langle\Big(\nabla^X_{\frac{\partial}{\partial z_j}}J\Big)\frac{\partial}{\partial z_k}, z'\bigg\rangle \\ &\qquad\times\Big\{\mathscr{P}\circ\big(\overline{z}''_s\overline{z}''_t z''_j z''_k\mathscr{P}\big)\Big\}(Z,Z') \\ &\approx \frac{\pi^2}{9}\bigg\langle\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_s}}J\Big)\frac{\partial}{\partial \overline{z}_t}, \overline{z}\bigg\rangle\bigg\langle\Big(\nabla^X_{\frac{\partial}{\partial z_j}}J\Big)\frac{\partial}{\partial z_k}, z'\bigg\rangle \\ &\qquad\times\mathscr{P}(Z,Z')\Big(\mathscr{P}\circ\big(\overline{z}_s\overline{z}_t z_j z_k\mathscr{P}\big)\Big)(0,0), \end{aligned} \tag{4.14}$$

where in the last equation we use $\mathscr{P}(0,0)=1$, since we need to compute the constant coefficient of $\mathscr{P}$ in $\mathscr{P}\circ\big(\overline{z}''_s\overline{z}''_t z''_j z''_k\mathscr{P}\big)$.

By (1.7) and (1.8), we get

$$\begin{aligned} (\overline{z}_s\overline{z}_t z_j z_k\mathscr{P})(Z,0) &= \frac{1}{4\pi^2}(z_jz_kb_sb_t\mathscr{P})(Z,0), \\ z_jz_kb_sb_t &= b_sb_tz_jz_k + 2\delta_{js}b_tz_k + 2\delta_{jt}b_sz_k \\ &\quad + 2\delta_{ks}b_tz_j + 2\delta_{kt}b_sz_j + 4\delta_{jt}\delta_{ks} + 4\delta_{js}\delta_{kt}. \end{aligned} \tag{4.15}$$

From Theorem 1.1 and (4.15), we get

$$\Big(\mathscr{P}\circ\big(\overline{z}_s\overline{z}_t z_j z_k\mathscr{P}\big)\Big)(0,0)=\frac{1}{4\pi^2}(4\delta_{jt}\delta_{ks}+4\delta_{js}\delta_{kt})\mathscr{P}(0,0)=\frac{1}{\pi^2}(\delta_{jt}\delta_{ks}+\delta_{js}\delta_{kt}).\tag{4.16}$$

From (4.7), (4.14) and (4.16), we obtain

$$(d_xd_yI_5)(0,0)=-\frac{1}{9}\mathscr{J}_{jir}\big(\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}+\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big)dz_r\wedge d\overline{z}_q.\tag{4.17}$$

By (3.15), (3.17) and (4.1), we get

$$\begin{aligned}I_6(Z,Z')&\approx-\frac{\pi^2}{9}\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_j}}J\right)z,\frac{\partial}{\partial z_l}\right\rangle\left\langle\left(\nabla^X_{\frac{\partial}{\partial\overline{z}_s}}J\right)\overline{z}',\frac{\partial}{\partial\overline{z}_t}\right\rangle\\&\quad\times\Big(\mathscr{P}\circ\big(z''_jz''_l\overline{z}''_s\overline{z}''_t\mathscr{P}\big)\Big)(Z,Z')\\&\approx-\frac{\pi^2}{9}\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_j}}J\right)z,\frac{\partial}{\partial z_k}\right\rangle\left\langle\left(\nabla^X_{\frac{\partial}{\partial\overline{z}_s}}J\right)\overline{z}',\frac{\partial}{\partial\overline{z}_t}\right\rangle\\&\quad\times\mathscr{P}(Z,Z')\Big(\mathscr{P}\circ\big(z_jz_k\overline{z}_s\overline{z}_t\mathscr{P}\big)\Big)(0,0).\end{aligned}\tag{4.18}$$

Thus by (4.7), (4.16) and (4.18), we get

$$\begin{aligned}(d_xd_yI_6)(0,0)&=-\frac{1}{9}\mathscr{J}_{jrk}\mathscr{J}_{\overline{s}\,\overline{q}\overline{t}}\big(\delta_{jt}\delta_{ks}+\delta_{js}\delta_{kt}\big)dz_r\wedge d\overline{z}_q\\&=-\frac{1}{9}\mathscr{J}_{jir}\big(\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}+\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big)dz_r\wedge d\overline{z}_q.\end{aligned}\tag{4.19}$$

4.2. Evaluation of $(d_x d_y I_2)(0,0)$: part I

Recall that by [14, Lemma 2.1] we have

(4.20)

$$
\begin{aligned}
\mathscr{O}_2\mathscr{P} = &\Bigg\{\frac{1}{3}b_ib_j\left\langle R^{TX}_{x_0}\left(\mathcal{R},\frac{\partial}{\partial\overline{z}_i}\right)\mathcal{R},\frac{\partial}{\partial\overline{z}_j}\right\rangle\\
&+\frac{1}{2}b_i\left[\sum_{|\alpha|=2}(\partial^\alpha R^L)\left(\mathcal{R},\frac{\partial}{\partial\overline{z}_i}\right)\frac{Z^\alpha}{\alpha!}\right]\\
&+\frac{4}{3}b_j\Bigg[\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial\overline{z}_i}\right)\mathcal{R},\frac{\partial}{\partial\overline{z}_j}\right\rangle\\
&-\left\langle R^{TX}\left(\mathcal{R},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial\overline{z}_i},\frac{\partial}{\partial\overline{z}_j}\right\rangle\Bigg]\\
&-2\pi\sqrt{-1}\left\langle(\nabla^X\nabla^X J)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i},\frac{\partial}{\partial\overline{z}_i}\right\rangle\\
&+4\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)\frac{\partial}{\partial\overline{z}_i},\frac{\partial}{\partial\overline{z}_j}\right\rangle\Bigg\}\mathscr{P}\\
&+\left[-\frac{1}{3}\mathscr{L}\left\langle R^{TX}\left(\mathcal{R},\frac{\partial}{\partial z_j}\right)\mathcal{R},\frac{\partial}{\partial\overline{z}_j}\right\rangle+\frac{4\pi^2}{9}\left|(\nabla^X_{\mathcal{R}}J)\mathcal{R}\right|^2+\Phi_{x_0}\right]\mathscr{P}.
\end{aligned}
$$

Set

(4.21)

$$
\begin{aligned}
I_{21}(Z,Z') &= \frac{1}{3}\left(\mathscr{L}^{-1}\mathscr{P}^\perp b_ib_j\left\langle R^{TX}_{x_0}\left(\mathcal{R},\frac{\partial}{\partial\overline{z}_i}\right)\mathcal{R},\frac{\partial}{\partial\overline{z}_j}\right\rangle\mathscr{P}\right)(Z,Z'),\\
I_{22}(Z,Z') &= \frac{1}{2}\left(\mathscr{L}^{-1}\mathscr{P}^\perp b_i\left[\sum_{|\alpha|=2}(\partial^\alpha R^L)\frac{Z^\alpha}{\alpha!}\left(\mathcal{R},\frac{\partial}{\partial\overline{z}_i}\right)\right]\mathscr{P}\right)(Z,Z'),\\
I_{23}(Z,Z') &= \frac{4}{3}\Bigg\{\mathscr{L}^{-1}b_j\Bigg\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial\overline{z}_i}\right)\mathcal{R}\\
&\qquad -R^{TX}\left(\mathcal{R},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial\overline{z}_i},\frac{\partial}{\partial\overline{z}_j}\Bigg\rangle\mathscr{P}\Bigg\}(Z,Z'),\\
I_{24}(Z,Z') &= -2\pi\sqrt{-1}\left(\mathscr{L}^{-1}\mathscr{P}^\perp\left\langle(\nabla^X\nabla^X J)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i},\frac{\partial}{\partial\overline{z}_i}\right\rangle\mathscr{P}\right)(Z,Z'),
\end{aligned}
$$

$$I_{25}(Z,Z') = -\frac{1}{3}\left(\mathscr{P}^{\perp}\mathscr{L}^{-1}\mathscr{L}\left\langle R^{TX}\left(\mathcal{R},\frac{\partial}{\partial z_i}\right)\mathcal{R},\frac{\partial}{\partial \overline{z}_i}\right\rangle\mathscr{P}\right)(Z,Z'),$$
$$I_{26}(Z,Z') = \frac{4\pi^2}{9}\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}\left|(\nabla^X_{\mathcal{R}}J)\mathcal{R}\right|^2\mathscr{P}\right)(Z,Z').$$

By Theorem 1.1, $\mathscr{P}^{\perp}\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_j}\right\rangle\mathscr{P} = \mathscr{P}^{\perp}\Phi_{x_0}\mathscr{P} = 0$. Thus (4.1), (4.20) and (4.21) yield

$$-I_2(Z,Z') = \sum_{j=1}^{6} I_{2j}(Z,Z'). \tag{4.22}$$

We evaluate first the contribution of I_{2j}, $j = 1,3,5,6$, in $(d_x d_y I_2)(0,0)$. We recall the following well-known symmetry properties of the curvature R^{TX}: for $U,V,W,Y \in TX$, we have

$$\begin{aligned}&\left\langle R^{TX}(U,V)W,Y\right\rangle = \left\langle R^{TX}(W,Y)U,V\right\rangle,\\ &R^{TX}(U,V)W + R^{TX}(V,W)U + R^{TX}(W,U)V = 0.\end{aligned} \tag{4.23}$$

Using (1.8) and (4.23), we have

$$\begin{aligned} & b_i b_j\left\langle R^{TX}\left(\mathcal{R},\frac{\partial}{\partial \overline{z}_i}\right)\mathcal{R},\frac{\partial}{\partial \overline{z}_j}\right\rangle\\ &= b_i b_j\left\langle R^{TX}\left(\frac{\partial}{\partial z_s},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_t},\frac{\partial}{\partial \overline{z}_j}\right\rangle z_s z_t\\ &\quad + 2b_i b_j\left\langle R^{TX}\left(\frac{\partial}{\partial z_s},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial \overline{z}_t},\frac{\partial}{\partial \overline{z}_j}\right\rangle z_s \overline{z}_t\\ &\quad + b_i b_j\left\langle R^{TX}\left(\frac{\partial}{\partial \overline{z}_s},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial \overline{z}_t},\frac{\partial}{\partial \overline{z}_j}\right\rangle \overline{z}_s \overline{z}_t.\end{aligned} \tag{4.24}$$

By (1.7) and (1.8), we get

$$\begin{aligned} z_s\overline{z}_t\mathscr{P}(Z,Z') &= z_s\left(\frac{b_t}{2\pi}+\overline{z}'_t\right)\mathscr{P}(Z,Z')\\ &= \left(\frac{b_t}{2\pi}z_s + \frac{\delta_{st}}{\pi} + z_s\overline{z}'_t\right)\mathscr{P}(Z,Z').\end{aligned} \tag{4.25}$$

By Theorem 1.1, (4.21), (4.24) and (4.25), we get

$$\begin{aligned}(4.26)\quad 3I_{21}(Z,Z') = &\frac{1}{8\pi}\left\langle R^{TX}\left(\frac{\partial}{\partial z_s},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_t},\frac{\partial}{\partial \overline{z}_j}\right\rangle\Big(b_ib_jz_sz_t\mathscr{P}\Big)(Z,Z')\\ &+\left\langle R^{TX}\left(\frac{\partial}{\partial z_s},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial \overline{z}_t},\frac{\partial}{\partial \overline{z}_j}\right\rangle\\ &\times\left[\left(\frac{b_ib_jb_t}{12\pi^2}z_s+\frac{b_ib_j}{4\pi}z_s\overline{z}'_t\right)\mathscr{P}\right](Z,Z')\\ &+\frac{1}{4\pi^2}\left\langle R^{TX}\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial \overline{z}_k},\frac{\partial}{\partial \overline{z}_j}\right\rangle\Big(b_ib_j\mathscr{P}\Big)(Z,Z')\\ &+I_{27}(Z,Z'),\end{aligned}$$

where

$$\begin{aligned}(4.27)\qquad I_{27}(Z,Z') = &\left\langle R^{TX}\left(\frac{\partial}{\partial \overline{z}_s},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial \overline{z}_t},\frac{\partial}{\partial \overline{z}_j}\right\rangle\\ &\times\Big(\mathscr{L}^{-1}\mathscr{P}^{\perp}b_ib_j\overline{z}_s\overline{z}_t\mathscr{P}\Big)(Z,Z').\end{aligned}$$

Note that by Theorem 1.1, (1.7) and (1.8),

$$\begin{aligned}(4.28)\qquad &4\pi^2\mathscr{L}^{-1}\mathscr{P}^{\perp}b_ib_j\overline{z}_s\overline{z}_t\mathscr{P}\\ &=\mathscr{L}^{-1}\mathscr{P}^{\perp}b_ib_j\big(b_s+2\pi\overline{z}'_s\big)\big(b_t+2\pi\overline{z}'_t\big)\mathscr{P}\\ &=\left[\frac{1}{16\pi}b_ib_jb_sb_t+\frac{1}{6}b_ib_j\big(b_s\overline{z}'_t+b_t\overline{z}'_s\big)+\frac{\pi}{2}b_ib_j\overline{z}'_s\overline{z}'_t\right]\mathscr{P}.\end{aligned}$$

Thus, from (1.7), (4.6) and (4.28), we get

$$(4.29)\qquad I_{27}(Z,Z')\approx 0.$$

From (1.7) and (1.8), we get

$$\begin{aligned}(4.30)\qquad &\Big(b_ib_j\mathscr{P}\Big)(Z,Z') = 4\pi^2\big(\overline{z}_i-\overline{z}'_i\big)\big(\overline{z}_j-\overline{z}'_j\big)\mathscr{P}(Z,Z'),\\ &\Big(b_ib_jz_s\overline{z}'_t\mathscr{P}\Big)(Z,Z') = \Big[-4\pi\delta_{js}\overline{z}'_t\big(\overline{z}_i-\overline{z}'_i\big)-4\pi\delta_{is}\overline{z}'_t(\overline{z}_j-\overline{z}'_j)\\ &\qquad\qquad+4\pi^2z_s\overline{z}'_t\big(\overline{z}_i-\overline{z}'_i\big)\big(\overline{z}_j-\overline{z}'_j\big)\Big]\mathscr{P}(Z,Z'),\\ &\Big(b_ib_jz_sz_t\mathscr{P}\Big)(Z,Z') = \Big[4\delta_{it}\delta_{js}-4\pi\delta_{js}z_t\big(\overline{z}_i-\overline{z}'_i\big)+4\delta_{jt}\delta_{is}-4\pi\delta_{jt}z_s\big(\overline{z}_i-\overline{z}'_i\big)\\ &\qquad\qquad-4\pi\delta_{is}z_t\big(\overline{z}_j-\overline{z}'_j\big)-4\pi\delta_{it}z_s\big(\overline{z}_j-\overline{z}'_j\big)\\ &\qquad\qquad+4\pi^2z_sz_t\big(\overline{z}_j-\overline{z}'_j\big)\big(\overline{z}_i-\overline{z}'_i\big)\Big]\mathscr{P}(Z,Z'),\end{aligned}$$

and

$$\begin{aligned}(4.31)\qquad &\Big(b_ib_jb_tz_s\mathscr{P}\Big)(Z,Z')\\ &=\Big[\big(-2\delta_{ts}b_ib_j-2\delta_{js}b_ib_t-2\delta_{is}b_jb_t+z_sb_ib_jb_t\big)\mathscr{P}\Big](Z,Z').\end{aligned}$$

By (1.7), (4.30) and (4.31), we get

$$(4.32)\qquad \big(d_xd_y(b_ib_j\mathscr{P})\big)(0,0)=0,\quad \big(d_xd_y(b_ib_jb_tz_s\mathscr{P})\big)(0,0)=0.$$

Substituting (3.11), (4.23), (4.29)–(4.32) into (4.26), we obtain

$$\begin{aligned}(4.33)\qquad (d_xd_yI_{21})(0,0)=-\frac{\sqrt{-1}}{3}\Big\langle & 2R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i}\\ &-R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_j}\Big\rangle\omega(x_0)\\ +\frac{1}{3}\Big\langle & 2R^{TX}\left(\frac{\partial}{\partial z_r},\frac{\partial}{\partial \overline{z}_j}\right)\frac{\partial}{\partial z_j}\\ &+R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r}\right)\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\Big\rangle dz_r\wedge d\overline{z}_q\\ +\frac{1}{3}\Big\langle & R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_r}\right)\frac{\partial}{\partial \overline{z}_j}\\ &+R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_j}\right)\frac{\partial}{\partial \overline{z}_r},\frac{\partial}{\partial \overline{z}_q}\Big\rangle d\overline{z}_r\wedge d\overline{z}_q.\end{aligned}$$

By (4.21),

$$\begin{aligned}(4.34)\qquad \frac{3}{4}I_{23}=\Big\langle & R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_s}\\ &-R^{TX}\left(\frac{\partial}{\partial z_s},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_j}\Big\rangle\mathscr{L}^{-1}b_jz_s\mathscr{P}\\ +\Big\langle & R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial \overline{z}_s}\\ &-R^{TX}\left(\frac{\partial}{\partial \overline{z}_s},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_j}\Big\rangle\mathscr{L}^{-1}b_j\overline{z}_s\mathscr{P}.\end{aligned}$$

By Theorem 1.1, (1.7) and (1.8),

$$\begin{aligned}(4.35)\quad \Big(\mathscr{L}^{-1}\mathscr{P}^{\perp}b_j z_s \mathscr{P}\Big)(Z,Z') &= \frac{1}{4\pi}\Big(b_j z_s \mathscr{P}\Big)(Z,Z')\\ &= \frac{1}{4\pi}\Big(-2\delta_{js} + 2\pi z_s\big(\overline{z}_j - \overline{z}'_j\big)\Big)\mathscr{P}(Z,Z').\end{aligned}$$

Note that by (1.7), $\overline{z}_s\mathscr{P} = \left(\frac{b_s}{2\pi} + \overline{z}'_s\right)\mathscr{P}$. Thus from Theorem 1.1, we get

$$\begin{aligned}(4.36)\qquad & \Big(\mathscr{L}^{-1}b_j\overline{z}_s\mathscr{P}\Big)(Z,Z')\\ &= \left[\left(\frac{b_j b_s}{16\pi^2} + \frac{b_j}{4\pi}\overline{z}'_s\right)\mathscr{P}\right](Z,Z')\\ &= \left[\frac{1}{4}\big(\overline{z}_j - \overline{z}'_j\big)\big(\overline{z}_s - \overline{z}'_s\big) + \frac{1}{2}\overline{z}'_s\big(\overline{z}_j - \overline{z}'_j\big)\right]\mathscr{P}(Z,Z').\end{aligned}$$

As in (4.32), we get

$$(4.37)\qquad \big(d_x d_y(\mathscr{L}^{-1}b_j\overline{z}_s\mathscr{P})\big)(0,0) = \frac{1}{2}d\overline{z}_j\wedge d\overline{z}_s.$$

From (3.11), (4.23), (4.34), (4.35) and (4.37), we get

$$\begin{aligned}(4.38)\qquad (d_x d_y I_{23})(0,0) = \frac{4\sqrt{-1}}{3}\Big\langle & R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i}\\ & - 2R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_j}\Big\rangle\omega(x_0)\\ - \frac{2}{3}\Big\langle & R^{TX}\left(\frac{\partial}{\partial z_r},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i}\\ & + 2R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_r}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_q}\Big\rangle dz_r\wedge d\overline{z}_q\\ - \frac{2}{3}\Big\langle & R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial \overline{z}_r}\\ & + R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_r}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_q}\Big\rangle d\overline{z}_r\wedge d\overline{z}_q.\end{aligned}$$

Clearly, by Theorem 1.1 and (4.21),

$$\begin{aligned}(4.39)\quad -3I_{25}(Z,Z') &= \left(\mathscr{P}^{\perp}\left\langle R^{TX}\left(\mathcal{R},\frac{\partial}{\partial z_i}\right)\mathcal{R},\frac{\partial}{\partial \overline{z}_i}\right\rangle\mathscr{P}\right)(Z,Z')\\ &= \left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \overline{z}_k}+R^{TX}\left(\frac{\partial}{\partial \overline{z}_k},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\right\rangle\\ &\quad\times\Big(\mathscr{P}^{\perp}\circ\big(z_j\overline{z}_k\mathscr{P}\big)\Big)(Z,Z')\\ &\quad+\left\langle R^{TX}\left(\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \overline{z}_k},\frac{\partial}{\partial \overline{z}_i}\right\rangle\\ &\quad\times\Big(\mathscr{P}^{\perp}\circ\big(\overline{z}_j\overline{z}_k\mathscr{P}\big)\Big)(Z,Z').\end{aligned}$$

From Theorem 1.1, (1.7) and (4.25), we get

$$\begin{aligned}(4.40)\qquad \Big(\mathscr{P}^{\perp}\circ\big(z_j\overline{z}_k\mathscr{P}\big)\Big)(Z,Z') &= \frac{1}{2\pi}\big(b_k z_j\mathscr{P}\big)(Z,Z')\\ &= \left(-\frac{1}{\pi}\delta_{jk}+z_j(\overline{z}_k-\overline{z}'_k)\right)\mathscr{P}(Z,Z'),\end{aligned}$$

and

$$\begin{aligned}(4.41)\qquad &\Big(\mathscr{P}^{\perp}\circ\big(\overline{z}_j\overline{z}_k\mathscr{P}\big)\Big)(Z,Z')\\ &= \left\{\mathscr{P}^{\perp}\left[\frac{1}{4\pi^2}b_jb_k+\frac{1}{2\pi}(b_j\overline{z}'_k+b_k\overline{z}'_j)\right]\mathscr{P}\right\}(Z,Z')\\ &= \Big((\overline{z}_j-\overline{z}'_j)(\overline{z}_k-\overline{z}'_k)+\overline{z}'_k(\overline{z}_j-\overline{z}'_j)+\overline{z}'_j(\overline{z}_k-\overline{z}'_k)\Big)\mathscr{P}(Z,Z')\\ &= \big(\overline{z}_j\overline{z}_k-\overline{z}'_j\overline{z}'_k\big)\mathscr{P}(Z,Z').\end{aligned}$$

As in [13, (8.3.56), (8.3.63)], we have

$$(4.42)\qquad \begin{aligned}&\left|\nabla^X J\right|^2=\sum_{i,j}\left|(\nabla^X_{e_i}J)e_j\right|^2=8\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_i}}J\right)\frac{\partial}{\partial z_j},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\right)\frac{\partial}{\partial \overline{z}_j}\right\rangle,\\ &\left\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_j}\right\rangle=\frac{1}{32}\left|\nabla^X J\right|^2.\end{aligned}$$

By (3.11), (4.23), (4.39), (4.40), (4.41) and (4.42), we get

$$
\begin{aligned}
(4.43)\quad (d_x d_y I_{25})(0,0) = &-\frac{1}{3}\Big\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial \overline{z}_k} \\
&+ R^{TX}\left(\frac{\partial}{\partial \overline{z}_k},\frac{\partial}{\partial z_i}\right)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\Big\rangle \\
&\times\left(2\sqrt{-1}\delta_{jk}\,\omega(x_0) - dz_j\wedge d\overline{z}_k\right) \\
= &\Big[-\frac{1}{96}\left|\nabla^X J\right|^2 \\
&+\frac{1}{3}\Big\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\Big\rangle\Big]2\sqrt{-1}\omega(x_0) \\
&+\frac{1}{3}\Big\langle R^{TX}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_r}\right)\frac{\partial}{\partial \overline{z}_i} \\
&- R^{TX}\left(\frac{\partial}{\partial z_r},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_q}\Big\rangle dz_r\wedge d\overline{z}_q.
\end{aligned}
$$

Then by Lemma 3.1 and (4.21), we get

$$
\begin{aligned}
(4.44)\qquad 9I_{26}(Z,Z') = 8\pi^2&\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_i}}J\right)\frac{\partial}{\partial z_j},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_s}}J\right)\frac{\partial}{\partial \overline{z}_t}\right\rangle \\
&\times\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_iz_j\overline{z}_s\overline{z}_t\mathscr{P}\right)(Z,Z').
\end{aligned}
$$

By Theorem 1.1, (1.7), (1.8) and (4.15), we get

$$
\begin{aligned}
(4.45)\qquad &\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_iz_jb_s\mathscr{P}\right)(Z,Z') \\
&=\frac{b_s}{4\pi}z_iz_j\mathscr{P}(Z,Z') \\
&=\frac{1}{4\pi}\big(-2\delta_{is}z_j-2\delta_{js}z_i+z_iz_jb_s\big)\mathscr{P}(Z,Z'), \\
(4.46)\qquad &\left(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_iz_jb_sb_t\mathscr{P}\right)(Z,Z') \\
&=\frac{1}{2\pi}\left(\frac{b_sb_t}{4}z_iz_j+\delta_{it}b_sz_j+\delta_{jt}b_sz_i+\delta_{is}b_tz_j+\delta_{js}b_tz_i\right)\mathscr{P}(Z,Z') \\
&=\frac{1}{2\pi}\bigg(-3\delta_{js}\delta_{it}-3\delta_{jt}\delta_{is}+\frac{1}{2}\delta_{it}z_jb_s+\frac{1}{2}\delta_{jt}z_ib_s \\
&\quad+\frac{1}{2}\delta_{is}z_jb_t+\frac{1}{2}\delta_{js}z_ib_t+\frac{1}{4}z_iz_jb_sb_t\bigg)\mathscr{P}(Z,Z').
\end{aligned}
$$

By (1.7), (4.45) and (4.46), we get

$$
\begin{aligned}
&\Big(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_iz_j\overline{z}_s\overline{z}_t\mathscr{P}\Big)(Z,Z') \\
&= \left(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_iz_j\left(\frac{b_s}{2\pi}+\overline{z}'_s\right)\big(\frac{b_t}{2\pi}+\overline{z}'_t\big)\mathscr{P}\right)(Z,Z') \\
&= \frac{1}{4\pi^2}\Big(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_iz_jb_sb_t\mathscr{P}\Big)(Z,Z') \\
&\quad + \frac{1}{2\pi}\Big(\mathscr{L}^{-1}\mathscr{P}^{\perp}\big(z_iz_j\overline{z}'_tb_s+z_iz_j\overline{z}'_sb_t\big)\mathscr{P}\Big)(Z,Z') \\
&= \frac{1}{4\pi^2}\Big\{-\frac{3}{2\pi}\delta_{it}\delta_{js}-\frac{3}{2\pi}\delta_{jt}\delta_{is}+\frac{1}{2}\delta_{it}z_j\big(\overline{z}_s-\overline{z}'_s\big) \\
&\qquad +\frac{1}{2}\delta_{jt}z_i\big(\overline{z}_s-\overline{z}'_s\big)+\frac{1}{2}\delta_{is}z_j\big(\overline{z}_t-\overline{z}'_t\big) \\
&\qquad +\frac{1}{2}\delta_{js}z_i\big(\overline{z}_t-\overline{z}'_t\big)+\frac{\pi}{2}z_iz_j\big(\overline{z}_t-\overline{z}'_t\big)\big(\overline{z}_s-\overline{z}'_s\big)\Big\}\mathscr{P}(Z,Z') \\
&\quad +\frac{1}{8\pi^2}\Big\{\big[-2\delta_{is}z_j-2\delta_{js}z_i+2\pi z_iz_j(\overline{z}_s-\overline{z}'_s)\big]\overline{z}'_t \\
&\qquad +\big[-2\delta_{it}z_j-2\delta_{jt}z_i+2\pi z_iz_j(\overline{z}_t-\overline{z}'_t)\big]\overline{z}'_s\Big\}\mathscr{P}(Z,Z').
\end{aligned}
\tag{4.47}
$$

By (3.11), (4.44) and (4.47), we get

$$
\begin{aligned}
&9(d_xd_yI_{26})(0,0) \\
&= -\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_i}}J\right)\frac{\partial}{\partial z_j},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_s}}J\right)\frac{\partial}{\partial \overline{z}_t}\right\rangle\Big[3(\delta_{it}\delta_{js}+\delta_{jt}\delta_{is})(-2\sqrt{-1})\omega(x_0) \\
&\quad +\delta_{it}dz_j\wedge d\overline{z}_s+\delta_{jt}dz_i\wedge d\overline{z}_s+\delta_{is}dz_j\wedge d\overline{z}_t+\delta_{js}dz_i\wedge d\overline{z}_t \\
&\quad +2\delta_{is}dz_j\wedge d\overline{z}_t+2\delta_{js}dz_i\wedge d\overline{z}_t+2\delta_{it}dz_j\wedge d\overline{z}_s+2\delta_{jt}dz_i\wedge d\overline{z}_s\Big].
\end{aligned}
\tag{4.48}
$$

By (4.48),

$$
\begin{aligned}
&(d_xd_yI_{26})(0,0) \\
&= \frac{2}{3}\sqrt{-1}\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_i}}J\right)\frac{\partial}{\partial z_j},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\right)\frac{\partial}{\partial \overline{z}_j}+\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}}J\right)\frac{\partial}{\partial \overline{z}_i}\right\rangle\omega(x_0) \\
&\quad -\frac{1}{3}\Big[\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_i}}J\right)\frac{\partial}{\partial z_r},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\right)\frac{\partial}{\partial \overline{z}_q}+\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}}J\right)\frac{\partial}{\partial \overline{z}_i}\right\rangle \\
&\quad +\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_r}}J\right)\frac{\partial}{\partial z_i},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\right)\frac{\partial}{\partial \overline{z}_q}+\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}}J\right)\frac{\partial}{\partial \overline{z}_i}\right\rangle\Big]dz_r\wedge d\overline{z}_q.
\end{aligned}
\tag{4.49}
$$

Note that for $U,V,W\in TX$, $\langle JU,V\rangle=\omega(U,V)$, thus (cf. [13, (8.3.48)]),

$$\big\langle(\nabla^X_U J)V,W\big\rangle+\big\langle(\nabla^X_V J)W,U\big\rangle+\big\langle(\nabla^X_W J)U,V\big\rangle=d\omega(U,V,W)=0. \tag{4.50}$$

From Lemma 3.1, (4.50) and $\big|\frac{\partial}{\partial z_j}\big|^2=\frac{1}{2}$, we have

$$\begin{aligned}&\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_i}}J\Big)\frac{\partial}{\partial z_r},\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}}J\Big)\frac{\partial}{\partial \overline{z}_i}\Big\rangle\\&=2\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_i}}J\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r}\Big\rangle\\&\quad\times\Big[\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\Big)\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\Big\rangle-\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}}J\Big)\frac{\partial}{\partial \overline{z}_i},\frac{\partial}{\partial \overline{z}_q}\Big\rangle\Big].\end{aligned} \tag{4.51}$$

When we sum (4.51) over $r=q$, we get by (4.42) (cf. [13, (8.3.58)]),

$$\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_i}}J\Big)\frac{\partial}{\partial z_q},\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}}J\Big)\frac{\partial}{\partial \overline{z}_i}\Big\rangle=\frac{1}{16}\big|\nabla^X J\big|^2. \tag{4.52}$$

By (4.7) and (4.51), we get

$$\begin{aligned}&\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_i}}J\Big)\frac{\partial}{\partial z_r},\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\Big)\frac{\partial}{\partial \overline{z}_q}+\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}}J\Big)\frac{\partial}{\partial \overline{z}_i}\Big\rangle\\&=2\mathscr{J}_{ijr}\big(2\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}-\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big).\end{aligned} \tag{4.53}$$

By Lemma 3.1, (4.50) and $\big|\frac{\partial}{\partial z_j}\big|^2=\frac{1}{2}$, we obtain

$$\begin{aligned}&\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_r}}J\Big)\frac{\partial}{\partial z_i},\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}}J\Big)\frac{\partial}{\partial \overline{z}_i}\Big\rangle\\&=2\Big[\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_i}}J\Big)\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r}\Big\rangle+\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_j}}J\Big)\frac{\partial}{\partial z_r},\frac{\partial}{\partial z_i}\Big\rangle\Big]\\&\quad\times\Big[\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\Big)\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\Big\rangle+\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}}J\Big)\frac{\partial}{\partial \overline{z}_q},\frac{\partial}{\partial \overline{z}_i}\Big\rangle\Big]\\&=4\mathscr{J}_{ijr}\big(\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}-\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big).\end{aligned} \tag{4.54}$$

By taking the conjugation of (4.51), we get

$$\begin{aligned}\Big\langle\Big(\nabla^X_{\frac{\partial}{\partial z_r}}J\Big)\frac{\partial}{\partial z_i},\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\Big)\frac{\partial}{\partial \overline{z}_q}\Big\rangle&=2\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}\big(\mathscr{J}_{ijr}-\mathscr{J}_{jir}\big)\\&=2\mathscr{J}_{ijr}\big(\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}-\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big).\end{aligned} \tag{4.55}$$

Substituting (4.42), (4.52), (4.53), (4.54) and (4.55) into (4.49) yields

$$(d_x d_y I_{26})(0,0) = \frac{\sqrt{-1}}{8}\left|\nabla^X J\right|^2 \omega(x_0) - \frac{2}{3}\mathscr{J}_{ijr}\big(5\mathscr{J}_{\bar{i}\,\bar{j}\bar{q}} - 4\mathscr{J}_{\bar{j}\,\bar{i}\bar{q}}\big)dz_r \wedge d\overline{z}_q. \tag{4.56}$$

4.3. Evaluation of $(d_x d_y I_2)(0,0)$: part II

We evaluate now the contribution of I_{22}, I_{24} in $(d_x d_y I_2)(0,0)$. The definitions of $\nabla^X\nabla^X J$ and R^{TX} imply that for $U,V,W,Y \in TX$ (cf. [13, (8.3.59)]),

$$\begin{aligned}
&\big(\nabla^X\nabla^X J\big)_{(U,V)} - \big(\nabla^X\nabla^X J\big)_{(V,U)} = \big[R^{TX}(U,V), J\big],\\
&\Big\langle\big(\nabla^X\nabla^X J\big)_{(Y,U)}V, W\Big\rangle + \Big\langle\big(\nabla^X\nabla^X J\big)_{(Y,V)}W, U\Big\rangle \\
&\qquad + \Big\langle\big(\nabla^X\nabla^X J\big)_{(Y,W)}U, V\Big\rangle = 0.
\end{aligned} \tag{4.57}$$

Recall that [13, (8.3.71)],

$$\begin{aligned}
&\sum_{|\alpha|=2}\big(\partial^\alpha R^L\big)\left(\mathcal{R}, \frac{\partial}{\partial \overline{z}_i}\right)\frac{Z^\alpha}{\alpha!}\\
&= -\sqrt{-1}\pi\left\langle\big(\nabla^X\nabla^X J\big)_{(\mathcal{R},\mathcal{R})}\mathcal{R}, \frac{\partial}{\partial\overline{z}_i}\right\rangle\\
&\quad - \frac{2\pi}{3}\left\langle R^{TX}(z,\overline{z})\mathcal{R}, \frac{\partial}{\partial\overline{z}_i}\right\rangle.
\end{aligned} \tag{4.58}$$

By (4.50) and (4.57), we get (cf. [13, (8.3.61)]): for $u_1, u_2, u_3 \in T^{(1,0)}X$, $\overline{v}_1, \overline{v}_2 \in T^{(0,1)}X$,

$$\begin{aligned}
&(\nabla^X\nabla^X J)_{(u_1,u_2)}u_3,\ (\nabla^X\nabla^X J)_{(\overline{v}_1,\overline{v}_2)}u_3 \in T^{(0,1)}X,\\
&(\nabla^X\nabla^X J)_{(u_1,\overline{v}_2)}u_3 \in T^{(1,0)}X,\\
&2\sqrt{-1}\left\langle(\nabla^X\nabla^X J)_{(u_1,\overline{v}_1)}u_2, \overline{v}_2\right\rangle = \left\langle(\nabla^X_{u_1}J)u_2, (\nabla^X_{\overline{v}_1}J)\overline{v}_2\right\rangle.
\end{aligned} \tag{4.59}$$

In particular, we have

(4.60)
$$\begin{aligned}
&\left\langle \big(\nabla^X\nabla^X J\big)_{(\overline{z},z)}\overline{z}, \frac{\partial}{\partial \overline{z}_i}\right\rangle = 0,\\
&\left\langle \big(\nabla^X\nabla^X J\big)_{(z,z)}u_1, \frac{\partial}{\partial \overline{z}_i}\right\rangle = \left\langle \big(\nabla^X\nabla^X J\big)_{(\overline{z},\overline{z})}u_1, \frac{\partial}{\partial \overline{z}_i}\right\rangle = 0,\\
&\left\langle \big(\nabla^X\nabla^X J\big)_{(\overline{z},z)}u_1, \frac{\partial}{\partial \overline{z}_i}\right\rangle = \left\langle \big(\nabla^X\nabla^X J\big)_{(z,\overline{z})}u_1 - [R^{TX}(z,\overline{z}), J]u_1, \frac{\partial}{\partial \overline{z}_i}\right\rangle\\
&\qquad = \left\langle \big(\nabla^X\nabla^X J\big)_{(z,\overline{z})}u_1, \frac{\partial}{\partial \overline{z}_i}\right\rangle.
\end{aligned}$$

By (4.57) and (4.60), we get

$$\begin{aligned}
\left\langle \big(\nabla^X\nabla^X J\big)_{(z,\overline{z})}\overline{z}, \frac{\partial}{\partial \overline{z}_i}\right\rangle &= \left\langle \big[R^{TX}(z,\overline{z}), J\big]\overline{z}, \frac{\partial}{\partial \overline{z}_i}\right\rangle\\
&= -2\sqrt{-1}\left\langle R^{TX}(z,\overline{z})\overline{z}, \frac{\partial}{\partial \overline{z}_i}\right\rangle.
\end{aligned} \tag{4.61}$$

By (4.57) and (4.59) we get (cf. [13, (8.3.62)]),

$$\begin{aligned}
&\left\langle \big(\nabla^X\nabla^X J\big)_{(u_1,u_2)}\overline{z}, \frac{\partial}{\partial \overline{z}_i}\right\rangle\\
&= \frac{1}{2\sqrt{-1}}\left\langle \big(\nabla^X_{u_1}J\big)u_2, \big(\nabla^X_{\overline{z}}J\big)\frac{\partial}{\partial \overline{z}_i} - \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\right)\overline{z}\right\rangle.
\end{aligned} \tag{4.62}$$

By (4.59), (4.60), (4.61) and (4.62), we get

$$\begin{aligned}
&-\pi\sqrt{-1}\left\langle \big(\nabla^X\nabla^X J\big)_{(\mathcal{R},\mathcal{R})}\mathcal{R}, \frac{\partial}{\partial \overline{z}_i}\right\rangle\\
&= -\frac{\pi}{2}\left\langle \big(\nabla^X_{z}J\big)z, 3\big(\nabla^X_{\overline{z}}J\big)\frac{\partial}{\partial \overline{z}_i} - \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\right)\overline{z}\right\rangle\\
&\quad - 2\pi\left\langle R^{TX}(z,\overline{z})\overline{z}, \frac{\partial}{\partial \overline{z}_i}\right\rangle - \pi\sqrt{-1}\left\langle \big(\nabla^X\nabla^X J\big)_{(\overline{z},\overline{z})}\overline{z}, \frac{\partial}{\partial \overline{z}_i}\right\rangle.
\end{aligned} \tag{4.63}$$

By (4.21), (4.58) and (4.63), we get

$$
\begin{aligned}
(4.64)\quad & \\
I_{22}(Z,Z') = & -\bigg[\frac{\pi}{4}\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_j}}J\right)\frac{\partial}{\partial z_k}, 3\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_s}}J\right)\frac{\partial}{\partial \overline{z}_i} - \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}}J\right)\frac{\partial}{\partial \overline{z}_s}\right\rangle \\
& + \frac{\pi}{3}\left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_s}\right)\frac{\partial}{\partial z_k},\frac{\partial}{\partial \overline{z}_i}\right\rangle\bigg]\Big(\mathscr{L}^{-1}b_i z_j z_k \overline{z}_s \mathscr{P}\Big)(Z,Z') \\
& - \frac{4\pi}{3}\left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_s}\right)\frac{\partial}{\partial \overline{z}_t},\frac{\partial}{\partial \overline{z}_i}\right\rangle\Big(\mathscr{L}^{-1}b_i z_j \overline{z}_s \overline{z}_t \mathscr{P}\Big)(Z,Z') \\
& - \frac{\pi}{2}\sqrt{-1}\bigg\{\mathscr{L}^{-1}b_i\left\langle\left(\nabla^X\nabla^X J\right)_{(\overline{z},\overline{z})}\overline{z},\frac{\partial}{\partial \overline{z}_i}\right\rangle\mathscr{P}\bigg\}(Z,Z').
\end{aligned}
$$

By (1.7) and (1.8), we get

$$
\begin{aligned}
(4.65)\qquad b_i z_j z_k \overline{z}_s \mathscr{P} &= b_i z_j z_k\left(\frac{b_s}{2\pi} + \overline{z}'_s\right)\mathscr{P} \\
&= \left(\frac{b_i b_s}{2\pi}z_j z_k + \frac{\delta_{js}}{\pi}b_i z_k + \frac{\delta_{ks}}{\pi}b_i z_j + b_i z_j z_k \overline{z}'_s\right)\mathscr{P}.
\end{aligned}
$$

Thus, by Theorem 1.1, (1.7), (1.8) and (4.65), as in (4.47), we get

$$
\begin{aligned}
(4.66)\qquad & \Big(\mathscr{L}^{-1}b_i z_j z_k \overline{z}_s \mathscr{P}\Big)(Z,Z') \\
&= \left(\frac{b_i b_s}{16\pi^2}z_j z_k + \frac{\delta_{js}}{4\pi^2}b_i z_k + \frac{\delta_{ks}}{4\pi^2}b_i z_j + \frac{b_i}{4\pi}z_j z_k \overline{z}'_s\right)\mathscr{P}(Z,Z') \\
&= \frac{1}{4\pi^2}\Big(- \delta_{js}\delta_{ik} + \pi\delta_{js}z_k\big(\overline{z}_i - \overline{z}'_i\big) - \delta_{ks}\delta_{ij} + \pi\delta_{ks}z_j\big(\overline{z}_i - \overline{z}'_i\big) \\
&\quad - \pi\delta_{ij}z_k\big(\overline{z}_s - \overline{z}'_s\big) - \pi\delta_{ik}z_j\big(\overline{z}_s - \overline{z}'_s\big) \\
&\quad + \pi^2 z_j z_k\big(\overline{z}_i - \overline{z}'_i\big)\big(\overline{z}_s - \overline{z}'_s\big)\Big)\mathscr{P}(Z,Z') \\
&\quad + \frac{1}{4\pi}\Big(- 2\delta_{ij}z_k - 2\delta_{ik}z_j + 2\pi z_j z_k\big(\overline{z}_i - \overline{z}'_i\big)\Big)\overline{z}'_s\mathscr{P}(Z,Z').
\end{aligned}
$$

By (4.6) and (4.66), note that the last line of (4.66) has also the term $z\overline{z}'$, we get

$$
\begin{aligned}
(4.67)\quad \Big(\mathscr{L}^{-1}b_i z_j z_k \overline{z}_s \mathscr{P}\Big)(Z,Z') \approx & -\bigg[\frac{1}{4\pi^2}\big(\delta_{js}\delta_{ik} + \delta_{ks}\delta_{ij}\big) \\
& + \frac{1}{4\pi}\big(\delta_{js}z_k\overline{z}'_i + \delta_{ks}z_j\overline{z}'_i + \delta_{ij}z_k\overline{z}'_s + \delta_{ik}z_j\overline{z}'_s\big)\bigg]\mathscr{P}(Z,Z').
\end{aligned}
$$

Again by (1.7) and (1.8),

$$b_i z_j \overline{z}_s \overline{z}_t \mathscr{P} = b_i z_j \left(\frac{b_s}{2\pi} + \overline{z}'_s\right)\left(\frac{b_t}{2\pi} + \overline{z}'_t\right)\mathscr{P} \tag{4.68}$$
$$= \left[\frac{b_i b_s b_t}{4\pi^2} z_j + \frac{1}{2\pi^2}\big(\delta_{js} b_i b_t + \delta_{jt} b_i b_s\big) + \frac{b_i}{2\pi}\big(b_s z_j + 2\delta_{js}\big)\overline{z}'_t + \frac{b_i}{2\pi}\big(b_t z_j + 2\delta_{jt}\big)\overline{z}'_s + b_i z_j \overline{z}'_s \overline{z}'_t\right]\mathscr{P}.$$

By Theorem 1.1 and (4.68), we get

$$\mathscr{L}^{-1} b_i z_j \overline{z}_s \overline{z}_t \mathscr{P} = \left[\frac{b_i b_s b_t}{48\pi^3} z_j + \frac{1}{16\pi^3}\big(\delta_{js} b_i b_t + \delta_{jt} b_i b_s\big) + \frac{b_i}{16\pi^2}\big(b_s z_j + 4\delta_{js}\big)\overline{z}'_t + \frac{b_i}{16\pi^2}\big(b_t z_j + 4\delta_{jt}\big)\overline{z}'_s + \frac{b_i}{4\pi} z_j \overline{z}'_s \overline{z}'_t\right]\mathscr{P}. \tag{4.69}$$

By (1.7), (1.8) and (4.6), we get

$$\left(\frac{b_i}{16\pi^2}\big(b_s z_j + 4\delta_{js}\big)\overline{z}'_t \mathscr{P}\right)(Z, Z') \approx \frac{1}{4\pi}\big(\delta_{js}\overline{z}_i - \delta_{ij}\overline{z}_s\big)\overline{z}'_t \mathscr{P}(Z, Z'). \tag{4.70}$$

By (4.32), (4.69) and (4.70), we obtain

$$\Big(d_x d_y\big(\mathscr{L}^{-1} b_i z_j \overline{z}_s \overline{z}_t \mathscr{P}\big)\Big)(0,0) = \frac{1}{4\pi}\big(\delta_{js} d\overline{z}_i - \delta_{ij} d\overline{z}_s\big) \wedge d\overline{z}_t + \frac{1}{4\pi}\big(\delta_{jt} d\overline{z}_i - \delta_{ij} d\overline{z}_t\big) \wedge d\overline{z}_s = \frac{1}{4\pi}\big(\delta_{js} d\overline{z}_i \wedge d\overline{z}_t + \delta_{jt} d\overline{z}_i \wedge d\overline{z}_s\big). \tag{4.71}$$

Finally, by Theorem 1.1, (1.7) and (1.8), as in (4.29), we get

$$\begin{aligned} &\Big(\mathscr{L}^{-1} b_i \overline{z}_j \overline{z}_s \overline{z}_t \mathscr{P}\Big)(Z, Z') \\ &= \left(\mathscr{L}^{-1} b_i \left(\frac{b_j}{2\pi} + \overline{z}'_j\right)\left(\frac{b_s}{2\pi} + \overline{z}'_s\right)\left(\frac{b_t}{2\pi} + \overline{z}'_t\right)\mathscr{P}\right)(Z, Z') \\ &\sim \left\{\mathscr{L}^{-1} b_i \left[\frac{b_j b_s b_t}{8\pi^3} + \frac{1}{4\pi^2}\big(b_j b_s \overline{z}'_t + b_j b_t \overline{z}'_s + b_s b_t \overline{z}'_j\big)\right]\mathscr{P}\right\}(Z, Z') \\ &\approx 0. \end{aligned} \tag{4.72}$$

From (3.11), (4.64), (4.67), (4.71) and (4.72), we obtain

$$
\begin{aligned}
(4.73)\qquad & (d_x d_y I_{22})(0,0) \\
&= \Bigg\{ \frac{1}{8\pi} \left\langle \left(\nabla^X_{\frac{\partial}{\partial z_j}} J\right) \frac{\partial}{\partial z_i}, \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}} J\right) \frac{\partial}{\partial \overline{z}_i} + \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J\right) \frac{\partial}{\partial \overline{z}_j} \right\rangle \\
&\quad + \frac{1}{12\pi} \left\langle R^{TX}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}\right) \frac{\partial}{\partial z_j} + R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_i}\right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle \Bigg\} \\
&\quad \times \big(-2\pi\sqrt{-1}\big)\omega(x_0) \\
&\quad + \frac{1}{8}\Bigg[\left\langle \left(\nabla^X_{\frac{\partial}{\partial z_j}} J\right) \frac{\partial}{\partial z_r}, \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}} J\right) \frac{\partial}{\partial \overline{z}_q} + \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}} J\right) \frac{\partial}{\partial \overline{z}_j} \right\rangle \\
&\quad + \left\langle \left(\nabla^X_{\frac{\partial}{\partial z_r}} J\right) \frac{\partial}{\partial z_j}, \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}} J\right) \frac{\partial}{\partial \overline{z}_q} + \left(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}} J\right) \frac{\partial}{\partial \overline{z}_j} \right\rangle \Bigg] dz_r \wedge d\overline{z}_q \\
&\quad + \frac{1}{6}\Bigg\langle R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}\right) \frac{\partial}{\partial z_r} \\
&\qquad + R^{TX}\left(\frac{\partial}{\partial z_r}, \frac{\partial}{\partial \overline{z}_j}\right) \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_q} \Bigg\rangle dz_r \wedge d\overline{z}_q \\
&\quad + \frac{1}{3}\Bigg\langle R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}\right) \frac{\partial}{\partial \overline{z}_r} \\
&\qquad + R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_r}\right) \frac{\partial}{\partial \overline{z}_j}, \frac{\partial}{\partial \overline{z}_q} \Bigg\rangle d\overline{z}_r \wedge d\overline{z}_q.
\end{aligned}
$$

Note that by (4.23),

$$
\begin{aligned}
(4.74)\qquad R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j}\right) \frac{\partial}{\partial z_r} &= R^{TX}\left(\frac{\partial}{\partial z_r}, \frac{\partial}{\partial \overline{z}_j}\right) \frac{\partial}{\partial z_j} \\
&\quad + R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_r}\right) \frac{\partial}{\partial \overline{z}_j}.
\end{aligned}
$$

By (4.23), (4.42), (4.52), (4.53), (4.54), (4.55), (4.73) and (4.74), we get

$$
\begin{aligned}
(4.75)\qquad & (d_x d_y I_{22})(0,0) \\
&= -\sqrt{-1}\Bigg[\frac{5}{96}\big|\nabla^X J\big|^2 + \frac{1}{3} \left\langle R^{TX}\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_i}\right) \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle \Bigg] \omega(x_0) \\
&\quad + \frac{1}{4} \mathscr{J}_{jir}\big(5\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}} - 4\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}\big) dz_r \wedge d\overline{z}_q +
\end{aligned}
$$

$$+\frac{1}{6}\Big\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r}\right)\frac{\partial}{\partial \overline{z}_j} + 2R^{TX}\left(\frac{\partial}{\partial z_r},\frac{\partial}{\partial \overline{z}_j}\right)\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_q}\Big\rangle dz_r\wedge d\overline{z}_q + \frac{1}{3}\Big\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_j}\right)\frac{\partial}{\partial \overline{z}_r} + R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_r}\right)\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\Big\rangle d\overline{z}_r\wedge d\overline{z}_q.$$

By (4.57), (4.59) and (4.60), we get (cf. [14, (2.28)]),

$$\left\langle (\nabla^X\nabla^X J)_{(\mathcal{R},\mathcal{R})}\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i}\right\rangle = 2\left\langle (\nabla^X\nabla^X J)_{(z,\overline{z})}\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_i}\right\rangle = -\sqrt{-1}\left\langle (\nabla^X_z J)\frac{\partial}{\partial z_i},(\nabla^X_{\overline{z}} J)\frac{\partial}{\partial \overline{z}_i}\right\rangle. \tag{4.76}$$

By (4.25) and (4.35), we get

$$\Big(\mathscr{L}^{-1}\mathscr{P}^{\perp}z_s\overline{z}_t\mathscr{P}\Big)(Z,Z') = -\frac{1}{4\pi^2}\Big(\delta_{st}-\pi z_s\big(\overline{z}_t-\overline{z}'_t\big)\Big)\mathscr{P}(Z,Z'). \tag{4.77}$$

By (3.11), (4.42), (4.54), (4.76) and (4.77), we obtain

$$\begin{aligned}(d_xd_yI_{24})(0,0) &= \left\langle\left(\nabla^X_{\frac{\partial}{\partial z_j}}J\right)\frac{\partial}{\partial z_i},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}}J\right)\frac{\partial}{\partial \overline{z}_i}\right\rangle(-\sqrt{-1})\omega(x_0)\\ &\quad+\frac{1}{2}\left\langle\left(\nabla^X_{\frac{\partial}{\partial z_s}}J\right)\frac{\partial}{\partial z_i},\left(\nabla^X_{\frac{\partial}{\partial \overline{z}_t}}J\right)\frac{\partial}{\partial \overline{z}_i}\right\rangle dz_s\wedge d\overline{z}_t\\ &= -\frac{\sqrt{-1}}{8}\big|\nabla^X J\big|^2\omega(x_0)\\ &\quad+2\mathscr{J}_{ijr}\big(\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}-\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big)dz_r\wedge d\overline{z}_q.\end{aligned} \tag{4.78}$$

Combining (4.22), (4.23), (4.33), (4.38), (4.42), (4.43), (4.56), (4.74), (4.75) and (4.78), we obtain

$$\begin{aligned}-(d_xd_yI_2)(0,0) &= \sqrt{-1}\left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right\rangle\omega(x_0)\\ &\quad-\frac{1}{2}\left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r}\right)\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\right\rangle dz_r\wedge d\overline{z}_q\\ &\quad-\frac{1}{12}\mathscr{J}_{jir}\big(\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}+4\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}\big)dz_r\wedge d\overline{z}_q.\end{aligned} \tag{4.79}$$

By Lemma 3.1, (4.50), (4.51), (4.54), (4.55),(4.57) and (4.62), we get (cf. [13, (8.3.63)])

$$
\begin{aligned}
(4.80)\qquad & \left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r}\right)\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\right\rangle \\
&= \frac{\sqrt{-1}}{2}\left\langle \left[R^{TX}\Big(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r}\Big),J\right]\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\right\rangle \\
&= \frac{\sqrt{-1}}{2}\left\langle \left[\big(\nabla^X\nabla^X J\big)_{(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_r})}-\big(\nabla^X\nabla^X J\big)_{(\frac{\partial}{\partial z_r},\frac{\partial}{\partial z_j})}\right]\frac{\partial}{\partial \overline{z}_j},\frac{\partial}{\partial \overline{z}_q}\right\rangle \\
&= \frac{1}{4}\left\langle \Big(\nabla^X_{\frac{\partial}{\partial z_j}}J\Big)\frac{\partial}{\partial z_r}-\Big(\nabla^X_{\frac{\partial}{\partial z_r}}J\Big)\frac{\partial}{\partial z_j},\ \Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_j}}J\Big)\frac{\partial}{\partial \overline{z}_q}-\Big(\nabla^X_{\frac{\partial}{\partial \overline{z}_q}}J\Big)\frac{\partial}{\partial \overline{z}_j}\right\rangle \\
&= \frac{1}{2}\mathscr{J}_{jri}\mathscr{J}_{\overline{j}\overline{q}\overline{i}}.
\end{aligned}
$$

Substituting (4.80) into (4.79) yields

$$
\begin{aligned}
(4.81)\qquad (d_xd_yI_2)(0,0) = &-\sqrt{-1}\left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right\rangle\omega(x_0) \\
&+\frac{1}{3}\mathscr{J}_{jir}\big(\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}+\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}\big)dz_r\wedge d\overline{z}_q.
\end{aligned}
$$

4.4. Proof of Theorem 3.3

By (4.4) and (4.81), we get

$$
(4.82)\qquad (d_xd_yI_4)(0,0) = (d_xd_yI_2)(0,0).
$$

Substituting (4.12), (4.13), (4.17), (4.19), (4.81) and (4.82) into (4.2), we finally obtain

$$
\begin{aligned}
(4.83)\qquad \big(d_xd_y\mathscr{F}_2\big)(0,0) &= -\frac{6}{9}\mathscr{J}_{jir}\big(\mathscr{J}_{\overline{i}\,\overline{j}\overline{q}}+\mathscr{J}_{\overline{j}\,\overline{i}\overline{q}}\big)dz_r\wedge d\overline{z}_q+2(d_xd_yI_2)(0,0) \\
&= -2\sqrt{-1}\left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right\rangle\omega(x_0).
\end{aligned}
$$

By [13, Theorem 8.3.4, Lemma 8.3.10],

$$
(4.84)\qquad 8\left\langle R^{TX}\left(\frac{\partial}{\partial z_j},\frac{\partial}{\partial \overline{z}_i}\right)\frac{\partial}{\partial z_i},\frac{\partial}{\partial \overline{z}_j}\right\rangle = r^X+\frac{1}{4}\big|\nabla^X J\big|^2 = 8\pi\boldsymbol{b}_1(x_0).
$$

The identities (4.83) and (4.84) yield Theorem 3.3. This concludes the proof of Theorem 0.1.

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