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Bochner Laplacian and Bergman kernel expansion of semipositive line bundles on a Riemann surface

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Abstract
We generalize the results of Montgomery (Commun Math Phys 168:651–675, 1995) for the Bochner Laplacian on high tensor powers of a line bundle. When specialized to Riemann surfaces, this leads to the Bergman kernel expansion for semipositive line bundles whose curvature vanishes at finite order. The proof exploits the relation of the Bochner Laplacian on tensor powers with the sub-Riemannian (sR) Laplacian.

Mathematics Subject Classification 53C17 · 58J50 · 32A25 · 53D50

1 Introduction

In [31] Montgomery studied the spectrum, and in particular the smallest eigenvalue, of the Bochner (magnetic) Laplacian on the tensor powers $L^k := L^{\otimes k}$ of a Hermitian line bundle L. He assumed that the underlying manifold is a Riemann surface and that the curvature of the line bundle vanishes transversally along a curve. The problem goes back at least to the work Simon et al. [2, 35] and Guillemin–Uribe [18] among others, who assumed the curvature is symplectic. The problem has since also been actively explored under different assumptions on the curvature. The first theorem in this article proves the most general such leading asymptotic for the smallest eigenvalue of the Bochner Laplacian on tensor powers.

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The holomorphic analog of the above is the study of the Bergman kernel of a holomorphic line bundle L on a complex manifold. The Bergman kernel is the Schwartz kernel of the projector from smooth sections of L onto holomorphic ones. The analysis of the Bergman kernel and holomorphic sections associated to tensor powers has important applications in complex geometry (see [14, 26]). When L is positive, the leading asymptotic for the Bergman kernel along the diagonal was first proved in [38] and later improved to a full expansion in [11, 39] using the Szegő kernel parametrix of [9]. Subsequently, a different geometric method for the expansion was developed in [12, 26, 27] inspired by the analytic localization technique of [8]. The problem of the expansion for semipositive line bundles is largely open. A second objective of this article is to give the first proof of the Bergman kernel expansion at vanishing points of the curvature. We achieve this for a semipositive line bundle L on a Riemann surface.

1.1 Statement of the main results

We now state our results more precisely. Let Y^{n-1} be a compact Riemannian manifold of dimension $n-1$ with complex Hermitian line bundle $\left(L, h^L\right)$ and vector bundle $\left(F, h^F\right)$. We equip these with unitary connections ∇^L, ∇^F to obtain the Bochner Laplacian

$$\Delta_k := \left(\nabla^{F\otimes L^k}\right)^* \nabla^{F\otimes L^k} : C^\infty\left(Y; F\otimes L^k\right) \to C^\infty\left(Y; F\otimes L^k\right), \quad k\in\mathbb{N}, \tag{1.1}$$

on tensor powers $F\otimes L^k$, where the adjoint is taken with respect to the natural L^2 metric. As the above is elliptic, self-adjoint and positive, one has a complete orthonormal basis $\left\{\psi_j^k\right\}_{j=1}^\infty$ of $L^2\left(Y; F\otimes L^k\right)$ consisting of its eigenvectors $\Delta_k\psi_j^k = \lambda_j(k)\,\psi_j^k$, with eigenvalues $0\leq\lambda_0\leq\lambda_1\ldots$. Denote by $R^L = \left(\nabla^L\right)^2 \in \Omega^2(Y; i\mathbb{R})$ the purely imaginary curvature form of the unitary connection ∇^L. The order of vanishing of R^L at a point $y\in Y$ is now defined[1]

$$r_y - 2 = \operatorname{ord}_y\left(R^L\right) := \min\left\{l \,|\, J^l\left(\Lambda^2 T^*Y\right) \ni j_y^l R^L \neq 0\right\}, \quad r_y \geq 2, \tag{1.2}$$

where $j^l R^L$ denotes the lth jet of the curvature. We shall assume that this order of vanishing is finite at all points of the manifold i.e.

$$r := \max_{y\in Y} r_y < \infty. \tag{1.3}$$

[1] The reason for this normalization, besides a simplification of resulting formulas, is the significance of r_y as the degree of nonholonomy of a relevant sR distribution (see Proposition 6).

The function $y \mapsto r_y$ being upper semi-continuous, gives a decomposition $Y = \bigcup_{j=2}^{r} Y_j$ of the manifold via

$$Y_j := \left\{y \in Y | r_y = j\right\} \quad \text{with each} \quad Y_{\leq j} := \bigcup_{j'=2}^{j} Y_{j'} \tag{1.4}$$

being open. Our first theorem is now the following.

Theorem 1 *Let $(L, h^L) \to \left(Y, g^{TY}\right)$, $(F, h^F) \to \left(Y, g^{TY}\right)$ be Hermitian line and vector bundles on a compact Riemannian manifold with unitary connections ∇^L, ∇^F. Assuming that the curvature R^L vanishes to finite order at all points, with maximal order r (1.3), the first eigenvalue $\lambda_0(k)$ of the Bochner Laplacian satisfies*

$$\lambda_0(k) \sim Ck^{2/r}, \quad \text{as } k \to \infty, \tag{1.5}$$

for some positive constant C. Moreover, the first eigenfunction concentrates on Y_r:

$$\left|\psi_0^k(y)\right| = O\left(k^{-\infty}\right); \; y \in Y_{\leq r-1}. \tag{1.6}$$

The leading constant above (1.5) can be identified

$$C = \inf_{y \in Y_r} \lambda_0\left(\Delta_y\right) \tag{1.7}$$

in terms of the bottom of the spectrum of certain model Laplacians $\Delta_y := \Delta_{g_y^{TY}, j^{r-2}R_y^L}$, depending on the metric g_y^{TY} and first non-vanishing jet tensors $j^{r-2}R_y^L$, defined on the tangent space T_yY at each $y \in Y$ (see Section A). The first case of the above theorem is $r = 2$, when the curvature R^L is non-vanishing, and can be found in [19]. Here the model Laplacian is a harmonic oscillator. The bottom of its spectrum is explicitly given $\lambda_0\left(\Delta_y\right) = \frac{1}{2}\text{tr}\sqrt{-J_y^2}$ in terms of the endomorphism $J_y : T_yY \to T_yY$, defined by the equation $g^{TY}\left(., J_y.\right) = R^L(., .)$. In [31] a particular case of $r = 3$, with Y a Riemann surface, is considered. It is surpising that the general case, despite being attempted, has been missed therein and in several references since then.

Without further hypotheses, the structure of the locus Y_r may be quite general; locally any closed subset of a hypersurface (see Sect. 3.0.1 below). To obtain further information on the small eigenvalues, we introduce additional assumptions. First, we assume $Y_r = \bigcup_{j=1}^{N} Y_{r,j}$ to be a union of embedded submanifolds of dimensions $d_j := \dim\left(Y_{r,j}\right)$. At points $y \in Y_r$, the first non-vanishing jet of the curvature $j_y^{r-2}R^L \in S^{r-2}T_y^*Y \otimes \Lambda^2 T_y^*Y$ may be thought of as an element of the product with the $(r-2)$th symmetric power. We say that the curvature R^L vanishes *non-degenerately* along Y_r if the following implication holds

$$i_v^s\left(j_y^{r-2}R^L\right) = 0, \; \forall s \leq r-2 \implies v \in T_yY_r, \tag{1.8}$$

where i^s above denotes the s-fold contraction of the symmetric part of $j_y^{r-2}R^L$. In Remark 9 below we note the following less invariant definition of the non-degeneracy condition (1.8) above in local coordinates: it is equivalent to assuming that the leading order part R_0^L in the Taylor expansion of the curvature at $y \in Y_r$ locally defines the same locus $Y_r = Y_r^0 := \left\{y \in Y | \text{ord}_y \left(R_0^L\right) = r - 2\right\}$ as (1.4). In Remark 9 we also note that our non-degeneracy condition (1.8) on the curvature is less restrictive than the assumption of Y_r being a 'magnetic well' for the curvature R^L that appears in earlier works [19].

Now set $d_j^{\max} := \max \left\{d_j\right\}_{j=1}^N$ and let $NY_{r,j} := TY_{r,j}^{\perp} \subset TY$ denote the normal bundle of each $Y_{r,j}$. Note that there is a natural density on each $NY_{r,j}$ coming from the metric. Denote by $\chi_{[c_1,c_2]}$ the characteristic function for $[c_1, c_2]$. In Sect. 3.2 we show that under the non-degeneracy hypothesis (1.8), the Schwartz kernel of the model Laplacian on the tangent space $\chi_{[c_1,c_2]}\left(\Delta_y\right)(v, v) = O\left(|v|^{-\infty}\right)$, $v \in NY_{r,j,y}$, is rapidly decaying, and thus integrable, in the normal directions.

Our next result is on the asymptotics for the Weyl counting function $N\left[c_1 k^{2/r}, c_2 k^{2/r}\right]$ for the number of eigenvalues of Δ_k in the given interval.

Theorem 2 *Let $(L, h^L) \to \left(Y, g^{TY}\right)$, $(F, h^F) \to \left(Y, g^{TY}\right)$ be Hermitian line and vector bundles on a compact Riemannian manifold with unitary connections ∇^L, ∇^F. Assuming $Y_r \subset Y$ (1.4) to be a union of embedded submanifolds along which the curvature vanishes non-degenerately (1.8), the counting function satisfies the asymptotics*

$$N\left[c_1 k^{2/r}, c_2 k^{2/r}\right] \sim k^{\frac{d_j^{max}}{r}} \sum_{d_j = d_j^{max}} \int_{NY_{r,j}} \chi_{[c_1,c_2]}\left(\Delta_y\right)(v, v), \quad as\ k \to \infty. \tag{1.9}$$

If further Y_r is a finite set of points (or $d_j^{max} = 0$), then the smallest eigenvalue of the Bochner Laplacian has a complete asymptotic expansion[2]

$$\lambda_0(k) = k^{2/r}\left[\sum_{l=0}^{N} \lambda_l k^{-l/r} + O\left(k^{-(2N+1)/r}\right)\right], \quad \forall N \in \mathbb{N}_0, \quad as\ k \to \infty. \tag{1.10}$$

Next, we consider the case when $\left(Y, h^{TY}\right)$ is a complex Hermitian manifold. The line and vector bundles $\left(L, h^L\right)$, $\left(F, h^F\right)$ are then assumed to be holomorphic. Taking ∇^L, ∇^F to be the Chern connections, one also has the associated Kodaira Laplacian

$$\Box_k^q : \Omega^{0,q}\left(Y; F \otimes L^k\right) \to \Omega^{0,q}\left(Y; F \otimes L^k\right), \quad 0 \le q \le m,$$

[2] The same result holds for the mth eigenvalue $\lambda_m(k)$ for any fixed $m \in \mathbb{N}_0$.

acting on tensor powers.[3] The first eigenvalue of the above is typically 0 with $\ker \Box_k^q = H^q\left(X; F \otimes L^k\right)$ being cohomological and corresponding to holomorphic sections. The Bergman kernel $\Pi_k^q\left(y, y'\right)$ is the Schwartz kernel of the orthogonal projector $\Pi_k^q : \Omega^{0,q}\left(Y; F \otimes L^k\right) \to \ker \Box_k^q$. Its value on the diagonal is given

$$\Pi_k^q\left(y, y\right) = \sum_{j=1}^{N_k^q} \left|s_j\left(y\right)\right|^2, \quad N_k^q := \dim H^q\left(X; F \otimes L^k\right), \tag{1.11}$$

in terms of an orthonormal basis $\left\{s_j\right\}_{j=1}^{N_k^q}$ of $H^q\left(X; F \otimes L^k\right)$, and thus controls pointwise norms of sections in $\ker \Box_k^q$ in the spirit of (1.6). To obtain the asymptotics of $\Pi_k^q\left(y, y\right)$, we specialize to the case of Riemann surface ($n - 1 = 2$). Furthermore, in addition to vanishing at finite order (1.3), the curvature is assumed to be semipositive: $R^L\left(w, \bar{w}\right) \geq 0$, for all $w \in T^{1,0}Y$. Under these assumptions one has $H^1\left(X; F \otimes L^k\right) = 0$ for k sufficiently large, with the asymptotics of the Bergman kernel $\Pi_k := \Pi_k^0$ being given by the following.

Theorem 3 *Let Y be a compact Riemann surface and $(L, h^L) \to Y$ a semipositive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F, h^F) \to Y$ be another Hermitian holomorphic vector bundle. Then the Bergman kernel $\Pi_k := \Pi_k^0$ has the pointwise asymptotic expansion on diagonal*

$$\Pi_k\left(y, y\right) = k^{2/r_y}\left[\sum_{j=0}^{N} c_j\left(y\right) k^{-2j/r_y}\right] + O\left(k^{-2N/r_y}\right), \quad \forall N \in \mathbb{N}. \tag{1.12}$$

Here c_j are sections of End (F), with the leading term $c_0\left(y\right) > 0$ being given in terms of the Bergman kernel of the model Kodaira Laplacian on the tangent space at y (A.9).

Note that at points where R^L is positive one has $r_y = 2$ and the above expansion recovers the usual Bergman kernel expansion at these points. The presence of fractional exponents, at points where the curvature vanishes, given in terms of the order of vanishing, represents a new feature. It would be desirable to have a more explicit formula for the leading term c_0 at vanishing points for the curvature. The final example 17 computes the leading term explicitly in the case of semipositive line bundles obtained from branched coverings. Finally, we note that unlike (1.6) the Bergman kernel expansion (1.12) does not exhibit any concentration phenomenon.

1.2 Background and commentary

The result of Theorem 1 was shown by Montgomery [31] in the case when Y is a Riemann surface and R^L vanishes to first order ($r = 3$) along a curve. The case of non-vanishing curvature ($r = 2$), and a special case of the expansion (1.10) for $r \geq 2$,

[3] Twisting by an additional bundle F is fairly standard in complex geometry, for instance one is often required to choose F to be the canonical bundle (see Proposition 14 below).

can be found in the work of Helffer–Mohamed [19]. The problem has since been explored in several further cases. All such previous works however are more restrictive in dimension, the curvature R^L or the geometry of the manifold and bundles. Our Theorem 1 is the most general leading asymptotic for the first Bochner eigenvalue. The only assumption is the finite order of vanishing of the curvature R^L and corresponds to Hörmander's condition on the unit circle of L.

The proof here uses the relation of the Bochner Laplacian with the sub-Riemannian (sR) Laplacian on the unit circle bundle of L, this is a manifestation of the semiclassical/microlocal correspondence in this context. Asymptotic bounds on the smallest eigenvalue can be obtained by replacing Guillemin–Uribe's use of the Melin inequality on the unit circle [18] by the subelliptic estimate of Rothschild–Stein [33]. The leading asymptotic (1.5) however requires understanding the sharp constant in the subelliptic estimate. Here we instead exploit a pointwise heat kernel expansion for the sR Laplacian [4, 24] on the circle bundle, this is also consistent with our method for the other announcements.

The first part of Theorem 2 is similarly the semiclassical analog of Weyl's law for hypoelliptic operators of Hörmander-type. The main difficulty here is the non-uniform nature of the relevant heat kernel expansion which does not immediately yield heat trace asymptotics. Prior results on hypoelliptic Weyl law's include the one by Mötivier [30] and the eigenvalue estimates of Fefferman–Phong [17].

The leading asymptotic of the Bergman kernel for positive line bundles on a compact complex manifold was first proved in [38] and later improved to a full expansion in [11, 39] as a consequence of the Boutet de Monvel–Sjöstrand parametrix [9]. Our geometric method here is closer to [12, 27] and we refer to [26] for a detailed account of this technique and its applications. The problem of the expansion for semipositive line bundles is well-known and largely unresolved, see [26, Problem 4.8] or [13] for the analogous problem for weakly pseudoconvex domains. Our final Theorem 3 is the first instance where the expansion has been proved at vanishing points of the curvature for surfaces, and this is yet unresolved in higher dimensions. A key step in our proof of Theorem 3, although one among several, is Corollary 15 below. This gives an $O\left(k^{2/r}\right)$ spectral gap for the Kodaira Laplacian on tensor powers by combining Theorem 1 with the method of [25]. Donnelly has earlier shown in [16] that the corresponding does not hold in higher dimensions as a counterexample to Siu's eigenvalue conjecture [36] (see Remark 16 below). Despite the counterexample, the problem of generalizing Theorem 3 to higher dimensions remains open, perhaps by circumventing the use of Corollary 15. Previously, [6] proved an asymptotic estimate for the Bergman kernel of semipositive line bundles. In [5] the expansion is proved on the positive part, and away from the augmented base locus, assuming the line bundle to be ample. In [21] the expansion is proved on the positive part when one twists by the canonical bundle (i.e. $F = K_Y$). The analogous problem of the boundary expansion for the Bergman kernel of weakly pseudoconvex domains in $\mathbb{C}^2$ has also been recently solved by the second author in [22], refining earlier estimates on Bergman kernels from [28, 32].

The analysis of holomorphic sections and the Bergman kernel for positive line bundles has several applications, particularly to the Tian–Yau–Donaldson program in Köhler geometry, Berezin–Toeplitz quantization, holomorphic torsion and its relation to Arakelov geometry, random holomorphic sections and the quantum Hall effect (see [26] for these references). Our Theorem 3 opens the way to extending these applications of Bergman kernels to the case of semipositive line bundles, which we plan to explore in a sequel to this article.

1.3 Organization of the article

The paper is organized as follows. In Sect. 2 we begin with some standard preliminaries on sub-Riemannian geometry and the sR Laplacian. In particular Sect. 2.1.1 gives a proof of the on-diagonal expansion for the sR heat kernel. In Sect. 3 we specialize to the case of sR structures on unit circle bundles. Here Sect. 3.1 proves Theorem 1 based on an analogous heat kernel expansion for the Bochner Laplacian on tensor powers Theorem 8. Next Sects. 3.2 and 3.3 prove the Weyl law and expansion of the first eigenvalue of Theorem 2 respectively. In Sect. 4 we come to the case of the Kodaira Laplacian on tensors powers of semipositive line bundles on a Riemann surface. Here we prove the Bergman kernel expansion Theorem 3 in Sect. 4.1.

2 sub-Riemannian geometry

Sub-Riemannian (sR) geometry is the study of (metric-)distributions in smooth manifolds. More precisely, let X^n be an n-dimensional, compact, oriented differentiable manifold X. Let $E^m \subset X$ be a rank m subbundle of the tangent bundle which is assumed to be *bracket generating:* sections of E generate all sections of TX under the Lie bracket. The subbundle E is further equipped with a metric g^E. We refer to the triple $\left(X, E, g^E\right)$ as a sub-Riemannian (sR) structure. Riemannian geometry corresponds to $E = TX$.

The obvious length function $l(\gamma) := \int_0^1 |\dot{\gamma}| \, dt$ may be defined on the set of *horizontal* paths of Sobolev regularity one connecting the two points $x_0, x_1 \in X$ as

$$\Omega_E(x_0, x_1) := \left\{ \gamma \in H^1([0,1]; X) \,|\, \gamma(0) = x_0,\ \gamma(1) = x_1,\ \dot{\gamma}(t) \in E_{\gamma(t)} \text{ a.e.} \right\}.$$

This also defines the sub-Riemannian distance function via

$$d^E(x_0, x_1) := \inf_{\gamma \in \Omega_E(x_0, x_1)} l(\gamma). \tag{2.1}$$

The Chow–Rashevskii theorem shows that this distance is finite, or that there exists a horizontal path connecting any two points on X, giving the manifold the structure of a metric space $\left(X, d^E\right)$.

The canonical flag of the distribution E at any point $x \in X$ is defined by

$$\underbrace{E_0(x)}_{=\{0\}} \subset \underbrace{E_1(x)}_{=E} \subset \ldots \subsetneq E_{r(x)}(x) = TX \tag{2.2}$$

where $E_{j+1} := E_j + \left[E_j, E_j\right],\ 0 \le j \le r(x) - 1$ denotes the span of the jth brackets. The number $r(x)$ is called the step or degree of nonholonomy of the distribution at x

and in general depends on the point $x \in X$. Furthermore, the ranks of the subspaces $E_j(x)$ might also might depend on $x \in X$ and E_j need not define a locally trivial vector bundles. The growth vector, weight vectors and Hausdorff dimension of the distribution at $x \in X$ are defined via

$$m^E(x) = \Big(\underbrace{m_0^E}_{:=0}, \underbrace{m_1^E}_{=m}, m_2^E, \ldots, \underbrace{m_r^E}_{=n}\Big), \quad \text{with } m_j^E(x) := \dim E_j(x), \tag{2.3}$$

$$w^E(x) = \left(w_1^E(x), \ldots, w_n^E(x)\right) := \Big(\underbrace{1, \ldots, 1}_{m_1^E \text{ times}}, \underbrace{2, \ldots 2}_{m_2^E - m_1^E \text{ times}}, \ldots, \underbrace{r, \ldots, r}_{m_r^E - m_{r-1}^E \text{ times}}\Big) \tag{2.4}$$

$$Q(x) := \sum_{j=1}^{m} j\left(m_j^E(x) - m_{j-1}^E(x)\right) = \sum_{j=1}^{n} w_j^E(x). \tag{2.5}$$

A point $x \in X$ is called regular if m_j^E's are locally constant functions near x or each distribution E_j is a locally trivial vector bundle near x. Mitchell's measure theorem shows that $Q(x)$ agrees with the Hausdorff dimension of $\left(X, d^E\right)$ as a metric space at a regular point $x \in X$. We call the distribution E *equiregular* if each point $x \in X$ is regular. Hence in the equiregular case each E_j is a subbundle of TX with $r(x)$, $m_j^E(x)$ and $Q(x)$ all being constants independent of x.

An important notion is that of a privileged coordinate system at x. To define this, fix a set of local orthonormal frame of vector fields $U_1, U_2, \ldots U_m$ for E near x. The E-order $\text{ord}_{E,x}(f)$ of a function $f \in C^\infty(X)$ at a point $x \in X$ is the maximum integer $s \in \mathbb{N}_0$ for which $\sum_{j=1}^m s_j = s$ implies that $\left(U_1^{s_1} \ldots U_m^{s_m} f\right)(x) = 0$. Similarly the E-order $\text{ord}_{E,x}(P)$ of a differential operator P at the point $x \in X$ is the maximum integer for which $\text{ord}_{E,x}(Pf) \geq \text{ord}_{E,x}(P) + \text{ord}_{E,x}(f)$ holds for each function $f \in C^\infty(X)$. One then has the obvious relation $\text{ord}_{E,x}(PQ) \geq \text{ord}_{E,x}(P) + \text{ord}_{E,x}(Q)$ for any pair of differential operators P, Q. A set of coordinates $(x_1, \ldots, x_n)$ near a point $x \in X$ is said to be *privileged* if each x_j has E-order $w_j^E(x)$ at x. A privileged coordinate system always exists near any point [3, pg. 36]. Furthermore, the coordinate system may be chosen such that each $\frac{\partial}{\partial x_j}$ equals the value of some bracket monomial in the generating vector fields at x. The E-order of the monomial x^α in privileged coordinates is clearly $w.\alpha$, while the defining vector fields U_j all have E-order -1. A basic vector field is one of the form $x^\alpha \partial_{x_j}$ for some j and has E-order $w.\alpha - w_j$. We may then use a Taylor expansion to write $U_j = \sum_{q=-1}^\infty \hat{U}_j^{(q)}$ with each vector field $\hat{U}_j^{(q)}$ being a sum of basic vector fields of E-order q. If one defines the rescaling/dilation $\delta_\varepsilon x = (\varepsilon^{w_1} x_1, \ldots, \varepsilon^{w_n} x_n)$ in privileged coordinates, the vector fields $\hat{U}_j^{(q)}$ are those appearing in the corresponding expansion $(\delta_\varepsilon)_* U_j = \sum_{q=-1}^\infty \varepsilon^q \hat{U}_j^{(q)}$ for the defining vector fields. A differential operator P on $\mathbb{R}^n$ is said to be E-homogeneous of $\text{ord}_E(P)$ iff $(\delta_\varepsilon)_* P = \varepsilon^{\text{ord}_E(P)} P$. It is clear that the product of two such homogeneous differential operators P_1, P_2 is homogeneous of $\text{ord}_E(P_1 P_2) = \text{ord}_E(P_1) + \text{ord}_E(P_2)$. The *nilpotentization* of the sR structure at an arbitrary $x \in X$ is the sR manifold given via $\hat{X} = \mathbb{R}^n$, $\hat{E} := \mathbb{R}\left[\hat{U}_1^{(-1)}, \ldots, \hat{U}_m^{(-1)}\right]$ with the metric $\hat{g}^E$ corresponding to the identification $\hat{U}_j^{(-1)} \mapsto (U_j)_x$. The nilpotentization $\hat{\mu}$ of a smooth measure μ at x is also defined as the leading part $\hat{\mu} = \hat{\mu}^{(0)}$ under the privileged coordinate expansion $(\delta_\varepsilon)_* \mu = \varepsilon^{Q(x)} \left[\sum_{q=0}^{\infty} \hat{\mu}^{(q)}\right]$. These nilpotentizations can be shown to be independent of the choice of privileged coordinates up to sR isometry [3, Ch. 5].

2.1 sR Laplacian

Here we define the sub-Riemannian (sR) Laplacian and state its basic properties. It shall be useful to define it as acting on sections of an auxiliary complex Hermitian vector bundle of rank l with connection (F, h^F, ∇^F). To define this first define the sR-gradient $\nabla^{g^E,F} s \in C^\infty(X; E \otimes F)$ of a section $s \in C^\infty(X; F)$ by the equation

$$h^{E,F}\left(\nabla^{g^E,F} s, v \otimes s'\right) := h^F\left(\nabla_v^F s, s'\right), \quad \forall v \in C^\infty(X; E), s' \in C^\infty(X; F), \tag{2.6}$$

where $h^{E,F} := g^E \otimes h^F$. Next, one defines the divergence or adjoint of this gradient. In the sR context, the lack of canonical volume form presents a difficulty in doing so, hence we shall choose an auxiliary non-vanishing volume form μ. The divergence $\left(\nabla^{g^E,F}\right)_\mu^* \omega \in C^\infty(X; F)$ of a section $\omega \in C^\infty(X; E \otimes F)$ is now defined to be the formal adjoint satisfying

$$\int \left\langle \left(\nabla^{g^E,F}\right)_\mu^* \omega, s \right\rangle \mu = -\int \left\langle \omega, \nabla^{g^E,F} s \right\rangle \mu, \quad \forall s \in C^\infty(X; F). \tag{2.7}$$

The *sR-Laplacian* acting on sections of F is then defined by the composition

$$\Delta_{g^E,F,\mu} := \left(\nabla^{g^E,F}\right)_\mu^* \circ \nabla^{g^E,F} : C^\infty(X; F) \to C^\infty(X; F). \tag{2.8}$$

In terms of a local orthonormal frame $\{U_j\}_{j=1}^m$ for E, we have the expression

$$\Delta_{g^E,F,\mu} s = \sum_{j=1}^m \left[-\left(\nabla_{U_j}^F\right)^2 s + \left(\nabla^{g^E} U_j\right)_\mu^* \nabla_{U_j}^F s \right] \tag{2.9}$$

with $\left(\nabla^{g^E} U_j\right)_\mu^*$ being the divergence of the vector field U_j with respect to μ. Changing the volume form μ changes the sR Laplacian (2.8) by a conjugate, up to a term of order zero.

The sR Laplacian $\Delta_{g^E,F,\mu}$ is non-negative and self-adjoint with respect to the obvious inner product $\langle s, s' \rangle = \int_X h^F(s, s') \mu$, $s, s' \in C^\infty(X; F)$. Its principal symbol is easily computed to be the Hamiltonian

$$\sigma = \sigma\left(\Delta_{g^E,F,\mu}\right)(x, \xi) = H^E(x, \xi) = \left|\xi|_E\right|^2 \in C^\infty\left(T^* X\right) \tag{2.10}$$

while its sub-principal symbol is zero. The characteristic variety

$$\Sigma_{\Delta_{g^E,F,\mu}} = \left\{(x,\xi) \in T^*X | \sigma\left(\Delta_{g^E,F,\mu}\right)(x,\xi) = 0\right\} = \left\{(x,\xi) \mid \xi|_E = 0\right\} =: E^\perp \tag{2.11}$$

is the annihilator of E. From the local expression (2.9) and the bracket generating condition on E, the Laplacian $\Delta_{g^E,F,\mu}$ is seen to be a sum of squares operator of Hörmander type [20]. It is then known to be hypoelliptic and satisfies the following optimal sub-elliptic estimate [33] with a gain of $\frac{1}{r}$ derivatives

$$\|\psi s\|^2_{H^{1/r}} \leq C\left[\left\langle \Delta_{g^E,F,\mu}\varphi s, \varphi s\right\rangle + \|\varphi s\|^2_{L^2}\right], \quad \forall s \in C^\infty(X; F) \tag{2.12}$$

for all $\varphi, \psi \in C_c^\infty(X)$, with $\varphi = 1$ on $\operatorname{spt}(\psi)$, and where $r := \sup_{x\in X} r(x)$ is the maximum step size of the distribution.

Thus on a compact manifold the sR Laplacian has a compact resolvent, a discrete spectrum of non-negative eigenvalues approaching infinity and a well-defined heat operator $e^{-t\Delta_{g^E,F,\mu}}$.

2.1.1 sR heat kernel

We shall now describe the asymptotics of the heat kernel $e^{-t\Delta_{g^E,F,\mu}}$. One first begins with the finite propagation speed for the sR wave equation [29]: the Schwartz kernel $K_t(x,y)$ of the half-wave operator $e^{it\sqrt{\Delta_{g^E,F,\mu}}}$ is supported

$$\operatorname{spt} K_t \subset \left\{(x,y) \left| d^E(x,y) \leq |t|\right.\right\} \tag{2.13}$$

in a $|t|$ size neighborhood of the diagonal measured with respect to the sR distance (2.1). From this one obtains a localization for the heat kernel. To state it, fix a Riemannian metric g^{TX} and a privileged coordinate ball $B_\varrho^{g^{TX}}(x)$, centered at a point x, of radius ϱ_x depending on x. Let $U_1, \ldots, U_m$ be a local orthonormal frame for E on this ball. Let $\chi \in C_c^\infty([-1,1]; [0,1])$ with $\chi = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Define the modified measure and vector fields on $\mathbb{R}^n$ via

$$\begin{aligned}
\tilde{\mu} &= \hat{\mu} + \chi\left(\frac{d^{g^{TX}}(x,x')}{\varrho_x}\right)(\mu - \hat{\mu}), \\
\tilde{U}_j &= U_j^{(-1)} + \chi\left(\frac{d^{g^{TX}}(x,x')}{\varrho_x}\right)\left(U_j - U_j^{(-1)}\right), \quad 1 \leq j \leq m,
\end{aligned} \tag{2.14}$$

in terms of the nilpotentization at x given by these privileged coordinates. These modified vector fields can be seen to be bracket generating for ϱ sufficiently small. The connection on F can be written $\nabla^F = d + A$, in terms of an orthonormal trivialization for F over the ball, where $A \in \Omega^1\left(B_\varrho^{g^{TX}}(x); \mathfrak{u}(l)\right)$, $A(0) = 0$, is a one form valued in the Lie algebra $\mathfrak{u}(l)$ of the unitary group. A modified sR metric $\tilde{g}^E$ on $\mathbb{R}^n$ is now defined by requiring the vector fields (2.14) to be orthonormal. While a modified connection on the trivial vector bundle of rank $\mathrm{rk}(F)$ on $\mathbb{R}^n$ is defined as $\tilde{\nabla}^F := d + \chi\left(\frac{d^E(x,x')}{\varrho}\right) A$. A formula similar to (2.9) now gives an sR Laplacian on $\mathbb{R}^n$ via

$$\tilde{\Delta}_{g^E,F,\mu} s = \sum_{j=1}^{m} \left[-\left(\tilde{\nabla}^F_{\tilde{U}_j}\right)^2 s + \left(\nabla^{\tilde{g}^E} \tilde{U}_j\right)^*_{\tilde{\mu}} \tilde{\nabla}^F_{\tilde{U}_j} s \right].$$

Being semi-bounded from below, it is essentially self-adjoint and has a well-defined heat kernel on $\mathbb{R}^n$ using functional calculus. Furthermore from the bracket generation of (2.14), it is of Hörmander type and satisfies a local sub-elliptic estimate (2.12). Next, an application of finite propagation speed for the wave operator (2.13) gives localization for the heat kernel for the sR Laplacian. Namely, there exist constants $\rho_{1,x}, C_x$ depending on x such that

$$e^{-t\Delta_{g^E,F,\mu}}\left(x,x'\right) \le C t^{-2nr-1} e^{-\frac{d^E(x,x')^2}{4t}} \tag{2.15}$$

$$e^{-t\Delta_{g^E,F,\mu}}\left(x,x'\right) - e^{-t\tilde{\Delta}_{g^E,F,\mu}}\left(x,x'\right) \le C_x e^{-\frac{\varrho_{1,x}^2}{16t}} \tag{2.16}$$

for $d^E\left(x,x'\right) \le \varrho_{1,x}$ and $t \le 1$.

We now have the following on diagonal expansion for the sR heat kernel.

Theorem 4 *There exist smooth sections $A_j \in C^\infty(X; End(F))$ such that*

$$\left[e^{-t\Delta_{g^E,F,\mu}}\right]_\mu (x,x) = \frac{1}{t^{Q(x)/2}} \left[A_0(x) + A_1(x)t + \cdots + A_N(x)t^N + O\left(t^N\right)\right] \tag{2.17}$$

$\forall x \in X, N \in \mathbb{N}$. *The leading term* $A_0 = \left[e^{-\hat{\Delta}_{\hat{g}^E,\hat{\mu}}}\right]_{\hat{\mu}}(0,0)$ *is a multiple of the identity and given in terms of the scalar heat kernel on the nilpotent approximation at* x.

Proof By (2.16) it suffices to demonstrate the expansion for the localized heat kernel $e^{-t\tilde{\Delta}_{g^E,F,\mu}}(0,0)$ at the point x. Next, the heat kernel of the rescaled sR-Laplacian

$$\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} := \varepsilon^2 (\delta_\varepsilon)_* \tilde{\Delta}_{g^E,F,\mu} \tag{2.18}$$

under the privileged coordinate dilation satisfies

$$e^{-t\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu}}\left(x,x'\right) = \varepsilon^{Q(x)} e^{-t\varepsilon^2\tilde{\Delta}_{g^E,F,\mu}}\left(\delta_\varepsilon x, \delta_\varepsilon x'\right). \tag{2.19}$$

Rearranging and setting $x = x' = 0,\ t = 1$; gives

$$\varepsilon^{-Q(x)} e^{-\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu}}(0,0) = e^{-\varepsilon^2\tilde{\Delta}_{g^E,F,\mu}}(0,0)$$

and it suffices to compute the expansion of the left-hand side above as the dilation $\varepsilon \to 0$. To this end, first note that the rescaled Laplacian has an expansion under the privileged coordinate dilation

$$\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} = \left(\sum_{j=0}^{N} \varepsilon^j \hat{\Delta}^{(j)}_{g^E,F,\mu} \right) + \varepsilon^{N+1} R^{(N)}_{\varepsilon}, \quad \forall N. \tag{2.20}$$

Here each $\hat{\Delta}^{(j)}_{g^E,F,\mu}$ is an ε-independent second-order differential operator on $\mathbb{R}^n$ of homogeneous E-order $j-2$. While each $R^{(N)}_{\varepsilon}$ is an ε-dependent second-order differential operator on $\mathbb{R}^n$ of E-order at least $N-1$. The coefficient functions of $\hat{\Delta}^{(j)}_{g^E,F,\mu}$ are polynomials of degree at most $j+2r$. While those of $R^{(N)}_{\varepsilon}$ are uniformly C^{∞}-bounded in ε. The first term is a scalar operator given in terms of the nilpotent approximation at x

$$\hat{\Delta}^{(0)}_{g^E,F,\mu} = \Delta_{\hat{g}^E,\hat{\mu};x} = \sum_{j=1}^{m} \left(\hat{U}^{(-1)}_j \right)^2 . \tag{2.21}$$

This expansion (2.20) along with the subelliptic estimate (2.12) now gives

$$\left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} - z \right)^{-1} - \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z \right)^{-1} = O_{H^s_{\text{loc}} \to H^{s+1/r-2}_{\text{loc}}} \left(\varepsilon \left| \text{Im} z \right|^{-2} \right),$$

$\forall s \in \mathbb{R}$. More generally, we let $I_j := \left\{ p = (p_0, p_1, \ldots) \,|\, p_\alpha \in \mathbb{N}, \sum p_\alpha = j \right\}$ denote the set of partitions of the integer j and define

$$\mathsf{C}^z_j := \sum_{p \in I_j} \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z \right)^{-1} \left[\prod_{\alpha} \hat{\Delta}^{(p_\alpha)}_{g^E,F,\mu} \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z \right)^{-1} \right]. \tag{2.22}$$

Then by repeated applications of the subelliptic estimate we have

$$\left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} - z \right)^{-1} - \sum_{j=0}^{N} \varepsilon^j \mathsf{C}^z_j = O_{H^s_{\text{loc}} \to H^{s+N(1/r-2)}_{\text{loc}}} \left(\varepsilon^{N+1} \left| \text{Im} z \right|^{-2N w_n - 2} \right), \tag{2.23}$$

$\forall s \in \mathbb{R}$. A similar expansion as (2.20) for $\left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} + 1 \right)^M \left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} - z \right)$, $M \in \mathbb{N}$, also gives

$$\left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} + 1 \right)^{-M} \left(\tilde{\Delta}^{\varepsilon}_{g^E,F,\mu} - z \right)^{-1} - \sum_{j=0}^{N} \varepsilon^j \mathsf{C}^z_{j,M}$$
$$= O_{H^s_{\text{loc}} \to H^{s+N(1/r-2)+\frac{M}{r}}_{\text{loc}}} \left(\varepsilon^{N+1} \left| \text{Im} z \right|^{-2N w_n - 2} \right) \tag{2.24}$$

where $\mathrm{C}^{z}_{j,M} = O_{H^{s}_{\mathrm{loc}} \to H^{s+N(1/r-2)+\frac{M}{r}}_{\mathrm{loc}}} \left(\varepsilon^{N+1} |\mathrm{Im} z|^{-2Nw_n-2}\right)$, $j = 0, \ldots, N$, with

$$\mathrm{C}^{z}_{0,M} = \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} + 1\right)^{-M} \left(\hat{\Delta}^{(0)}_{g^E,F,\mu} - z\right)^{-1}.$$

For $M \gg 0$ sufficiently large, Sobolev's inequality gives an expansion for the corresponding Schwartz kernels of (2.24) in $C^0\left(\mathbb{R}^n \times \mathbb{R}^n\right)$. The heat kernel expansion now follows by plugging the resolvent expansion into the Helffer–Sjöstrand formula (see [15, Ch. 8, eq. 8.3] for this formula and the notion of an analytic continuation used therein). Finally, to see that the expansion only involves even powers of ε, or that (2.17) has no half-integer powers of t, note that the operators $\hat{\Delta}^{(j)}_{g^E,F,\mu}$ in the expansion (2.20) change sign by $(-1)^j$ under the rescaling δ_{-1}. Thus the Schwartz kernel for C^{z}_{j} (2.22) then changes sign by $(-1)^j$ under this change of variables giving $\mathrm{C}^{z}_{j}(0,0) = 0$ for j odd. □

The above proof similarly gives an expansion for functions of the Laplacian

$$\left[\varphi\left(t\Delta_{g^E,F,\mu}\right)\right]_{\mu}(x,x) = \frac{1}{t^{Q(x)/2}}\left[A^{\varphi}_0(x) + A^{\varphi}_1(x)\,t + \cdots + A^{\varphi}_N(x)\,t^N + O\left(t^N\right)\right], \tag{2.25}$$

$\forall \varphi \in \mathcal{S}(\mathbb{R})$. As usual, the same proof gives a point-wise, near-off diagonal expansion for the heat kernel and its derivatives: i.e. an asymptotic expansion for $\left[\varphi\left(t\Delta_{g^E,F,\mu}\right)\right]_{\mu}\left(\delta_{t^{\frac{1}{2}}}x, \delta_{t^{\frac{1}{2}}}x'\right)$, as $t \to 0$, on the chosen privileged coordinate ball in the C^∞-norm on the product. This is only a matter of different substitutions in (2.19) and in the Helffer–Sjöstrand formula for φ in (2.24).

However both the above and the expansion Theorem 4 hold only pointwise along the diagonal. In particular, the leading order $Q(x)$ is in general a function of the point x on the diagonal. This hence does not immediately give heat trace or spectral function asymptotics for the sR Laplacian as the expansion might not be uniform or integrable in x. In the equiregular case, where $Q(x) = Q$ is constant, a uniform set of privileged coordinates, privileged at each point in a neighborhood of x, may be chosen in the proof. This gives the uniformity of the expansion in x and one obtains the asymptotics for the Weyl counting function $N(\lambda)$, for the number of eigenvalues of $\Delta_{g^E,F,\mu}$ below λ.

Theorem 5 *For an equiregular sR manifold case there is a heat trace expansion*

$$tr\, e^{-t\Delta_{g^E,F,\mu}} = \frac{1}{t^{Q/2}}\left[a_0 + a_1 t + \cdots + a_N t^N + O\left(t^N\right)\right],$$

$\forall N \in \mathbb{N}$, *with leading term given by* $a_0 = \int_X \left[e^{-\hat{\Delta}_{\hat{g}^E,\hat{\mu}}}\right]_{\hat{\mu}}(0,0)\,\mu$. *Thus the Weyl counting function satisfies*

$$N(\lambda) = \frac{\lambda^{Q/2}(1+o(1))}{\Gamma(Q/2+1)} \int_X \left[e^{-\hat{\Delta}_{\hat{g}^E,\hat{\mu}}}\right]_{\hat{\mu}}(0,0)\,\mu.$$

The above two theorems are by now well known [4, 24, 30, 37], with the investigation of the small time heat kernel asymptotics having begun in [7]. The above proof is based on the analytic localization technique [8] combined with the use of sR geometric privileged coordinate dilations.

3 Bochner Laplacian on tensor powers

A natural place where sub-Riemannian structures arise is on unit circle bundles. To be precise, let us consider $\left(X, E, g^{E}\right)$ a corank 1 sR structure on an n-dimensional manifold X. We assume that there is a free S^1 action on X with respect to which the sR structure is invariant and transversal: the generator $e \in C^{\infty}(TX)$ of the action and E are transversal at each point. The quotient $Y := X/S^1$ is then a manifold with a Riemannian metric g^{TY} induced from g^E. Equivalently, the natural projection $\pi : X \rightarrow Y$ is a principal S^1 bundle with connection given by the horizontal distribution E. Let $L := X \times_{\rho} S^1 \rightarrow Y$ be the Hermitian line bundle associated to the standard one-dimensional representation ρ of S^1 with induced connection ∇^L and curvature R^L. Since the distribution is of corank 1, the growth vector at x is simply a function of the step $r(x)$ and given by $m^E(x) = \Big(0, \underbrace{n-1, n-1, \ldots, n-1}_{r(x)-1 \text{ times}}, n\Big)$ (2.3). Equivalently, the canonical flag (2.2) is given by

$$E_j(x) = \begin{cases} E; & 1 \leq j \leq r(x) - 1 \\ TX; & j = r(x) \end{cases}.$$

Also, note that the weight vector at x is $w^E(x) = \Big(\underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}}, r(x)\Big)$, while the Hausdorff dimension is given by $Q(x) = n - 1 + r(x)$. On account of the S^1 invariance, each of $m^E(x)$, $r(x)$ and $Q(x)$ descend to functions on the base manifold Y. The degree of nonholonomy $r(x)$ at x is now characterized in terms of the order of vanishing of the curvature R^L as below.

Proposition 6 *The degree of nonholonomy of an S^1 invariant sR structure*

$$r(x) - 2 = ord\left(R^L\right) := \min\left\{ l \,|\, j^l_{\pi(x)}\left(R^L\right) \neq 0 \right\} \tag{3.1}$$

is given in terms of the order of vanishing of the curvature R^L on the base.

Proof In terms of local coordinates on Y and a local orthonormal section $\mathbb{1}$ for L, we may write $\nabla^L = d + ia^L$; $a^L \in \Omega^1(Y)$, while $E = \ker\left[d\theta + a^L\right]$ with θ being the induced coordinate on each fiber of X. The proposition now follows on noting $\left[U_i, U_j\right] = \left(da^L\right)_{ij} \partial_\theta = R^L_{ij} \partial_\theta$ for the local generating vector fields $U_j := \partial_{y_j} - a^L_j \partial_\theta$, $1 \leq j \leq n-1$. Repeated brackets among the U_j's are then given in terms of derivatives of the curvature R^L. □

Thus we see that the bracket generating condition is equivalent to the curvature R^L having a finite order of vanishing at each point of Y.

3.0.1. *Structure of* Y_r. As noted before, the function $y \mapsto r_y$ (1.2) is upper semi-continuous and gives a decomposition of the manifold $Y = \bigcup_{j=2}^{r} Y_j$; $Y_j := \{y \in Y | r_y = j\}$ with each $Y_{\leq j} := \bigcup_{j'=0}^{j} Y_{j'}$ being open. We next address the local structure of Y_r, the locus of highest vanishing order for the curvature.

Proposition 7 *The subset $Y_r \subset Y$ is locally any closed subset of a hypersurface.*

Proof First, express the curvature $R^L = R^L_{ij} dy_i \wedge dy_j$ in some coordinates centered at $y \in Y_r$. By definition, Y_r is described by equations of the following type near y

$$\partial_y^{\alpha} R^L_{ij} = 0, \quad \forall i, j = 1, 2, \ldots, n-1, \ \alpha \in \mathbb{N}_0^{n-1}, |\alpha| \leq r-3, \quad \text{while} \tag{3.2}$$

$$\partial_y^{\alpha_0} R^L_{i_0 j_0} \neq 0, \quad \text{for some } i_0, j_0 = 1, 2, \ldots, n-1, \ \alpha_0 \in \mathbb{N}_0^{n-1}, |\alpha_0| = r-2. \tag{3.3}$$

The second equation (3.3) implies that one of the functions $\partial_y^{\alpha} R^L_{ij}$, $|\alpha| = r-3$, has a non-zero differential and cuts out a hypersurface.

Conversely, let $S \subset \{0\} \times \mathbb{B}^{n-2}_{y_2,\ldots,y_{n-1}} \subset \mathbb{R}^{n-1}_{y_1,y_2,\ldots,y_{n-1}}$ be any closed subset of the $n-2$ dimensional ball, sitting inside the hypersurface $\{y_1 = 0\}$ in $n-1$ dimensions. By an application of the Whitney extension theorem, there exists a smooth function $f(y_2, \ldots, y_{n-1}) \in C^{\infty}\left(\mathbb{R}^{n-2}_{y_2,\ldots,y_{n-1}}\right)$ such that $S = \{0\} \times \left\{f_{y_3} = \ldots = f_{y_{n-1}} = 0\right\}$. The closed two form

$$R^L = d\left[-f dy_1 + \frac{1}{2} y_1^2 dy_2\right] = \left(y_1 + f_{y_2}\right) dy_1 dy_2 + \sum_{j=3}^{n-1} f_{y_j} dy_1 dy_j$$

is the curvature of some connection on the trivial line bundle over the ball. This curvature form satisfies $r = 3$ with $Y_3 = \left\{y_1 + f_{y_2} = f_{y_3} = \cdots = f_{y_{n-1}} = 0\right\}$. The local structure of Y_3 near the origin is now the same as S under the diffeomorphism $(y_1, y_2 \ldots, y_{n-1}) \mapsto \left(y_1 + f_{y_2}, y_2, \ldots, y_{n-1}\right)$. □

3.1 Smallest eigenvalue

The unit circle bundle of L being X, the pullback $\mathbb{C} \cong \pi^* L \to X$ is canonically trivial via the identification $\pi^* L \ni (x, l) \mapsto x^{-1} l \in \mathbb{C}$. Pick an auxiliary complex Hermitian vector bundle with connection $\left(F, h^F, \nabla^F\right)$ on Y and we denote by the same notation its pullback to X. Pulling back sections then gives the identification

$$C^{\infty}(X; F) = \oplus_{k \in \mathbb{Z}} C^{\infty}\left(Y; F \otimes L^k\right). \tag{3.4}$$

Each summand on the right-hand side above corresponds to an eigenspace of ∇_e^F with eigenvalue $-ik$. While horizontal differentiation d^H on the left corresponds to differentiation with respect to the tensor product connection∇^{L^k} on the right-hand side above. Pick an invariant density μ_X on X inducing a density μ_Y on Y. This now defines the sR Laplacian Δ_{g^E,F,μ_X} acting on sections of F. By invariance the sR Laplacian commutes $\left[\Delta_{g^E,F,\mu_X}, e\right] = 0$ with the generator of the circle action and hence preserves the decomposition (3.4). It acts via

$$\Delta_{g^E,F,\mu_X} = \oplus_{k\in\mathbb{Z}}\Delta_k \tag{3.5}$$

on each component where Δ_k is the Bochner Laplacian (1.1) on the tensor powers $F \otimes L^k$, with adjoint being taken with respect to μ_Y.

Next, we show that the heat kernel expansion for the sR Laplacian Theorem 4 gives a corresponding heat kernel expansion for the Bochner Laplacian.

Theorem 8 *The heat kernel of the Bochner Laplacian Δ_k has the following pointwise expansion on the diagonal*

$$e^{-\frac{t}{k^{2/r}}\Delta_k}(y,y) = \begin{cases} k^{(n-1)/r}\left[\sum_{j=0}^{N} a_{2j}(y;t)\,k^{-2j/r} + O\left(k^{-(N+1)/r}\right)\right]; & y \in Y_r \\ O\left(k^{-\infty}\right); & y \in Y_{\leq r-1} \end{cases} \tag{3.6}$$

with leading coefficient $a_0(y;t) = e^{-t\Delta_y}(0,0)$ being the heat kernel of the model operator (A.3) on the tangent space.

Proof The Fourier decomposition for the Laplacians (3.5), gives the corresponding relation

$$e^{-T\Delta_k}(y_1,y_2) = \left[\int d\theta\, e^{-T\Delta_{g^E,F,\mu_X}}\left(l_{y_1}, l_{y_2}e^{i\theta}\right)e^{-ik\theta}\right] l_{y_1}\otimes l_{y_2}^{*} \tag{3.7}$$

between the heat kernels with l_{y_1}, l_{y_2} denoting two unit elements in the fibers of L above y_1, y_2 respectively. We again note that the kernels are computed with respect to the densities μ_X, μ_Y chosen before. The above relation together with (2.15) first gives

$$e^{-\frac{1}{k^{2/r}}\Delta_k}(y_1,y_2) = c_{\varepsilon,N}k^{-N}, \quad \forall N \in \mathbb{N}, \tag{3.8}$$

when $d(y_1,y_2) > \varepsilon > 0$.

Choosing a coordinate system centered at a point $y \in Y$ and a local orthonormal section $\mathtt{l}$ of L gives an induced coordinate system on the unit circle bundle near x. It is easy to see that this induced coordinate system is privileged at each point on the fiber above y.

Next, using (3.7) with $T = \varepsilon^2 t$ and $y_1 = y_2$ belonging to this coordinate chart, one has

$$e^{-\varepsilon^2 t\Delta_k}(y_1,y_1) = \left[\int d\delta_\varepsilon\theta'\, e^{-\varepsilon^2 t\Delta_{g^E,F,\mu_X}}\left(\mathtt{l}(y_1), \mathtt{l}(y_1)\,e^{i\delta_\varepsilon\theta'}\right)e^{-ik\delta_\varepsilon\theta'}\right]$$

where δ_ε denotes the privileged coordinate dilation as before. Now setting $y_1 = \varepsilon \mathsf{y} = \delta_\varepsilon \mathsf{y}$, the Eqs. (2.19), (2.20) in the proof of Theorem 4 give an expansion for the integrand above

$$\begin{aligned} & e^{-\varepsilon^2 t \Delta_k} (\delta_\varepsilon \mathsf{y}, \delta_\varepsilon \mathsf{y}) \\ & = \int d\delta_\varepsilon \theta' e^{-ik\delta_\varepsilon \theta'} \varepsilon^{-Q(y)} \left[\sum_{j=0}^{N} a_{2j} \left(\mathsf{y}, \theta'; t \right) \varepsilon^{2j} + \frac{\varepsilon^{2N+1}}{t^{Q(y)/2}} R_{N+1} \left(\mathsf{y}, \theta'; t \right) \right] \end{aligned} \tag{3.9}$$

uniformly in $t \leq 1$ and $\mathsf{y} \in B_R(0)$, $\forall R > 0$. A slight difference above being that the coefficients $a_j \left(\mathsf{y}, \theta'; t \right)$ above are computed with respect to the model nilpotent sR Laplacian $\hat{\Delta}_y \coloneqq \hat{\Delta}_{g_y^{TY}, j^{r-2} R_y^L}$ (A.5) on the product $S^1_\theta \times \mathbb{R}^{n-1}$ rather than (2.21) on Euclidean space. In particular, the leading term is $a_0 \left(\mathsf{y}, \theta'; t \right) = e^{-t\hat{\Delta}_y} \left(\mathsf{y}, 0; \mathsf{y}, \theta' \right)$. Now set $\varepsilon = k^{-\frac{1}{r}}$ and $r_1(y) \coloneqq 1 - \frac{r(y)}{r}$ to obtain

$$\begin{aligned} & e^{-\frac{t}{k^{2/r}} \Delta_k} \left(k^{-\frac{1}{r}} \mathsf{y}, k^{-\frac{1}{r}} \mathsf{y} \right) \\ & = \int d\delta_{k^{-1/r}} \theta' e^{-ik^{r_1(y)} \theta'} k^{Q(y)/r} \left[\sum_{j=0}^{N} a_{2j} \left(\mathsf{y}, \theta'; t \right) k^{-2j/r} + O\left(k^{-(2N+1)/r} \right) \right] \\ & = \begin{cases} k^{(n-1)/r} \left[\sum_{j=0}^{N} a_{2j} (\mathsf{y}; t) k^{-2j/r} + O\left(k^{-(2N+1)/r} \right) \right]; & y \in Y_r \\ O\left(k^{-\infty} \right); & y \in Y_{\leq r-1} \end{cases} \end{aligned} \tag{3.10}$$

following a stationary phase expansion in θ'. Finally, setting $y = y_1 = \mathsf{y}$ in (3.10) proves the theorem.

Above we again note that the remainders are uniform for $\mathsf{y} \in B_R(0)$, $\forall R > 0$. The first coefficient is given by the model Laplacian on the tangent space $\Delta_y \coloneqq \Delta_{g_y^{TY}, j^{r-2} R_y^L}$ via

$$a_0 (\mathsf{y}; t) = \int d\theta' e^{-i\theta'} e^{-t\hat{\Delta}_y} \left(\mathsf{y}, 0; \mathsf{y}, \theta' \right) = e^{-t\Delta_y} (\mathsf{y}, \mathsf{y})$$

by (A.7). While the general coefficient has the form

$$\begin{aligned} a_{2j} (\mathsf{y}; t) &= -\frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\rho}(z) \, \mathrm{C}^z_{2j} (\mathsf{y}, \mathsf{y}) \, dz d\bar{z} \\ \mathrm{C}^z_{2j} &= \sum_{p \in I_{2j}} \left(\Delta_y - z \right)^{-1} \left[\prod_{\alpha} \Delta_{p_\alpha} \left(\Delta_y - z \right)^{-1} \right] \end{aligned} \tag{3.11}$$

as in (2.22), for some set of second-order differential operators Δ_j, $j = 1, 2, \ldots$, (see also (3.9) below). Above $\tilde{\rho}$ denotes an almost analytic continuation of $\rho \in \mathcal{S}(\mathbb{R})$ satisfying $\rho(x) = e^{-tx}$, $x \geq 0$. □

We now show how the heat kernel expansion immediately proves our first Theorem 1.

Proof of Theorem 1 We first give a short argument for asymptotic bounds on the smallest eigenvalue

$$C_1 k^{2/r} - C_1 \leq \lambda_0(k) \leq C[1 + o(1)] k^{2/r}. \tag{3.12}$$

The upper bound follows easily from a min-max argument. Namely by the min-max principle for self-adjoint operators applied to the model operator Δ_y on the tangent space at $y \in Y_r$, there exists $\tilde{\psi} \in C_c^\infty(\mathbb{R}^{n-1})$, $\|\psi\| = 1$ such that $\left\langle \Delta_y \tilde{\psi}, \tilde{\psi} \right\rangle \leq \lambda_0(\Delta_y) + \varepsilon$, for each $\varepsilon > 0$. Furthermore, the model operator arises as the leading term $\left(\delta_{k^{-1/r}}\right)_* \Delta_k = k^{2/r}\left[\Delta_y + O\left(k^{-1/r}\right)\right]$ under the rescaling $\delta_{k^{-1/r}} y := k^{-1/r} y$ in geodesic coordinates centered at y (cf. also Sect. 3.3 below). From the min–max principle for Δ_k one then obtains

$$\frac{\lambda_0(k)}{k^{2/r}} \leq k^{-2/r} \left\langle \Delta_k \tilde{\psi}_0^k, \tilde{\psi}_0^k \right\rangle \leq \lambda_0(\Delta_y) + o(1)$$

for $\tilde{\psi}_0^k := k^{(n-1)/r} \left(\delta_{k^{-1/r}}\right)^* \tilde{\psi}$. The upper bound (3.12) now follows. For the lower bound, we combine the trick of Guillemin–Uribe with the Rothschild–Stein subelliptic estimate (2.12) on the circle bundle to obtain

$$C_1 \left\| \partial_\theta^{1/r} s \right\|^2 \leq C_1 \|s\|_{H^{1/r}}^2 \leq \left[\left\langle \Delta_{g^E, F, \mu_X} s, s \right\rangle + \|s\|_{L^2}^2 \right],$$

$\forall s \in C^\infty(X; F)$. Letting $s = \pi^* \psi_0^k$ be the pullback of the orthonormal eigenfunction ψ_0^k of Δ_k gives $C_1 k^{2/r} \leq (\lambda_0(k) + 1)$ as required. To obtain the leading asymptotic (1.5) in Theorem 1 however one needs to show $C_1 = C$ in (3.12). This requires a closer look at the Rothschild–Stein subelliptic estimate (2.12) and in particular identifying the sharp constant therein.

We instead take an alternate route via the heat kernel, this is also consistent with our proofs of the other two theorems in the introduction. First for any $0 < t_1 < t_2$, $y \in Y_r$ and $R > 0$, one has the following estimate at leading order using (3.10)

$$\begin{aligned} \frac{\lambda_0(k)}{k^{2/r}} &\leq \frac{1}{(t_2 - t_1)} \ln \left(\frac{\int_{B_R(0)} d\left(k^{-\frac{1}{r}} \mathsf{y}\right) e^{-\frac{t_1}{k^{2/r}} \Delta_k} \left(k^{-\frac{1}{r}} \mathsf{y}, k^{-\frac{1}{r}} \mathsf{y}\right)}{\int_{B_R(0)} d\left(k^{-\frac{1}{r}} \mathsf{y}\right) e^{-\frac{t_2}{k^{2/r}} \Delta_k} \left(k^{-\frac{1}{r}} \mathsf{y}, k^{-\frac{1}{r}} \mathsf{y}\right)} \right) \\ &= \frac{1}{(t_2 - t_1)} \ln \left(\frac{\int_{B_R(0)} d\mathsf{y} e^{-t_1 \Delta_y}(\mathsf{y}, \mathsf{y}) + O\left(k^{-1/r}\right)}{\int_{B_R(0)} d\mathsf{y} e^{-t_2 \Delta_y}(\mathsf{y}, \mathsf{y}) + O\left(k^{-1/r}\right)} \right) \\ &= \frac{1}{(t_2 - t_1)} \ln \left(\frac{\int_{B_R(0)} d\mathsf{y} e^{-t_1 \Delta_y}(\mathsf{y}, \mathsf{y})}{\int_{B_R(0)} d\mathsf{y} e^{-t_1 \Delta_y}(\mathsf{y}, \mathsf{y})} \right) + O\left(k^{-1/r}\right). \end{aligned} \tag{3.13}$$

This already gives an upper bound on the first eigenvalue. To identify the constant (1.7) one takes the limit as $t_1 \to t_2$ to obtain

$$\frac{\lambda_0(k)}{k^{2/r}} \leq \frac{\int_{B_R(0)} d\mathsf{y} \left[\Delta_y e^{-t_1 \Delta_y}\right](\mathsf{y}, \mathsf{y})}{\int_{B_R(0)} d\mathsf{y} e^{-t_1 \Delta_y}(\mathsf{y}, \mathsf{y})} + O\left(k^{-1/r}\right),$$

$\forall t_1 > 0$. Using Proposition 19 of Section A, this gives $\limsup_{k \to \infty} \frac{\lambda_0(k)}{k^{2/r}} \leq \lambda_0\left(\Delta_y\right) + \varepsilon, \forall \varepsilon > 0$, $y \in Y_r$, and hence

$$\limsup_{k \to \infty} \frac{\lambda_0(k)}{k^{2/r}} \leq \inf_{y \in Y_r} \lambda_0\left(\Delta_y\right). \tag{3.14}$$

For the lower bound on $\lambda_0(k)$, first note that as in (2.25) one may prove an on diagonal expansion

$$\varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)(y, y) = k^{(n-1)/r}\left[a_0^{\varphi}(x) + a_1^{\varphi}(x) k^{-1/r} + \ldots + a_N^{\varphi}(x) k^{-N/r} + O\left(k^{-(N+1)/r}\right)\right]$$

$\forall \varphi \in \mathcal{S}(\mathbb{R})$, and where the coefficient a_j^{φ} has the form (3.11) with $\tilde{\rho}$ replaced with an analytic continuation of φ. Next note that each of the terms C_{2j}^{z} (3.11) is holomorphic in z for $\text{Re} z < C := \inf_{y \in Y_r} \lambda_0\left(\Delta_y\right)$. This gives $\varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)(y, y) = O\left(k^{-N}\right), \forall N \in \mathbb{N}$, uniformly in $y \in Y$, for $\varphi \in C_c^{\infty}(-\infty, C)$. Thus

$$\varphi\left(\frac{\lambda_0(k)}{k^{2/r}}\right) \leq \text{tr}\, \varphi\left(\frac{1}{k^{2/r}}\Delta_k\right) = O\left(k^{-N}\right)$$

and hence

$$\inf_{y \in Y_r} \lambda_0\left(\Delta_y\right) \leq \liminf_{k \to \infty} \frac{\lambda_0(k)}{k^{2/r}}. \tag{3.15}$$

From (3.14), (3.15) we have (1.5).

The estimate on the eigenfunction (1.6) then follows from $\left|\psi_0^k(y)\right|^2 \leq e^{\frac{\lambda_0(k)}{k^{2/r}}} e^{-\frac{1}{k^{2/r}}\Delta_k}(y, y)$ on using (3.6) and (1.5). □

3.2 Weyl law

In this subsection and the next, we shall prove Theorem 2 assuming $Y_r = \bigcup_{j=1}^{N} Y_{r,j}$ to be a union of embedded submanifolds, of dimensions $d_j := \dim\left(Y_{r,j}\right)$, along which the curvature R^L vanishes non-degenerately (1.8). Before proceeding, the following remark on our non-degeneracy hypothesis is in order.

Remark 9 (Non-degeneracy hypothesis) The non-degeneracy hypothesis (1.8) can be described more explicitly in local coordinates. Namely, if we choose a coordinate system

$$\Big(\underbrace{y_1, \ldots, y_{d_j}}_{=y'}; \underbrace{y_{d_j+1}, \ldots, y_{n-1}}_{=y''}\Big)$$

near $y \in Y_{r,j}$ in which $Y_{r,j} = \{y'' = 0\}$ is given by the vanishing of the last $n-1-d_j$ of these coordinates, then the curvature can be Taylor expanded as

$$R^L = \underbrace{\sum_{|\alpha|=r-2} \sum_{p,q=1}^{n} R_{pq,\alpha} \left(y''\right)^{\alpha} dy_p dy_q}_{=R_0^L} + O\left(\left(y''\right)^{r-1}\right). \tag{3.16}$$

The non-degeneracy condition (1.8) is now seen to be equivalent to the implication

$$\left(\partial^{\beta} R_0^L\right)(y) = 0, \ \forall |\beta| < r-2 \iff y'' = 0. \tag{3.17}$$

That is, the $(r-2)$-order vanishing locus $Y_r = Y_r^0 := \left\{y \in Y | \mathrm{ord}_y \left(R_0^L\right) = r-2\right\}$ is locally the same for the curvature R^L and its leading part R_0^L. An example of a curvature that is not non-degenerate in this sense is $R^L = \left(y_1^2 + y_2^4\right) dy_1 dy_2$. Here $r = 4$, the leading part of the curvature is $R_0^L = y_1^2 dy_1 dy_2$, while $\{0\} = Y_4 \neq Y_4^0 = \{y_1 = 0\}$.

A more restrictive condition, that is common in the literature and satisfied in the Montgomery case [31], is that the curvature R^L defines a 'magnetic well' at Y_r [19]. This assumes the existence of positive constants $C_1, C_2 > 0$ for which the curvature satisfies

$$C_1 d^g (y, Y_r)^{r-2} \leq \left|R^L(y)\right| \leq C_2 d^g (y, Y_r)^{r-2}, \quad \forall y \in Y, \tag{3.18}$$

with d^g denoting the Riemannian distance above. It is easy to see that the above (3.18) is stronger than and implies our non-degeneracy hypothesis (3.17). Examples of curvatures in dimension two that are non-degenerate (3.17) without defining a magnetic well (3.18) are $R^L = y_1 y_2 dy_1 dy_2$ (normal crossing), $y_1 y_2 (y_1 + y_2) dy_1 dy_2$, $y_1 y_2 \left(y_1^2 - y_2^2\right) dy_1 dy_2$ (multiple crossings), $y_1 \left(y_2 - y_1^2\right) dy_1 dy_2$ (tangential crossing), $y_1 \left(y_1^3 - y_2^2\right) dy_1 dy_2$ (cuspidal vanishing) and $y_1 \left(y_1^{k+1} \pm y_2^2\right) dy_1 dy_2$ ($A_k^{\pm}$ singularity). While in higher dimension a general class of examples is given by curvatures of the form $R^L = f df \wedge y_1 dy_1 \in \Omega^2\left(\mathbb{R}^{n-1}\right)$, for $f = y_2 \ldots y_{n-1} g (y_2, \ldots, y_{n-1})$, with g being any homogeneous polynomial. The vanishing set $Y_{\geq 3}$ for these curvatures includes $\{0\} \times V[g]$ for the variety $V[g] := \{g = 0\} \subset \mathbb{R}^{n-2}_{y_2,\ldots,y_n}$ corresponding to the arbitrary homogeneous polynomial g. While the highest order vanishing locus $Y_r = \{0\}$ is the origin for the above.

By a standard Tauberian argument, the first part of Theorem 2 on the asymptotics of the Weyl counting function now follows from the following heat trace expansion.

Theorem 10 *Assume that* $Y_r = \bigcup_{j=1}^{N} Y_{r,j}$ *is a union of embedded submanifolds, of dimensions* $d_j := \dim\left(Y_{r,j}\right)$*, along which the curvature* R^L *vanishes non-degenerately (1.8). For any* $f \in C^\infty(Y)$*, the heat trace of the Bochner Laplacian satisfies the asymptotics*

$$tr\left[f e^{-\frac{t}{k^{2/r}}\Delta_k}\right] = \sum_{j=1}^{N}\left\{\sum_{s=0}^{M} k^{(d_j-2s)/r}\left[\int_{NY_{r,j}} a_{j,s}(f;t)\right] + O\left(k^{(d_j-2M-1)/r}\right)\right\} \tag{3.19}$$

$\forall M \in \mathbb{N}$, $t \leq 1$. *Moreover, the leading terms above are given by*

$$a_{j,0}(f;t) = f|_{Y_{r,j}}\, e^{-t\Delta_y}(v,v)\,, \quad v \in N_y Y_{r,j}, \tag{3.20}$$

in terms of the pullback to the normal bundle of $f|_{Y_{r,j}}$.

Proof By Theorem 8 it suffices to consider f supported in a sufficiently small neighborhood of a given point $y \in Y_{r,j}$. We then again choose a coordinate system near y in which $Y_{r,j}$ is given by the vanishing of the last $n-1-d_j$ of the coordinates and in which the curvature has the Taylor expansion (3.16). We may further assume the coordinate vector fields $\left\{\partial_{y_j}\right\}_{j=1}^{n-1}$ to be orthonormal at y. The model operator (A.4) on the tangent space

$$\Delta_y = -\sum_{|\alpha|=r-2}\sum_{p,q=1}^{n}\left(\partial_{y_p} + \frac{i}{r} y^q \left(y''\right)^\alpha R_{pq,\alpha}\right)^2,$$

is given in terms of this leading part of the curvature. Below it shall also be useful to define the model semiclassical k-Bochner Laplacian

$$\Delta^{\text{mod}}_{y;\mathsf{k}} := -\sum_{|\alpha|=r-2}\sum_{p,q=1}^{n}\left(\partial_{y_p} + \frac{i\mathsf{k}}{r} y^q \left(y''\right)^\alpha R_{pq,\alpha}\right)^2, \quad \forall \mathsf{k} > 0, \tag{3.21}$$

corresponding to the leading part of the curvature in (3.16).

Next from (3.10) one has

$$e^{-\frac{1}{k^{2/r}}\Delta_k}(\delta_\varepsilon \mathsf{y}, \delta_\varepsilon \mathsf{y}) = k^{(n-1)/r}\left[\sum_{j=0}^{N} a_{2j}\left(\varepsilon k^{1/r}\mathsf{y}; t\right) k^{-2j/r} + O\left(k^{-(2N+1)/r}\right)\right],$$

$$\text{with} \quad a_0\left(\varepsilon k^{1/r}\mathsf{y}; t\right) = e^{-t\Delta_y}\left(\varepsilon k^{1/r}\mathsf{y}, \varepsilon k^{1/r}\mathsf{y}\right), \tag{3.22}$$

uniformly for $k^{-1/r} \geq \varepsilon$ and $\mathsf{y} \in B_1(0)$. Furthermore, substituting $t = \frac{1}{\varepsilon^2 k^{2/r}}$ in (3.9) we obtain

$$e^{-\frac{1}{k^{2/r}}\Delta_k}\left(\delta_\varepsilon \mathsf{y}, \delta_\varepsilon \mathsf{y}\right) = \varepsilon^{-(n-1)} \int d\theta' e^{-ik\varepsilon^r\theta'} \left[\sum_{j=0}^{N} a_{2j}\left(\mathsf{y}, \theta'; \frac{1}{\varepsilon^2 k^{2/r}}\right)\varepsilon^{2j} \right. \tag{3.23}$$

$$\left. + \frac{\varepsilon^{2N+1}}{\left(\varepsilon k^{1/r}\right)^{n-1+r}} R_{2N+1}\left(\mathsf{y}, \theta'; \frac{1}{\varepsilon^2 k^{2/r}}\right)\right],$$

$$a_0\left(\mathsf{y}, \theta'; \frac{1}{\varepsilon^2 k^{2/r}}\right) = e^{-\frac{1}{\varepsilon^2 k^{2/r}}\hat{\Delta}_y}\left(\mathsf{y}, 0; \mathsf{y}, \theta'\right) \tag{3.24}$$

uniformly for $k \in \mathbb{N}$, $k^{-1/r} \leq \varepsilon$ and $\mathsf{y} \in B_1(0)$. The leading term above is identified with the heat kernel

$$e^{-\frac{1}{\mathsf{k}^{2/r}}\Delta^{\text{mod}}_{y;\mathsf{k}}}(\mathsf{y}, \mathsf{y}) = \int d\theta' e^{-ik\varepsilon^r\theta'} a_0\left(\mathsf{y}, \theta'; \frac{1}{\varepsilon^2 k^{2/r}}\right),$$

of the model k-Bochner Laplacian (3.21) for $\mathsf{k} := k\varepsilon^r$. One next chooses

$$\mathsf{y} = \Big(\underbrace{0, \ldots, 0}_{=y'}; \underbrace{y_{d_j+1}, \ldots, y_{n-1}}_{=y''}\Big), \ \left|y''\right| = 1,$$

of the given form so that $\text{ord}_{\mathsf{y}}\left(R^L_0\right) < r - 2$ by (3.17). Then

$$e^{-\frac{1}{\mathsf{k}^{2/r}}\Delta^{\text{mod}}_{y;\mathsf{k}}}(\mathsf{y}, \mathsf{y}) = e^{-\Delta_y}\left(\mathsf{k}^{1/r}\mathsf{y}, \mathsf{k}^{1/r}\mathsf{y}\right) = O\left(\mathsf{k}^{-\infty}\right), \tag{3.25}$$

follows by a stationary phase type argument as in Theorem 8. A similar argument applied to the subsequent terms in (3.24), which are given by convolution integrals with the leading part, shows that $\int d\theta' e^{-ik\varepsilon^r\theta'} a_{2j}\left(\mathsf{y}, \theta'; \frac{1}{\varepsilon^2 k^{2/r}}\right) = O\left(\mathsf{k}^{-\infty}\right)$, $\forall j$. In particular, the terms of (3.23), (3.24) are integrable in ε for fixed k. Thus (3.22), (3.24), (3.25) and a Taylor expansion for f near $\mathsf{y} = 0$ combine to give (3.19). □

3.3 Expansion for the smallest eigenvalue

In this subsection we prove the second part of Theorem 2 on the expansion for the first eigenvalue $\lambda_0(k)$, assuming non-degeneracy (1.8) and when Y_r is a finite set of points. The same argument as below, with a minor modification, also gives an expansion for the mth eigenvalue $\lambda_m(k)$ for any fixed $m \in \mathbb{N}_0$.

Before proceeding, we note a short argument showing that a weaker version of the second part (1.10) of Theorem 2 is immediate from its first part (1.9). Namely, when Y_r is a finite set of points (or $d^{\max}_j = 0$), the number of eigenvalues for the Bochner Laplacian $N\left[c_1 k^{2/r}, c_2 k^{2/r}\right]$, for $c_1 < C < c_2$, has a limit as $k \to \infty$ by (1.9). Furthermore, by Theorem 10 the functional traces $\operatorname{tr}\varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)$, $\varphi \in C_c^\infty(c_1, c_2)$, involving the eigenvalues in this interval, have expansions in powers of $k^{-1/r}$. Thus for k sufficiently large $\lambda_0(k)$ is a root of the polynomial $p_{k^{-1/r}}(\lambda) := \Pi_{\lambda_j(k)\in[c_1 k^{2/r}, c_2 k^{2/r}]}\left(\lambda - \lambda_j(k)\right)$ of a fixed degree in λ. The coefficients of this polynomial can be written in terms of the functional traces and hence have expansions in powers of $k^{-1/r}$. By an application of analytic perturbation theory for polynomial roots [23, Ch. 2.2], the smallest eigenvalue $\lambda_0(k)$ has an expansion in powers of $k^{-1/Mr}$, where $M \in \mathbb{N}$ is the multiplicity of one of the roots of $p_0(\lambda)$.

The above argument is however insufficient to obtain an expansion in powers of $k^{-1/r}$. Below we instead show that $\lambda_0(k)$ is an eigenvalue of a family of self-adjoint matrices $A_{k^{-1/r}}$, of fixed rank, whose entries admit expansions in $k^{-1/r}$. One may then apply analytic perturbation theory for self-adjoint matrices. This requires working at the level of eigenfunctions and our technique again partly borrows from [8, Ch. 9].

We first need some terminology. Let $\varrho < \min\left\{\frac{1}{2}, \frac{1}{2} i_{g^{TY}}\right\}$ be smaller than half the injectivity radius $i_{g^{TY}}$ of $\left(Y, g^{TY}\right)$. Choose a geodesic coordinate system on a ball $B_{2\varrho}(y)$ centered at $y \in Y_r$. Below it shall also be useful to choose ϱ small enough so that the balls $\left\{B_{2\varrho}(y)\right\}_{y\in Y_r}$ are disjoint. Choose local trivializations $\mathbb{1}, \left\{s_j\right\}_{j=1}^{\operatorname{rank}(F)}$ of L, F over $B_{2\varrho}(y)$ that are parallel with respect to ∇^L, ∇^F respectively along geodesics starting at the origin. The Bochner Laplacian can be written in this local frame and coordinates as $\Delta_k = \left(\nabla^{F\otimes L^k}\right)^* \nabla^{F\otimes L^k}$ where

$$\begin{aligned}
\nabla^{F\otimes L^k} &= d + a^F + k a^L \\
a_p^L &= \int_0^1 d\rho \left(\rho y^q R_{pq}^L(\rho x)\right), \\
a_p^F &= \int_0^1 d\rho \left(\rho y^q R_{pq}^F(\rho x)\right),
\end{aligned} \tag{3.26}$$

With $\chi \in C_c^\infty([-1,1];[0,1])$ with $\chi = 1$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we define the modified connections on $\mathbb{R}^{n-1}$ via

$$\begin{aligned}
\tilde{\nabla}^F &= d + \chi\left(\frac{|y|}{2\varrho}\right) a^F \\
\tilde{\nabla}^L &= d + \left[\underbrace{\int_0^1 d\rho\, \rho y^k \left(\tilde{R}^L\right)_{jk}(\rho y)}_{=\tilde{a}_j^L}\right] dy_j, \quad \text{where} \\
\tilde{R}^L &= \chi\left(\frac{|y|}{2\varrho}\right) R^L + \left[1 - \chi\left(\frac{|y|}{2\varrho}\right)\right] R_0^L.
\end{aligned} \tag{3.27}$$

Further, we choose a modified metric $\tilde{g}^{TY}$ which is Euclidean outside $B_{2\varrho}(y)$ and agrees with g^{TY} on $B_{\varrho}(y)$. This defines the modified Bochner Laplacian $\tilde{\Delta}_k := \left(\tilde{\nabla}^{F\otimes L^k}\right)^* \tilde{\nabla}^{F\otimes L^k}$ agreeing with $\Delta_k = \tilde{\Delta}_k$ on the geodesic ball $B_{\varrho}(y)$.

A dilation as before is now defined via $\delta_{k^{-1/r}} y := \left(k^{-1/r} y_1, \ldots, k^{-1/r} y_{n-1}\right)$ and we consider the rescaled Bochner Laplacian

$$\triangle := k^{-2/r}\left(\delta_{k^{-1/r}}\right)_* \tilde{\Delta}_k. \tag{3.28}$$

Using a Taylor expansion and (3.27), the rescaled Bochner Laplacian has an expansion

$$\triangle = \left(\sum_{j=0}^{N} k^{-j/r} \triangle_j\right) + k^{-2(N+1)/r} \mathrm{E}_{N+1}, \quad \forall N. \tag{3.29}$$

$$\text{where each } \triangle_j = a_{j;pq}(y)\,\partial_{y_p}\partial_{y_q} + b_{j;p}(y)\,\partial_{y_p} + c_j(y) \tag{3.30}$$

is a k-independent, self-adjoint, second-order differential operator while each

$$\mathrm{E}_j = \sum_{|\alpha|=N+1} y^{\alpha}\left[a^{\alpha}_{j;pq}(y;k)\,\partial_{y_p}\partial_{y_q} + b^{\alpha}_{j;p}(y;k)\,\partial_{y_p} + c^{\alpha}_j(y;k)\right] \tag{3.31}$$

is a k-dependent self-adjoint, second-order differential operator on $\mathbb{R}^{n-1}$. Furthermore the functions appearing in (3.30) are polynomials with degrees satisfying

$$\begin{gathered}\deg a_j = j, \quad \deg b_j \le j + r - 1, \quad \deg c_j \le j + 2r - 2 \\ \deg b_j - (j-1) = \deg c_j - j = 0 \pmod 2\end{gathered} \tag{3.32}$$

and whose coefficients involve

$$\begin{aligned} a_j &: \text{ atmost } j-2 \text{ derivatives of } R^{TY} \\ b_j, c_j &: \text{ atmost } j-2 \text{ derivatives of } R^F, R^{TY} \text{ and atmost } j+r-2 \text{ derivatives of } R^L \end{aligned} \tag{3.33}$$

The coefficients $a^{\alpha}_{j;pq}(y;k), b^{\alpha}_{j;p}(y;k), c^{\alpha}_j(y;k)$ of (3.31) are moreover uniformly C^{∞} bounded in k. The leading term of (4.19) is computed

$$\triangle_0 = \Delta_y := \Delta_{g^{TY}_y, j^{r-2}R^L_y} \tag{3.34}$$

in terms of the model Bochner Laplacian on the tangent space TY (A.3). We shall see below that these operators (3.30) are the same as those appearing in (3.11).

Next, in our chosen coordinates and trivialization, the curvature R^L again has a Taylor expansion as in (3.16) with the non-degeneracy condition (1.8) being equivalent to (3.17). If Y_r is further a finite set of points then the model operator (3.34) at $y \in Y_r$ has a discrete spectrum, $\mathrm{EssSpec}\left(\Delta_y\right) = \emptyset$, by Proposition 18 in Section A. We then set $\lambda_{0,y} < \lambda_{1,y}$ to be the two smallest eigenvalues of Δ_y and $E_{0,y} := \ker\left[\Delta_y - \lambda_{0,y}\right]$ the smallest eigenspace. Any normalized $\tilde{\psi} \in E_{0,y}$ defines a quasimode

$$\widetilde{\psi}_k(y) := \chi\left(\frac{2|y|}{\varrho}\right) \underbrace{k^{(n-1)/2r}\tilde{\psi}\left(k^{1/r}y\right)}_{=k^{(n-1)/2r}\delta^*_{k^{-1/r}}\tilde{\psi}} \in C^\infty\left(Y; F\otimes L^k\right), \quad \text{satisfying}$$

$$\begin{aligned} \left\|\widetilde{\psi}_k\right\| &= 1 + o(1) \\ \Delta_k\widetilde{\psi}_k &= k^{2/r}\lambda_{0,y}\widetilde{\psi}_k + O_{L^2}\left(k^{1/r}\right). \end{aligned} \tag{3.35}$$

And we define $\tilde{E}_{0,y}$ to be the span of the quasimodes corresponding to an orthonormal basis of $E_{0,y}$. Finally set $\bar{\lambda}_0 := \min_{y\in Y_r}\lambda_{0,y}$, $\bar{Y}_r := \left\{y \in Y_r | \lambda_{0,y} = \bar{\lambda}_0\right\} \subset Y_r$ and $\bar{\lambda}_1 := \min\left\{\lambda_{1,y} | y \in \bar{Y}_r\right\} \cup \left\{\lambda_{0,y} | y \in Y_r \backslash \bar{Y}_r\right\} > \bar{\lambda}_0$. Further set $\tilde{E}_0 := \oplus_{y\in\bar{Y}_r}\tilde{E}_{0,y} \subset C^\infty\left(Y; F\otimes L^k\right)$ and $\tilde{E}_0^\perp$ to be its L^2 orthogonal complement.

We now have the following proposition.

Proposition 11 *There exist $c > 0$, $k_0 \in \mathbb{N}$ such that*

$$\left|\left\langle \Delta_k\tilde{\psi}, \tilde{\psi}\right\rangle - \bar{\lambda}_0 k^{2/r}\right| \le ck^{1/r} \tag{3.36}$$

$$\langle \Delta_k\psi, \psi\rangle \ge \frac{1}{2}\left(\bar{\lambda}_0 + \bar{\lambda}_1\right)k^{2/r} \tag{3.37}$$

for each $k > k_0$ and $\tilde{\psi} \in \tilde{E}_0$, $\psi \in C^\infty\left(Y; F\otimes L^k\right) \cap \tilde{E}_0^\perp$ of unit L^2-norm.

Proof The first Eq. (3.36) follows easily from construction (3.35).

For (3.37), we first set $\chi_y\psi := \chi\left(\frac{d^{g^{TY}}(.,y)}{\varrho}\right)\psi$, with $d^{g^{TY}}$ being the Riemannian distance, for each $y \in Y_r$ and split

$$\psi = \underbrace{\left(\sum_{y\in Y_r}\chi_y\right)\psi}_{=\psi_1} + \underbrace{\left(1 - \sum_{y\in Y_r}\chi_y\right)\psi}_{=\psi_2}.$$

Now since the ψ_2 is compactly supported away from Y_r, an argument similar to (3.12) gives

$$\langle \Delta_k\psi_2, \psi_2\rangle \ge \left[c_1 k^{2/(r-1)} - c_2\right]\|\psi_2\|^2 \tag{3.38}$$

for some constants $c_1, c_2 > 0$ depending only on ϱ. Next since $\chi_y\psi$, $y \in \bar{Y}_r$, has compact support in $B_\varrho(y)$, we may decompose

$$k^{-(n-1)/2r}\left(\delta^{-1}_{k^{-1/r}}\right)^*\chi_y\psi = \underbrace{\psi_y^0}_{\in\ker\left[\Delta_0-\bar{\lambda}_0\right]} + \underbrace{\psi_y^+}_{\in\ker\left[\Delta_0-\bar{\lambda}_0\right]^\perp}.$$

Clearly ψ_y^0 is orthogonal to ψ_y^+ and $\triangle_0\psi_y^+$ while $\left\langle \triangle_0\psi_y^+, \psi_y^+\right\rangle \geq \bar{\lambda}_1 \left\|\psi_y^+\right\|^2$ by definition. Furthermore, $\chi_y\psi \perp \tilde{E}_{0,y}$ by construction and hypothesis. Hence we may compute

$$\begin{aligned}\left\langle \chi_y\psi, k^{(n-1)/2r}\delta^*_{k^{-1/r}}\tilde{\psi}\right\rangle &= \left\langle \chi_y\psi, (1-\chi)\, k^{(n-1)/2r}\delta^*_{k^{-1/r}}\tilde{\psi}\right\rangle \\ &= \left\langle k^{-(n-1)/2r}\left(\delta^{-1}_{k^{-1/r}}\right)^*\chi_y\psi, \left[1-\chi\left(k^{-1/r}y\right)\right]\tilde{\psi}\right\rangle \\ &= o\,(1)\,\|\chi_y\psi\|\end{aligned}$$

for any normalized $\tilde{\psi} \in E_{0,y}$. This in turn gives $\left\|\psi_y^0\right\| = o\,(1)\,\|\chi_y\psi\|$, $\left\|\psi_y^+\right\| = [1-o\,(1)]\,\|\chi_y\psi\|$ and hence

$$\begin{aligned}\left\langle \triangle_0 k^{-n/2r}\left(\delta^{-1}_{k^{-1/r}}\right)^*\chi_y\psi, k^{-n/2r}\left(\delta^{-1}_{k^{-1/r}}\right)^*\chi_y\psi\right\rangle &= \left\langle \triangle_0\psi_y^0, \psi_y^0\right\rangle + \left\langle \triangle_0\psi_y^+, \psi_y^+\right\rangle \\ &\geq \bar{\lambda}_1\left\|\psi_y^+\right\|^2 \geq \left[\bar{\lambda}_1 - o\,(1)\right]\|\chi_y\psi\|^2.\end{aligned}$$

On account of the rescaling (3.28), (3.29), (3.34) we then have

$$\left\langle \Delta_k\chi_y\psi, \chi_y\psi\right\rangle \geq k^{2/r}\left[\bar{\lambda}_1 - o\,(1)\right]\|\chi_y\psi\|^2. \tag{3.39}$$

Finally, with $\chi_1 = \sum_{y\in\bar{Y}_r}\chi_y$ we estimate

$$\begin{aligned}\left\|\nabla^{F\otimes L^k}\psi\right\| &\geq \rho\left\|\chi_1\nabla^{F\otimes L^k}\psi\right\| + (1-\rho)\left\|(1-\chi_1)\,\nabla^{F\otimes L^k}\psi\right\| \\ &= \rho\left\|-d\chi_1\psi + \nabla^{F\otimes L^k}\chi_1\psi\right\| + (1-\rho)\left\|d\chi_1\psi + \nabla^{F\otimes L^k}(1-\chi_1)\,\psi\right\| \\ &= \rho\left\|\nabla^{F\otimes L^k}\chi_1\psi\right\| + (1-\rho)\left\|\nabla^{F\otimes L^k}(1-\chi_1)\,\psi\right\| - O\,(1)\,\|\psi\| \\ &\geq \rho k^{1/r}\left[\bar{\lambda}_1 - o\,(1)\right]^{1/2}\|\chi_1\psi\|\end{aligned}$$

$$+ (1-\rho)\left[c_1 k^{2/(r-1)} - c_2\right]^{1/2} \|(1-\chi_1)\,\psi\| - O(1)\,\|\psi\|$$
$$\geq \frac{1}{2}\left(\bar{\lambda}_0 + \bar{\lambda}_1\right)^{1/2} k^{1/r}\,\|\psi\|$$

for $k \gg 0$ by (3.38) and (3.39). □

Following the above proposition, the min–max principle for eigenvalues gives

$$\mathrm{Spec}\,(\Delta_k) \subset \underbrace{\left[\bar{\lambda}_0 k^{2/r} - ck^{1/r}, \bar{\lambda}_0 k^{2/r} + ck^{1/r}\right]}_{\mathcal{I}_k :=} \cup \left[\frac{1}{2}\left(\bar{\lambda}_0 + \bar{\lambda}_1\right) k^{2/r}, \infty\right). \tag{3.40}$$

Next, choose $\alpha \in \left(\bar{\lambda}_0, \frac{\bar{\lambda}_0+\bar{\lambda}_1}{2}\right)$. And let $\Gamma = \{|z| = \alpha\}$ and $\varphi \in C_c(0, \alpha)$, with $\varphi = 1$ near $\bar{\lambda}_0$, define a circular contour in the complex plane and a cutoff function respectively. The resolvent $\left(\frac{1}{k^{2/r}}\Delta_k - z\right)^{-1}$ then exists for $z \in \Gamma$, $k \gg 0$ and one may define via

$$P_0 := \frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{k^{2/r}}\Delta_k - z\right)^{-1} = \varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)$$

the spectral projection onto the span of the Δ_k-eigenspaces with eigenvalue in the first interval $\mathcal{I}_k$ of (3.40). Finally, (3.36) and (3.37) imply that

$$P_0 : \tilde{E}_0 \xrightarrow{\sim} E_0 := \bigoplus\left\{\ker(\Delta_k - \lambda) : \lambda \in \mathcal{I}_k\right\} \tag{3.41}$$

is an isomorphism for $k \gg 0$. We now have the following.

Theorem 12 *For any two quasimodes $\widetilde{\psi}_k, \widetilde{\psi}'_k \in \tilde{E}_0$ (3.35), the inner product*

$$\langle\widetilde{\psi}_k, \Delta_k P_0 \widetilde{\psi}'_k\rangle = k^{2/r}\sum_{j=0}^{N} \tilde{c}_j k^{-j/r} + O\left(k^{(1-N)/r}\right) \tag{3.42}$$

has an asymptotic expansion for some $\tilde{c}_j \in \mathbb{R}$, $j = 0, 1, \ldots$.

Proof For two quasimodes $\widetilde{\psi}_k, \widetilde{\psi}'_k$ localized at two different points of $\bar{Y}_r$ one has $\langle\widetilde{\psi}_k, \Delta_k P_0 \widetilde{\psi}'_k\rangle = O\left(k^{-\infty}\right)$ following a similar off-diagonal decay for the kernel of $\varphi\left(\frac{1}{k^{2/r}}\Delta_k\right)$ as (3.8). We now consider two $\widetilde{\psi}_k, \widetilde{\psi}'_k \in \tilde{E}_{0,y}$ of the form (3.35) localized at the same point $y \in \bar{Y}_r$. In this case, first a finite propagation argument as in (2.16) gives

$$\langle\widetilde{\psi}_k, \Delta_k P_0 \widetilde{\psi}'_k\rangle = \left\langle\widetilde{\psi}_k, \tilde{\Delta}_k \varphi\left(\frac{1}{k^{2/r}}\tilde{\Delta}_k\right)\widetilde{\psi}'_k\right\rangle + O\left(k^{-\infty}\right), \quad \text{while}$$

$$\frac{1}{k^{2/r}}\tilde{\Delta}_k\varphi\left(\frac{1}{k^{2/r}}\tilde{\Delta}_k\right)\left(y,y'\right)=k^{(n-1)/r}\Delta\varphi\left(\Delta\right)\left(k^{1/r}y,k^{1/r}y'\right) \tag{3.43}$$

follows by a similar rescaling as in (2.19). We now obtain an expansion for the right-hand side above by a resolvent expansion for Δ similar to (2.23). Namely, let

$$I_j:=\left\{p=(p_0,p_1,\ldots)\,|\,p_\alpha\in\mathbb{N},\sum p_\alpha=j\right\}$$

denote the set of partitions of the integer j and define

$$\mathrm{C}_j^z:=\sum_{p\in I_j}\left(\Delta_0-z\right)^{-1}\left[\prod_\alpha\Delta_{p_\alpha}\left(\Delta_0-z\right)^{-1}\right].$$

Then by repeated applications of the local elliptic estimate we have

$$\left(\Delta-z\right)^{-1}-\sum_{j=0}^{N}k^{-j/r}\mathrm{C}_j^z=O_{H^s_{\mathrm{loc}}\to H^{s+2}_{\mathrm{loc}}}\left(k^{-(N+1)/r}\left|\mathrm{Im}z\right|^{-2rN-2}\right), \tag{3.44}$$

for each $N\in\mathbb{N}$, $s\in\mathbb{R}$. Plugging the above expansion into the Helffer–Sjöstrand formula then gives

$$\Delta\varphi\left(\Delta\right)-\sum_{j=0}^{N}k^{-j/r}\mathrm{C}_j^\varphi=O_{H^s_{\mathrm{loc}}\to H^{s+2}_{\mathrm{loc}}}\left(k^{-(N+1)/r}\right) \tag{3.45}$$

$\forall N\in\mathbb{N}$ and for some k-independent $\mathrm{C}_j^\varphi\in L^2\left(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\right)$, $j=0,1,\ldots$. A similar argument as (2.24), replacing (3.44) by the resolvent expansion for $\left(\Delta+1\right)^{-M}\left(\Delta-z\right)^{-1}$, shows that the last expansion above is valid in $C^l\left(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\right)$, $\forall l\in\mathbb{N}$. Hence plugging (3.45) into (3.43) finally gives

$$\langle\widetilde{\psi}_k,\Delta_kP_0\widetilde{\psi}_k'\rangle-k^{2/r}\left(\sum_{j=0}^{N}c_jk^{-j/r}\right)=O\left(k^{-(N-1)/r}\right)$$

$\forall N\in\mathbb{N}$, with $\tilde{c}_j:=\left\langle\tilde{\psi},\mathrm{C}_j^\varphi\tilde{\psi}'\right\rangle$ as required. □

The proof of Theorem 2 now follows from the above and is summarized below.

Proof of Theorem 2 As noted before, the first part of the theorem regarding the Weyl law (1.9) follows from Theorem 10 by a Tauberian argument.

For the second part of the theorem regarding the expansion for $\lambda_0(k)$, note from (3.41) that the low lying eigenvalues of Δ_k are given by $\mathrm{Spec}\left(\Delta_k|_{E_0}\right)=\mathrm{Spec}\left(\Delta_k|_{P_0\tilde{E}_0}\right)$ for $k\gg0$. But since the matrix coefficients of $\Delta_k|_{P_0\tilde{E}_0}$ were just shown to have an expansion in Theorem 12, the expansion for the smallest eigenvalue

$\lambda_0(k)$ now follows by an application of standard perturbation theory for self-adjoint matrices as in [23, Ch. 2.6]. □

Remark 13 (Spectrum and abnormals) Our Theorems 1 and 2 proved in this section are generalizations of the results in [31]. This latter article seems to have been motivated by describing a correspondence between the asymptotics of sR Laplace spectrum and the phenomenon of singular or abnormal geodesies in sR geometry, and claims to have achieved this goal. However our generalization of its results here shows that this is not the case, as indeed the concentration of the eigenfunction ultimately occurs on the locus Y_r where the Hausdorff dimension is maximized. And this in general has little if anything to do with abnormals. As a reference for the first spectral study of abnormals in sR geometry we refer instead to the recent article [34] of the second author.

4 Kodaira Laplacian on tensor powers

In this final section, we shall prove the Bergman kernel expansion in Theorem 3. Thus we now specialize to the case when Y is a complex Hermitian manifold with integrable complex structure J. For the arguments of this section, we shall further need to restrict to the two dimensional case, when Y is a Riemann surface (see Remark 16). The metric g^{TY} is induced from the Hermitian metric on the complex tangent space $T_{\mathbb{C}}Y = T^{1,0}Y$. Further (L, h^L), (F, h^F) are chosen to be a Hermitian, holomorphic line and vector bundles respectively. We denote by ∇^L, ∇^F the corresponding Chern connections. The curvature R^L of ∇^L is a $(1, 1)$ form which is further assumed to be semipositive

$$R^L(w, \bar{w}) \geq 0, \quad \forall w \in T^{1,0}Y. \tag{4.1}$$

We also assume as before that the curvature R^L vanishes at finite order at any point of Y. We note that semipositivity implies that the order of vanishing $r_y - 2 \in 2\mathbb{N}_0$ of the curvature R^L at any point y is even. Semipositivity and finite order of vanishing imply that there are points where the curvature is positive, the set where the curvature is positive is in fact an open dense set. Hence $\deg L = \int_Y \frac{i}{2\pi} R^L > 0$, so that L is ample.

Denote by $\left(\Omega^{0,*}\left(X; F \otimes L^k\right); \bar{\partial}_k\right)$ the Dolbeault complex and define the Kodaira Laplace and Dirac operators acting on $\Omega^{0,*}\left(X; F \otimes L^k\right)$ via

$$\Box_k := \frac{1}{2}(D_k)^2 = \bar{\partial}_k \bar{\partial}_k^* + \bar{\partial}_k^* \bar{\partial}_k \tag{4.2}$$

$$D_k := \sqrt{2}\left(\bar{\partial}_k + \bar{\partial}_k^*\right). \tag{4.3}$$

Clearly, D_k interchanges while $\Box_k$ preserves $\Omega^{0,0/1}$. We denote $D_k^{\pm} = D_k|_{\Omega^{0,0/1}}$ and $\Box_k^{0/1} = \Box_k|_{\Omega^{0,0/1}}$. The Clifford multiplication endomorphism $c : TY \to \mathrm{End}\left(\Lambda^{0,*}\right)$ is defined via $c(v) := \sqrt{2}\left(v^{1,0} \wedge - i_{v^{0,1}}\right)$, $v \in TY$, and extended multiplicatively to the entire exterior algebra $\Lambda^* TY$.

Denote by $\nabla^{TY}, \nabla^{T^{1,0}Y}$ the Levi-Civita and Chern connections on the real and holomorphic tangent spaces as well as by $\nabla^{T^{0,1}Y}$ the induced connection on the anti-holomorphic tangent space. Denote by Θ the real $(1,1)$ form defined by contraction of the complex structure with the metric $\Theta(.,.) = g^{TY}(J.,.)$. This is clearly closed $d\Theta = 0$, or Y is Köhler, and the complex structure is parallel $\nabla^{TY}J = 0$ or $\nabla^{TY} = \nabla^{T^{1,0}Y} \oplus \nabla^{T^{1,0}Y}$.

With the induced tensor product connection on $\Lambda^{0,*} \otimes F \otimes L^k$ being denoted via $\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}$, the Kodaira Dirac operator (4.3) is now given by the formula

$$D_k = c \circ \nabla^{\Lambda^{0,*}\otimes F\otimes L^k}. \tag{4.4}$$

Next, we denote by R^F the curvature of ∇^F and by κ the scalar curvature of g^{TY}. Define the following endomorphisms of $\Lambda^{0,*}$

$$\begin{aligned} \omega\left(R^F\right) &:= R^F(w,\bar{w})\,\bar{w}i_{\bar{w}}, \quad \tau^F := R^F(w,\bar{w}) \\ \omega\left(R^L\right) &:= R^L(w,\bar{w})\,\bar{w}i_{\bar{w}}, \quad \tau^L := R^L(w,\bar{w}) \\ \omega(\kappa) &:= \kappa\bar{w}i_{\bar{w}}, \end{aligned} \tag{4.5}$$

in terms of an orthonormal section w of $T^{1,0}Y$. The Lichnerowicz formula for the above Dirac operator [26, Thm. 1.4.7] simplifies for a Riemann surface and is given by

$$\begin{aligned} 2\square_k = D_k^2 &= \left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \nabla^{\Lambda^{0,*}\otimes F\otimes L^k} \\ &\quad + k\left[2\omega\left(R^L\right) - \tau^L\right] + \left[2\omega\left(R^F\right) - \tau^F\right] + \frac{1}{2}\omega(\kappa). \end{aligned} \tag{4.6}$$

We now have the following.

Proposition 14 *Let Y be a compact Riemann surface, $(L, h^L) \to Y$ a semipositive line bundle whose curvature R^L vanishes to finite order at any point. Let $(F, h^F) \to Y$ be a Hermitian holomorphic vector bundle. Then there exist constants $c_1, c_2 > 0$, such that*

$$\|D_k s\|^2 \geq \left(c_1 k^{2/r} - c_2\right)\|s\|^2$$

for all $s \in \Omega^{0,1}\left(Y; F \otimes L^k\right)$.

Proof Writing $s = |s|\,\bar{w} \in \Omega^{0,1}\left(Y; F \otimes L^k\right)$ in terms of a local orthonormal section $\bar{w}$ gives

$$\left\langle\left[2\omega\left(R^L\right) - \tau^L\right]s, s\right\rangle = R^L(w,\bar{w})\,|s|^2 \geq 0 \tag{4.7}$$

from (4.1), (4.5). This gives

$$\begin{aligned}\left\langle D_k^2 s, s\right\rangle &= \left\langle \left[\left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \nabla^{\Lambda^{0,*}\otimes F\otimes L^k} + k\left[2\omega\left(R^L\right) - \tau^L\right]\right.\right. \\ &\quad \left.\left. + \left[2\omega\left(R^F\right) - \tau^F\right] + \frac{1}{2}\omega\left(\kappa\right)\right] s, s\right\rangle \\ &\geq \left\langle \left(\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \nabla^{\Lambda^{0,*}\otimes F\otimes L^k} s, s\right\rangle - c_0 \left\|s\right\|^2 \\ &\geq \left(c_1 k^{2/r} - c_2\right)\left\|s\right\|^2\end{aligned}$$

from Theorem 1, (4.6) and (4.7). □

We now derive as a corollary a spectral gap property for Kodaira Dirac and Laplace operators D_k, $\Box_k$ corresponding to Theorem 1.

Corollary 15 *Under the hypotheses of Proposition 14 there exist constants $c_1, c_2 > 0$, such that Spec $(\Box_k) \subset \{0\} \cup \left[c_1 k^{2/r} - c_2, \infty\right)$ for each k. Moreover,* $\ker D_k^- = 0$ *and $H^1\left(Y; F\otimes L^k\right) = 0$ for k sufficiently large.*

Proof From Proposition 14, it is clear that

$$\text{Spec}\left(\Box_k^1\right) \subset \left[c_1 k^{2/r} - c_2, \infty\right) \tag{4.8}$$

for some $c_1, c_2 > 0$ giving the second part of the corollary. Moreover, the eigenspaces of $D_k^2\big|_{\Omega^{0,0/1}}$ with non-zero eigenvalue being isomorphic by Mckean-Singer, the first part also follows. □

The vanishing $H^1\left(Y; F\otimes L^k\right) = 0$ for k sufficiently large also gives

$$\dim H^0\left(Y; F\otimes L^k\right) = \chi\left(Y; F\otimes L^k\right) = k\left[\text{rk}\left(F\right)\int_Y c_1\left(L\right)\right] + \int_Y c_1\left(F\right) + 1 - g, \tag{4.9}$$

by Riemann-Roch, with $\chi\left(Y; F\otimes L^k\right)$, $ch\left(F\otimes L^k\right)$, Td (Y), g denoting the holomorphic Euler characteristic, Chern character, Todd genus and genus of Y respectively.

Remark 16 Our proof of the last two results Proposition 14 and Corollary 15 follows [18, 25] from the positive case. In the semipositive case however the proof only works on a Riemann surface, since in higher dimensions there are more components to the $\left[2\omega\left(R^L\right) - \tau^L\right]$ term (4.7) in the Lichnerowicz formula (4.6) which semipositivity is insufficient to control.

Indeed, Donnelly has shown a counterexample to the existence of a spectral gap for semipositive line bundles in higher dimensions [16]. In the same paper [16, Cor. 3.3], Donnelly has also observed that on a Riemann surface the Kodaira Laplacian satisfies an $O(1)$ spectral gap: Spec $(\Box_k) \subset \{0\} \cup [c, \infty)$, for some $c > 0$, by using its equivalence with the closed range hypothesis for the Kohn Laplacian $\Box_b$ on the unit circle. It is however crucial for our proof of the Bergman kernel expansion Theorem 3 that the size of the spectral gap is $O\left(k^{2/r}\right)$, as in Corollary 15, or that it is at least growing with k.

4.1 Bergman kernel expansion

We now investigate the asymptotics of the Bergman kernel. Recall that this is the Schwartz kernel $\Pi_k\left(y_1, y_2\right)$ of the projector onto the kernel of $\Box_k$ with respect to the L^2 inner product given by the metrics g^{TY}, h^F and h^L. Alternately, if $\left\{s_j\right\}_{j=1}^{N_k^0}$ denotes an orthonormal basis of eigensections of $H^0\left(X; F \otimes L^k\right)$ then

$$\Pi_k\left(y_1, y_2\right) := \sum_{j=1}^{N_k} s_j\left(y_1\right) \otimes s_j\left(y_2\right)^* . \tag{4.10}$$

We wish to describe the asymptotics of $\Pi_k(y, y)$ along the diagonal in $Y \times Y$.

Next as in Sect. 3.3, we fix a geodesic coordinate system centered at $y \in Y$. By using parallel transport of an orthonormal basis $\{w\}$, $\{l\}$, $\left\{f_j\right\}_{j=1}^{\mathrm{rk}(F)}$ for $T_y^{1,0}Y$, L_y, F with respect to $\nabla^{T^{1,0}Y}$, ∇^L, ∇^F respectively we obtain a local orthonormal trivialization for the corresponding bundles over a geodesic ball $B_{2\varrho}(y)$. In this frame and coordinate system, the connection on the tensor product $\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}$ again has a similar expression as (3.26).

We now define a modified frame $\{\tilde{w}\}$ on $T^{1,0}\mathbb{R}^2$ which agrees with $\{w\}$ on $B_\varrho(y)$ and with $\left\{\frac{1}{\sqrt{2}}\left(\partial_{x_1} + i\partial_{x_2}\right)\right\}$ outside $B_{2\varrho}(y)$. Also define the modified metric $\tilde{g}^{TY}$ and almost complex structure $\tilde{J}$ on $\mathbb{R}^2$ to be standard in this frame and hence agreeing with g^{TY}, J on $B_\varrho(y)$. The Christoffel symbol of the corresponding modified induced connection on $\Lambda^{0,*}$now satisfies $\tilde{a}^{\Lambda^{0,*}} = 0$ outside $B_{2\varrho}(y)$.

Further we may as before define the modified connections $\tilde{\nabla}^F$, $\tilde{\nabla}^L$ (3.27) as well as the corresponding tensor product connection $\tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}$ which agrees with $\nabla^{\Lambda^{0,*}\otimes F\otimes L^k}$ on $B_\varrho(y)$. Clearly the curvature of the modified connection $\tilde{\nabla}^L$ is given by $\tilde{R}^L$ (3.27). This modified curvature can also be seen to be semipositive and vanishing to order $r_y - 2$ for ϱ sufficiently small. We now define the modified Kodaira Dirac operator on $\mathbb{R}^2$ by the similar formula $\tilde{D}_k = c \circ \tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}$, agreeing with D_k on $B_\varrho(y)$ by (4.4). This has a similar Lichnerowicz formula

$$\tilde{D}_k^2 = 2\tilde{\Box}_k := \underbrace{\left(\tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}\right)^* \tilde{\nabla}^{\Lambda^{0,*}\otimes F\otimes L^k}}_{=\tilde{\Delta}_k} + k\left[2\omega\left(\tilde{R}^L\right) - \tilde{\tau}^L\right]$$
$$+ \left[2\omega\left(\tilde{R}^F\right) - \tilde{\tau}^F\right] + \frac{1}{2}\omega\left(\tilde{\kappa}\right) \tag{4.11}$$

with the adjoint being taken with respect to the metric $\tilde{g}^{TY}$ and corresponding volume form. Also the endomorphisms $\tilde{R}^F$, $\tilde{\tau}^F$, $\tilde{\tau}^L$ and $\omega\left(\tilde{\kappa}\right)$ are the obvious modifications of

(4.5) defined using the curvatures of $\tilde{\nabla}^F$, $\tilde{\nabla}^L$ and $\tilde{g}^{TY}$ respectively. The above (4.11) again agrees with $\Box_k$ on $B_\varrho(y)$ while the endomorphisms $\tilde{R}^F, \tilde{\tau}^F, \omega(\tilde{\kappa})$ all vanish outside $B_\varrho(y)$. Being semi-bounded below (4.11) is essentially self-adjoint. A similar argument as Corollary 15 gives a spectral gap

$$\mathrm{Spec}\left(\tilde{\Box}_k\right) \subset \{0\} \cup \left[c_1 k^{2/r_y} - c_2, \infty\right). \tag{4.12}$$

To repeat some parts of the argument, first note that by construction the localized Bochner Laplacian in (4.11) is the rescaled model Laplacian $\tilde{\Delta}_k = k^{2/r}\mathscr{R}\Delta_0\mathscr{R}^{-1}$ on the complement of a compact ball $B_\varrho(y)^c$. Using the global subelliptic estimate for the model Laplacian (A.6), one obtains $c_1, c_2 > 0$ such that

$$\left\langle \tilde{\Delta}_k s, s\right\rangle \geq \left(c_1 k^{2/r} - c_2\right)\|s\|^2 \tag{4.13}$$

for each $s \in C_c^\infty\left(B_\varrho(y)^c\right)$ supported outside the ball. Combinig this with the local subelliptic estimate on the compact ball $B_\varrho(y)$ one obtains (4.13) for each $s \in C_c^\infty\left(\mathbb{R}^2\right)$ and hence for all s in $\mathrm{Dom}\left(\tilde{\Delta}_k\right)$ as an unbouded operator on L^2. The spectral gap (4.12) for $\tilde{\Box}_k$ now again follows by the Lichnerwicz formula as in the proof of 15.

By elliptic regularity, the projector $\tilde{\Pi}_k$ from $L^2\left(\mathbb{R}^2; \Lambda_y^{0,*} \otimes F_y \otimes L_y^{\otimes k}\right)$ onto $\ker\left(\tilde{\Box}_k\right)$ then has a smooth Schwartz kernel with respect to the Riemannian volume of $\tilde{g}^{TY}$.

We are now ready to prove the Bergman kernel expansion Theorem 3.

Proof of Theorem 3 First choose $\varphi \in \mathcal{S}(\mathbb{R}_s)$ satisfying $\varphi(0) = 1$. For $c > 0$, set $\varphi_1(s) = 1_{[c,\infty)}(s)\varphi(s)$. On account of the spectral gap Corollary 15, and as φ_1 decays at infinity, we have

$$\begin{aligned} \varphi(\Box_k) - \Pi_k = \varphi_1(\Box_k) \quad \text{with} \\ \left\|\Box_k^a \varphi_1(\Box_k)\right\|_{L^2 \to L^2} = O\left(k^{-\infty}\right) \end{aligned} \tag{4.14}$$

$\forall a \in \mathbb{N}$. Combining the above with semiclassical Sobolev and elliptic estimates gives

$$|\varphi(\Box_k) - \Pi_k|_{C^l(Y\times Y)} = O\left(k^{-\infty}\right), \tag{4.15}$$

$\forall l \in \mathbb{N}_0$. Next, we may write $\varphi(\Box_k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi\Box_k}\hat{\varphi}(\xi)\,d\xi$ via Fourier inversion. Since $\Box_k = \tilde{\Box}_k$ on $B_\varrho(y)$, we may use a finite propagation argument as in (2.16) to conclude

$$\varphi(\Box_k)\left(y', y\right) = \varphi\left(\tilde{\Box}_k\right)\left(y', 0\right) + O\left(k^{-\infty}\right)$$

for $y' \in B_{\frac{\varrho}{2}}(y)$. Next, since the operator $\tilde{\Box}_k$ also satisfies a spectral gap (4.12), similar arguments as above for the localized Kodaira Laplacian give $\left\| \tilde{\Box}_k^a \varphi_1 \left(\tilde{\Box}_k\right)\right\|_{L^2_{\mathrm{loc}} \to L^2_{\mathrm{loc}}} = O\left(k^{-\infty}\right)$. And there after local elliptic regularity gives $\left|\varphi\left(\tilde{\Box}_k\right) - \tilde{\Pi}_k\right|_{C^l_{\mathrm{loc}}(Y\times Y)} = O\left(k^{-\infty}\right)$ similar to (4.15). Thus we now have a localization of the Bergman kernel

$$\begin{aligned}\Pi_k\left(.,y\right) &= O\left(k^{-\infty}\right), \quad \text{on } B_\varrho\left(y\right)^c\\ \Pi_k\left(.,y\right) - \tilde{\Pi}_k\left(.,0\right) &= O\left(k^{-\infty}\right), \quad \text{on } B_\varrho\left(y\right).\end{aligned} \tag{4.16}$$

It thus suffices to consider the Bergman kernel of the model Kodaira Laplacian (4.11) on $\mathbb{R}^2$.

Next, with the dilation $\delta_{k^{-1/r}} y = \left(k^{-1/r} y_1, \ldots, k^{-1/r} y_{n-1}\right)$ as in Sect. 3.3, the rescaled Kodaira Laplacian

$$\boxdot := k^{-2/r_y}\left(\delta_{k^{-1/r}}\right)_* \tilde{\Box}_k \tag{4.17}$$

satisfies

$$\varphi\left(\frac{\tilde{\Box}_k}{k^{2/r_y}}\right)\left(y,y'\right) = k^{2/r_y}\varphi\left(\boxdot\right)\left(yk^{1/r_y}, y'k^{1/r_y}\right) \tag{4.18}$$

for $\varphi \in C_c^\infty\left(\mathbb{R}\right)$ as in (3.43). Using a Taylor expansion via (3.27), the rescaled Kodaira Laplacian again has an expansion

$$\boxdot = \left(\sum_{j=0}^{N} k^{-j/r_y}\boxdot_j\right) + k^{-2(N+1)/r_y}\mathrm{E}_{N+1}, \ \forall N, \tag{4.19}$$

$$\text{where each } \boxdot_j = a_{j;pq}\left(y\right)\partial_{y_p}\partial_{y_q} + b_{j;p}\left(y\right)\partial_{y_p} + c_j\left(y\right) \tag{4.20}$$

is a k-independent self-adjoint, second-order differential operator while each

$$\mathrm{E}_j = \sum_{|\alpha|=N+1} y^\alpha\left[a^\alpha_{j;pq}\left(y;k\right)\partial_{y_p}\partial_{y_q} + b^\alpha_{j;p}\left(y;k\right)\partial_{y_p} + c^\alpha_j\left(y;k\right)\right] \tag{4.21}$$

is a k-dependent self-adjoint, second-order differential operator on $\mathbb{R}^2$. Furthermore the functions appearing in (4.20) are again polynomials with degrees satisfying the same conditions in (3.32) and (3.33). While the coefficients $a^\alpha_{j;pq}\left(y;k\right), b^\alpha_{j;p}\left(y;k\right)$, $c^\alpha_j\left(y;k\right)$ of (4.21) are uniformly C^∞ bounded in k. Using (3.27), (A.4), (A.9) and (A.10) the leading term of (4.19) is computed

$$\boxdot_0 = \Box_y := \Box_{g^{TY}, j_y^{r_y-2}R^L, J^{TY}} \tag{4.22}$$

in terms of the model Kodaira Laplacian on the tangent space TY (A.9).

Next, we obtain an expansion for the right-hand side of (4.18) by a resolvent expansion for $\boxdot$ as in (3.45). Namely, we let $I_j := \left\{p = (p_0, p_1, \ldots) \,|\, p_\alpha \in \mathbb{N}, \sum p_\alpha = j\right\}$ denote the set of partitions of the integer j and define

$$\mathrm{C}_j^z = \sum_{p \in I_j} (z - \boxdot_0)^{-1} \left[\Pi_\alpha \left[\boxdot_{p_\alpha} (z - \boxdot_0)^{-1}\right]\right]. \tag{4.23}$$

Then by repeated applications of the local elliptic estimate using (4.19) we have

$$(z - \boxdot)^{-1} - \left(\sum_{j=0}^{N} k^{-j/r_y} \mathrm{C}_j^z\right) = O_{H^s_{\mathrm{loc}} \to H^{s+2}_{\mathrm{loc}}} \left(k^{-(N+1)/r_y} \left|\mathrm{Im} z\right|^{-2Nr_y - 2}\right), \tag{4.24}$$

for each $N \in \mathbb{N}$, $s \in \mathbb{R}$. Plugging the above into the Helffer–Sjöstrand formula gives

$$\varphi(\boxdot) - \sum_{j=0}^{N} k^{-j/r_y} \mathrm{C}_j^\varphi = O_{H^s_{\mathrm{loc}} \to H^{s+2}_{\mathrm{loc}}} \left(k^{-(N+1)/r_y}\right) \tag{4.25}$$

$\forall l, N \in \mathbb{N}_0$ and for some k-independent $\mathrm{C}_j^\varphi \in C^\infty\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, $j = 0, 1, \ldots$. The leading term $\mathrm{C}_0^\varphi = \varphi(\boxdot_0) = \varphi\left(\Box_y\right)$ is given in terms of the modal Kodaira Laplacian. Again a similar argument as (2.24), replacing (4.24) by the resolvent expansion for $(\boxdot + 1)^{-M} (z - \boxdot)^{-1}$, shows that (4.25) is valid at the level of kernels in $C^l\left(\mathbb{R}^2 \times \mathbb{R}^2\right)$, $\forall l \in \mathbb{N}$. Finally choosing φ supported, and equal to one, near 0 gives (1.12) from the spectral gap (4.12) as well as (4.18) and (4.25). The leading term $c_0(y) = \mathrm{C}_0(0, 0) = \Pi_0(0, 0) := \Pi^{\boxdot_0}(0, 0)$ is seen to be the model Bergman kernel on the tangent space. See the argument in Section A at the bottom of page 29 for the positivity $c_0(y) > 0$. From this identification of $c_0(y)$ with the model kernel one sees that it has a locally smooth extension $c_{0,r_y}\left(y'\right)$ for y' near y, depending only on the type r_y at y. However such an extension might have nothing to do with the Bergman kernel at points y' other than y. Finally, to show that there are no odd powers of k^{-j/r_y}, one again notes that the operators $\boxdot_j$ (4.20) change sign by $(-1)^j$ under $\delta_{-1}x := -x$. Thus the Schwartz kernel for C_j^z (4.23) changes sign by $(-1)^j$ giving $\mathrm{C}_j^z(0, 0) = 0$ for j odd. □

We end by giving an example where semipositive bundles arise and where the first term of the Bergman kernel expansion (1.12) above can further be made explicit.

Example 17 (Branched coverings) Consider $f : Y \to Y_0$ a branched covering of a Riemann surface Y_0 with branch points $\{y_1, \ldots, y_M\} \subset Y$. The Hermitian holomorphic line bundle on Y is pulled back $\left(L, h^L\right) = \left(f^* L_0, f^* h^{L_0}\right)$ from one on Y_0. If $\left(L_0, h^{L_0}\right)$ is assumed positive, then $\left(L, h^L\right)$ is semipositive with curvature vanishing at the branch points. In particular, near a branch point $y \in Y$ of local degree $\frac{r}{2}$ one may find holomorphic geodesic coordinate such that the curvature is given by $R^L = \frac{r^2}{4} (z\bar{z})^{r/2-1} R^{L_0}_{f(y)} + O\left(|z|^{r-1}\right)$. The leading term of (1.12) is given by the model Bergman kernel $\Pi^{\boxdot_0}(0,0)$ of the operator $\boxdot_0 = bb^\dagger$, $b^\dagger = 2\partial_{\bar{z}} + a$, $a = \frac{r}{4}z(z\bar{z})^{r/2-1} R^{L_0}_{f(y)}$. An orthonormal basis for $\ker(\boxdot_0)$ is then seen to be

$$s_\alpha(z) := \left(\frac{1}{2\pi}\frac{r}{\Gamma\left(\frac{2(\alpha+1)}{r}\right)}\left[R^{L_0}_{f(y)}\right]^{\frac{2(\alpha+1)}{r}}\right)^{1/2} z^\alpha e^{-\Phi}, \quad \alpha \in \mathbb{N}_0, \quad \text{with}$$

$$\Phi(z) := \frac{1}{4}(z\bar{z})^{r/2} R^{L_0}_{f(y)}.$$

Since $s_\alpha(0) = 0$ for $\alpha \geq 1$, the model Bergman kernel at the origin is now computed

$$c_0(y) = \Pi_0(0,0) := \Pi^{\boxdot_0}(0,0) = |s_0(0)|^2 = \frac{1}{2\pi}\frac{r}{\Gamma\left(\frac{2}{r}\right)}\left[R^{L_0}_{f(y)}\right]^{\frac{2}{r}}$$

at the vanishing or branch point y.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data sharing Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A: Model operators

Here we define certain model Bochner, Kodaira Laplacians and Dirac operators acting on a vector space V. The Bochner Laplacian is intrinsically associated to a triple $\left(V, g^V, R^V\right)$ consisting of a metric g^V and a non-vanishing tensor $0 \neq R^V \in S^{r-2}V^* \otimes \Lambda^2 V^*$, $r \geq 2$. While the Kodaira Laplacian depends on an additional complex structure J^V on V. Throughout the article, the vector space $V = T_y Y$ is the tangent space of the manifold, with $g^V = g^{TY}_y$ the Riemannian metric, $J^V = J_y$ the complex structure and $R^V = j^{r-2}R^L_y$ the first non-vanishing jet of the auxiliary curvature R^L at a point $y \in Y$.

We say that the tensor R^V is nondegenerate if the following is satisfied

$$S^{r-s-2}V^* \otimes \Lambda^2 V^* \ni i_v^s \left(R^V\right) = 0,\ \forall s \leq r-2 \implies T_yY \ni v = 0. \quad \text{(A.1)}$$

Above i^s denotes the s-fold contraction of the symmetric part of R^V.

For $v_1 \in V$, $v_2 \in T_{v_1}V = V$, contraction of the antisymmetric part, denoted by ι, of R^V gives $\iota_{v_2}R^V \in S^{r-2}V^* \otimes V^*$. The contraction may then be evaluated $\left(\iota_{v_2}R^V\right)\left(v_1^r\right)$ at $v_1^r := v_1^{\odot r} \otimes v_1$ for $v_1 \in V$ and hence viewed as a homogeneous degree $r-1$ polynomial function on V. The tensor R^V now determines a one form $a^{R^V} \in \Omega^1(V)$ via

$$a_{v_1}^{R^V}(v_2) := \int_0^1 d\rho \left(\iota_{v_2}R^V\right)\left((\rho v_1)^r\right) = \frac{1}{r}\left(\iota_{v_2}R^V\right)\left(v_1^r\right), \quad \text{(A.2)}$$

which we may view as a unitary connection $\nabla^{R^V} = d + ia^{R^V}$ on a trivial Hermitian vector bundle E of arbitrary rank over V. The curvature of this connection is clearly R^V now viewed as a homogeneous degree $r-2$ polynomial function on V valued in Λ^2V^*. One now defines the model Bochner Laplacian, intrinsically associated to the tuple $\left(V, g^V, R^V\right)$, via

$$\Delta_0 = \Delta_{g^V,R^V} := \left(\nabla^{R^V}\right)^* \nabla^{R^V} : C^\infty(V;E) \to C^\infty(V;E). \quad \text{(A.3)}$$

depending on the pair $\left(g^V, R^V\right)$. An orthonormal basis $\{e_1, e_2, \ldots, e_n\}$, determines components $R_{pq,\alpha} := R^V\left(e^{\odot\alpha}; e_p, e_q\right)$, $\alpha \in \mathbb{N}_0^{n-1}$, $|\alpha| = r-2$, as well as linear coordinates $(y_1, \ldots, y_n)$ on V. The connection form in these coordinates is given by $a_p^{R^V} = \frac{i}{r}y^q y^\alpha R_{pq,\alpha}$. While the model Laplacian (A.3) is given

$$\Delta_0 = -\sum_{q=1}^n \left(\partial_{y_p} + \frac{i}{r}y^q y^\alpha R_{pq,\alpha}\right)^2. \quad \text{(A.4)}$$

As in (3.5), the above may now be related to the nilpotent sR Laplacian on the product $S_\theta^1 \times V$ given by

$$\hat{\Delta}_0 = \hat{\Delta}_{g^V,R^V} := -\sum_{q=1}^n \left(\partial_{y_p} + \frac{i}{r}y^q y^\alpha R_{pq,\alpha}\partial_\theta\right)^2, \quad \text{(A.5)}$$

and corresponding to the sR structure $\left(S_\theta^1 \times V, \ker\left(d\theta + a^{R^V}\right), \pi^* g^V, d\theta \text{vol} g^V\right)$ where the sR metric corresponds to g^V under the natural projection $\pi : S_\theta^1 \times V \to V$. Note that the above differs from the usual nilpotent approximation of the sR Laplacian since it acts on the product with S^1. Following [33, Part III, Sec. 16], the above satisfies a subelliptic estimate: there exists $C > 0$ such that

$$\|s\|^2_{H^{1/r}} \leq C\left[\left\langle \hat{\Delta}_0 s, s\right\rangle + \|s\|^2_{L^2}\right], \quad \forall s \in C^\infty_c\left(S^1_\theta \times V\right). \tag{A.6}$$

As (3.7), the heat kernels of (A.3), (A.5) are now related

$$e^{-t\Delta_0}\left(y, y'\right) = \int e^{-i\theta} e^{-t\hat{\Delta}_0}\left(y, 0; y', \theta\right) d\theta. \tag{A.7}$$

Next, assume that the vector space V is of even dimension and additionally is equipped with an orthogonal endomorphism $J^V \in O(V)$; $\left(J^V\right)^2 = -1$. This gives rise to a linear integrable almost complex structure on V, a decomposition $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$ into $\pm i$ eigenspaces of J and a Clifford multiplication endomorphism $c : V \to \text{End}\left(\Lambda^* V^{0,1}\right)$. We further assume that R^V is a $(1, 1)$ form with respect to J that is $S^k V^* \ni R^V(w_1, w_2) = 0$, $\forall w_1, w_2 \in V^{1,0}$. The $(0, 1)$ part of the connection form (A.2) then gives a holomorphic structure on the trivial Hermitian line bundle $\mathbb{C}$ with holomorphic derivative $\bar{\partial}_{\mathbb{C}} = \bar{\partial} + \left(a^V\right)^{0,1}$. One may now define the model Kodaira Dirac and Laplace operators, intrinsically associated to the tuple $\left(V, g^V, R^V, J^V\right)$, via

$$D_0 = D_{g^V, R^V, J^V} := \sqrt{2}\left(\bar{\partial}_{\mathbb{C}} + \bar{\partial}^*_{\mathbb{C}}\right) \tag{A.8}$$

$$\Box_0 = \Box_{g^V, R^V, J^V} := \frac{1}{2}\left(D_{g^V, R^V, J^V}\right)^2 \tag{A.9}$$

acting on $C^\infty\left(V; \Lambda^* V^{0,1}\right)$. The above (A.3), (A.9) are related by the Lichnerowicz formula

$$\Box_0 = \Delta_0 + c\left(R^V\right) \tag{A.10}$$

where $c\left(R^V\right) = \sum_{p<q} R_{pq}^{i_1 \ldots i_{r-2}} y_{i_1 \ldots} y_{i_{r-2}} c\left(e_p\right) c\left(e_q\right)$. We may choose a complex orthonormal basis $\left\{w_j\right\}_{j=1}^m$ of $V^{1,0}$ that diagonalizes the tensor R^V: $R^V\left(w_i, \bar{w}_j\right) = \delta_{ij} R_{j\bar{j}}$; $R_{i\bar{j}} \in S^{r-2}V^*$. This gives complex coordinates on V in which (A.9) may be written as

$$\Box_0 = \sum_{q=1}^{\dim V/2} b_j b_j^\dagger + 2\left(\partial_{z_j} a_j + \partial_{\bar{z}_j} \bar{a}_j\right) \bar{w}_j i_{\bar{w}_j}$$

$$\text{where} \quad b_j := -2\partial_{z_j} + \bar{a}_j, \quad b_j^\dagger = 2\partial_{\bar{z}_j} + a_j, \quad \text{for} \quad a_j = \frac{1}{r} R_{j\bar{j}} z_j, \tag{A.11}$$

with each $R_{j\bar{j}}(z)$, $1 \leq j \leq \dim V/2$, being a real homogeneous function of order $r - 2$.

Being symmetric with respect to the standard Euclidean density and semi-bounded below, both Δ_0 and $\Box_0$ are essentially self-adjoint on L^2. We shall need the following information regarding their spectrum.

Proposition 18 *There exists $c > 0$ such that Spec $(\Delta_0) \subset [c, \infty)$. For R^V satisfying the non-degeneracy condition (A.1) one has EssSpec $(\Delta_0) = \emptyset$. While for $\dim V = 2$ with $R^V(w, \bar{w}) \geq 0$ semipositive one has Spec $(\Box_0) \subset \{0\} \cup [c, \infty)$.*

Proof The proof of the first part is similar to that of (3.12). Introduce the deformed Laplacian $\Delta_k := \Delta_{g^V, kR^V}$ obtained by rescaling the tensor R^V. From (A.4) $\Delta_k = k^{2/r}\mathscr{R}\Delta_0\mathscr{R}^{-1}$ are conjugate under the rescaling $\mathscr{R} : C^\infty(V; E) \to C^\infty(V; E)$, $(\mathscr{R}u)(x) := u\left(yk^{1/r}\right)$ implying

$$\begin{aligned} \operatorname{Spec}(\Delta_k) &= k^{2/r}\operatorname{Spec}(\Delta_0) \\ \operatorname{EssSpec}(\Delta_k) &= k^{2/r}\operatorname{EssSpec}(\Delta_0) \end{aligned} \tag{A.12}$$

By an argument similar to (3.12), one has $\operatorname{Spec}(\Delta_k) \subset \left[c_1 k^{2/r} - c_2, \infty\right)$ for some $c_1, c_2 > 0$ for $R^V \neq 0$. From here $\operatorname{Spec}(\Delta_0) \subset [c, \infty)$ follows. Next, under the non-degeneracy condition, the order of vanishing of the homogeneous curvature R^V (of the homogeneous connection a^{R^V} (A.2)) is seen to be maximal at the origin: $\operatorname{ord}_y\left(R^V\right) < r - 2$ for $y \neq 0$. Following a similar sub-elliptic estimate (2.12) on $V \times S^1_\theta$ as in (3.12), we have

$$k^{2/(r-1)}\|u\|^2 \leq C\left[\langle \Delta_k u, u\rangle + \|u\|^2_{L^2}\right], \quad \forall u \in C^\infty_c(V \setminus B_1(0)),$$

holds on the complement of the unit ball centered at the origin. Combining the above with Persson's characterization of the essential spectrum [1, Ch. 3]

$$\operatorname{EssSpec}(\Delta_k) = \sup_R \inf_{\substack{\|u\|=1 \\ u \in C^\infty_c(V \setminus B_R(0))}} \langle \Delta_k u, u\rangle,$$

we have $\operatorname{EssSpec}(\Delta_k) \subset \left[c_1 k^{2/(r-1)} - c_2, \infty\right)$. From here and using (A.12), $\operatorname{EssSpec}(\Delta_0) = \emptyset$ follows.

For the final part, similarly set $\Box_k := \Box_{g^V, kR^V, J^V}$ and note that $k^{2/r}\operatorname{Spec}(\Box_0) = \operatorname{Spec}(\Box_k) \subset \{0\} \cup \left[c_1 k^{2/r} - c_2, \infty\right)$ by an argument similar to Corollary 15. □

Next, the heat $e^{-t\Delta_0}$, $e^{-t\Box_0}$ and wave $e^{it\sqrt{\Delta_0}}$, $e^{it\sqrt{\Box_0}}$ operators being well-defined by functional calculus, a finite propagation type argument as in (2.15) gives $\varphi(\Delta_0)(., 0) \in \mathcal{S}(V)$, $\varphi(\Box_0)(., 0) \in \mathcal{S}(V)$ are of Schwartz class for $\varphi \in \mathcal{S}(\mathbb{R})$. When $\operatorname{EssSpec}(\Delta_0) = \emptyset$ any eigenfunction of Δ_0 also lies in $\mathcal{S}(V)$. Finally, under the hypothesis of 18, the Schwartz kernel $\Pi_0(., 0) \in \mathcal{S}(V)$ of the projector $\Pi_0 = \Pi^{\Box_0}$ onto the kernel of $\Box_0$ is also seen to be of Schwartz class, on choosing φ supported close to the origin.

The constant $a_0 := \Pi_0(0, 0)$ is also the leading term in the boundary expansion $\Pi_D(z, z) \sim a_0(-\rho)^{-2-\frac{2}{r}}$ for the Bergman kernel of a weakly pseudoconvex finite type domain $D := \{\rho < 0\} \subset \mathbb{C}^2$ as $z \to x' \in \partial D$ a point on its boundary [22, Thm. 2]. Here $r = r\left(x'\right)$ is the typer of the point on the boundary. In this case, [10, Thm. 2] proved the lower bound $\Pi_D(z, z) \geq c(-\rho)^{-2-\frac{2}{r}}$ for some $c > 0$. Thus $a_0 > 0$.

We now state another proposition regarding the heat kernel of Δ_0. Below we denote $\lambda_0(\Delta_0) := \inf \mathrm{Spec}(\Delta_0)$.

Proposition 19 *For each $\varepsilon > 0$ there exist $t, R > 0$ such that the integral of the heat kernel satisfies*

$$\frac{\int_{B_R(0)} dx \left[\Delta_0 e^{-t\Delta_0}\right](x,x)}{\int_{B_R(0)} dx\, e^{-t\Delta_0}(x,x)} \leq \lambda_0(\Delta_0) + \varepsilon$$

Proof Setting $P := \Delta_0 - \lambda_0(\Delta_0)$ it suffices to show

$$\frac{\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \leq \varepsilon$$

for some $t, R > 0$. With $\Pi^P_{[0,x]}$ denoting the spectral projector onto $[0,x]$, we split the numerator

$$\begin{aligned}&\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x)\\ &\quad= \int_{B_R(0)} dx \left[\Pi^P_{[0,4\varepsilon]} Pe^{-tP}\right](x,x) + \int_{B_R(0)} dx \left[\left(1 - \Pi^P_{[0,4\varepsilon]}\right) Pe^{-tP}\right](x,x).\end{aligned}$$

From $P \geq 0$, $\Pi^P_{[0,4\varepsilon]} Pe^{-tP} \leq 4\varepsilon e^{-tP}$ and $\left(1 - \Pi^P_{[0,4\varepsilon]}\right) Pe^{-tP} \leq ce^{-3\varepsilon t}$, $\forall t \geq 1$, we may bound

$$\frac{\int_{B_R(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \leq 4\varepsilon + \frac{ce^{-3\varepsilon t} R^{n-1}}{\int_{B_R(0)} dx\, e^{-tP}(x,x)} \tag{A.13}$$

$\forall R, t \geq 1$. Next, as $0 \in \mathrm{Spec}(P)$ there exists $\|\psi_\varepsilon\|_{L^2} = 1$, $\|P\psi_\varepsilon\|_{L^2} \leq \varepsilon$. It now follows that $\left\|\psi_\varepsilon - \Pi^P_{[0,2\varepsilon]}\psi_\varepsilon\right\| \leq \frac{1}{2}$ and hence

$$\begin{aligned}\frac{1}{2} = -\frac{1}{4} + \int_{B_{R_\varepsilon}(0)} dx\, |\psi_\varepsilon(x)|^2 &\leq \int_{B_{R_\varepsilon}(0)} dx \left|\int dy \Pi^P_{[0,2\varepsilon]}(x,y)\, \psi_\varepsilon(y)\right|^2\\ &\leq \int_{B_{R_\varepsilon}(0)} dx \left(\int dy \Pi^P_{[0,2\varepsilon]}(x,y)\, \Pi^P_{[0,2\varepsilon]}(y,x)\right) = \int_{B_{R_\varepsilon}(0)} dx \Pi^P_{[0,2\varepsilon]}(x,x),\end{aligned}$$

for some $R_\varepsilon > 0$, using $\left(\Pi^P_{[0,2\varepsilon]}\right)^2 = \Pi^P_{[0,2\varepsilon]}$ and Cauchy–Schwartz. This gives

$$\int_{B_{R_\varepsilon}(0)} dx\, e^{-tP}(x,x) \geq \frac{e^{-2\varepsilon t}}{2}, \quad t > 1.$$

Plugging this last inequality into (A.13) gives

$$\frac{\int_{B_{R_\varepsilon}(0)} dx \left[Pe^{-tP}\right](x,x)}{\int_{B_{R_\varepsilon}(0)} dx\, e^{-tP}\,(x,x)} \le 4\varepsilon + ce^{-\varepsilon t} R_\varepsilon^{n-1}$$

from which the theorem follows on choosing t large. □

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