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Geometric Quantization Results for Semi-positive Line Bundles on a Riemann Surface

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Abstract

In earlier work (Marinescu and Savale in Math Ann. https://doi.org/10.1007/s00208-023-02750-3, 2023) the authors proved the Bergman kernel expansion for semi-positive line bundles over a Riemann surface whose curvature vanishes to at most finite order at each point. Here we explore the related results and consequences of the expansion in the semi-positive case including: Tian's approximation theorem for induced Fubini-Study metrics, leading-order asymptotics and composition for Toeplitz operators, asymptotics of zeroes for random sections, and the asymptotics of holomorphic torsion.

Keywords Semi-positive line bundle · Bergman kernel expansion · Tian's approximation theorem · Berezin-Toeplitz quantization · Zeros for random holomorphic sections · Holomorphic torsion

Mathematics Subject Classification 53C17 · 58J50 · 32A25 · 53D50

Dedicated to the memory of Nessim Sibony.

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1 Introduction

Geometric quantization is a procedure to relate classical observables (smooth functions) on a phase space (a symplectic manifold) to quantum observables (bounded linear operators) on the corresponding quantum space (sections of a line bundle). In the case when the line bundle in question is positive, and consequently the underlying manifold Kähler, a well-known quantization recipe is that of Berezin-Toeplitz [5, 28, 34]. Showing the validity of the quantization procedure involves proving that it has the right properties in the semiclassical limit. Key to the proof is the analysis of the semiclassical limit of the Bergman kernel [14, 16, 29, 30, 38]. In earlier work [32] the authors proved the Bergman kernel expansion in the case when the underlying line bundle is only semi-positive, with curvature vanishing at finite order at each point, on a Riemann surface. It is the purpose of this article to explore the corresponding applications of the expansion therein to results in geometric quantization in the semi-positive case. These include the Tian's approximation theorem for induced Fubini-Study metrics, leading-order asymptotics and composition for Toeplitz operators, asymptotics of zeroes for random sections, and the asymptotics of holomorphic torsion.

We now state our results more precisely. Let Y^2 be a compact Riemannian surface equipped with an integrable complex structure J and Hermitian metric h^{TY}. Consider holomorphic, Hermitian line and vector bundles $\left(L, h^L\right), \left(F, h^F\right)$ on Y and let ∇^L, ∇^F be the corresponding Chern connections. Denote by $R^L = \left(\nabla^L\right)^2 \in \Omega^2\left(Y; i\mathbb{R}\right)$ the corresponding curvature of the line bundle. The order of vanishing of R^L at a point $y \in Y$ is now defined

$$r_y - 2 = \text{ord}_y\left(R^L\right) := \min\left\{l | J^l\left(\Lambda^2 T^* Y\right) \ni j^l_y R^L \neq 0\right\}, \quad r_y \geq 2, \tag{1.1}$$

where $j^l R^L$ denotes the lth jet of the curvature. We shall assume that this order of vanishing is finite at any point of the manifold i.e.,

$$r := \max_{y \in Y} r_y < \infty. \tag{1.2}$$

The function $y \mapsto r_y$ being upper semi-continuous then gives a decomposition of the manifold $Y = \bigcup_{j=2}^r Y_j$; $Y_j := \left\{y \in Y | r_y = j\right\}$ with each $Y_{\leq j} := \bigcup_{j'=0}^{j} Y_{j'}$ being open. Furthermore, the curvature is assumed to be semi-positive: $R^L\left(w, \bar{w}\right) \geq 0$, for all $w \in T^{1,0}Y$.

Associated to the above one has the Kodaira Laplacian

$$\Box^q_k : \Omega^{0,q}\left(Y; F \otimes L^k\right) \to \Omega^{0,q}\left(Y; F \otimes L^k\right), \quad 0 \leq q \leq 1,$$

acting on tensor powers. The kernel of the Kodaira Laplacian $\ker \Box^q_k = H^q\left(X; F \otimes L^k\right)$ is cohomological and corresponds to holomorphic sections. The Bergman kernel $\Pi^q_k\left(y, y'\right)$ is the Schwartz kernel of the orthogonal projector $\Pi^q_k : \Omega^{0,q}\left(Y; F \otimes L^k\right)$

$\to \ker \Box_k^q$. Its value on the diagonal is

$$\Pi_k^q(y,y) = \sum_{j=1}^{N_k^q} |s_j(y)|^2, \quad N_k^q := \dim H^q\left(X; F \otimes L^k\right),$$

for an orthonormal basis $\{s_j\}_{j=1}^{N_k^q}$ of $H^q\left(X; F \otimes L^k\right)$. Under these assumptions one has $H^1\left(X; F \otimes L^k\right) = 0$ for $k \gg 0$. We now first recall our theorem from [32] on the asymptotics of the Bergman kernel $\Pi_k := \Pi_k^0$.

Theorem 1.1 [32, Theorem 3] *Let Y be a compact Riemann surface and $(L, h^L) \to Y$ a semi-positive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F, h^F) \to Y$ be another Hermitian holomorphic vector bundle. Then the Bergman kernel $\Pi_k := \Pi_k^0$ has the pointwise asymptotic expansion on diagonal*

$$\Pi_k(y,y) = k^{2/r_y}\left[\sum_{j=0}^{N} c_j(y)\, k^{-2j/r_y}\right] + O\left(k^{-2N/r_y}\right), \quad \forall N \in \mathbb{N}. \tag{1.3}$$

Here c_j are sections of $End(F)$, with the leading term $c_0(y) = \Pi^{g_y^{TY}, j_y^{r_y-2}R^L, J_y^{TY}}(0,0) > 0$ being given in terms of the Bergman kernel of the model Kodaira Laplacian on the tangent space at y (A.8).

To explain our first consequence of the above, note that the cohomology $H^0(Y; F \otimes L^k)$ is endowed with an L^2 product induced by h^{TY}, h^L and h^F. This induces a Fubini-Study metric ω_{FS} on the projective space $\mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right]$. The Kodaira map is now defined

$$\begin{aligned} \Phi_k : Y &\to \mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right], \\ \Phi_k(y) &:= \left\{s \in H^0\left(Y; F \otimes L^k\right) | s(y) = 0\right\}. \end{aligned} \tag{1.4}$$

It is well known that the map is holomorphic. We now have the semi-positive version of Tian's approximation theorem.

Theorem 1.2 *Let Y be a compact Riemann surface and (L, h^L), (F, h^F) be holomorphic Hermitian line bundles on Y such that (L, h^L) is semi-positive and its curvature vanishes at most at finite order. Then the Fubini-Study forms induced by the Kodaira map (1.4) converge uniformly on Y to the curvature R^L of the line bundle with speed $k^{-1/3}$*

$$\left\| \frac{1}{k}\Phi_k^* \omega_{FS} - \frac{i}{2\pi} R^L \right\|_{C^0(Y)} = O\left(k^{-1/3}\right)$$

as $k \to \infty$.

For the next application we consider the Toeplitz quantization of functions on Y, or more generally sections of F. The Toeplitz operator $T_{f,k}$ operator corresponding to a section $f \in C^{\infty}(Y; \mathrm{End}(F))$ is defined via

$$T_{f,k} : C^{\infty}\left(Y; F \otimes L^k\right) \to C^{\infty}\left(Y; F \otimes L^k\right)$$
$$T_{f,k} := \Pi_k f \Pi_k, \tag{1.5}$$

where f denotes the operator of pointwise composition by f. Each Toeplitz operator above further maps $H^0\left(Y; F \otimes L^k\right)$ to itself. A generalized Toeplitz operator, see 5.6, acting on $H^0\left(Y; F \otimes L^k\right)$ is defined as one having an asymptotic expansion in k^{-1} with coefficients being the Toeplitz operators (1.5) as above. Our next result is now as follows.

Theorem 1.3 *Let (L, h^L) and (F, h^F) be Hermitian holomorphic line bundles on a compact Riemann surface Y and assume that (L, h^L) is semi-positive line bundle and its curvature R^L vanishes to finite order at any point. Given $f, g \in C^{\infty}(Y; End(F))$, the Toeplitz operators (5.1) satisfy*

$$\lim_{k\to\infty} \left\| T_{f,k} \right\| = \|f\|_{\infty} := \sup_{\substack{y\in Y \\ u \in F_y \setminus 0}} \frac{|f(y)u|_{h^F}}{|u|_{h^F}}, \tag{1.6}$$

$$T_{f,k}T_{g,k} = T_{fg,k} + O_{L^2\to L^2}\left(k^{-1/r}\right). \tag{1.7}$$

Moreover, the space of generalized Toeplitz operators supported on the subset Y_2 where the curvature is positive form an algebra under operator addition and composition.

For our next result, we consider the asymptotics of zeroes of random sections associated to tensor powers. To state the result first note that the natural L^2 metric on $H^0\left(Y; F \otimes L^k\right)$ gives rise to a probability density μ_k on the sphere

$$SH^0\left(Y; F \otimes L^k\right) := \left\{ s \in H^0\left(Y; F \otimes L^k\right) \mid \|s\| = 1 \right\},$$

of finite dimension $\chi\left(Y; F \otimes L^k\right) - 1$ (2.15). We now define the product probability space $(\Omega, \mu) := \left(\Pi_{k=1}^{\infty} SH^0\left(Y; F \otimes L^k\right), \Pi_{k=1}^{\infty}\mu_k\right)$. To a random sequence of sections $s = (s_k)_{k\in\mathbb{N}} \in \Omega$ given by this probability density, we then associate the random sequence of zero divisors $Z_{s_k} = \{s_k = 0\}$ and view it as a random sequence of currents of integration in the space $\Omega'_{0,0}(Y)$ of currents of bidimension $(0, 0)$. Note that we can introduce a large class of probability measures as in [4] on the space of holomorphic sections for which our results still hold.

We now have the following.

Theorem 1.4 *Let (L, h^L) and (F, h^F) be Hermitian holomorphic line bundles on a compact Riemann surface Y and assume that (L, h^L) is semi-positive line bundle and its curvature R^L vanishes to finite order at any point. Then for μ-almost all*

$s = (s_k)_{k \in \mathbb{N}} \in \Omega$, *the sequence of currents*

$$\frac{1}{k} Z_{s_k} \rightharpoonup \frac{i}{2\pi} R^L$$

converges weakly to the semi-positive curvature form.

Our final result concerns the asymptotics of holomorphic torsion. Below $\tau^L := R^L(w, \bar{w})$ in terms of an orthonormal section w of $T^{1,0}Y$.

Theorem 1.5 *Let (L, h^L) and (F, h^F) be Hermitian holomorphic line bundles on a compact Riemann surface Y and assume that (L, h^L) is semi-positive line bundle and its curvature R^L vanishes to finite order at any point. The holomorphic torsion satisfies the asymptotics*

$$\ln \mathcal{T}_k := -\frac{1}{2}\zeta_k'(0) = -k \ln k \int_Y \left[\frac{\tau^L}{8\pi}\right] - k \int_Y \left[\frac{\tau^L}{8\pi} \ln\left(\frac{\tau^L}{2\pi}\right)\right] + o(k)$$

as $k \to \infty$.

All of our results above are well known in the case when the line bundle L is positive. In the positive case, the leading term of the Bergman kernel expansion Theorem 1.1 was first shown in [37] and thereafter improved to a full expansion in [14, 38] as a consequence of the Boutet de Monvel–Sjöstrand parametrix [13] for the Szegő kernel of a strongly pseudoconvex CR manifold. Subsequently a different geometric method for the expansion was developed in [16, 29] inspired by the analytic localization method of [7]. The application of the Bergman kernel to induced Fubini-Study metrics Theorem 1.2 is also found in [37] in the positive case. The construction of the full Toeplitz algebra, along with the properties of Toeplitz operators, was first done in [11] as an application of the Boutet de Monvel–Guillemin calculus of Toeplitz operators [12]. The equidistribution result for random sections in the positive case was first done in [36], and [18, 19] also gave the speed of convergence of the zero divisors. Finally, the asymptotics of holomorphic torsion for positive line bundles is due to Bismut-Vasserot [8].

In the semi-positive case our results are mostly new. The Bergman kernel expansion Theorem 1.1 was shown by the authors in their earlier work [32]. The corresponding problem for the Szegő kernel of a weakly pseudoconvex CR manifold in dimension three was solved by the second author in [24]. The expansion proved in [32, Theorem 3] is, however, only pointwise along the diagonal. In order to obtain the approximation for Fubini-Study metrics Theorem 1.2 one needs to prove uniform estimates on the Bergman kernel and its derivatives. The composition for Toeplitz operators supported on the subset where the curvature is positive in Theorem 1.3 was shown earlier by the first author in [23, Theorem 1.4] under the assumption of a small spectral gap for the Kodaira Laplacian. A more general result, than the equidistribution for zeroes of a random holomorphic section of a semi-positive line bundle, was obtained in [18, Sec. 4] using L^2 estimates for the $\bar{\partial}$-equation of a modified positive metric.

The paper is organized as follows. In Sect. 2 we begin with some standard preliminaries. These include the relevant spectral gap properties for the Bochner and

Kodaira Laplacians in Sects. 2.1 and 2.2, respectively. In Sect. 3 we recall the proof of the pointwise Bergman kernel expansion from [32]. In Sect. 3.1 we further derive uniform estimates on semi-positive Bergman kernels that are necessary for the applications in this article. In Sect. 4 we use the uniform Bergman kernel estimates to prove the semi-positive version of Tian's theorem Theorem 1.2. In Sect. 5 we prove the analogous expansion for the kernel of a Toeplitz operator and the corresponding theorem Theorem 1.3 on Toeplitz quantization. In Sect. 6 we prove the equidistribution result Theorem 1.4 for random sections. In the final Sect. 7 we prove the asymptotic result for holomorphic torsion Theorem 1.5. The final appendix Section A describes facts on model Laplacians and Bergman kernels that are used throughout the article.

2 Preliminaries

Here we begin with some preliminary notions. Let Y be a compact Riemann surface. It is equipped with an integrable complex structure J and Hermitian metric h^{TY} on its complex tangent space. Also denote by g^{TY} the associated Riemannian metric on TY. Next let (L, h^L), (F, h^F) be an auxiliary pair of Hermitian, holomorphic bundles where L is of rank one. We denote by ∇^L, ∇^F the corresponding Chern connections and R^L, R^F their corresponding curvatures. The order of vanishing r_y of the curvature R^L at a point $y \in Y$ is now defined as in (1.1). And we assume that the curvature R^L vanishes at finite order at any point of Y, i.e.,

$$r := \max_{y \in Y} r_y < \infty. \tag{2.1}$$

The curvature R^L of ∇^L is a $(1, 1)$ form which is further assumed to be semi-positive

$$\begin{aligned} iR^L(v, Jv) &\geq 0, \quad \forall v \in TY \quad \text{or equivalently} \\ R^L(w, \bar{w}) &\geq 0, \quad \forall w \in T^{1,0}Y. \end{aligned} \tag{2.2}$$

We note that semi-positivity implies that the order of vanishing $r_y - 2 \in 2\mathbb{N}_0$ of the curvature R^L at any point y is even. Semi-positivity and finite order of vanishing imply that there are points where the curvature is positive (the set where the curvature is positive is in fact an open dense set). Hence

$$\deg L = \int_Y c_1(L) = \int_Y \frac{i}{2\pi} R^L > 0,$$

so that L is ample.

2.1 sR and Bochner Laplacians

Associated to the above data one has the Bochner Laplacian on tensor powers defined by

$$\Delta_k := \left(\nabla^{F\otimes L^k}\right)^* \nabla^{F\otimes L^k} : C^\infty\left(Y; F\otimes L^k\right) \to C^\infty\left(Y; F\otimes L^k\right), \tag{2.3}$$

for each $k \in \mathbb{N}$, with the adjoint above being taken with respect to the corresponding metrics and the Riemannian volume form.

Each Bochner Laplacian (2.3) above is the Fourier mode of a sub-Riemannian (sR) Laplacian on the unit circle bundle of L. To elaborate, denote by $X = S^1L \to Y$ the unit circle bundle of the line bundle L. Further let $E := HX \subset TX$ be the horizontal distribution induced by ∇^L. The distribution carries the metric $g^E = \pi^* g^{TY}$ pulled back from the base. We also denote by the same notation the pullback of $\left(F, h^F, \nabla^F\right)$ from Y to X. The finite order of vanishing for the curvature R^L in (1.2) is equivalent to the *bracket generating* condition for the distribution E: the Lie brackets in $C^\infty(E)$ generates all vector fields $C^\infty(TX)$ [32, Prop. 6]. As such the triple $\left(X, E\subset TX, g^E\right)$ is a *sub-Riemannian (sR) manifold*. Furthermore the maximum order of vanishing for the curvature r (1.2) is then the degree of non-holonomy of the distribution E, i.e., the number of brackets required to generate the missing vertical direction. A volume form on X is defined via $\mu_X := \mu_{g^{TY}} \wedge e^*$ with $\mu_{g^{TY}}$ denoting the Riemannian volume form on Y and e^* being the dual one form to the generating $e \in C^\infty(TX)$ of the circle action on X.

The sub-Riemannian Laplacian on X

$$\begin{aligned}\Delta_{g^E,\mu_X} &: C^\infty(X; F) \to C^\infty(X; F)\\ \Delta_{g^E,\mu_X} &:= \left(\nabla^{g^E,F}\right)^*_{\mu_X} \circ \nabla^{g^E,F}\end{aligned} \tag{2.4}$$

being the composition of the sR gradient defined via

$$\begin{aligned}\nabla^{g^E,F} &: C^\infty(X; E) \to C^\infty(X; E\otimes F),\\ h^{E,F}\left(\nabla^{g^E,F}s, v\otimes s'\right) &:= h^F\left(\nabla^F_v s, s'\right),\end{aligned} \tag{2.5}$$

for all $v \in C^\infty(X; E)$, $s' \in C^\infty(X; F)$, where $h^{E,F} := g^E \otimes h^F$, with its adjoint taken with respect to μ_X. Under the bracket generating condition, the sR Laplacian satisfies the sharp subelliptic estimate of Rothschild and Stein with a gain of $\frac{1}{r}$ derivatives

$$\|\psi s\|^2_{H^{1/r}} \le C\left[\left\langle \Delta_{g^E,F,\mu}\varphi s, \varphi s\right\rangle + \|\varphi s\|^2_{L^2}\right], \quad \forall s \in C^\infty(X; F) \tag{2.6}$$

for all $\varphi, \psi \in C^\infty_c(X)$, with $\varphi = 1$ on the support spt (ψ), and where r is again given by (1.2) and corresponds to the maximum step size of the distribution E.

Next, the unit circle bundle of L being X, the pullback $\mathbb{C} \cong \pi^* L \to X$ is canonically trivial via the identification $\pi^* L \ni (x, l) \mapsto x^{-1} l \in \mathbb{C}$. Pulling back sections then gives the identification

$$C^\infty (X; F) = \oplus_{k \in \mathbb{Z}} C^\infty \left(Y; F \otimes L^k \right). \tag{2.7}$$

Each summand on the right-hand side corresponds to an eigenspace of ∇_e^F with eigenvalue $-ik$. While horizontal differentiation d^H on the left corresponds to differentiation with respect to the tensor product connection∇^{L^k} on the right-hand side. Pick an invariant density μ_X on X inducing a density μ_Y on Y. This now defines the sR Laplacian Δ_{g^E, F, μ_X} acting on sections of F. By invariance the sR Laplacian commutes $\left[\Delta_{g^E, F, \mu_X}, e\right] = 0$ with the generator of the circle action and hence preserves the decomposition (2.7). It acts via

$$\Delta_{g^E, F, \mu_X} = \oplus_{k \in \mathbb{Z}} \Delta_k \tag{2.8}$$

on each component where Δ_k is the Bochner Laplacian (2.3) on the tensor powers $F \otimes L^k$, with adjoint being taken with respect to $\mu_{g^{TY}}$.

Using the description of the Bochner Laplacian as the Fourier mode of the sR Laplacian (2.8), in [32, Thm. 1] a general leading asymptotic result for the first positive eigenvalues was proved. Here we recall a simple argument for its lower bound.

Proposition 2.1 *There exist constants $c_1, c_2 > 0$, such that one has Spec $(\Delta_k) \subset \left[c_1 k^{2/r} - c_2, \infty\right)$ for each k.*

Proof The subelliptic estimate (2.6) on the circle bundle is

$$\left\| \partial_\theta^{1/r} s \right\|^2 \leq \|s\|_{H^{1/r}}^2 \leq C \left[\left\langle \Delta_{g^E, F, \mu_X} s, s \right\rangle + \|s\|_{L^2}^2 \right], \quad \forall s \in C^\infty (X; F).$$

Letting $s = \pi^* s'$ be the pullback of an orthonormal eigenfunction s' of Δ_k with eigenvalue λ on the base gives $k^{2/r} \leq C (\lambda + 1)$ as required. ◻

2.2 Kodaira Laplacian and its Spectral Gap

Related to the Bochner Laplacian (2.3) is the Kodaira Laplacian on tensor powers. Namely, with $\left(\Omega^{0,*}\left(X; F \otimes L^k\right); \bar{\partial}_k\right)$ denoting the Dolbeault complex the Kodaira Laplace and Dirac operators acting on $\Omega^{0,*}\left(X; F \otimes L^k\right)$ are defined

$$\Box_k := \frac{1}{2} (D_k)^2 = \bar{\partial}_k \bar{\partial}_k^* + \bar{\partial}_k^* \bar{\partial}_k \tag{2.9}$$

$$D_k := \sqrt{2} \left(\bar{\partial}_k + \bar{\partial}_k^* \right). \tag{2.10}$$

Clearly, D_k interchanges while $\Box_k$ preserves $\Omega^{0,0/1}$. We denote $D_k^{\pm} = D_k|_{\Omega^{0,0/1}}$ and $\Box_k^{0/1} = \Box_k|_{\Omega^{0,0/1}}$. The Clifford multiplication endomorphism $c : TY \to$

End $\left(\Lambda^{0,*}\right)$ is defined via $c(v) := \sqrt{2}\left(v^{1,0} \wedge -i_{v^{0,1}}\right)$, $v \in TY$, and extended to the entire exterior algebra $\Lambda^{*}TY$ via $c(1) = 1$, $c(v_1 \wedge v_2) := c(v_1)\,c(v_2)$, for orthonormal $v_1, v_2 \in TY$.

Denote by $\nabla^{TY}, \nabla^{T^{1,0}Y}$ the Levi-Civita and Chern connections on the real and holomorphic tangent spaces as well as by $\nabla^{T^{0,1}Y}$ the induced connection on the anti-holomorphic tangent space. Denote by Θ the real $(1,1)$ form defined by contraction of the complex structure with the metric $\Theta(.,.) = g^{TY}(J.,.)$. This is clearly closed $d\Theta = 0$ (or Y is Kähler) and the complex structure is parallel $\nabla^{TY}J = 0$ or $\nabla^{TY} = \nabla^{T^{1,0}Y} \oplus \nabla^{T^{1,0}Y}$.

With the induced tensor product connection on $\Lambda^{0,*} \otimes F \otimes L^k$ being denoted via $\nabla^{\Lambda^{0,*}\otimes F \otimes L^k}$, the Kodaira Dirac operator (2.10) is now given by the formula

$$D_k = c \circ \nabla^{\Lambda^{0,*}\otimes F \otimes L^k}.$$

Next we denote by R^F the curvature of ∇^F and by κ the scalar curvature of g^{TY}. Define the following endomorphisms of $\Lambda^{0,*}$

$$\begin{aligned}
\omega\left(R^F\right) &:= R^F(w, \bar{w})\,\bar{w} i_{\bar{w}} \\
\omega\left(R^L\right) &:= R^L(w, \bar{w})\,\bar{w} i_{\bar{w}} \\
\omega(\kappa) &:= \kappa \bar{w} i_{\bar{w}} \\
\tau^F &:= R^F(w, \bar{w}) \\
\tau^L &:= R^L(w, \bar{w})
\end{aligned} \tag{2.11}$$

in terms of an orthonormal section w of $T^{1,0}Y$. The Lichnerowicz formula for the above Dirac operator ([29] Thm 1.4.7) simplifies for a Riemann surface and is given by

$$\begin{aligned}
2\Box_k = D_k^2 &= \left(\nabla^{\Lambda^{0,*}\otimes F \otimes L^k}\right)^* \nabla^{\Lambda^{0,*}\otimes F \otimes L^k} \\
&\quad + k\left[2\omega\left(R^L\right) - \tau^L\right] + \left[2\omega\left(R^F\right) - \tau^F\right] + \frac{1}{2}\omega(\kappa).
\end{aligned} \tag{2.12}$$

We now have the following.

Proposition 2.2 *Let Y be a compact Riemann surface, $(L, h^L) \to Y$ a semi-positive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F, h^F) \to Y$ be a Hermitian holomorphic vector bundle. Then there exist constants $c_1, c_2 > 0$, such that*

$$\|D_k s\|^2 \geq \left(c_1 k^{2/r} - c_2\right)\|s\|^2$$

for all $s \in \Omega^{0,1}\left(Y; F \otimes L^k\right)$.

Proof Writing $s = |s|\,\bar{w} \in \Omega^{0,1}\left(Y; F \otimes L^k\right)$ in terms of a local orthonormal section $\bar{w}$ gives

$$\left\langle\left[2\omega\left(R^L\right) - \tau^L\right]s, s\right\rangle = R^L\left(w, \bar{w}\right)|s|^2 \geq 0 \tag{2.13}$$

from (2.2), (2.11). This gives

$$\begin{aligned}\|D_k s\|^2 &= \left\langle D_k^2 s, s\right\rangle \\ &= \Big\langle\Big[\left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \nabla^{\Lambda^{0,*}\otimes F\otimes L^k} + k\left[2\omega\left(R^L\right) - \tau^L\right] \\ &\quad + \left[2\omega\left(R^F\right) - \tau^F\right] + \frac{1}{2}\omega\left(\kappa\right)\Big]s, s\Big\rangle \\ &\geq \left\langle\left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \nabla^{\Lambda^{0,*}\otimes F\otimes L^k} s, s\right\rangle - c_0\|s\|^2 \geq \left(c_1 k^{2/r} - c_2\right)\|s\|^2\end{aligned}$$

from Proposition 2.1, (2.12), and (2.13). □

We now derive as a corollary a spectral gap property for Kodaira Dirac/Laplace operators $D_k, \Box_k$ corresponding to Proposition 2.1.

Corollary 2.3 *Under the hypotheses of Proposition 2.2 there exist constants $c_1, c_2 > 0$, such that Spec $(\Box_k) \subset \{0\} \cup \left[c_1 k^{2/r} - c_2, \infty\right)$ for each k. Moreover,* ker $D_k^- = 0$ *and $H^1\left(Y; F \otimes L^k\right) = 0$ for k sufficiently large.*

Proof From Proposition 2.2, it is clear that

$$\text{Spec}\left(\Box_k^1\right) \subset \left[c_1 k^{2/r} - c_2, \infty\right) \tag{2.14}$$

for some $c_1, c_2 > 0$ giving the second part of the corollary. Moreover, the eigenspaces of $D_k^2\big|_{\Omega^{0,0/1}}$ with non-zero eigenvalue being isomorphic by Mckean-Singer, the first part also follows. □

Since L is ample, we know also by the Kodaira-Serre vanishing theorem that $H^1\left(Y; F \otimes L^k\right)$ vanishes for k sufficiently large. If F is also a line bundle this follows from the well-known fact that for a line bundle E on Y we have $H^1(Y; E) = 0$ whenever deg $E > 2g - 2$. It is, however, interesting to have a direct analytic proof. Of course, the vanishing theorem for a semi-positive line bundle works only in dimension one, see Remark 2.4.

The vanishing $H^1\left(Y; F \otimes L^k\right) = 0$ for k sufficiently large gives

$$\begin{aligned}\dim H^0\left(Y; F \otimes L^k\right) &= \chi\left(Y; F \otimes L^k\right) \\ &= \int_Y ch\left(F \otimes L^k\right)\text{Td}\left(Y\right) \\ &= k\left[\text{rk}\left(F\right)\int_Y c_1\left(L\right)\right] + \int_Y c_1\left(F\right) + 1 - g,\end{aligned} \tag{2.15}$$

by Riemann-Roch, with $\chi\left(Y; F \otimes L^k\right)$, $ch\left(F \otimes L^k\right)$, $\mathrm{Td}\,(Y)$, g denoting the holomorphic Euler characteristic, Chern character, Todd genus, and genus of Y, respectively.

Remark 2.4 The argument for Proposition 2.2 breaks down in higher dimensions since there are more components to $\left[2\omega\left(R^L\right) - \tau^L\right]$ in the Lichnerowicz formula (2.12) where semi-positivity is insufficient to control. Indeed, there is a known counterexample to the existence of a spectral gap for semi-positive line bundles in higher dimensions due to Donnelly [20].

3 Bergman Kernel Expansion

In this section we now first recall the expansion for the Bergman kernel proved in [32, Sec 4.1]. First recall that the Bergman kernel is the Schwartz kernel $\Pi_k\,(y_1, y_2)$ of the projector onto the nullspace of $\Box_k$

$$\Pi_k : C^\infty\left(Y; F \otimes L^k\right) \to \ker\left(\Box_k|_{C^\infty(Y;F\otimes L^k)}\right), \tag{3.1}$$

with respect to the L^2 inner product given by the metrics g^{TY}, h^F, and h^L. Alternately, if $s_1, s_2, \ldots, s_{N_k}$ denotes an orthonormal basis of eigensections of $H^0\left(X; F \otimes L^k\right)$ then

$$\Pi_k\,(y_1, y_2) = \sum_{j=1}^{N_k} s_j\,(y_1) \otimes s_j\,(y_2)^*\,. \tag{3.2}$$

We wish to describe the asymptotics of Π_k along the diagonal in $Y \times Y$.

Consider $p \in Y$, and fix orthonormal bases $\{e_1, e_2\,(= Je_1)\}$, $\{l\}$, $\left\{f_j\right\}_{j=1}^{\mathrm{rk}(F)}$ for T_pY, L_p, F, respectively, and let $\left\{w := \frac{1}{\sqrt{2}}\,(e_1 - ie_2)\right\}$ be the corresponding orthonormal frame for $T_y^{1,0}Y$. Using the exponential map from this basis obtain a geodesic coordinate system on a geodesic ball $B_{2\varrho}\,(p)$. Further parallel transport these bases along geodesic rays using the connections $\nabla^{T^{1,0}Y}$, ∇^L, ∇^F to obtain orthonormal frames for $T^{1,0}Y$, L, F on $B_{2\varrho}\,(p)$. In this frame and coordinate system, the connection on the tensor product again has the expression

$$\begin{aligned}
\nabla^{\Lambda^{0,*}\otimes F\otimes L^k} &= d + a^{\Lambda^{0,*}} + a^F + ka^L \\
a_j^{\Lambda^{0,*}} &= \int_0^1 d\rho\left(\rho y^k R_{jk}^{\Lambda^{0,*}}\,(\rho y)\right) \\
a_j^F &= \int_0^1 d\rho\left(\rho y^k R_{jk}^F\,(\rho y)\right) \\
a_j^L &= \int_0^1 d\rho\left(\rho y^k R_{jk}^L\,(\rho y)\right)
\end{aligned} \tag{3.3}$$

in terms of the curvatures of the respective connections. We now define a modified frame $\{\tilde{e}_1, \tilde{e}_2\}$ on $\mathbb{R}^2$ which agrees with $\{e_1, e_2\}$ on $B_\varrho(p)$ and with $\left\{\partial_{x_1}, \partial_{x_2}\right\}$ outside $B_{2\varrho}(p)$. Also define the modified metric $\tilde{g}^{TY}$ and almost complex structure $\tilde{J}$ on $\mathbb{R}^2$ to be standard in this frame and hence agreeing with g^{TY}, J on $B_\varrho(p)$. The Christoffel symbol of the corresponding modified induced connection on $\Lambda^{0,*}$now satisfies

$$\tilde{a}^{\Lambda^{0,*}} = 0 \quad \text{outside } B_{2\varrho}(p).$$

With $r_y - 2 \in 2\mathbb{N}_0$ being the order of vanishing of the curvature R^L as before, we may Taylor expand the curvature as

$$R^L = \underbrace{\sum_{|\alpha|=r-2} R_{pq,\alpha} y^\alpha dy_p dy_q}_{=R_0^L} + O\left(y^{r-1}\right) \quad \text{with} \tag{3.4}$$

$$iR_0^L(e_1, e_2) \geq 0. \tag{3.5}$$

Further we may define the modified connections $\tilde{\nabla}^F$, $\tilde{\nabla}^L$ via

$$\begin{aligned}
\tilde{\nabla}^F &= d + \chi\left(\frac{|y|}{2\varrho}\right) a^F \\
\tilde{\nabla}^L &= d + \left[\underbrace{\int_0^1 d\rho\, \rho y^k \left(\tilde{R}^L\right)_{jk}(\rho y)}_{=\tilde{a}_j^L}\right] dy_j, \quad \text{where} \\
\tilde{R}^L &= \chi\left(\frac{|y|}{2\varrho}\right) R^L + \left[1 - \chi\left(\frac{|y|}{2\varrho}\right)\right] R_0^L.
\end{aligned} \tag{3.6}$$

as well as the corresponding tensor product connection $\tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}$ which agrees with $\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}$ on $B_\varrho(p)$. Clearly the curvature of the modified connection $\tilde{\nabla}^L$ is given by $\tilde{R}^L$(3.6) and is semi-positive by (3.5). Equation (3.6) also gives $\tilde{R}^L = R_0^L + O\left(\varrho^{r_y-1}\right)$ and that the $\left(r_y - 2\right)$-th derivative/jet of $\tilde{R}^L$ is non-vanishing at all points on $\mathbb{R}^2$ for

$$0 < \varrho < c\left|j^{r_y-2} R^L(y)\right|. \tag{3.7}$$

Here c is a uniform constant depending on the C^{r-2} norm of R^L. We now define the modified Kodaira Dirac operator on $\mathbb{R}^2$ by the similar formula

$$\tilde{D}_k = c \circ \tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}, \tag{3.8}$$

agreeing with D_k on $B_\varrho(p)$. This has a similar Lichnerowicz formula

$$\tilde{D}_k^2 = 2\tilde{\Box}_k := \left(\tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k} + k\left[2\omega\left(\tilde{R}^L\right) - \tilde{\tau}^L\right] \tag{3.9}$$

$$+\left[2\omega\left(\tilde{R}^F\right) - \tilde{\tau}^F\right] + \frac{1}{2}\omega\left(\tilde{\kappa}\right) \tag{3.10}$$

the adjoint being taken with respect to the metric $\tilde{g}^{TY}$ and corresponding volume form. Also the endomorphisms $\tilde{R}^F, \tilde{\tau}^F, \tilde{\tau}^L$, and $\omega(\tilde{\kappa})$ are the obvious modifications of (2.11) defined using the curvatures of $\tilde{\nabla}^F, \tilde{\nabla}^L$, and $\tilde{g}^{TY}$, respectively. The above-mentioned (3.9) again agrees with $\Box_k$ on $B_\varrho(p)$ while the endomorphisms $\tilde{R}^F, \tilde{\tau}^F, \omega(\tilde{\kappa})$ all vanish outside $B_\varrho(p)$. Being semi-bounded from below (3.9) is essentially self-adjoint. A similar argument as Corollary 2.3 gives a spectral gap

$$\mathrm{Spec}\left(\tilde{\Box}_k\right) \subset \{0\} \cup \left[c_1 k^{2/r_y} - c_2, \infty\right). \tag{3.11}$$

Thus for $k \gg 0$, the resolvent $\left(\tilde{\Box}_k - z\right)^{-1}$ is well defined in a neighborhood of the origin in the complex plane. On account on the local elliptic estimate, the projector $\tilde{\Pi}_k$ from $L^2\left(\mathbb{R}^2; \Lambda_y^{0,*} \otimes F_y \otimes L_y^{\otimes k}\right)$ onto $\ker\left(\tilde{\Box}_k\right)$ then has a smooth Schwartz kernel with respect to the Riemannian volume of $\tilde{g}^{TY}$.

We are now ready to prove the Bergman kernel expansion Theorem 1.1, the procedure is similar to [16].

Proof of Theorem 1.1 First choose $\varphi \in \mathcal{S}(\mathbb{R}_s)$ even satisfying $\hat{\varphi} \in C_c\left(-\frac{\varrho}{2}, \frac{\varrho}{2}\right)$ and $\varphi(0) = 1$. For $c > 0$, set $\varphi_1(s) = 1_{[c,\infty)}(s)\varphi(s)$. On account of the spectral gap Corollary 2.3, and as φ_1 decays at infinity, we have

$$\varphi(D_k) - \Pi_k = \varphi_1(D_k) \quad \text{with}$$
$$\left\|D_k^a \varphi_1(D_k)\right\|_{L^2 \to L^2} = O\left(k^{-\infty}\right) \tag{3.12}$$

for $a \in \mathbb{N}$. Combining the above with semiclassical Sobolev and elliptic estimates gives

$$|\varphi(D_k) - \Pi_k|_{C^l(Y\times Y)} = O\left(k^{-\infty}\right), \tag{3.13}$$

for all $l \in \mathbb{N}_0$. Next we may write $\varphi(D_k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi D_k}\hat{\varphi}(\xi)\,d\xi$ via Fourier inversion. Since $D_k = \tilde{D}_k$ on $B_\varrho(p)$ and $\hat{\varphi} \in C_c\left(-\frac{\varrho}{2}, \frac{\varrho}{2}\right)$, we may use a finite propagation argument to conclude

$$\varphi(D_k)(., y) = \varphi\left(\tilde{D}_k\right)(., 0).$$

By similar estimates as (3.12) for $\tilde{D}_k$ we now have a localization of the Bergman kernel

$$\begin{aligned} \Pi_k(., y) &= O\left(k^{-\infty}\right), \quad \text{on } B_\varrho(p)^c \\ \Pi_k(., y) - \tilde{\Pi}_k(., 0) &= O\left(k^{-\infty}\right), \quad \text{on } B_\varrho(p). \end{aligned} \tag{3.14}$$

It thus suffices to consider the Bergman kernel of the model Kodaira Laplacian (3.9) on $\mathbb{R}^2$.

Next with the rescaling/dilation $\delta_{k^{-1/r}} y = \left(k^{-1/r} y_1, \ldots, k^{-1/r} y_{n-1}\right)$, the rescaled Kodaira Laplacian

$$\boxdot := k^{-2/r_y} \left(\delta_{k^{-1/r}}\right)_* \tilde{\Box}_k \tag{3.15}$$

satisfies

$$\varphi\left(\frac{\tilde{\Box}_k}{k^{2/r_y}}\right)\left(y, y'\right) = k^{2/r_y} \varphi\left(\boxdot\right)\left(y k^{1/r_y}, y' k^{1/r_y}\right) \tag{3.16}$$

for $\varphi \in \mathcal{S}(\mathbb{R})$. Using a Taylor expansion via (3.6), (3.8) the rescaled Dirac operator has an expansion

$$\boxdot = \left(\sum_{j=0}^{N} k^{-j/r_y} \boxdot_j\right) + k^{-2(N+1)/r_y} \mathrm{E}_{N+1}, \ \forall N. \tag{3.17}$$

Here each

$$\boxdot_j = a_{j;pq}(y)\, \partial_{y_p} \partial_{y_q} + b_{j;p}(y)\, \partial_{y_p} + c_j(y) \tag{3.18}$$

is a (k-independent) self-adjoint, second-order differential operator while each

$$\mathrm{E}_j = \sum_{|\alpha|=N+1} y^\alpha \left[a^\alpha_{j;pq}(y;k)\, \partial_{y_p} \partial_{y_q} + b^\alpha_{j;p}(y;k)\, \partial_{y_p} + c^\alpha_j(y;k)\right] \tag{3.19}$$

is a k-dependent self-adjoint, second-order differential operator on $\mathbb{R}^2$. Furthermore the functions appearing in (3.18) are polynomials with degrees satisfying

$$\begin{aligned} &\deg a_j = j,\ \deg b_j \leq j + r_y - 1, \deg c_j \leq j + 2r_y - 2 \\ &\deg b_j - (j-1) = \deg c_j - j = 0 \pmod 2 \end{aligned}$$

and whose coefficients involve

$$\begin{aligned} a_j &: \leq j - 2 \text{ derivatives of } R^{TY} \\ b_j &: \leq j - 2 \text{ derivatives of } R^F, R^{\Lambda^{0,*}} \end{aligned}$$

$$\begin{aligned} &\leq j + r - 2 \text{ derivatives of } R^L \\ c_j :\, &\leq j - 2 \text{ derivatives of } R^F, R^{\Lambda^{0,*}} \\ &\leq j + r - 2 \text{ derivatives of } R^L \end{aligned}$$

while the coefficients $a^{\alpha}_{j;pq}(y;k)$, $b^{\alpha}_{j;p}(y;k)$, $c^{\alpha}_{j}(y;k)$ of (3.19) are uniformly (in k) C^∞ bounded. Using (3.3), (A.4), (A.8), and (A.9) the leading term of (3.17) is computed

$$\boxdot_0 = \boxdot_{g^{TY}, j_y^{r_y-2} R^L, J^{TY}} \tag{3.20}$$

in terms of the model Kodaira Laplacian on the tangent space TY (A.8).

In light of the spectral gap (3.11), equation (3.16) specializes to

$$\tilde{\Pi}_k\left(y', y\right) = k^{2/r_y} \Pi^{\boxdot}\left(y' k^{1/r_y}, y k^{1/r_y}\right) \tag{3.21}$$

as a relation between the Bergman kernels of $\tilde{\boxdot}_k, \boxdot$. Next, the expansion (3.17) along with local elliptic estimates gives

$$(\boxdot - z)^{-1} - (\boxdot_0 - z)^{-1} = O_{H^s_{\text{loc}} \to H^{s+2}_{\text{loc}}}\left(k^{-1/r_y} |\text{Im} z|^{-2}\right)$$

for each $s \in \mathbb{R}$. More generally, we let $I_j := \left\{p = (p_0, p_1, \ldots) \,|\, p_\alpha \in \mathbb{N}, \sum p_\alpha = j\right\}$ denote the set of partitions of the integer j and define

$$\mathrm{C}^z_j = \sum_{p \in I_j} (z - \boxdot_0)^{-1}\left[\Pi_\alpha\left[\boxdot_{p_\alpha}(z - \boxdot_0)^{-1}\right]\right]. \tag{3.22}$$

Then by repeated applications of the local elliptic estimate using (3.17) we have

$$(z - \boxdot)^{-1} - \left(\sum_{j=0}^{N} k^{-j/r_y} \mathrm{C}^z_j\right) = O_{H^s_{\text{loc}} \to H^{s+2}_{\text{loc}}}\left(k^{-(N+1)/r_y} |\text{Im} z|^{-2Nr_y - 2}\right), \tag{3.23}$$

for each $N \in \mathbb{N}$, $s \in \mathbb{R}$. A similar expansion as (3.17) for the operator $(\boxdot + 1)^M (\boxdot - z)$, $M \in \mathbb{N}$, also gives

$$\begin{aligned} &(\boxdot + 1)^{-M} (\boxdot - z)^{-1} - \sum_{j=0}^{N} k^{-j/r_y} \mathrm{C}^z_{j,M} \\ &\quad = O_{H^s_{\text{loc}} \to H^{s+2+2M}_{\text{loc}}}\left(k^{-(N+1)/r_y} |\text{Im} z|^{-2Nr_y - 2}\right) \end{aligned} \tag{3.24}$$

for operators $\mathsf{C}^{z}_{j,M} = O_{H^s_{\mathrm{loc}} \to H^{s+2+2M}_{\mathrm{loc}}}\left(k^{-(N+1)/r_y} |\mathrm{Im} z|^{-2Nr_y-2}\right)$, $j = 0, \dots, N$, with

$$\mathsf{C}^{z}_{0,M} = \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} + 1\right)^{-M} \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z\right)^{-1}.$$

For $M \gg 0$ sufficiently large, Sobolev's inequality gives an expansion for the corresponding Schwartz kernels in (3.24) in $C^l(K)$, for all $l \in \mathbb{N}_0$ and compact subset $K \subset \mathbb{R}^2 \times \mathbb{R}^2$. Next, plugging the above resolvent expansion into the Helffer–Sjöstrand formula [17, eq. 8.3] as before gives

$$\left| \varphi(\boxdot) - \sum_{j=0}^{N} k^{-j/r_y} \mathsf{C}^{\varphi}_j \right|_{C^l(K)} = O\left(k^{-(N+1)/r_y}\right)$$

for all $l, N \in \mathbb{N}_0$ and for some (k-independent) $\mathsf{C}^{\varphi}_j \in C^\infty(K)$, $j = 0, 1, \dots$, with leading term $\mathsf{C}^{\varphi}_0 = \varphi(\boxdot_0) = \varphi\left(\boxdot_{g^{TY}, j_y^{r_y-2} R^L, J^{TY}}\right)$. As φ was chosen supported near 0, the spectral gap property (3.11) gives

$$\left| \Pi^{\boxdot} - \sum_{j=0}^{N} k^{-j/r_y} \mathsf{C}_j \right|_{C^l(K)} = O\left(k^{-(N+1)/r_y}\right) \tag{3.25}$$

for some $\mathsf{C}_j \in C^\infty(K)$, $j = 0, 1, \dots$, with leading term $\mathsf{C}_0 = \Pi^{\boxdot_{g^{TY}, j_y^{r_y-2} R^L, J^{TY}}}$. The expansion is now a consequence of (3.13), (3.14), and (3.21). Finally, in order to show that there are no odd powers of k^{-j/r_y}, one again notes that the operators $\boxdot_j$ (3.18) change sign by $(-1)^j$ under $\delta_{-1} x := -x$. Thus the integral expression (3.22) corresponding to $\mathsf{C}^{z}_j(0,0)$ changes sign by $(-1)^j$ under this change of variables and must vanish for j odd. □

Next we show that a pointwise expansion on the diagonal also exists for derivatives of the Bergman kernel. In what follows we denote by $j^l s / j^{l-1} s \in S^l T^* Y \otimes E$ the component of the l-th jet of a section $s \in C^\infty(E)$ of a Hermitian vector bundle E that lies in the kernel of the natural surjection $J^l(E) \to J^{l-1}(E)$.

Theorem 3.1 *For each $l \in \mathbb{N}_0$, the l-th jet of the on-diagonal Bergman kernel has a pointwise expansion*

$$j^l [\Pi_k(y,y)] / j^{l-1} [\Pi_k(y,y)] = k^{(2+l)/r_y} \left[\sum_{j=0}^{N} c_j(y) k^{-2j/r_y} \right] + O\left(k^{-(2N-l-1)/r_y}\right), \tag{3.26}$$

for all $N \in \mathbb{N}$, *in* $j^l End(F)/j^{l-1}End(F) = S^l T^*Y \otimes End(F)$, *with the leading term*

$$c_0(y) = j^l \left[\Pi^{g_y^{TY}, j_y^{r_y-2} R^L, J_y^{TY}}(0,0) \right] / j^{l-1} \left[\Pi^{g_y^{TY}, j_y^{r_y-2} R^L, J_y^{TY}}(0,0) \right]$$

being given in terms of the l-th jet of the Bergman kernel of the Kodaira Laplacian (A.8) *on the tangent space at* y.

Proof The proof is a modification of the previous. First note that a similar localization

$$\Pi_k(\mathsf{y}, \mathsf{y}) - \tilde{\Pi}_k(\mathsf{y}, \mathsf{y}) = O\left(k^{-\infty}\right), \tag{3.27}$$

to (3.14) is valid in C^l, for all $l \in \mathbb{N}_0$, and for y in a uniform neighborhood of y. Next differentiating (3.21) with $y = y'$ gives

$$\partial_{\mathsf{y}}^{\alpha} \tilde{\Pi}_k(\mathsf{y}, \mathsf{y}) = k^{(2+|\alpha|)/r_y} \partial_{\mathsf{y}}^{\alpha} \Pi^{\boxdot}\left(\mathsf{y}k^{1/r_y}, \mathsf{y}k^{1/r_y}\right), \tag{3.28}$$

for all $\alpha \in \mathbb{N}_0^2$. Finally, the expansion (3.25) being valid in C^l, for all $l \in \mathbb{N}_0$, may be differentiated and plugged into the above with $\mathsf{y} = 0$ to give the theorem. □

Remark 3.2 The expansion (1.3) is the same as the positive case on Y_2 (points where $r_y = 2$) and furthermore uniform in any C^l-topology on compact subsets of Y_2 cf. [29, Theorem 4.1.1]. In particular the first two coefficients for $y \in Y_2$ are given by

$$c_0(y) = \Pi^{g_y^{TY}, j_y^0 R^L, J_y^{TY}}(0,0) = \frac{1}{2\pi}\tau^L$$
$$c_1(y) = \frac{1}{16\pi}\tau^L\left[\kappa - \Delta \ln \tau^L + 4\tau^F\right].$$

The derivative expansion on Y_2 is also known to satisfy $c_0 = c_1 = \ldots = c_{\left[\frac{l-1}{2}\right]} = 0$ (i.e., begins at the same leading order k [29, Theorem 4.1.1]) with the leading term given by

$$c_{\left[\frac{l+1}{2}\right]}(y) = \frac{1}{2\pi} j^l \tau^L / \frac{1}{2\pi} j^{l-1}\tau^L.$$

3.1 Uniform Estimates on the Bergman Kernel

The expansions for the Bergman kernel Theorem 1.1 and its derivatives Theorem 3.1 are not uniform in the point on the diagonal. For applications in the later sections we need to give uniform estimates on the Bergman kernel. Below we set $C_{r_1} := \inf_{|R^V|=1} \Pi^{g^V, R^V, J^V}(0,0)$ for each $0 \neq R^V \in S^{r_1-2}V^* \otimes \Lambda^2 V^*$, $r_1 \geq 2$. Furthermore, the Bergman kernel $\Pi^{g_y^{TY}, j_y^0 R^L, J_y^{TY}}(0,0)$ of the model operator (A.8) is extended (continuously) by zero from Y_2 to Y.

Lemma 3.3 *The Bergman kernel satisfies*

$$\begin{aligned}\left[\inf_{y\in Y_r}\Pi^{g_y^{TY},j_y^{r-2}R^L,J_y^{TY}}(0,0)\right][1+o(1)]k^{2/r}&\leq\Pi_k(y,y)\\ &\leq\left[\sup_{y\in Y}\Pi^{g_y^{TY},j_y^0R^L,J_y^{TY}}(0,0)\right][1+o(1)]k,\end{aligned}\tag{3.29}$$

with the $o(1)$ terms being uniform in $y\in Y$.

Proof Note that Theorem 1.1 already shows that there exists constants $C_0, C_1, C_2 \dots$ such that

$$\Pi_k(y,y)\geq C_{r_y-2}\left(\left|j^{r_y-2}R^L\right|k\right)^{2/r_y}-c_y,\tag{3.30}$$

for all $y\in Y$, with

$$c_y=c\left(\left|j^{r_y-2}R^L(y)\right|^{-1}\right)=O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1),\tag{3.31}$$

being a (y-dependent) constant given in terms of the norm of the first non-vanishing jet. The norm of this jet affects the choice of ϱ needed for (3.7), which in turn affects the C^∞-norms of the coefficients of (3.19) via (3.6). We first show that this estimate extends to a small ($\left|j^{r_y-2}R^L(y)\right|$-dependent) size neighborhood of y. To this end, for any $\varepsilon>0$ there exists a uniform constant c_ε depending only on ε and $\left\|R^L\right\|_{C^r}$ such that

$$\left|j^{r_y-2}R^L(\mathsf{y})\right|\geq(1-\varepsilon)\left|j^{r_y-2}R^L(y)\right|,\tag{3.32}$$

for all $\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$.

We begin by rewriting the model Kodaira Laplacian $\tilde{\square}_k$ (3.9) near y in terms of geodesic coordinates centered at y. In the region

$$\mathsf{y}\in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)\cap\left\{C_0\left(\left|j^0R^L(\mathsf{y})\right|k\right)\geq k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)\right\}\tag{3.33}$$

a rescaling of $\tilde{\square}_k$ by $\delta_{k^{-1/2}}$, now centered at y, shows

$$\begin{aligned}\Pi_k(\mathsf{y},\mathsf{y})&=k\Pi^{g_{\mathsf{y}}^{TY},j_{\mathsf{y}}^0R^L,J_{\mathsf{y}}^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &=k\left|j^0R^L(\mathsf{y})\right|\Pi^{g_{\mathsf{y}}^{TY},\frac{j_{\mathsf{y}}^0R^L}{\left|j^0R^L(\mathsf{y})\right|},J_{\mathsf{y}}^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\\ &\geq k^{2/r_y}\Pi^{g_y^{TY},j_y^{r_y-2}R^L,J_y^{TY}}(0,0)+O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}(1)\end{aligned}\tag{3.34}$$

as in (3.30). The first line above follows as in the Bergman kernel expansion Theorem 1.1, along with its leading coefficient. The last line follows from (3.33) together with (3.32). Now, in the region

$$\begin{aligned} \mathsf{y} \in B_{c_\varepsilon |j^{r_\mathsf{y}-2}R^L|}(y) \cap \Big\{ & C_1 \left(\left| j^1 R^L(\mathsf{y}) / j^0 R^L(\mathsf{y}) \right| k \right)^{2/3} \\ & \geq k^{2/r_\mathsf{y}} \Pi^{g_\mathsf{y}^{TY}, j_\mathsf{y}^{r_\mathsf{y}-2}R^L, J_\mathsf{y}^{TY}}(0,0) \\ & \geq C_0 \left(\left| j^0 R^L(\mathsf{y}) \right| k \right) \Big\} \end{aligned}$$

a rescaling of $\tilde{\Box}_k$ by $\delta_{k^{-1/3}}$ centered at y similarly shows

$$\begin{aligned} \Pi_k(\mathsf{y},\mathsf{y}) &= k^{2/3} \left[1 + O\left(k^{2/r-2/3}\right) \right] \Pi^{g_\mathsf{y}^{TY}, j_\mathsf{y}^1 R^L / j_\mathsf{y}^0 R^L, J_\mathsf{y}^{TY}}(0,0) \\ &\quad + O_{\left|j^{r_\mathsf{y}-2}R^L(y)\right|^{-1}}(1) \\ &= k^{2/3} \left[1 + O\left(k^{2/r-2/3}\right) \right] \left| j_\mathsf{y}^1 R^L / j_\mathsf{y}^0 R^L \right|^{2/3} \Pi^{g_\mathsf{y}^{TY}, \frac{j_\mathsf{y}^1 R^L / j_\mathsf{y}^0 R^L}{\left| j_\mathsf{y}^1 R^L / j_\mathsf{y}^0 R^L \right|}, J_\mathsf{y}^{TY}}(0,0) \\ &\quad + O_{\left|j^{r_\mathsf{y}-2}R^L(y)\right|^{-1}}(1) \qquad (3.35) \\ &\geq (1-\varepsilon) k^{2/r_\mathsf{y}} \Pi^{g_\mathsf{y}^{TY}, j_\mathsf{y}^{r_\mathsf{y}-2}R^L, J_\mathsf{y}^{TY}}(0,0) + O_{\left|j^{r_\mathsf{y}-2}R^L(y)\right|^{-1}}(1) \qquad (3.36) \end{aligned}$$

Next, in the region

$$\begin{aligned} \mathsf{y} \in B_{c_\varepsilon |j^{r_\mathsf{y}-2}R^L|}(y) \cap \Big\{ & C_2 \left(\left| j^2 R^L(\mathsf{y}) / j^1 R^L(\mathsf{y}) \right| k \right)^{1/2} \\ & \geq k^{2/r_\mathsf{y}} \Pi^{g_\mathsf{y}^{TY}, j_\mathsf{y}^{r_\mathsf{y}-2}R^L, J_\mathsf{y}^{TY}}(0,0) \\ & \geq \max \left[C_0 \left(\left| j^0 R^L(\mathsf{y}) \right| k \right), C_1 \left(\left| j^1 R^L(\mathsf{y}) / j^0 R^L(\mathsf{y}) \right| k \right)^{2/3} \right] \Big\} \end{aligned}$$

a rescaling of $\tilde{\Box}_k$ by $\delta_{k^{-1/4}}$ centered at y shows

$$\begin{aligned} \Pi_k(\mathsf{y},\mathsf{y}) &= k^{1/2} \left[1 + O\left(k^{2/r-1/2}\right) \right] \Pi^{g_\mathsf{y}^{TY}, j_\mathsf{y}^2 R^L / j_\mathsf{y}^1 R^L, J_\mathsf{y}^{TY}}(0,0) + O_{\left|j^{r_\mathsf{y}-2}R^L(y)\right|^{-1}}(1) \\ &= k^{1/2} \left[1 + O\left(k^{2/r-1/2}\right) \right] \left| j_\mathsf{y}^2 R^L / j_\mathsf{y}^1 R^L \right|^{1/2} \Pi^{g_\mathsf{y}^{TY}, \frac{j_\mathsf{y}^2 R^L / j_\mathsf{y}^1 R^L}{\left| j_\mathsf{y}^2 R^L / j_\mathsf{y}^1 R^L \right|}, J_\mathsf{y}^{TY}}(0,0) \\ &\quad + O_{\left|j^{r_\mathsf{y}-2}R^L(y)\right|^{-1}}(1) \\ &\geq (1-\varepsilon) k^{2/r_\mathsf{y}} \Pi^{g_\mathsf{y}^{TY}, j_\mathsf{y}^{r_\mathsf{y}-2}R^L, J_\mathsf{y}^{TY}}(0,0) + O_{\left|j^{r_\mathsf{y}-2}R^L(y)\right|^{-1}}(1) \qquad (3.37) \end{aligned}$$

Continuing in this fashion, we are finally left with the region

$$\mathsf{y} \in B_{c_\varepsilon \left| j^{r_y-2} R^L \right|}(y) \cap \left\{ k^{2/r_y} \Pi^{g_y^{TY}, j_y^{r_y-2} R^L, J_y^{TY}}(0,0) \right.$$
$$\left. \geq \max\left[C_0 \left(\left| j^0 R^L(\mathsf{y}) \right| k \right), \ldots, C_{r_y-3} \left(\left| j^{r_y-3} R^L(\mathsf{y}) / j^{r_y-4} R^L(\mathsf{y}) \right| k \right)^{2/(r_y-1)} \right] \right\}.$$

In this region we have

$$\left| j^{r_y-2} R^L(\mathsf{y}) / j^{r_y-3} R^L(\mathsf{y}) \right| \geq (1-\varepsilon) \left| j^{r_y-2} R^L(y) \right| + O\left(k^{2/r_y - 2/(r_y-1)} \right)$$

following (3.32) with the remainder being uniform. A rescaling by $\delta_{k^{-1/r_y}}$ then giving a similar estimate in this region, we have finally arrived at

$$\Pi_k(\mathsf{y}, \mathsf{y}) \geq (1-\varepsilon) k^{2/r_y} \Pi^{g_y^{TY}, j_y^{r_y-2} R^L, J_y^{TY}}(0,0) + O_{\left| j^{r_y-2} R^L(y) \right|^{-1}}(1)$$

for all $\mathsf{y} \in B_{c_\varepsilon \left| j^{r_y-2} R^L \right|}(y)$.

Finally a compactness argument finds a finite set of points $\{y_j\}_{j=1}^N$ such that the corresponding $B_{c_\varepsilon \left| j^{r_{y_j}-2} R^L \right|}(y_j)$'s cover Y. This gives a uniform constant $c_{1,\varepsilon} > 0$ such that

$$\Pi_k(y, y) \geq (1-\varepsilon) \left[\inf_{y \in Y_r} \Pi^{g_y^{TY}, j_y^{r-2} R^L, J_y^{TY}}(0,0) \right] k^{2/r} - c_{1,\varepsilon}$$

for all $y \in Y$, $\varepsilon > 0$ proving the lower bound (3.29). The argument for the upper bound is similar. □

We now prove a second lemma giving a uniform estimate on the derivatives of the Bergman kernel. Again below, the model Bergman kernel $\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)$ and its relevant ratio

$$\frac{\left| \left[j^l \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}} \right](0,0) \right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)}$$

are extended (continuously) by zero from $\left\{ y | j_y^1 R^L / j_y^0 R^L \neq 0 \right\}$ to Y.

Lemma 3.4 *The l-th jet of the Bergman kernel satisfies*

$$\left| j^l [\Pi_k(y, y)] \right| \leq k^{l/3} [1 + o(1)] \left[\sup_{y \in Y} \frac{\left| \left[j^l \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}} \right](0,0) \right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)} \right] \Pi_k(y, y)$$

with the $o(1)$ term being uniform in $y \in Y$.

Proof The proof follows a similar argument as the previous lemma. Given $\varepsilon > 0$ we find a uniform c_ε such that (3.32) holds for each $y \in Y$ and $\mathsf{y} \in B_{c_\varepsilon |j^{r_y - 2} R^L|}(y)$. Then rewrite the model Kodaira Laplacian $\tilde{\Box}_k$ (3.9) near y in terms of geodesic coordinates centered at y. Again, let the constants $C_0, C_1, C_2 \ldots$ be such that (3.30) holds. In the region

$$\mathsf{y} \in B_{c_\varepsilon |j^{r_y-2} R^L|}(y) \cap \left\{ C_0 \left(\left| j^0 R^L(\mathsf{y}) \right| k \right) \geq k^{2/r_y} \Pi^{g_y^{TY}, j_y^{r_y-2} R^L, J_y^{TY}}(0,0) \right\}$$

a rescaling of $\tilde{\Box}_k$ by $\delta_{k^{-1/2}}$, now centered at y, shows

$$\partial^\alpha \Pi_k(\mathsf{y}, \mathsf{y}) = \frac{k}{2\pi} \left(\partial^\alpha \tau^L(\mathsf{y}) \right) + O_{|j^{r_y-2} R^L(y)|^{-1}}(1)$$

following remark 3.2 as $r_{\mathsf{y}} = 2$. Dividing the above by (3.34) gives

$$\begin{aligned}
\frac{|\partial^\alpha \Pi_k(\mathsf{y}, \mathsf{y})|}{\Pi_k(\mathsf{y}, \mathsf{y})} &\leq \frac{\left| \partial^\alpha \tau^L(\mathsf{y}) \right|}{\tau^L(\mathsf{y})} + O_{|j^{r_y-2} R^L(y)|^{-1}}\left(k^{-1}\right) \\
&\leq k^{|\alpha|/3} \left[\sup_{y \in Y} \frac{\left| \left[j^{|\alpha|} \Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}} \right](0,0) \right|}{\Pi^{g_y^{TY}, j_y^1 R^L / j_y^0 R^L, J_y^{TY}}(0,0)} \right] \Pi_k(\mathsf{y}, \mathsf{y}) \\
&\quad + O_{|j^{r_y-2} R^L(y)|^{-1}}\left(k^{-1}\right)
\end{aligned}$$

Next, in the region

$$\begin{aligned}
\mathsf{y} \in B_{c_\varepsilon |j^{r_y-2} R^L|}(y) \cap \Big\{ & C_1 \left(\left| j^1 R^L(\mathsf{y}) / j^0 R^L(\mathsf{y}) \right| k \right)^{2/3} \\
& \geq k^{2/r_y} \Pi^{g_y^{TY}, j_y^{r_y-2} R^L, J_y^{TY}}(0,0) \geq C_0 \left(\left| j^0 R^L(\mathsf{y}) \right| k \right) \Big\}
\end{aligned}$$

a rescaling of $\tilde{\Box}_k$ by $\delta_{k^{-1/3}}$ centered at y similarly shows

$$\begin{aligned}
\partial^\alpha \Pi_k(\mathsf{y}, \mathsf{y}) = k^{(2+|\alpha|)/3} & \left[1 + O\left(k^{2/r - 2/3}\right) \right] \left[\partial^\alpha \Pi^{g_{\mathsf{y}}^{TY}, j_{\mathsf{y}}^1 R^L / j_{\mathsf{y}}^0 R^L, J_{\mathsf{y}}^{TY}} \right](0,0) \\
& + O_{|j^{r_y-2} R^L(y)|^{-1}}\left(k^{(1+|\alpha|)/3}\right)
\end{aligned}$$

as in Theorem 3.1. Dividing this by (3.36) gives

$$\begin{aligned}
\frac{|\partial^\alpha \Pi_k(\mathsf{y}, \mathsf{y})|}{\Pi_k(\mathsf{y}, \mathsf{y})} \leq k^{|\alpha|/3} (1+\varepsilon) & \frac{\left| \left[\partial^\alpha \Pi^{g_{\mathsf{y}}^{TY}, j_{\mathsf{y}}^1 R^L / j_{\mathsf{y}}^0 R^L, J_{\mathsf{y}}^{TY}} \right](0,0) \right|}{\left[\Pi^{g_{\mathsf{y}}^{TY}, j_{\mathsf{y}}^1 R^L / j_{\mathsf{y}}^0 R^L, J_{\mathsf{y}}^{TY}} \right](0,0)} \\
& + O_{|j^{r_y-2} R^L(y)|^{-1}}\left(k^{(|\alpha|-1)/3}\right)
\end{aligned}$$

$$\leq k^{|\alpha|/3}(1+\varepsilon)\left[\sup_{y\in Y}\frac{\left|\left[j^{|\alpha|}\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}(0,0)}\right] + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(k^{(|\alpha|-1)/3}\right).$$

Continuing in this fashion as before eventually gives

$$\frac{|\partial^\alpha \Pi_k(\mathsf{y},\mathsf{y})|}{\Pi_k(\mathsf{y},\mathsf{y})} \leq k^{|\alpha|/3}(1+\varepsilon)\left[\sup_{y\in Y}\frac{\left|\left[j^{|\alpha|}\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}(0,0)}\right] + O_{\left|j^{r_y-2}R^L(y)\right|^{-1}}\left(k^{(|\alpha|-1)/3}\right)$$

for all $y \in Y$, $\mathsf{y} \in B_{c_\varepsilon\left|j^{r_y-2}R^L\right|}(y)$, for all $\alpha \in \mathbb{N}_0^2$. By compactness one again finds a uniform $c_{1,\varepsilon}$ such that

$$\frac{|\partial^\alpha \Pi_k(y,y)|}{\Pi_k(y,y)} \leq k^{|\alpha|/3}(1+\varepsilon)\left[\sup_{y\in Y}\frac{\left|\left[j^{|\alpha|}\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}\right](0,0)\right|}{\Pi^{g_y^{TY},j_y^1R^L/j_y^0R^L,J_y^{TY}}(0,0)}\right] + c_{1,\varepsilon}$$

for all $y \in Y$, proving the lemma. □

4 Induced Fubini-Study Metrics

A theorem of Tian [37], with improvements in [14, 38] (see also [29, S 5.1.2, S 5.1.4]), asserts that the induced Fubini-Study metrics by Kodaira embeddings given by kth tensor powers of a positive line bundle converge to the curvature of the bundle as k goes to infinity. In this Section we will give a generalization for semi-positive line bundles on compact Riemann surfaces.

Let us review first Tian's theorem. Let (Y, J, g^{TY}) be a compact Hermitian manifold, (L, h^L), (F, h^F) be holomorphic Hermitian line bundles such that (L, h^L) is positive. We endow $H^0(Y; F \otimes L^k)$ with the L^2 product induced by g^{TY}, h^L, and h^F. This induces a Fubini-Study metric ω_{FS} on the projective space $\mathbb{P}\left[H^0\left(Y; F\otimes L^k\right)^*\right]$ and a Fubini-Study metric h_{FS} on $\mathcal{O}(1) \to \mathbb{P}\left[H^0\left(Y; F\otimes L^k\right)^*\right]$ (see [29, S 5.1]). Since (L, h^L) is positive the Kodaira embedding theorem shows that the Kodaira maps $\Phi_k : Y \to \mathbb{P}\left[H^0\left(Y; F\otimes L^k\right)^*\right]$ (see (4.7)) are embeddings for $k \gg 0$. Moreover, the Kodaira map induces a canonical isomorphism $\Theta_k : F \otimes L^k \to \Phi_k^*\mathcal{O}(1)$ and we have (see e.g., [29, (5.1.15)])

$$(\Theta_k^* h_{FS})(y) = \Pi_k(y,y)^{-1} h^{F\otimes L^k}(y), \quad y \in Y. \tag{4.1}$$

This implies immediately (see e.g., [29, (5.1.50)])

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \frac{i}{2\pi}R^L = \frac{i}{2\pi k}R^F - \frac{i}{2\pi k}\bar{\partial}\partial \ln \Pi_k(y, y). \tag{4.2}$$

Applying now the Bergman kernel expansion in the positive case one obtains Tian's theorem, which asserts that we have

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \frac{i}{2\pi}R^L = O\left(k^{-1}\right), \quad k \to \infty, \quad \text{in any } C^\ell\text{-topology}. \tag{4.3}$$

Let us also consider the convergence of the induced Fubini-Study metric $\Theta_k^* h_{FS}$ to the initial metric h^L. For this purpose we fix a metric h_0^L on L with positive curvature. We can then express $h^L = e^{-\varphi}h_0^L$, $\Theta_k^* h_{FS} = e^{-\varphi_k}(h_0^L)^k \otimes h^F$, where $\varphi, \varphi_k \in C^\infty(Y)$ are the global potentials of the metrics h and $\Theta_k^* h_{FS}$ with respect to h_0^L and $(h_0^L)^k \otimes h^F$. Note that

$$R^{(L,h^L)} = R^{(L,h_0^L)} + \partial\bar{\partial}\varphi, \quad R^{(L^k,\Theta_k^* h_{FS})} = kR^{(L,h_0^L)} + R^{(F,h^F)} + \partial\bar{\partial}\varphi_k,$$

and $\frac{i}{2\pi}R^{(L,\Theta_k^* h_{FS})} = \Phi_k^*\omega_{FS}$. Then (4.1) can be written as

$$\frac{1}{k}\varphi_k(y) - \varphi(y) = \frac{1}{k}\ln \Pi_k(y, y), \quad y \in Y. \tag{4.4}$$

We obtain by (1.3) that

$$\left|\frac{1}{k}\varphi_k - \varphi\right|_{C^0(Y)} = O\big(k^{-1}\ln k\big), \quad k \to \infty, \tag{4.5}$$

that is, the normalized potentials of the Fubini-Study metric converge uniformly on Y to the potential of the initial metric h^L with speed $k^{-1}\ln k$. Moreover,

$$\left|\frac{1}{k}\partial\varphi_k - \partial\varphi\right|_{C^0(Y)} = O\big(k^{-1}\big), \quad \left|\frac{1}{k}\partial\bar{\partial}\varphi_k - \partial\bar{\partial}\varphi\right|_{C^0(Y)} = O\big(k^{-1}\big), \quad k \to \infty, \tag{4.6}$$

and we get the same bound $O\big(k^{-1}\big)$ for higher derivatives, obtaining again (4.3). Note that if g^{TY} is the metric associated to $\omega = \frac{i}{2\pi}R^L$, then we have a bound $O(k^{-2})$ in (4.3) and (4.6).

We return now to our situation and consider that Y is a compact Riemann surface and (L, h^L), (F, h^F) be holomorphic Hermitian line bundles on Y such that (L, h^L) is semi-positive and its curvature vanishes at finite order. An immediate consequence of Lemma 3.3 is that the base locus

$$\mathrm{Bl}\left(F \otimes L^k\right) := \left\{y \in Y | s(y) = 0, \ s \in H^0\left(Y; F \otimes L^k\right)\right\} = \emptyset$$

is empty for $k \gg 0$. This shows that the subspace

$$\Phi_{k,y} := \left\{ s \in H^0\left(Y; F \otimes L^k\right) | s(y) = 0 \right\} \subset H^0\left(Y; F \otimes L^k\right),$$

is a hyperplane for each $y \in Y$. One may identify the Grassmannian $\mathbb{G}\left(d_k - 1; H^0\left(Y; F \otimes L^k\right)\right)$, $d_k := \dim H^0\left(Y; F \otimes L^k\right)$, with the projective space $\mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right]$ by sending a non-zero dual element in $H^0\left(Y; F \otimes L^k\right)^*$ to its kernel. This now gives a well-defined Kodaira map

$$\begin{aligned} &\Phi_k : Y \to \mathbb{P}\left[H^0\left(Y; F \otimes L^k\right)^*\right], \\ &\Phi_k(y) := \left\{ s \in H^0\left(Y; F \otimes L^k\right) | s(y) = 0 \right\}. \end{aligned} \tag{4.7}$$

It is well known that the map is holomorphic.

Theorem 4.1 *Let Y be a compact Riemann surface and (L, h^L), (F, h^F) be holomorphic Hermitian line bundles on Y such that (L, h^L) is semi-positive and its curvature vanishes at most at finite order. Then the normalized potentials of the Fubini-Study metric converge uniformly on Y to the potential of the initial metric h^L with speed $k^{-1}\ln k$ as in (4.5). Moreover,*

$$\left| \frac{1}{k}\partial\varphi_k - \partial\varphi \right|_{C^0(Y)}, \ \left| \frac{1}{k}\bar{\partial}\varphi_k - \bar{\partial}\varphi \right|_{C^0(Y)} = O\left(k^{-2/3}\right), \ \ k \to \infty, \tag{4.8}$$

and

$$\left| \frac{1}{k}\partial\bar{\partial}\varphi_k - \partial\bar{\partial}\varphi \right|_{C^0(Y)} = O\left(k^{-1/3}\right), \ \ k \to \infty, \tag{4.9}$$

especially

$$\frac{1}{k}\Phi_k^*\omega_{FS} - \frac{i}{2\pi}R^L = O\left(k^{-1/3}\right), \ \ k \to \infty, \tag{4.10}$$

uniformly on Y. On compact sets of Y_2 the estimates (4.3) and (4.6) hold.

Proof The proof follows from (4.2), (4.4), and the uniform estimate of Lemma 3.4 on the derivatives of the Bergman kernel. □

As we noted before, the bundle L satisfying the hypotheses of Theorem 4.1 is ample, so for $k \gg 0$ the Kodaira map is an embedding and the induced Fubini-Study forms $\frac{1}{k}\Phi_k^*\omega_{FS}$ are indeed metrics on Y. Due to the possible degeneration of the curvature R^L the rate of convergence in (4.10) is slower than in the positive case (4.3).

One can easily prove a generalization of Theorem 4.1 for vector bundles (F, h^F) of arbitrary rank (see [29, S 5.1.4] for the case of a positive bundle (L, h^L)). We have then Kodaira maps $\Phi_k : Y \to \mathbb{G}\left(\mathrm{rk}(F); H^0\left(Y; F \otimes L^k\right)^*\right)$ into the Grassmannian of $\mathrm{rk}(F)$-dimensional linear spaces of $H^0\left(Y; F \otimes L^k\right)^*$ and we introduce the Fubini-Study metric on the Grassmannian as the curvature of the determinant bundle of the dual of the tautological bundle (cf. [29, (5.1.6)]). Then by following the proof of [29, Theorem 5.1.17] and using Lemma 3.4 we obtain

$$\frac{1}{k}\Phi_k^* \omega_{FS} - \mathrm{rk}(F)\,\frac{i}{2\pi} R^L = O\left(k^{-1/3}\right), \quad k \to \infty, \tag{4.11}$$

uniformly on Y.

5 Toeplitz Operators

A generalization of the projector (3.1) and Bergman kernel (3.2) is given by the notion of a Toeplitz operator. The Toeplitz operator $T_{f,k}$ operator corresponding to a section $f \in C^\infty(Y; \mathrm{End}(F))$ is defined via

$$T_{f,k} : C^\infty\left(Y; F \otimes L^k\right) \to C^\infty\left(Y; F \otimes L^k\right), \quad T_{f,k} := \Pi_k f \Pi_k, \tag{5.1}$$

where f denotes the operator of pointwise composition by f. Each Toeplitz operator above further maps $H^0\left(Y; F \otimes L^k\right)$ to itself.

We now prove the expansion for the kernel of a Toeplitz operator generalizing Theorem 1.1. For positive line bundles the analogous result was proved in [15, Theorem 2] for compact Kähler manifolds and $F = \mathbb{C}$ and in [29, Lemma 7.2.4 and (7.4.6)], [31, Lemma 4.6], in the symplectic case.

Theorem 5.1 *Let Y be a compact Riemann surface, $(L, h^L) \to Y$ a semi-positive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F, h^F) \to Y$ be a Hermitian holomorphic vector bundle. Then the kernel of the Toeplitz operator (5.1) has an on-diagonal asymptotic expansion*

$$T_{f,k}(y, y) = k^{2/r_y}\left[\sum_{j=0}^{N} c_j(f, y)\, k^{-2j/r_y}\right] + O\left(k^{-2N/r_y}\right), \quad \forall N \in \mathbb{N}$$

where the coefficients $c_j(f, \cdot)$ are sections of End(F) with leading term

$$c_0(f, y) = \Pi^{g_y^{TY}, R_y^{TY}, J_y^{TY}}(0, 0)\, f(y).$$

Proof Firstly from the definition (5.1) and the localization/rescaling properties (3.14), (3.21) one has

$$\begin{aligned} T_{f,k}(y, y) &= \int_Y dy'\, \Pi_k\left(y, y'\right) f\left(y'\right) \Pi_k\left(y', y\right) \\ &= \int_{B_\varepsilon(y)} dy'\, \tilde{\Pi}_k\left(0, y'\right) f\left(y'\right) \tilde{\Pi}_k\left(y', 0\right) + O\left(k^{-\infty}\right) \end{aligned}$$

$$
\begin{aligned}
&= \int_{B_\varepsilon(y)} dy'\, k^{4/r_y} \Pi^{\boxdot}\left(0, y' k^{1/r_y}\right) f\left(y'\right) \Pi^{\boxdot}\left(y' k^{1/r_y}, 0\right) + O\left(k^{-\infty}\right) \\
&= \int_{k^{1/r_y} B_\varepsilon(y)} dy'\, k^{2/r_y} \Pi^{\boxdot}\left(0, y'\right) f\left(y' k^{-1/r_y}\right) \Pi^{\boxdot}\left(y', 0\right) + O\left(k^{-\infty}\right).
\end{aligned} \tag{5.2}
$$

Next as in Section A, $\varphi\left(\boxdot\right)(.,0) \in \mathcal{S}(V)$ for $\varphi \in \mathcal{S}(\mathbb{R})$ in the Schwartz class via a finite propagation argument. Thus plugging (3.25) and a Taylor expansion

$$
f\left(y' k^{-1/r_y}\right) = \sum_{|\alpha| \le N+1} \frac{1}{\alpha!} \left(y'\right)^\alpha k^{-\alpha/r_y} f^{(\alpha)}(0) + O\left(k^{-(N+1)/r_y}\right)
$$

into (5.2) above gives the result with the leading term again coming from (3.20). Finally and as in the proof of Theorem 1.1, there are no odd powers of k^{-j/r_y} as the corresponding coefficients are given by odd integrals (the integrands change sign by $(-1)^j$ under $\delta_{-1} x := -x$) which are zero. □

We now show that the Toeplitz operators (5.1) can be composed up to highest order generalizing the results of [11] in the Kähler case and $F = \mathbb{C}$ and [29, Theorems 7.4.1–2], [31, Theorems 1.1 and 4.19] in the symplectic case.

Theorem 5.2 *Given $f, g \in C^\infty(Y; End(F))$, the Toeplitz operators (5.1) satisfy*

$$
\lim_{k \to \infty} \left\| T_{f,k} \right\| = \|f\|_\infty := \sup_{\substack{y \in Y \\ u \in F_y \setminus 0}} \frac{|f(y) u|_{h^F}}{|u|_{h^F}}, \tag{5.3}
$$

$$
T_{f,k} T_{g,k} = T_{fg,k} + O_{L^2 \to L^2}\left(k^{-1/r}\right). \tag{5.4}
$$

Proof The first part of (5.3) is similar to the positive case. Firstly, $\left\|T_{f,k}\right\| \le \|f\|_\infty$ is clear from the definition (5.1). For the lower bound, let us consider $y \in Y_2$ where the curvature is non-vanishing and $u \in F_y$, $|u|_{h^F} = 1$. It follows from the proof of [29, Theorem 7.4.2] (see also [2, Proposition 5.2, (5.40), Remark 5.7]) that

$$
|f(y)(u)|_{h^F} + O_{y,u}\left(k^{-1/2}\right) \le \left\|T_{f,k}\right\|. \tag{5.5}
$$

If $\|f\|_\infty = |f(y_0)(u_0)|_{h^F}$ is attained at a point $y_0 \in Y_2$, it follows immediately from (5.5) that

$$
\|f\|_\infty + O\left(k^{-1/2}\right) \le \left\|T_{f,k}\right\|,
$$

so one obtains the lower bound. Next let $\|f\|_\infty = |f(y_0)(u_0)|_{h^F}$ be attained at $y_0 \in Y \setminus Y_2$, a vanishing point of the curvature. As $Y \setminus Y_2 \subset Y$ is open and dense one may find for any $\varepsilon > 0$ a point $y_\varepsilon \in Y \setminus Y_2$ and $u_\varepsilon \in F_{y_\varepsilon}$, $|u_\varepsilon|_{h^F} = 1$, with

$\|f\|_\infty - \varepsilon \leq |f(y_\varepsilon)(u_\varepsilon)|_{h^F}$. Combined with (5.5) this gives

$$\|f\|_\infty - \varepsilon + O_\varepsilon\left(k^{-1/2}\right) \leq \left\|T_{f,k}\right\|, \quad \text{and}$$
$$\|f\|_\infty - \varepsilon \leq \liminf_{k\to\infty} \left\|T_{f,k}\right\|.$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|f\|_\infty \leq \liminf_{k\to\infty} \left\|T_{f,k}\right\|$ proving the lower bound.

Next, to prove the composition expansion (5.4) it suffices to prove a uniform kernel estimate

$$\left\|\left[T_{f,k}T_{g,k} - T_{fg,k}\right](.,y)\right\|_{L^2} = O\left(k^{-1/r}\right), \quad \forall y \in Y.$$

To this end we again compute in geodesic chart centered at y

$$\begin{aligned}
T_{f,k}T_{g,k}(.,0) &= \int_{Y\times Y} dy_1 dy_2\, \Pi_k(.,y_1)\, f(y_1)\, \Pi_k(y_1,y_2)\, g(y_2)\, \Pi_k(y_2,0) \\
&= O_{L^2}\left(k^{-\infty}\right) + \int_{B_\varepsilon(y_2)} dy_1 \int_{B_\varepsilon(y)} dy_2\, \tilde{\Pi}_k \\
&\quad (.,y_1)\, f(y_1)\, \tilde{\Pi}_k(y_1,y_2)\, g(y_2)\, \tilde{\Pi}_k(y_2,0) \\
&= O_{L^2}\left(k^{-\infty}\right) + \int_{B_\varepsilon(y_2)} dy_1 \int_{B_\varepsilon(y)} dy_2 k^{6/r_y} \left\{\Pi^{\boxdot}\left(k^{1/r_y}.,k^{1/r_y}y_1\right)\right. \\
&\qquad \left. f(y_1)\, \Pi^{\boxdot}\left(k^{1/r_y}y_1, k^{1/r_y}y_2\right) g(y_2)\, \Pi^{\boxdot}\left(k^{1/r_y}y_2, 0\right)\right\} \\
&= O_{L^2}\left(k^{-\infty}\right) + \int_{k^{1/r_y}B_\varepsilon(y_2)} dy_1 \int_{k^{1/r_y}B_\varepsilon(y)} dy_2 k^{2/r_y} \left\{\Pi^{\boxdot}\left(k^{1/r_y}.,y_1\right)\right. \\
&\qquad \left. f\left(y_1 k^{-1/r_y}\right) \Pi^{\boxdot}(y_1,y_2)\, g\left(y_2 k^{-1/r_y}\right) \Pi^{\boxdot}(y_2,0)\right\} \\
&= O_{L^2}\left(k^{-1/r_y}\right) + \int_{k^{1/r_y}B_\varepsilon(y_2)} dy_1 \int_{k^{1/r_y}B_\varepsilon(y)} dy_2 k^{2/r_y} \left\{\Pi^{\boxdot}\left(k^{1/r_y}.,y_1\right)\right. \\
&\qquad \left. \Pi^{\boxdot}(y_1,y_2)\, fg\left(y_2 k^{-1/r_y}\right) \Pi^{\boxdot}(y_2,0)\right\} \\
&= O_{L^2}\left(k^{-1/r_y}\right) + \int_{B_\varepsilon(y_2)} dy_1 \int_{B_\varepsilon(y)} dy_2\, \tilde{\Pi}_k(.,y_1)\, \tilde{\Pi}_k(y_1,y_2) \\
&\quad fg(y_2)\, \tilde{\Pi}_k(y_2,0) \\
&= O_{L^2}\left(k^{-1/r_y}\right) + T_{fg,k}
\end{aligned}$$

with all remainders being uniform in $y \in Y$. Above we have again used the localization/rescaling properties (3.14), (3.21). As well as the first-order Taylor expansion $f\left(y_1 k^{-1/r_y}\right) = f\left(y_2 k^{-1/r_y}\right) + O_{\|f\|_{C^1}}\left(|y_1 - y_2|\, k^{-1/r_y}\right)$ and the off-diagonal decay of $\Pi^{\boxdot}(.,y_2) \in \mathcal{S}\left(\mathbb{R}^2\right)$. □

Remark 5.3 Similar to the previous Remark 3.2, we can recover the usual algebra properties of Toeplitz operators when f, g are compactly supported on the set Y_2

where the curvature R^L is positive. In particular we define a generalized Toeplitz operator to be a sequence of operators $T_k : L^2(Y, F \otimes L^k) \longrightarrow L^2(Y, F \otimes L^k)$, $k \in \mathbb{N}$, such that there exist $K \Subset Y_2$, $h_j \in C_c^\infty(K; \mathrm{End}(F))$, $C_j > 0$, $j = 0, 1, 2, \ldots$ satisfying

$$\left\| T_k - \sum_{j=0}^{N} k^{-j} T_{h_j,k} \right\| \leq C_N k^{-N-1}, \quad \forall N \in \mathbb{N}. \tag{5.6}$$

Then this class is closed under composition and one may define a formal star product on $C_c^\infty(Y_2)\left[\left[k^{-1}\right]\right]$, via

$$f *_{k^{-1}} g = \sum_{j=0}^{\infty} C_j(f, g) k^{-j} \in C_c^\infty(Y_2)\left[\left[k^{-1}\right]\right] \quad \text{where}$$
$$T_{f,k} \circ T_{g,k} \sim \sum_{j=0}^{\infty} T_{C_j(f,g)} k^{-j},$$

(cf. [11, 15, 31]). Furthermore

$$T_{f,k} \circ T_{g,k} = T_{fg,k} + O_{L^2 \to L^2}\left(k^{-1}\right)$$
$$[T_{f,k}, T_{g,k}] = \frac{i}{k} T_{\{f,g\},k} + O_{L^2 \to L^2}(k^{-2})$$

for all $f, g \in C_c^\infty(Y_2; \mathrm{End}(F))$, with $\{\cdot, \cdot\}$ being the Poisson bracket on the Kähler manifold (Y_2, iR^L).

Finally we address the asymptotics of the spectral measure of the Toeplitz operator (5.1), called Szegő-type limit formulas [12, 21]. The spectral measure of $T_{f,k}$ is defined via

$$u_{f,k}(s) := \sum_{\lambda \in \mathrm{Spec}(T_{f,k})} \delta(s - \lambda) \in \mathcal{S}'(\mathbb{R}_s). \tag{5.7}$$

We now have the following asymptotic formula.

Theorem 5.4 *The spectral measure* (5.7) *satisfies*

$$u_{f,k} \sim \frac{k}{2\pi} f_* R^L \tag{5.8}$$

in the distributional sense as $k \to \infty$.

Proof Since $\mathrm{Spec}\left(T_{f,k}\right) \subset \left[-\|f\|_\infty, \|f\|_\infty\right]$ by (5.3), equation (5.8) is equivalent to

$$\mathrm{tr}\, \varphi\left(T_{f,k}\right) = \sum_{\lambda \in \mathrm{Spec}(T_{f,k})} \varphi(\lambda) \sim \frac{k}{2\pi} \int_Y [\varphi \circ f] R^L,$$

for all $\varphi \in C_c^\infty\left(-\|f\|_\infty - 1, \|f\|_\infty + 1\right)$. We first prove that the trace of a Toeplitz operator (5.1) satisfies the asymptotics

$$\operatorname{tr} T_{f,k} \sim \frac{k}{2\pi}\int_Y f R^L. \tag{5.9}$$

To this end first note that the expansion of Theorem 5.1 is uniform on compact subsets $K \subset Y_2$ while $\left|T_{f,k}(y,y)\right| = O(k)$ uniformly in $y \in Y$ as in Lemma 3.3. Furthermore, as in [32, Proposition 7], $Y_{\geq 3}$ is a closed subset of a hypersurface and has measure zero. Let $K_j \subset Y_2$, $j = 1, 2, \ldots$, be a sequence of compact subsets satisfying $K_j \subset K_{j+1}$, $\cap_{j=1}^\infty K_j^c = Y_{\geq 3}$. One may then breakup the trace integral

$$\begin{aligned} \frac{1}{k}\operatorname{tr} T_{f,k} &= \frac{1}{k}\int_{K_j} \operatorname{tr} T_{f,k}(y,y) + \frac{1}{k}\int_{Y\setminus K_j} \operatorname{tr} T_{f,k}(y,y) \\ &= \frac{1}{2\pi}\int_{K_j} f R^L + O_j\left(\frac{1}{k}\right) + O\left(\mu\left(Y \setminus K_j\right)\right) \end{aligned}$$

from which (5.9) follows on knowing $\frac{1}{2\pi}\int_{K_j} f R^L \to \frac{1}{2\pi}\int_Y f R^L$, $\mu\left(Y \setminus K_j\right) \to 0$ as $j \to \infty$.

Following this one has

$$\operatorname{tr} T_{f,k}^l = \operatorname{tr} T_{f^l,k} + O_f\left(k^{1-1/r}\right)$$

for all $l \in \mathbb{N}$ from (2.15), (5.4). A polynomial approximation of the compactly supported function $\varphi \in C_c^\infty\left(-\|f\|_\infty - 1, \|f\|_\infty + 1\right)$ then gives

$$\begin{aligned} \operatorname{tr} \varphi\left(T_{f,k}\right) &= \operatorname{tr} T_{\varphi\circ f,k} + o(k) \\ &= \frac{k}{2\pi}\int_Y [\varphi \circ f]\, R^L + o(k) \end{aligned}$$

by (5.9) as required. □

The analogous result for projective manifolds endowed with the restriction of the hyperplane bundle was originally proved in [12, Theorem 13.13], [21] and for arbitrary positive line bundles in [6], see also [27]. In [23, Theorem 1.6] the asymptotics (5.9) are proved for a semiclassical spectral function of the Kodaira Laplacian on an arbitrary manifold.

5.1 Branched Coverings

We now consider Toeplitz operators and their composition in a particular case of semi-positive line bundles. Namely, those that arise from pullbacks along branched coverings. Here $f : Y \to Y_0$ is a branched covering of a Riemann surface Y_0 with branch points $\{y_1, \ldots, y_M\} \subset Y$. The Hermitian holomorphic line bundle on Y is pulled back $(L, h^L) = (f^* L_0, f^* h^{L_0})$ from one on Y_0. If (L_0, h^{L_0}) is assumed positive, then (L, h^L) is semi-positive with curvature vanishing at the branch points. In particular, near a branch point $y \in Y$ of local degree $\frac{r}{2}$ one may find holomorphic geodesic coordinate such that the curvature is given by $R^L = \frac{r^2}{4} (z\bar{z})^{r/2--1} R^{L_0}_{f(y)} + O(y^{r-1})$. We set in the following $R_0 =: R^{L_0}_{f(y)}$. The leading term of (1.3) is given by the model Bergman kernel $\Pi^{\boxdot_0}(0, 0)$ of the operator

$$\boxdot_0 = bb^\dagger, \quad \text{for} \tag{5.10}$$

$$b = -2\partial_z + \bar{a} \tag{5.11}$$

$$b^\dagger = 2\partial_{\bar{z}} + a,$$

$$a = \frac{r}{4} z (z\bar{z})^{r/2-1} R_0. \tag{5.12}$$

We first compute this model Bergman kernel.

Lemma 5.5 *The model Bergman kernel corresponding to the model operator* (5.10) *at a branch point is given by*

$$\Pi^{\boxdot_0}(z, z') = \frac{r e^{-2[\Phi(z)+\Phi(z')]} R_0^{\frac{2}{r}}}{2\pi} G\left(R_0^{\frac{2}{r}} z\overline{z'}\right) \quad \textit{where} \tag{5.13}$$

$$\Phi(z) := \frac{1}{4} (z\bar{z})^{r/2} R_0 \quad \textit{and} \tag{5.14}$$

$$G(x) := \sum_{\alpha=0}^{\frac{r}{2}-1} \frac{x^\alpha}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)} + x^{\frac{r}{2}-1} e^{x^{\frac{r}{2}}} \left[\sum_{\alpha=0}^{\frac{r}{2}-2} \frac{\Gamma\left(\frac{2(\alpha+1)}{r}\right) - \Gamma\left(\frac{2(\alpha+1)}{r}, x^{\frac{r}{2}}\right)}{\Gamma\left(\frac{2(\alpha+1)}{r}\right) / \left(\frac{2(\alpha+1)}{r} - 1\right)}\right] \tag{5.15}$$

is given in terms of the incomplete gamma function.

Proof From the formulas (5.12), an orthonormal basis for $\ker(\boxdot_0)$ is easily found to be

$$s_\alpha := \left(\frac{1}{2\pi} \frac{r}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)} R_0^{\frac{2(\alpha+1)}{r}}\right)^{1/2} z^\alpha e^{-\Phi}, \quad \alpha \in \mathbb{N}_0,$$

$$\text{with} \quad \Phi := \frac{1}{4} (z\bar{z})^{r/2} R_0.$$

From here the model Bergman kernel is computed

$$\begin{aligned} \Pi^{\boxdot_0}(z, z') &= \sum_{\alpha \in \mathbb{N}_0} s_\alpha(z) \overline{s_\alpha(z')} \\ &= \frac{1}{2\pi} \sum_{\alpha \in \mathbb{N}_0} \frac{r}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)} R_0^{\frac{2(\alpha+1)}{r}} \left(z\overline{z'}\right)^\alpha e^{-2\Phi}. \end{aligned} \tag{5.16}$$

To compute the above in a closed form, consider the series

$$F(y) := \sum_{\alpha=0}^{\infty} \frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)}$$
$$= \sum_{\alpha=0}^{s-1} \frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)} + \underbrace{\sum_{\alpha=s}^{\infty} \frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)}}_{F_0(y):=},$$

for $s = \frac{r}{2}$. Differentiating the second term in the series gives $F_0'(y) = F_0(y) + \sum_{\alpha=0}^{s-2} \left(\frac{\alpha+1}{s} - 1\right) \frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)}$, which is an ODE that can be solved with the initial condition $F_0(0) = 0$ to give

$$F_0(y) = \sum_{\alpha=s}^{\infty} \frac{y^{\frac{\alpha+1}{s}-1}}{\Gamma\left(\frac{\alpha+1}{s}\right)} = e^y \left[\sum_{\alpha=0}^{s-2} \frac{\Gamma\left(\frac{\alpha+1}{s}\right) - \Gamma\left(\frac{\alpha+1}{s}, y\right)}{\Gamma\left(\frac{\alpha+1}{s}\right) / \left(\frac{\alpha+1}{s} - 1\right)}\right]$$
$$\text{in terms of } \Gamma(a, z) := \int_z^{\infty} t^{a-1} e^{-t} dt, \quad \mathrm{Re}(z) > 0, \tag{5.17}$$

the incomplete gamma function. Thus in particular we have computed $F(y) := y^{\frac{1}{s}-1} G\left(y^{\frac{1}{s}}\right)$ (5.15). Finally noting from (5.16) that

$$\Pi^{\boxdot_0}\left(z, z'\right) = \frac{r e^{-2\Phi} R_0^{\frac{2}{r}}}{2\pi} x^{s-1} F\left(x^s\right),$$

for $x = R_0^{\frac{2}{r}} z\overline{z'}$, completes the proof. □

This gives the first term of the expansion

$$c_0(y) = \Pi^{\boxdot_0}(0, 0) = \frac{1}{2\pi} \frac{r}{\Gamma\left(\frac{2}{r}\right)} R_0^{\frac{2}{r}}$$

at the vanishing/branch point y in this example.

6 Random Sections

In this section we generalize the results of [36] to the semi-positive case considered here. Let us consider Hermitian holomorphic line bundles (L, h^L) and (F, h^F) on a compact Riemann surface Y. To state the result first note that the natural metric on $H^0\left(Y; F \otimes L^k\right)$ arising from g^{TY}, h^F, and h^L gives rise to a probability density μ_k

on the sphere

$$SH^0\left(Y; F \otimes L^k\right) := \left\{s \in H^0\left(Y; F \otimes L^k\right) \mid \|s\| = 1\right\},$$

of finite dimension $\chi\left(Y; F \otimes L^k\right) - 1$ (2.15). We now define the product probability space $(\Omega, \mu) := \left(\Pi_{k=1}^{\infty} SH^0\left(Y; F \otimes L^k\right), \Pi_{k=1}^{\infty}\mu_k\right)$. To a random sequence of sections $s = (s_k)_{k\in\mathbb{N}} \in \Omega$ given by this probability density, we then associate the random sequence of zero divisors $Z_{s_k} = \{s_k = 0\}$ and view it as a random sequence of currents of integration in $\Omega'_{0,0}(Y)$. We now have the following.

Theorem 6.1 *Let (L, h^L) and (F, h^F) be Hermitian holomorphic line bundles on a compact Riemann surface Y and assume that (L, h^L) is semi-positive line bundle and its curvature R^L vanishes to finite order at any point. Then for μ-almost all $s = (s_k)_{k\in\mathbb{N}} \in \Omega$, the sequence of currents*

$$\frac{1}{k} Z_{s_k} \rightharpoonup \frac{i}{2\pi} R^L$$

converges weakly to the semi-positive curvature form.

Proof The proof follows [29, Theorem 5.3.3] with some modifications which we point out below. With Φ_k denoting the Kodaira map (4.7), we first have

$$\mathbb{E}\left[Z_{s_k}\right] = \Phi_k^*\left(\omega_{FS}\right) \tag{6.1}$$

as in [29, Theorem 5.3.3]. For a given $\varphi \in \Omega^{0,0}(Y)$, one has

$$\left\langle \frac{1}{k} Z_{s_k} - \frac{i}{2\pi} R^L, \varphi \right\rangle = \left\langle \frac{1}{k} Z_{s_k} - \frac{1}{k}\Phi_k^*\left(\omega_{FS}\right), \varphi \right\rangle + O\left(k^{-1/3}\|\varphi\|_{C^0}\right)$$

following (4.10) and it thus suffices to show $Y^{\varphi}(s_k) \to 0$, μ-almost surely with

$$Y^{\varphi}(s_k) := \left\langle \frac{1}{k} Z_{s_k} - \frac{1}{k}\Phi_k^*\left(\omega_{FS}\right), \varphi \right\rangle$$

being the given random variable. But (6.1) gives

$$\begin{aligned}\mathbb{E}\left[\left|Y^{\varphi}(s_k)\right|^2\right] &= \frac{1}{k^2}\mathbb{E}\left[\left\langle Z_{s_k}, \varphi\right\rangle^2\right] - \frac{1}{k^2}\mathbb{E}\left[\left\langle \Phi_k^*\left(\omega_{FS}\right), \varphi\right\rangle^2\right]\\ &= O\left(k^{-2}\right)\end{aligned}$$

as in [29, Theorem 5.3.3]. Thus $\int_{\Omega} d\mu \left[\sum_{k=1}^{\infty}\left|Y^{\varphi}(s_k)\right|^2\right] < \infty$ proving the theorem. □

The above result may be alternatively obtained using L^2 estimates for the $\bar{\partial}$-equation of a modified positive metric as in [18, S 4].

Example 6.2 (Random polynomials) The last theorem has an interesting specialization to random polynomials. To this end, let $Y = \mathbb{CP}^1 = \mathbb{C}^2_w \setminus \{0\} / \mathbb{C}^*$ with homogeneous coordinates $[w_0 : w_1]$. A semi-positive curvature form for each even $r \geq 2$ is given by

$$\begin{aligned}\omega_r :=& \frac{i}{2\pi} \partial\bar{\partial} \ln \left(|w_0|^r + |w_1|^r \right) \\ =& \frac{i}{2\pi} \frac{r^2}{4} \frac{|z|^{r-2}}{\left(1 + |z|^r\right)^2} dz \wedge d\bar{z}, \quad \text{for } z = \frac{w_0}{w_1} \neq 0,\end{aligned} \tag{6.2}$$

which can be seen to have two vanishing points at the north/south poles of order $r-2$. This is the curvature form on the hyperplane line bundle $L = \mathcal{O}(1)$ for the metric with potential $\varphi = \ln \left(|w_0|^r + |w_1|^r \right)$. An orthogonal basis for $H^0 \left(X, L^k \right)$ is given by $s_\alpha := z^\alpha$, $0 \leq \alpha \leq k$, in terms of the affine coordinate $z = w_0/w_1$ on the chart $\{w_1 \neq 0\}$ and a $\mathbb{C}^*$ invariant trivialization of L. The normalization is now given by

$$\begin{aligned}\|s_\alpha\|^2 &= \frac{1}{2\pi} \frac{r^2}{4} \int_{\mathbb{C}} \frac{|z|^{2\alpha + r - 2}}{\left(1 + |z|^r\right)^{k+2}} \\ &= \frac{1}{\frac{2}{r}(k+1) \binom{k}{\frac{2}{r}\alpha}}\end{aligned}$$

with the binomial coefficient

$$\binom{k}{\frac{2}{r}\alpha} = \frac{\Gamma(k+1)}{\Gamma\left(\frac{2}{r}\alpha + 1\right) \Gamma\left(k - \frac{2}{r}\alpha\right)}$$

given in terms of the Gamma function. We have now arrived at the following.

Corollary 6.3 *For each even $r \geq 2$, let*

$$p_k(z) = \sum_{\alpha=0}^{k} c_\alpha \sqrt{\binom{k}{\frac{2}{r}\alpha}} z^\alpha$$

be a random polynomial of degree k with the coefficients c_α being standard i.i.d. Gaussian variables. The distribution of its roots converges in probability

$$\frac{1}{k} Z_{p_k} \rightharpoonup \frac{1}{2\pi} \frac{r^2}{4} \frac{|z|^{r-2}}{\left(1 + |z|^r\right)^2}.$$

The above theorem interpolates between the case of $SU(2)$/elliptic polynomials ($r = 2$) [10] and the case of Kac polynomials ($r = \infty$) [22, 26, 35]. For recent results on the distribution of zeroes of more general classes of random polynomials we refer to [3, 9, 25].

7 Holomorphic Torsion

In this section we give an asymptotic result for the holomorphic torsion of the semi-positive line bundle L generalizing that of [8] (see also [29, S 5.5]). First recall that the holomorphic torsion of L is defined in terms of the zeta function

$$\zeta_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{tr}\left[e^{-t\Box_k^1}\right], \quad \mathrm{Re}(s) > 1. \tag{7.1}$$

The above converges absolutely and defines a holomorphic function of $s \in \mathbb{C}$ in this region. It possesses a meromorphic extension to $\mathbb{C}$ with no pole at zero and the holomorphic torsion is defined to be $\mathcal{T}_k := \exp\left\{-\frac{1}{2}\zeta_k'(0)\right\}$.

Next, with $\tau^L, \omega\left(R^L\right)$ as in (2.11) and $t > 0$, set

$$R_t(y) := \begin{cases} \frac{1}{2\pi}\tau^L\left(1 - e^{-t\tau^L}\right)^{-1} e^{-t\omega(R^L)}; & \tau^L(y) > 0 \\ \frac{1}{2\pi}\frac{1}{t}; & \tau^L(y) = 0. \end{cases} \tag{7.2}$$

Note that the above defines a smooth endomorphism $R_t(y) \in C^\infty\left(Y; \mathrm{End}\left(\Lambda^{0,*}\right)\right)$. Further, let $A_j \in C^\infty\left(Y; \mathrm{End}\left(\Lambda^{0,*}\right)\right)$ be such that

$$\rho_t^N := R_t(y) - \sum_{j=-1}^{N} A_j(y)\, t^j = O\left(t^{N+1}\right). \tag{7.3}$$

We now have the following uniform small time asymptotic expansion for the heat kernel [29, Theorem 5.5.9].

Proposition 7.1 *There exist $A_{k,j} \in C^\infty\left(Y; End\left(\Lambda^{0,*}\right)\right)$, $j = -1, 0, 1, \ldots$, satisfying $A_{k,j} - A_j = O\left(k^{-1}\right)$, such that for each $t > 0$*

$$\left| k^{-1} e^{-\frac{t}{2k} D_k^2}(y, y) - \sum_{j=-1}^{N} A_{k,j}(y)\, t^j - \rho_t^N \right| = O\left(t^{N+1} k^{-1}\right) \tag{7.4}$$

uniformly in $y \in Y$, $k \in \mathbb{N}$.

We now prove the asymptotic result for holomorphic torsion. Below we denote by $x \ln x$ the continuous extension of this function from $\mathbb{R}_{>0}$ to $\mathbb{R}_{\geq 0}$ (i.e., taking the value zero at the origin).

Theorem 7.2 *The holomorphic torsion satisfies the asymptotics*

$$\ln \mathcal{T}_k := -\frac{1}{2}\zeta_k'(0) = -k \ln k \int_Y \left[\frac{\tau^L}{8\pi}\right] - k \int_Y \left[\frac{\tau^L}{8\pi} \ln\left(\frac{\tau^L}{2\pi}\right)\right] + o(k)$$

as $k \to \infty$.

Proof First define the rescaled zeta function $\tilde{\zeta}_k(s) := \frac{k^{-1}}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \text{tr}\left[e^{-\frac{t}{k}\Box_k^1}\right] = k^{-1}k^s \zeta_k(s)$ satisfying

$$\zeta_k'(0) = k\tilde{\zeta}_k'(0) - (k \ln k)\,\tilde{\zeta}_k(0)\,. \tag{7.5}$$

With $a_{k,j} := \int_Y \text{tr}\left[A_{k,j}\right] dy$, $j = -1, 0, \ldots$, and the analytic continuation of the zeta function being given in terms of the heat trace, one has

$$\tilde{\zeta}_k(0) = a_{k,0} \to \int_Y dy\, \text{tr}\left[A_0\right], \tag{7.6}$$

$$\begin{aligned}\tilde{\zeta}_k'(0) = &\underbrace{\int_0^T dt\, t^{-1}\left\{k^{-1}\text{tr}\left[e^{-\frac{t}{k}\Box_k^1}\right] - a_{k,-1}t^{-1} - a_{k,0}\right\}}_{=\int_0^T dt\, t^{-1}\rho_t^0 + O\left(\frac{T}{k}\right)} \\ &+ \int_T^\infty dt\, t^{-1}k^{-1}\text{tr}\left[e^{-\frac{t}{k}\Box_k^1}\right] \\ &- a_{k,-1}T^{-1} + \Gamma'(1)\,a_{k,0}\end{aligned} \tag{7.7}$$

following (7.4).

Choosing $T = k^{1-2/r}$, gives

$$\begin{aligned}t^{-1}k^{-1}\text{tr}\left[e^{-\frac{t}{k}\Box_k^1}\right] &\leq e^{-\frac{(t-1)}{k}\left[c_1 k^{2/r} - c_2\right]} t^{-1}k^{-1}\text{tr}\left[e^{-\frac{1}{k}\Box_k^1}\right] \\ &\leq Ct^{-1}k^{-1}e^{-\frac{(t-1)}{k}\left[c_1 k^{2/r} - c_2\right]}, \quad t \geq T,\end{aligned}$$

on account of (2.14), (7.4). The integral on $[T, \infty)$ of the last expression is uniformly bounded in k. By dominated convergence we have as $k \to \infty$,

$$\begin{aligned}\tilde{\zeta}_k'(0) &\longrightarrow \int_Y dy\, \alpha(y)\,, \quad \text{where} \\ \alpha(y) &:= \int_0^T dt\, t^{-1}\left\{\text{tr}\left[R_t(y)\right] - \text{tr}\left[A_{-1}\right]t^{-1} - \text{tr}\left[A_0\right]\right\} \\ &\quad + \int_T^\infty dt\, t^{-1}\text{tr}\left[R_t(y)\right] \\ &\quad - \text{tr}\left[A_{-1}\right]t^{-1} + \Gamma'(1)\,\text{tr}\left[A_0\right].\end{aligned} \tag{7.8}$$

Finally, using (7.2) one has

$$\begin{aligned}\text{tr}\left[A_0\right] &= -\frac{\tau^L}{4\pi} \\ \alpha(y) &= \frac{\tau^L}{4\pi}\ln\left(\frac{\tau^L}{2\pi}\right)\end{aligned} \tag{7.9}$$

with again the extension of the function $x \ln x$ to the origin being given by continuity to be zero as before. The proposition now follows from putting together (7.5), (7.6), (7.7), (7.8), and (7.9). □

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Appendix A: Model operators

Here we define certain model Bochner/Kodaira Laplacians and Dirac operators acting on a vector space V. First the Bochner Laplacian is intrinsically associated to a triple $\left(V, g^V, R^V\right)$ with metric g^V and tensor $0 \neq R^V \in S^{r-2}V^* \otimes \Lambda^2 V^*$, $r \geq 2$. We say that tensor R^V is non-degenerate if

$$S^{r-s-2}V^* \otimes \Lambda^2 V^* \ni i_v^s\left(R^V\right) = 0, \ \forall s \leq r-2 \implies T_yY \ni v = 0. \quad \text{(A.1)}$$

Above i^s denotes the s-fold contraction of the symmetric part of R^V.

For $v_1 \in V, v_2 \in T_{v_1}V = V$, contraction of the antisymmetric part (denoted by ι) of R^V gives $\iota_{v_2}R^V \in S^{r-2}V^* \otimes V^*$. The contraction may then be evaluated $\left(\iota_{v_2}R^V\right)(v_1)$ at $v_1 \in V$, i.e., viewed as a homogeneous degree $r-1$ polynomial function on V. The tensor R^V now determines a one form $a^{R^V} \in \Omega^1(V)$ via

$$a_{v_1}^{R^V}(v_2) := -\int_0^1 d\rho \left(\iota_{v_2}R^V\right)(\rho v_1) = -\frac{1}{r}\left(\iota_{v_2}R^V\right)(v_1), \quad \text{(A.2)}$$

which we may view as a unitary connection $\nabla^{R^V} = d + ia^{R^V}$ on a trivial Hermitian vector bundle E of arbitrary rank over V. The curvature of this connection is clearly R^V now viewed as a homogeneous degree $r-2$ polynomial function on V valued in $\Lambda^2 V^*$. This now gives the model Bochner Laplacian

$$\Delta_{g^V,R^V} := \left(\nabla^{R^V}\right)^* \nabla^{R^V} : C^\infty(V; E) \to C^\infty(V; E). \quad \text{(A.3)}$$

An orthonormal basis $\{e_1, e_2, \ldots, e_n\}$, determines components $R_{pq,\alpha} := R^V\left(e^{\odot\alpha}; e_p, e_q\right) \neq 0$, $\alpha \in \mathbb{N}_0^{n-1}$, $|\alpha| = r-2$, as well as linear coordinates $(y_1, \ldots, y_n)$ on V. The connection form in these coordinates is given by $a_p^{R^V} = \frac{i}{r} y^q y^\alpha R_{pq,\alpha}$.

While the model Laplacian (A.3) is given

$$\Delta_{g^V,R^V} = -\sum_{q=1}^{n}\left(\partial_{y_p} + \frac{i}{r} y^q y^\alpha R_{pq,\alpha}\right)^2. \tag{A.4}$$

As in (2.8), the above may now be related to the (nilpotent) sR Laplacian on the product $S^1_\theta \times V$ given by

$$\hat{\Delta}_{g^V,R^V} := -\sum_{q=1}^{n}\left(\partial_{y_p} + \frac{i}{r} y^q y^\alpha R_{pq,\alpha}\partial_\theta\right)^2, \tag{A.5}$$

and corresponding to the sR structure $\left(S^1_\theta \times V, \ker\left(d\theta + a^{R^V}\right), \pi^* g^V, d\theta \mathrm{vol} g^V\right)$ where the sR metric corresponds to g^V under the natural projection $\pi : S^1_\theta \times V \to V$. Note that the above differs from the usual nilpotent approximation of the sR Laplacian since it acts on the product with S^1. The heat kernels of (A.3), (A.5) are now related

$$e^{-t\Delta_{g^V,R^V}}\left(y, y'\right) = \int e^{-i\theta} e^{-t\hat{\Delta}_{g^V,R^V}}\left(y, 0; y', \theta\right) d\theta. \tag{A.6}$$

Next, assume that the vector space V of even dimension and additionally equipped with an orthogonal endomorphism $J^V \in O(V)$; $\left(J^V\right)^2 = -1$. This gives rise to a (linear) integrable almost complex structure on V, a decomposition $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ into $\pm i$ eigenspaces of J and a Clifford multiplication endomorphism $c : V \to \mathrm{End}\left(\Lambda^* V^{0,1}\right)$. We further assume that R^V is a $(1, 1)$ form with respect to J (i.e., $S^k V^* \ni R^V(w_1, w_2) = 0$, for all $w_1, w_2 \in V^{1,0}$). The $(0, 1)$ part of the connection form (A.2) then gives a holomorphic structure on the trivial Hermitian line bundle $\mathbb{C}$ with holomorphic derivative $\bar{\partial}_{\mathbb{C}} = \bar{\partial} + \left(ia^V\right)^{0,1}$. One may now define the Kodaira Dirac and Laplace operators, intrinsically associated to the tuple $\left(V, g^V, R^V, J^V\right)$, via

$$D_{g^V,R^V,J^V} := \sqrt{2}\left(\bar{\partial}_{\mathbb{C}} + \bar{\partial}^*_{\mathbb{C}}\right) \tag{A.7}$$

$$\square_{g^V,R^V,J^V} := \frac{1}{2}\left(D_{g^V,R^V,J^V}\right)^2 \tag{A.8}$$

acting on $C^\infty\left(V; \Lambda^* V^{0,1}\right)$. The above (A.3), (A.8) are related by the Lichnerowicz formula

$$\square_{g^V,R^V,J^V} = \Delta_{g^V,R^V} + c\left(iR^V\right) \tag{A.9}$$

where $c\left(R^V\right) = \sum_{p<q} R^{i_1\ldots i_{r-2}}_{pq} y_{i_1}\ldots y_{i_{r-2}} c\left(e_p\right) c\left(e_q\right)$.

Being symmetric with respect to the standard Euclidean density and semi-bounded below, both Δ_{g^V,R^V} and $\square^V$ are essentially self-adjoint on L^2. The domains of their unique self-adjoint extensions are

$$\mathrm{Dom}\left(\Delta_{g^V,R^V}\right) = \left\{\psi \in L^2 | \Delta_{g^V,R^V}\psi \in L^2\right\},$$
$$\mathrm{Dom}\left(\Box_{g^V,R^V,J^V}\right) = \left\{\psi \in L^2 | \Box_{g^V,R^V,J^V}\psi \in L^2\right\},$$

respectively. We shall need the following information regarding their spectrum.

Proposition A.1 *For some $c > 0$, one has Spec $\left(\Delta_{g^V,R^V}\right) \subset [c,\infty)$. For R^V satisfying the non-degeneracy condition (A.1) one has EssSpec $\left(\Delta_{g^V,R^V}\right) = \emptyset$. Finally, for dim$V = 2$ with $R^V(w,\bar{w}) \geq 0$, for all $w \in V^{1,0}$ semi-positive one has Spec $\left(\Box_{g^V,R^V,J^V}\right) \subset \{0\} \cup [c,\infty)$.*

Proof The proof is similar to those of Proposition 2.1 and Corollary 2.3. Introduce the deformed Laplacian $\Delta_k := \Delta_{g^V,kR^V}$ obtained by rescaling the tensor R^V. From (A.4) $\Delta_k = k^{2/r}\mathscr{R}\Delta_{g^V,R^V}\mathscr{R}^{-1}$ are conjugate under the rescaling $\mathscr{R} : C^\infty(V;E) \to C^\infty(V;E)$, $(\mathscr{R}u)(x) := u\left(yk^{1/r}\right)$ implying

$$\begin{aligned}\mathrm{Spec}\left(\Delta_k\right) &= k^{2/r}\mathrm{Spec}\left(\Delta_{g^V,R^V}\right)\\ \mathrm{EssSpec}\left(\Delta_k\right) &= k^{2/r}\mathrm{EssSpec}\left(\Delta_{g^V,R^V}\right)\end{aligned} \tag{A.10}$$

By an argument similar to Proposition 2.1, one has Spec $(\Delta_k) \subset \left[c_1k^{2/r} - c_2,\infty\right)$ for some $c_1, c_2 > 0$ for $R^V \neq 0$. From here Spec $\left(\Delta_{g^V,R^V}\right) \subset [c,\infty)$ follows. Next, under the non-degeneracy condition, the order of vanishing of the curvature homogeneous curvature R^V (of the homogeneous connection a^{R^V} (A.2)) is seen to be maximal at the origin: $\mathrm{ord}_y\left(R^V\right) < r-2$ for $y \neq 0$. Following a similar subelliptic estimate (2.6) on $V \times S^1_\theta$ as in Proposition 2.1, we have

$$k^{2/(r-1)}\|u\|^2 \leq C\left[\langle\Delta_k u, u\rangle + \|u\|^2_{L^2}\right], \quad \forall u \in C_c^\infty\left(V \setminus B_1(0)\right),$$

holds on the complement of the unit ball centered at the origin. Combining the above with Persson's characterization of the essential spectrum (cf. [1, 33] Ch. 3)

$$\mathrm{EssSpec}\left(\Delta_k\right) = \sup_R \inf_{\substack{\|u\|=1\\ u\in C_c^\infty(V\setminus B_R(0))}} \langle\Delta_k u, u\rangle,$$

we have EssSpec $(\Delta_k) \subset \left[c_1k^{2/(r-1)} - c_2,\infty\right)$. From here and using (A.10), EssSpec $\left(\Delta_{g^V,R^V}\right) = \emptyset$ follows.

The proof of the final part is similar following $k^{2/r}\mathrm{Spec}\left(\Box_{g^V,R^V,J^V}\right) = \mathrm{Spec}\left(\Box_{g^V,kR^V,J^V}\right) = \mathrm{Spec}\left(\Box_k\right) \subset \{0\} \cup \left[c_1k^{2/r} - c_2,\infty\right)$, $\Box_k := \Box_{g^V,kR^V,J^V}$, by an argument similar to Corollary 2.3. □

Next, the heat $e^{-t\Delta_{g^V,R^V}}$, $e^{-t\Box_{g^V,R^V,J^V}}$ and wave $e^{it\sqrt{\Delta_{g^V,R^V}}}$, $e^{it\sqrt{\Box_{g^V,R^V,J^V}}}$ operators being well defined by functional calculus, a finite propagation type argument as in [32, eqs. 2.15, 2.16] gives $\varphi\left(\Delta_{g^V,R^V}\right)(.,0) \in \mathcal{S}(V)$, $\varphi\left(\Box_{g^V,R^V,J^V}\right)(.,0) \in \mathcal{S}(V)$ are in the Schwartz class for $\varphi \in \mathcal{S}(\mathbb{R})$. Further, when $\text{EssSpec}\left(\Delta_{g^V,R^V}\right) = \emptyset$ any eigenfunction of Δ_{g^V,R^V} also lies in $\mathcal{S}(V)$. Finally, on choosing φ supported close to the origin, the Schwartz kernel $\Pi^{g^V,R^V,J^V}(.,0) \in \mathcal{S}(V)$ of the projector Π^{g^V,R^V,J^V} onto the kernel of $\Box_{g^V,R^V,J^V}$ is also of Schwartz class.

We now state another proposition regarding the heat kernel of Δ_{g^V,R^V}. Below we denote $\lambda_0\left(\Delta_{g^V,R^V}\right) := \inf \text{Spec}\left(\Delta_{g^V,R^V}\right)$.

Proposition A.2 *For each $\varepsilon > 0$ there exist $t, R > 0$ such that the heat kernel*

$$\frac{\int_{B_R(0)} dx \left[\Delta_{g^V,R^V} e^{-t\Delta_{g^V,R^V}}\right](x,x)}{\int_{B_R(0)} dx\, e^{-t\Delta_{g^V,R^V}}(x,x)} \leq \lambda_0\left(\Delta_{g^V,R^V}\right) + \varepsilon$$

Proof Setting $P := \Delta_{g^V,R^V} - \lambda_0\left(\Delta_{g^V,R^V}\right)$ it suffices to show

$$\frac{\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \leq \varepsilon$$

for some $t, R > 0$. With $\Pi^P_{[0,x]}$ denoting the spectral projector onto $[0,x]$, we split the numerator

$$\begin{aligned}\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x) &= \int_{B_R(0)} dx \left[\Pi^P_{[0,4\varepsilon]} Pe^{-tP}\right](x,x)\\ &+ \int_{B_R(0)} dx \left[\left(1 - \Pi^P_{[0,4\varepsilon]}\right) Pe^{-tP}\right](x,x).\end{aligned}$$

From $P \geq 0$, $\Pi^P_{[0,4\varepsilon]} Pe^{-tP} \leq 4\varepsilon e^{-tP}$ and $\left(1 - \Pi^P_{[0,4\varepsilon]}\right) Pe^{-tP} \leq ce^{-3\varepsilon t}$, for all $t \geq 1$, we may bound

$$\frac{\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \leq 4\varepsilon + \frac{ce^{-3\varepsilon t} R^{n-1}}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \tag{A.11}$$

for all $R, t \geq 1$. Next, as $0 \in \text{Spec}(P)$ there exists $\|\psi_\varepsilon\|_{L^2} = 1$, $\|P\psi_\varepsilon\|_{L^2} \leq \varepsilon$. It now follows that $\left\|\psi_\varepsilon - \Pi^P_{[0,2\varepsilon]}\psi_\varepsilon\right\| \leq \frac{1}{2}$ and hence

$$\begin{aligned}\frac{1}{2} = -\frac{1}{4} + \int_{B_{R_\varepsilon}(0)} dx \left|\psi_\varepsilon(x)\right|^2 &\leq \int_{B_{R_\varepsilon}(0)} dx \left|\int dy \Pi^P_{[0,2\varepsilon]}(x,y)\,\psi_\varepsilon(y)\right|^2\\ \leq \int_{B_{R_\varepsilon}(0)} dx \left(\int dy \Pi^P_{[0,2\varepsilon]}(x,y)\,\Pi^P_{[0,2\varepsilon]}(y,x)\right) &= \int_{B_{R_\varepsilon}(0)} dx \Pi^P_{[0,2\varepsilon]}(x,x),\end{aligned}$$

for some $R_\varepsilon > 0$, using $\left(\Pi^P_{[0,2\varepsilon]}\right)^2 = \Pi^P_{[0,2\varepsilon]}$ and Cauchy–Schwarz. This gives

$$\int_{B_{R_\varepsilon}(0)} dx\, e^{-tP}(x,x) \geq \frac{e^{-2\varepsilon t}}{2}, \quad t > 1.$$

Plugging this last inequality into (A.11) gives

$$\frac{\int_{B_{R_\varepsilon}(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_{R_\varepsilon}(0)} dx\, e^{-tP}(x,x)} \leq 4\varepsilon + ce^{-\varepsilon t} R_\varepsilon^{n-1}$$

from which the theorem follows on choosing t large. □

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