

Scaling asymptotics of heat kernels of line bundles

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Dedicated to Professor Duong H. Phong on the occasion of his 60th birthday

ABSTRACT. We consider a general Hermitian holomorphic line bundle L on a compact complex manifold M and let \square_p^q be the Kodaira Laplacian on $(0, q)$ forms with values in L^p . We study the scaling asymptotics of the heat kernel $\exp(-u\square_p^q/p)(x, y)$.

The main result is a complete asymptotic expansion for the semi-classically scaled heat kernel $\exp(-u\square_p^q/p)(x, x)$ along the diagonal. It is a generalization of the Bergman/Szegö kernel asymptotics in the case of a positive line bundle, but no positivity is assumed. We give two proofs, one based on the Hadamard parametrix for the heat kernel on a principal bundle and the second based on the analytic localization of the Dirac-Dolbeault operator.

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1. Introduction

Let (M, J) be a complex manifold with complex structure J , and complex dimension n . Let L and E be two holomorphic vector bundles on M such that $\text{rk}(L) = 1$; the bundle E plays the role of an auxiliary twisting bundle. We fix Hermitian metrics h^L, h^E on L, E . Let L^p denote the p th tensor power of L . The purpose of this article is to prove scaling asymptotics of various heat kernels on $L^p \otimes E$ as $p \rightarrow \infty$. We present the scaling asymptotics from two points of view. The first one (Theorem 1.1) gives scaling asymptotics of the Kodaira heat

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kernels and is based on the analytic localization technique of Bismut-Lebeau [5], adaptating the arguments from [19, §1.6, §4.2]. The second (Theorem 1.2) gives scaling asymptotics of the heat kernels associated to the Bochner Laplacian, and is an adaptation of the Szegö kernel asymptotics of [20]. It is based on lifting sections of L^p to equivariant functions on the associated principal S^1 bundle $X_h \rightarrow M$, and obtaining scaling asymptotics of heat kernels from Fourier analysis of characters and stationary phase asymptotics. Either method can be applied to any of the relevant heat kernels and it seems to us of some interest to compare the methods. We refer to [6, 19–21] for background from both points of view of analysis on higher powers of line bundles.

To state our results, we need to introduce some notation. Let ∇^E, ∇^L be the holomorphic Hermitian connections on $(E, h^E), (L, h^L)$. Let R^L, R^E be the curvatures of ∇^L, ∇^E . Let g^{TM} be a J -invariant Riemannian metric on M , i. e., $g^{TM}(Ju, Jv) = g^{TM}(u, v)$ for all $x \in M$ and $u, v \in T_x M$. Set

$$(1.1) \quad \omega := \frac{\sqrt{-1}}{2\pi} R^L, \quad \Theta(\cdot, \cdot) := g^{TM}(J\cdot, \cdot).$$

Then ω, Θ are real $(1, 1)$ -forms on M , and ω is the Chern-Weil representative of the first Chern class $c_1(L)$ of L . The Riemannian volume form dv_M of (TM, g^{TM}) is $\Theta^n/n!$. We will identify the 2-form R^L with the Hermitian matrix $\hat{R}^L \in \text{End}(T^{(1,0)}M)$ defined by

$$(1.2) \quad \langle \hat{R}^L W, \bar{Y} \rangle = R^L(W, \bar{Y}), \quad W, Y \in T^{(1,0)}M.$$

The curvature R^L acts as a derivation $\omega_d \in \text{End}(\Lambda(T^{*(0,1)}M))$ on $\Lambda(T^{*(0,1)}M)$. Namely, let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}M$ with dual frame $\{w^j\}_{j=1}^n$. Set

$$(1.3) \quad \omega_d = - \sum_{l,m} R^L(w_l, \bar{w}_m) \bar{w}^m \wedge i_{\bar{w}_l}, \quad \tau(x) = \sum_j R^L(w_j, \bar{w}_j).$$

Consider the Dolbeault-Dirac operator

$$(1.4) \quad D_p = \sqrt{2} \left(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *} \right),$$

and the Kodaira Laplacian

$$(1.5) \quad \square_p = \frac{1}{2} D_p^2 = \bar{\partial}^{L^p \otimes E} \bar{\partial}^{L^p \otimes E, *} + \bar{\partial}^{L^p \otimes E, *} \bar{\partial}^{L^p \otimes E}.$$

For $p \in \mathbb{N}$, we denote by

$$(1.6) \quad E_p^j := \Lambda^j(T^{*(0,1)}M) \otimes L^p \otimes E, \quad E_p = \bigoplus_j E_p^j,$$

and let h_p the induced Hermitian metric on E_p .

The operator $D_p^2 = 2\square_p$ is a second order elliptic differential operator with principal symbol $\sigma(D_p^2)(\xi) = |\xi|^2 \text{Id}_{E_p}$ for $\xi \in T_x^*M, x \in M$. The heat operator $\exp(-uD_p^2)$ is well defined for $u > 0$. Let $\exp(-uD_p^2)(x, x')$, where $x, x' \in M$, be its smooth kernel with respect to the Riemannian volume form $dv_M(x')$. Then

$$(1.7) \quad \exp(-uD_p^2)(x, x') \in (E_p)_x \otimes (E_p)_{x'}^*,$$

especially

$$(1.8) \quad \exp(-uD_p^2)(x, x) \in \text{End}(E_p)_x = \text{End}(\Lambda(T^{*(0,1)}M) \otimes E)_x,$$

where we use the canonical identification $\text{End}(L^p) = \mathbb{C}$ for any line bundle L on M . Note that D_p^2 preserves the \mathbb{Z} -grading of the Dolbeault complex $\Omega^{0,\bullet}(M, L^p \otimes E)$, so

$$(1.9) \quad \exp(-uD_p^2|_{\Omega^{0,j}})(x, x') = \sum_{k=1}^{\infty} e^{-u\lambda_{k,p}^j} \varphi_{k,p}^j(x) \otimes \varphi_{k,p}^j(x')^* \in (E_p^j)_x \otimes (E_p^j)_{x'}^*,$$

where $\{\lambda_{k,p}^j : k \geq 1\}$ is the spectrum of $D_p^2|_{\Omega^{0,j}}$ and $\{\varphi_{k,p}^j : k \geq 1\}$ is an orthonormal basis of $L^2(M, E_p^j)$ consisting of eigensections of $D_p^2|_{\Omega^{0,j}}$ with $D_p^2\varphi_{k,p}^j = \lambda_{k,p}^j\varphi_{k,p}^j$, cf. [19, (D.1.7)]. Thus

$$\exp(-uD_p^2)(x, x) \in \bigoplus_j \text{End}(\Lambda^j(T^{*(0,1)}M) \otimes E)_x.$$

We will denote by \det the determinant on $T^{(1,0)}M$. The following gives the scaling asymptotics for the Kodaira-Laplacian heat kernel.

THEOREM 1.1. *Assume that M is compact. For $T > 0$, and any $k, m \in \mathbb{N}$ we have as $p \rightarrow \infty$*

$$(1.10) \quad \exp\left(-\frac{u}{p}D_p^2\right)(x, x) = \sum_{r=0}^m \left(\frac{p}{u}\right)^{n-r} e_{\infty r}(u, x) + \left(\frac{p}{u}\right)^{n-m-1} R_{m+1}\left(\frac{u}{p}, u, x\right)$$

uniformly for $0 < u < T$ and $x \in M$, in the \mathcal{C}^k -norm on $\mathcal{C}^\infty(M, \text{End}(\Lambda(T^{*(0,1)}M) \otimes E))$, i. e., the reminder term $R_{m+1}(\frac{u}{p}, u, x)$ is uniformly bounded for $0 < u < T$, $x \in M$, $p \in \mathbb{N}^*$. For any $r \in \mathbb{N}$, the coefficient $e_{\infty r}(u, x)$ is smooth at $u = 0$ and the principal term is given by

$$(1.11) \quad e_{\infty 0}(u, x) = \frac{1}{(2\pi)^n} \frac{\det(u\dot{R}_x^L) \exp(2u\omega_{d,x})}{\det(1 - \exp(-2u\dot{R}_x^L))} \otimes \text{Id}_E.$$

The leading term of the scaling asymptotics has been known for some time in connection with the Demailly holomorphic Morse inequalities [12]. Bismut [4] and Demailly [13] used the heat kernel to prove these inequalities, based on the principal term of the scaling asymptotics above. The new feature of Theorem 1.1 is the complete asymptotic expansion in the \mathcal{C}^∞ sense, and the computability of the coefficients. It is a kind of generalization, in terms of both statement and proof, of the Bergman/Szegö kernel expansion on the diagonal given in [11], [20] in the case of positive Hermitian holomorphic line bundles. The main feature of the heat kernel expansion is its generality: it does not require that (L, h^L) be a positive line bundle, nor even that (M, Θ) be a Kähler manifold. In the general case, the Bergman/Szegö kernel is difficult to analyze and the heat kernel is a good substitute. Note that for $u > 0$ fixed, Theorem 1.1 was obtained in [14, (1.4)], [19, (4.2.4)].

Let us give another form of the principal term (1.11) in order to recover Demailly's formula [13, Theorem 4.4]. Let us choose $\{w_j\}_{j=1}^n$ to be an orthonormal basis of $T^{(1,0)}M$ such that

$$(1.12) \quad \dot{R}^L(x) = \text{diag}(\alpha_1(x), \dots, \alpha_n(x)) \in \text{End}(T_x^{(1,0)}M).$$

The elements $\alpha_1(x), \dots, \alpha_n(x)$ are called the eigenvalues of R^L with respect to Θ . Then

$$(1.13) \quad \omega_d(x) = - \sum_j \alpha_j(x) \bar{w}^j \wedge i\bar{w}_j, \quad \tau(x) = \sum_j \alpha_j(x).$$

We have by [19, (1.6.4)]

$$(1.14) \quad e_{\infty 0}(u, x) = u^n \prod_{j=1}^n \frac{\alpha_j(x) (1 + (\exp(-2u\alpha_j(x)) - 1)\bar{w}^j \wedge i\bar{w}_j)}{2\pi(1 - \exp(-2u\alpha_j(x)))} \otimes \text{Id}_E .$$

Here we use the following convention: if an eigenvalue $\alpha_j(x)$ of \dot{R}_x^L is zero, then its contribution to $\det(\dot{R}_x^L) / \det(1 - \exp(-2u\dot{R}_x^L))$ is $1/(2u)$.

Remark that the operator $D_p^2 = 2\Box_p$ preserves the \mathbb{Z} -grading of the Dolbeault complex $\Omega^{0,\bullet}(M, L^p \otimes E)$. We will denote by \Box_p^q the restriction of \Box_p to $\Omega^{0,q}(M, L^p \otimes E)$. We set

$$(1.15) \quad e_p^q(u, x) = \text{Tr} \exp\left(-\frac{2u}{p}\Box_p^q\right)(x, x) = \text{Tr}_q \exp\left(-\frac{u}{p}D_p^2\right)(x, x) .$$

where Tr_q is the trace of an operator acting on E_p^q . By taking the trace Tr_q of (1.10) we obtain

$$(1.16) \quad e_p^q(u, x) = \sum_{r=0}^m \binom{p}{u}^{n-r} e_{\infty r}^q(u, x) + \binom{p}{u}^{n-m-1} R_{m+1}^q\left(\frac{u}{p}, u, x\right) ,$$

where

$$e_{\infty r}^q(u, x) = \text{Tr}_q e_{\infty r}(u, x) , \quad R_{m+1}^q\left(\frac{u}{p}, u, x\right) = \text{Tr}_q R_{m+1}\left(\frac{u}{p}, u, x\right) .$$

We obtain thus from (1.14),

$$(1.17) \quad e_{\infty 0}^q(u, x) = \text{rk}(E)(4\pi)^{-n} \left(\sum_{|J|=q} e^{u(\alpha_{\mathbb{C}J} - \alpha_J)} \right) \prod_{j=1}^n \frac{u\alpha_j(x)}{\sinh(u\alpha_j(x))} .$$

We use the following notation for a multi-index $J \subset \{1, \dots, n\}$:

$$\alpha_J = \sum_{j \in J} \alpha_j , \quad \mathbb{C}J = \{1, \dots, n\} \setminus J .$$

It is understood that

$$\frac{\alpha}{\sinh \alpha u} = \frac{1}{u} , \quad \text{when } \alpha = 0 .$$

1.1. Scaling asymptotics of the heat kernel of the associated principal bundle. We now state a closely result of the scaling asymptotics of the heat kernel for the Bochner Laplacian $\nabla_p^* \nabla_p$. The method also applies to the Kodaira Laplacian but we only present it in this case. For simplicity, we do not twist by a vector bundle E .

As above, we denote by (M, Θ) a compact complex n -manifold with Hermitian metric Θ , with volume form $dv_M = \frac{\Theta^n}{n!}$, and let $(L, h^L) \rightarrow M$ be a holomorphic line bundle with curvature R^L . Let ∇_p denote the Chern connection associated to h^L on L^p .

Denote by L^* the dual line bundle and let D_h^* be the unit disc bundle of L^* with respect to the dual metric h^{L^*} . The boundary $X = X_h = \partial D_h^*$ is then a principal S^1 bundle $\pi : X \rightarrow M$ over M . The powers L^p of L are the line bundles $L^p = X \times_{\chi_p} \mathbb{C}$ associated to the characters $\chi_p(e^{i\theta}) = e^{ip\theta}$ of S^1 . Sections s of L^p naturally lift to L^* as equivariant functions $\hat{s}(\lambda)(x) = \lambda(s(\pi(x)))$, and the lifting map identifies $L^2(M, L^p)$ with the space $L_p^2(X)$ of equivariant functions on X transforming by

$e^{ip\theta}$ under the S^1 action on X , which we denote by $e^{i\theta} \cdot x$. The Chern connection induces an S^1 -invariant vertical 1-form β , defining a connection on TX (see §2.1).

We define the Bochner Laplacian Δ^{L^p} on L^p by $\Delta^{L^p} = \nabla_p^* \nabla_p$ where $*$ is taken with respect to dv_M . Under the lifting identification $L^2(M, L^p) \simeq L_p^2(X)$, Δ^{L^p} corresponds to restriction to $L_p^2(X)$ of the horizontal Laplacian $\Delta_H = d_H^* d_H$, where d_H is the horizontal differential on X for the connection β .

The lifting identification induces an identification of heat kernels, which takes the following form on the diagonal: Let $x \in X, z \in M$ and $\pi(x) = z$. Then

$$(1.18) \quad \exp\left(-\frac{u}{p} \Delta^{L^p}\right)(z, z) = \int_{S^1} e^{-(u/p)\Delta_H}(e^{i\theta}x, x)e^{-ip\theta} d\theta.$$

Using this formula, we prove

THEOREM 1.2. *Assume that M is compact. With the above notations and assumptions, there exist smooth coefficients $e_{\infty,r}^H(u, z)$ so that for $T > 0$, and any $k, m \in \mathbb{N}$ we have as $p \rightarrow \infty$*

$$(1.19) \quad \exp\left(-\frac{u}{p} \Delta^{L^p}\right)(z, z) = \sum_{r=0}^m \left(\frac{p}{u}\right)^{n-r} e_{\infty,r}^H(u, z) + \left(\frac{p}{u}\right)^{n-m-1} R_{m+1}\left(\frac{u}{p}, u, z\right).$$

uniformly for $0 < u < T$ and $z \in M$, in the \mathcal{C}^k -norm on $\mathcal{C}^\infty(M)$, i. e., the reminder term $R_{m+1}(\frac{u}{p}, u, z)$ is uniformly bounded for $0 < u < T, z \in M, p \in \mathbb{N}^$.*

In view of (1.18) we could also state this result as giving the scaling asymptotics of the p th Fourier component of the horizontal heat kernel. As discussed in §3.1, the reason for using (1.18), is that there exists a rather concrete Hadamard style parametrix for $e^{-u\Delta_H}(x, y)$, involving the Hadamard heat kernel coefficients Φ_j of a principal bundle, computed in [3, Theorem 5.8]. All the properties stated in the theorem follow from standard facts about the stationary phase method and from the properties of the coefficients Φ_j . The principal term is given by (cf. (1.1)–(1.11)):

$$(1.20) \quad e_{\infty 0}^H(u, x) = \frac{1}{(2\pi)^n} \frac{\det(u\dot{R}_x^L) \exp(-u\tau)}{\det(1 - \exp(-2u\dot{R}_x^L))}.$$

Recall that τ is the trace of the curvature R^L defined in (1.3). The subleading term is given by

$$(1.21) \quad \left(\frac{p}{4\pi u}\right)^n \left[u\Phi_1(x, 2u) + \frac{\partial^2}{\partial\theta^2} \Phi_0(x, i\theta + 2u) \Big|_{\theta=0} \right].$$

Let us compare the expansions (1.10) of the Kodaira Laplacian and (1.19) of the Bochner Laplacian. Note that by Lichnerowicz formula (4.3) we have $D_p^2 = \Delta^{L^p} - p\tau + \mathcal{O}(1)$ on $\Omega^{0,0}(M, L^p)$. Consider the rescaled operator $\widetilde{\mathcal{L}}_2^{0,u}$ corresponding to $\frac{u}{p}\Delta^{L^p}$ analogous to $\mathcal{L}_2^{0,u}$ as in §4; the analogue of (4.15) is $\widetilde{\mathcal{L}}_2^{0,u} = -\sum_i (\nabla_{0,u,e_i})^2$. Thus the difference between (1.20) and (1.11) for $(0, 0)$ -forms is the factor $\exp(-u\tau)$.

If one uses the Lichnerowicz formula to express the Kodaira Laplacian in terms of the horizontal (Bochner) Laplacian, one may then apply the Duhamel formula to express the heat kernel of the Kodaira Laplacian in terms of that of the Bochner Laplacian. Alternatively, one may go through the parametrix construction as in [3, Theorem 5.8] but with the Kodaira Laplacian. The transport equations change because of the extra curvature term. We omit the details since we are already

giving a proof of Theorem 1.1 by another method. We also leave to the reader the adaptation of the analytic localization proof of Theorem 1.1 to obtain Theorem 1.2.

1.2. Relation to the holomorphic Morse inequalities. The original application of the scaling asymptotics was to estimating dimensions $h^q(L^p \otimes E) := \dim H^q(M, L^p \otimes E)$ of holomorphic sections [4, 12, 13, 19]. We follow the exposition of [19, 21].

Let $M(q) \subset M$ be the subset in which $\sqrt{-1}R^L$ has precisely q negative eigenvalues and $n - q$ positive eigenvalues. Set $M(\leq q) = \bigcup_{i=0}^q M(i)$, $M(\geq q) = \bigcup_{i=q}^n M(i)$.

The holomorphic Morse inequalities of J.-P. Demailly [12] give asymptotic estimates for the alternating sums of the dimensions $h^q(L^p \otimes E)$ as $p \rightarrow \infty$.

THEOREM 1.3. *Let M be a compact complex manifold with $\dim M = n$, and let (L, h^L) , (E, h^E) be Hermitian holomorphic vector bundles on M , $\text{rk}(L) = 1$. As $p \rightarrow \infty$, the following strong Morse inequalities hold for every $q = 0, 1, \dots, n$:*

$$(1.22) \quad \sum_{j=0}^q (-1)^{q-j} h^j(L^p \otimes E) \leq \text{rk}(E) \frac{p^n}{n!} \int_{M(\leq q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n),$$

with equality for $q = n$ (asymptotic Riemann-Roch-Hirzebruch formula).

Moreover, we have the weak Morse inequalities

$$(1.23) \quad h^q(L^p \otimes E) \leq \text{rk}(E) \frac{p^n}{n!} \int_{M(q)} (-1)^q \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n + o(p^n).$$

It was observed by J.-M. Bismut [4] that the leading order scaling asymptotics of the heat kernel could be used to simplify the proof of these inequalities. Bismut’s probability arguments were replaced by classical heat kernel methods by J.-P. Demailly [13] and by T. Bouche [7, 8]. Since one obviously has

$$(1.24) \quad h^q(L^p \otimes E) \leq \int_M e_p^q(u, x) dv_M$$

for any u , we can let $p \rightarrow \infty$ to obtain

$$(1.25) \quad \limsup_{p \rightarrow \infty} p^{-n} h^q(L^p \otimes E) \leq \int_M u^{-n} e_{\infty 0}^q(u, x) dv_M$$

and then let $u \rightarrow \infty$ to obtain the weak Morse inequalities,

$$(1.26) \quad \limsup_{p \rightarrow \infty} p^{-n} h^q(L^p \otimes E) \leq \text{rk}(E) (-1)^q \frac{1}{n!} \int_{M(q)} \left(\frac{\sqrt{-1}}{2\pi} R^L \right)^n.$$

In the last step we used that

$$(1.27) \quad \text{As } u \rightarrow +\infty, \quad \frac{e^{u\alpha} u\alpha}{\sinh u\alpha} = \frac{2u\alpha}{1 - e^{-2u\alpha}} \sim \begin{cases} 2u\alpha + O(ue^{-2\alpha u}), & \alpha > 0, \\ 1, & \alpha = 0, \\ O(ue^{-2|\alpha|u}), & \alpha < 0. \end{cases}$$

In the case of $q = 0$ we have:

$$(1.28) \quad e_{\infty 0}^0(u, x) \sim \text{rk}(E) \begin{cases} (4\pi)^{-n} (2u)^r \prod_{i, \alpha_i(x) > 0} \alpha_i(x), & \alpha_i(x) \geq 0 \ \forall i, r = \#\{i : \alpha_i(x) > 0\}, \\ 1 & \alpha_i(x) = 0, \ \forall i, \\ 0 & \exists i : \alpha_i(x) < 0. \end{cases}$$

For general q the asymptotics depends in a more complicated way on the eigenvalues of $\sqrt{-1}R^L$. Assume first that $x \in M(q)$ so $\sqrt{-1}R^L$ is non-degenerate at x and let $J_-(x)$ denote the set of q indices for which $\alpha_j(x) < 0$, resp. $J_+(x)$ denote the set of indices for which $\alpha_j(x) > 0$. The only term $\alpha_{\mathbf{c}_J} - \alpha_J$ which makes a non-trivial asymptotic contribution is the one for which $J = J_-(x)$. Hence

$$(1.29) \quad \begin{aligned} u^{-n} e_{\infty 0}^q(u, x) &\sim \text{rk}(E) (4\pi)^{-n} u^{-n} \prod_j e^{u|\alpha_j(x)|} \frac{u\alpha_j(x)}{\sinh(u\alpha_j(x))} \\ &\sim \text{rk}(E) (-1)^q \prod_j \frac{\alpha_j(x)}{2\pi}. \end{aligned}$$

Now assume that the curvature is degenerate at x , with n_- negative eigenvalues, n_0 zero eigenvalues and n_+ positive eigenvalues. Since we must change the sign of q eigenvalues and since any negative eigenvalue causes the whole product to vanish in the $u \rightarrow \infty$ limit, the asymptotics are trivial unless $n_- \leq q$ and $n_- + n_0 \geq q$. If $n_- = q$ there is only one term in the J sum, namely where $J = J_-(x)$. If $n_- < q$ then we may choose any $q - n_-$ indices of zero eigenvalues to flip. There are $\binom{n_0}{q-n_-}$ such indices. Hence in the degenerate case with $n_- \leq q$, $n_- + n_0 \geq q$ we have

$$(1.30) \quad \begin{aligned} u^{-n} e_{\infty 0}^q(u, x) &\sim \text{rk}(E) u^{-n} \binom{n_0}{q-n_-} (4\pi)^{-n} \prod_{j \in J_+(x) \cup J_-(x)} e^{u|\alpha_j(x)|} \frac{u\alpha_j(x)}{\sinh u\alpha_j(x)} \\ &\sim \text{rk}(E) u^{-n_0} \binom{n_0}{q-n_-} (4\pi)^{-n} (-1)^{n_-} \prod_{j \in J_+(x) \cup J_-(x)} (2\alpha_j(x)). \end{aligned}$$

Thus, in order to obtain the local Morse inequalities, one first takes the limit $p \rightarrow \infty$ and then $u \rightarrow \infty$. A natural question is whether one can let $u \rightarrow \infty, p \rightarrow \infty$ simultaneously in the scaling asymptotics of Theorem 1.1. Suppose that $\sqrt{-1}R^L$ has rank $\leq n - s$ at all points. Then one would conjecture that $h^q(L^p \otimes E) \leq \varepsilon(p)^s p^{n-s}$ as $p \rightarrow \infty$ where $\varepsilon(p)$ is any function such that $\varepsilon(p) \uparrow \infty$ as $p \rightarrow \infty$. Let $u(p) = \frac{p}{\varepsilon(p)}$. Suppose that $u(p)/p = 1/\varepsilon(p)$ could be used as a small parameter in the expansion of Theorem 1.1. The principal term is of order $(\frac{p}{u(p)})^n u(p)^{n-s} = \varepsilon(p)^s p^{n-s}$ and one would hope that the remainder is of order $\varepsilon(p)^{s+1} p^{n-1-s}$. Our remainder estimate in Theorem 1.1 is not sharp enough for this application.

Let us close by recalling the proof of the *strong Morse inequalities* (1.22) (cf. [19, §1.7]). As before, we denote by $\text{Tr}_q \exp\left(-\frac{u}{p} D_p^2\right)$ the trace of $\exp\left(-\frac{u}{p} D_p^2\right)$

acting on $\Omega^{0,q}(M, L^p \otimes E)$. Then we have (using notation (1.15))

$$(1.31) \quad \text{Tr}_q \exp\left(-\frac{u}{p} D_p^2\right) = \int_M e_p^q(u, x) dv_M(x).$$

By a linear algebra argument involving the spectral spaces [19, Lemma 1.7.2] we have for any $u > 0$ and $q \in \mathbb{N}$ with $0 \leq q \leq n$,

$$(1.32) \quad \sum_{j=0}^q (-1)^{q-j} h^j(L^p \otimes E) \leq \sum_{j=0}^q (-1)^{q-j} \text{Tr}_j \exp\left(-\frac{u}{p} D_p^2\right),$$

with equality for $q = n$. Note that in the notation of (1.3),

$$(1.33) \quad \exp(-2u\alpha_j(x_0)\bar{w}^j \wedge i_{\bar{w}_j}) = 1 + (\exp(-2u\alpha_j(x_0)) - 1)\bar{w}^j \wedge i_{\bar{w}_j}.$$

We denote by $\text{Tr}_{\Lambda^{0,q}}$ the trace on $\Lambda^q(T^{*(0,1)}M)$. By (1.33),

$$(1.34) \quad \text{Tr}_{\Lambda^{0,q}}[\exp(2u\omega_d)] = \sum_{j_1 < j_2 < \dots < j_q} \exp\left(-2u \sum_{i=1}^q \alpha_{j_i}(x)\right).$$

Thus by (1.10),

$$\frac{\det(\dot{R}^L/(2\pi))}{\det(1 - \exp(-2u\dot{R}^L))} \text{Tr}_{\Lambda^{0,q}}[\exp(2u\omega_d)]$$

is uniformly bounded in $x \in M$, $u > 1$, $0 \leq q \leq n$. Hence for any $x_0 \in M$, $0 \leq q \leq n$,

$$(1.35) \quad \lim_{u \rightarrow \infty} \frac{\det(\dot{R}^L/(2\pi)) \text{Tr}_{\Lambda^{0,q}}[\exp(2u\omega_d)]}{\det(1 - \exp(-2u\dot{R}^L))}(x_0) = \mathbf{1}_{M(q)}(x_0)(-1)^q \det\left(\frac{\dot{R}^L}{2\pi}\right)(x_0),$$

where $\mathbf{1}_{M(q)}$ is the characteristic function of $M(q)$. From Theorem 1.1, (1.31) and (1.32), we have

$$(1.36) \quad \begin{aligned} & \limsup_{p \rightarrow \infty} p^{-n} \sum_{j=0}^q (-1)^{q-j} h^j(L^p \otimes E) \\ & \leq \text{rk}(E) \int_M \frac{\det(\dot{R}^L/(2\pi)) \sum_{j=0}^q (-1)^{q-j} \text{Tr}_{\Lambda^{0,j}}[\exp(2u\omega_d)]}{\det(1 - \exp(-2u\dot{R}^L))} dv_M(x), \end{aligned}$$

for any q with $0 \leq q \leq n$ and any $u > 0$. Using (1.35), (1.36) and dominated convergence, we get

$$(1.37) \quad \limsup_{p \rightarrow \infty} p^{-n} \sum_{j=0}^q (-1)^{q-j} h^j(L^p \otimes E) \leq (-1)^q \text{rk}(E) \int_{\bigcup_{i=0}^q M(i)} \det\left(\frac{\dot{R}^L}{2\pi}\right)(x) dv_M(x).$$

But (1.12) entails

$$(1.38) \quad \det\left(\frac{\dot{R}^L}{2\pi}\right)(x) dv_M(x) = \prod_j \frac{\alpha_j(x)}{2\pi} dv_M(x) = \frac{1}{n!} \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n.$$

Relations (1.37), (1.38) imply (1.22).

Let us finally mention that the original proof of Demailly of the holomorphic Morse inequalities is based on the asymptotics of the spectral function of the

Kodaira Laplacian \square_p^q as $p \rightarrow \infty$, the semiclassical Weyl law, cf. [12], [19, Theorem 3.2.9]; see also [16] for the lower terms of the asymptotics of the spectral function.

1.3. Organization and notation. We end the introduction with some remarks on the organization and notation.

We first prove Theorem 1.2. The proof involves the construction of a Hadamard parametrix for the heat kernel of a principal S^1 bundle adapted from [3], and in particular involves the connection, distance function and volume form on the S^1 bundle $X_h \rightarrow M$. The geometric background is presented in §2, and then the proof of Theorem 1.2 is given in §3. The main complication is that the heat kernel must be analytically continued to L^* , which is an important but relatively unexplored aspect of heat kernels. The proof of Theorem 1.1 is then given in §4.

We now record some basic definitions and notations.

1.3.1. *The complex Laplacian.* Let F be a holomorphic Hermitian vector bundle over a complex manifold M . Let $\Omega^{r,q}(M, F)$ be the space of smooth (r, q) -forms on X with values in F . Since F is holomorphic the operator $\bar{\partial}^F : \mathcal{C}^\infty(M, F) \rightarrow \Omega^{0,1}(M, F)$ is well defined. A connection ∇^F on F is said to be a holomorphic connection if $\nabla_U^F s = i_U(\bar{\partial}^F s)$ for any $U \in T^{(0,1)}M$ and $s \in \mathcal{C}^\infty(X, F)$. It is well-known that there exists a unique holomorphic Hermitian connection ∇^E on (F, h^F) , called the *Chern connection*.

The operator $\bar{\partial}^F$ extends naturally to $\bar{\partial}^F : \Omega^{\bullet,\bullet}(M, F) \rightarrow \Omega^{\bullet,\bullet+1}(M, F)$ and $(\bar{\partial}^F)^2 = 0$. Let ∇^F be the Chern connection on (F, h^F) . Then we have a decomposition of ∇^F after bidegree

$$(1.39) \quad \begin{aligned} \nabla^F &= (\nabla^F)^{1,0} + (\nabla^F)^{0,1}, \quad (\nabla^F)^{0,1} = \bar{\partial}^F, \\ (\nabla^F)^{1,0} : \Omega^{\bullet,\bullet}(M, F) &\rightarrow \Omega^{\bullet+1,\bullet}(M, F). \end{aligned}$$

The Kodaira Laplacian is defined by:

$$(1.40) \quad \square^F = [\bar{\partial}^F, \bar{\partial}^{F,*}].$$

Recall that the Bochner Laplacian Δ^F associated to a connection ∇^F on a bundle F is defined (in terms of a local orthonormal frame $\{e_i\}$ of TM) by

$$(1.41) \quad \Delta^F := - \sum_{i=1}^{2n} \left((\nabla_{e_i}^F)^2 - \nabla_{\nabla_{e_i}^{TM} e_i}^F \right),$$

where ∇^{TM} is the Levi-Civita connection on (TM, g^{TM}) . Moreover, $\Delta^F = (\nabla^F)^* \nabla^F$, where the adjoint of ∇^F is taken with respect to dv_M (cf. [19, (1.3.19), (1.3.20)]).

1.3.2. *Notational appendix.*

- (M, J) is a complex manifold with complex structure J .
- g^{TM} is a J -invariant Riemannian metric on M , $\Theta(\cdot, \cdot) := g^{TM}(J\cdot, \cdot)$.
- dv_M is the Riemannian volume form of (TM, g^{TM}) .
- (L, h^L, ∇^L) is a holomorphic line bundle with Hermitian metric h^L and Chern connection ∇^L ; (E, h^E, ∇^E) is a holomorphic vector bundle E with Hermitian metric h^E and Chern connection ∇^E on M .

- $\omega := \frac{\sqrt{-1}}{2\pi} R^L$; $\dot{R}^L \in \text{End}(T^{(1,0)}M)$ is the associated Hermitian endomorphism (1.2); the derivation ω_d and the trace $\tau(x)$ of R^L are defined in (1.3).
- Dolbeault-Dirac operator $D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *})$.
- Kodaira Laplacian $\square_p = \frac{1}{2}D_p^2 = \bar{\partial}^{L^p \otimes E} \bar{\partial}^{L^p \otimes E, *} + \bar{\partial}^{L^p \otimes E, *} \bar{\partial}^{L^p \otimes E}$.
- \square_p^q is the resctriction of \square_p to $\Omega^{0,q}(M, L^p \otimes E)$.
- ∇_p is the Chern connection of (L^p, h^{L^p}) . The Bochner Laplacian Δ^{L^p} on L^p is $\Delta^{L^p} = \nabla_p^* \nabla_p$ where $*$ is taken with respect to dv_M .
- $X_h = X = \partial D_h^*$ where $D_h^* \subset L^*$ is the unit co-disc bundle. Then Δ^{L^p} can be identified with the horizontal Laplacian Δ_H on X .

2. Heat kernels on the principal S^1 bundle

In this section, we prepare for the proof of Theorem 1.2 by reviewing the geometry of the principal S^1 bundle $X_h \rightarrow M$ associated to a Hermitian holomorphic line bundle. The geometry is discussed in more detail in [9, 10, 20] but mainly under the assumption that (L, h^L) is a positive Hermitian line bundle. We do not make this assumption in Theorem 1.2.

As above, we denote by (M, Θ) a compact complex n -manifold with a Hermitian metric Θ and then $dv_M = \frac{\Theta^n}{n!}$ is its Riemannian volume form. We consider a Hermitian holomorphic line bundle $(L, h^L) \rightarrow M$ with curvature R^L . Denote the eigenvalues of \dot{R}^L relative to Θ by $\alpha_1(x), \dots, \alpha_n(x)$ (cf. (1.12)).

Denote by L^* the dual line bundle and let D_h^* be the unit disc bundle of L^* with respect to the dual metric h^{L^*} . The boundary $X = \partial D_h^*$ is then a principal S^1 bundle $\pi : X \rightarrow M$ over M , and we denote the S^1 action by $e^{i\theta} \cdot x$. We may express the powers L^p of L as $L^p = X \times_{\chi_p} \mathbb{C}$ where $\chi_p(e^{i\theta}) = e^{ip\theta}$. Sections $L^2(M, L^p)$ of L^p can be naturally identified with the space $L_p^2(X)$ equivariant functions on X transforming by $e^{ip\theta}$ under the S^1 action on X .

REMARK 2.1. The condition that (L, h^L) be a positive line bundle is equivalent to the condition that D_h^* be a strictly pseudo-convex domain in L^* , or equivalently that X_h be a strictly pseudo-convex CR manifold. These assumptions are used in [10] to construct parametrices for the Szegö kernel, but are not necessary to construct parametrices for the heat kernels.

2.1. Geometry on a circle bundle. The Hermitian metric h^L on L induces a connection $\beta = (h^L)^{-1} \partial h^L$ on the S^1 bundle $X \rightarrow M$, which is invariant under the S^1 action and satisfies $\beta(\frac{\partial}{\partial \theta}) = 1$. Here, $\frac{\partial}{\partial \theta}$ denotes the generator of the action. We denote by $V_x = \mathbb{R} \frac{\partial}{\partial \theta}$ its span in $T_x X$ and by $H_x = \ker \beta$ its horizontal complement. We further equip X with the *Kaluza-Klein bundle metric* G , defined by declaring the vertical and horizontal spaces orthogonal, by equipping the vertical space with the S^1 invariant metric and by equipping the horizontal space with the lift of the metric g^{TM} . More precisely, it is defined by the conditions: $H \perp V$, $\pi_* : (H_x, G_x) \rightarrow (T_{\pi(x)}M, g_{\pi(x)}^{TM})$ is an isometry and $|\frac{\partial}{\partial \theta}|_G = 1$. The volume form of the Kaluza-Klein metric on X will be denoted by dv .

Set $H_x^* := V_x^o$, and $V_x^* = H_x^o$. Here, F^o denotes the annihilator of a subspace $F \subset E$ in the dual space E^* . By definition we have $V_x^* = \mathbb{R}\beta_x$. The vertical, resp. horizontal components of $\nu \in T_x^*X$ are given by:

$$(2.1) \quad \nu_V := \left\langle \nu, \frac{\partial}{\partial \theta} \right\rangle \beta_x, \quad \nu_H = \nu - \nu_V.$$

There also exists a natural pullback map $\pi^* : T^*M \rightarrow T^*X$. It is obvious that $\pi^*T_x^*M \subset V_x^o$ and as the two sides have the same dimension we see that $\pi^*T_x^*M = H_x^*X$. We write π_* with a slight abuse of notation for the inverse map $\pi_* : H_x^*X \rightarrow T_x^*M$.

This duality can also be defined by the metric G , which induces isomorphisms $\tilde{G}_x : T_xX \rightarrow T_x^*X$, $\tilde{G}(X) = G(X, \cdot)$. We note that by definition of G , $\beta_x = G(\frac{\partial}{\partial \theta}, \cdot)$ hence $\tilde{G} : V_x \rightarrow V_x^*$. Similarly, $\tilde{G}_x : H_x \rightarrow H_x^*$.

2.2. Distance function on X . To construct a parametrix for the heat kernel, we will need a formula for the square $d^2(x, y)$ of the Kaluza-Klein geodesic distance function on a neighborhood of the diagonal of $X \times X$.

We first describe the geodesic flow and exponential map of (X, G) . It is convenient to identify $TX \equiv T^*X$ as above, and to consider the (co-)geodesic flow on T^*X . This is the Hamiltonian flow of the metric function $|\xi|_G^2 = |\xi_H|_G^2 + |\xi_V|^2$. We note that $|\xi_V|^2 = \langle \xi, \frac{\partial}{\partial \theta} \rangle^2$ and that $|\xi_H|_G^2 = |\pi_*\xi_H|_{g_{TM}}^2$. Henceforth we put $p_\theta(x, \xi) := \langle \xi, \frac{\partial}{\partial \theta} \rangle$. The Hamiltonian flow of p_θ is the lift to T^*X of the S^1 action on X , i.e., $V^u(x, \xi) = (e^{iu}x, e^{iu}\xi)$ where $e^{it}\xi$ denotes derived action of S^1 on X . We also denote the Hamiltonian flow of $|\xi_H|_G^2$ by $G_H^t(x, \xi)$.

Let $U \subset M$ denote a trivializing open set for $X \rightarrow M$, and let $\mu : U \rightarrow X$ denote a local unitary frame. Also, let z_i, \bar{z}_i denote local coordinates in U . Together with μ they induced local coordinates (z_i, \bar{z}_i, θ) on $\pi^{-1}(U) \sim U \times S^1$ defined by $x = e^{i\theta} \cdot \mu(z, \bar{z})$. Thus, z_i is the pull-back π^*z_i while θ depends on a slice of the S^1 action. They induce local coordinates $(z, p_z, \theta, p_\theta)$ on $T^*(\pi^{-1}(U)) \subset T^*X$ by the rule $\eta_x = p_z dz + p_\theta d\theta$. Here we simplify the notation (z, \bar{z}) for local coordinates to z until we need to emphasize the complex structure. As a pullback, $dz \in H_z^*$ and the notation $p_z dz$ for a form η on M and its pullback to X are compatible. We also note that the canonical symplectic form σ on T^*X is given by $\sigma = dz \wedge dp_z + d\theta \wedge dp_\theta$. Hence the Hamiltonian vector field Ξ_{p_θ} of p_θ is given by $\Xi_{p_\theta} = \frac{\partial}{\partial \theta}$.

It follows that the Poisson bracket $\{|\xi_H|_G, p_\theta\} = \frac{\partial}{\partial \theta}|\xi_H|_G = 0$, so p_θ is a constant of the geodesic motion and $\tilde{G}^u = \tilde{G}_H^u \circ V^u = V^u \circ \tilde{G}_H^u$. In local coordinates the Hamiltonian $|\xi|_G^2$ has the form

$$(2.2) \quad |\xi|_G^2 = H(z, p_z, p_\theta) + p_\theta^2$$

and the equations of the Hamiltonian flow of $|\xi|_G^2$ are:

$$(2.3) \quad \begin{cases} \dot{z}_j = \partial H / \partial p_z^j \\ \dot{p}_{z_k} = \frac{\partial}{\partial z_k} H \\ \dot{\theta} = 2p_\theta \\ \dot{p}_\theta = 0. \end{cases}$$

These equations decouple and we see that

$$(2.4) \quad G^u(z, p_z, \theta, p_\theta) = G_H^u(z, p_z, \theta, p_\theta) \cdot e^{i2up_\theta}.$$

Let $d_M(x, y)$ be the distance on M from $\pi(x)$ to $\pi(y)$. We claim:

LEMMA 2.2. *In a neighborhood of the diagonal in $X \times X$, the distance function satisfies the identity*

$$(2.5) \quad d(x, e^{i\theta_0}x) = \theta_0^2$$

Indeed, by definition $d(x, y)^2 = |\xi|_G^2 = |\xi_H|^2 + p_\theta^2$ where $\exp_x \xi = y$. The identity holds because the fiber is totally geodesic.

2.3. Volume density of the Kaluza-Klein metric. The heat kernel parametrix also involves the volume form of (X, G) . Hence we consider Jacobi fields and volume distortion under the geodesic flow. Following [3, Section 5.1], we define

$$(2.6) \quad J(x, a) : T_x X \rightarrow T_{\exp_x a} X$$

as the derivative $d_a \exp_x$ of the exponential map of the Kaluza-Klein metric at $a \in T_x X$. We identify $H_x \equiv T_x M$ and $V_x \equiv \mathbb{R}$. For completeness we sketch the proof.

PROPOSITION 2.3 ([3, Theorem 5.4]). *Let $a = a \frac{\partial}{\partial \theta}$. The map $J(x, a)$ preserves the subspaces H_x and V_x . Moreover*

$$J(x, a)|_{H_x} = \frac{1 - e^{-\tau(\omega_x \cdot a)/2}}{\tau(\omega_x \cdot a)/2},$$

$$J(x, a)|_{V_x} = \text{Id}.$$

Here, τ is defined as in the proof of [3, Proposition 5.1]. Namely, for a vector space V , $\tau : \bigwedge^2 V \simeq so(V)$ is defined by

$$(2.7) \quad \langle \tau(\alpha)e_i, e_j \rangle = 2\langle \alpha, e_i \wedge e_j \rangle.$$

PROOF. We need to compute $\frac{d}{du}|_{u=0} \exp_x(a + u\xi_H)$. Let $Y(s) = \frac{d}{du}|_{u=0} \exp_x s(a + u\xi_H)$. Then $Y(s)$ is a Jacobi field along $\exp_x(sa)$ and $J(x, a)\xi_H = Y(1)$. Hence

$$\frac{D^2}{ds^2} Y + R(T, Y)T = 0$$

where $T = \frac{\partial}{\partial s}$ is the tangent vector to $\exp_x(sa)$. It is easy to see that $Y(s)$ is horizontal, hence that

$$R(T, Y)T = \frac{1}{4}\tau(\omega_x \cdot a)^2 Y.$$

Using a parallel horizontal frame along $\exp_x(sa)$ we identify $Y(s)$ with a curve in H_x and find that

$$\left(\frac{\partial}{\partial s} + \tau(\omega \cdot a)/4\right)^2 y(s) = (\tau(\omega \cdot a)/4)^2 y(s).$$

The formula follows. □

We then have

COROLLARY 2.4 ([3, Corollary 5.5]). *We have*

$$\det J(x, a) = j_H(\tau(\omega_x \cdot a)/2), \text{ where } j_H(A) = \det\left(\frac{\sinh(A/2)}{A/2}\right).$$

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2 for the heat kernels associated to the Bochner Laplacians by $\Delta^{L^p} = \nabla_p^* \nabla_p$ on L^p where $*$ is taken with respect to dv_M . As discussed in the introduction, we use (1.18) to analyze heat kernels on L^p by identifying them as Fourier coefficients of heat kernels on X .

There is a natural lift of sections of L^p to equivariant functions on X . Under the identification $L^2(M, L^p) \simeq L^2_p(X)$, Δ^{L^p} corresponds to the horizontal Laplacian $\Delta_H = d_H^* d_H$, where d_H is the horizontal differential on X . We then add the vertical Laplacian to define the (Kaluza-Klein) metric Laplacian $\Delta^X = (\frac{\partial}{\partial \theta})^2 + \Delta_H$ on X .

Since $[(\frac{\partial}{\partial \theta})^2, \Delta_H] = 0$ we have

$$(3.1) \quad e^{-u\Delta^{L^p}} = e^{u(\frac{\partial}{\partial \theta})^2} e^{-u\Delta^X}.$$

This shows that the equation (1.18) is correct, i. e., if $\pi(x) = z$,

$$(3.2) \quad \exp\left(-\frac{u}{p}\Delta^{L^p}\right)(z, z) = e^{up} \int_{S^1} e^{-(u/p)\Delta^X}(e^{i\theta}x, x)e^{-ip\theta} d\theta.$$

REMARK 3.1. The purpose of adding the vertical term $(\frac{\partial}{\partial \theta})^2$ is that there exists a simple parametrix for $e^{-u\Delta^X}$. Without adding the vertical term, the horizontal heat kernel is much more complicated and reflects the degeneracies of the horizontal curvature. The model case of the Heisenberg sub-Laplacian only applies directly when the line bundle is positive.

The Fourier formula (3.2) shows that from a spectral point of view, the addition of $(\frac{\partial}{\partial \theta})^2$ is harmless. But from a Brownian motion point of view it is drastic and it is responsible for the necessity of analytically continuing the heat kernel to L^* . For further remarks see §3.7.

3.1. Parametrics for heat kernels. We will use the construction in [3] of a parametrix for the heat kernel on a principal S^1 bundle $\pi : X \rightarrow M$. For remainder estimates, we use the off-diagonal estimates of Kannai [17], which apply to these and more general heat kernel constructions.

To get oriented, let us review the general Hadamard (-Minakshisundaram-Pleijel) parametrix construction on a general Riemannian manifold. Let (Y, g) denote a complete Riemannian manifold of dimension m and let Δ_g denote its (positive) Laplacian. For x close enough to y , the heat kernel of $e^{-u\Delta_g}$ has an asymptotic expansion as $u \rightarrow 0$ of the form

$$(3.3) \quad e^{-u\Delta_g}(x, y) \sim \frac{1}{(4\pi u)^{m/2}} e^{-d(x,y)^2/4u} \sum_{j=0}^{\infty} v_j(x, y)u^j$$

where $d(x, y)$ denotes the distance between x and y and where v_j are coefficients satisfying certain well-known transport equations. More precisely, choose a cut-off function $\psi(d(x, y)^2)$, equal one in a neighborhood U of the diagonal. Then there exist smooth locally defined functions $v_j(x, y)$ such that

$$(3.4) \quad H_M(u, x, y) := \psi(d(x, y)^2) \frac{1}{(4\pi u)^{m/2}} e^{-d(x,y)^2/4u} \sum_{j=0}^M v_j(x, y)u^j,$$

is a parametrix for $e^{-u\Delta_g}(x, y)$, i. e., in U we have

$$(3.5) \quad \left(\frac{\partial}{\partial u} + \Delta_g\right)H_M(u, x, y) = \psi(d(x, y)^2) \frac{1}{(4\pi u)^{m/2}} e^{-d(x, y)^2/4u} \Delta_g v_M(x, y) u^M.$$

It follows by the off-diagonal estimates of Kannai ([17, (3.11)]) that

$$(3.6) \quad e^{-u\Delta_g}(x, y) - H_M(u, x, y) = O(u^{M-m/2}) e^{-d(x, y)^2/4u}$$

for (u, x, y) satisfying: $u < \text{inj}(x)$ and $y \in B(x, \text{inj}(x))$. Here, $\text{inj}(x)$ is the injectivity radius at x and $B(x, r)$ is the geodesic ball of radius r centered at x .

3.2. Heat kernels on S^1 bundles and \mathbb{R} bundles. We apply the heat kernel parametrix construction to the principal S^1 bundle $X = X_h \rightarrow M$ equipped with the Kaluza-Klein metric. As a special case where $M = \text{pt}$, this gives a heat kernel parametrix on the circle S^1 . It is standard to express the heat kernel on S^1 as the projection over \mathbb{Z} of the heat kernel on \mathbb{R} , since the distance squared is globally defined on \mathbb{R} but not on S^1 . It thus simplifies the analysis of the distance function to consider the principal \mathbb{R} bundle $\tilde{\pi} : \tilde{X} \rightarrow M$, where \tilde{X} is the fiberwise universal cover of X . We thus express the heat kernel on X as the projection to X of the heat kernel $e^{-u\Delta_{\tilde{X}}}(\tilde{x} + i\theta, \tilde{x})$ on \tilde{X} , where \tilde{X} is equipped with the Kaluza-Klein metric, so that $p : \tilde{X} \rightarrow X$ is a Riemannian \mathbb{Z} -cover. We use additive notation for the \mathbb{R} action on the fiber of \tilde{X} , i. e., in place of the S^1 action $e^{i\theta} \cdot x$ on X we write $\tilde{x} + i\theta$ on \tilde{X} .

A key point in the formula (3.2) is that in Theorem 1.2 we only use the heat kernel at points (x, y) where $\pi(x) = \pi(y)$. The same is true when we lift to \tilde{X} . Although we need to construct a heat kernel parametrix off the diagonal, it is only evaluated at such off-diagonal points. Hence it is sufficient to use a base cut-off, of the form $\psi(d_M(x, y)^2)$ where as above, $d_M(x, y)$ is the distance on M from $\pi(x)$ to $\pi(y)$. This cutoff is identically equal to one on points (x, y) on the same fiber.

The following proposition is adapted from [3, Theorems 2.30].

PROPOSITION 3.2. *There exist smooth functions $\tilde{\Phi}_\ell$ on $\tilde{X} \times i\mathbb{R}$ such that*

$$e^{-u\Delta_{\tilde{X}}}(\tilde{x}, \tilde{x} + i\theta) = (4\pi u)^{-(n+1/2)} \sum_{\ell=0}^M u^\ell \tilde{\Phi}_\ell(\tilde{x}, i\theta) \tilde{j}(\tilde{x}, i\theta)^{-1/2} e^{-|\theta|^2/4u} + R_M(\tilde{x}, i\theta)$$

where $\tilde{j}(\tilde{x}, i\theta)$ is the volume density $j(\tilde{x}, \tilde{y})$ at $\tilde{y} = \tilde{x} + i\theta$ in normal coordinates centered at \tilde{x} and where

$$\begin{aligned} \tilde{\Phi}_0(\tilde{x}, i\theta) &= 1, \\ R_M(\tilde{x}, i\theta) &\ll (4\pi u)^{-(n+1/2+M)} e^{-|\theta|^2/4u}. \end{aligned}$$

PROOF. By [3, Theorem 2.30], and by [17, (3.11)], there exist smooth $v_\ell(\tilde{x}, \tilde{y})$ such that in U ,

$$(3.7) \quad e^{-u\Delta_{\tilde{X}}}(\tilde{x}, \tilde{y}) = H_M(\tilde{x}, \tilde{y}) + R_M(\tilde{x}, \tilde{y}), \quad \text{where}$$

$$H_M(\tilde{x}, \tilde{y}) = (4\pi u)^{-(n+1/2)} \psi(d_M(\tilde{x}, \tilde{y})^2) e^{-d(\tilde{x}, \tilde{y})^2/4u} \sum_{\ell=0}^M u^\ell v_\ell(\tilde{x}, \tilde{y}) j(\tilde{x}, \tilde{y})^{-1/2}$$

with $R_M(\tilde{x}, \tilde{y}) \ll (4\pi u)^{-(n+1/2)} e^{-d(\tilde{x}, \tilde{y})^2/4u} u^{M+1}$. The v_ℓ solve the transport equations,

$$v_\ell(\tilde{x}, \tilde{y}) = - \int_0^1 s^{\ell-1} (B_{\tilde{x}} v_{\ell-1})(\tilde{x}_s, \tilde{y}) ds.$$

Here, $B = j^{\frac{1}{2}} \circ \Delta^{\tilde{X}} j^{-\frac{1}{2}}$. Now put $\tilde{y} = \tilde{x} + i\theta$ and put $\Phi_\ell(\tilde{x}, i\theta) = v_\ell(\tilde{x}, \tilde{x} + i\theta)$. The stated estimate follows from the off-diagonal estimates (3.6) of [17]. \square

We now project the heat kernel on \tilde{X} to X to get:

PROPOSITION 3.3. *The heat kernel on X is given by*

$$(3.8) \quad e^{-u\Delta^X}(x, e^{i\theta}x) = \sum_{n \in \mathbb{Z}} e^{-u\Delta^{\tilde{X}}}(\tilde{x}, \tilde{x} + i\theta + in).$$

Here, $p(\tilde{x}) = x$. Moreover,

$$(3.9) \quad e^{-u\Delta^X}(x, e^{i\theta}x) = H_M(u, x, e^{i\theta}x) + R_M(x, \theta), \quad \text{where}$$

$$H_M(u, x, e^{i\theta}x) = (4\pi u)^{-(n+1/2)} \times$$

$$\left[\sum_{\ell=0}^M \sum_{n \in \mathbb{Z}} e^{-(\theta+n)^2/4u} u^\ell \tilde{\Phi}_\ell(\tilde{x}, i\theta + in) \tilde{j}(\tilde{x}, i\theta + in)^{-1/2} \right],$$

and where for \tilde{x} with $\pi(\tilde{x}) = x$,

$$R_M(\tilde{x}, i\theta) \ll (4\pi u)^{-(n+1/2+M)} \sum_{n \in \mathbb{Z}} e^{-|\theta+n|^2/4u}.$$

PROOF. Both statements hold because $\tilde{X} \rightarrow X$ is locally isometric, and therefore the heat kernel and parametrix on X are Poincaré series in those on \tilde{X} . \square

3.3. Stationary phase calculation of the asymptotics. We now use the heat kernel parametrix (3.9) to calculate the scaled heat kernel asymptotics by the stationary phase method. Our calculation of the coefficients is based on Theorem 5.8 of [3]. We therefore rewrite Proposition 3.3 in the form stated there.

REMARK 3.4. The notation in [3] for the ‘Hadamard’ coefficients Φ_j is somewhat different in their Theorems 2.26 and 5.8. In the latter, the volume half-density factor in the $q_t(x, y)$ factor in [3, Theorem 2.26] is absorbed into the Φ_j of [3, Theorem 5.8], and therefore $\Phi_0(y, y)$ changes from I to $\det^{-\frac{1}{2}}(J(x, a))$ in [3, Theorem 5.8]. Since we are using their computation of the heat kernel expansion, we follow their notational conventions.

We thus rewrite Proposition 3.3 in the form of [3, Theorem 5.8] and combine with (3.2) to obtain,

$$(3.10) \quad \exp\left(-\frac{u}{p} \nabla_p^* \nabla_p\right)(z, z) \sim \left(\frac{p}{4\pi u}\right)^{n+\frac{1}{2}} e^{pu} \int_{\mathbb{R}}^{asympt} \sum_{j=0}^{\infty} p^{-j} u^j \Phi_j(x, i\theta) e^{-p\theta^2/4u} e^{-ip\theta} d\theta,$$

where (cf. (2.3))

$$(3.11) \quad \Phi_0(x, a) = (\det J(x, a))^{-\frac{1}{2}}$$

and where $\int_{\mathbb{R}}^{asympt}$ is the notation of [3, (5.5)] for the asymptotic expansion of the integral. The integral is an oscillatory integral with complex phase

$$(3.12) \quad -|\theta|^2/4u - i\theta,$$

with a single non-degenerate critical point at $\theta = -2ui$ with constant Hessian. We would like to apply the method of stationary phase to the integral, but the critical point is complex and in particular does not lie in the contour of integration. This is not surprising: the integral must be exponentially decaying to balance the factor of e^{pu} in front of it. Therefore, we must deform the contour to $|z| = 2t$. However, then we no longer have the heat kernel in the real domain, but rather the analytic continuation of the heat kernel of $e^{-t\Delta^X}$ in the fiber direction. Thus we first need to discuss the analytic continuation of the heat kernel in the fiber.

3.4. Analytic continuation of heat kernels. In this section, we analyze the analytic continuation of the kernel $e^{-u\Delta^X}(e^{i\theta}x, x)$ and its Hadamard parametrix in the $e^{i\theta}$ variable from $S^1 \times X$ to $\mathbb{C}^* \times X$. Despite the fact that the metric h^L is only \mathcal{C}^∞ and not real analytic, the heat kernel always has an analytic continuation in the variable $e^{i\theta}$, as the next Proposition shows.

The S^1 action $e^{i\theta} \cdot x$ on X extends to a holomorphic action of \mathbb{C}^* on L^* which we denote by $e^z \cdot \mu$ for $\mu \in L^*$. When $\mu = x \in X$ we denote it more simply by $e^z \cdot x$. We also write $z = t + i\theta$ with $t = 2u$ when the heat kernel is at time u .

PROPOSITION 3.5. *The kernel $e(u, x, \theta) := e^{-u\Delta^X}(x, e^{i\theta}x)$ on $X \times \mathfrak{g}$ extends for each $(u, x) \in \mathbb{R}^+ \times X$ to an entire function $e(u, x, z) := e^{-u\Delta^X}(x, e^z x)$ on $\mathfrak{g}_{\mathbb{C}}$, with $\mathfrak{g} = \text{Lie}(S^1) \simeq \mathbb{R}$.*

PROOF. This is most easily seen using the Fourier/eigenfunction expansion of the heat kernel,

$$(3.13) \quad e^{-u\Delta^X}(x, y) = \sum_{p \in \mathbb{Z}} \sum_{j=1}^{\infty} e^{-\lambda_{pj}u} \phi_{pj}(x) \bar{\phi}_{pj}(y).$$

Thus

$$(3.14) \quad e(u, x, \theta) = e^{-u\Delta^X}(x, e^{i\theta}x) = \sum_{p \in \mathbb{Z}} \sum_{j=1}^{\infty} e^{ip\theta} e^{-\lambda_{pj}u} |\phi_{pj}(x)|^2,$$

hence the analytic continuation must be given by

$$(3.15) \quad e^{-u\Delta^X}(x, e^z x) = \sum_{p \in \mathbb{Z}} \sum_{j=1}^{\infty} e^{pz} e^{-\lambda_{pj}u} |\phi_{pj}(x)|^2.$$

The only question is whether the sum convergences uniformly to a holomorphic function of z . As above, we write $\Delta^X = \Delta_H + \frac{\partial^2}{\partial \theta^2}$. Since $[\Delta_H, \frac{\partial^2}{\partial \theta^2}] = 0$, the eigenvalues have the form $\lambda_{pj} = p^2 + \mu_{pj}$ with $\{\mu_{pj}\} \subset \mathbb{R}^+$ the spectrum of the horizontal Laplacian Δ_H on the space $L_p^2(X)$. Thus

$$(3.16) \quad e^{-u\Delta^X}(x, e^z x) = \sum_{p \in \mathbb{Z}} e^{pz} e^{-p^2u} e^{-uL^p}(x, x),$$

where $e^{-uL^p}(x, y) := \int_{S^1} e^{-ip\theta} e^{-u\Delta_H}(x, e^{i\theta}x) d\theta$. But

$$|e^{-uL^p}(x, x)| = \left| \int_{S^1} e^{-ip\theta} e^{-u\Delta_H}(x, e^{i\theta}x) d\theta \right| \leq \int_{S^1} e^{-u\Delta_H}(x, e^{i\theta}x) d\theta = e_0(u, x),$$

where $e_0(u, x)$ is a continuous function of (u, x) . The proposition follows from the fact that $\sum_{p \in \mathbb{Z}} e^{pz} e^{-p^2 u}$ converges uniformly on compact sets in $|z|$. \square

3.5. Analytic continuation of parametrices. Next we consider the analytic continuation of the parametrix. We first observe that the connection β extends to L^* by the requirement that it be \mathbb{C}^* invariant. Thus, $T_\ell L^* = H_\ell \oplus V_\ell$ where $V_\ell = \mathbb{C} \frac{\partial}{\partial \lambda}$ where $\lambda \cdot x$ denotes the \mathbb{C}^* action. Using the metric G we may identify TX with T^*X and similarly decompose T^*X and T^*L^* into horizontal and vertical spaces. The vertical space V_ℓ^* is spanned by α_ℓ .

PROPOSITION 3.6. *The fiber distance squared function $d^2(\tilde{x}, \tilde{x} + i\theta)$ admits an analytic extension in θ to \mathbb{C} satisfying*

$$(3.17) \quad d(\tilde{x}, \tilde{x} + i\theta + i\lambda)^2 = (i\theta + i\lambda)^2.$$

Moreover, the Hadamard coefficients $v_j(\tilde{x}, \tilde{x} + i\theta)$ (3.7) admit holomorphic extensions to $i\theta + i\lambda$.

PROOF. As mentioned above, the holomorphic continuation of the S^1 action is the action of \mathbb{C}^* on T^*L^* . The first statement about the distance function is obvious since the distance squared function on the fiber is real analytic (this is why we lifted the heat kernel from X to \tilde{X}).

The second statement is also obvious for $v_0 = 1$. We then prove it for the higher v_k 's inductively, using the formula

$$(3.18) \quad v_{k+1}(x, y) = \int_0^1 s^k B_x v_k(x_s, y) ds$$

The geodesic x_s from x to y stays in the ‘domain of holomorphy’. Moreover, $B_x = j^{\frac{1}{2}} \Delta^X j^{-\frac{1}{2}}$ so it suffices to show that Δ^X admits a fiberwise holomorphic continuation. But clearly, the fiber analytic extension of Δ^X is $\Delta_H + (\lambda \frac{\partial}{\partial \lambda})^2$. Hence $v_{k+1}(x, e^z y)$ is well-defined and holomorphic in z . Since $j \neq 0$ for such (x, y) it possesses a holomorphic square root and inverse. \square

COROLLARY 3.7. *The functions $\Phi_i(x, \theta)$ on $\tilde{X} \times \mathbb{R}$ extend to holomorphic functions $\Phi_i(x, z)$ on $X \times \mathbb{C}$. Since $e^{-u\Delta^X}(x, e^{i\theta} \cdot x)$ and its parametrices admit analytic continuations, it follows that the remainder $R_M(u, x, e^{i\theta} \cdot x)$ admits an analytic continuation.*

The main lemma in the proof of Theorem 1.2 is the following expression for the scaled, analytically continued heat kernel. Note that the parameter u appears twice: once as the time parameter and once as the dilation factor in L^* .

LEMMA 3.8.

$$e^{-(u/p)\Delta^{\tilde{X}}}(\tilde{x}, \tilde{x} + \theta - 2iu) = C_n \left(\frac{p}{u}\right)^{\frac{\dim X}{2}} e^{p(i\theta+2u)^2/4u} \sum_{k=0}^M \left(\frac{u}{p}\right)^k \tilde{\Phi}_k(\tilde{x}, i\theta + 2u) + R_M(u/p, \tilde{x}, i\theta - 2iu + \tilde{x})$$

where $R_M(u/p, \tilde{x}, i\theta - 2iu + \tilde{x}) = O\left(e^{-p\theta^2/u} \left(\frac{u}{p}\right)^{-\frac{\dim X}{2} + M} e^{pu}\right)$.

PROOF. We first consider the remainder in the real domain. For simplicity we write points of \tilde{X} as x rather than \tilde{x} . The first goal is to obtain a Duhamel type formula (see (3.26)) for R_M . The derivation of this formula is valid for Laplacians

on all Riemannian manifolds (X, G) , and we therefore use the general notation Δ for a Laplacian and H_M for the Mth Hadamard parametrix.

We first note that the remainder

$$(3.19) \quad R_M(u, x, y) := e^{-u\Delta}(x, y) - H_M(u, x, y)$$

solves the initial value problem

$$(3.20) \quad \begin{cases} (\frac{\partial}{\partial u} - \Delta)R_M(u, x, y) = A_M(u, x, y) + B_M(d\psi, x, y), \\ R_M(0, x, y) = 0. \end{cases}$$

where

$$(3.21) \quad A_M(u, x, y) = (4\pi u)^{-\left(\frac{\dim X}{2} + M\right)} \psi(d_M(x, y)^2) e^{-d(x, y)^2/4u} j(x, y)^{-1/2} \Delta_x u_M(x, y)$$

and where $B_M(d\psi, x, y)$ is the sum of the terms in which at least one derivative falls on ψ . Put

$$G(u, x, y) = \phi(d_M(x, y)^2) \frac{1}{(4\pi u)^{\frac{\dim X}{2}}} e^{-d(x, y)^2/4u} j(x, y)^{-\frac{1}{2}}$$

where ϕ is supported in a neighborhood of the diagonal, with $\phi \equiv 1$ on $\text{supp} \psi$, and put

$$(3.22) \quad R_M(u, x, y) = G(u, x, y) S_M(u, x, y).$$

The equation for R_M then becomes

$$(3.23) \quad \begin{cases} G(u, x, y)^{-1} \left(\frac{\partial}{\partial u} - \Delta^X \right) G(u, x, y) S_M(u, x, y) \\ \quad = u^M (\psi(d_M(x, y)^2) \Delta_x^X v_M(x, y) + b_M(d\psi, x)), \\ S_M(0, x, y) = 0. \end{cases}$$

One easily calculates (cf. [3, Proposition 2.24]) that

$$G(u, x, y)^{-1} \left(\frac{\partial}{\partial u} - \Delta^X \right) G(u, x, y) = \frac{\partial}{\partial u} + u^{-1} \nabla_{\mathcal{R}} + j^{\frac{1}{2}} \Delta j^{-\frac{1}{2}}.$$

Here, $\nabla_{\mathcal{R}}$ is the directional derivative along the radial vector field from x .

Multiplying through by u to regularize the equation, and changing variables to $t = \log u$, we get

$$(3.24) \quad \begin{cases} \left(\frac{\partial}{\partial t} + \nabla_{\mathcal{R}} + t B_x \right) S_M(t, x, y) = e^{Mt} (\psi(d_M(x, y)^2) \Delta_x v_M(x, y) + b_M(d\psi, x)), \\ S_M(-\infty, x, y) = 0. \end{cases}$$

The solution is given by

$$(3.25) \quad S_M(u, x, y) = \int_0^u \int_X e^{-(u-s)\Delta^X}(x, a) A_M(s, a, y) dv(a) ds \\ + \int_0^u \int_X e^{-(u-s)\Delta^X}(x, a) B_M(s, a, y) dv(a) ds.$$

We now specialize to \tilde{X} and $\Delta^{\tilde{X}}$ or equivalently X and Δ^X . Our goal is to estimate the analytic continuation of the remainder. When dealing with the parametrix, it is convenient to work on \tilde{X} since its distance-squared function is real

analytic along the fibers. When estimating the remainder R_M it is convenient to work on X because it is compact.

In the case of X we obtain the Duhamel type formula,

$$(3.26) \quad R_M(u, x, e^{i\theta}x) = \int_0^u \int_X e^{-(u-s)\Delta^X}(x, y)A_M(s, y, e^{i\theta}x)dv(y)ds \\ + \int_0^u \int_X e^{-(u-s)\Delta^X}(x, y)B_M(s, y, e^{i\theta}x)dv(y)ds.$$

The same formula is valid on \tilde{X} but there we write the \mathbb{R} action additively.

We observe that since $\psi(d_M(x, y)^2)$ is constant along the fibers of $X \rightarrow M$, both $A_M(s, y, e^{i\theta}x)$ and $B_M(s, y, e^{i\theta}x)$ admit holomorphic continuations in the variable $e^{i\theta}$. As above, we continue to e^z with $z = 2u + i\theta$ for the heat kernel at time u . For instance,

$$(3.27) \quad A_M(u, y, e^z x) \\ = (4\pi u)^{-\left(\frac{\dim X}{2} + M\right)} \sum_{n \in \mathbb{Z}} \psi(d_M(\tilde{x}, \tilde{y} + n)^2) e^{d(\tilde{y}, e^z \tilde{x} + n)^2 / 4u} \Delta_{\tilde{y}}^{\tilde{X}} \\ \times v_M(\tilde{y}, e^z \tilde{x} + n) j(\tilde{y}, e^z \tilde{x} + n)^{-1/2}.$$

It follows that the analytic continuation of $R_M(u, x, e^{i\theta}x)$ may be expressed as

$$(3.28) \quad R_M(u, x, e^z x) = \int_0^u \int_X e^{-(u-s)\Delta^X}(x, y)A_M(s, y, e^z x)dv(y)ds \\ + \int_0^u \int_X e^{-(u-s)\Delta^X}(e^z x, y)B_M(s, y, x)dv(y)ds.$$

Dilating the time variable and setting $z = 2u + i\theta$ gives

$$(3.29) \quad R_M(u/p, x, e^{i\theta+2u}x) = \frac{1}{p} \int_0^u \int_X e^{-((u-s)/p)\Delta^X}(x, y)A_M(s/p, y, e^{i\theta+2u}x)dv(y)ds \\ + \int_0^{u/p} \int_X e^{-((u-s)/p)\Delta^X}(x, y)B_M(s/p, y, e^{i\theta+2u}x)dv(y)ds.$$

The desired estimate on R_M would follow if we could establish that

$$(3.30) \quad (i) \left| \int_0^u \int_X e^{-((u-s)/p)\Delta^X}(x, y)A_M(s/p, y, e^{i\theta+2u}x)dv(y)ds \right| \ll \left(\frac{u}{p}\right)^{M+1} e^{pu}, \text{ and} \\ (ii) \left| \int_0^u \int_X e^{-((u-s)/p)\Delta^X}(x, y)B_M(s/p, y, e^{i\theta+2u}x)dv(y)ds \right| \ll \left(\frac{u}{p}\right)^{M+1} e^{pu}.$$

We establish (3.30 (i)) using the explicit Gaussian formula

$$(3.31) \quad \begin{aligned} A_M(s/p, y, e^{i\theta+2u}x) &= 4\pi \left(\frac{s}{p}\right)^{-\left(\frac{\dim X}{2}+M\right)} \sum_{n \in \mathbb{Z}} e^{pd(y, e^{i\theta+2u}x+n)^2/4s} \\ &\quad \times \Delta_{\tilde{y}}^{\tilde{X}} v_M(\tilde{y}, e^{i\theta+2u}\tilde{x}+n) j(\tilde{y}, e^{i\theta+2u}\tilde{x}+n)^{-1/2} \end{aligned}$$

and the Gaussian upper bound

$$(3.32) \quad e^{-u\Delta^X}(u, x, y) \leq G(u, x, y),$$

of Kannai [17]. They give that (3.29 (i)) is bounded by

$$(3.33) \quad \begin{aligned} &\ll \left(\frac{1}{p}\right)^{-\frac{\dim X}{2}+M+1} \sup_{(x,y) \in X \times X} \left| \Delta_{\tilde{y}}^{\tilde{X}} v_M(\tilde{y}, e^{i\theta+2u}\tilde{x}) \right| \\ &\int_0^u \int_X s^M G((u-s)/p, x, y) |G(s/p, y, e^{-i\theta+2u}x)| dv(y) ds. \end{aligned}$$

Here, $G(s, x, e^{\bar{z}}y) = s^{-\frac{\dim X}{2}} e^{-d(x, e^{\bar{z}}y)^2/4s}$. Its modulus is then equal to $|G(s, x, e^{\bar{z}}y)| = s^{-\frac{\dim X}{2}} e^{-\operatorname{Re} d(x, e^{\bar{z}}y)^2/4s}$. We can asymptotically estimate the resulting integral

$$\int_X \exp\left(-p(d(x, y)^2/4(u-s) + \operatorname{Re} d(x, e^{\overline{i\theta+2u}y})^2/4s)\right) dv(y)$$

by the stationary phase method. We have,

$$(3.34) \quad \operatorname{Re} d(x, e^{i\theta+2t}y)^2 = d(x, e^{i\theta}y)^2 - 4t^2.$$

Hence critical points occur at y such that

$$(3.35) \quad \nabla_y d(x, y)^2/4(u-s) = -\nabla_y d(y, e^{-i\theta}x)^2/4s.$$

Now $\nabla_y d(x, y)^2$ is tangent to the geodesic from x to y and $\nabla_y d(y, e^{-i\theta}x)^2$ is tangent to the geodesic from y to $e^{-i\theta}x$. Since they are multiples, it follows that y must lie along the minimizing geodesic from x to $e^{-i\theta}x$. This is just the curve $\gamma(u) = e^{iu}x, u \in [0, \theta]$. Moreover, $d(x, y)/(u-s) = d(y, e^{-i\theta}x)/s$. Hence we have

$$u/(u-s) = (\theta-u)/s \iff u\left(\frac{1}{u-s} + \frac{1}{s}\right) = -\frac{\theta}{s} \iff u = -\frac{(u-s)\theta}{u}.$$

Hence the critical locus is given by: $y_s = e^{i\frac{(u-s)}{u}\theta}x$. The value of the phase along the critical locus equals

$$(3.36) \quad \begin{aligned} &d(x, e^{i\frac{(u-s)}{u}\theta}x)^2/4(u-s) + d(x, e^{-i\theta}e^{i\frac{(u-s)}{u}\theta}x)^2/4s - 4u^2/4s \\ &= d(x, e^{i\theta}x)^2/4u - 4u^2/4s = \frac{\theta^2}{u} - \frac{u^2}{s}. \end{aligned}$$

Also, the transverse Hessian of the phase equals $\frac{1}{u-s} + \frac{1}{s} = \frac{u}{s(u-s)}$. Raising it to the power $-\frac{\dim X}{2}$ cancels the factors of $s^{-\frac{\dim X}{2}}$ and $(u-s)^{-\frac{\dim X}{2}}$ and leaves $u^{-\frac{\dim X}{2}}$.

Hence

$$\begin{aligned}
 & \int_0^u \int_X G((u-s)/p, x, y) s^M |G(s/p, y, e^{-i\theta+2u}x)| dv(y) ds \\
 (3.37) \quad & \sim u^{-\frac{\dim X}{2}} e^{-p\theta^2/u} \int_0^u e^{pu^2/s} s^M ds \\
 & \ll u^{-\frac{\dim X}{2}} e^{-p\theta^2/u} e^{pu} u^{M+1}.
 \end{aligned}$$

This completes the proof of Lemma 3.8. □

3.6. Completion of proof of Theorem 1.2. We now complete the proof of Theorem 1.2. We begin with the oscillatory integral (3.9) with complex phase (3.12) with a single non-degenerate critical point at $\theta = -2ui$ and with constant Hessian. Since the critical point is complex, we deform the contour to $|z| = 2t$. Thus, we have (with $z = e^{i\theta} \in S^1$)

$$\begin{aligned}
 e_p^H(u, x) & := e^{up} \frac{1}{2\pi i} \int_{|z|=1} e^{-(u/p)\Delta^X}(zx, x) z^{-p} \frac{dz}{z} \\
 (3.38) \quad & = e^{up} \frac{1}{2\pi i} \int_{|z|=e^{2u}} e^{-(u/p)\Delta^X}(zx, x) z^{-p} \frac{dz}{z} \\
 & = e^{up} \int_0^{2\pi} e^{-(u/p)\Delta^X}(e^{i\theta+2u}x, x) e^{-p(i\theta+2u)} d\theta.
 \end{aligned}$$

We now plug in the Poincaré series formula of Proposition 3.3 and unfold the sum over \mathbb{Z} to get

$$(3.39) \quad e_p^H(u, x) = e^{up} \int_{\mathbb{R}} e^{-(u/p)\Delta^{\bar{X}}}(e^{i\theta+2u}x, x) e^{-p(i\theta+2u)} d\theta.$$

We then substitute the parametrix for $e^{-(u/p)\Delta^{\bar{X}}}(e^{i\theta+2u}x, x)$ with remainder from Lemma 3.8 for $e^{-(u/p)\Delta^X}(e^{i\theta+2u}x, x)$. Using the notation of [3], we obtain,

$$\begin{aligned}
 (3.40) \quad & e_p^H(u, x) \\
 & = \left(\frac{p}{4\pi u}\right)^{n+\frac{1}{2}} e^{pu} \int_{\mathbb{R}}^{asympt} \left(\sum_{\ell=0}^M p^{-\ell} u^\ell \Phi_\ell(x, i\theta + 2u) e^{-p(\theta-2ui)^2/4u} e^{-p(i\theta+2u)} + R_M\right) d\theta \\
 & = \left(\frac{p}{4\pi u}\right)^{n+\frac{1}{2}} \int_{\mathbb{R}}^{asympt} \left(\sum_{\ell=0}^M p^{-\ell} u^\ell \Phi_\ell(x, i\theta + 2u) e^{-p\theta^2/4u} + R_M\right) d\theta.
 \end{aligned}$$

The integral is now a standard Gaussian integral with complex phase (3.12)

$$(3.41) \quad -(\theta - 2ui)^2/4u - (i\theta + 2u),$$

which has a unique critical point on the line of integration at $\theta = 0$. We may neglect the remainder term if we only want to expand to order M in p^{-1} and apply the

method of stationary phase (see [15, Theorem 7.7.5]) to obtain,

(3.42)

$$\begin{aligned}
 e_p^H(u, x) &\sim \left(\frac{p}{4\pi u}\right)^{n+\frac{1}{2}} |p/u|^{-\frac{1}{2}} \sum_{k=0}^{M-1} \sum_{\ell=0}^{\infty} p^{-\ell-k} u^\ell \frac{1}{k!} \left[\frac{\partial}{\partial \theta}\right]^{2k} \Phi_\ell(x, i\theta + 2u) \Big|_{\theta=0} \\
 &= \left(\frac{p}{4\pi u}\right)^n \sum_{k=0}^{M-1} \sum_{\ell=0}^{\infty} p^{-\ell-k} u^\ell \frac{1}{k!} \left[\frac{\partial}{\partial \theta}\right]^{2k} \Phi_\ell(x, i\theta + 2u) \Big|_{\theta=0}.
 \end{aligned}$$

All the properties stated in the theorem follow from standard facts about the stationary phase method and from the properties of the coefficients Φ_j of [3, Theorem 5.8]. By (3.10), the principal term is given by $\left(\frac{p}{4\pi u}\right)^n \Phi_0(x, 2u)$ or equivalently, by using (3.11) and Proposition 2.3,

(3.43)

$$\left(\frac{p}{4\pi u}\right)^n \Phi_0(x, 2u) = \left(\frac{p}{4\pi u}\right)^n (\det J(x, 2u))^{-\frac{1}{2}} = \left(\frac{p}{4\pi u}\right)^n \det \left(\frac{1 - e^{-u\tau(\omega_x)}}{u\tau(\omega_x)}\right)^{-\frac{1}{2}}$$

This is compatible with (1.20) because for the determinant of functions of $\tau(\omega_x)$ on TM we have

$$\begin{aligned}
 (3.44) \quad \det \left(\frac{1 - e^{-u\tau(\omega_x)}}{u\tau(\omega_x)}\right)^{-\frac{1}{2}} &= \det \left(\frac{e^{u\tau(\omega_x)/2} - e^{-u\tau(\omega_x)/2}}{u\tau(\omega_x)}\right)^{-\frac{1}{2}} \\
 &= \det \Big|_{T^{(1,0)}M} \left(\frac{u\tau(\omega_x)}{e^{u\tau(\omega_x)/2} - e^{-u\tau(\omega_x)/2}}\right),
 \end{aligned}$$

and because the factor of 2 in (2.7) is not used in the definition of \dot{R}^L . To explain the last equality and to clarify the notation between (3.43)–(3.44) and (1.11), we recall that $\tau(\omega_x) = 2\dot{R}_x^L$. If we diagonalize \dot{R}_x^L as in (1.12) as an element of $\text{End}(TM)$, then under the decomposition $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M$, $\dot{R}_x^L = \text{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)$. Here we extend \dot{R}^L to an element of $\text{End}(TM \otimes_{\mathbb{R}} \mathbb{C})$ by the formula (1.2), so on $T^{(0,1)}M$ we obtain the negative of the eigenvalues on $T^{(1,0)}M$, by the antisymmetry of the 2-form R^L . Hence $\det(e^{-u\tau(\omega_x)/2}) = 1$. We refer [3, p. 152] for a similar calculation.

The subleading term is given by (1.21). This completes the proof of Theorem 1.2.

REMARK 3.9. In the Kähler case and for a quantum line bundle L , i.e., if $\Theta = \omega = \frac{\sqrt{-1}}{2\pi} R^L$, then a precise formula for $e_{\infty 1}(u, x)$ in terms of curvature was obtained by Dai, Liu and Ma in [14, (5.14)]:

$$\begin{aligned}
 (3.45) \quad e_{\infty 1}(u, x) &= \frac{-u^{n-1}}{3(1 - e^{-4\pi u})^n} \left[\frac{u}{2} - \frac{u}{2 \tanh^2(2\pi u)} \right. \\
 &\quad \left. - \frac{2}{\sinh^2(2\pi u)} \left(\frac{-3}{32\pi} \sinh(4\pi u) + \frac{u}{8} \right) \right] r_x^M,
 \end{aligned}$$

with r^M the scalar curvature of (M, g^{TM}) . If Θ and (L, h^L) are arbitrary, a corresponding formula should follow from (1.21) or from an adaptation of [14, (5.14)], but it is certainly more involved than (3.45). For the calculation of the second

coefficient of the expansion of the Bergman kernel for non-positive line bundles see [18].

3.7. Further remarks. The method of completing the square to convert the horizontal Laplacian to the full Laplacian on X is quite drastic because it replaces the horizontal Brownian motion of the original problem with the free Brownian motion on X . It is a natural question to ask if one can improve Theorem 1.2 if one has parametrices for the horizontal heat kernels. The rest of the argument would apply.

In certain model cases, Beals-Greiner-Gaveau construct parametrices for heat kernels of sub-Laplacians [1, 2]. In the case of a positive line bundle, there should exist a parametrix locally modeled on that of the Heisenberg group, although we are not aware of a construction at this level of generality. Even so it would not be useful for the main problems of this article, i.e., for Hermitian line bundles which are not positive. In the case of a positive line bundle one can construct a parametrix for the Szegő kernel directly (see [10]; see also [19] for results on the relation between heat kernels and Szegő kernels). In more general cases, it seems that the heat kernels have only rarely been constructed.

In situations where one can construct parametrices for the horizontal heat kernels, it seems plausible that one could gain better control over the u dependence of the remainder term $R_r(p, u, x)$. The original motivation of this article was to investigate whether there exists a joint asymptotic expansion in (u, p) which would allow one to set $u = p^\alpha$ or ideally $u = cp$ in the asymptotics. One observes that the expansion occurs in powers of $\frac{u}{p}$ and this seems to be the natural Planck constant for the problem. In particular, it would be natural to try to respect the Heisenberg scaling in which $\frac{\partial}{\partial \theta}$ is of weight 2. But the coefficients and remainder we obtain by completing the square are not functions of $\frac{u}{p}$, and we have little control over the remainder $R_r(p, u, x)$, which might be of exponential growth (or worse) in u . This reflects the fact that we must analytically continue far out into L^* to make up for the brutal addition of $(\frac{\partial}{\partial \theta})^2$. We would probably not have to continue so far out in L^* if we add the first power $\frac{\partial}{\partial \theta}$ as the Heisenberg scaling would suggest.

4. Proof by localization and rescaling of the Dolbeault-Dirac operator

Before going further let us recall some differential-geometric notions. Let ∇^{TM} be the Levi-Civita connection on TM and $\tilde{\nabla}^{TM}$ the connection on TM defined by $\tilde{\nabla}^{TM} = \nabla^{T^{(1,0)}M} \oplus \nabla^{T^{(0,1)}M}$, where $\nabla^{T^{(1,0)}M}$ is the Chern connection on $T^{(1,0)}M$ and $\nabla^{T^{(0,1)}M}$ is its conjugate (see [19, (1.2.35)]). We set $S = \tilde{\nabla}^{TM} - \nabla^{TM}$.

We denote by ∇^B the Bismut connection [19, (1.2.61)] on TM . It preserves the complex structure on TM by [19, Lemma 1.2.10], thus, as in [19, (1.2.43)], it induces a natural connection ∇^B on $\Lambda(T^{*(0,1)}M)$ which preserves the \mathbb{Z} -grading. Let $\nabla^{B, \Lambda^{0, \bullet}}$, $\nabla^{B, \Lambda^{0, \bullet} \otimes L^p \otimes E}$ be the connections on $\Lambda(T^{*(0,1)}M)$, $\Lambda(T^{*(0,1)}M) \otimes L^p \otimes E$, defined by

$$(4.1) \quad \begin{aligned} \nabla^{B, \Lambda^{0, \bullet}} &= \nabla^B + \langle S(\cdot)w_j, \bar{w}_j \rangle, \\ \nabla^{B, \Lambda^{0, \bullet} \otimes L^p \otimes E} &= \nabla^{B, \Lambda^{0, \bullet}} \otimes 1 + 1 \otimes \nabla^{L^p \otimes E}, \end{aligned}$$

where $\{w_j\}_{j=1}^n$ is a local orthonormal frame of $T^{(1,0)}M$ (cf. [19, (1.4.27)]).

Let Φ_E be the smooth self-adjoint section of $\text{End}(\Lambda(T^{*(0,1)}M) \otimes E)$ on M defined by

$$(4.2) \quad \Phi_E = \frac{1}{4}r^M + {}^c(R^E + \frac{1}{2}R^{\det}) + \frac{\sqrt{-1}}{2} {}^c(\bar{\partial}\partial\Theta) - \frac{1}{8}|(\partial - \bar{\partial})\Theta|^2,$$

cf. [19, (1.3.32), (1.6.20)]. The endomorphism Φ_E appears as the difference between the Bochner Laplacian (cf. (1.41)) associated to the Bismut connection $\nabla^{B,\Lambda^{0,\bullet} \otimes L^p \otimes E}$ and the Dirac operator, cf. [19, Theorem 1.4.7]:

$$(4.3) \quad D_p^2 = \Delta^{B,\Lambda^{0,\bullet} \otimes L^p \otimes E} + \Phi_E + p {}^cR^L.$$

We start by noting the following analogue of [19, Proposition 1.6.4]. Let $f : \mathbb{R} \rightarrow [0, 1]$ be a smooth even function such that

$$(4.4) \quad f(v) = \begin{cases} 1 & \text{for } |v| \leq \varepsilon/2, \\ 0 & \text{for } |v| \geq \varepsilon. \end{cases}$$

For $u > 0, a \in \mathbb{C}$, set

$$(4.5) \quad \begin{aligned} \mathbf{F}_u(a) &= \int_{-\infty}^{+\infty} e^{iva} \exp\left(-\frac{v^2}{2}\right) f(\sqrt{uv}) \frac{dv}{\sqrt{2\pi}}, \\ \mathbf{G}_u(a) &= \int_{-\infty}^{+\infty} e^{iva} \exp\left(-\frac{v^2}{2}\right) (1 - f(\sqrt{uv})) \frac{dv}{\sqrt{2\pi}}. \end{aligned}$$

The functions $\mathbf{F}_u(a), \mathbf{G}_u(a)$ are even holomorphic functions. The restrictions of $\mathbf{F}_u, \mathbf{G}_u$ to \mathbb{R} lie in the Schwartz space $\mathcal{S}(\mathbb{R})$. Clearly,

$$(4.6) \quad \mathbf{F}_u(vD_p) + \mathbf{G}_u(vD_p) = \exp\left(-\frac{v^2}{2} D_p^2\right).$$

For $x, x' \in M$ let $\mathbf{F}_u(vD_p)(x, x'), \mathbf{G}_u(vD_p)(x, x')$ be the smooth kernels associated to $\mathbf{F}_u(vD_p), \mathbf{G}_u(vD_p)$, calculated with respect to the Riemannian volume form $dv_M(x')$. Let $B^M(x, \varepsilon)$ be the open ball in M with center x and radius ε .

PROPOSITION 4.1. *For any $m \in \mathbb{N}, T > 0, \varepsilon > 0$, there exists $C > 0$ such that for any $x, x' \in M, p \in \mathbb{N}^*, 0 < u < T$,*

$$(4.7) \quad \left| \mathbf{G}_{u/p}(\sqrt{u/p} D_p)(x, x') \right|_{\mathcal{C}^m} \leq C \exp\left(-\frac{\varepsilon^2 p}{32u}\right).$$

Here the \mathcal{C}^m norm is induced by $\nabla^L, \nabla^E, \nabla^{B,\Lambda^{0,\bullet}}$ and h^L, h^E, g^{TM} .

The kernel $\mathbf{F}_{u/p}(\sqrt{u/p} D_p)(x, \cdot)$ only depends on the restriction of D_p to $B^M(x, \varepsilon)$, and is zero outside $B^M(x, \varepsilon)$.

This follows from the proof of [19, Proposition 1.6.4], in particular from [19, (1.6.16)] with $\zeta = 1$, since under our assumption any polynomial in p, u^{-1} will be absorbed by the factor $\exp(-\frac{\varepsilon^2 p}{32u})$. The second assertion of follows by using (4.5), the finite propagation speed of the wave operator, cf. [19, Theorem D.2.1 and (D.2.17)].

Thus the problem on the asymptotic expansion of $\exp(-\frac{u}{p} D_p^2)(x, x)$, for $0 < u < T$ and $p \in \mathbb{N}$, is a local problem and only depends on the restriction of D_p to $B^M(x, \varepsilon)$.

To analyze the local problem, we fix $x_0 \in M$ and work on $M_0 := \mathbb{R}^{2n} \simeq T_{x_0} M$. From now on, we identify $B^{T_{x_0} M}(0, 4\varepsilon)$ with $B^M(x_0, 4\varepsilon)$ by the exponential map. For $Z \in B^{T_{x_0} M}(0, 4\varepsilon)$, we identify

$$E_Z \cong E_{x_0}, \quad L_Z \cong L_{x_0}, \quad \Lambda(T_Z^{*(0,1)} M) \cong \Lambda(T_{x_0}^{*(0,1)} M),$$

by parallel transport with respect to the connections $\nabla^E, \nabla^L, \nabla^{B, \Lambda^{0, \bullet}}$ along the curve $[0, 1] \ni u \mapsto uZ$. Thus on $B^M(x_0, 4\varepsilon)$, we have the following identifications of Hermitian bundles

$$(E, h^E) \cong (E_{x_0}, h^{E_{x_0}}), (L, h^L) \cong (L_{x_0}, h^{L_{x_0}}), (\Lambda(T^{*(0,1)}M), h^{\Lambda^{0, \bullet}}) \cong (\Lambda(T_{x_0}^{*(0,1)}M), h^{\Lambda_{x_0}^{0, \bullet}}) \\ (E_p, h_p) \cong (E_{p, x_0}, h^{E_{p, x_0}}),$$

where the bundles on the right-hand side are trivial Hermitian bundles.

Let $\Gamma^E, \Gamma^L, \Gamma^{B, \Lambda^{0, \bullet}}$ be the corresponding connection forms of ∇^E, ∇^L and $\nabla^{B, \Lambda^{0, \bullet}}$ on $B^M(x_0, 4\varepsilon)$. Then $\Gamma^E, \Gamma^L, \Gamma^{B, \Lambda^{0, \bullet}}$ are skew-adjoint with respect to $h^{E_{x_0}}, h^{L_{x_0}}, h^{\Lambda_{x_0}^{0, \bullet}}$.

Let $\rho : \mathbb{R} \rightarrow [0, 1]$ be a smooth even function such that

$$(4.8) \quad \rho(v) = 1 \text{ if } |v| < 2; \quad \rho(v) = 0 \text{ if } |v| > 4.$$

Denote by ∇_U the ordinary differentiation operator on $T_{x_0}M$ in the direction U . From the above discussion,

$$(4.9) \quad \nabla^{E_{p, x_0}} = \nabla + \rho\left(\frac{1}{\varepsilon}|Z|\right)\left(p\Gamma^L + \Gamma^E + \Gamma^{B, \Lambda^{0, \bullet}}\right)(Z),$$

defines a Hermitian connection on $(E_{p, x_0}, h^{E_{p, x_0}})$ on $\mathbb{R}^{2n} \simeq T_{x_0}M$ where the identification is given by

$$(4.10) \quad \mathbb{R}^{2n} \ni (Z_1, \dots, Z_{2n}) \longmapsto \sum_i Z_i e_i \in T_{x_0}M.$$

Here $\{e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j), e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \bar{w}_j)\}_j$ is an orthonormal basis of $T_{x_0}M$.

Let g^{TM_0} be a metric on $M_0 := \mathbb{R}^{2n}$ which coincides with g^{TM} on $B^{T_{x_0}M}(0, 2\varepsilon)$, and $g^{T_{x_0}M}$ outside $B^{T_{x_0}M}(0, 4\varepsilon)$. Let dv_{M_0} be the Riemannian volume form of (M_0, g^{TM_0}) . Let $\Delta^{E_{p, x_0}}$ be the Bochner Laplacian associated to $\nabla^{E_{p, x_0}}$ and g^{TM_0} on M_0 . Set

$$(4.11) \quad \mathbf{L}_{p, x_0} = \Delta^{E_{p, x_0}} - p\rho\left(\frac{1}{\varepsilon}|Z|\right)(2\omega_{d, Z} + \tau_Z) + \rho\left(\frac{1}{\varepsilon}|Z|\right)\Phi_{E, Z}.$$

Then \mathbf{L}_{p, x_0} is a self-adjoint operator with respect to the L^2 scalar product induced by $h^{E_{p, x_0}}, g^{TM_0}$ on M_0 . Moreover, \mathbf{L}_{p, x_0} coincides with D_p^2 on $B^{T_{x_0}M}(0, 2\varepsilon)$. By using (4.3) we obtain the analogue of Proposition 4.1 for $\sqrt{\frac{u}{p}\mathbf{L}_{p, x_0}}$. Thus by using the finite propagation speed for the wave operator we get

$$(4.12) \quad \left| \exp\left(-\frac{u}{2p}D_p^2\right)(x_0, x_0) - \exp\left(-\frac{u}{2p}\mathbf{L}_{p, x_0}\right)(0, 0) \right| \leq C \exp\left(-\frac{\varepsilon^2 p}{32u}\right).$$

Let dv_{TM} be the Riemannian volume form on $(T_{x_0}M, g^{T_{x_0}M})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$(4.13) \quad dv_{M_0}(Z) = \kappa(Z)dv_{TM}(Z),$$

with $k(0) = 1$.

Set $\mathbf{E}_{x_0} := (\Lambda(T^{*(0,1)}M) \otimes E)_{x_0}$. For $s \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$, $Z \in \mathbb{R}^{2n}$ and $t = \frac{\sqrt{u}}{\sqrt{p}}$, set

$$(4.14) \quad (S_t s)(Z) = s(Z/t), \\ \nabla_{t, u} = S_t^{-1} t \kappa^{1/2} \nabla^{E_{p, x_0}} \kappa^{-1/2} S_t, \\ \mathcal{L}_2^{t, u} = S_t^{-1} \kappa^{1/2} t^2 \mathbf{L}_{p, x_0} \kappa^{-1/2} S_t.$$

Note that in [19, (1.6.27)] we used the scaling parameter $t = \frac{1}{\sqrt{p}}$. In the present situation we wish to obtain an expansion in the variable $\frac{u}{t}$, so we need to rescale the coordinates by setting $t = \frac{\sqrt{u}}{\sqrt{p}}$. Put

$$(4.15) \quad \begin{aligned} \nabla_{0,u,\bullet} &= \nabla_{\bullet} + \frac{u}{2} R_{x_0}^L(Z, \cdot), \\ \mathcal{L}_2^{0,u} &= - \sum_i (\nabla_{0,u,e_i})^2 - 2u\omega_{d,x_0} - u\tau_{x_0}. \end{aligned}$$

Then we have the following analogue of [19, Theorem 4.1.7].

THEOREM 4.2. *There exist polynomials $\mathcal{A}_{i,j,r}$ (resp. $\mathcal{B}_{i,r}, \mathcal{C}_r$) in the variables Z and in u , where $r \in \mathbb{N}, i, j \in \{1, \dots, 2n\}$, with the following properties:*

- *their coefficients are polynomials in R^{TM} (resp. $R^{TM}, R^{B,\Lambda^0,\bullet}, R^E, R^{\det}, d\Theta, R^L$) and their derivatives at x_0 up to order $r - 2$ (resp. $r - 2, r - 2, r - 2, r - 2, r - 1, r$),*
- *$\mathcal{A}_{i,j,r}$ is a homogeneous polynomial in Z of degree r and does not depend on u , the degree in Z of $\mathcal{B}_{i,r}$ is $\leq r + 1$ (resp. the degree of \mathcal{C}_r in Z is $\leq r + 2$), and has the same parity as $r - 1$ (resp. r), the degree in u of $\mathcal{B}_{i,r}$ is ≤ 1 , and the degree in u of \mathcal{C}_r is ≤ 2 ,*
- *if we denote by*

$$(4.16) \quad \mathcal{O}_{u,r} = \mathcal{A}_{i,j,r} \nabla_{e_i} \nabla_{e_j} + \mathcal{B}_{i,r}(u) \nabla_{e_i} + \mathcal{C}_r(u),$$

then

$$(4.17) \quad \mathcal{L}_2^{t,u} = \mathcal{L}_2^{0,u} + \sum_{r=1}^m t^r \mathcal{O}_{u,r} + \mathcal{O}(t^{m+1}),$$

and there exists $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}, t \leq 1, 0 < u < T$, the derivatives of order $\leq k$ of the coefficients of the operator $\mathcal{O}(t^{m+1})$ are dominated by $Ct^{m+1}(1 + |Z|)^{m'}$.

Set $g_{ij}(Z) = g^{TM_0}(e_i, e_j)(Z) = \langle e_i, e_j \rangle_Z$ and let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z))$. We observe that $pt = \frac{u}{t}$, thus the analogue of [19, (4.1.34)] reads

$$(4.18) \quad \begin{aligned} \nabla_{t,u,\bullet} &= \kappa^{1/2}(tZ) \left(\nabla_{\bullet} + t \Gamma_{tZ}^{A_0} + \frac{u}{t} \Gamma_{tZ}^{L_0} + t \Gamma_{tZ}^{E_0} \right) \kappa^{-1/2}(tZ), \\ \mathcal{L}_2^{t,u} &= - g^{ij}(tZ) \left(\nabla_{t,u,e_i} \nabla_{t,u,e_j} - t \Gamma_{ij}^k(tZ) \nabla_{t,u,e_k} \right) \\ &\quad - 2u \omega_{0,d,tZ} - u \tau_{0,tZ} + t^2 \Phi_{E_0,tZ}. \end{aligned}$$

Comparing with [19, (4.1.34)], the term of $\nabla_{t,u,\bullet}$ involving u is $\frac{u}{t} \Gamma_{tZ}^{L_0}$ instead of $\frac{1}{t} \Gamma_{tZ}^{L_0}$ therein.

Theorem 4.2 follows by taking the Taylor expansion of (4.18). Using Theorem 4.2 we see that [19, Theorems 1.6.7–1.6.10] (or more precisely [19, Theorems 4.1.9–4.1.14] with the contour $\delta \cup \Delta$ replaced by the contour Γ from [19, Theorems 1.6.7–1.6.10]) hold uniformly for $0 < u < T$.

Thus we get the following analogue of [19, Theorem 4.2.8] in normal coordinates.

THEOREM 4.3. *There exists $C'' > 0$ such that for any $k, m, m' \in \mathbb{N}$, there exists $C > 0$ such that if $t \in]0, 1], 0 < u < T, Z, Z' \in T_{x_0}M,$*

$$(4.19) \quad \sup_{|\alpha|+|\alpha'| \leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\exp(-\mathcal{L}_2^{t,u}) - \sum_{r=0}^k J_{r,u} t^r \right) (Z, Z') \right|_{\mathcal{C}^{m'}(X)} \leq C t^{k+1} (1 + |Z| + |Z'|)^{M_{k+1,m,m'}} \exp(-C''|Z - Z'|^2).$$

Note that we use the operator $\mathcal{L}_2^{t,u}$ and we rescale the coordinates by the factor $t = \frac{\sqrt{u}}{\sqrt{p}}$, thus the factor u in the right-hand side of the second equation of [19, (4.2.30)] is 1 here. Moreover, we have (cf. also [19, (1.6.61)])

$$(4.20) \quad J_{0,u}(Z, Z') = \exp(-\mathcal{L}_2^{0,u})(Z, Z').$$

We infer from (4.15) (compare [19, (1.6.68)]) that

$$(4.21) \quad \exp(-\mathcal{L}_2^{0,u})(0, 0) = \frac{1}{(2\pi)^n} \frac{\det(u\dot{R}_{x_0}^L) \exp(2u\omega_{d,x_0})}{\det(1 - \exp(-2u\dot{R}_{x_0}^L))} \otimes \text{Id}_E.$$

The analogue of [19, (1.6.66), (4.2.37)] is that for $Z, Z' \in T_{x_0}M,$

$$(4.22) \quad \exp\left(-\frac{u}{p}L_{p,x_0}\right)(Z, Z') = \left(\frac{p}{u}\right)^n \exp(-\mathcal{L}_2^{t,u})\left(\frac{Z}{t}, \frac{Z'}{t}\right) \kappa^{-1/2}(Z)\kappa^{-1/2}(Z').$$

By taking $Z = Z' = 0$ in Theorem 4.3, and using (4.22), we get the analogue of [19, (4.2.39)],

$$(4.23) \quad \left| \left(\frac{u}{p}\right)^n \exp\left(-\frac{u}{p}D_p^2\right)(x_0, x_0) - \sum_{r=0}^k J_{r,u}(0, 0) \left(\frac{p}{u}\right)^{-r/2} \right|_{\mathcal{C}^{m'}(X)} \leq C \left(\frac{p}{u}\right)^{-\frac{k+1}{2}}.$$

Finally, by the same argument as in the proof of [19, (4.2.40)], we get for any $r \in \mathbb{N},$

$$(4.24) \quad J_{2r+1,u}(0, 0) = 0.$$

Relations (4.20)–(4.24) yield Theorem 1.1 with $e_{\infty 0}(u, x_0)$ given by (1.11).

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