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Bergman kernel functions associated to measures supported on totally real submanifolds

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Abstract. We prove that the Bergman kernel function associated to a smooth measure supported on a piecewise-smooth maximally totally real submanifold K in $\mathbb{C}^n$ is of polynomial growth. For example, this holds in dimension one if K is a finite union of transverse Jordan arcs in $\mathbb{C}$. Our bounds are sharp when K is smooth. We give an application to the equidistribution of the zeros of random polynomials, which extends a result of Shiffman–Zelditch to the higher-dimensional setting.

1. Introduction

Let K be a non-pluripolar compact subset in $\mathbb{C}^n$, i.e., K is not contained in $\{\varphi = -\infty\}$ for any plurisubharmonic (psh) function φ on $\mathbb{C}^n$, which is not identically $-\infty$. Let μ be a probability measure whose support is non-pluripolar and is contained in K, and let Q be a real continuous function on K. Let $\mathcal{P}_k$ be the space of restrictions to K of complex polynomials of degree at most k on $\mathbb{C}^n$. The scalar product

$$\langle s_1, s_2\rangle_{L^2(\mu,kQ)} := \int_K s_1 \bar{s}_2 e^{-2kQ}\, d\mu$$

induces the $L^2(\mu, kQ)$-norm on $\mathcal{P}_k$. *The Bergman kernel function of order* k associated to μ with weight Q is defined by

$$B_k(x) := \sup_{s\in\mathcal{P}_k} |s(x)e^{-kQ(x)}|^2/\|s\|^2_{L^2(\mu,kQ)}$$

for $x \in K$. Equivalently, if $(s_1, \dots, s_{d_k})$ (here d_k denotes the dimension of $\mathcal{P}_k$) is an orthonormal basis of $\mathcal{P}_k$ with respect to the $L^2(\mu, kQ)$-norm, then

$$B_k(x) = \sum_{j=1}^{d_k} |s_j(x)|^2 e^{-2kQ(x)}.$$

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When $Q \equiv 0$, we say that B_k is *unweighted*. In this case ($Q \equiv 0$), the inverse of B_k is known as *the Christoffel function* in the literature on orthogonal polynomials. In practice, we also use a modified version of the Bergman kernel function as follows:

$$\widetilde{B}_k(x) := \sup_{s \in \mathcal{P}_k} |s(x)|^2 / \|s\|^2_{L^2(\mu, kQ)}$$

for $x \in \mathbb{C}^n$. The advantage of $\widetilde{B}_k$ is that it is well defined on $\mathbb{C}^n$.

The asymptotics of the Bergman kernel function (or its inverse, the Christoffel function) is essential for many applications in (higher-dimensional or not) real analysis including approximation theory, random matrix theory, etc. There is an immense literature on such asymptotics. We refer to [7,9,16,24,26,36,46,49,67,71], to cite a just few, for an overview on this very active research field.

Most standard settings are measures supported on concrete domains on $\mathbb{R}^n \subset \mathbb{C}^n$ (such as balls or simplexes in $\mathbb{R}^n$) or in the unit ball in $\mathbb{C}^n$. Considering measures on $\mathbb{C}^n$ whose support are not necessarily in $\mathbb{R}^n$ are also important in many applications; e.g., one can consult [7,40,67] where the authors consider measures supported on finite unions of piecewise smooth Jordan curves $\mathbb{C}$ or domains in $\mathbb{C}$ bounded by Jordan curves. We refer to the end of this section for a concrete application to the equidistribution of zeros of random polynomials.

All of settings mentioned above are particular cases of a more natural situation where our measures are supported on piecewise-smooth domains in a generic Cauchy–Riemann submanifold K in $\mathbb{C}^n$. This is the context in which we will work on in this paper.

We underline that, in view of potential applications, it is important to work with piecewise-smooth compact sets K (rather than only smooth ones). In what follows, by a (convex) *polyhedron* in $\mathbb{R}^M$, we mean a subset in $\mathbb{R}^M$ which is the intersection of a finite number of closed half-hyperplanes in $\mathbb{R}^M$.

Definition 1.1. A subset K of a real M-dimensional smooth manifold Y is called a nondegenerate $\mathscr{C}^5$ piecewise-smooth submanifold of dimension m if, for every point $p \in K$, there exists a local chart (W_p, Ψ) of Y such that Ψ is a $\mathscr{C}^5$-diffeomorphism from W_p to the unit ball of $\mathbb{R}^M$ and $\Psi(K \cap W_p)$ is the intersection with the unit ball of a finite union of convex polyhedra of the same dimension m.

A point $p \in K$ is said to be *a regular point* of K if the above local chart (W_p, Ψ_p) can be chosen such that $\Psi_p(K \cap W_p)$ is the intersection of the unit ball with an m-dimensional vector subspace in $\mathbb{R}^M$; in other words, K is an m-dimensional submanifold locally near p. The regular part of K is the set of regular points of K. *The singular part* of K is the complement of the regular part of K in K. Hence if K is a smooth manifold with boundary, then the boundary of K is the singular part of K and its complement in K is the regular part of K.

Now let Y be a complex manifold of dimension n and let K be a nondegenerate $\mathscr{C}^5$ piecewise-smooth submanifold of Y. Since Y is a complex manifold, its real tangent spaces have a natural complex structure J. We say that K is *Cauchy–Riemann (or CR for short) generic* in the sense of Cauchy–Riemann geometry if, for every $p \in K$ and every sequence of regular points $(p_m)_m \subset K$ approaching to p, any limit space of the sequence of tangent spaces of K at p_m is not contained in a complex hyperplane of the (real) tangent space at p of Y (equivalently, if E is a limit space of the sequence $(T_{p_m} K)_{m \in \mathbb{N}}$ of tangent spaces at p_m, then we have $E + JE = T_pY$, where T_pY is the real tangent space of Y at p).

For a CR generic K, note that the space $T_pK \cap JT_pK$ (p is a regular point in K) is invariant under J and hence has a complex structure induced by J. In this case, the complex dimension of $T_pK \cap JT_pK$ is the same for every p and is called *the CR dimension* of K. If r denotes the CR dimension of K, then $r = \dim K - n$. Thus the dimension of a generic K is at least n.

If K is CR generic and $\dim K = n$, then K is said to be (maximally) *totally real*, and it is locally the graph of a smooth function over a small ball centered at $0 \in \mathbb{R}^n$ which is tangent at 0 to $\mathbb{R}^n$. Examples of piecewise-smooth totally real submanifolds are polygons in $\mathbb{C}$ or boundaries of polygons in $\mathbb{C}$, and polyhedra of dimension n in $\mathbb{R}^n \subset \mathbb{C}^n$.

A notion playing an important role in the study of Bergman kernel functions is the following extremal function:

$$V_{K,Q} := \sup\{\psi \in \mathcal{L}(\mathbb{C}^n) : \psi \leq Q \text{ on } K\},$$

where $\mathcal{L}(\mathbb{C}^n)$ is the set of psh functions ψ on $\mathbb{C}^n$ such that $\psi(z) - \log|z|$ is bounded at infinity on $\mathbb{C}^n$. If $Q \equiv 0$, we put $V_K := V_{K,0}$.

Since K is non-pluripolar, the upper semi-continuous regularization $V^*_{K,Q}$ of $V_{K,Q}$ belongs to $\mathcal{L}(\mathbb{C}^n)$. The function $V_{K,Q}$ is always lower semi-continuous; see the comment right after Lemma 3.4 or [44, Corollary 5.1.3]. If $V_{K,Q} = V^*_{K,Q}$ (or equivalently, $V_{K,Q}$ is continuous), then we say (K, Q) is *regular*. A stronger notion is the following: we say that K is *locally regular* if, for every $z \in K$, there is an open neighborhood U of z in $\mathbb{C}^n$ such that, for every increasing sequence of psh functions $(u_j)_j$ on U with $u_j \leq 0$ on $K \cap U$, then $(\sup_j u_j)^* \leq 0$ on $K \cap U$. Observe that K is locally regular if, for every $z \in K$, there exists a small ball $B(z, r)$ centered at z in $\mathbb{C}^n$ such that $V_{K\cap B(z,r)}$ is continuous. Moreover, the following properties are equivalent:

(i) K is a locally regular set,

(ii) (K, Q) is regular for every continuous function Q on K,

(iii) (K, Q_0) is regular for $Q_0(z) := \frac{1}{2}\log(1 + |z|^2)$.

The (only) nontrivial implication (iii) to (i) was proved in [63, Theorem A]; see Remark 2.2 below for clarifications and [28, Proposition 6.1] for an earlier result showing that (i) is equivalent to (ii). We refer also to [54, Theorem 1.2] for generalizations.

We note however that there is an example of a compact set K in $\mathbb{C}^n$ such that K is not locally regular but (K, Q) for $Q \equiv 0$ is regular; see [61, Proposition 8.1]. One can consult [54, Section 5] for a survey of examples of locally regular sets. The following result answers the question raised in [9, Remark 1.8].

Theorem 1.2. *Let K be a compact generic Cauchy–Riemann nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$. Then K is locally regular.*

Note that it was known that K is locally regular if K is smooth real analytic; see, e.g., [9, Corollary 1.7]. After our paper appeared on arXiv, Viêt-Anh Nguyên informed us that Theorem 1.2 follows from his result [55, Theorem 1.1] provided that K is smooth.

A remark about the required smoothness is in order. In Theorem 1.2, we need $\mathcal{C}^5$-smoothness because we use results from [66]. For the proofs of the other main theorems, we use results from [65], for which only $\mathcal{C}^3$-smoothness is sufficient. For the sake of simplicity of presentation, we use $\mathcal{C}^5$-smoothness throughout.

The measure μ is said to be a *Bernstein–Markov measure* (with respect to (K, Q)) if, for every $\epsilon > 0$, there exists $C > 0$ such that

$$\sup_K |s|^2 e^{-2kQ} \le C e^{\epsilon k} \|s\|^2_{L^2(\mu,kQ)}$$

for every $s \in \mathcal{P}_k$. In other words, the Bergman kernel function of order k grows at most subexponentially, i.e., $\sup_K B_k = O(e^{\epsilon k})$ as $k \to \infty$ for every $\epsilon > 0$.

For some examples of Bernstein–Markov measures and criteria checking this condition, we refer to [16]. However, apart from few explicit geometric situations, there are not many (geometric) examples of Bernstein–Markov measures in higher dimensions. This is the motivation for our next main result giving a large geometric class of Bernstein–Markov measures.

Theorem 1.3. *Let K be a compact generic Cauchy–Riemann nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$. Let μ be a finite measure supported on K such that there exist constants $\tau > 0$, $r_0 > 0$ satisfying $\mu(\mathbb{B}(z,r) \cap K) \ge r^\tau$ for every $z \in K$, $r \le r_0$ (where $\mathbb{B}(z,r)$ denotes the ball of radius r centered at z in $\mathbb{C}^n$). Then, for every continuous function Q on K, μ is a Bernstein–Markov measure with respect to (K, Q).*

To the best of our knowledge, the above result was only known when K is real analytic. We are not aware of any results of this kind in the previous literature for maximally totally real submanifolds. A measure μ_0 on K is said to be a smooth volume form on K if μ_0 is given by a smooth volume form locally at every regular point in K, and if, for every singular point p in K, there is a local chart (Ψ, W_p) as in Definition 1.1, such that $\Psi(K \cap W_p) = \bigcup_{j=1}^s P_j$, where P_j's are polyhedra in $\mathbb{C}^n$ of the same dimension for every $1 \le j \le s$, and the restriction of μ_0 to P_j is a smooth volume form on P_j for $1 \le j \le s$.

Let Leb_K be now a smooth volume form on K. Then, for any $M > 0$, the measure $\mu = |z - z_0|^M \,\mathrm{Leb}_K$ satisfies the hypothesis of Theorem 1.3. Here is our next main result.

Theorem 1.4. *Let K be a compact generic Cauchy–Riemann nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$ of dimension n_K. Let Q be a Hölder continuous function of Hölder exponent $\alpha \in (0,1)$ on K, and let Leb_K be a smooth volume form on K, and $\mu = \rho \,\mathrm{Leb}_K$, where $\rho \ge 0$ and $\rho^{-\lambda} \in L^1(\mathrm{Leb}_K)$ for some constant $\lambda > 0$. Then we have*

$$\sup_K B_k \le C k^{2n_K(\lambda+1)/(\alpha\lambda)},$$

for some constant $C > 0$ independent of k.

We would like to point out that the regularity of weights affects considerably the growth of the Bergman kernel function; cf. [11, Remark 3.2]. One can also consult [11] or [31, Theorem 3.6] for polynomial upper bounds for $\mu = \mathrm{Leb}_{\mathbb{C}^n}$ (the Lebesgue measure on $\mathbb{C}^n$), and $K = \mathbb{C}^n$ or K to be the closure of a relatively compact open subset with $\mathcal{C}^2$-smooth boundary in $\mathbb{C}^n$, respectively.

With the exception of [12], there have been only a few papers on the polynomial growth of Bergman kernel functions associated with measures on real submanifolds in higher dimensions. Known upper bounds on B_k were proved mostly based on special geometric structures of the compact $K \subset \mathbb{R}^n$ (see, e.g., [46, 49]). Such a method is not useful in dealing with general situations as in Theorem 1.4. In [12], it was supposed that K is smooth real algebraic in $\mathbb{R}^m$ or the closure of a bounded convex open subset in $\mathbb{R}^m$, and their arguments use this hypothesis in an essential way.

Note that $\mu = |z - z_0|^M \operatorname{Leb}_K$ satisfies the hypothesis of Theorem 1.4 for any $M > 0$. In general, it is not possible to bound B_k from below by a polynomial in k; see Remark 3.7.

By [12, Theorems 2 and 4], if K is the closure of a bounded open convex subset in $\mathbb{R}^n$ and $Q \equiv 0$ and μ is the restriction of the Lebesgue measure on $\mathbb{R}^n$ to K, then $k^{-n} B_n$ is bounded and bounded away from 0 on a fixed compact subset in the interior of K (the behavior of B_k at boundary points is more complicated). On the other hand, for general μ on such K, by [24], the upper bound for B_k on K cannot be $O(k^n)$ in general. To be precise, it was proved there that if K is a smooth Jordan curve in $\mathbb{C}$, and μ_0 is the arc measure on K, and $\mu = (z - z_0)^\alpha \mu_0$ for some constant $\alpha > 0$, then $B_k(z_0) \approx k^{1+\alpha}$ as $k \to \infty$. One can see also [47] for a similar asymptotic in the case where K is the closure of the unit ball in $\mathbb{R}^n$.

We note that, by [32, Corollary 2.13], if (K, μ, Q) is as in the hypothesis of Theorem 1.4), then the triple (K, μ, Q) is 1-Bernstein–Markov in the sense that, for every constant $0 < \delta \le 1$, there exists a constant $C > 0$ such that

$$\sup_K |s|^2 e^{-2kQ} \le C e^{Ck^{1-\delta}} \|s\|^2_{L^2(\mu, kQ)}$$

for every $s \in \mathcal{P}_k$. This is much weaker than our bound. Nevertheless, [32, Corollary 2.13] is applicable to a broader class of K.

If K is smooth (has no boundary) and $Q \in \mathcal{C}^{1,\delta}(K)$ for some constant $\delta > 0$ (e.g., K is the unit circle in $\mathbb{C}$ as in a classical setting), we obtain sharp bounds which have potential applications in studying sampling or interpolation problems of multivariate polynomials on maximally totally real sets in $\mathbb{C}^n$. The case where K is compact smooth real algebraic was considered in [12]. Here is our next main result.

Theorem 1.5. *Let K be a maximally totally real $\mathcal{C}^5$-submanifold without boundary in $\mathbb{C}^n$. Let μ be a smooth volume form on K regarded as a measure on $\mathbb{C}^n$. Let $Q \in \mathcal{C}^{1,\delta}(K)$ for some constant $\delta > 0$. Let B_k be the Bergman kernel function associated to μ with weight Q. Then there exists $C > 0$ such that, for every $k \ge 0$,*

$$\sup_K B_k \le C k^n.$$

When K is smooth compact real algebraic of dimension n in $\mathbb{R}^m$, it was proved in [12] that $B_k/k^n \approx 1$ as $k \to \infty$. The proof of the upper bound for B_k in [12] relies crucially on the algebraicity of K. Our approach to Theorems 1.4 and 1.5 is different and is based on constructions of analytic discs partly attached to K, subharmonic functions on unit discs, and fine regularity of extremal plurisubharmonic envelopes associated to K.

In dimension one, we refer to [40, Theorem 4.3] for a similar bound in the case where K is analytic, and to [24, 67] and references therein for asymptotics of B_k (which behaves like k at regular points, but is more complicated at singular points).

We are not aware of any estimates for general smooth maximally totally real submanifolds in higher dimensions in the previous literature, that are similar to those given in Theorem 1.5 (except for the real algebraic case in [12] mentioned above). We refer to [1, 45, 60] for more precise bounds in the special case when K is a convex subset in $\mathbb{R}^n$.

Next, we present a convergence result, which is a consequence of Theorem 1.4.

Theorem 1.6. *Let K be a compact generic Cauchy–Riemann nondegenerate $\mathscr{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$. Let Q be a Hölder continuous function on K and let μ be as in Theorem* 1.4*. Then there exists $C > 0$ such that, for every $k \geq 1$, we have*

$$\Big\| \frac{1}{2k} \log \widetilde{B}_k - V_{K,Q} \Big\|_{\mathscr{C}^0(\mathbb{C}^n)} \leq C \frac{\log k}{k}.$$

We note that we also obtain a version of the (Bernstein–)Markov inequality for maximally totally real submanifolds, which may be useful elsewhere; see Theorem 3.13 below.

Zeros of random polynomials. We give an application of the above results to the study of equidistribution of zeros of random polynomials.

Let K be a non-pluripolar set in $\mathbb{C}^n$ and let μ be a probability measure on $\mathbb{C}^n$ such that the support of μ is contained in K and is non-pluripolar. Let Q be a continuous weight on K. Let $\mathscr{P}_k(K)$ be the space of restrictions of complex polynomials of degree at most k in $\mathbb{C}^n$ to K. Let $d_k := \dim \mathscr{P}_k(K)$, and let $s_1, \dots, s_{d_k}$ be an orthonormal basis of $\mathscr{P}_k(K)$ with respect to the $L^2(\mu, kQ)$-scalar product. Consider the random polynomial

$$p_k := \sum_{j=1}^{d_k} \alpha_j s_j, \tag{1.1}$$

where α_j are complex i.i.d. random variables. The study of zeros of random polynomials has a long history. A very classical example may be the Kac polynomial where $n = 1$, and $p_j = z^j$.

The distribution of zeros of more general random polynomials associated to orthonormal polynomials (as in (1.1)) was considered in [65] by observing that $1, z, \dots, z^k$ form an orthonormal basis of the restriction of the space of polynomials in $\mathbb{C}$ to $\mathbb{S}^1$ with respect to the L^2-norm induced by the Haar measure μ_0 on $\mathbb{S}^1$. In this setting, the necessary and sufficient conditions for the distribution of α_j so that the zeros of p are equidistributed almost surely or in probability with respect to the Lebesgue measure μ_0 on the unit circle as $k \to \infty$ are known; see [14, 25, 41, 43]. We also refer to [5, Sections 4 and 5] for explicit examples for distributions of zeros of random polynomials and numerical simulations.

There are many works (in one or higher dimension) following [65], to cite just a few, [3, 4, 6, 13, 15]. In all of these works, it seems to us that the issue of large deviation type estimates for the equidistribution of zeros of random polynomials has not been studied in any great detail. As will become clear in our proof below, the new ingredient needed for such an estimate is a quantitative convergence rate of $1/(2k) \log \widetilde{B}_k$ to the extremal function associated to K. This is what we obtained in Theorem 1.6. To state our result, we need some hypothesis on μ and the distribution of the random variables α_j.

Assume now that, for any $j = 1, \dots, d_k$, the distribution of α_j is $f \,\mathrm{Leb}_{\mathbb{C}}$, where f is a nonnegative bounded Borel function on $\mathbb{C}$ satisfying the following mild regularity property: there exists $C > 0$ such that, for every $r > 0$, we have

$$\int_{|z|>r} |f| \,\mathrm{Leb}_{\mathbb{C}} \leq C/r^2. \tag{1.2}$$

This condition was introduced in [13,15]. We want to study the distribution of zeros of $p_k \in \mathscr{P}_k$ as $k \to \infty$. We denote by $[p_k = 0]$ the current of integration along the zero divisor $\mathrm{Div}(p_k)$ of p_k. If $n = 1$, then $[p_k = 0]$ is the sum of Dirac masses at zeros of p_k, counted with multiplicities.

If (K, Q, μ) is Bernstein–Markov, it was proved in [15, Theorem 4.2] that, almost surely,

$$k^{-1}[p_k = 0] \to dd^c \log|V^*_{K,Q}|, \quad k \to \infty, \tag{1.3}$$

where the convergence is the weak one between currents. In other words, for every smooth form Φ of degree $(2n-2)$ with compact support in $\mathbb{C}^n$, one has

$$k^{-1}\int_{\mathrm{Div}(p_k)} \Phi \to \int_{\mathbb{C}^n} dd^c \log|V^*_{K,Q}| \wedge \Phi, \quad k \to \infty.$$

Theorem 1.3 above thus provides us a large class of measures for which the equidistribution of zeros of p holds.

Our goal now is to obtain a rate of convergence in (1.3). To this end, it is reasonable to ask for finer regularity on μ and of the distribution of α_j. We do not try to make the most optimal condition. Here is our hypothesis.

(H1) $|f(z)| \le |z|^{-3}$ for $|z|$ sufficiently large.

(H2) Let K be a nondegenerate $\mathscr{C}^5$ piecewise-smooth generic Cauchy–Riemann submanifold of $\mathbb{C}^n$, and let Q be a Hölder continuous function on K. Let $\mu = \rho\,\mathrm{Leb}_K$, where $\rho^{-\lambda} \in L^1(\mathrm{Leb}_K)$ for some constant $\lambda > 0$.

Condition (H1) ensures that (1.2) holds, and the joint-distribution of $\alpha_1, \dots, \alpha_{d_k}$ is dominated by the Fubini–Study volume form $\omega_{\mathrm{FS}}^{d_k}$ on $\mathbb{C}^{d_k}$, where ω_{FS} is the Fubini–Study form on $\mathbb{P}^{d_k} \supset \mathbb{C}^{d_k}$. Obviously, the Gaussian random variables satisfy this condition.

Condition (H2) is a natural extension of the classical setting of Kac polynomials where K is the unit circle in $\mathbb{C}$. In fact, in [65], the authors considered the setting where μ is the surface area on a closed analytic curve in $\mathbb{C}$ that bounds a simply connected domain Ω in $\mathbb{C}$, or μ is the restriction of the Lebesgue measure on $\mathbb{C}$ to Ω. This setting is relevant to random matrix theory as already pointed out in [65]. We refer to [14,58,59] for partial generalizations (without quantitative estimates) to domains with smooth boundary in $\mathbb{C}$. We would also like to mention that, in some cases, certain large deviation type estimates for random polynomials in dimension one were known; see [38, Theorem 10] for polynomial error terms, and [29, Theorem 1.1], [35, Theorem 3.10] for exponential error terms. To the best of our knowledge, there has been no quantitative generalization of the results in [65] to higher dimension. It has been commented in the latter paper that their method does not seem to have a simple generalization to the case of higher dimension.

We now recall the following notion of distance on the space of currents. For every $\beta \ge 0$, and T, S closed positive currents of bi-degree (m, m) on the complex projective space $\mathbb{P}^n$, define

$$\mathrm{dist}_{-\beta}(T, S) := \sup_{\Phi : \|\Phi\|_{\mathscr{C}^{[\beta], \beta-[\beta]}} \le 1} |\langle T - S, \Phi\rangle|,$$

where $[\beta]$ denotes the greatest integer less than or equal to β, and Φ is a smooth form of degree $(2n-m)$ on $\mathbb{P}^n$. It is a standard fact that the distance $\mathrm{dist}_{-\beta}$ for $\beta > 0$ induces the weak topology on the space of closed positive currents (see for example [34, Proposition 2.1.4]). We have the following interpolation inequality: for every $0 < \beta_1 \le \beta_2$, there exists $c_{\beta_1,\beta_2} > 0$ such that

$$\mathrm{dist}_{-\beta_2} \le \mathrm{dist}_{-\beta_1} \le c_{\beta_1,\beta_2}[\mathrm{dist}_{-\beta_2}]^{\beta_1/\beta_2}; \tag{1.4}$$

see [34, Lemma 2.1.2] or [50, 68].

Note that the currents $[p = 0]$ and $dd^c V_{K,Q}$ extend trivially through the hyperplane at infinity $\mathbb{P}^n \backslash \mathbb{C}^n$ to be closed positive currents of bi-degree $(1,1)$ on $\mathbb{P}^n$ (this is due to the correspondence (2.1)). Hence one can consider $\mathrm{dist}_{-\beta}$ between $k^{-1}[p = 0]$ and $dd^c V_{K,Q}$ as closed positive currents on $\mathbb{P}^n$.

Theorem 1.7 (Large deviation type estimate). *Assume that* (H1) *and* (H2) *are satisfied. Then, for every $M \geq 1$, there exists $C_M > 0$ such that, for every k,*

$$\mathscr{P}_k\Big\{(\alpha_1, \dots, \alpha_{d_k}) \in \mathbb{C}^{d_k} : \mathrm{dist}_{-2}(k^{-1}[p_k = 0], dd^c V_{K,Q}) \geq \frac{C_M \log k}{k}\Big\} \leq C_M k^{-M}, \tag{1.5}$$

where $\mathscr{P}_k$ denotes the joint-distribution of $\alpha_1, \dots, \alpha_{d_k}$.

By (1.4), one obtains similar estimates for $\mathrm{dist}_{-\beta}$ with $0 < \beta \leq 2$ as in Theorem 1.7. We do not know if the right-hand side of (1.5) is sharp.

We now state a direct consequence of Theorem 1.7 which is a higher-dimensional generalization of [65, Theorems 1 and 2]; see also Theorem 1.9 below. Denote by $\mathbb{E}_k(k^{-1}[p = 0])$ the expectation of the random normalized currents $k^{-1}[p = 0]$. For a sequence $(S_k)_{k\geq 1}$ of currents in $\mathbb{C}^n$, we write $S_k = O(k^{-1}\log k)$, $k \to \infty$, if each S_k is of order 0 and degree m, and there exists $C > 0$ such that, for every smooth form Φ of degree $(2n - m)$ with compact support in $\mathbb{C}^n$ with $\|\Phi\|_{\mathscr{C}^2} \leq 1$, and any $k \geq 1$, we have

$$|\langle S_k, \Phi\rangle| \leq C\frac{\log k}{k}.$$

Corollary 1.8. *Assume that* (H1) *and* (H2) *are satisfied. Then we have*

$$\mathbb{E}_k(k^{-1}[p_k = 0]) = dd^c V_{K,Q} + O\Big(\frac{\log k}{k}\Big).$$

Note that, in the case where α_j are Gaussian variables, the decay rate obtained in [65] is $O(k^{-1})$, and that this error term is optimal in dimension one (this can be seen by carefully examining the calculations in [65, Proposition 3.3]).

In order to have an appropriate notion of correlation of zeros in higher dimensions (where varieties of zeros are not necessarily discrete sets), we reformulate the equidistribution property of zeros of random polynomials in the following way. Let L be a complex algebraic curve in $\mathbb{C}^n$. Since zero varieties of generic polynomials intersect transversely L, almost surely, the number of intersection points (without counting multiplicities) of the random hypersurface $\{p = 0\}$ and L is exactly $k \deg L$ by Bézout's theorem. Define

$$\mu_{k,L} := \frac{1}{k \deg L}\sum_{j=1}^{k \deg L} \delta_{z_j},$$

where $z_1, \dots, z_{k \deg L}$ are zeros of p on L. Let $[L]$ be the current of integration along L. Since $V_{K,Q}$ is bounded, the product

$$\mu_L := \frac{1}{\deg L} dd^c V_{K,Q} \wedge [L]$$

is a well-defined measure supported on L (it is simply $dd^c(V_{K,Q}|_L)$ if L is smooth).

Theorem 1.9. *Assume that* (H1) *and* (H2) *are satisfied. Then, for every $M \geq 1$, there exists $C_M > 0$ so that, for every k,*

$$\mathscr{P}_k\Big\{(\alpha_1, \dots, \alpha_{d_k}) \in \mathbb{C}^{d_k} : \operatorname{dist}_{-2}(\mu_{k,L}, \mu_L) \geq C_M \frac{\log k}{k}\Big\} \leq C_M k^{-M}.$$

In particular, the measure $\mu_{k,L}$ converges weakly to μ_L as $k \to \infty$.

Now, since the zero sets of p_k on L are discrete and equidistributed with respect to μ_L as $k \to \infty$, one can ask as in [65] how they are correlated (if scaled appropriately). Nevertheless, such questions seem to be still out of reach in the higher-dimensional setting. Finally, we note that one can even consider L to be a transcendental curve in $\mathbb{C}^n$. In this case, generic polynomials p still intersect L transversely asymptotically (see [42]); the issue of equidistribution is however more involved.

2. Bergman kernel functions associated to a line bundle

The results mentioned in the introduction have their direct generalizations in the context of complex geometry where $\mathbb{C}^n$ is replaced by a compact Kähler manifold. Working in such a generality will make the presentation more clear and enlarge the range of applicability of the theory. We will now describe the setting.

Let X be a projective manifold of dimension n. Let (L, h_0) be an ample line bundle equipped with a Hermitian metric h_0 whose Chern form ω is positive. Let K be a compact non-pluripolar subset in X. Let μ be a probability measure on X such that the support of μ is non-pluripolar and is contained in K. Let h be a Hermitian metric on $L|_K$ such that $h = e^{-2\phi} h_0$, where ϕ is a continuous function on K. For $s_1, s_2 \in H^0(X, L)$, we define

$$\langle s_1, s_2 \rangle := \int_X \langle s_1, s_2 \rangle_h \, d\mu.$$

Since $\operatorname{Supp}\mu$ is non-pluripolar, the last scalar product defines a norm called $L^2(\mu, h)$-norm on $H^0(X, L)$. Let $k \in \mathbb{N}$. We obtain induced Hermitian metric h^k on L^k and a similar norm $L^2(\mu, h^k)$ on $H^0(X, L^k)$. Put $d_k := \dim H^0(X, L^k)$. Let $\{s_1, \dots, s_{d_k}\}$ be an orthonormal basis of $H^0(X, L^k)$ with respect to $L^2(\mu, h^k)$. The *Bergman kernel function of order k associated to* (L, h, μ) is

$$B_k(x) := \sum_{j=1}^{d_k} |s_j(x)|^2_{h^k} = \sup\{|s(x)|^2_{h^k} : s \in H^0(X, L^k),\ \|s\|_{L^2(\mu, h^k)} = 1\}, \quad x \in K.$$

When μ is a volume form on X and $h = h_0$, the Bergman kernel function is an object of great importance in complex geometry; see [51] for a comprehensive study.

The setting considered in the introduction corresponds to the case where $X = \mathbb{P}^n$ and $(L, h_0) = (\mathcal{O}(1), h_{\mathrm{FS}})$ is the hyperplane line bundle on $\mathbb{P}^n$ endowed with the Fubini–Study metric. We consider $\mathbb{C}^n$ as an open subset in $\mathbb{P}^n$ and the weight Q corresponds to

$$\phi + \frac{1}{2}\log(1 + |z|^2).$$

Recall that there is a natural identification between $\mathscr{L}(\mathbb{C}^n)$ and the set of ω_{FS}-psh functions on $\mathbb{P}^n$ (where ω_{FS} denotes the Fubini–Study form on $\mathbb{P}^n$) given by

$$u \leftrightarrow u - \frac{1}{2}\log(1+|z|^2), \quad u \in \mathscr{L}(\mathbb{C}^n). \tag{2.1}$$

Another well-known example is the case where K is the unit sphere in $\mathbb{R}^n$ (here $n \geq 2$; see, e.g., [52]) and X is the complexification of K, i.e., $K = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ which is considered as usual a compact subset of $X := \{z_0^2 + z_1^2 + \cdots + z_n^2 = 1\} \subset \mathbb{P}^n$. The line bundle L on X is the restriction of the hyperplane bundle $\mathscr{O}(1) \to \mathbb{P}^n$ to X. We remark that, in this case, $H^0(X, L^k)$ is equal to the restriction of the space of $H^0(\mathbb{P}^n, \mathscr{O}(k))$ to X. Hence the restriction of $H^0(X, L^k)$ to K is that of the space of complex polynomials in $\mathbb{C}^n$ to K. To see this, notice that X is a smooth hypersurface in $\mathbb{P}^n$. Consider the standard exact sequence of sheaves

$$0 \to \mathscr{O}(k - \deg X) \to \mathscr{O}(k) \to \mathscr{O}(k)|_X \to 0,$$

where the second arrow is the multiplication by a section of $\mathscr{O}(\deg X)$ whose zero divisor is equal to X. We thus obtain a long exact sequence of cohomology spaces

$$\begin{aligned} 0 \to H^0(\mathbb{P}^n, \mathscr{O}(k - \deg X)) \to H^0(\mathbb{P}^n, \mathscr{O}(1)) \to H^0(\mathbb{P}^n, \mathscr{O}(k)|_X) \\ \to H^1(\mathbb{P}^n, \mathscr{O}(k - \deg X)) \to \cdots . \end{aligned}$$

In this sequence, $H^0(\mathbb{P}^n, \mathscr{O}(k)|_X)$ is isomorphic to $H^0(X, \mathscr{O}(k)|_X)$, and by the Kodaira–Nakano vanishing theorem, we have $H^1(\mathbb{P}^n, \mathscr{O}(k - \deg X)) = 0$; see [39, page 156]. As above, the weight Q on K in the spherical model corresponds to $\phi - \frac{1}{2}\log(1+|z|^2)|_X$ in the setting $(K, X, \mathscr{O}(1)|_X)$.

The measure μ is said to be a *Bernstein–Markov measure* (with respect to (K, ϕ, L)) if, for every $\epsilon > 0$, there exists $C = C(\epsilon) > 0$ such that

$$\sup_K |s|^2_{h^k} \leq C e^{\epsilon k} \|s\|^2_{L^2(\mu, h^k)}$$

for every $s \in H^0(X, L^k)$. In other words, the Bergman kernel function of order k grows at most subexponentially, i.e., $\sup_K B_k = O(e^{\epsilon k})$ as $k \to \infty$ for every $\epsilon > 0$. Theorem 1.3 is a particular case of the following result.

Theorem 2.1. *Let K be a compact nondegenerate $\mathscr{C}^5$ piecewise-smooth Cauchy–Riemann generic submanifold of X. Then, for every continuous function ϕ on K, if μ is a finite measure whose support is equal to K such that there exist constants $\tau > 0$, $r_0 > 0$ satisfying $\mu(\mathbb{B}(z, r) \cap K) \geq r^\tau$ for every $z \in K$, and every $r \leq r_0$ (where $\mathbb{B}(z, r)$ denotes the ball of radius r centered at z induced by a fixed smooth Riemannian metric on X), then μ is a Bernstein–Markov measure with respect to (K, ϕ, L).*

Let

$$\phi_K := \sup\{\psi \in \mathrm{PSH}(X, \omega) : \psi \leq \phi \text{ on } K\}.$$

Since K is non-pluripolar, the function ϕ_K^* is a bounded ω-psh function. If $\phi_K = \phi_K^*$, then we say (K, ϕ) is *regular*. A stronger notion is the following: we say that K is *locally regular* if, for every $z \in K$, there is an open neighborhood U of z such that, for every increasing sequence of psh functions $(u_j)_j$ on U with $u_j \leq 0$ on $K \cap U$, then $(\sup_j u_j)^* \leq 0$ on $K \cap U$.

Remark 2.2. Let $\phi_{\mathrm{FS}}(z) := -\frac{1}{2}\log(1+|z|^2)$, $z \in \mathbb{C}^n$. Let $X = \mathbb{P}^n$ and let $\omega := \omega_{\mathrm{FS}}$ be the Fubini–Study form on $\mathbb{P}^n$. Recall that $dd^c\phi_{\mathrm{FS}} = -\omega_{\mathrm{FS}}$ and if $u \in \mathcal{L}(\mathbb{C}^n)$, then $u + \phi_{\mathrm{FS}}$ belongs to $\mathrm{PSH}(\mathbb{P}^n, \omega_{\mathrm{FS}})$. It follows that $(\phi_{\mathrm{FS}})_K = V_K + \phi_{\mathrm{FS}}$ on $\mathbb{C}^n$. More generally, for every ϕ, one has

$$\phi_K = V_{K,\phi-\phi_{\mathrm{FS}}} + \phi_{\mathrm{FS}} \tag{2.2}$$

on $\mathbb{C}^n$. It was proved in [63] that, for every non-pluripolar compact set K in $\mathbb{C}^n$, there holds the following: K is locally regular if and only if $(\phi_1)_K$ is continuous for $\phi_1 \equiv 0$ on K, which in turn is equivalent to the fact that $V_{K,-\phi_{\mathrm{FS}}}$ is continuous (by (2.2)). We refer to [54, Theorem 1.2] for generalizations.

The following result answers the question raised in [9, Remark 1.8].

Theorem 2.3. *Every compact nondegenerate $\mathscr{C}^5$ piecewise-smooth Cauchy–Riemann generic submanifold of X is locally regular.*

Note that Theorem 1.2 is a direct consequence of the above result. It was shown in [9, Corollary 1.7] that K is locally regular if K is smooth real analytic. Theorem 2.1 is actually a direct consequence of Theorem 2.3 and the criterion [16, Proposition 3.4] giving a sufficient condition for measures to be Bernstein–Markov.

The Monge–Ampère current $(dd^c\phi_K^* + \omega)^n$ is called *the equilibrium measure* associated to (K, ϕ). It is well known that this measure is supported on K. By [9, Theorem B], one has

$$d_k^{-1} B_k \mu \to (dd^c\phi_K^* + \omega)^n, \quad k \to \infty,$$

provided that μ is a Bernstein–Markov measure associated to (K, ϕ, L). The last property suggests that the Bergman kernel function B_k cannot behave too wildly at infinity.

Theorem 2.4. *Let K be a compact nondegenerate $\mathscr{C}^5$ piecewise-smooth Cauchy–Riemann generic submanifold of X. Let n_K be the dimension of K. Let ϕ be a Hölder continuous function of Hölder exponent $\alpha \in (0,1)$ on K, let Leb_K be a smooth volume form on K, and $\mu = \rho\,\mathrm{Leb}_K$, where $\rho \geq 0$ and $\rho^{-\lambda} \in L^1(\mathrm{Leb}_K)$ for some constant $\lambda > 0$. Then there exists a constant $C > 0$ such that, for every k,*

$$\sup_K B_k \leq C k^{2n_K(\lambda+1)/(\alpha\lambda)}.$$

Note that, by the proof of [30, Theorem 1.3] or [31, Theorem 3.6], for every Hölder continuous function ϕ_1 on X, and $\mu_1 := \omega^n$, the Bergman kernel function of order k associated to (X, μ_1, ϕ_1) grows at most polynomially on K as $k \to \infty$; see also [8, Theorem 3.1] for the case where ϕ_1 is smooth.

Theorem 2.5. *Assume that the following two conditions hold:*

(i) *K is maximally totally real and has no singularity (i.e., K is smooth, without boundary),*

(ii) *$\phi \in \mathscr{C}^{1,\delta}(K)$ for some constant $\delta > 0$.*

Then there exists $C > 0$ such that, for every k and every $x \in K$, $B_k(x) \leq C k^n$ holds.

Consider the case when $X = \mathbb{P}^n$, $L := \mathcal{O}(1)$, $h_0 = h_{\mathrm{FS}}$ is the Fubini–Study metric on $\mathcal{O}(1)$, and K is a smooth maximally totally real compact submanifold in $\mathbb{C}^n \subset \mathbb{P}^n$, Q is a continuous function on K, and $h := e^{-2\phi} h_0$ on K, where $\phi := Q - \frac{1}{2}\log(1 + |z|^2)$. Observe that ϕ is in $\mathscr{C}^{1,\delta}(K)$ if Q is so. In this case, the hypotheses of Theorem 2.5 are fulfilled. Thus Theorem 2.5 implies Theorem 1.5.

As a consequence of Theorem 2.4, we obtain the following estimate generalizing Theorem 1.6.

Theorem 2.6. *Let K be a compact nondegenerate $\mathscr{C}^5$ piecewise-smooth generic submanifold of X. Let ϕ be a Hölder continuous function on K. Let μ be a smooth volume form on K. Then we have*

$$\Big\| \frac{1}{2k} \log \widetilde{B}_k - \phi_K \Big\|_{\mathscr{C}^0(X)} = O\Big(\frac{\log k}{k}\Big),$$

as $k \to \infty$, where

$$\widetilde{B}_k := e^{2k\phi} B_k = \sup\{|s(x)|^2_{h_0^k} : s \in H^0(X, L^k),\ \|s\|_{L^2(\mu, h^k)} = 1\}.$$

3. Bernstein–Markov property for totally real submanifolds

In the first part of this section, we prove Theorem 2.3, and hence Theorem 2.1 according to the comments in the paragraph after Theorem 2.3. In the second part of the section, assuming that Theorem 2.4 holds, we prove Theorem 2.6.

3.1. Local regularity. Let X be a compact Kähler manifold of dimension n with a Kähler form ω. Let K be a compact non-pluripolar subset on X and let ϕ be a continuous function on K. Recall that

$$\phi_K := \sup\{\psi : \psi\ \omega\text{-psh},\ \psi \le \phi \text{ on } K\}.$$

As K is non-pluripolar, we have $\phi_K < \infty$. Hence ϕ_K^* is a bounded ω-psh function on X.

When $K = X$ and $\phi \in \mathscr{C}^{1,1}$, it was proved in [10, 22, 66] that $\phi_K \in \mathscr{C}^{1,1}$. In general, the best regularity for ϕ_K is Hölder one; see Theorem 3.9 and Remark 3.10. One can check that if K is locally regular, then (K, ϕ) is regular for every ϕ.

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. *An analytic disc f* in X is a holomorphic mapping from $\mathbb{D}$ to X which is continuous up to the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. For an interval $I \subset \partial\mathbb{D}$, *$f$ is said to be I-attached to a subset $E \subset X$* if $f(I) \subset E$. Fix a Riemannian metric on X and denote by $\operatorname{dist}(\cdot,\cdot)$ the distance induced by it. For $x \in X$ and $r \in \mathbb{R}^+$, let $\mathbb{B}(x, r)$ be the ball of radius r centered at x with respect to the fixed metric. Here is the crucial property for us showing the existence of well-behaved analytic discs partly attached to a generic Cauchy–Riemann submanifold.

Proposition 3.1 ([70, Proposition 2.5]). *Let K be a compact generic nondegenerate $\mathscr{C}^5$ piecewise-smooth submanifold of X. Then there are positive constants c_0, r_0 and $\theta_0 \in (0, \pi/2)$ such that, for any $a_0 \in K$ and any $a \in \mathbb{B}(a_0, r_0) \backslash \{a_0\}$, there is a $\mathscr{C}^2$ analytic disc $f : \overline{\mathbb{D}} \to X$ such that f is $[e^{-i\theta_0}, e^{i\theta_0}]$-attached to K, $\operatorname{dist}(f(1), a_0) \le c_0\delta$, $\delta = \operatorname{dist}(a, a_0)$, $\|f\|_{\mathscr{C}^2} \le c_0$, and there is $z^* \in \mathbb{D}$ so that $|1 - z^*| \le \sqrt{c_0\delta}$ and $f(z^*) = a$. Moreover, if a_0 is in a fixed compact subset K' of the regular part of K, then we have $|1 - z^*| \le c_0\delta$.*

It was stated that $f \in \mathscr{C}^1(\overline{\mathbb{D}})$ instead of $f \in \mathscr{C}^2(\overline{\mathbb{D}})$ in [70, Proposition 2.5]. But the latter regularity is indeed clear from the construction in the proof of [70, Proposition 2.5]. Note that the compactness of X is not necessary in the above result. In particular, if $K \Subset U$ for some open subset U of X, then the analytic disc f can be chosen to lie entirely in U. Here is a slight improvement of Proposition 3.1.

Proposition 3.2. *Assume that one of the following assumptions hold:*

(1) *K is a compact generic $\mathscr{C}^5$ smooth submanifold with $\mathscr{C}^5$ smooth boundary such that the boundary of K is also generic,*

(2) *K is a union of a finite number of compact sets as in* (1).

Then there are positive constants c_0, r_0 and $\theta_0 \in (0, \pi/2)$ such that, for any $a_0 \in K$ and any $a \in \mathbb{B}(a_0, r_0)\backslash\{a_0\}$, there exists a $\mathscr{C}^2$ analytic disc $f : \overline{\mathbb{D}} \to X$ such that f is $[e^{-i\theta_0}, e^{i\theta_0}]$-attached to K, $\operatorname{dist}(f(1), a_0) \leq c_0\delta$ *with* $\delta = \operatorname{dist}(a, a_0)$, $\|f\|_{\mathscr{C}^2} \leq c_0$, *and there is $z^* \in \mathbb{D}$ so that $|1 - z^*| \leq c_0\delta$, $f(z^*) = a$.*

Proof. If K fulfills one of the conditions (1) or (2), then K can be covered by a finite number of sets K_j such that, for every j, there exist an open subset U_j in X and a smooth family $(K_{js})_{s\in S_j}$ of $\mathscr{C}^5$ smooth generic CR submanifolds K_{js} in U_j such that K_{js} is $\mathscr{C}^5$ smooth without boundary in U_j, for every s, and satisfies $K_j = \bigcup_{s\in S_j} K_{js}$. Now the desired assertion follows directly from Proposition 3.1 applied to each K_{js} and points in U_j correspondingly. We note that the constants c_0, r_0, θ_0 can be chosen independent of $s \in S_j$ because, as shown in the proof of [70, Proposition 2.5], they depend only on bounds on $\mathscr{C}^3$-norm of diffeomorphisms defining local charts in K_{js} (see [70, Lemma 4.1]); these bounds are independent of $s \in S_j$ because the family $(K_{js})_{s\in S_j}$ is smooth. □

Examples of compact sets K satisfying the hypothesis of Proposition 3.2 include the union of a finite number of smooth Jordan arcs in $\mathbb{C}$, regardless of their configuration, or the closure of an open subset with smooth boundary in X.

Lemma 3.3. *Let $\theta_0 \in (0, \pi/6)$, $\beta \in (0, 1)$ and let $c > 0$ be a constant. Let ψ be a subharmonic function on $\mathbb{D}$. Assume that*

$$\limsup_{z\in\mathbb{D}\to e^{i\theta}} \psi(z) \leq c|\theta|^\beta \quad \text{for } \theta \in (-\theta_0, \theta_0) \quad \text{and} \quad \sup_{\mathbb{D}} \psi \leq c.$$

Then there exists a constant C depending only on (θ_0, β, c) so that, for any $z \in \mathbb{D}$, we have

$$\psi(z) \leq C|1 - z|^\beta. \tag{3.1}$$

Moreover, if $\limsup_{z\in\mathbb{D}\to e^{i\theta}} \psi(z) \leq g(e^\theta)$ *for some function $g \in \mathscr{C}^{1,\delta}$ on $[e^{-i\theta_0}, e^{i\theta_0}]$ and for some $\delta > 0$ so that $g(1) = 0$, then*

$$\psi(z) \leq C|1 - z| \tag{3.2}$$

for some constant C independent of $z \in \mathbb{D}$.

Proof. The desired inequality (3.1) is essentially contained in [70, Lemma 2.6]. The hypothesis of continuity up to boundary of ψ in the last lemma is superfluous, and the proof there still works in our current setting. Note that the proof of [70, Lemma 2.6] does not work for $\beta = 1$ because the harmonic extension of a Lipschitz function on $\partial\mathbb{D}$ is not necessarily Lipschitz on $\overline{\mathbb{D}}$. However, since the harmonic extension of a $\mathscr{C}^{1,\delta}$ function on $\partial\mathbb{D}$ to $\mathbb{D}$ is also $\mathscr{C}^{1,\delta}$ on $\overline{\mathbb{D}}$ (see, e.g., [37, page 41]), we obtain (3.2). □

End of the proof of Theorem 2.3. Let $a_0 \in K$ and let $\mathbb{B}$ be a small ball of X around a_0. Consider an increasing sequence $(u_j)_j$ of psh functions bounded uniformly from above on $\mathbb{B}$ such that $u_j \le 0$ on $K \cap \mathbb{B}$. We need to check that $(\sup_j u_j)^* \le 0$ on $K \cap \mathbb{B}$. Now, we will essentially follow arguments from the proof of [70, Theorem 2.3]. Let $\mathbb{B}'$ be a relatively compact subset of $\mathbb{B}$ containing a_0. We will check that there exists a constant $C > 0$ such that, for every $a \in \mathbb{B}'$, we have

$$u_j(a) \le C \operatorname{dist}(a, K)^{1/5}. \tag{3.3}$$

The desired assertion is deduced from the last inequality by taking $\operatorname{dist}(a, K) \to 0$. It remains to check (3.3).

Let a_0' be a point in K such that $\operatorname{dist}(a, a_0') = \operatorname{dist}(a, K)$. Put

$$\delta := \operatorname{dist}(a, K) = \operatorname{dist}(a, a_0').$$

By Proposition 3.1, there exists an analytic disc $f : \mathbb{D} \to \mathbb{B}$ continuous up to boundary and $z_a \in \mathbb{D}$ with $|z_a - 1| \le C\delta^{1/2}$ such that $f(z_a) = a$ and $\operatorname{dist}(f(1), a_0') \le C\delta$, and

$$f([e^{-i\theta_0}, e^{i\theta_0}]) \subset K,$$

for some constants C and θ_0 independent of a.

Put $v_j := u_j \circ f$. Since $u_j \le 0$ on $\mathbb{B} \cap K$ and $f([e^{-i\theta_0}, e^{i\theta_0}]) \subset K$, we get $v_j(e^{i\theta}) \le 0$ for $\theta \in [-\theta_0, \theta_0]$. Moreover, since u_j is uniformly bounded from above, there is a constant M such that $v_j \le M$ for every j. This allows us to apply Lemma 3.3 for $\beta = 1 - \epsilon$ (for some constant $\epsilon > 0$ small) and c big enough. We infer that $v_j(z) \lesssim |1 - z|^{1/2}$. Substituting $z = z_a$ in the last inequality gives

$$u_j(a) = u_j(f(z_a)) = v_j(z_a) \lesssim |1 - z_a|^{(1-\epsilon)/2} \lesssim \delta^{(1-\epsilon)/4}.$$

Hence (3.3) follows by choosing $\epsilon := 1/5$. The proof is finished. □

Let h_0, h, ϕ be as in the previous section. Recall that the Chern form of h_0 is equal to ω. Define

$$\phi_{K,k} := \sup\{k^{-1} \log|\sigma|_{h_0} : \sigma \in H^0(X, L^k),\ \sup_K(|\sigma|_{h_0^k} e^{-k\phi}) \le 1\}.$$

Clearly, $\phi_{K,k} \le \phi_K$. We recall the following well-known fact.

Lemma 3.4. *The sequence $(\phi_{K,k})_k$ increases pointwise to ϕ_K as $k \to \infty$.*

As a direct consequence of the above lemma, we see that ϕ_K is lower semi-continuous.

Proof. Since we could not find a proper reference, we present detailed arguments here. We just need to use Demailly's analytic approximation. Since ϕ_K is bounded, without loss of generality, we can assume that $\phi_K < 0$. Clearly, $\phi_{K,k} \le \phi_K$. Fix $a \in X$. Let δ be a positive constant. Let ψ be a negative ω-psh function with $\psi \le \phi$ on K such that $\psi(a) \ge \phi_K(a) - \delta$. Let $\epsilon \in (0,1)$. Observe $dd^c(1-\epsilon)\psi + \omega \ge \epsilon\omega$. This allows us to apply [27, Theorem 14.21] to ψ. Let $(\sigma_j)_j$ be an orthonormal basis of $H^0(X, L^k)$ with respect to L^2-norm generated by the Hermitian metric $h_{\epsilon,\psi,k} := e^{-k(1-\epsilon)\psi} h_0^k$ and ω^n. Set

$$\psi_{\epsilon,k} := \frac{1}{k} \log \sum_{j=1}^{d_k} |\sigma_j|_{h_0^k}.$$

Then $\psi_{\epsilon,k} \ge (1-\epsilon)\psi$, and ψ_k converges pointwise to $(1-\epsilon)\psi$ as $k \to \infty$. Note that

$$\psi_{\epsilon,k} = \sup\Big\{\frac{1}{k}\log|\sigma|_{h_0^k} : \sigma \in H^0(X, L^k) : \|\sigma\|_{L^2(\omega^n, h_{\epsilon,\psi,k})} = 1\Big\}.$$

Let $(\psi^{(N)})_N$ be a sequence of continuous functions decreasing to ψ as $N \to \infty$. By Hartog's lemma applied to $(\psi_{1/N,k})_{k\in\mathbb{N}}$, we see that there is a sequence $(k_N)_N \subset \mathbb{N}$ increasing to ∞ such that

$$(1 - 1/N)\psi \le \psi_{1/N,k_N} \le (1-1/N)\psi^{(N)} + 1/N.$$

Consequently, $\psi_{1/N,k_N}$ converges pointwise to ψ as $N \to \infty$.

Recall that $\psi \le \phi$ on K. By Hartog's lemma again and the continuity of ϕ, for every constant $\delta' > 0$ and N large enough, we have

$$\psi_{1/N,k_N} \le \phi + \delta'$$

on K. It follows that

$$k_N^{-1}\log|\sigma|_{h_0^{k_N}} \le \phi + \delta'$$

for every $\sigma \in H^0(X, L^{k_N})$ with $\|\sigma\|_{L^2(\omega^n, h_{1/N,\psi,k_N})} = 1$. We deduce that

$$\phi_{K,k_N} \ge k_N^{-1}\log|\sigma|_{h_0^{k_N}} - \delta'$$

for such σ. In other words, $\phi_{K,k_N} \ge \psi_{1/N,k_N} - \delta'$ for N big enough. Letting $N \to \infty$ gives

$$\liminf_{N\to\infty} \phi_{K,k_N}(a) \ge \lim_{N\to\infty} \psi_{1/N,k_N}(a) - \delta' = \psi(a) - \delta' \ge \phi_K(a) - \delta' - \delta.$$

Letting δ, δ' tend to 0 yields that $\liminf_{N\to\infty}\phi_{K,k_N}(a) = \phi_K(a)$. Hence $\phi_{K,k_N} \to \phi_K$ as $N \to \infty$. We have actually shown that, for every sequence $(k'_N)_N \subset \mathbb{N}$ converging to ∞, there is a subsequence $(k_N)_N$ such that ϕ_{K,k_N} converges to ϕ_K. Thus the desired assertion follows. □

Using arguments from [17, Lemma 3.2] and Lemma 3.4 gives the following.

Lemma 3.5. *If (K, ϕ) is regular, then $\phi_{K,k}$ converges uniformly to ϕ_K as $k \to \infty$.*

Proof. For readers' convenience, we briefly recall the proof here. Since $\phi_K = \phi_K^*$, we see that ϕ_K is upper semi-continuous. This combined with the fact that ϕ_K is already lower semi-continuous gives that ϕ_K is continuous. By Lemma 3.4, we have the pointwise convergence of $\phi_{K,k}$ to ϕ_K. Using the envelop defining ϕ_K, observe next that

$$k\phi_{K,k} + m\phi_{K,m} \le (k+m)\phi_{K,k+m} \tag{3.4}$$

for every k, m. We fix a Riemannian metric d on X. Let $\epsilon > 0$. Since X is compact, ϕ_K is uniformly continuous on X. Hence there exists a constant $\delta > 0$ such that $d(\phi_K(x), \phi_K(y)) \leq \epsilon$ if $d(x, y) \leq \delta$ for every $x, y \in X$. Fix $x_0 \in X$. Let $k_0 > 0$ be a natural number such that, for $k \geq k_0$, we have $d(\phi_K(x_0), \phi_{K,k}(x_0)) \leq \epsilon$. Since the line bundle L is positive, $\phi_{K,r}$ is continuous for r big enough. Hence, without loss of generality, we can assume that $\phi_{K,r}$ is continuous for every r, for only big r matters for us. By shrinking δ if necessary, we obtain that, for every $1 \leq r \leq k_0$, one has $d(\phi_{K,r}(x), \phi_{K,r}(y)) \leq \epsilon$ if $d(x, y) \leq \delta$. Write $k = k_0 r + s$ for $0 \leq s \leq k_0 - 1$. Using this and (3.4) yields

$$k\phi_{K,k} \geq rk_0\phi_{K,k_0} + s\phi_{K,s}.$$

It follows that

$$\phi_{K,k} \geq r\frac{k_0}{k}\phi_{K,k_0} + \frac{s}{k}\phi_{K,s}.$$

Thus

$$\phi_{K,k}(x) - \phi_K(x) \geq \frac{rk_0}{k}(\phi_{K,k_0}(x) - \phi_K(x)) - \Big(1 - r\frac{k_0}{k}\Big)\phi_K(x) + \frac{s}{k}\phi_{K,s}(x).$$

The choice of δ now implies that there exists a constant $C > 0$ such that the right-hand side is bounded from below by $-3\epsilon - Ck_0/k$ if $d(x, x_0) \leq \delta$. Since $\phi_{K,k} \leq \phi_K$, we obtain the uniform convergence of $\phi_{K,k}$ to ϕ_K. □

Put

$$\widetilde{\phi}_{K,k} := \frac{1}{2k}\log \widetilde{B}_k = \frac{1}{2k}\log \sum_{j=1}^{d_k}|s_j|^2_{h_0^k}.$$

Proposition 3.6. *Assume that (K, ϕ) is regular and (K, μ, ϕ) satisfies the Bernstein–Markov property. Then we have*

$$\|\widetilde{\phi}_{K,k} - \phi_K\|_{\mathscr{C}^0(X)} \to 0, \quad \textit{as } k \to \infty. \tag{3.5}$$

In particular,

$$\lim_{k\to\infty} \widetilde{B}_k^{1/k} = e^{2\phi_K}. \tag{3.6}$$

Note that the limit in (3.6) is independent of μ. We refer to [7, Lemma 2.8] for more information in the case $K \subset \mathbb{C}^n \subset X = \mathbb{P}^n$.

Proof. When $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$, this is [17, Lemma 3.4]. The arguments there work for our setting. We reproduce here the proof for the readers' convenience. It suffices to check the first desired property (3.5). Observe that

$$\sup_K(|s|^2_{h_0^k}e^{-2k\phi}) = \sup_K|s|^2_{h^k} \leq \big(\sup_K B_k\big)\|s\|^2_{L^2(\mu,k\phi)}. \tag{3.7}$$

Combining this with the Bernstein–Markov property, we see that, for every $\epsilon > 0$, there holds

$$\sup_K(|s|^2_{h_0^k}e^{-2k\phi}) \leq e^{\epsilon k}\|s\|^2_{L^2(\mu,k\phi)} \tag{3.8}$$

for every $s \in H^0(X, L^k)$. Observe also that

$$|s|_{h_0^k} \leq \sup_K (|s|_{h_0^k} e^{-k\phi}) e^{k\phi_{K,k}} \tag{3.9}$$

on X. Applying the last inequality to $s := s_j$ and using (3.8), we infer that

$$\frac{1}{2k} \log \widetilde{B}_k \leq \epsilon + \phi_{K,k}.$$

In other words, $\widetilde{\phi}_{K,k} \leq \epsilon + \phi_{K,k}$ on X. On the other hand, if $\sup_K(|s|_{h_0^k} e^{-k\phi}) \leq 1$, then

$$\|s\|_{L^2(\mu,k\phi)} \leq C$$

for some constant C independent of k. It follows that

$$\widetilde{B}_k \geq C^{-1} e^{2k\phi_{K,k}}.$$

Consequently, $\widetilde{\phi}_{K,k} \geq \phi_{K,k} + O(k^{-1})$. Thus, using Lemma 3.5, we obtain the desired assertion. This finishes the proof. □

Remark 3.7. Recall $\phi_K \leq \phi$ on K. If $x \in K$ is a point so that $\phi_K(x) < \phi(x)$, then by Proposition 3.6, we see that $B_k^{-1}(x)$ grows exponentially as $k \to \infty$. Consider now the case where $K = X$ and ϕ is not an ω-psh function. In this case, there exists $x \in X$ with $\phi_X(x) < \phi(x)$, and hence B_k becomes exponentially small as $k \to \infty$.

3.2. Hölder regularity of extremal plurisubharmonic envelopes. Let $\alpha \in (0, 1]$ and let Y be a metric space. For every $f\colon Y \to \mathbb{C}$, we define

$$\|f\|_{\mathscr{C}^{0,\alpha}} := \sup_{x,y \in Y, x \neq y} \frac{|f(x) - f(y)|}{(d(x,y))^\alpha}.$$

We denote by $\mathscr{C}^{0,\alpha}(Y)$ the space of functions on Y of finite $\mathscr{C}^{0,\alpha}$-norm. If $0 < \alpha < 1$, then for simplicity, we will sometimes write $\mathscr{C}^\alpha$ for $\mathscr{C}^{0,\alpha}$. The following notion introduced in [31] will play a crucial role for us.

Definition 3.8. For $\alpha \in (0, 1]$ and $\alpha' \in (0, 1]$, a non-pluripolar compact K is said to be $(\mathscr{C}^{0,\alpha}, \mathscr{C}^{0,\alpha'})$-regular if, for any positive constant C, the set

$$\{\phi_K : \phi \in \mathscr{C}^{0,\alpha}(K) \text{ and } \|\phi\|_{\mathscr{C}^{0,\alpha}(K)} \leq C\}$$

is a bounded subset of $\mathscr{C}^{0,\alpha'}(X)$.

The following provides examples for the last notion.

Theorem 3.9 ([70, Theorem 2.3]). *Let $\alpha \in (0, 1)$. Then any compact generic nondegenerate $\mathscr{C}^5$ piecewise-smooth submanifold K of X is $(\mathscr{C}^{0,\alpha}, \mathscr{C}^{0,\alpha/2})$-regular. Moreover, if K has no singularity, then K is $(\mathscr{C}^{0,\alpha}, \mathscr{C}^{0,\alpha})$-regular.*

The next remark follows immediately from Proposition 3.2 and the proof of [70, Theorem 2.3].

Remark 3.10. If K is as in Proposition 3.2, then K is also $(\mathscr{C}^{0,\alpha}, \mathscr{C}^{0,\alpha})$-regular for $\alpha \in (0,1)$. The union of a finite number of open subsets with smooth boundary in X is an example of such K.

If $K = X$ and $\phi \in \mathscr{C}^{0,1}$, it was shown in [22] that $\phi_X \in \mathscr{C}^{0,1}$; hence X is $(\mathscr{C}^{0,1}, \mathscr{C}^{0,1})$-regular; see also [10, 22, 66] for more information. In the case where $K = X$ or K is an open subset with smooth boundary in X, it was proved in [31] that K is $(\mathscr{C}^{0,\alpha}, \mathscr{C}^{0,\alpha})$-regular for $\alpha \in (0,1)$. This was extended for K as in the statement of Theorem 3.9 in [70, Theorem 2.3]; see also [48]. We do not know if Theorem 3.9 holds for $\alpha = 1$. Here is a partial result whose proof is exactly as of [70, Theorem 2.3] by using (3.2) instead of (3.1) (and noting that the analytic disc in Proposition 3.1 is $\mathscr{C}^2(\overline{\mathbb{D}})$, hence in particular, is $\mathscr{C}^{1,\delta}$ for some $\delta \in (0,1]$).

Theorem 3.11. *Let $\delta \in (0,1)$, $C_1 > 0$ be constants. Let K be a compact generic $\mathscr{C}^5$ smooth submanifold (without boundary) of X. There exists a constant $C_2 > 0$ such that, for every $\phi \in \mathscr{C}^{1,\delta}(K)$ with $\|\phi\|_{\mathscr{C}^{1,\delta}} \le C_1$, then $\|\phi_K\|_{\mathscr{C}^{0,1}} \le C_2$.*

We mention at this point an example in [62] of a domain K with $\mathscr{C}^0$ boundary, but (K, ϕ) is not regular even for $\phi = 0$. Applying Theorem 3.11 to $X = \mathbb{P}^n$, we obtain the following result that implies [64, Conjecture 6.2] as a special case.

Theorem 3.12. *Let K be a compact generic nondegenerate $\mathscr{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$. Then $V_K \in \mathscr{C}^{1/2}(\mathbb{C}^n)$. Additionally, if K has no singularity, $V_K \in \mathscr{C}^{0,1}(\mathbb{C}^n)$.*

Recall that

$$V_K = \sup\{\psi \in \mathscr{L}(\mathbb{C}^n) : \psi \le 0 \text{ on } K\}.$$

We note that the fact that $V_K \in \mathscr{C}^{0,1}(\mathbb{C}^n)$ when K has no singularity was proved in [64] (and, as can be seen from the above discussion, this property also follows essentially from [70]). As a direct consequence of Theorem 3.12, we record here a Bernstein–Markov type inequality of independent interest.

Theorem 3.13. *Let K be a compact generic nondegenerate $\mathscr{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$. Then there exists a constant $C > 0$ such that, for every complex polynomial p on $\mathbb{C}^n$, we have*

$$\|\nabla p\|_{L^\infty(K)} \le C(\deg p)^2 \|p\|_{L^\infty(K)}. \tag{3.10}$$

If additionally K has no singularity (e.g., $K = \mathbb{S}^{2n-1}$), then

$$\|\nabla p\|_{L^\infty(K)} \le C \deg p \|p\|_{L^\infty(K)}.$$

Note that the exponent of $\deg p$ is optimal as it is well known for the classical Markov and Bernstein inequalities in dimension one. The above result was known when K is algebraic in $\mathbb{R}^n$; see [12, 19]. We emphasize that inequalities similar to those in Theorem 3.13 also hold for other situations (with the same proof), for example, $K = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and considering K as a maximally totally real submanifold in the complexification of $\mathbb{S}^{n-1}$.

Markov (or Bernstein) type inequalities are a subject of great interest in approximation theory. There is a large literature on this topic, e.g., [12, 18–21, 23, 56, 72], to cite just a few.

Proof. Observe first that V_K is $\mathcal{C}^{0,1}$ if K has no singularity or $\mathcal{C}^{1/2}$ in general by Theorem 3.12 (applied to $X = \mathbb{P}^n$) and Remark 2.2. Let p be a complex polynomial in $\mathbb{C}^n$. Put $k := \deg p$. Since $\frac{1}{k}(\log|p| - \log \max_K|p|)$ is a candidate in the envelope defining V_K, we get

$$|p| \le e^{kV_K} \max_K |p|$$

on $\mathbb{C}^n$. We use the same notation C to denote a constant depending only on K, n. Let

$$a = (a_1, \dots, a_n) \in K \subset \mathbb{C}^n.$$

Let $r > 0$ be a small constant. Consider the analytic disc $D_a := (a_1 + r\mathbb{D}, a_2, \dots, a_n)$. Applying the Cauchy formula to the restriction of p to D_a shows that

$$|\partial_{z_1} p(a)| \le r^{-1} \max_{D_a} |p| \le r^{-1} (\max_K |p|) \max_{D_a} e^{kV_K}.$$

Since $\phi_K = 0$ on K, using $\mathcal{C}^{1/2}$ regularity of V_K gives

$$|\partial_{z_1} p(a)| \le r^{-1} \max_{D_a} |p| \le r^{-1} (\max_K |p|) e^{Ckr^{1/2}}$$

for some constant $C > 0$ independent of p and a. Choosing $r = k^{-2}$ in the last inequality yields

$$|\partial_{z_1} p(a)| \le Ck^2 \max_{D_a} |p|.$$

Similarly, we also get

$$|\partial_{z_j} p(a)| \le Ck^2 \max_{D_a} |p|$$

for every $1 \le j \le n$. Hence the first desired inequality (3.10) for general K. When K has no singularity, the arguments are similar. This finishes the proof. □

Here is a quantitative version of Lemma 3.4.

Proposition 3.14. *Let K be a compact generic nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold of X. Let ϕ be a Hölder continuous function on K. Then we have*

$$\|\phi_{K,k} - \phi_K\|_{\mathscr{C}^0(X)} = O\Big(\frac{\log k}{k}\Big).$$

Proof. The desired estimate was proved for $K = X$ in [30, Corollary 4.4]. For the general case, we use the proof of [70, Theorem 2.3] (and [31]). Let $\widetilde{\phi}$ be the continuous extension of ϕ to X as in the proof of [70, Theorem 2.3]. It was shown there that $\phi_K = \widetilde{\phi}_X$. On the other hand, one can check directly that $\phi_{K,k} \ge \widetilde{\phi}_{X,k}$. Hence we get

$$|\phi_{K,k} - \phi_K| = \phi_K - \phi_{K,k} \le \widetilde{\phi}_X - \widetilde{\phi}_{X,k} = O\Big(\frac{\log k}{k}\Big)$$

for every k. This implies the conclusion. □

End of the proof of Theorem 2.6. The desired estimate is deduced directly using Proposition 3.14, Theorem 2.4, and following the same arguments as in the proof of Proposition 3.6.

We just briefly recall here how to do it. Firstly, as in the proof of Proposition 3.6, we have

$$\widetilde{\phi}_{K,k} - \phi_{K,k} \geq O(k^{-1}).$$

It remains to bound from above $\widetilde{\phi}_{K,k} - \phi_{K,k}$. Combining the polynomial upper bound for B_k in Theorem 2.4 and (3.7), one gets, for some constants $C, N > 0$ independent of k,

$$\sup_K (|s|_{h_0^k} e^{-k\phi}) \leq C k^N \|s\|_{L^2(\mu,k\phi)}$$

for every $s \in H^0(X, L^k)$. This coupled with (3.9) yields $|s|_{h_0^k} \leq k^N e^{k\phi_{K,k}}$ on X. It follows that

$$\widetilde{\phi}_{K,k} = \frac{1}{2k} \log \widetilde{B}_k \leq \phi_{K,k} + N \frac{\log k}{k}.$$

This finishes the proof. □

4. Polynomial growth of Bergman kernel functions

This section is devoted to the proof of Theorems 2.4 and 2.5.

4.1. Families of analytic discs attached to K. The goal of this part is to construct suitable families of analytic discs partly attached to K. In the literature, there are various constructions of families of analytic discs partly attached to generic Cauchy–Riemann submanifolds for different purposes (e.g., see [2,53]). Although the main tool is usually a modified Bishop equation, depending on each problem, one has to prove some additional properties of the family in consideration. In our case, we need to have very good quantitative properties of the families of analytic discs. Our construction below is based on [69]. It is not clear whether this can be deduced from other previous work on analytic discs (such as [57]) since, in our setting, we are dealing with a piecewise-smooth submanifold K rather than a smooth one.

We do not need all of properties of the family of analytic discs given in [69]. For the convenience of the reader, we briefly recall the construction shown below. We will only consider the case where $\dim K = n$ in this section. Here is our result giving the desired family of analytic discs.

Theorem 4.1. *Let $C_0 > 0$ be a constant. Then there exist constants $C > 0$, $r_0 > 0$ and $\theta_0 \in (0, \pi/2)$ such that, for every $0 < t < r_0$, the following properties are satisfied. Let p_0 be a regular point of K of distance at least t/C_0 to the singularity of K, and let W_{p_0} be a local chart around p_0 in X such that p_0 corresponds to the origin 0 in $\mathbb{C}^n$. Then there is a $\mathscr{C}^2$ map $F : \overline{\mathbb{D}} \times \mathbb{B}_{n-1} \to W_{p_0}$ such that the following properties are fulfilled.*

(i) *$F(\cdot, y)$ is holomorphic for every $y \in \mathbb{B}_{n-1}$, and $\|DF(\xi, y)\| \leq Ct$ for every $\xi \in \mathbb{D}$, $y \in \mathbb{B}_{n-1}$.*

(ii) *$F(e^{i\theta}, y) \in K$ for $-\theta_0 \leq \theta \leq \theta_0$ and $y \in \mathbb{B}_{n-1}$, and $|F(1, y)| \leq t/C_0$,*

(iii) *Let G denote the restriction of F to $[e^{-i\theta_0}, e^{i\theta_0}] \times \mathbb{B}_{n-1}$. Then G is bijective onto its image, and the image of G is contained in $\mathbb{B}(p_0, t/C_0) \cap K$ (here $\mathbb{B}(p_0, t/C_0)$ denotes the ball centered at p_0 of radius t/C_0 in X), and $C^{-1} t^n \leq |\det DG(e^{i\theta}, y)| \leq C t^n$ for every $-\theta_0 \leq \theta \leq \theta_0$ and $y \in \mathbb{B}_{n-1}$.*

We proceed with the proof of Theorem 4.1. Denote by $z = x + iy$ the complex variable on $\mathbb{C}$ and by $\xi = e^{i\theta}$ the variable on $\partial\mathbb{D}$. For any $m \in \mathbb{N}$ and $r > 0$, let $\mathbb{B}_m(0, r)$ be the Euclidean ball centered at 0 of radius r of $\mathbb{R}^m$, and for $r = 1$, we write $\mathbb{B}_m$ for $\mathbb{B}_m(0, 1)$. Let Z be a compact submanifold with or without boundary of $\mathbb{R}^m$. The Euclidean metric on $\mathbb{R}^m$ induces a metric on Z. For $\beta \in (0, 1]$ and $k \in \mathbb{N}$, let $\mathcal{C}^{k,\beta}(Z)$ be the space of real-valued functions on Z which are differentiable up to the order k and whose k-th derivatives are Hölder continuous of order β. For any tuple $v = (v_0, \ldots, v_m)$ consisting of functions in $\mathcal{C}^{k,\beta}(Z)$, we define its $\mathcal{C}^{k,\beta}$-norm to be the maximum of the ones of its components.

Let u_0 be a continuous function on $\partial\mathbb{D}$. Let

$$\mathcal{C}u_0(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(e^{i\theta})\frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta,$$

which is a holomorphic function on $\mathbb{D}$. Recall that the real part of $\mathcal{C}u_0$ is u_0. Let $\mathcal{T}u_0(z)$ denote the imaginary part of $\mathcal{C}u_0(z)$. Put $\mathcal{T}_1 u_0 := \mathcal{T}u_0 - \mathcal{T}u_0(1)$. For basic properties of $\mathcal{T}_1$, one can consult [2, 53].

We now go back to our current situation with X. We endow X with an arbitrary Riemannian metric. For $\theta_0 \in (0, \pi)$, let $[e^{-i\theta_0}, e^{i\theta_0}]$ denote the arc of $\partial\mathbb{D}$ of arguments from $-\theta_0$ to θ_0. Let p_0 be a regular point in K and let r_{p_0} denote the distance of p_0 to the singular part of K. Recall that we assume in this section that $\dim K = n$.

Lemma 4.2. *There exist a constant $c_K > 1$ depending only on (K, X) and a local chart (W_{p_0}, Ψ) around p_0, where $\Psi\colon W_{p_0} \to \mathbb{B}_{2n}$ is biholomorphic with $\Psi(p_0) = 0$ such that the two following conditions hold:*

(i) *we have*

$$\|\Psi\|_{\mathcal{C}^5} \le c_K, \quad \|\Psi^{-1}\|_{\mathcal{C}^5} \le c_K,$$

(ii) *there is a $\mathcal{C}^3$ map h from $\overline{\mathbb{B}}_n$ to $\mathbb{R}^n$ so that $h(0) = Dh(0) = 0$, and*

$$\Psi(K \cap W_{p_0}) \supset \{(\mathbf{x}, h(\mathbf{x})) : \mathbf{x} \in \overline{\mathbb{B}}_n(0, r_{p_0}/c_K)\},$$

where the canonical coordinates on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ are denoted by $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, and

$$\|h\|_{\mathcal{C}^3} \le c_K.$$

Note that h is indeed $\mathcal{C}^5$ (because K is so), but $\mathcal{C}^3$ is sufficient for our purpose in what follows.

Proof. The existence of local coordinates so that $h(0) = Dh(0) = 0$ is standard; see [2] or [70, Lemma 4.1]. Perhaps, one needs to explain a bit about the radius r_{p_0}/c_K: the existence of c_K comes from the fact that, for every singular point a in K, there are an open neighborhood U of a in X and sets $A_j \Subset B_j \subset U$ for $1 \le j \le m$ such that B_j is $\mathcal{C}^5$ smooth generic CR submanifold of dimension $\dim K$ in U, and

$$K \cap U = \bigcup_{j=1}^{m} (A_j \cap U),$$

and A_j is the closure in U of a relatively compact open subset in B_j, and ∂A_j (in U) is contained in the singularity of K. Thus, by applying the standard local coordinates to points in B_j, we obtain the existence of c_K. This finishes the proof. □

From now on, we only use the local coordinates introduced in Lemma 4.2 and identify points in W_{p_0} with those in $\mathbb{B}_{2n}$ via Ψ.

Lemma 4.3 ([70, Lemma 3.1]). *There exists a function $u_0 \in \mathscr{C}^\infty(\partial\mathbb{D})$ such that*

$$u_0(e^{i\theta}) = 0 \quad \text{for } \theta \in [-\pi/2, \pi/2] \quad \text{and} \quad \partial_x u_0(1) = -1.$$

In what follows, we identify $\mathbb{C}^n$ with $\mathbb{R}^n + i\mathbb{R}^n$. Let u_0 be a function described in Lemma 4.3. Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in \overline{\mathbb{B}}_{n-1} \subset \mathbb{R}^{n-1}$. Define $\boldsymbol{\tau}_1^* := (1, \boldsymbol{\tau}_1) \in \mathbb{R}^n$ and $\boldsymbol{\tau}_2^* := (0, \boldsymbol{\tau}_2) \in \mathbb{R}^n$ and $\boldsymbol{\tau} := (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$. Let t be a positive number in $(0, 1]$ which plays a role as a scaling parameter in the equation (4.1) below.

In order to construct an analytic disc partly attached to K, it suffices to find a map

$$U : \partial\mathbb{D} \to \mathbb{B}_n \subset \mathbb{R}^n,$$

which is Hölder continuous, satisfying the following Bishop-type equation:

$$U_{\boldsymbol{\tau},t}(\xi) = t\boldsymbol{\tau}_2^* - \mathcal{T}_1(h(U_{\boldsymbol{\tau},t}))(\xi) - t\mathcal{T}_1 u_0(\xi)\boldsymbol{\tau}_1^*, \tag{4.1}$$

where the Hilbert transform $\mathcal{T}_1$ is extended to a vector-valued function by acting on each component. The existence of solution of the last equation is a standard fact in the Cauchy–Riemann geometry.

Proposition 4.4 ([69, Proposition 3.3]). *There are a positive number $t_1 \in (0, 1)$ and a real number $c_1 > 0$ satisfying the following property: for any $t \in (0, t_1]$ and any $\boldsymbol{\tau} \in \overline{\mathbb{B}}_{n-1}^2$, equation* (4.1) *has a unique solution $U_{\boldsymbol{\tau},t}$ which is $\mathscr{C}^{2,1/2}$ in $(\xi, \boldsymbol{\tau})$ and such that*

$$\|D^j_{(\xi,\boldsymbol{\tau})} U_{\boldsymbol{\tau},t}\|_{\mathscr{C}^{1/2}(\partial\mathbb{D})} \le c_1 t$$

for any $\boldsymbol{\tau} \in \overline{\mathbb{B}}_{n-1}^2$ and $j = 0, 1$.

From now on, we consider $t < \min\{t_1, C_0 r_{p_0}\}$ (hence the distance from p_0 to the singularity of K is at least t/C_0). Let $U_{\boldsymbol{\tau},t}$ be the unique solution of (4.1). For simplicity, we use the same notation $U_{\boldsymbol{\tau},t}(z)$ to denote the harmonic extension of $U_{\boldsymbol{\tau},t}(\xi)$ to $\mathbb{D}$. Let $P_{\boldsymbol{\tau},t}(z)$ be the harmonic extension of $h(U_{\boldsymbol{\tau},t}(\xi))$ to $\mathbb{D}$. Recall the following result.

Lemma 4.5 ([69, Lemma 3.4]). *There exists a constant c_2 so that, for every $t \in (0, t_1]$ and every $(z, \boldsymbol{\tau}) \in \overline{\mathbb{D}} \times \overline{\mathbb{B}}_{n-1}^2$, we have*

$$\|D^j_{(z,\boldsymbol{\tau})} U_{\boldsymbol{\tau},t}(z)\| \le c_2 t \quad \text{and} \quad \|D^j_{(z,\boldsymbol{\tau})} P_{\boldsymbol{\tau},t}(z)\| \le c_2 t^2$$

for $j = 0, 1$.

We note that the hypothesis that $D^2 h(0) = 0$ was required in [69], but it is actually superfluous in the proof of Lemma 4.5. Define

$$F(z, \boldsymbol{\tau}, t) := U_{\boldsymbol{\tau},t}(z) + iP_{\boldsymbol{\tau},t}(z) + itu_0(z)\boldsymbol{\tau}_1^*,$$

which is a family of analytic discs parametrized by $(\boldsymbol{\tau}, t)$. Compute

$$F(1, \boldsymbol{\tau}, t) = U(\boldsymbol{\tau}, t)(1) + iP_{\boldsymbol{\tau},t}(1) = t\boldsymbol{\tau}_2^* + h(t\boldsymbol{\tau}_2^*).$$

Hence if $|\boldsymbol{\tau}_2|$ is small enough, we see that

$$|F(1,\boldsymbol{\tau},t)| \le t|\boldsymbol{\tau}_2| < r_{p_0}/(2c_K).$$

This combined with Lemma 4.5 yields that

$$|U_{\boldsymbol{\tau},t}(e^{i\theta})| \le |\theta||U_{\boldsymbol{\tau},t}(1)| \le |\theta| r_{p_0}/(2c_K) < r_{p_0}/c_K \tag{4.2}$$

if θ and $\boldsymbol{\tau}_2$ are small enough. Now the defining formula of F and the fact that $u_0 \equiv 0$ on $[e^{-i\pi/2}, e^{i\pi/2}]$ imply that

$$F(\xi,\boldsymbol{\tau},t) = U_{\boldsymbol{\tau},t}(\xi) + iP_{\boldsymbol{\tau},t}(\xi) = U_{\boldsymbol{\tau},t}(\xi) + ih(U_{\boldsymbol{\tau},t}(\xi)) \in K$$

by (4.2) if θ and $\boldsymbol{\tau}_2$ are small enough. In other words, there is a small constant θ_0 such that if $|\boldsymbol{\tau}_2| < \theta_0$, then F is $[e^{-i\theta_0}, e^{i\theta_0}]$-attached to K.

Proposition 4.6. *By decreasing θ_0 and t_1 if necessary, we obtain the following property: for every $\boldsymbol{\tau}_1 \in \overline{\mathbb{B}}_{n-1}$, the map $F(\cdot,\boldsymbol{\tau}_1,\cdot,t)\colon [e^{-i\theta_0}, e^{i\theta_0}] \times \overline{\mathbb{B}}_{n-1}(0,\theta_0) \to K$ is a diffeomorphism onto its image, and*

$$C^{-1}t^n \le \|\det DF(\cdot,\boldsymbol{\tau}_1,\cdot,t)\|_{L^\infty} \le Ct^n$$

for some constant $C > 0$ independent of $t, \boldsymbol{\tau}_1$.

Proof. The desired assertion was implicitly obtained in the proof of [69, Proposition 3.5]. We present here complete arguments for readers' convenience. Recall

$$F(e^{i\theta},\boldsymbol{\tau}_1,t) = U_{\boldsymbol{\tau},t}(e^{i\theta}) + ih(U_{\boldsymbol{\tau},t}(e^{i\theta}).$$

By the Cauchy–Riemann equations, we have

$$\partial_y U_{\boldsymbol{\tau},t}(1) = -t\partial_x u_0(1)\boldsymbol{\tau}_1^* - \partial_x P_{\boldsymbol{\tau},t}(1) = t\boldsymbol{\tau}_1^* - \partial_x P_{\boldsymbol{\tau},t}(1).$$

The last term is $O(t^2)$ by Lemma 4.5. Thus the first component of $\partial_y U_{\boldsymbol{\tau},t}(1)$ is greater than $t/2$ provided that $t \le t_2$ small enough. A direct computation gives $\partial_y U_{\boldsymbol{\tau},t}(1) = \partial_\theta U_{\boldsymbol{\tau},t}(1)$. Consequently, the first component of

$$\partial_\theta F(e^{i\theta},\boldsymbol{\tau}_1,t) = \partial_\theta U_{\boldsymbol{\tau},t}(1) + i\partial_\theta h(U_{\boldsymbol{\tau},t}(1))$$

is greater than $t/2$ for $t \le t_2$ (note that $Dh(0) = 0$). Moreover, as computed above, we have

$$F(1,\boldsymbol{\tau}_1,t) = t\boldsymbol{\tau}_2^* + h(t\boldsymbol{\tau}_2^*).$$

Thus $D_{\boldsymbol{\tau}_2,\theta}F(1,\boldsymbol{\tau},t)$ is a nondegenerate matrix whose determinant satisfies the desired inequalities if $|\theta| < \theta_0$ is small enough. The proof is finished. □

End of the proof of Theorem 4.1. Let θ_0 be as above and smaller than θ_1 and suppose that $M > |\theta_0|^{-1}$ is a big constant. Fix a parameter $\boldsymbol{\tau}_1$ and define

$$F_t(\xi,\boldsymbol{\tau}_2) := F(\xi,\boldsymbol{\tau}_1,\boldsymbol{\tau}_2/M,t).$$

By the above results, we see that the family F_t satisfies all of required properties (because $|\boldsymbol{\tau}_2/M| < \theta_0$). This finishes the proof. □

4.2. Upper bound of Bergman kernel functions. We start with the following useful estimate in one dimension.

Lemma 4.7. *Let $\beta \in (0,1)$. Then there exists a constant $C_\beta > 0$ such that, for every $\theta_0 \in (0,\pi]$ and every constant $M > 0$, and every subharmonic function g on $\mathbb{D}$ such that g is continuous up to $\partial\mathbb{D}$ and $|g(\xi)| \le M$ for $\xi \in \partial\mathbb{D}\backslash\{e^{i\theta} : -\theta_0 \le \theta \le \theta_0\}$, we have*

$$g(z) \le C_\beta\Big[|1-z|^\beta \theta_0^{-1} M + (1-|z|)^{-1}\int_{-\theta_0}^{\theta_0} g(e^{i\theta})\, d\theta\Big].$$

Proof. Let $\theta_1 \in [0, 2\pi)$. Put

$$I := \{e^{i\theta} : -\theta_0 \le \theta \le \theta_0\}, \quad I' := \{e^{i\theta} : -\theta_0/4 \le \theta \le \theta_0/4\}.$$

Let g_1 be the harmonic function on $\mathbb{D}$ such that $g_1 \in \mathcal{C}^{0,1}(\partial\mathbb{D})$, and $g_1(\xi) = M$ for $\xi \in \partial\mathbb{D}\backslash I$ and $g_1 \equiv 0$ on I'. Observe that $\|g_1\|_{\mathcal{C}^{0,1}(\partial\mathbb{D})} \le C\theta_0^{-1} M$ for some constant $C > 0$ independent of M, θ_1, θ_0.

By a classical result (see [37, page 41] or [70, (3.4)]) on harmonic functions on the unit disc, we have

$$\|g_1\|_{\mathcal{C}^{0,\beta}(\mathbb{D})} \lesssim \|g_1\|_{\mathcal{C}^{0,\beta}(\partial\mathbb{D})} \lesssim \theta_0^{-1} M$$

for every $\beta \in (0,1)$. As a result, we get

$$g_1(z) = g_1(z) - g_1(1) \lesssim |1-z|^\beta \|g_1\|_{\mathcal{C}^{0,\beta}(\partial\mathbb{D})} \lesssim |1-z|^\beta \theta_0^{-1} M. \tag{4.3}$$

Let g_2 be the harmonic function on $\mathbb{D}$ such that $g_2(e^{i\theta}) = 0$ for $|\theta| > \theta_0$, and $g_2(e^{i\theta}) = g(e^{i\theta})$ for $\theta \in [-\theta_0, \theta_0]$. Observe that $g \le g_1 + g_2$ because the latter function is harmonic and greater than or equal to g on the boundary of $\mathbb{D}$. Using Poisson's formula, we see that

$$g_2(z) \le (1-|z|)^{-1}\int_{-\pi}^{\pi} g_2(e^{i\theta})\, d\theta = (1-|z|)^{-1}\int_{-\theta_0}^{\theta_0} g(e^{i\theta})\, d\theta.$$

Summing this and (4.3) gives the desired assertion. The proof is finished. □

Let K be a $\mathcal{C}^5$ smooth (without boundary) maximally totally real submanifold in X. Let $s \in H^0(X, L^k)$ with $\|s\|_{L^2(\mu,h^k)} = 1$ and $M := \sup_K |s|^2_{h^k}$. Let $p_0 \in K$. Consider a local chart $(U, \mathbf{z})$ around p_0 with coordinates $\mathbf{z}$, and p_0 corresponds to the origin 0 in $\mathbb{C}^n$. Shrinking U if necessary, we can assume also that L is trivial on U.

We trivialize (L, h_0) over U such that $h_0 = e^{-\psi}$ for some psh function ψ on U with $\psi(0) = -\phi(0)$ (we implicitly fix a local holomorphic frame on $L|_U$ so that one can identify Hermitian metrics on $L|_U$ with functions on U), and identify s with a holomorphic function g_s on U. Thus $h = e^{-\phi}h_0 = e^{-\phi-\psi}$ on U. In particular, $h(0) = 1$ on U.

Lemma 4.8. *There exists a constant $C_1 > 0$ independent of k, s, p_0 such that*

$$\sup_{\{|\mathbf{z}|\le 1/k\}} |s(\mathbf{z})|^2_{h^k} \le C_1 M, \tag{4.4}$$

and

$$C_1^{-1}|s(\mathbf{z})|^2_{h^k} \le |g_s(\mathbf{z})|^2 \le C_1|s(\mathbf{z})|_{h^k} \le C_1^2 M \tag{4.5}$$

for $\mathbf{z} \in \mathbb{B}(0, k^{-1})$. *Moreover, for every constant* $\epsilon > 0$, *there exists a constant* $c_\epsilon > 0$ *independent of* k, s, p_0 *such that* $|g_s(0)|^2 \le |g_s(\mathbf{z})|^2 + \epsilon M$ *for* $|\mathbf{z}| \le 1/(c_\epsilon k)$.

Proof. The desired inequality (4.5) follows immediately from (4.4) and the equalities

$$|g_s(\mathbf{z})|^2 = |s(\mathbf{z})|^2_{h^k} e^{k(\psi(\mathbf{z})+\phi(\mathbf{z}))}, \quad \psi(0) + \phi(0) = 0.$$

Recall that

$$\phi_K := \sup\{\psi\ \omega\text{-psh} : \psi \le \phi \text{ on } K\}.$$

Note that $\phi_K \le \phi$ on K. By hypothesis, $\phi \in \mathscr{C}^{1,\delta}(K)$ for some constant $\delta > 0$. This combined with Theorem 3.11 and the fact that K has no singularity yields that ϕ_K is Lipschitz. Using the fact that $k^{-1} \log|s|_{h_0^k}$ is ω-psh, we get the Bernstein–Walsh inequality

$$|s|^2_{h_0^k} \le \big(\sup_K |s|^2_{h^k}\big) e^{2k\phi_K} = M e^{2k\phi_K}.$$

Hence $|s|^2_{h^k} \le M e^{2k(\phi_K - \phi)}$. This combined with the Lipschitz property of ϕ_K and ϕ (and also the property that $\phi_K(0) - \phi(0) \le 0$ on K) yields

$$\sup_{\{\mathbf{z}:|\mathbf{z}|\le 1/k\}} |s(\mathbf{z})|^2_{h^k} \le C_1 M$$

for some constant C_1 independent of k, s, p_0. Hence (4.4) also follows.

Arguing as in the proof of Theorem 3.13, one obtains the following version of Bernstein–Markov inequality: for $\mathbf{z} \in \mathbb{B}(0, k^{-1}/2)$, there holds

$$|\nabla g_s(\mathbf{z})| \lesssim k \sup_{\mathbb{B}(0,k^{-1})} |g_s| \lesssim k M^{1/2}. \tag{4.6}$$

Consequently, for $\mathbf{z} \in \mathbb{B}(0, k^{-1}/c_\epsilon)$ with c_ϵ big enough, we get

$$|g_s(\mathbf{z}) - g_s(0)| \le |\mathbf{z}| \sup_{\mathbb{B}(0,k^{-1}/C_0)} |\nabla g_s| \le \epsilon^{1/2} M^{1/2}.$$

This finishes the proof. □

Proof of the upper bound in Theorem 2.5. Let $s \in H^0(X, L^k)$ with $\|s\|_{L^2(\mu,h^k)} = 1$. Let $M := \|s\|^2_{L^\infty(K,h^k)}$. We need to prove that $M \lesssim k^n$.

Let $p_0 \in K$ and consider a local chart $(U, \mathbf{z})$ around p_0 with coordinates $\mathbf{z}$; the point p_0 corresponds to 0 in the local chart $(U, \mathbf{z})$. Let $\epsilon > 0$ be a small constant to be chosen later. Let C_1, c_ϵ be the constants in Lemma 4.8. Let $A \ge C_1^2 \epsilon^{-2}$ be a big constant. Using the Lipschitz continuity of ϕ yields

$$\int_{\{|\mathbf{z}|\le k^{-1}\}} |g_s|^2\, d\mu = \int_{\{|\mathbf{z}|\le k^{-1}\}} |s|^2_{h^k} e^{k(\psi+\phi)}\, d\mu \lesssim 1 \tag{4.7}$$

(uniformly in s). Now let $F : \mathbb{D} \times Y \to X$ be the family of analytic discs in Theorem 4.1 associated to $C_0 := 2c_\epsilon$ for p_0, and $t := k^{-1}$, where $Y := \mathbb{B}_{n-1}$. Let $\theta_0 \in (0, \pi/2)$ be the constant in the last theorem.

Let $g := |g_s \circ F|^2$. Put $\mathbf{z}_y := F(1 - 1/A, y)$. By expressing

$$\mathbf{z}_y = F(1 - 1/A, y) - F(1, y) + F(1, y),$$

one gets $|\mathbf{z}_y| \leq 1/(c_\epsilon k)$ if A is big enough. Applying Lemma 4.7 to $g(\cdot, y)$ and $\beta = 1/2$ and using (4.5) yield

$$|g_s(\mathbf{z}_y)|^2 = g(\xi_y, y) \leq C'\Big(A^{-1/2}M + A\int_{-\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta\Big)$$

for some constant $C' > 0$ independent of s, p_0, k. This combined with Lemma 4.8 yields

$$|g_s(0)|^2 \leq \epsilon M + C'A^{-1/2}M + C'A\int_{\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta.$$

Integrating the last inequality over $y \in Y$ gives

$$\begin{aligned}|g_s(0)|^2 &\leq \epsilon M + C'A^{-1/2}M + C'A(\mathrm{vol}(Y))^{-1}\int_Y \mathrm{vol}_y \int_{-\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta \\ &\leq 2\epsilon M + C'A(\mathrm{vol}(Y))^{-1}\int_Y \mathrm{vol}_y \int_{-\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta\end{aligned}$$

if $A \geq \epsilon^{-2}C'^2$. By properties of F and (4.7), the second term in the right-hand side of the last inequality is $\lesssim k^n$. We infer that $|g_s(0)|^2 \leq 2\epsilon M + AC_2k^n$, where C_2 is a constant independent of k, s, p_0. Consequently, by (4.5), one gets $|s(p_0)|^2_{h^k} \leq 2C_1\epsilon M + AC_2C_1k^n$ for every $p_0 \in K$. By choosing $\epsilon := 1/(4C_1)$ and A big enough as required, one gets

$$M \leq M/2 + AC_1C_2k^n.$$

Thus the desired upper bound for M follows. The proof is finished. □

We now proceed with the proof of Theorem 2.4.

End of the proof of Theorem 2.4 *for* $\dim K = n$. We assume $\dim K = n$. We will explain how to treat the case $\dim K \geq n$ later. Let $s \in H^0(X, L^k)$ with $\|s\|_{L^2(\mu,h^k)} = 1$. Put $M := \sup_K |s|_{h^k}$. Let $\alpha \in (0,1)$ be a Hölder exponent of ϕ. We have $\phi_K \in \mathscr{C}^{\alpha/2}$ (note $\alpha < 1$). We follow essentially the scheme of the proof for the upper bound of B_k in Theorem 2.5.

Denote by $K_{k,\alpha}$ the set of points in K of distance at least $k^{-2/\alpha}$ to the singular part of K, and $U_{k,\alpha}$ the set of points in X of distance at most $k^{-2/\alpha}$ to K. As in the proof of Lemma 4.8, one has

$$\sup_{p\in U_{k,\alpha}} |s(p)|^2_{h^k} \leq CM, \quad M \leq C \sup_{p\in K_{k,\alpha}} |s(p)|^2_{h^k} \tag{4.8}$$

for some constant $C \geq 4$ independent of k, s. In particular, it is sufficient to estimate $|s(p)|^2_{h^k}$ for $p \in K_{k,\alpha}$.

Let $p_0 \in K_{k,\alpha}$ (hence p_0 is a regular point of K, and the ball $\mathbb{B}(p_0, k^{-2/\alpha}) \cap K$ lies entirely in the regular part of K), and consider a local chart $(U, \mathbf{z})$ around p_0 with coordinates $\mathbf{z}$; the point p_0 corresponds to 0 in the local chart $(U, \mathbf{z})$.

We trivialize (L, h_0) over U such that $h_0 = e^{-\psi}$ for some smooth psh function ψ on U with $\psi(0) = -\phi(0)$ (we implicitly fix a local holomorphic frame on $L|_U$ so that one can identify Hermitian metrics on $L|_U$ with functions on U), and identify s with a holomorphic function g_s on U. Thus $h = e^{-\phi}h_0 = e^{-\phi-\psi}$ on U. In particular, $h(0) = 1$ on U. Using the

Hölder continuity of ϕ yields

$$\int_{\{|\mathbf{z}|\le k^{-2/\alpha}\}} |g_s|^2\, d\mu = \int_{\{|\mathbf{z}|\le k^{-2/\alpha}\}} |s|^2_{h^k} e^{k(\psi+\phi)}\, d\mu \lesssim 1$$

(uniformly in s, k). As in the proof of Lemma 4.8, a Bernstein–Walsh type inequality implies that, by increasing C if necessary (independent of p_0, k, s), there holds

$$C^{-1}|s(\mathbf{z})|^2_{h^k} \le |g_s(\mathbf{z})|^2 = |s(\mathbf{z})|^2_{h^k} e^{k(\psi(\mathbf{z})+\phi(\mathbf{z}))} \le CM \tag{4.9}$$

for $\mathbf{z} \in \mathbb{B}(0, k^{-2/\alpha})$. Let $\lambda' := \lambda/(1+\lambda)$. One also sees that there is a constant $C_0 > 0$ independent of k, s, p_0 such that

$$|g_s(0)|^2 \le |g_s(\mathbf{z})|^2 + M/C^{2/\lambda'} \tag{4.10}$$

if $|\mathbf{z}| \le 2k^{-2/\alpha}/C_0$. Since $\mu = \rho\,\mathrm{Leb}_K$, where $\rho^{-\lambda} \in L^1(\mathrm{Leb}_K)$, applying Hölder inequality to $|g_s|^{2\lambda'} = (|g_s|^{2\lambda'}\rho^{\lambda'})(\rho^{-\lambda'})$ gives

$$\int_{\{|\mathbf{z}|\le k^{-2/\alpha}\}} |g_s|^{2\lambda'}\, d\,\mathrm{Leb}_K \lesssim \Big(\int_{\{|\mathbf{z}|\le k^{-2/\alpha}\}} |g_s|^2\, d\mu\Big)^{\lambda'} \lesssim 1. \tag{4.11}$$

Let $A \ge C^6$ be a constant. Now let $F\colon \mathbb{D}\times Y \to X$ be the family of analytic discs in Theorem 4.1 associated to C_0 for $p_0 = 0$, and $t := k^{-2/\alpha}$.

Let $g := |g_s \circ f|^{2\lambda'}$. Put $\mathbf{z}_y := F(1-1/A, y)$, which is at most $2k^{-2/\alpha}/C_0$ by properties of F if A is big enough. Applying Lemma 4.7 to $g(\cdot, y)$ and $\beta = 1/2$ yield

$$|g_s(\mathbf{z}_y)|^{2\lambda'} = g(\xi_y, y) \le CM^{\lambda'}/A^{1/2} + AC\int_{-\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta$$

(again, we increase C if necessary independently of s, p_0, k). By this and (4.10), we get

$$\begin{aligned} |g_s(0)|^{2\lambda'} &\le M^{\lambda'}/C^2 + CM^{\lambda'}/A^{1/2} + AC\int_{-\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta \\ &\le 2M^{\lambda'}/C^2 + AC\int_{-\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta. \end{aligned}$$

Integrating the last inequality over $y \in Y$ gives

$$|g_s(0)|^{\lambda'} \le 2M^{\lambda'}/C^2 + AC(\mathrm{vol}(Y))^{-1}\int_Y \mathrm{vol}_y \int_{-\theta_0}^{\theta_0} g(e^{i\theta}, y)\, d\theta. \tag{4.12}$$

The second term in the right-hand side of the last inequality is bounded by

$$\begin{aligned} I_k &:= \int_{\mathbb{B}(0,k^{-2/\alpha})\cap K} |g_s(x)|^{2\lambda'} |\det DG|^{-1}\, d\,\mathrm{Leb}_K \\ &\le k^{2n/\alpha}\int_{\mathbb{B}(0,k^{-2/\alpha})\cap K} |g_s(x)|^{2\lambda'}\, d\,\mathrm{Leb}_K\,. \end{aligned}$$

Using this, (4.12) and (4.11) gives

$$(g_s(0))^{2\lambda'} \le 2M^{\lambda'}/C^2 + AC_2 k^{2n/\lambda'}$$

for some constant C_2 big enough (independent of k, s, p_0). This combined with (4.9) gives

$$|s(p_0)|_{h^k}^{2\lambda'} \leq 2M^{\lambda'}/C + ACC_2k^{2n/\lambda'}$$

for every $p_0 \in K_k$. By this and (4.8), we obtain

$$M^{\lambda'} \lesssim k^{2n/\alpha}$$

by choosing C big enough. Hence $M \lesssim k^{2n/(\alpha\lambda')}$. This finishes the proof. □

We now explain how to prove Theorem 2.4 when K is not necessarily totally real.

End of the proof of Theorem 2.4 *for* $\dim K \geq n$. As in the case of $\dim K = n$, it suffices to work with points in K of distance at least $k^{-2/\alpha}/C_0$ to the singularity of K for some $C_0 > 0$ big enough. Let p_0 be such a point, and let $s \in H^0(X, L^k)$ with $\|s\|_{L^2(\mu,h^k)} = 1$. Our goal is to show that $|s(p_0)|_{h^k} \leq Ck^{2n_K(\lambda+1)/(\alpha\lambda)}$ for some constant C independent of s and k. We identify s with a holomorphic function g_s on a small open neighborhood U of p_0 as usual. Hence, as above, one gets

$$\int_{\mathbb{B}(p_0,k^{-2/\alpha})\cap K} |g_s|^{2\lambda'} d\,\mathrm{Leb}_K \lesssim 1, \tag{4.13}$$

where λ' is as in the case of $\dim K = n$.

Let r be the CR dimension of K. Recall that $\dim K = n + r$. Using standard local coordinates near p_0 on K, one sees that, by shrinking U if necessary, there are holomorphic local coordinates $(\mathbf{z}_1, \mathbf{z}_2) \in U \subset \mathbb{C}^{n-r} \times \mathbb{C}^r$ such that K is locally given by the graph $\operatorname{Im} \mathbf{z}_1 = h(\operatorname{Re} \mathbf{z}_1, \mathbf{z}_2)$ for $h \in \mathscr{C}^5$. In particular, for every vector $v \in \mathbb{R}^r$ (small enough), the real linear subspace $\operatorname{Im} z_2 = v$ intersects K at a generic CR $\mathscr{C}^5$ smooth submanifold K_v in U. Put $g_{s,v} := g_s|_{K_v}$. Let $C_0 > 0$ be a big constant to be chosen later. By Fubini's theorem and (4.13), we obtain

$$\int_{|v|\leq k^{-2/\alpha}} d\,\mathrm{Leb}_{\mathbb{R}^r} \int_{K_v} |g_{s,v}|^{2\lambda'} d\,\mathrm{Leb}_{K_v} \lesssim 1.$$

It follows that there exists v with $|v| \leq k^{-2/\alpha}/C_0$ so that

$$\int_{K_v} |g_{s,v}|^{2\lambda'} d\,\mathrm{Leb}_{K_v} \lesssim k^{-2r/\alpha}.$$

Applying the proof of the case where $\dim K = n$ to K_v, we see that

$$|s(0, h(0, 0, v), 0, v)|_{h^k}^2 \lesssim k^{2n_K/(\alpha\lambda')}, \tag{4.14}$$

where we write $(\mathbf{z}_1, \mathbf{z}_2) = (\operatorname{Re} \mathbf{z}_1, \operatorname{Im} \mathbf{z}_1, \operatorname{Re} \mathbf{z}_2, \operatorname{Im} \mathbf{z}_2)$, and p_0 is identified with 0 in these local coordinates. On the other hand, since $|v| \leq k^{-2/\alpha}/C_0$, using a version of Bernstein–Markov inequality (similar to (4.6)) yields

$$|s(0)|_{h^k}^2 \leq |s(0, h(0, 0, v), 0, v)|_{h^k}^2 + \frac{1}{2} \sup_K |s|_{h^k}^2$$

if C_0 is big enough. This combined with (4.14) gives the desired upper bound. The proof is finished. □

Remark 4.9. The proof of Theorem 2.4 actually shows the following local estimate. Let U be an open subset in X, and let μ_U be the restriction of μ to U. Define

$$B_{k,U} := \sup_{s \in H^0(U,L^k)} \frac{|s|^2_{h^k}}{\|s\|^2_{L^2(\mu_U,h^k)}}.$$

Then, for every $U' \Subset U$, there exists $C' > 0$ such that

$$\sup_{K \cap U'} B_{k,U} \le C' k^{2n_K(\lambda+1)/(\alpha\lambda)}.$$

5. Zeros of random polynomials

In this section, we prove Theorem 1.7. Let $\mathrm{Leb}_{\mathbb{C}^m}$ be the Lebesgue measure on $\mathbb{C}^m$ for $m \ge 1$, and we denote by $\|\cdot\|$ the standard Euclidean norm on $\mathbb{C}^m$. Let $\omega_{\mathrm{FS},m}$ be the Fubini–Study form on $\mathbb{P}^m$, and let $\Omega_{\mathrm{FS},m} := \omega^m_{\mathrm{FS},m}$ be the Fubini–Study volume form on $\mathbb{P}^m$. We always embed $\mathbb{C}^m$ in $\mathbb{P}^m$. We recall the following key lemma.

Lemma 5.1 ([33, Proposition A.3 and Corollary A.5]). *There exist constants $C, \lambda > 0$ such that, for every $k \ge 0$, and every $\omega_{\mathrm{FS},k}$-psh function u on $\mathbb{P}^k$ with $\int_X u\Omega_{\mathrm{FS},k} = 0$, then*

$$u \le C(1 + \log k), \qquad \int_{\{u<-t\}} \Omega_{\mathrm{FS},k} \le Cke^{-\lambda t}$$

for every $t \ge 0$.

The essential point is that the constants in the above result are uniformly in the dimension k of $\mathbb{P}^k$. In our applications, the dimension k will tend to ∞. Let d_k be the dimension of the space of polynomials of degree at most k on $\mathbb{C}^n$. Note that $d_k \approx k^n$.

Let f be a bounded Borel function on $\mathbb{C}$ such that there is a constant $C_0 > 0$ for which, for every $r > 1$, we have

$$\int_{|z|\ge r} f d\,\mathrm{Leb}_{\mathbb{C}} \le C_0/r^2.$$

Let $a_1, a_2, \dots, a_{d_k}$ be complex-valued i.i.d random variables whose distribution is $f\,\mathrm{Leb}_{\mathbb{C}}$. Assume furthermore that the joint distribution of $(a_1, \dots, a_{d_k})$ satisfies

$$\mathscr{P}_k := f(z_1)\dots f(z_{d_k})\,\mathrm{Leb}(z_1) \otimes \dots \otimes \mathrm{Leb}(z_{d_k}) \le C_0\Omega_{\mathrm{FS},d_k}$$

on $\mathbb{C}^{d_k}$.

Let $p^{(d_k)} := (p_1, \dots, p_{d_k})$ be an orthonormal basis of $\mathcal{P}_k(\mathbb{C}^n)$. Let L be a complex algebraic subvariety of dimension $m \ge 1$ in $\mathbb{C}^n$. Note that the topological closure of L in $\mathbb{P}^n$ is an algebraic subvariety in $\mathbb{P}^n$. Observe that

$$\omega^m_{\mathrm{FS},n} \le C\omega^m,$$

where ω is the standard Kähler form on $\mathbb{C}^n$, and $C > 0$ is a constant. Fix a compact A of volume $\mathrm{vol}(A) := \int_A \omega^m_{\mathrm{FS},n} > 0$ in $\mathbb{C}^n$. We start with a version of [15, Lemma 2.4] with more or less the same proof. We use the Euclidean norm on $\mathbb{C}^{d_k}$.

Lemma 5.2. *Let $M \geq 1$ be a constant. Let E_k be the set of $(a_1, \dots, a_{d_k}) \in \mathbb{C}^{d_k}$ such that*

$$\int_A \Big(\log \Big| \sum_{j=1}^{d_k} a_j p_j(z) \Big| - \frac{1}{2} \log \sum_{j=1}^{d_k} |p_j(z)|^2 \Big) \omega^m_{\mathrm{FS},n} \geq 2M \operatorname{vol}(A) \log d_k.$$

Let E'_k be the set of $a^{(d_k)}$ so that $\|a^{(d_k)}\| \geq d_k^{2M}$. Then we have $\mathscr{P}_k(E_k \cup E'_k) \leq C d_k^{-3M}$ for some constant $C > 0$ independent of k and M.

Proof. Put $a^{(d_k)} := (a_1, \dots, a_{d_k})$, and

$$I_k(a_1, \dots, a_{d_k}) := \int_A \Big(\log \Big| \sum_{j=1}^{d_k} a_j p_j(z) \Big| - \frac{1}{2} \log \sum_{j=1}^{d_k} |p_j(z)|^2 \Big) \omega^m_{\mathrm{FS},n}.$$

Observe

$$\Big| \sum_{j=1}^{d_k} a_j p_j(z) \Big| \leq \|a^{(d_k)}\| \Big(\sum_{j=1}^{d_k} |p_j(z)|^2 \Big)^{1/2}.$$

It follows that, for $a^{(d_k)} \in E_k$, one has

$$2M \operatorname{vol}(A) \log d_k \leq I_k(a^{(d_k)}) \leq \log \|a^{(d_k)}\| \operatorname{vol}(A).$$

This implies that, for each k, there exists $1 \leq j_k \leq d_k$ such that $|a_{j_k}| \geq d_k^{(4M-1)/2}$. We infer that

$$\begin{aligned} \mathscr{P}_k(E_k) &\leq \mathscr{P}_k(a^{(d_k)} : |a_j| \geq d_k^{(4M-1)/2} \text{ for some } 1 \leq j \leq d_k) \\ &\leq d_k \int_{|a_1| \geq d_k^{(4M-1)/2}} f \, \mathrm{Leb}_C \leq C_0 d_k^{-4M+1} \leq C_0 d_k^{-3M}. \end{aligned}$$

Similarly, we also get $\mathscr{P}_k(E'_k) \lesssim d_k^{-3M}$. This finishes the proof. □

Put $a^{(d_k)} := (a_1, \dots, a_{d_k})$ and $p^{(d_k)} := (p_1, \dots, p_{d_k})$. Define

$$\varphi(a^{(d_k)}) := \int_{z \in L} \log \frac{\big| \sum_{j=1}^{d_k} a_j p_j(z) \big|}{(\|a^{(d_k)}\|^2 + 1)^{1/2} \|p^{(d_k)}(z)\|} \omega^m_{\mathrm{FS},n}(z).$$

Observe that $\varphi \leq 0$ on $\mathbb{C}^{d_k}$. We put

$$I_\varphi := \int_{\mathbb{C}^{d_k}} \varphi(a^{(d_k)}) \Omega_{\mathrm{FS},d_k}.$$

Lemma 5.3. *There exists a constant $C > 0$ such that, for every $k \geq 1$, we have*

$$I_\varphi \geq -C \log d_k.$$

Proof. Let I be the right-hand side of the desired inequality. By Fubini's theorem and the transitivity of the unitary group on $\mathbb{C}^{d_k}$, one has

$$I_\varphi = \int_{z \in L} \omega^m_{\mathrm{FS},n} \int_{\mathbb{C}^{d_k}} \varphi(a^{(d_k)}) \Omega_{\mathrm{FS},d_k} = \int_{z \in L} \omega^m_{\mathrm{FS},n} \int_{\mathbb{C}^{d_k}} \log \frac{|a_1|}{(\|a\|^2 + 1)^{1/2}} \Omega_{\mathrm{FS},d_k}.$$

The function

$$\psi := \log \frac{|a_1|}{(\|a^{(d_k)}\|^2 + 1)^{1/2}}$$

is ω_{FS,d_k}-psh on $\mathbb{P}^{d_k}$ (where ω_{FS,d_k} is the Fubini–Study form on $\mathbb{P}^{d_k}$). Let

$$\psi' := \psi - \int_{\mathbb{C}^{d_k}} \psi \Omega_{\mathrm{FS},d_k}.$$

Thus $\int_{\mathbb{P}^{d_k}} \psi' \Omega_{\mathrm{FS},d_k} = 0$ and ψ' is ω_{FS,d_k}-psh. Applying Lemma 5.1 to ψ' gives

$$\psi' \leq c(1 + \log d_k)$$

for some constant $c > 0$ independent of k, ψ'. Consequently,

$$\psi(a^{(d_k)}) \leq c(1 + \log d_k) + \int_{\mathbb{C}^{d_k}} \psi \Omega_{\mathrm{FS},d_k}$$

for every $a^{(d_k)} \in \mathbb{C}^m$. In particular, for $a^{(d_k)} = (1, 0, \dots, 0)$, we obtain

$$\int_{\mathbb{C}^{d_k}} \psi \Omega_{\mathrm{FS},d_k} \geq -c(1 + \log d_k) - 1.$$

Thus the desired inequality follows. $\square$

End of the proof of Theorems 1.7 *and* 1.9. Let $\varphi' := \varphi - I_\varphi$. We have

$$\int_{\mathbb{C}^{d_k}} \varphi' \Omega_{\mathrm{FS},d_k} = 0.$$

By Lemma 5.1 again, there exist $c, \alpha > 0$ independent of k such that

$$\int_{\{\varphi' \leq -t\}} \Omega_{\mathrm{FS},d_k} \leq c d_k e^{-\alpha t}.$$

Combining this with Lemma 5.3 yields

$$\int_{\{\varphi \leq -C \log d_k - t\}} \Omega_{\mathrm{FS},d_k} \leq c d_k e^{-\alpha t}.$$

Let $M \geq 1$ be a constant. Choosing $t := 4M\alpha^{-1} \log d_k$, where $C > 0$ big enough gives

$$\int_{\{\varphi \leq -(C+3/\alpha) \log d_k\}} \Omega_{\mathrm{FS},d_k} \leq c d_k^{-3M}. \tag{5.1}$$

Let E_k' be the set of $a^{(d_k)}$ such that $\|a^{(d_k)}\| \leq d_k^{2M}$ and $\varphi \geq -(C + 4M/\alpha) \log d_k$. Combining (5.1) and Lemma 5.2, we obtain that

$$\mathscr{P}_k(\mathbb{C}^{d_k} \backslash E_k') \lesssim 2c d_k^{-3M}$$

(we increase c if necessary). On the other hand, by the definition of E_k' and φ, we see that the set of $a^{(d_k)}$ such that

$$\int_{z \in L} \log \frac{\left|\sum_{j=1}^{d_k} a_j p_j(z)\right|}{\|p^{(d_k)}(z)\|} \omega_{\mathrm{FS},n}^m(z) \geq -C_M' \log d_k$$

(for some constant $C'_M > 0$ big enough independent of k) contains E'_k. It follows that

$$\mathscr{P}_k\left\{a^{(d_k)} : \int_{z\in L} \log \frac{\left|\sum_{j=1}^{d_k} a_j p_j(z)\right|}{\|p^{(d_k)}(z)\|}\Omega_{\mathrm{FS},n}(z) \le -C'_M \log d_k\right\} \le 2c d_k^{-3M}.$$

This together with Lemma 5.2 implies that there exists a Borel set F_k such that

$$\mathscr{P}_k(F_k) \le 3c d_k^{-3M},$$

and for $a^{(d_k)} \notin F_k$, one has $\|a^{(d_k)}\| \le d_k^{2M}$, and

$$\int_{z\in L} \log \frac{\left|\sum_{j=1}^{d_k} a_j p_j(x)\right|}{\|p^{(d_k)}(z)\|}\omega^m_{\mathrm{FS},n}(z) \ge -C'_M \log d_k.$$

Consequently, for $p = \sum_{j=1}^{d_k} a_j p_j$, $\psi_k := 1/(2k)\log \widetilde{B}_k$ and $a^{(d_k)} \notin F_k$, there holds

$$-C'_M \frac{\log k}{k} \le \int_L (k^{-1}\log|p^{(d_k)}| - \psi_k)\omega^m_{\mathrm{FS},n}.$$

Moreover, $\|a^{(d_k)}\| \le d_k^{2M}$, one has

$$\begin{aligned}\left|k^{-1}\log|p| - \psi_k\right| &= 2\max\{k^{-1}\log|p|, \psi_k\} - \psi_k - k^{-1}\log|p| \\ &\le \psi_k - k^{-1}\log|p| + 2Mk^{-1}\log d_k.\end{aligned}$$

It follows that

$$\int_L \left|k^{-1}\log|p| - \psi_k\right|\omega^m_{\mathrm{FS},n} \le C''_M \frac{\log d_k}{k}$$

for some constant $C''_M > 0$ independent of k. This together with Theorem 1.6 implies

$$\int_L \left|k^{-1}\log|p| - V_{K,Q}\right|\omega^m_{\mathrm{FS},n} \le C''_M \frac{\log k}{k}$$

by increasing C''_M if necessary. It follows that, for $a^{(d_k)} \notin F_k$,

$$\mathrm{dist}_{-2}(k^{-1}[p=0]\wedge[L], dd^c V_{K,Q}\wedge[L]) \lesssim \int_L \left|k^{-1}\log|p| - \psi_k\right|\omega^m_{\mathrm{FS},n} \le C''_M \frac{\log k}{k}.$$

Here recall that we define dist_{-2} by considering $[p=0]\wedge[L]$ and $dd^c V_{K,Q}\wedge[L]$ as currents on $\mathbb{P}^m$. This finishes the proof. □

We end the paper with some explicit examples to which our main results apply.

Example 5.4. Let K be the unit ball in $\mathbb{C}^n$. Then we have $V_K(z) = \log^+\|z\|$ (where $\log^+ t := \max\{\log t, 0\}$ for $t > 0$); see [44, Example 5.1.1]. Hence we see that V_K is Lipschitz but not continuously differentiable.

Example 5.5. Let $K = [-1, 1]$ in $\mathbb{C}$. By [44, Corollary 5.4.5], we get

$$V_K(z) = \log|z + \sqrt{z^2-1}|$$

on $\mathbb{C}$, where the square root is chosen such that

$$|z + \sqrt{z^2-1}| \ge 1.$$

One sees that $V_K \in \mathscr{C}^{1/2}(X)\backslash\mathscr{C}^{1/2+\epsilon}$ for any $\epsilon \in (0, 1/2)$. In higher dimension, similar observations also work for $K = [-1, 1]^n \subset \mathbb{C}^n$ and V_K.

Example 5.6. Let K be the unit polydisc in $\mathbb{C}^n$. Thus, by Example 5.4 and [44, Theorem 5.1.8], one obtains $V_K(z) = \max_{1\le j\le n} \log^+|z_j|$, where $z = (z_1, \dots, z_n)$. Let μ be the restriction of the Lebesgue measure on $\mathbb{C}^n$ to K. For $J = (j_1, \dots, j_n) \in \mathbb{N}^n$, put

$$J := j_1 + \cdots + j_n, \quad z^J := z^{j_1} \cdots z^{j_n}.$$

Observe that $(c_J z^J)_{J,|J|\le k}$ forms an orthonormal basis of $\mathcal{P}_k(K)$, where $c_J > 0$ is a constant such that the norm of $c_J z^J$ is equal to 1. Let

$$p_k := \sum_{J,|J|\le k} \alpha_J c_J z^J,$$

where α_J are independent complex Gaussian random variables of mean 0 and variance 1. Hence the hypothesis of Theorem 1.7 (and Corollary 1.8) is fulfilled for p_k. It follows that these results apply to this setting to give an asymptotic of the expectation of zeros of p_k with an explicit estimate on the error term.

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