GENERALIZED BERGMAN KERNELS ON SYMPLECTIC MANIFOLDS

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ABSTRACT. We study the near diagonal asymptotic expansion of the generalized Bergman kernel of the renormalized Bochner-Laplacian on high tensor powers of a positive line bundle over a compact symplectic manifold. We show how to compute the coefficients of the expansion by recurrence and give a closed formula for the first two of them. As consequence, we calculate the density of states function of the Bochner-Laplacian and establish a symplectic version of the convergence of the induced Fubini-Study metric. We also discuss generalizations of the asymptotic expansion for non-compact or singular manifolds as well as their applications. Our approach is inspired by the analytic localization techniques of Bismut-Lebeau.

0. INTRODUCTION

The Bergman kernel for complex projective manifolds is the smooth kernel of the orthogonal projection from the space of smooth sections of a positive line bundle L on the space of holomorphic sections of L, or, equivalently, on the kernel of the Kodaira-Laplacian $\square^L = \overline{\partial}^L \overline{\partial}^{L*} + \overline{\partial}^{L*} \overline{\partial}^L$ on L. It is studied in [46, 41, 50, 17, 6, 32, 49, 31], in various generalities, establishing the diagonal asymptotic expansion for high powers of L. Moreover, the coefficients in the diagonal asymptotic expansion encode geometric information about the underlying complex projective manifolds. This diagonal asymptotic expansion plays a crucial role in the recent work of Donaldson [27] where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow-Mumford stability.

In [20], Dai, Liu and Ma studied the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator associated to a positive line bundle on a compact symplectic manifold, and related it to that of the corresponding heat kernel. As a by product, they gave a new proof of the above results. This approach is inspired by Local Index Theory, especially by the analytic localization techniques of Bismut-Lebeau [4, §11].

Another natural generalization of the operator $\square^L$ in symplectic geometry was initiated by Guillemin and Uribe [29]. In this very interesting short paper, they introduce a renormalized Bochner–Laplacian (cf. (0.4)) which is exactly $2\square^L$ in the Kähler case. The asymptotic of the spectrum of the renormalized Bochner–Laplacian on L^p when $p \to \infty$ is studied in various generalities in [9, 15, 29] by applying the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [13], and in [33] as a direct application of Lichnerowicz formula.

Of course, there exists also a replacement of the $\overline{\partial}$–operator and of the notion of holomorphic section based on a construction of Boutet de Monvel–Guillemin

[13] of a first order pseudodifferential operator D_b which mimic the $\overline{\partial}_b$ operator on the circle bundle associated to L. However, D_b is neither canonically defined nor unique. This point of view was adopted in a series of papers [10, 42, 7].

In this paper, we will study the asymptotic expansion of the generalized Bergman kernel of the renormalized Bochner-Laplacian, namely the smooth kernel of the projection on its bound states as $p \to \infty$. The advantage of this approach is that the renormalized Bochner-Laplacian has geometric meaning and is canonically defined. Moreover, it does not require the passage to the associated circle bundle as we can work directly on the base manifold. Let's explain our results in detail.

Let (X, ω) be a compact symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that $\frac{\sqrt{-1}}{2\pi} R^L = \omega$, where $R^L = (\nabla^L)^2$ is the curvature of (L, ∇^L). Let (E, h^E) be a Hermitian vector bundle on X with Hermitian connection ∇^E and curvature R^E. Let g^{TX} be a Riemannian metric on X and $\mathbf{J} : TX \longrightarrow TX$ be the skew–adjoint linear map which satisfies the relation

$$(0.1) \qquad \omega(u, v) = g^{TX}(\mathbf{J}u, v) \quad \text{for} \quad u, v \in TX.$$

Let J be an almost complex structure such that $g^{TX}(J\cdot, J\cdot) = g^{TX}(\cdot, \cdot)$, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and that $\omega(\cdot, J\cdot)$ defines a metric on TX. Then J commutes with $\mathbf{J}$, and $-J\mathbf{J} \in \text{End}(TX)$ is positive, thus $J = \mathbf{J}(-\mathbf{J}^2)^{-1/2}$.

We introduce the Levi-Civita connection ∇^{TX} on (TX, g^{TX}) with its curvature R^{TX} and scalar curvature r^X. Let $\nabla^X J \in T^*X \otimes \text{End}(TX)$ be the covariant derivative of J induced by ∇^{TX}. Let $\Delta^{L^p \otimes E}$ be the induced Bochner-Laplacian acting on $\mathscr{C}^\infty(X, L^p \otimes E)$. We fix a smooth Hermitian section Φ of $\text{End}(E)$ on X. Let $\{e_i\}_i$ be an orthonormal frame of (TX, g^{TX}). Set

$$(0.2) \qquad \tau(x) = -\pi \, \text{Tr}_{|TX}[J\mathbf{J}] = \frac{\sqrt{-1}}{2} R^L(e_j, Je_j) > 0,$$

$$(0.3) \qquad \mu_0 = \inf_{u \in T_x X, \, x \in X} \sqrt{-1} R_x^L(u, Ju)/|u|_{g^{TX}}^2 > 0,$$

$$(0.4) \qquad \Delta_{p,\Phi} = \Delta^{L^p \otimes E} - p\tau + \Phi.$$

By [33, Cor. 1.2] (also cf. [9, 29, 15, 5]), there exists $C_L > 0$ (which can be estimated precisely by using the $\mathscr{C}^0$-norms of R^{TX}, R^E, R^L, $\nabla^X J$ and Φ cf. [33, p.656−658]) independent of p such that

$$(0.5) \qquad \text{Spec} \, \Delta_{p,\Phi} \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty[.$$

where we denote by $\text{Spec}(A)$ the spectrum of any operator A.

Let $\mathcal{H}_p$ be the eigenspace of $\Delta_{p,\Phi}$ with the eigenvalues in $[-C_L, C_L]$. Then for p large enough, again by [33, Cor. 1.2] (also cf. [9, 29] when E is trivial and $\mathbf{J} = J$)

$$(0.6) \quad \dim \mathcal{H}_p = d_p = \int_X \text{ch}(L^p \otimes E) \, \text{Td}(TX)$$

$$= \text{rk}(E) \int_X \frac{c_1(L)^n}{n!} p^n + \int_X \left(c_1(E) + \frac{\text{rk}(E)}{2} c_1(TX) \right) \frac{c_1(L)^{n-1}}{(n-1)!} p^{n-1} + \mathscr{O}(p^{n-2}),$$

where $\mathrm{ch}(\cdot), c_1(\cdot), \mathrm{Td}(\cdot)$ are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles (TX is a complex vector bundle with complex structure J).

Let $\{S_i^p\}_{i=1}^{d_p}$ be any orthonormal basis of $\mathcal{H}_p$ with respect to the inner product (1.1) such that $\Delta_{p,\Phi} S_i^p = \lambda_{i,p} S_i^p$. For $q \in \mathbb{N}$, we define $B_{q,p} \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ as follows,

$$(0.7) \qquad B_{q,p}(x) = \sum_{i=1}^{d_p} \lambda_{i,p}^q S_i^p(x) \otimes (S_i^p(x))^*,$$

here we denote by $\lambda_{i,p}^0 = 1$. Clearly, $B_{q,p}(x)$ does not depend on the choice of $\{S_i^p\}$. Let $\det \mathbf{J}$ be the determinant function of $\mathbf{J}_x \in \mathrm{End}(T_x X)$. A corollary of Theorem 1.19 is one of our main results:

Theorem 0.1. *There exist smooth coefficients $b_{q,r}(x) \in \mathrm{End}(E)_x$ which are polynomials in R^{TX}, R^E (and R^L, Φ), their derivatives of order $\leqslant 2(r+q)-1$ (resp. $2(r+q)$), and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x, with*

$$(0.8) \qquad b_{0,0} = (\det \mathbf{J})^{1/2} \, \mathrm{Id}_E,$$

such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x \in X$, $p \in \mathbb{N}$,

$$(0.9) \qquad \left| \frac{1}{p^n} B_{q,p}(x) - \sum_{r=0}^{k} b_{q,r}(x) p^{-r} \right|_{\mathscr{C}^l} \leqslant C_{k,l} \, p^{-k-1}.$$

Moreover, the expansion is uniform in the following sense: for any fixed $k, l \in \mathbb{N}$, assume that the derivatives of g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ with order $\leqslant 2n + 2k + 2q + l + 4$ run over a set bounded in the $\mathscr{C}^l$–norm taken with respect to the parameter $x \in X$ and, moreover, g^{TX} runs over a set bounded below. Then the constant $C_{k,l}$ is independent of the data; and the $\mathscr{C}^l$-norm in (0.9) includes also the derivatives on the parameters.

By derivatives with respect to the parameters we mean directional derivatives in the spaces of all appropriate g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ (on which $B_{q,p}$ and $b_{q,r}$ implicitly depend).

We calculate further the coefficients $b_{0,1}$ and $b_{q,0}$, $q \geqslant 1$ as follows[1].

Theorem 0.2. *If $J = \mathbf{J}$, then for $q \geqslant 1$,*

$$(0.10) \qquad b_{0,1} = \frac{1}{8\pi}\left[r^X + \frac{1}{4}|\nabla^X J|^2 + 2\sqrt{-1}R^E(e_j, Je_j) \right],$$

$$(0.11) \qquad b_{q,0} = \left(\frac{1}{24}|\nabla^X J|^2 + \frac{\sqrt{-1}}{2}R^E(e_j, Je_j) + \Phi \right)^q.$$

Let us check our formulas with the help of the Atiyah-Singer formula (0.6). Let $T^{(1,0)}X = \{v \in TX \otimes_\mathbb{R} \mathbb{C}; Jv = \sqrt{-1}v\}$ be the almost complex tangent bundle on X and let $P^{1,0} = \frac{1}{2}(1 - \sqrt{-1}J)$ be the natural projection from $TX \otimes_\mathbb{R} \mathbb{C}$ onto $T^{(1,0)}X$.

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[1]Here $|\nabla^X J|^2 = \sum_{ij} |(\nabla_{e_i}^X J)e_j|^2$ which is two times the corresponding $|\nabla^X J|^2$ from [34].

Then $\nabla^{1,0} = P^{1,0}\nabla^{TX}P^{1,0}$ is a Hermitian connection on $T^{(1,0)}X$, and the Chern-Weil representative of $c_1(TX)$ is $c_1(T^{(1,0)}X, \nabla^{1,0}) = \frac{\sqrt{-1}}{2\pi}\operatorname{Tr}_{|T^{(1,0)}X}(\nabla^{1,0})^2$. By (1.98),

(0.12) $$(\nabla^{1,0})^2 = P^{1,0}\Big[R^{TX} - \frac{1}{4}(\nabla^X J)\wedge(\nabla^X J)\Big]P^{1,0}.$$

Thus if $J = \mathbf{J}$, then by (0.12), (2.13), (2.15), (2.21) and (2.22),

(0.13) $$\big\langle c_1(T^{(1,0)}X, \nabla^{1,0}), \omega\big\rangle = \frac{1}{4\pi}\Big(r^X + \frac{1}{4}|\nabla^X J|^2\Big).$$

Therefore, by integrating over X the expansion (0.9) for $k = 1$ we obtain (0.6), so (0.10) is compatible with (0.6).

Theorem 0.1 for $q = 0$ and (0.10) generalize the results of [17, 50, 32] and [49] to the symplectic case. The term $r^X + \frac{1}{4}|\nabla^X J|^2$ in (0.10) is called the Hermitian scalar curvature in the literature [28, Chap. 10] and is a natural substitute for the Riemannian scalar curvature in the almost-Kähler case. It was used by Donaldson [26] to define the moment map on the space of compatible almost-complex structures. We can view (0.11) as an extension and refinement of the results of [11], [29, §5] about the density of states function of $\Delta_{p,\Phi}$ (cf. Remark 3.2 for the details).

In [20], Dai, Liu and Ma also focused on the *full off-diagonal* asymptotic expansion (cf. [20, Theorem 4.18]) which is needed to study the Bergman kernel on orbifolds, and the only small eigenvalue of the operator is 0 when $p \to \infty$, thus they had the key equation [20, (4.89)]. In the current situation, we have small eigenvalues (cf. (0.5)) and we are interested to prove Theorem 1.19, that is, the *near diagonal* expansion of the generalized Bergman kernels. This result is enough for most of applications. At first, the spectral gap (0.5) and the finite propagation speed of solutions of hyperbolic equations allow to localize the problem. Then we will combine the Sobolev norm estimates as in [20] and a formal power series trick to obtain Theorem 1.19, and in this way, we get a method to compute the coefficients (cf. (1.113), (1.117)) which is new also in the case of [20].

In a forthcoming paper [35], we will find the *full off-diagonal* asymptotic expansion of the generalized Bergman kernels by combining the results here and in [20], and as a direct application, we will study the Toeplitz operators on symplectic manifolds and Donaldson Theorem [25] for the Kodaira map Φ_p (3.14).

Let us provide a short road-map of the paper. In Section 1, we prove Theorem 0.1. In Section 2, we compute the coefficients $b_{q,r}$, and thus establish Theorem 0.2. In Section 3, we explain some applications of our results. Among others, we give a symplectic version of the convergence of the induced Fubini-Study metric [46], and we show how to handle the first-order pseudo-differential operator D_b of Boutet de Monvel and Guillemin [13], which was studied extensively by Shiffman and Zelditch [42], and the operator $\overline{\partial} + \overline{\partial}^*$ when X is Kähler but $\mathbf{J} \neq J$. We include also generalizations for non-compact or singular manifolds and as a consequence we obtain an unified treatment of the convergence of the induced Fubini–Study metric, the holomorphic Morse inequalities and the characterization of Moishezon spaces. Some results of this paper have been announced in [34]. We refer also the readers to our forthcoming book [36] for our approach.

1. GENERALIZED BERGMAN KERNELS

As pointed out in Introduction, we will apply the strategy of the proof in [20]. However, we have small eigenvalues when $p \to \infty$ (cf. (0.5)), thus we cannot use directly the key equation [20, (4.89)] to get a *full off-diagonal* asymptotic expansion of the generalized Bergman kernels. After localizing the problem, we will adapt the Sobolev norm estimates developed in [20] to our problem in Section 1.3. To complete the proof of Theorem 0.1, we need to prove the vanishing of the coefficients $F_{q,r}$ ($r < 2q$) in the expansion (1.77). We will introduce a formal power series trick to overcome this difficulty and give a method to compute the coefficients in (0.9). The ideas used here are inspired by the technique of Local Index Theory, especially by [4, §10, 11].

This Section is organized as follows. In Section 1.1, we explain that the asymptotic expansion of the generalized Bergman kernel $P_{q,p}(x, x')$ is local on X by using the spectral gap (0.5) and the finite propagation speed of solutions of hyperbolic equations. In Section 1.2, we obtain an asymptotic expansion of $\Delta_{p,\Phi}$ in normal coordinates. In Section 1.3, we study the uniform estimate of the generalized Bergman kernels of the renormalized Bochner-Laplacian $\mathscr{L}_t$. In Section 1.4, we study the Bergman kernel of the limit operator $\mathscr{L}_0$. In Section 1.5, we compute some coefficients $F_{q,r}$ ($r \leqslant 2q$) in the asymptotic expansion in Theorem 1.13. Finally, in Section 1.6, we prove Theorem 0.1.

1.1. **Localization of the problem.** Let a^X be the injectivity radius of (X, g^{TX}). We fix $\varepsilon \in (0, a^X/4)$. We denote by $B^X(x, \varepsilon)$ and $B^{T_x X}(0, \varepsilon)$ the open balls in X and $T_x X$ with center x and radius ε, respectively. Then the map $T_x X \ni Z \to \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_x X}(0, \varepsilon)$ on $B^X(x, \varepsilon)$ for $\varepsilon \leqslant a^X$. From now on, we identify $B^{T_x X}(0, \varepsilon)$ with $B^X(x, \varepsilon)$ for $\varepsilon \leqslant a^X$.

Let $\langle \, , \, \rangle_{L^p \otimes E}$ be the metric on $L^p \otimes E$ induced by h^L and h^E and dv_X be the Riemannian volume form of (TX, g^{TX}). The L^2–scalar product on $\mathscr{C}^\infty(X, L^p \otimes E)$, the space of smooth sections of $L^p \otimes E$, is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_{L^p \otimes E} \, dv_X(x) \,. \tag{1.1}$$

We denote the corresponding norm with $\| \cdot \|_{L^2}$.

Let ∇^{TX} be the Levi–Civita connection of the metric g^{TX} and $\nabla^{L^p \otimes E}$ be the connection on $L^p \otimes E$ induced by ∇^L and ∇^E. Let $\{e_i\}_i$ be an orthonormal frame of TX. Then the Bochner-Laplacian on $L^p \otimes E$ is given by

$$\Delta^{L^p \otimes E} = -\sum_i \left[(\nabla_{e_i}^{L^p \otimes E})^2 - \nabla_{\nabla_{e_i}^{TX} e_i}^{L^p \otimes E} \right]. \tag{1.2}$$

Let $P_{\mathcal{H}_p}$ be the orthonormal projection from $\mathscr{C}^\infty(X, L^p \otimes E)$ onto $\mathcal{H}_p$, the span of eigensections of $\Delta_{p,\Phi} = \Delta^{L^p \otimes E} - p\tau + \Phi$ corresponding to eigenvalues in $[-C_L, C_L]$.

Definition 1.1. The smooth kernel of $(\Delta_{p,\Phi})^q P_{\mathcal{H}_p}$, $q \geqslant 0$ (where $(\Delta_{p,\Phi})^0 = 1$), with respect to $dv_X(x')$ is denoted $P_{q,p}(x, x')$ and is called a *generalized Bergman kernel* of $\Delta_{p,\Phi}$.

The kernel $P_{q,p}(x, x')$ is a section of $\pi_1^*(L^p \otimes E) \otimes \pi_2^*(L^p \otimes E)^*$ over $X \times X$, where π_1 and π_2 are the projections of $X \times X$ on the first and second factor. Using the notations of (0.7) we can write

$$\boxed{0.42} \quad (1.3) \qquad P_{q,p}(x, x') = \sum_{i=1}^{d_p} \lambda_{i,p}^q S_i^p(x) \otimes (S_i^p(x'))^* \in (L^p \otimes E)_x \otimes (L^p \otimes E)_{x'}^*.$$

Since $L_x^p \otimes (L_x^p)^*$ is canonically isomorphic to $\mathbb{C}$, the restriction of $P_{q,p}$ to the diagonal $\{(x, x) : x \in X\}$ can be identified to $B_{q,p} \in \mathscr{C}^\infty(X, E \otimes E^*) = \mathscr{C}^\infty(X, \mathrm{End}(E))$.

Let $f : \mathbb{R} \to [0, 1]$ be a smooth even function such that $f(v) = 1$ for $|v| \leqslant \varepsilon/2$, and $f(v) = 0$ for $|v| \geqslant \varepsilon$. Set

$$\boxed{0c3} \quad (1.4) \qquad F(a) = \left(\int_{-\infty}^{+\infty} f(v) dv \right)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v) dv.$$

Then $F(a)$ is an even function and lies in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$. Let $\widetilde{F}$ be the holomorphic function on $\mathbb{C}$ such that $\widetilde{F}(a^2) = F(a)$. The restriction of $\widetilde{F}$ to $\mathbb{R}$ lies in the Schwartz space $\mathcal{S}(\mathbb{R})$. Then there exists $\{c_j\}_{j=1}^\infty$ such that for any $k \in \mathbb{N}$, the function

$$\boxed{0c5} \quad (1.5) \qquad F_k(a) = \widetilde{F}(a) - \sum_{j=1}^k c_j a^j \widetilde{F}(a),$$

verifies

$$\boxed{0c6} \quad (1.6) \qquad F_k^{(i)}(0) = 0 \quad \text{for any } 0 < i \leqslant k.$$

$\boxed{0t3.0}$ **Proposition 1.2.** *For any $k, m \in \mathbb{N}$, there exists $C_{k,m} > 0$ such that for $p \geqslant 1$*

$$\boxed{0c7} \quad (1.7) \qquad \left| F_k\left(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)(x, x') - P_{0,p}(x, x') \right|_{\mathscr{C}^m(X \times X)} \leqslant C_{k,m} p^{-\frac{k}{2} + 2(2m+2n+1)}.$$

Here the $\mathscr{C}^m$ norm is induced by ∇^L, ∇^E, h^L, h^E and g^{TX}.

Proof. By (1.4), for any $m \in \mathbb{N}$, there exists $C'_{k,m} > 0$ such that

$$\boxed{c9} \quad (1.8) \qquad \sup_{a \in \mathbb{R}} |a|^m |F_k(a)| \leqslant C'_{k,m}.$$

Set

$$\boxed{0c8} \quad (1.9) \qquad G_{k,p}(a) = 1_{[\sqrt{p}\mu_0, +\infty[}(a) F_k(a), \quad H_{k,p}(a) = 1_{[0, \frac{C_L}{\sqrt{p}}]}(|a|) F_k(a),$$

By (0.5), for p big enough,

$$\boxed{0c9} \quad (1.10) \qquad F_k\left(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\right) = G_{k,p}\left(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\right) + H_{k,p}\left(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\right).$$

As X is compact, there exist $\{x_i\}_{i=1}^r$ such that $\{U_i = B^X(x_i, \varepsilon)\}_{i=1}^r$ is a covering of X. We identify $B^{T_{x_i}X}(0, \varepsilon)$ with $B^X(x_i, \varepsilon)$ by the exponential map as above. We identify $(L^p \otimes E)_Z$ for $Z \in B^{T_{x_i}X}(0, \varepsilon)$ to $(L^p \otimes E)_{x_i}$ by parallel transport with respect to the connection $\nabla^{L^p \otimes E}$ along the curve $\gamma_Z : [0, 1] \ni u \to \exp_{x_i}^X(uZ)$. Let $\{e_i\}_i$ be an orthonormal basis of $T_{x_i}X$. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TX} along the above curve. Let Γ^E, Γ^L be the corresponding connection forms of ∇^E, ∇^L with respect to any fixed frame for E, L which is parallel along the curve

γ_Z under the trivialization on U_i. Denote by ∇_U is the ordinary differentiation operator on $T_{x_i}X$ in the direction U. Then

$\boxed{\text{c10}}$ (1.11)
$$\nabla^{L^p \otimes E}_{e_j} = \nabla_{e_j} + p\Gamma^L(e_j) + \Gamma^E(e_j).$$

Let φ_i be a partition function associated to $\{U_i\}$. We define an Sobolev norm on the l-th Sobolev space $H^l(X, L^p \otimes E)$ by

$\boxed{\text{c11}}$ (1.12)
$$\|s\|^2_{H^l_p} = \sum_i \sum_{k=0}^{l} \sum_{i_1 \cdots i_k = 1}^{2n} \|\nabla_{e_{i_1}} \cdots \nabla_{e_{i_k}}(\varphi_i s)\|^2_{L^2}$$

Then by $(\overset{|0.2}{0.2})$, $(\overset{|0c1}{1.2})$, $(\overset{|c10}{1.11})$, there exists $C > 0$ such that for $p \geqslant 1$, $s \in H^2(X, L^p \otimes E)$,

$\boxed{\text{c12}}$ (1.13)
$$\|s\|_{H^2_p} \leqslant C(\|\Delta_{p,\Phi}s\|_{L^2} + p^2\|s\|_{L^2}).$$

Let Q be a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in U_i, then by $(\overset{|c12}{1.13})$ and $[\Delta_{p,\Phi}, Q]$ is a differential operator of order $m + 1$, we get

$\boxed{\text{c13}}$ (1.14)
$$\begin{aligned}\|Qs\|_{H^2_p} &\leqslant C(\|\Delta_{p,\Phi}Qs\|_{L^2} + p^2\|Qs\|_{L^2}) \\ &\leqslant C(\|Q\Delta_{p,\Phi}s\|_{L^2} + p^2\|s\|_{H^{m+1}_p} + p^2\|Qs\|_{L^2}).\end{aligned}$$

This means

$\boxed{\text{c17}}$ (1.15)
$$\|s\|_{H^{2m+2}_p} \leqslant C_m p^{4m+4} \sum_{j=0}^{m+1} \|\Delta^j_{p,\Phi}s\|_{L^2}.$$

Moreover for $\mathbf{G}_{k,p} = G_{k,p}$ or $H_{k,p}$, $\langle \Delta^{m'}_{p,\Phi}\mathbf{G}_{k,p}(\frac{1}{\sqrt{p}}\Delta_{p,\Phi})Qs, s'\rangle = \langle s, Q^*\mathbf{G}_{k,p}(\frac{1}{\sqrt{p}}\Delta_{p,\Phi})\Delta^{m'}_{p,\Phi}s'\rangle$, so from $(\overset{|0c6}{1.6})$, $(\overset{|c9}{1.8})$, we know that for $l, m' \in \mathbb{N}$, there exist $C, C' > 0$ such that for $p > 1$,

$\boxed{\text{c18}}$ (1.16)
$$\begin{aligned}\left\|\Delta^{m'}_{p,\Phi}G_{k,p}\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big)Qs\right\|_{L^2} &\leqslant Cp^{-l}\|s\|_{L^2}, \\ \left\|\Delta^{m'}_{p,\Phi}\Big(H_{k,p}\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big) - P_{\mathcal{H}_p}\Big)Qs\right\|_{L^2} &\leqslant C'p^{2m-\frac{k}{2}}\|s\|_{L^2}.\end{aligned}$$

We deduce from $(\overset{|c17}{1.15})$ and $(\overset{|c18}{1.16})$ that if P, Q are differential operators with compact support in U_i, U_j respectively, then for any $l \in \mathbb{N}$, there exists $C > 0$ such that for $p > 1$,

$\boxed{\text{c19}}$ (1.17)
$$\begin{aligned}\left\|PG_{k,p}\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big)Qs\right\|_{L^2} &\leqslant Cp^{-l}\|s\|_{L^2}, \\ \left\|P\Big(H_{k,p}\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big) - P_{\mathcal{H}_p}\Big)Qs\right\|_{L^2} &\leqslant Cp^{2(m+m')-\frac{k}{2}}\|s\|_{L^2}.\end{aligned}$$

On $U_i \times U_j$, we use Sobolev inequality, we know for any $l \in \mathbb{N}$, there exists $C > 0$ such that for $p > 1$,

$\boxed{\text{c20}}$ (1.18)
$$\begin{aligned}\left|G_{k,p}\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big)(x,x')\right|_{\mathscr{C}^m} &\leqslant C_{l,m}p^{-l}, \\ \left|\Big(H_{k,p}\big(\tfrac{1}{\sqrt{p}}\Delta_{p,\Phi}\big) - P_{0,p}\Big)(x,x')\right|_{\mathscr{C}^m} &\leqslant Cp^{2(2m+2n+1)-\frac{k}{2}}.\end{aligned}$$

By $(\overset{|0c9}{1.10})$ and $(\overset{|c20}{1.18})$, we get our Proposition $\overset{|0t3.0}{1.2}$. $\square$

Using (1.4), (1.5) and the finite propagation speed [19, §7.8], [45, §4.4], it is clear that for $x, x' \in X$, $F_k\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)(x, \cdot)$ only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$, and $F_k\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)(x, x') = 0$, if $d(x, x') \geqslant \varepsilon p^{-\frac{1}{4}}$. This means that the asymptotic of $\Delta_{p,\Phi}^q P_{\mathcal{H}_p}(x, \cdot)$ when $p \to +\infty$, modulo $\mathscr{O}(p^{-\infty})$ (i.e. terms whose $\mathscr{C}^m$ norm is $\mathscr{O}(p^{-l})$ for any $l, m \in \mathbb{N}$), only depends on the restriction of $\Delta_{p,\Phi}$ to $B^X(x, \varepsilon p^{-\frac{1}{4}})$.

1.2. Rescaling and a Taylor expansion of the operator $\Delta_{p,\Phi}$.

We fix $x_0 \in X$. From now on, we identify $B^{T_{x_0}X}(0, \varepsilon)$ with $B^X(x_0, \varepsilon)$. For $Z \in B^{T_{x_0}X}(0, \varepsilon)$ we identify L_Z, E_Z and $(L^p \otimes E)_Z$ to L_{x_0}, E_{x_0} and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and $\nabla^{L^p \otimes E}$ along the curve $\gamma_Z : [0, 1] \ni u \to \exp_{x_0}^X(uZ)$. Let $\{e_i\}_i$ be an oriented orthonormal basis of $T_{x_0}X$, and let $\{e^i\}_i$ be its dual basis.

For $\varepsilon > 0$ small enough, we will extend the geometric objects from $B^{T_{x_0}X}(0, \varepsilon)$ to $\mathbb{R}^{2n} \simeq T_{x_0}X$ where the identification is given by

$$(Z_1, \cdots, Z_{2n}) \in \mathbb{R}^{2n} \longrightarrow \sum_i Z_i e_i \in T_{x_0}X \tag{1.19}$$

such that $\Delta_{p,\Phi}$ is the restriction of a renormalized Bochner-Laplacian on $\mathbb{R}^{2n}$ associated to a Hermitian line bundle with positive curvature. In this way, we replace X by $\mathbb{R}^{2n}$.

At first, we denote by L_0, E_0 the trivial bundles with fiber L_{x_0}, E_{x_0} on $X_0 = \mathbb{R}^{2n}$. We still denote by ∇^L, ∇^E, h^L etc. the connections and metrics on L_0, E_0 on $B^{T_{x_i}X}(0, 4\varepsilon)$ induced by the above identification. Then h^L, h^E is identified to the constant metrics $h^{L_0} = h^{L_{x_0}}$, $h^{E_0} = h^{E_{x_0}}$.

Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$\rho(v) = 1 \ \text{if} \ |v| < 2; \quad \rho(v) = 0 \ \text{if} \ |v| > 4. \tag{1.20}$$

Let $\varphi_\varepsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the map defined by $\varphi_\varepsilon(Z) = \rho(|Z|/\varepsilon)Z$. Then $\Phi_0 = \Phi \circ \varphi_\varepsilon$ is a smooth self-adjoint section of $\text{End}(E_0)$ on X_0. Let $g^{TX_0}(Z) = g^{TX}(\varphi_\varepsilon(Z))$, $J_0(Z) = J(\varphi_\varepsilon(Z))$ be the metric and complex structure on X_0. Set $\nabla^{E_0} = \varphi_\varepsilon^* \nabla^E$. Then ∇^{E_0} is the extension of ∇^E on $B^{T_{x_0}X}(0, \varepsilon)$. If $\mathcal{R} = \sum_i Z_i e_i = Z$ denotes radial vector field on $\mathbb{R}^{2n}$, we define the Hermitian connection ∇^{L_0} on (L_0, h^{L_0}) by

$$\nabla^{L_0}|_Z = \varphi_\varepsilon^* \nabla^L + \frac{1}{2}(1 - \rho^2(|Z|/\varepsilon))R_{x_0}^L(\mathcal{R}, \cdot). \tag{1.21}$$

Then we calculate easily that its curvature $R^{L_0} = (\nabla^{L_0})^2$ is

$$
\begin{aligned}
R^{L_0}(Z) &= \varphi_\varepsilon^* R^L + \frac{1}{2}d\Big(\big(1 - \rho^2(|Z|/\varepsilon)\big)R_{x_0}^L(\mathcal{R}, \cdot)\Big) \\
&= \Big(1 - \rho^2(|Z|/\varepsilon)\Big)R_{x_0}^L + \rho^2(|Z|/\varepsilon)R_{\varphi_\varepsilon(Z)}^L \\
&\quad - (\rho\rho')(|Z|/\varepsilon)\frac{Z_i e^i}{\varepsilon|Z|} \wedge \big[R_{x_0}^L(\mathcal{R}, \cdot) - R_{\varphi_\varepsilon(Z)}^L(\mathcal{R}, \cdot)\big].
\end{aligned}
\tag{1.22}
$$

Thus R^{L_0} is positive in the sense of (0.2) for ε small enough, and the corresponding constant μ_0 for R^{L_0} is bigger than $\frac{4}{5}\mu_0$. From now on, we fix ε as above.

Let $\Delta_{p,\Phi_0}^{X_0} = \Delta^{L_0^p \otimes E_0} - p\tau_0 - \Phi_0$ be the renormalized Bochner-Laplacian on X_0 associated to the above data, as in (0.4). Observe that R^{L_0} is uniformly positive on

$\mathbb{R}^{2n}$, so by the relations (3.2), (3.11) and (3.12) in [33, p. 656–658], we know that (0.5) still holds for $\Delta_{p,\Phi_0}^{X_0}$. Especially, there exists $C_{L_0} > 0$ such that

| `1c17` | (1.23) | $\operatorname{Spec} \Delta_{p,\Phi_0}^{X_0} \subset [-C_{L_0}, C_{L_0}] \cup [\frac{8}{5}p\mu_0 - C_{L_0}, +\infty[.$ |

We note that $\Delta_{p,\Phi_0}^{X_0}$ has not necessarily discrete spectrum.

Let S_L be an unit vector of L_{x_0}. Using S_L and the above discussion, we get an isometry $E_0 \otimes L_0^p \simeq E_{x_0}$. Let $P_{0,\mathcal{H}_p}$ be the spectral projection of $\Delta_{p,\Phi_0}^{X_0}$ from $\mathscr{C}^\infty(X_0, L_0^p \otimes E_0) \simeq \mathscr{C}^\infty(X_0, E_{x_0})$ corresponding to the interval $[-C_{L_0}, C_{L_0}]$, and let $P_{0,q,p}(x,x')$ $(q \geqslant 0)$ be the smooth kernels of $P_{0,q,p} = (\Delta_{p,\Phi_0}^{X_0})^q P_{0,\mathcal{H}_p}$ (we set $(\Delta_{p,\Phi_0}^{X_0})^0 = 1$) with respect to the volume form $dv_{X_0}(x')$. The following Proposition shows that $P_{q,p}$ and $P_{0,q,p}$ are asymptotically close on $B^{T_{x_0}X}(0,\varepsilon)$ in the $\mathscr{C}^\infty$–topology, as $p \to \infty$.

| `p3.2` | **Proposition 1.3.** *For any $l,m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $x,x' \in B^{T_{x_0}X}(0,\varepsilon)$,* |

| `1c19` | (1.24) | $|(P_{0,q,p} - P_{q,p})(x,x')|_{\mathscr{C}^m} \leqslant C_{l,m}p^{-l}.$ |

Proof. Using (1.4) and (1.23), we know that for $x,x' \in B^{T_{x_0}X}(0,\varepsilon)$,

| `1c20` | (1.25) | $\left| F_k\left(\frac{1}{\sqrt{p}}\Delta_{p,\Phi}\right)(x,x') - P_{0,0,p}(x,x') \right|_{\mathscr{C}^m} \leqslant C_{k,m}p^{-\frac{k}{2}+2(m+n+1)}.$ |

Thus from (1.7) and (1.25) for k big enough, we infer (1.24) for $q = 0$; Now from the definition of $P_{0,q,p}$ and $P_{q,p}$, we get (1.24) from (1.11) and (1.24) for $q = 0$. $\qquad\square$

It suffices therefore to study the kernel $P_{0,q,p}$ and for this purpose we rescale the operator $\Delta_{p,\Phi_0}^{X_0}$. Let dv_{TX} be the Riemannian volume form of $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

| `c22` | (1.26) | $dv_{X_0}(Z) = \kappa(Z)dv_{TX}(Z),$ |

with $\kappa(0) = 1$. Denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U, and set $\partial_i = \nabla_{e_i}$. If $\alpha = (\alpha_1, \cdots, \alpha_{2n})$ is a multi-index, set $Z^\alpha = Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$. We also denote by $(\partial^\alpha R^L)_{x_0}$ the tensor $(\partial^\alpha R^L)_{x_0}(e_i,e_j) = \partial^\alpha(R^L(e_i,e_j))_{x_0}$. Denote by $t = \frac{1}{\sqrt{p}}$. For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$ and $Z \in \mathbb{R}^{2n}$, set

| `c27` | (1.27) | $\begin{aligned} (S_t s)(Z) &= s(Z/t), \quad \nabla_t = t S_t^{-1} \kappa^{\frac{1}{2}} \nabla^{L_0^p \otimes E_0} \kappa^{-\frac{1}{2}} S_t, \\ \mathscr{L}_t &= S_t^{-1} \frac{1}{p} \kappa^{\frac{1}{2}} \Delta_{p,\Phi_0}^{X_0} \kappa^{-\frac{1}{2}} S_t. \end{aligned}$ |

The operator $\mathscr{L}_t$ is the rescaled operator, which we now develop in Taylor series.

| `t3.3` | **Theorem 1.4.** *There exist polynomials $\mathcal{A}_{i,j,r}$ (resp. $\mathcal{B}_{i,r}, \mathcal{C}_r$) $(r \in \mathbb{N}, i,j \in \{1, \cdots, 2n\})$ in Z with the following properties:* |

– their coefficients are polynomials in R^{TX} (resp. R^{TX}, R^L, R^E, Φ) and their derivatives at x_0 up to order $r-2$ (resp. $r-1$, r, $r-1$, r),

– $\mathcal{A}_{i,j,r}$ is a monomial in Z of degree r, the degree in Z of $\mathcal{B}_{i,r}$ (resp. $\mathcal{C}_r$) has the same parity with $r-1$ (resp. r),

– if we denote by

| `0c35` | (1.28) | $\mathcal{O}_r = \mathcal{A}_{i,j,r}\nabla_{e_i}\nabla_{e_j} + \mathcal{B}_{i,r}\nabla_{e_i} + \mathcal{C}_r,$ |

then

$\boxed{\texttt{c30}}$ (1.29)
$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathcal{O}(t^{m+1}).$$

and there exists $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $t \leqslant 1$ the derivatives of order $\leqslant k$ of the coefficients of the operator $\mathcal{O}(t^{m+1})$ are dominated by $Ct^{m+1}(1+|Z|)^{m'}$. Moreover

$\boxed{\texttt{c31}}$ (1.30)

$$\mathscr{L}_0 = -\sum_j \left(\nabla_{e_j} + \frac{1}{2} R_{x_0}^L(Z, e_j) \right)^2 - \tau_{x_0},$$

$$\mathcal{O}_1(Z) = -\frac{2}{3}(\partial_j R^L)_{x_0}(\mathcal{R}, e_i) Z_j \left(\nabla_{e_i} + \frac{1}{2} R_{x_0}^L(\mathcal{R}, e_i) \right) - \frac{1}{3}(\partial_i R^L)_{x_0}(\mathcal{R}, e_i) - (\nabla_{\mathcal{R}} \tau)_{x_0},$$

$$\mathcal{O}_2(Z) = \frac{1}{3} \left\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j \right\rangle_{x_0} \left(\nabla_{e_i} + \frac{1}{2} R_{x_0}^L(\mathcal{R}, e_i) \right) \left(\nabla_{e_j} + \frac{1}{2} R_{x_0}^L(\mathcal{R}, e_j) \right)$$

$$+ \left[\frac{2}{3} \left\langle R_{x_0}^{TX}(\mathcal{R}, e_j)e_j, e_i \right\rangle_{x_0} - \left(\frac{1}{2} \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} + R_{x_0}^E \right)(\mathcal{R}, e_i) \right] \left(\nabla_{e_i} + \frac{1}{2} R_{x_0}^L(\mathcal{R}, e_i) \right)$$

$$- \frac{1}{4} \nabla_{e_i} \left(\sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!}(\mathcal{R}, e_i) \right) - \frac{1}{9} \sum_i \left[\sum_j (\partial_j R^L)_{x_0}(\mathcal{R}, e_i) Z_j \right]^2$$

$$- \frac{1}{12} \left[\mathscr{L}_0, \left\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_i \right\rangle_{x_0} \right] - \sum_{|\alpha|=2} (\partial^\alpha \tau)_{x_0} \frac{Z^\alpha}{\alpha!} + \Phi_{x_0}.$$

Proof. Set $g_{ij}(Z) = g^{TX}(e_i, e_j)(Z) = \langle e_i, e_j \rangle_Z$ and let $(g^{ij}(Z))$ be the inverse of the matrix $(g_{ij}(Z))$. By [1, Proposition 1.28], the Taylor expansion of $g_{ij}(Z)$ with respect to the basis $\{e_i\}$ to order r is a polynomial of the Taylor expansion of R^{TX} to order $r-2$, moreover

$\boxed{\texttt{0c30}}$ (1.31)
$$g_{ij}(Z) = \delta_{ij} + \frac{1}{3} \left\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_j \right\rangle_{x_0} + \mathcal{O}(|Z|^3),$$

$$\kappa(Z) = |\det(g_{ij}(Z))|^{1/2} = 1 + \frac{1}{6} \left\langle R_{x_0}^{TX}(\mathcal{R}, e_i)\mathcal{R}, e_i \right\rangle_{x_0} + \mathcal{O}(|Z|^3).$$

If Γ_{ij}^l is the connection form of ∇^{TX} with respect to the basis $\{e_i\}$, we have $(\nabla_{e_i}^{TX} e_j)(Z) = \Gamma_{ij}^l(Z)e_l$. Owing to (1.31),

$\boxed{\texttt{0c31}}$ (1.32)
$$\Gamma_{ij}^l(Z) = \frac{1}{2} g^{lk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})(Z)$$
$$= \frac{1}{3} \left[\left\langle R_{x_0}^{TX}(\mathcal{R}, e_j)e_i, e_l \right\rangle_{x_0} + \left\langle R_{x_0}^{TX}(\mathcal{R}, e_i)e_j, e_l \right\rangle_{x_0} \right] + \mathcal{O}(|Z|^2).$$

Now by (1.2),

$\boxed{\texttt{c32}}$ (1.33)
$$\Delta_{p,\Phi} = -g^{ij}(\nabla_{e_i}^{L^p \otimes E} \nabla_{e_j}^{L^p \otimes E} - \nabla_{\nabla_{e_i}^{TX} e_j}^{L^p \otimes E}) - p\tau + \Phi.$$

so from (1.27) and (1.33) we infer the expression

$\boxed{\texttt{0c37}}$ (1.34)
$$\mathscr{L}_t = -g^{ij}(tZ) \left[\nabla_{t,e_i} \nabla_{t,e_j} - t\Gamma_{ij}^l \nabla_{t,e_l} \right](tZ) - \tau(tZ) + t^2 \Phi(tZ).$$

Let Γ^E, Γ^L be the connection forms of ∇^E and ∇^L with respect to any fixed frames for E, L which are parallel along the curve γ_Z under our trivializations on $B^{T_{x_0}X}(0, \varepsilon)$.

(1.27) yields on $B^{T_{x_0}X}(0, \varepsilon/t)$

(1.35) $\qquad \nabla_{t,e_i}|_Z = \kappa^{\frac{1}{2}}(tZ)\Big(\nabla_{e_i} + \frac{1}{t}\Gamma^L(e_i)(tZ) + t\Gamma^E(e_i)(tZ)\Big)\kappa^{-\frac{1}{2}}(tZ).$

Let $\Gamma^\bullet = \Gamma^E, \Gamma^L$ and $R^\bullet = R^E, R^L$, respectively. By [1, Proposition 1.18] the Taylor coefficients of $\Gamma^\bullet(e_j)(Z)$ at x_0 to order r are only determined by those of $R^\bullet$ to order $r-1$, and

(1.36) $\qquad \displaystyle\sum_{|\alpha|=r}(\partial^\alpha\Gamma^\bullet)_{x_0}(e_j)\frac{Z^\alpha}{\alpha!} = \frac{1}{r+1}\sum_{|\alpha|=r-1}(\partial^\alpha R^\bullet)_{x_0}(\mathcal{R}, e_j)\frac{Z^\alpha}{\alpha!}.$

Owing to (1.31), (1.36)

(1.37) $\quad \mathscr{L}_t = -\Big(\delta_{ij} - \frac{t^2}{3}\big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\big\rangle + \mathscr{O}(t^3)\Big)\kappa^{\frac{1}{2}}(tZ)\Big\{$

$\qquad \Big[\nabla_{e_i} + \Big(\frac{1}{2}R^L_{x_0} + \frac{t}{3}(\partial_k R^L)_{x_0}Z_k + \frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}R^E_{x_0}\Big)(\mathcal{R}, e_i) + \mathscr{O}(t^3)\Big]$

$\qquad \Big[\nabla_{e_j} + \Big(\frac{1}{2}R^L_{x_0} + \frac{t}{3}(\partial_k R^L)_{x_0}Z_k + \frac{t^2}{4}\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}\frac{Z^\alpha}{\alpha!} + \frac{t^2}{2}R^E_{x_0}\Big)(\mathcal{R}, e_j) + \mathscr{O}(t^3)\Big]$

$\qquad -t\Gamma^l_{ij}(tZ)\Big(\nabla_{e_l} + \frac{1}{2}R^L_{x_0}(\mathcal{R}, e_l) + \mathscr{O}(t)\Big)\Big\}\kappa^{-\frac{1}{2}}(tZ)$

$\qquad -\tau_{x_0} - t(\nabla_\mathcal{R}\tau)_{x_0} - t^2\sum_{|\alpha|=2}(\partial^\alpha\tau)_{x_0}\frac{Z^\alpha}{\alpha!} + t^2\Phi_{x_0} + \mathscr{O}(t^3).$

Relations (1.31) and (1.34)–(1.37) settle our Theorem. $\qquad\qquad\qquad \square$

1.3. Uniform estimate of the generalized Bergman kernels.

We shall estimate the Sobolev norm of the resolvent of $\mathscr{L}_t$ so we introduce the following norms. We denote by $\langle\cdot,\cdot\rangle_{0,L^2}$ and $\|\cdot\|_{0,L^2}$ the scalar product and the L^2 norm on $\mathscr{C}^\infty(X_0, E_{x_0})$ induced by g^{TX_0}, h^{E_0} as in (1.1). For $s \in \mathscr{C}^\infty(X_0, E_{x_0})$, set

(1.38) $\qquad\qquad \displaystyle \|s\|^2_{t,0} = \|s\|^2_0 = \int_{\mathbb{R}^{2n}}|s(Z)|^2_{h^{E_{x_0}}}dv_{TX}(Z),$

$\qquad\qquad\qquad \displaystyle \|s\|^2_{t,m} = \sum_{l=0}^{m}\sum_{i_1,\cdots,i_l=1}^{2n}\|\nabla_{t,e_{i_1}}\cdots\nabla_{t,e_{i_l}}s\|^2_{t,0}.$

We denote by $\langle s', s\rangle_{t,0}$ the inner product on $\mathscr{C}^\infty(X_0, E_{x_0})$ corresponding to $\|\cdot\|^2_{t,0}$. Let H^m_t be the Sobolev space of order m with norm $\|\cdot\|_{t,m}$. Let H^{-1}_t be the Sobolev space of order -1 and let $\|\cdot\|_{t,-1}$ be the norm on H^{-1}_t defined by $\|s\|_{t,-1} = \sup_{0\neq s'\in H^1_t}|\langle s, s'\rangle_{t,0}|/\|s'\|_{t,1}$. If $A \in \mathscr{L}(H^m, H^{m'})$ $(m, m' \in \mathbb{Z})$, we denote by $\|A\|^{m,m'}_t$ the norm of A with respect to the norms $\|\cdot\|_{t,m}$ and $\|\cdot\|_{t,m'}$.

Remark 1.5. Note that $\Delta^{X_0}_{p,\Phi_0}$ is self-adjoint with respect to $\|\cdot\|_0$, thus by (1.26), (1.27), (1.38), $\mathscr{L}_t$ is a formal self adjoint elliptic operator with respect to $\|\ \|_0$, and is a smooth family of operators with the parameter $x_0 \in X$. Thus $\mathscr{L}_0$ and $\mathcal{O}_r$ are also formal self-adjoint with respect to $\|\cdot\|_0$. This will simplify the computation of

the coefficients $b_{0,1}$ in (0.11) (cf. §2.3) and explains why we prefer to conjugate with $\kappa^{1/2}$ comparing to [20, (3.38)].

`tu1` **Theorem 1.6.** *There exist constants $C_1, C_2, C_3 > 0$ such that for $t \in]0,1]$ and any $s, s' \in \mathscr{C}_0^\infty(\mathbb{R}^{2n}, E_{x_0})$,*

`u1` (1.39)
$$\langle \mathscr{L}_t s, s \rangle_{t,0} \geqslant C_1 \|s\|_{t,1}^2 - C_2 \|s\|_{t,0}^2,$$
$$|\langle \mathscr{L}_t s, s' \rangle_{t,0}| \leqslant C_3 \|s\|_{t,1} \|s'\|_{t,1}.$$

Proof. Relations (0.4) and (1.2) yield

`1u1` (1.40)
$$\langle \Delta_{p,\Phi} s, s \rangle_{0,L^2} = \|\nabla^{L_0^p \otimes E_0} s\|_{0,L^2}^2 - \langle (p\tau - \Phi) s, s \rangle_{0,L^2}.$$

Thus from (1.27), (1.38) and (1.40) we get

`1u2` (1.41)
$$\langle \mathscr{L}_t s, s \rangle_{t,0} = \|\nabla_t s\|_{t,0}^2 - \langle (S_t^{-1} \tau - t^2 \Phi) s, s \rangle_{t,0}.$$

which implies (1.39). $\qquad\qquad\square$

Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$.

`tu4` **Theorem 1.7.** *There exists $t_0 > 0$ such that the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \delta$, $t \in]0,t_0]$, and there exists $C > 0$ such that for $t \in]0,t_0]$, $\lambda \in \delta$, and $x_0 \in X$,*

`ue2` (1.42)
$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{0,0} \leqslant C, \qquad\qquad \|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant C.$$

Proof. By (1.23), (1.27), for t small enough,

`1u3` (1.43)
$$\operatorname{Spec} \mathscr{L}_t \subset \big[-C_{L_0} t^2, C_{L_0} t^2\big] \cup \big[\mu_0, +\infty\big[.$$

Thus the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ exists for $\lambda \in \delta$ and t small enough, and we get the first inequality of (1.42). By (1.39) $(\lambda_0 - \mathscr{L}_t)^{-1}$ exists for $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leqslant -2C_2$, and $\|(\lambda_0 - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{C_1}$. Now,

`ue7` (1.44)
$$(\lambda - \mathscr{L}_t)^{-1} = (\lambda_0 - \mathscr{L}_t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathscr{L}_t)^{-1}(\lambda_0 - \mathscr{L}_t)^{-1}.$$

Thus for $\lambda \in \delta$, from (1.44), we get

`ue8` (1.45)
$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,0} \leq \frac{1}{C_1}\left(1 + \frac{4}{\mu_0}|\lambda - \lambda_0|\right).$$

Changing the last two factors in (1.44) and applying (1.45) we get

`ue9` (1.46)
$$\|(\lambda - \mathscr{L}_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{C_1^2}\left(1 + \frac{4}{\mu_0}|\lambda - \lambda_0|\right) \leqslant C.$$

The proof of our Theorem is complete. $\qquad\qquad\square$

`tu5` **Proposition 1.8.** *Take $m \in \mathbb{N}^*$. There exists $C_m > 0$ such that for $t \in]0,1]$, $Q_1, \cdots, Q_m \in \{\nabla_{t,e_i}, Z_i\}_{i=1}^{2n}$ and $s, s' \in C_0^\infty(X_0, E_{x_0})$,*

`ue11` (1.47)
$$\left|\langle [Q_1, [Q_2, \ldots, [Q_m, \mathscr{L}_t]]\ldots]s, s' \rangle_{t,0}\right| \leqslant C_m \|s\|_{t,1} \|s'\|_{t,1}.$$

Proof. Note that $[\nabla_{t,e_i}, Z_j] = \delta_{ij}$, hence (1.34) implies that $[Z_j, \mathscr{L}_t]$ verifies (1.47). On the other hand, we obtain from (1.27)

(1.48) $$[\nabla_{t,e_i}, \nabla_{t,e_j}] = \left(R^{L_0}(tZ) + t^2 R^{E_0}(tZ)\right)(e_i, e_j).$$

Thus from (1.34) and (1.48), we know that $[\nabla_{t,e_k}, \mathscr{L}_t]$ has the same structure as $\mathscr{L}_t$ for $t \in]0,1]$, i.e. $[\nabla_{t,e_k}, \mathscr{L}_t]$ has the same type as

(1.49) $$\sum_{ij} a_{ij}(t, tZ)\nabla_{t,e_i}\nabla_{t,e_j} + \sum_i b_i(t, tZ)\nabla_{t,e_i} + c(t, tZ),$$

and $a_{ij}(t, Z), b_i(t, Z), c(t, Z)$ and their derivatives in Z are uniformly bounded for $Z \in \mathbb{R}^{2n}, t \in [0,1]$. Moreover they are polynomials in t.

If $(\nabla_{t,e_i})^*$ is the adjoint of ∇_{t,e_i} with respect to $\langle \cdot, \cdot \rangle_{t,0}$, (1.38) yields

(1.50) $$(\nabla_{t,e_i})^* = -\nabla_{t,e_i} - t(\kappa^{-1}(e_i\kappa))(tZ).$$

Thus by (1.49) and (1.50), (1.47) is verified for $m = 1$.

By recurrence, it transpires that $[Q_1, [Q_2, \ldots, [Q_m, \mathscr{L}_t]]\ldots]$ has the same structure (1.49) as $\mathscr{L}_t$, so from (1.50) we get the required assertion. $\qquad\square$

Theorem 1.9. *For any $t \in]0, t_0]$, $\lambda \in \delta$, $m \in \mathbb{N}$, the resolvent $(\lambda - \mathscr{L}_t)^{-1}$ maps H_t^m into H_t^{m+1}. Moreover for any $\alpha \in \mathbb{Z}^{2n}$, there exists $C_{\alpha,m} > 0$ such that for $t \in]0,1]$, $\lambda \in \delta$, $s \in \mathscr{C}^\infty(X_0, E_{x_0})$,*

(1.51) $$\|Z^\alpha(\lambda - \mathscr{L}_t)^{-1}s\|_{t,m+1} \leqslant C_{\alpha,m}\sum_{\alpha' \leqslant \alpha}\|Z^{\alpha'}s\|_{t,m}.$$

Proof. For $Q_1, \cdots, Q_m \in \{\nabla_{t,e_i}\}_{i=1}^{2n}$, $Q_{m+1}, \cdots, Q_{m+|\alpha|} \in \{Z_i\}_{i=1}^{2n}$, we can express $Q_1 \cdots Q_{m+|\alpha|}(\lambda - \mathscr{L}_t)^{-1}$ as a linear combination of operators of the type

(1.52) $$[Q_1[Q_2, \ldots [Q_{m'}, (\lambda - \mathscr{L}_t)^{-1}]]\ldots]Q_{m'+1}\cdots Q_{m+|\alpha|}, \quad m' \leqslant m + |\alpha|.$$

Let $\mathscr{R}_t$ be the family of operators $\mathscr{R}_t = \{[Q_{j_1}[Q_{j_2}, \ldots [Q_{j_l}, \mathscr{L}_t]]\ldots]\}$. Clearly, any commutator $[Q_1[Q_2, \ldots [Q_{m'}, (\lambda - \mathscr{L}_t)^{-1}]]\ldots]$ is a linear combination of operators of the form

(1.53) $$(\lambda - \mathscr{L}_t)^{-1}R_1(\lambda - \mathscr{L}_t)^{-1}R_2\cdots R_{m'}(\lambda - \mathscr{L}_t)^{-1}$$

with $R_1, \cdots, R_{m'} \in \mathscr{R}_t$.

From Proposition 1.8 we deduce that the norm $\|\cdot\|_t^{1,-1}$ of the operators $R_j \in \mathscr{R}_t$ is uniformly bounded by C. By Theorem 1.7 there exists $C > 0$, such that the norm $\|\cdot\|_t^{0,1}$ of operators (1.53) is dominated by C. $\qquad\square$

The next step is to convert the estimates for the resolvent into estimates for the spectral projection $\mathcal{P}_{0,t} : (\mathscr{C}^\infty(X_0, E_{x_0}), \|\cdot\|_0) \to (\mathscr{C}^\infty(X_0, E_{x_0}), \|\cdot\|_0)$ of $\mathscr{L}_t$ corresponding to the interval $[-C_{L_0}t^2, C_{L_0}t^2]$. Let $\mathcal{P}_{q,t}(Z, Z') = \mathcal{P}_{q,t,x_0}(Z, Z')$, (with $Z, Z' \in X_0$, $q \geqslant 0$) be the smooth kernel of $\mathcal{P}_{q,t} = (\mathscr{L}_t)^q\mathcal{P}_{0,t}$ (we set $(\mathscr{L}_t)^0 = 1$) with respect to $dv_{TX}(Z')$. Note that $\mathscr{L}_t$ is a family of differential operators on $T_{x_0}X$ with coefficients in $\mathrm{End}(E)_{x_0}$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of TX on X. Then we can view $\mathcal{P}_{q,t}(Z, Z')$ as a smooth section of $\pi^*(\mathrm{End}(E))$ over $TX \times_X TX$ by identifying a section $S \in \mathscr{C}^\infty(TX \times_X TX, \pi^*\mathrm{End}(E))$ with the family $(S_x)_{x \in X}$, where $S_x = S|_{\pi^{-1}(x)}$. Let $\nabla^{\mathrm{End}(E)}$ be the connection on

$\operatorname{End}(E)$ induced by ∇^E. Then $\nabla^{\pi^* \operatorname{End}(E)}$ induces naturally a $\mathscr{C}^m-$norm of S for the parameter $x_0 \in X$.

$\boxed{\text{tue8}}$ **Theorem 1.10.** *For any $m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$, such that for $t \in]0, t_0]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

$\boxed{\text{ue15}}$ (1.54)
$$\sup_{|\alpha|, |\alpha'|, r \leqslant m} \left| \frac{\partial^{|\alpha| + |\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} \mathcal{P}_{q,t}(Z, Z') \right|_{\mathscr{C}^{m'}(X)} \leqslant C.$$

Here $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$ norm for the parameter $x_0 \in X$.

Proof. By (1.43), for any $k \in \mathbb{N}^*$, $q \geqslant 0$,

$\boxed{\text{1ue15}}$ (1.55)
$$\mathcal{P}_{q,t} = (\mathscr{L}_t)^q \mathcal{P}_{0,t} = \frac{1}{2\pi i} \binom{q+k-1}{k-1}^{-1} \int_\delta \lambda^{q+k-1} (\lambda - \mathscr{L}_t)^{-k} d\lambda.$$

For $m \in \mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\{\nabla_{t,e_{i_1}} \cdots \nabla_{t,e_{i_j}}\}_{j \leqslant m}$. From Theorem 1.9, we deduce that if $Q \in \mathcal{Q}^m$, there is $C_m > 0$ such that

$\boxed{\text{ue18}}$ (1.56)
$$\|Q(\lambda - \mathscr{L}_t)^{-m}\|_t^{0,0} \leqslant C_m, \quad \text{for all } \lambda \in \delta.$$

Observe that $\mathscr{L}_t$ is self–adjoint with respect to $\|\cdot\|_{t,0}$, so after taking the adjoint of (1.56), we have

$\boxed{\text{ue20}}$ (1.57)
$$\|(\lambda - \mathscr{L}_t)^{-m} Q\|_t^{0,0} \leqslant C_m.$$

From (1.55), (1.56) and (1.57), we obtain

$\boxed{\text{ue21}}$ (1.58)
$$\|Q \mathcal{P}_{q,t} Q'\|_t^{0,0} \leqslant C_m, \quad \text{for } Q, Q' \in \mathcal{Q}^m.$$

Let $|\cdot|_{(\sigma),m}$ be the usual Sobolev norm on $\mathscr{C}^\infty(B^{T_{x_0}X}(0, \sigma+1), E_{x_0})$ induced by $h^{E_{x_0}}$ and the volume form $dv_{TX}(Z)$ as in (1.38). Let $\|A\|_{(\sigma),m}$ be the operator norm of A with respect to $|\cdot|_{(\sigma),m}$. Observe that by (1.35), (1.38), for $m > 0$, there exists $C_\sigma > 0$ such that for $s \in \mathscr{C}^\infty(X_0, E_{x_0})$, $\operatorname{supp} s \subset B^{T_{x_0}X}(0, \sigma+1)$,

$\boxed{\text{1ue21}}$ (1.59)
$$\frac{1}{C_\sigma} \|s\|_{t,m} \leqslant |s|_{(\sigma),m} \leqslant C_\sigma \|s\|_{t,m}.$$

Now (1.58) and (1.59) together with Sobolev's inequalities imply

$\boxed{\text{ue23}}$ (1.60)
$$\sup_{|Z|, |Z'| \leqslant \sigma} |Q_Z Q'_{Z'} \mathcal{P}_{q,t}(Z, Z')| \leqslant C, \quad \text{for } Q, Q' \in \mathcal{Q}^m.$$

Thanks to (1.35) and (1.60) estimate (1.54) holds for $r = m' = 0$.
To obtain (1.54) for $r \geqslant 1$ and $m' = 0$, note that from (1.55),

$\boxed{\text{ue28}}$ (1.61)
$$\frac{\partial^r}{\partial t^r} \mathcal{P}_{q,t} = \frac{1}{2\pi i} \binom{q+k-1}{k-1}^{-1} \int_\delta \lambda^{q+k-1} \frac{\partial^r}{\partial t^r} (\lambda - \mathscr{L}_t)^{-k} d\lambda, \quad \text{for } k \geqslant 1.$$

Set

$\boxed{\text{ue29}}$ (1.62)
$$I_{k,r} = \left\{ (\mathbf{k}, \mathbf{r}) = (k_i, r_i) \middle| \sum_{i=0}^j k_i = k + j, \ \sum_{i=1}^j r_i = r, \ k_i, r_i \in \mathbb{N}^* \right\}.$$

Then there exist $a_{\mathbf{r}}^{\mathbf{k}} \in \mathbb{R}$ such that

(1.63)
$$A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) = (\lambda - \mathscr{L}_t)^{-k_0} \frac{\partial^{r_1} \mathscr{L}_t}{\partial t^{r_1}} (\lambda - \mathscr{L}_t)^{-k_1} \cdots \frac{\partial^{r_j} \mathscr{L}_t}{\partial t^{r_j}} (\lambda - \mathscr{L}_t)^{-k_j},$$
$$\frac{\partial^r}{\partial t^r} (\lambda - \mathscr{L}_t)^{-k} = \sum_{(\mathbf{k}, \mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t).$$

We claim that $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)$ is well defined and for any $m \in \mathbb{N}$, $k > 2(m + r + 1)$, $Q, Q' \in \mathcal{Q}^m$, there exist $C > 0$, $N \in \mathbb{N}$ such that for $\lambda \in \delta$,

(1.64)
$$\|Q A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) Q' s\|_{t,0} \leq C \sum_{|\beta| \leq 2r} \|Z^\beta s\|_{t,0}.$$

In fact, by (1.34), $\frac{\partial^r}{\partial t^r} \mathscr{L}_t$ is combination of $\frac{\partial^{r_1}}{\partial t^{r_1}} (g^{ij}(tZ)) \left(\frac{\partial^{r_2}}{\partial t^{r_2}} \nabla_{t,e_i}\right) \left(\frac{\partial^{r_3}}{\partial t^{r_3}} \nabla_{t,e_j}\right)$, $\frac{\partial^{r_1}}{\partial t^{r_1}} (d(tZ))$, $\frac{\partial^{r_1}}{\partial t^{r_1}} (d_i(tZ)) \left(\frac{\partial^{r_2}}{\partial t^{r_2}} \nabla_{t,e_i}\right)$. Now $\frac{\partial^{r_1}}{\partial t^{r_1}} (d(tZ))$ (resp. $\frac{\partial^{r_1}}{\partial t^{r_1}} \nabla_{t,e_i}$) ($r_1 \geq 1$), are functions of the type as $d'(tZ)Z^\beta$, $|\beta| \leq r_1$ (resp. $r_1 + 1$) and $d'(Z)$ and its derivatives on Z are bounded smooth functions on Z.

Let $\mathscr{R}_t'$ be the family of operators of the type

$$\mathscr{R}_t' = \{[f_{j_1} Q_{j_1}, [f_{j_2} Q_{j_2}, \ldots [f_{j_l} Q_{j_l}, \mathscr{L}_t]] \ldots]\}$$

with f_{j_i} smooth bounded (with its derivatives) functions and $Q_{j_i} \in \{\nabla_{t,e_l}, Z_l\}_{l=1}^{2n}$.

Now for the operator $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) Q'$, we will move first all the term Z^β in $d'(tZ)Z^\beta$ as above to the right hand side of this operator, to do so, we always use the commutator trick, i.e., each time, we consider only the commutation for Z_i, not for Z^β with $|\beta| > 1$. Then $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) Q'$ is as the form $\sum_{|\beta| \leq 2r} L_\beta^t Q_\beta'' Z^\beta$, and Q_β'' is obtained from Q' and its commutation with Z^β. Now we move all the terms ∇_{t,e_i} in $\frac{\partial^{r_j} \mathscr{L}_t}{\partial t^{r_j}}$ to the right hand side of the operator L_β^t. Then as in the proof of Theorem 1.9, we get finally that $Q A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) Q'$ is as the form $\sum_{|\beta| \leq 2r} \mathscr{L}_\beta^t Z^\beta$ where $\mathscr{L}_\beta^t$ is a linear combination of operators of the form

$$Q(\lambda - \mathscr{L}_t)^{-k_0'} R_1 (\lambda - \mathscr{L}_t)^{-k_1'} R_2 \cdots R_{l'} (\lambda - \mathscr{L}_t)^{-k_{l'}'} Q''' Q'',$$

with $R_1, \cdots, R_{l'} \in \mathscr{R}_t'$, $Q''' \in \mathcal{Q}^l$, $Q'' \in \mathcal{Q}^m$, $|\beta| \leq 2r$, and Q'' is obtained from Q' and its commutation with Z^β. By the argument as in (1.56) and (1.57), as $k > 2(m + r + 1)$, we can split the above operator to two parts

$$Q(\lambda - \mathscr{L}_t)^{-k_0'} R_1 (\lambda - \mathscr{L}_t)^{-k_1'} R_2 \cdots R_i (\lambda - \mathscr{L}_t)^{-k_i''};$$
$$(\lambda - \mathscr{L}_t)^{-(k_i' - k_i'')} \cdots R_{l'} (\lambda - \mathscr{L}_t)^{-k_{l'}'} Q''' Q'',$$

and the $\| \quad \|_t^{0,0}$-norm of each part is bounded by C for $\lambda \in \delta$. Thus the proof of (1.64) is complete.

By (1.61), (1.63) and the above argument, we get the estimate (1.54) with $m' = 0$.

Finally, for any vector U on X,

(1.65)
$$\nabla_U^{\pi^* \operatorname{End}(E)} \mathcal{P}_{q,t} = \frac{1}{2\pi i} \binom{q + k - 1}{k - 1}^{-1} \int_\delta \lambda^{q+k-1} \nabla_U^{\pi^* \operatorname{End}(E)} (\lambda - \mathscr{L}_t)^{-k} d\lambda.$$

Now we use a similar formula as (1.63) for $\nabla_U^{\pi^* \operatorname{End}(E)} (\lambda - \mathscr{L}_t)^{-k}$ by replacing $\frac{\partial^{r_1} \mathscr{L}_t}{\partial t^{r_1}}$ by $\nabla_U^{\pi^* \operatorname{End}(E)} \mathscr{L}_t$, and remark that $\nabla_U^{\pi^* \operatorname{End}(E)} \mathscr{L}_t$ is a differential operator on $T_{x_0} X$ with

the same structure as $\mathscr{L}_t$. Then by the above argument, we conclude that (1.54) holds for $m' \geqslant 1$.   □

For k big enough, set

$$ue31 \quad (1.66) \qquad F_{q,r} = \frac{1}{2\pi i\, r!} \binom{q+k-1}{k-1}^{-1} \int_\delta \lambda^{q+k-1} \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda,0)d\lambda,$$

$$F_{q,r,t} = \frac{1}{r!}\frac{\partial^r}{\partial t^r}\mathcal{P}_{q,t} - F_{q,r}.$$

Let $F_{q,r}(Z,Z')$ $(Z,Z' \in T_{x_0}X)$ be the smooth kernel of $F_{q,r}$ with respect to $dv_{TX}(Z')$. Then $F_{q,r} \in \mathscr{C}^\infty(TX \times_X TX, \pi^*\operatorname{End}(E))$. Certainly, as $t \to 0$, the limit of $\|\ \ \|_{t,m}$ exists, and we denote it by $\|\ \ \|_{0,m}$.

Theorem 1.11. *For any* $r \geq 0$, $k > 0$, *there exists* $C > 0$ *such that for* $t \in [0,t_0]$, $\lambda \in \delta$,

$$ue32 \quad (1.67) \qquad \left\|\left(\frac{\partial^r \mathscr{L}_t}{\partial t^r} - \frac{\partial^r \mathscr{L}_t}{\partial t^r}\Big|_{t=0}\right)s\right\|_{t,-1} \leqslant Ct \sum_{|\alpha|\leqslant r+3} \|Z^\alpha s\|_{0,1},$$

$$\left\|\left(\frac{\partial^r}{\partial t^r}(\lambda - \mathscr{L}_t)^{-k} - \sum_{(\mathbf{k},\mathbf{r})\in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda,0)\right)s\right\|_{0,0} \leqslant Ct \sum_{|\alpha|\leqslant 4r+3} \|Z^\alpha s\|_{0,0}.$$

Proof. Note that by (1.35), (1.38), for $t \in [0,1]$, $k \geqslant 1$,

$$1ue35 \quad (1.68) \qquad \|s\|_{t,0} = \|s\|_{0,0}, \quad \|s\|_{t,k} \leqslant C \sum_{|\alpha|\leqslant k} \|Z^\alpha s\|_{0,k}.$$

An application of Taylor expansion for (1.34) leads to the following estimate for compactly supported s, s':

$$ue33 \quad (1.69) \qquad \left|\left\langle\left(\frac{\partial^r \mathscr{L}_t}{\partial t^r} - \frac{\partial^r \mathscr{L}_t}{\partial t^r}\Big|_{t=0}\right)s, s'\right\rangle_{0,0}\right| \leqslant Ct\|s'\|_{t,1} \sum_{|\alpha|\leqslant r+3} \|Z^\alpha s\|_{0,1}.$$

Thus we get the first inequality of (1.67). Note that

$$ue34 \quad (1.70) \qquad (\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1} = (\lambda - \mathscr{L}_t)^{-1}(\mathscr{L}_t - \mathscr{L}_0)(\lambda - \mathscr{L}_0)^{-1}.$$

After taking the limit, we know that Theorems 1.7, 1.8, 1.9 still hold for $t = 0$. Now from Theorem 1.9 for $\mathscr{L}_0$, (1.69) and (1.70),

$$0ue34 \quad (1.71) \qquad \left\|\left((\lambda - \mathscr{L}_t)^{-1} - (\lambda - \mathscr{L}_0)^{-1}\right)s\right\|_{0,0} \leqslant Ct \sum_{|\alpha|\leqslant 3} \|Z^\alpha s\|_{0,0}.$$

Note that $\nabla_{0,e_j} = \nabla_{e_j} + \frac{1}{2}R_{x_0}^L(\mathcal{R}, e_j)$ by (1.35). If we denote by $\mathscr{L}_{\lambda,t} = \lambda - \mathscr{L}_t$, then

$$0ue35 \quad (1.72) \quad A_{\mathbf{r}}^{\mathbf{k}}(\lambda,t) - A_{\mathbf{r}}^{\mathbf{k}}(\lambda,0) = \sum_{i=1}^{j} \mathscr{L}_{\lambda,t}^{-k_0}\cdots\left(\frac{\partial^{r_i}\mathscr{L}_t}{\partial t^{r_i}} - \frac{\partial^{r_i}\mathscr{L}_t}{\partial t^{r_i}}\Big|_{t=0}\right)\mathscr{L}_{\lambda,0}^{-k_i}\cdots\mathscr{L}_{\lambda,0}^{-k_j}$$

$$+ \sum_{i=0}^{j} \mathscr{L}_{\lambda,t}^{-k_0}\cdots\left(\mathscr{L}_{\lambda,t}^{-k_i} - \mathscr{L}_{\lambda,0}^{-k_i}\right)\left(\frac{\partial^{r_{i+1}}\mathscr{L}_t}{\partial t^{r_{i+1}}}\Big|_{t=0}\right)\cdots\mathscr{L}_{\lambda,0}^{-k_j}.$$

From the discussion after (1.64), formulas (1.42), (1.63) and (1.71), we get the second inequality of (1.67).   □

tue12 **Theorem 1.12.** *For $\sigma > 0$, there exists $C > 0$ such that for $t \in]0,1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

ue42 (1.73)
$$\left| F_{q,r,t}(Z,Z') \right| \leqslant C t^{1/2(2n+1)}.$$

Proof. By (1.61), (1.66) and (1.67), there exists $C > 0$ such that for $t \in]0,1]$,

ue39 (1.74)
$$\| F_{q,r,t} \|_{(\sigma),0} \leqslant Ct.$$

Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function with compact support, equal 1 near 0, such that $\int_{T_{x_0}X} \phi(Z) dv_{TX}(Z) = 1$. Take $\nu \in]0,1]$. By the proof of Theorem 1.10 and (1.66), $F_{q,r}$ verifies the similar inequality as in (1.54) with $r = 0$. Thus by (1.54), there exists $C > 0$ such that if $|Z|, |Z'| \leqslant \sigma$, $U, U' \in E_{x_0}$,

ue43 (1.75) $\left| \langle F_{q,r,t}(Z,Z')U, U' \rangle - \int_{T_{x_0}X \times T_{x_0}X} \langle F_{q,r,t}(Z - W, Z' - W')U, U' \rangle \right.$
$$\left. \frac{1}{\nu^{4n}} \phi(W/\nu) \phi(W'/\nu) dv_{TX}(W) dv_{TX}(W') \right| \leqslant C\nu |U||U'|.$$

On the other hand, by (1.74),

ue44 (1.76) $\left| \int_{T_{x_0}X \times T_{x_0}X} \langle F_{q,r,t}(Z - W, Z' - W')U, U' \rangle \right.$
$$\left. \frac{1}{\nu^{4n}} \phi(W/\nu) \phi(W'/\nu) dv_{TX}(W) dv_{TX}(W') \right| \leqslant C t \frac{1}{\nu^{2n}} |U||U'|.$$

By taking $\nu = t^{1/2(2n+1)}$, we obtain (1.73). $\qquad \square$

Finally, we obtain the following off-diagonal estimate for the kernel of $\mathcal{P}_{q,t}$.

tue14 **Theorem 1.13.** *For $k, m, m' \in \mathbb{N}$, $\sigma > 0$, there exists $C > 0$ such that if $t \in]0,1]$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \sigma$,*

0ue45 (1.77)
$$\sup_{|\alpha|, |\alpha'| \leqslant m} \left| \frac{\partial^{|\alpha| + |\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\mathcal{P}_{q,t} - \sum_{r=0}^{k} F_{q,r} t^r \right) (Z,Z') \right|_{\mathscr{C}^{m'}(X)} \leqslant C t^{k+1}.$$

Proof. By (1.66), and (1.73),

0ue47 (1.78)
$$\frac{1}{r!} \frac{\partial^r}{\partial t^r} \mathcal{P}_{q,t} |_{t=0} = F_{q,r}.$$

Now by Theorem 1.10 and (1.66), $F_{q,r}$ has the same estimate as $\frac{1}{r!} \frac{\partial^r}{\partial t^r} \mathcal{P}_{q,t}$, in (1.54). Again from (1.54), (1.66), and the Taylor expansion

0ue46 (1.79)
$$G(t) - \sum_{r=0}^{k} \frac{1}{r!} \frac{\partial^r G}{\partial t^r}(0) t^r = \frac{1}{k!} \int_0^t (t - t_0)^k \frac{\partial^{k+1} G}{\partial t^{k+1}}(t_0) dt_0,$$

we have (1.77). $\qquad \square$

s3.4

1.4. **Bergman kernel of $\mathscr{L}_0$.** The almost complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. We denote by $\det_{\mathbb{C}}$ the determinant function on the complex bundle $T^{(1,0)}X$. Set

0ue62   (1.80) $$\mathscr{J} = -2\pi\sqrt{-1}\mathbf{J}.$$

By (0.1), $\mathscr{J} \in \operatorname{End}(T^{(1,0)}X)$ is positive, and $\mathscr{J}$ acting on TX is skew-adjoint. For any tensor ψ on X, we denote by $\nabla^X\psi$ the covariant derivative of ψ induced by ∇^{TX}. Thus $\nabla^X\mathscr{J}, \nabla^X J \in T^*X \otimes \operatorname{End}(TX)$, $\nabla^X\nabla^X\mathscr{J} \in T^*X \otimes T^*X \otimes \operatorname{End}(TX)$. We also adopt the convention that all tensors will be evaluated at the base point $x_0 \in X$, and most of the time, we will omit the subscript x_0.

Let P^N be the orthogonal projection from $(L^2(\mathbb{R}^{2n}, E_{x_0}), \| \ \|_0 = \| \ \|_{t,0})$ onto $N = \operatorname{Ker}\mathscr{L}_0$, and let $P^N(Z, Z')$ be the smooth kernel of P^N with respect to $dv_{T_{x_0}X}(Z)$. Then $P^N(Z, Z')$ is the the Bergman kernel of $\mathscr{L}_0$. For $Z, Z' \in T_{x_0}X$, we have

ue62   (1.81) $$P^N(Z, Z') = \frac{\det_{\mathbb{C}}\mathscr{J}_{x_0}}{(2\pi)^n}\exp\Big(-\frac{1}{4}\big\langle (\mathscr{J}_{x_0}^2)^{1/2}(Z - Z'), (Z - Z')\big\rangle + \frac{1}{2}\langle\mathscr{J}_{x_0}Z, Z'\rangle\Big).$$

Now we discuss the eigenvalues and eigenfunctions of $\mathscr{L}_0$ in more precise way. We choose $\{w_i\}_{i=1}^n$ an orthonormal basis of $T_{x_0}^{(1,0)}X$, such that

0ue52   (1.82) $$\mathscr{J}_{x_0} = \operatorname{diag}(a_1, \cdots, a_n) \in \operatorname{End}(T_{x_0}^{(1,0)}X),$$

with $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, and let $\{w^j\}_{j=1}^n$ be its dual basis. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \ldots, n$ forms an orthonormal basis of $T_{x_0}X$. We use the coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ induced by $\{e_i\}$ as in (1.19) and in what follows we also introduce the complex coordinates $z = (z_1, \cdots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Thus $Z = z + \overline{z}$, and $w_i = \sqrt{2}\frac{\partial}{\partial z_i}$, $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial\overline{z}_i}$. We will also identify z to $\sum_i z_i\frac{\partial}{\partial z_i}$ and $\overline{z}$ to $\sum_i \overline{z}_i\frac{\partial}{\partial\overline{z}_i}$ when we consider z and $\overline{z}$ as vector fields. Remark that

g0   (1.83) $$\left|\frac{\partial}{\partial z_i}\right|^2 = \left|\frac{\partial}{\partial\overline{z}_i}\right|^2 = \frac{1}{2}, \quad \text{so that } |z|^2 = |\overline{z}|^2 = \frac{1}{2}|Z|^2.$$

It is very useful to rewrite $\mathscr{L}_0$ by using the creation and annihilation operators. Set

0g1   (1.84) $$\nabla_{0,\cdot} = \nabla_\cdot + \frac{1}{2}R_{x_0}^L(\mathcal{R}, \cdot), \quad b_i = -2\nabla_{0,\frac{\partial}{\partial z_i}}, \quad b_i^+ = 2\nabla_{0,\frac{\partial}{\partial\overline{z}_i}}, \quad b = (b_1, \cdots, b_n).$$

Then by (1.80), and (1.82), we have

g1   (1.85) $$b_i = -2\frac{\partial}{\partial z_i} + \frac{1}{2}a_i\overline{z}_i, \quad b_i^+ = 2\frac{\partial}{\partial\overline{z}_i} + \frac{1}{2}a_i z_i,$$

and for any polynomial $g(z, \overline{z})$ on z and $\overline{z}$,

g2   (1.86) $$[b_i, b_j^+] = b_i b_j^+ - b_j^+ b_i = -2a_i\delta_{ij},$$
$$[b_i, b_j] = [b_i^+, b_j^+] = 0,$$
$$[g(z, \overline{z}), b_j] = 2\frac{\partial}{\partial z_j}g(z, \overline{z}), \quad [g(z, \overline{z}), b_j^+] = -2\frac{\partial}{\partial\overline{z}_j}g(z, \overline{z}).$$

By (0.2), (1.82), $\tau_{x_0} = \sum_i a_i$. Thus from (1.30), (1.82), (1.84)-(1.86),

$$\boxed{\text{g3}} \quad (1.87) \qquad \mathscr{L}_0 = \sum_i b_i b_i^+.$$

$\boxed{\text{r3.4}}$ *Remark* 1.14. Let $L = \mathbb{C}$ be the trivial holomorphic line bundle on $\mathbb{C}^n$ with the canonical section 1. Let h^L be the metric on L defined by $|1|_{h^L}(z) := e^{-\frac{1}{4}\sum_{j=1}^n a_j|z_j|^2} =:$ $h(z)$ for $z \in \mathbb{C}^n$. Let $g^{T\mathbb{C}^n}$ be the euclidian metric on $\mathbb{C}^n$. Then $\mathscr{L}_0$ is twice the corresponding Kodaira-Laplacian $\overline{\partial}^{L*}\overline{\partial}^L$ under the trivialization of L by using the unit section $e^{\frac{1}{4}\sum_{j=1}^n a_j|z_j|^2} 1$. Indeed, let $\overline{\partial}^*$ be the adjoint of the Dolbeault operator $\overline{\partial}$ associated to L with the trivial metric on L. In fact, under the canonical trivialization by 1,

$$\boxed{\text{ag3}} \quad (1.88) \qquad \overline{\partial}^L = \overline{\partial}, \quad \overline{\partial}^{L*} = h^{-2}\overline{\partial}^* h^2.$$

Set

$$\boxed{\text{ag4}} \quad (1.89) \qquad \overline{\partial}_h = h\overline{\partial}h^{-1}, \quad \overline{\partial}_h^* = h^{-1}\overline{\partial}^* h.$$

Then

$$b_j^+ = 2[i_{\frac{\partial}{\partial \overline{z}_j}}, \overline{\partial}_h], \quad b_j = [\overline{\partial}_h^*, d\overline{z}_j \wedge],$$

$$\boxed{\text{ag5}} \quad (1.90)$$

$$\overline{\partial}_h = \frac{1}{2}\sum_j d\overline{z}_j \wedge b_j^+, \quad \overline{\partial}_h^* = \sum_j i_{\frac{\partial}{\partial \overline{z}_j}} b_j.$$

Under the trivialization by $h^{-1} \cdot 1$, we know the Kodaira-Laplacian $\overline{\partial}^{L*}\overline{\partial}^L + \overline{\partial}^L\overline{\partial}^{L*}$ is $h(\overline{\partial}^{L*}\overline{\partial}^L + \overline{\partial}^L\overline{\partial}^{L*})h^{-1} = \overline{\partial}_h^*\overline{\partial}_h + \overline{\partial}_h\overline{\partial}_h^*$, and its restriction on functions is $\frac{1}{2}\mathscr{L}_0$.

$\boxed{\text{t3.4}}$ **Theorem 1.15.** *The spectrum of the restriction of $\mathscr{L}_0$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\boxed{\text{g4}} \quad (1.91) \qquad \mathrm{Spec}\, \mathscr{L}_0|_{L^2(\mathbb{R}^{2n})} = \left\{ 2\sum_{i=1}^n \alpha_i a_i \,:\, \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \right\}$$

and an orthogonal basis of the eigenspace of $2\sum_{i=1}^n \alpha_i a_i$ is given by

$$\boxed{\text{g5}} \quad (1.92) \qquad b^\alpha\Big(z^\beta \exp\big(-\frac{1}{4}\sum_i a_i|z_i|^2\big)\Big), \quad \text{with } \beta \in \mathbb{N}^n.$$

Proof. At first $z^\beta \exp\big(-\frac{1}{4}\sum_i a_i|z_i|^2\big)$, $\beta \in \mathbb{N}^n$ are annihilated by b_i^+ $(1 \leq i \leq n)$, thus they are in the kernel of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$. Now, by (1.86), (1.92) are eigenfunctions of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$ with eigenvalue $2\sum_{i=1}^n \alpha_i a_i$. But the span of functions (1.92) includes all the rescaled Hermite polynomials multiplied by $\exp\big(-\frac{1}{4}\sum_i a_i|z_i|^2\big)$, which is an orthogonal basis of $L^2(\mathbb{R}^{2n})$ by [44, §6]. Thus the eigenfunctions in (1.92) are all the eigenfunctions of $\mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$. The proof of Theorem 1.15 is complete. $\square$

Especially an orthonormal basis of $\mathrm{Ker}\, \mathscr{L}_0|_{L^2(\mathbb{R}^{2n})}$ is

$$\boxed{\text{g6}} \quad (1.93) \qquad \Big(\frac{a^\beta}{(2\pi)^n 2^{|\beta|}\beta!}\prod_{i=1}^n a_i\Big)^{1/2} z^\beta \exp\Big(-\frac{1}{4}\sum_{j=1}^n a_j|z_j|^2\Big), \quad \beta \in \mathbb{N}^n.$$

From (1.93), we recover (1.81):

g7   (1.94) $$P^N(Z, Z') = \frac{1}{(2\pi)^n} \prod_{i=1}^n a_i \ \exp\Big(-\frac{1}{4}\sum_i a_i\big(|z_i|^2 + |z_i'|^2 - 2z_i\overline{z}_i'\big)\Big).$$

Recall that the operators $\mathcal{O}_1, \mathcal{O}_2$ were defined in (1.30). Theorem 1.16 below is crucial in proving the vanishing result of $F_{q,r}$ (cf. Theorem 1.18).

t3.5   **Theorem 1.16.** *We have the relation*

g8   (1.95) $$P^N\mathcal{O}_1 P^N = 0.$$

Proof. From (0.1), for $U, V, W \in TX$, $\langle(\nabla_U^X \mathbf{J})V, W\rangle = (\nabla_U^X \omega)(V, W)$, thus

g9   (1.96) $$\langle(\nabla_U^X \mathbf{J})V, W\rangle + \langle(\nabla_V^X \mathbf{J})W, U\rangle + \langle(\nabla_W^X \mathbf{J})U, V\rangle = d\omega(U, V, W) = 0.$$

By (0.1) and (0.2),

g10   (1.97) $$\begin{aligned} R^L(U, V) &= \langle \mathcal{J}U, V\rangle, \\ (\nabla_U^X R^L)(V, W) &= \langle(\nabla_U^X \mathcal{J})V, W\rangle, \\ \nabla_U\tau &= -\frac{\sqrt{-1}}{2} \operatorname{Tr}|_{TX}[\nabla_U^X(J\mathcal{J})]. \end{aligned}$$

As $J, \mathcal{J} \in \operatorname{End}(TX)$ are skew-adjoint and commute, $\nabla_U^X J, \nabla_U^X \mathcal{J}$ are skew-adjoint and $\nabla_U^X(J\mathcal{J})$ is symmetric. From $J^2 = -\operatorname{Id}$, we know that

g11   (1.98) $$J(\nabla^X J) + (\nabla^X J)J = 0,$$

thus $\nabla_U^X J$ exchanges $T^{(1,0)}X$ and $T^{(0,1)}X$. From (1.96) and (1.97), we have

g12   (1.99) $$\begin{aligned} (\nabla_{\mathcal{R}}\tau)_{x_0} &= -2\sqrt{-1}\Big\langle(\nabla_{\mathcal{R}}^X(J\mathcal{J}))\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle = 2\Big\langle(\nabla_{\mathcal{R}}^X \mathcal{J})\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle, \\ (\partial_i R^L)_{x_0}(\mathcal{R}, e_i) &= 2\Big\langle(\nabla_{\frac{\partial}{\partial z_i}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle + 2\Big\langle(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial z_i}\Big\rangle \\ &= 4\Big\langle(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial z_i}\Big\rangle - 2\Big\langle(\nabla_{\mathcal{R}}^X \mathcal{J})\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle. \end{aligned}$$

From (1.30), (1.86), (1.97) and (1.99), we infer

g13   (1.100) $$\begin{aligned} \mathcal{O}_1 &= -\frac{2}{3}\Big[\Big\langle(\nabla_{\mathcal{R}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial z_i}\Big\rangle b_i^+ - \Big\langle(\nabla_{\mathcal{R}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle b_i \\ &\qquad + 2\Big\langle(\nabla_{\frac{\partial}{\partial z_i}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle + 2\Big\langle(\nabla_{\mathcal{R}}^X \mathcal{J})\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle\Big] \\ &= -\frac{2}{3}\Big[\Big\langle(\nabla_{\mathcal{R}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial z_i}\Big\rangle b_i^+ - b_i\Big\langle(\nabla_{\mathcal{R}}^X \mathcal{J})\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}\Big\rangle\Big]. \end{aligned}$$

Note that by (1.85), (1.94),

g14   (1.101) $$(b_i^+ P^N)(Z, Z') = 0, \quad (b_i P^N)(Z, Z') = a_i(\overline{z}_i - \overline{z}_i')P^N(Z, Z').$$

We learn from (1.101) that for any polynomial $g(z, \overline{z})$ in $z, \overline{z}$ we can write $g(z, \overline{z})P^N(Z, Z')$ as sums of $b^\beta g_\beta(z, \overline{z}')P^N(Z, Z')$ with $g_\beta(z, \overline{z}')$ polynomials in $z, \overline{z}'$. By Theorem 1.15,

g15   (1.102) $$P^N b^\alpha g(z, \overline{z})P^N = 0, \quad \text{for } |\alpha| > 0,$$

and relations (1.100)–(1.102) yield the desired relation (1.95). $\qquad\square$

s3.5

1.5. **Evaluation of** $F_{q,r}$. For $s \in \mathbb{R}$, let $[s]$ denote the greatest integer which is less than or equal to s. Let $f(\lambda, t)$ be a formal power series with values in $\operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0}))$

c54 (1.103)
$$f(\lambda, t) = \sum_{r=0}^{\infty} t^r f_r(\lambda), \quad f_r(\lambda) \in \operatorname{End}(L^2(\mathbb{R}^{2n}, E_{x_0})).$$

By (1.29), consider the equation of formal power series for $\lambda \in \delta$,

c55 (1.104)
$$\left(-\mathscr{L}_0 + \lambda - \sum_{r=1}^{\infty} t^r \mathcal{O}_r\right) f(\lambda, t) = \operatorname{Id}_{L^2(\mathbb{R}^{2n}, E_{x_0})}.$$

Let $N^\perp$ be the orthogonal space of N in $L^2(\mathbb{R}^{2n}, E_{x_0})$, and $P^{N^\perp}$ be the orthogonal projection from $L^2(\mathbb{R}^{2n}, E_{x_0})$ to $N^\perp$. We decompose $f(\lambda, t)$ according the splitting $L^2(\mathbb{R}^{2n}, E_{x_0}) = N \oplus N^\perp$,

c56 (1.105)
$$g_r(\lambda) = P^N f_r(\lambda), \quad f_r^\perp(\lambda) = P^{N^\perp} f_r(\lambda).$$

Using (1.105) and identifying the powers of t in (1.104), we find that

$$g_0(\lambda) = \frac{1}{\lambda} P^N, \quad f_0^\perp(\lambda) = (\lambda - \mathscr{L}_0)^{-1} P^{N^\perp},$$

c57 (1.106)
$$f_r^\perp(\lambda) = (\lambda - \mathscr{L}_0)^{-1} \sum_{j=1}^{r} P^{N^\perp} \mathcal{O}_j f_{r-j}(\lambda),$$

$$g_r(\lambda) = \frac{1}{\lambda} \sum_{j=1}^{r} P^N \mathcal{O}_j f_{r-j}(\lambda).$$

t3.6 **Lemma 1.17.** *For* $r \in \mathbb{N}$, $\lambda^{[\frac{r}{2}]+1} g_r(\lambda)$, $\lambda^{[\frac{r+1}{2}]} f_r^\perp(\lambda)$ *are holomorphic functions for* $|\lambda| \leqslant \mu_0/4$ *and*

c81 (1.107)
$$(\lambda^{r+1} g_{2r})(0) = (P^N \mathcal{O}_2 P^N - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 P^N)^r P^N.$$

Proof. By (1.106) we know that Lemma 1.17 is true for $r = 0$. Assume that Lemma 1.17 is true for $r \leqslant m$. Now, by Theorem 1.15, (1.106) and the recurrence assumption, it follows that $\lambda^{[\frac{m}{2}]+1} f_{m+1}^\perp(\lambda)$ is holomorphic for $|\lambda| \leqslant \mu_0/4$, and

c82 (1.108)
$$\lambda^{[\frac{m+1}{2}]+1} g_{m+1}(\lambda) = \lambda^{[\frac{m+1}{2}]} \sum_{i=1}^{m+1} P^N \mathcal{O}_i \Big[g_{m+1-i}(\lambda) + f_{m+1-i}^\perp(\lambda) \Big].$$

By our recurrence, $\lambda^{[\frac{m+1}{2}]-1} f_{m+1-j}^\perp(\lambda)$, $\lambda^{[\frac{m+1}{2}]-1} g_{m-j}(\lambda)$, $\lambda^{[\frac{m+1}{2}]} f_m^\perp(\lambda)$ are holomorphic for $|\lambda| \leqslant \mu_0/4$, $j \geqslant 2$. Thus by Theorem 1.16, and (1.106) and (1.108), $\lambda^{[\frac{m+1}{2}]+1} g_{m+1}(\lambda)$ is also holomorphic for $|\lambda| \leqslant \mu_0/4$, and

c84 (1.109)
$$(\lambda^{[\frac{m+1}{2}]+1} g_{m+1})(0) = \Big(\lambda^{[\frac{m+1}{2}]} (P^N \mathcal{O}_1 f_m^\perp + P^N \mathcal{O}_2 g_{m-1}) \Big)(0)$$
$$= -P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 \Big(\lambda^{[\frac{m+1}{2}]} (g_{m-1} + f_{m-1}^\perp) \Big)(0) + (\lambda^{[\frac{m+1}{2}]} P^N \mathcal{O}_2 g_{m-1})(0).$$

If m is odd, then by (1.109) and recurrence assumption,

$$
\boxed{\text{c85}} \quad (1.110) \quad (\lambda^{[\frac{m+1}{2}]+1} g_{m+1})(0) = P^N(-\mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 + \mathcal{O}_2) P^N(\lambda^{[\frac{m-1}{2}]+1} g_{m-1})(0)
$$
$$
= (P^N \mathcal{O}_2 P^N - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 P^N)^{[\frac{m+1}{2}]} P^N.
$$

The proof of Lemma 1.17 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

$\boxed{\text{0t3.6}}$ **Theorem 1.18.** *There exist $J_{q,r}(Z, Z')$ polynomials in Z, Z' with the same parity as r and $\deg J_{q,r}(Z,Z') \leq 3r$, whose coefficients are polynomials in R^{TX}, R^E (and R^L, Φ) and their derivatives of order $\leqslant r-1$ (resp. r), and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x_0, such that*

$$
\boxed{\text{c86}} \quad (1.111) \qquad\qquad F_{q,r}(Z, Z') = J_{q,r}(Z, Z') P^N(Z, Z').
$$

Moreover,

$$
F_{0,0} = P^N,
$$
$$
\boxed{\text{1c52}} \quad (1.112) \qquad F_{q,r} = 0, \quad \textit{for } q > 0, r < 2q,
$$
$$
F_{q,2q} = \left(P^N \mathcal{O}_2 P^N - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 P^N \right)^q P^N \quad \textit{for } q > 0.
$$

Proof. Recall that $\mathcal{P}_{q,t} = (\mathscr{L}_t)^q \mathcal{P}_{0,t}$. By (1.55), $\mathcal{P}_{q,t} = \frac{1}{2\pi i} \int_\delta \lambda^q (\lambda - \mathscr{L}_t)^{-1} d\lambda$. Thus by (1.61), (1.66) and (1.105),

$$
\boxed{\text{c90}} \quad (1.113) \qquad F_{q,r} = \frac{1}{2\pi i} \int_\delta \lambda^q g_r(\lambda) d\lambda + \frac{1}{2\pi i} \int_\delta \lambda^q f_r^\perp(\lambda) d\lambda.
$$

From Lemma 1.17 and (1.113), we get (1.112). Generally, from Theorems 1.4, 1.15, Remark 1.5, (1.94), (1.106), (1.113) and the residue formula, we conclude that $F_{q,r}$ has the form (1.111). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

From Theorems 1.15, 1.16, (1.106), (1.113) and the residue formula, we can get $F_{q,r}$ by using the operators $\mathscr{L}_0^{-1}$, P^N, $P^{N^\perp}$, $\mathcal{O}_k(k \leq r)$. This give us a direct method to compute $F_{q,r}$ in view of Theorem 1.15. In particular, we get [2]

$$
F_{0,1} = - P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^{N^\perp} \mathscr{L}_0^{-1} \mathcal{O}_1 P^N,
$$
$$
F_{0,2} = \frac{1}{2\pi i} \int_\delta \left[(\lambda - \mathscr{L}_0)^{-1} P^{N^\perp} (\mathcal{O}_1 f_1 + \mathcal{O}_2 f_0)(\lambda) + \frac{1}{\lambda} P^N (\mathcal{O}_1 f_1 + \mathcal{O}_2 f_0)(\lambda) \right] d\lambda
$$
$$
\boxed{\text{g51}} \quad (1.114) \qquad = \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 P^N - \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_2 P^N
$$
$$
+ P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^N \mathcal{O}_2 \mathscr{L}_0^{-1} P^{N^\perp}
$$
$$
+ P^{N^\perp} \mathscr{L}_0^{-1} \mathcal{O}_1 P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N^\perp} - P^N \mathcal{O}_1 \mathscr{L}_0^{-2} P^{N^\perp} \mathcal{O}_1 P^N.
$$

---

[2] The formula $F_{0,2}$ in [34, (20)] missed the last two terms here which are zero at $(0,0)$ if $\mathbf{J} = J$, cf. Section 2.3.

`s3.6`

1.6. **Proof of Theorem 0.1.** Recall that $P_{0,q,p} = (\Delta^{X_0}_{p,\Phi_0})^q P_{0,\mathcal{H}_p}$. By (1.26), (1.27), for $Z, Z' \in \mathbb{R}^{2n}$,

`0c53` (1.115) $$P_{0,q,p}(Z, Z') = t^{-2n-2q} \kappa^{-\frac{1}{2}}(Z) \mathcal{P}_{q,t}(Z/t, Z'/t) \kappa^{-\frac{1}{2}}(Z').$$

By (1.24), (1.115), Proposition 1.3, Theorems 1.13 and 1.18, we get the following main technical result of this paper, the near off-diagonal expansion of the generalized Bergman kernels:

`t3.8` **Theorem 1.19.** *For $k, m, m' \in \mathbb{N}$, $k \geqslant 2q$, $\sigma > 0$, there exists $C > 0$ such that if $p \geqslant 1$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leq \sigma/\sqrt{p}$,*

`1c53` (1.116) $$\sup_{|\alpha|+|\alpha'| \leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_{q,p}(Z, Z') \right. \right.$$

$$\left. \left. - \sum_{r=2q}^{k} F_{q,r}(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') p^{-\frac{r}{2}+q} \right) \right|_{\mathscr{C}^{m'}(X)} \leqslant C p^{-\frac{k-m}{2}+q}.$$

Set now $Z = Z' = 0$ in (1.116). By Theorem 1.18, we obtain (0.9) and

`1c54` (1.117) $$b_{q,r}(x_0) = F_{q,2r+2q}(0,0).$$

Hence (0.8) follows from (1.81) and (1.117). The statement about the structure of $b_{q,r}$ follows from Theorems 1.15 and 1.18. To prove the uniformity part of Theorem 0.1, we notice that in the proof of Theorem 1.10, we only use the derivatives of the coefficients of $\mathscr{L}_t$ with order $\leqslant 2n + 2m + m' + 2$. Thus by (1.79), the constants in Theorems 1.10, 1.12 and 1.13 are uniformly bounded, if with respect to a fixed metric g_0^{TX}, the $\mathscr{C}^{2n+2m+m'+4}$–norms on X of the data (g^{TX}, h^L, ∇^L, h^E, ∇^E, J and Φ) are bounded, and g^{TX} is bounded below. Moreover, taking derivatives with respect to the parameters we obtain a similar equation as (1.65), where $x_0 \in X$ plays now a role of a parameter. Thus the $\mathscr{C}^{m'}$–norm in (1.116) can also include the parameters if the $C^{m'}$–norms (with respect to the parameter $x_0 \in X$) of the derivatives of above data with order $\leqslant 2n + 2m + 2k + m' + 4$ are bounded. Hence we can take $C_{k,l}$ in (0.9) independent of g^{TX} under our condition. This achieves the proof of Theorem 0.1.

2. COMPUTING THE COEFFICIENTS $b_{q,r}$

`s4`

In principle, Theorem 1.15, the equations (1.106), (1.113) and the residue formula give us a direct method to calculate $b_{q,r}$ by recurrence. Actually, it is computable for the first few terms $b_{q,r}$ in (0.9) in this way. This Section is organized as follows. In Section 2.1, we will give a simplified formula for $\mathcal{O}_2 P^N$ without the assumption $\mathbf{J} = J$. In Sections 2.2, 2.3, we will compute $b_{q,0}$ and $b_{0,1}$ under the assumption $\mathbf{J} = J$, thus proving Theorem 0.2.

In this Section, we use the notation in Section 1.4, and all tensors will be evaluated at the base point $x_0 \in X$. Recall that the operators $\mathcal{O}_1, \mathcal{O}_2$ were defined in (1.30).

s4.1

2.1. A formula for $\mathcal{O}_2 P^N$. We will use the following Lemma to evaluate $b_{q,r}$ in (0.9).

t4.1 **Lemma 2.1.** *The following relation holds:*

g17 (2.1)
$$
\begin{aligned}
\mathcal{O}_2 P^N = \Big\{ &\frac{1}{3} b_i b_j \Big\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle + \frac{1}{2} b_i \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} (\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \frac{Z^\alpha}{\alpha!} \\
&+ \frac{4}{3} b_j \Big[\Big\langle R^{TX}(\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_i}) \mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle - \Big\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle \Big] + R^E(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) b_i \\
&+ \Big\langle (\nabla^X \nabla^X \mathcal{J})_{(\mathcal{R}, \mathcal{R})} \tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle + 4 \Big\langle R^{TX}(\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j}) \tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle \Big\} P^N \\
&+ \Big(-\frac{1}{3} \mathscr{L}_0 \Big\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j}) \mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle + \frac{1}{9} |(\nabla_{\mathcal{R}}^X \mathcal{J}) \mathcal{R}|^2 - \sum_{|\alpha|=2} (\partial^\alpha \tau)_{x_0} \frac{Z^\alpha}{\alpha!} + \Phi \Big) P^N.
\end{aligned}
$$

Proof. Set

g18 (2.2)
$$
\begin{aligned}
I_1 = &\frac{1}{2} \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} (\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \frac{Z^\alpha}{\alpha!} b_i \\
&- \frac{1}{2} \tfrac{\partial}{\partial \overline{z}_i} \Big(\sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} (\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \frac{Z^\alpha}{\alpha!} \Big) - \frac{1}{2} \tfrac{\partial}{\partial z_i} \Big(\sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} (\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \frac{Z^\alpha}{\alpha!} \Big), \\
I_2 = &\frac{1}{3} \Big\langle R_{x_0}^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle b_i b_j \\
&- \frac{4}{3} \Big[\Big\langle R_{x_0}^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle + \Big\langle R_{x_0}^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle \Big] b_j.
\end{aligned}
$$

From (1.30), (1.80), (1.85), (1.86), (1.97), (2.2), and since $\mathcal{J}$ is purely imaginary,

g19 (2.3)
$$
\begin{aligned}
\mathcal{O}_2 = &I_1 + I_2 - \frac{1}{3} \Big[\mathscr{L}_0, \Big\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j}) \mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle \Big] + R^E(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) b_i \\
&+ \frac{1}{3} \Big[\Big\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial z_i}) \mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle (-2 b_j b_i^+ - 2 a_i \delta_{ij}) + \Big\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle b_i^+ b_j^+ \Big] \\
&+ \Big(\frac{2}{3} \Big\langle R^{TX}(\mathcal{R}, e_j) e_j, \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle - \Big(\frac{1}{2} \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0} \frac{Z^\alpha}{\alpha!} + R^E \Big) (\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i}) \Big) b_i^+ \\
&+ \frac{1}{9} |(\nabla_{\mathcal{R}}^X \mathcal{J}) \mathcal{R}|^2 - \sum_{|\alpha|=2} (\partial^\alpha \tau)_{x_0} \frac{Z^\alpha}{\alpha!} + \Phi.
\end{aligned}
$$

In normal coordinates, $(\nabla_{e_i}^{TX} e_j)_{x_0} = 0$, so from (1.32), at x_0,

g20 (2.4)
$$
\begin{aligned}
\nabla_{e_j} \nabla_{e_i} \langle \mathcal{J} e_k, e_l \rangle &= \Big\langle (\nabla_{e_j}^X \nabla_{e_i}^X \mathcal{J}) e_k + \mathcal{J} (\nabla_{e_j}^{TX} \nabla_{e_i}^{TX} e_k), e_l \Big\rangle + \Big\langle \mathcal{J} e_k, \nabla_{e_j}^{TX} \nabla_{e_i}^{TX} e_l \Big\rangle \\
&= \Big\langle (\nabla_{e_j}^X \nabla_{e_i}^X \mathcal{J}) e_k, e_l \Big\rangle - \frac{1}{3} \Big\langle R^{TX}(e_j, e_i) e_k + R^{TX}(e_j, e_k) e_i, \mathcal{J} e_l \Big\rangle \\
&\quad + \frac{1}{3} \Big\langle R^{TX}(e_j, e_i) e_l + R^{TX}(e_j, e_l) e_i, \mathcal{J} e_k \Big\rangle.
\end{aligned}
$$

From (1.97), (2.4),

$$\boxed{\text{g21}} \quad (2.5) \quad \begin{aligned} \sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}(e_k,e_l)\frac{Z^\alpha}{\alpha!} &= \frac{1}{2}(\nabla_{e_j}\nabla_{e_i}\langle\mathcal{J}e_k,e_l\rangle)_{x_0}Z_iZ_j \\ &= \frac{1}{2}\left\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}e_k,e_l\right\rangle + \frac{1}{6}\left[\left\langle R^{TX}(\mathcal{R},e_l)\mathcal{R},\mathcal{J}e_k\right\rangle - \left\langle R^{TX}(\mathcal{R},e_k)\mathcal{R},\mathcal{J}e_l\right\rangle\right]. \end{aligned}$$

Thus

$$\boxed{\text{g22}} \quad (2.6) \quad \sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}(\mathcal{R},e_l)\frac{Z^\alpha}{\alpha!} = \frac{1}{2}\left\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},e_l\right\rangle + \frac{1}{6}\left\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},e_l\right\rangle.$$

From (1.86), (1.94), (2.2) and (2.6), we know that

$$\boxed{\text{g23}} \quad (2.7) \quad \begin{aligned} I_1 = {} & \frac{1}{2}b_i\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}(\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i})\frac{Z^\alpha}{\alpha!} \\ & + \frac{1}{12}\left[\frac{\partial}{\partial z_i}\left\langle R^{TX}_{x_0}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle - \frac{\partial}{\partial\bar{z}_i}\left\langle R^{TX}_{x_0}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\tfrac{\partial}{\partial z_i}\right\rangle\right] \\ & + \frac{1}{4}\left[\frac{\partial}{\partial z_i}\left\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle - \frac{\partial}{\partial\bar{z}_i}\left\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\tfrac{\partial}{\partial z_i}\right\rangle\right]. \end{aligned}$$

The definition of $\nabla^X\nabla^X\mathcal{J}$, R^{TX} and (1.96) imply, for $U,V,W,Y\in TX$,

$$\boxed{\text{g24}} \quad (2.8) \quad \begin{aligned} &\left\langle R^{TX}(U,V)W,Y\right\rangle = \left\langle R^{TX}(W,Y)U,V\right\rangle, \\ & R^{TX}(U,V)W + R^{TX}(V,W)U + R^{TX}(W,U)V = 0, \\ & (\nabla^X\nabla^X\mathcal{J})_{(U,V)} - (\nabla^X\nabla^X\mathcal{J})_{(V,U)} = [R^{TX}(U,V),\mathcal{J}], \\ & \left\langle(\nabla^X\nabla^X\mathcal{J})_{(Y,U)}V,W\right\rangle + \left\langle(\nabla^X\nabla^X\mathcal{J})_{(Y,V)}W,U\right\rangle + \left\langle(\nabla^X\nabla^X\mathcal{J})_{(Y,W)}U,V\right\rangle = 0. \end{aligned}$$

Note that $\mathcal{J},(\nabla^X\nabla^X\mathcal{J})_{(Y,U)}$ are skew-adjoint, by (1.82) and (2.8),

$$\boxed{\text{g25}} \quad (2.9) \quad \begin{aligned} & \frac{\partial}{\partial z_i}\left\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle - \frac{\partial}{\partial\bar{z}_i}\left\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\mathcal{R},\tfrac{\partial}{\partial z_i}\right\rangle \\ & = \left\langle 2(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\tfrac{\partial}{\partial z_i} + 2(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\frac{\partial}{\partial z_i})}\mathcal{R} + \left[R^{TX}(\tfrac{\partial}{\partial z_i},\mathcal{R}),\mathcal{J}\right]\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle \\ & \quad - \left\langle 2(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\frac{\partial}{\partial\bar{z}_i})}\mathcal{R} + \left[R^{TX}(\tfrac{\partial}{\partial\bar{z}_i},\mathcal{R}),\mathcal{J}\right]\mathcal{R},\tfrac{\partial}{\partial z_i}\right\rangle \\ & = 4\left\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle + \left\langle 2a_iR^{TX}(\mathcal{R},\tfrac{\partial}{\partial z_i})\mathcal{R} - R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle, \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial z_i}\left\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle - \frac{\partial}{\partial\bar{z}_i}\left\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\mathcal{R},\tfrac{\partial}{\partial z_i}\right\rangle \\ & = 2\left\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle + 2a_i\left\langle R^{TX}(\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i})\mathcal{R},\tfrac{\partial}{\partial z_i}\right\rangle \\ & \quad + \left\langle R^{TX}(\tfrac{\partial}{\partial z_i},\mathcal{J}\mathcal{R})\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle - \left\langle R^{TX}(\tfrac{\partial}{\partial\bar{z}_i},\mathcal{J}\mathcal{R})\mathcal{R},\tfrac{\partial}{\partial z_i}\right\rangle \\ & = 3\left\langle R^{TX}(\mathcal{R},\mathcal{J}\mathcal{R})\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle + 2a_i\left\langle R^{TX}(\mathcal{R},\tfrac{\partial}{\partial z_i})\mathcal{R},\tfrac{\partial}{\partial\bar{z}_i}\right\rangle. \end{aligned}$$

Thus by (2.7)-(2.9),

(2.10) $\quad I_1 = \frac{1}{2} b_i \sum_{|\alpha|=2} (\partial^\alpha R^L)_{x_0}(\mathcal{R}, \frac{\partial}{\partial \overline{z}_i}) \frac{Z^\alpha}{\alpha!} + \left\langle (\nabla^X \nabla^X \mathcal{J})_{(\mathcal{R},\mathcal{R})} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} \right\rangle$

$$+ \frac{2}{3} a_i \left\langle R^{TX}(\mathcal{R}, \frac{\partial}{\partial z_i})\mathcal{R}, \frac{\partial}{\partial \overline{z}_i} \right\rangle.$$

Now by (1.86), (2.2) and (2.8),

(2.11) $\quad I_2 = \frac{4}{3} b_j \left[\left\langle R^{TX}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i})\mathcal{R}, \frac{\partial}{\partial \overline{z}_j} \right\rangle - \left\langle R^{TX}(\mathcal{R}, \frac{\partial}{\partial z_i})\frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle \right]$

$$+ \frac{1}{3} b_i b_j \left\langle R^{TX}(\mathcal{R}, \frac{\partial}{\partial \overline{z}_i})\mathcal{R}, \frac{\partial}{\partial \overline{z}_j} \right\rangle - \frac{8}{3} \left\langle R^{TX}(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_i})\frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle$$

$$+ \frac{4}{3} \left[\left\langle R^{TX}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i})\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j} \right\rangle - \left\langle R^{TX}(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_i})\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle \right]$$

$$= \frac{4}{3} b_j \left[\left\langle R^{TX}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i})\mathcal{R}, \frac{\partial}{\partial \overline{z}_j} \right\rangle - \left\langle R^{TX}(\mathcal{R}, \frac{\partial}{\partial z_i})\frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle \right]$$

$$+ \frac{1}{3} b_i b_j \left\langle R^{TX}(\mathcal{R}, \frac{\partial}{\partial \overline{z}_i})\mathcal{R}, \frac{\partial}{\partial \overline{z}_j} \right\rangle + 4 \left\langle R^{TX}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j})\frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle.$$

Finally (1.87), (1.101), (2.3), (2.10) and (2.11), yield (2.1). $\qquad\square$

From (1.97), (1.100), (1.102), (1.112) and (2.1), follows

(2.12) $\quad F_{1,2} = J_{1,2} P^N = \left(2R^E(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}) + 4 \left\langle R^{TX}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j})\frac{\partial}{\partial \overline{z}_i}, \frac{\partial}{\partial \overline{z}_j} \right\rangle + \Phi \right) P^N$

$$+ P^N \left(\left\langle (\nabla^X \nabla^X \mathcal{J})_{(\mathcal{R},\mathcal{R})} \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} \right\rangle + \frac{\sqrt{-1}}{4} \mathrm{Tr}_{|TX} \left(\nabla^X \nabla^X (J\mathcal{J}) \right)_{(\mathcal{R},\mathcal{R})} \right.$$

$$\left. + \frac{1}{9} |(\nabla^X_{\mathcal{R}} \mathcal{J})\mathcal{R}|^2 + \frac{4}{9} \left\langle (\nabla^X_{\mathcal{R}} \mathcal{J})\mathcal{R}, \frac{\partial}{\partial z_i} \right\rangle b_i^+ \mathscr{L}_0^{-1} b_i \left\langle (\nabla^X_{\mathcal{R}} \mathcal{J})\mathcal{R}, \frac{\partial}{\partial \overline{z}_i} \right\rangle \right) P^N.$$

2.2. The coefficients $b_{q,0}$.

In the rest of this Section we assume that $\mathbf{J} = J$. A very useful observation is that (1.96), (1.98) imply

(2.13) $\quad$ $\mathcal{J} = -2\pi\sqrt{-1} J$ and $a_i = 2\pi$ in (1.82), $\tau = 2\pi n$. $\nabla^X_U J$ is skew-adjoint and the tensor $\left\langle (\nabla^X_\cdot J)\cdot, \cdot \right\rangle$ is of the type $(T^{*(1,0)}X)^{\otimes 3} \oplus (T^{*(0,1)}X)^{\otimes 3}$.

Before computing $b_{q,0}$, we establish the relation between the scalar curvature r^X and $|\nabla^X J|^2$.

Lemma 2.2.

(2.14) $$r^X = 8 \left\langle R^{TX}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_j})\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_i} \right\rangle - \frac{1}{4} |\nabla^X J|^2.$$

Proof. By (2.13),

(2.15) $$|\nabla^X J|^2 = 4 \left\langle (\nabla^X_{\frac{\partial}{\partial z_i}} J) e_j, (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J) e_j \right\rangle = 8 \left\langle (\nabla^X_{\frac{\partial}{\partial z_i}} J)\frac{\partial}{\partial z_j}, (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J)\frac{\partial}{\partial \overline{z}_j} \right\rangle.$$

By (1.83), (1.96) and (2.13),

$$
\begin{aligned}
\boxed{\text{g31}} \quad (2.16) \quad \Big\langle (\nabla^X_{\frac{\partial}{\partial z_j}} J)\tfrac{\partial}{\partial z_i}, (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J)\tfrac{\partial}{\partial \overline{z}_j} \Big\rangle &= 2\Big\langle (\nabla^X_{\frac{\partial}{\partial z_j}} J)\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_k} \Big\rangle \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J)\tfrac{\partial}{\partial \overline{z}_j}, \tfrac{\partial}{\partial \overline{z}_k} \Big\rangle \\
&= 2\Big\langle (\nabla^X_{\frac{\partial}{\partial z_i}} J)\tfrac{\partial}{\partial z_k} - (\nabla^X_{\frac{\partial}{\partial z_k}} J)\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j} \Big\rangle \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J)\tfrac{\partial}{\partial \overline{z}_k}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle \\
&= \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J)\tfrac{\partial}{\partial z_k}, (\nabla^X_{\frac{\partial}{\partial z_i}} J)\tfrac{\partial}{\partial \overline{z}_k} \Big\rangle - \Big\langle (\nabla^X_{\frac{\partial}{\partial z_k}} J)\tfrac{\partial}{\partial z_i}, (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J)\tfrac{\partial}{\partial \overline{z}_k} \Big\rangle .
\end{aligned}
$$

By (2.15) and (2.16),

$$
\boxed{\text{g32}} \quad (2.17) \quad \Big\langle (\nabla^X_{\frac{\partial}{\partial \overline{z}_j}} J)\tfrac{\partial}{\partial z_i}, (\nabla^X_{\frac{\partial}{\partial z_i}} J)\tfrac{\partial}{\partial \overline{z}_j} \Big\rangle = \frac{1}{16}|\nabla^X J|^2 .
$$

Now, from (1.98), we get

$$
\boxed{\text{g34}} \quad (2.18) \quad (\nabla^X \nabla^X J)_{(U,V)} J + (\nabla^X_U J)\circ(\nabla^X_V J) + (\nabla^X_V J)\circ(\nabla^X_U J) + J(\nabla^X \nabla^X J)_{(U,V)} = 0.
$$

thus from (2.8), (2.13) and (2.18), for $u_1, u_2, u_3 \in T^{(1,0)}X$, $\overline{v}_1, \overline{v}_2 \in T^{(0,1)}X$,

$$
\boxed{\text{g35}} \quad (2.19) \quad
\begin{aligned}
&(\nabla^X \nabla^X J)_{(u_1,u_2)} u_3,\ (\nabla^X \nabla^X J)_{(\overline{v}_1,\overline{v}_2)} u_3 \in T^{(0,1)}X, \quad (\nabla^X \nabla^X J)_{(u_1,\overline{v}_2)} u_3 \in T^{(1,0)}X, \\
&2\sqrt{-1}\,\big\langle (\nabla^X \nabla^X J)_{(u_1,\overline{v}_1)} u_2, \overline{v}_2 \big\rangle = \big\langle (\nabla^X_{u_1} J) u_2, (\nabla^X_{\overline{v}_1} J)\overline{v}_2 \big\rangle .
\end{aligned}
$$

Formulas (2.8) and (2.19) yield

$$
\begin{aligned}
\boxed{\text{0g35}} \quad (2.20) \quad \big\langle (\nabla^X \nabla^X J)_{(u_1,u_2)}\overline{v}_1, \overline{v}_2 \big\rangle &= -\big\langle (\nabla^X \nabla^X J)_{(u_1,\overline{v}_1)}\overline{v}_2, u_2 \big\rangle - \big\langle (\nabla^X \nabla^X J)_{(u_1,\overline{v}_2)} u_2, \overline{v}_1 \big\rangle \\
&= \frac{1}{2\sqrt{-1}}\big\langle (\nabla^X_{u_1} J) u_2, (\nabla^X_{\overline{v}_1} J)\overline{v}_2 - (\nabla^X_{\overline{v}_2} J)\overline{v}_1 \big\rangle .
\end{aligned}
$$

From (2.8), (2.17) and (2.20), we deduce

$$
\begin{aligned}
\boxed{\text{g36}} \quad (2.21) \quad \Big\langle R^{TX}(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j})\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle &= \frac{\sqrt{-1}}{2}\Big\langle [R^{TX}(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j}), J]\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle \\
&= \frac{\sqrt{-1}}{2}\Big\langle \big((\nabla^X \nabla^X J)_{(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j})} - (\nabla^X \nabla^X J)_{(\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_i})}\big)\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle \\
&= \frac{1}{4}\Big\langle (\nabla^X_{\frac{\partial}{\partial z_i}} J)\tfrac{\partial}{\partial z_j}, (\nabla^X_{\frac{\partial}{\partial \overline{z}_i}} J)\tfrac{\partial}{\partial \overline{z}_j} \Big\rangle = \frac{1}{32}|\nabla^X J|^2 .
\end{aligned}
$$

The scalar curvature r^X of (X, g^{TX}) is

$$
\begin{aligned}
\boxed{\text{g38}} \quad (2.22) \quad r^X &= -\big\langle R^{TX}(e_i, e_j) e_i, e_j \big\rangle = -4\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i}, e_j)\tfrac{\partial}{\partial \overline{z}_i}, e_j \Big\rangle \\
&= -8\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j})\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial \overline{z}_j} \Big\rangle - 8\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial \overline{z}_j})\tfrac{\partial}{\partial \overline{z}_i}, \tfrac{\partial}{\partial z_j} \Big\rangle .
\end{aligned}
$$

In conclusion, relations (2.21) and (2.22) imply (2.14). $\qquad\square$

From (1.100) and (2.13) we know

$$
\boxed{\text{g41}} \quad (2.23) \quad \mathcal{O}_1 = \frac{2}{3} b_i \Big\langle (\nabla^X_{\overline{z}} \mathcal{J})\overline{z}, \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle - \frac{2}{3}\Big\langle (\nabla^X_z \mathcal{J}) z, \tfrac{\partial}{\partial z_i} \Big\rangle b_i^+ .
$$

Hence by (1.86), (1.101), (2.13) and (2.23),

$$
\boxed{\text{g42}} \quad (2.24) \quad
\begin{aligned}
(\mathcal{O}_1 P^N)(Z, Z') &= \frac{2}{3} \Big(b_i \Big\langle (\nabla_{\overline{z}}^X \mathcal{J})\overline{z}, \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle P^N \Big)(Z, Z') \\
&= \frac{2}{3} \Big\{ \Big(\frac{b_i b_j}{2\pi} \Big\langle (\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X \mathcal{J})\overline{z}', \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle + b_i \Big\langle (\nabla_{\overline{z}'}^X \mathcal{J})\overline{z}', \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle \Big) P^N \Big\}(Z, Z').
\end{aligned}
$$

By Theorem 1.15, (1.102) and (2.24), we have

$$
\boxed{\text{0g42}} \quad (2.25) \quad
\begin{aligned}
&(\mathscr{L}_0^{-1} P^{N\perp} \mathcal{O}_1 P^N)(Z, Z') \\
&\quad = \frac{2}{3} \Big\{ \Big(\frac{b_i b_j}{16\pi^2} \Big\langle (\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X \mathcal{J})\overline{z}', \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle + \frac{b_i}{4\pi} \Big\langle (\nabla_{\overline{z}'}^X \mathcal{J})\overline{z}', \tfrac{\partial}{\partial \overline{z}_i} \Big\rangle \Big) P^N \Big\}(Z, Z'), \\
&P^N \mathcal{O}_1 \mathscr{L}_0^{-1} P^{N\perp} \mathcal{O}_1 P^N = -\frac{2}{3} P^N \Big\langle (\nabla_z^X \mathcal{J})z, \tfrac{\partial}{\partial z_k} \Big\rangle b_k^+ \mathscr{L}_0^{-1} P^{N\perp} \mathcal{O}_1 P^N.
\end{aligned}
$$

(1.83), (1.86), (1.101), (2.23) and (2.25) imply

$$
\boxed{\text{g43}} \quad (2.26) \quad
\begin{aligned}
&\frac{2}{3} \Big(\Big\langle (\nabla_z^X \mathcal{J})z, \tfrac{\partial}{\partial z_k} \Big\rangle b_k^+ \mathscr{L}_0^{-1} P^{N\perp} \mathcal{O}_1 P^N \Big)(Z, Z') = \frac{4}{9} \Big\{ \Big\langle (\nabla_z^X \mathcal{J})z, \tfrac{\partial}{\partial z_k} \Big\rangle \\
&\quad \times \Big(-\frac{b_i}{4\pi} \Big\langle (\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X \mathcal{J})\tfrac{\partial}{\partial \overline{z}_k} + (\nabla_{\frac{\partial}{\partial \overline{z}_k}}^X \mathcal{J})\tfrac{\partial}{\partial \overline{z}_i}, \overline{z}' \Big\rangle + \Big\langle (\nabla_{\overline{z}'}^X \mathcal{J})\overline{z}', \tfrac{\partial}{\partial \overline{z}_k} \Big\rangle \Big) P^N \Big\}(Z, Z') \\
&\quad = \Big\{ \Big[-\frac{b_i}{9\pi} \Big\langle (\nabla_z^X \mathcal{J})z, \tfrac{\partial}{\partial z_k} \Big\rangle \Big\langle (\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X \mathcal{J})\tfrac{\partial}{\partial \overline{z}_k} + (\nabla_{\frac{\partial}{\partial \overline{z}_k}}^X \mathcal{J})\tfrac{\partial}{\partial \overline{z}_i}, \overline{z}' \Big\rangle \\
&\quad\quad - \frac{2}{9\pi} \Big\langle (\nabla_z^X \mathcal{J})\tfrac{\partial}{\partial z_i} + (\nabla_{\frac{\partial}{\partial z_i}}^X \mathcal{J})z, \tfrac{\partial}{\partial z_k} \Big\rangle \Big\langle (\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X \mathcal{J})\tfrac{\partial}{\partial \overline{z}_k} + (\nabla_{\frac{\partial}{\partial \overline{z}_k}}^X \mathcal{J})\tfrac{\partial}{\partial \overline{z}_i}, \overline{z}' \Big\rangle \\
&\quad\quad\quad + \frac{2}{9} \Big\langle (\nabla_z^X \mathcal{J})z, (\nabla_{\overline{z}'}^X \mathcal{J})\overline{z}' \Big\rangle \Big] P^N \Big\}(Z, Z').
\end{aligned}
$$

Thanks to (1.98), (1.101), (2.13), (2.15) and (2.16) we obtain

$$
\boxed{\text{g44}} \quad (2.27) \quad
\begin{aligned}
&\frac{1}{9} \Big| (\nabla_{\mathcal{R}}^X \mathcal{J})\mathcal{R} \Big|^2 P^N(Z, Z') = \frac{8\pi^2}{9} \Big\langle (\nabla_z^X J)z, (\nabla_{\overline{z}}^X J)\overline{z} \Big\rangle P^N(Z, Z') \\
&= \frac{8\pi^2}{9} \Big\{ \Big\langle (\nabla_z^X J)z, \frac{b_i b_j}{4\pi^2} (\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j} + \frac{b_i}{2\pi} \Big[(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\overline{z}' + (\nabla_{\overline{z}'}^X J)\tfrac{\partial}{\partial \overline{z}_i} \Big] + (\nabla_{\overline{z}'}^X J)\overline{z}' \Big\rangle P^N \Big\}(Z, Z') \\
&= \frac{8\pi^2}{9} \Big\{ \Big[\Big\langle \frac{b_i b_j}{4\pi^2} (\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j} + \frac{b_i}{2\pi} \Big((\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\overline{z}' + (\nabla_{\overline{z}'}^X J)\tfrac{\partial}{\partial \overline{z}_i} \Big), (\nabla_z^X J)z \Big\rangle \\
&\quad + \frac{b_i}{2\pi^2} \Big\langle (\nabla_{\frac{\partial}{\partial z_j}}^X J)z + (\nabla_z^X J)\tfrac{\partial}{\partial z_j}, (\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j} + (\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\tfrac{\partial}{\partial \overline{z}_i} \Big\rangle \\
&\quad + \frac{1}{\pi} \Big\langle (\nabla_{\frac{\partial}{\partial z_i}}^X J)z + (\nabla_z^X J)\tfrac{\partial}{\partial z_i}, (\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\overline{z}' + (\nabla_{\overline{z}'}^X J)\tfrac{\partial}{\partial \overline{z}_i} \Big\rangle \\
&\quad + \Big\langle (\nabla_z^X J)z, (\nabla_{\overline{z}'}^X J)\overline{z}' \Big\rangle + \frac{3}{16\pi^2} |\nabla^X J|^2 \Big] P^N \Big\}(Z, Z').
\end{aligned}
$$

Taking into account ($\overset{\text{g24}}{2.8}$), ($\overset{\text{g35}}{2.19}$) and $\left\langle [R^{TX}(\overline{z}, z), J]\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} \right\rangle = 0$, we get

$$\overset{\boxed{\text{g45}}}{\text{(2.28)}} \quad \left\langle (\nabla^X \nabla^X J)_{(\mathcal{R}, \mathcal{R})}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} \right\rangle = \left\langle (\nabla^X \nabla^X J)_{(z, \overline{z})}\frac{\partial}{\partial z_i} + (\nabla^X \nabla^X J)_{(\overline{z}, z)}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} \right\rangle$$
$$= -\sqrt{-1}\left\langle (\nabla_z^X J)\frac{\partial}{\partial z_i}, (\nabla_{\overline{z}}^X J)\frac{\partial}{\partial \overline{z}_i} \right\rangle.$$

From ($\overset{\text{g14}}{1.101}$), ($\overset{\text{g30}}{2.15}$) and ($\overset{\text{g45}}{2.28}$),

$$\overset{\boxed{\text{g46}}}{\text{(2.29)}} \quad \left\langle (\nabla^X \nabla^X \mathcal{J})_{(\mathcal{R}, \mathcal{R})}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i} \right\rangle P^N(Z, Z') = -2\pi \left\langle (\nabla_z^X J)\frac{\partial}{\partial z_i}, (\nabla_{\overline{z}}^X J)\frac{\partial}{\partial \overline{z}_i} \right\rangle P^N(Z, Z')$$
$$= -2\pi\left\{ \left\langle (\nabla_z^X J)\frac{\partial}{\partial z_i}, \frac{b_j}{2\pi}(\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\frac{\partial}{\partial \overline{z}_i} + (\nabla_{\overline{z}'}^X J)\frac{\partial}{\partial \overline{z}_i} \right\rangle P^N\right\}(Z, Z')$$
$$= -\left\{ \left[\left\langle b_j(\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\frac{\partial}{\partial \overline{z}_i} + 2\pi(\nabla_{\overline{z}'}^X J)\frac{\partial}{\partial \overline{z}_i}, (\nabla_z^X J)\frac{\partial}{\partial z_i} \right\rangle + \frac{1}{4}|\nabla^X J|^2 \right] P^N\right\}(Z, Z').$$

Recall that the polynomial $J_{q,2q}(Z, Z')$ was defined in ($\overset{\text{c86}}{1.111}$). From $J\mathcal{J} = 2\pi\sqrt{-1}$, ($\overset{\text{g15}}{1.102}$), ($\overset{\text{0g27}}{2.12}$), ($\overset{\text{g36}}{2.21}$) and ($\overset{\text{g44}}{2.23}$)-($\overset{\text{g46}}{2.29}$), $J_{1,2}(Z, Z')$ is a polynomial on $z, \overline{z}'$, and each monomial of $J_{1,2}$ has the same degree in z and $\overline{z}'$; moreover

$$\overset{\boxed{\text{g48}}}{\text{(2.30)}} \quad J_{1,2}(0,0) = \frac{1}{24}|\nabla^X J|_{x_0}^2 + 2R_{x_0}^E\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}\right) + \Phi_{x_0}.$$

Using ($\overset{\text{g7}}{1.94}$), ($\overset{\text{1c52}}{1.112}$) and the recurrence, we infer that each monomial of $J_{q,2q}$ has the same degree in z and $\overline{z}'$, and

$$\overset{\boxed{\text{g49}}}{\text{(2.31)}} \quad J_{q,2q}(0,0) = (J_{1,2}(0,0))^q.$$

In view of ($\overset{\text{g7}}{1.94}$), ($\overset{\text{c86}}{1.111}$), ($\overset{\text{1c54}}{1.117}$), ($\overset{\text{g48}}{2.30}$) and ($\overset{\text{g49}}{2.31}$) we obtain ($\overset{\text{0.7}}{0.11}$).

s4.3

2.3. **The coefficient** $b_{0,1}$. By ($\overset{\text{1c54}}{1.117}$), we need to compute $F_{0,2}(0,0)$. By ($\overset{\text{g42}}{2.24}$) and ($\overset{\text{0g42}}{2.25}$), we know that

$$\overset{\boxed{\text{0g52}}}{\text{(2.32)}} \quad (\mathcal{O}_1 P^N)(Z, 0) = 0, \quad (\mathcal{L}_0^{-1} P^{N^\perp}\mathcal{O}_1 P^N)(0, Z') = 0.$$

Thus the first and last two terms in ($\overset{\text{g51}}{1.114}$) are zero at $(0,0)$. Thus we only need to compute $-(\mathcal{L}_0^{-1} P^{N^\perp}\mathcal{O}_2 P^N)(0,0)$, since the third and fourth terms in ($\overset{\text{g51}}{1.114}$) are adjoint of the first two terms by Remark $\overset{\text{0t3.3}}{1.5}$.

Let $h_i(z)$ and $f_{ij}(z)$, $(i, j = 1, \cdots, n)$ be arbitrary polynomials in z. By Theorem $\overset{\text{t3.4}}{1.15}$, ($\overset{\text{g2}}{1.86}$), ($\overset{\text{g7}}{1.94}$) and ($\overset{\text{g14}}{1.101}$), we have

$$\overset{\boxed{\text{g52}}}{\text{(2.33)}} \quad (b_i h_i P^N)(0,0) = -2\frac{\partial h_i}{\partial z_i}(0), \quad (b_i b_j f_{ij} P^N)(0,0) = 4\frac{\partial^2 f_{ij}}{\partial z_i \partial z_j}(0),$$
$$(\mathcal{L}_0^{-1} b_i f_{ij} b_j P^N)(0,0) = -\frac{1}{2\pi}\frac{\partial^2 f_{ij}}{\partial z_i \partial z_j}(0).$$

Owing to Theorem 1.15, (2.15), (2.17), (2.27) and (2.33),

$$
\text{(2.34)} \quad -\frac{1}{9}\Big(\mathscr{L}_0^{-1}P^{N\perp}\big|(\nabla_{\mathcal{R}}^X\mathcal{J})\mathcal{R}\big|^2 P^N\Big)(0,0) = -\frac{8}{9}\bigg\{\Big[\frac{b_ib_j}{32\pi}\Big\langle(\nabla_z^X J)z,(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j}\Big\rangle
$$

$$
+\frac{b_i}{8\pi}\Big\langle(\nabla_{\frac{\partial}{\partial z_j}}^X J)z+(\nabla_z^X J)\tfrac{\partial}{\partial z_j},(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j}+(\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle\Big]P^N\bigg\}(0,0)
$$

$$
=\frac{1}{9\pi}\Big\langle(\nabla_{\frac{\partial}{\partial z_i}}^X J)\tfrac{\partial}{\partial z_j}+(\nabla_{\frac{\partial}{\partial z_j}}^X J)\tfrac{\partial}{\partial z_i},(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j}+2(\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle=\frac{1}{16\pi}|\nabla^X J|^2,
$$

and by Theorem 1.15, (2.29) and (2.33),

$$
\text{(2.35)} \quad -\Big(\mathscr{L}_0^{-1}P^{N\perp}\Big\langle(\nabla^X\nabla^X\mathcal{J})_{(\mathcal{R},\mathcal{R})}\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle P^N\Big)(0,0)
$$

$$
=\Big(\frac{b_j}{4\pi}\Big\langle(\nabla_z^X J)\tfrac{\partial}{\partial z_i},(\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle P^N\Big)(0,0)=-\frac{1}{16\pi}|\nabla^X J|^2.
$$

Observe that by (1.101), for a polynomial $g(z)$ in z, the constant term of $\frac{1}{P^N}\frac{b^\alpha}{2^{|\alpha|}}g(z)P^N$ is the constant term of $(\tfrac{\partial}{\partial z})^\alpha g$. Thus in the term $-\mathscr{L}_0^{-1}P^{N\perp}\mathcal{O}_2 P^N$, by (1.101), the contribution of $\frac{1}{2}b_i\sum_{|\alpha|=2}(\partial^\alpha R^L)_{x_0}(\mathcal{R},\tfrac{\partial}{\partial \overline{z}_i})\frac{Z^\alpha}{\alpha!}$ in $\mathcal{O}_2$ consists of the terms whose total degree of b_i and $\overline{z}_j$ is same as the degree of z. Hence we only need to consider the contribution from the terms where the degree of z is 2. By (2.6), (2.13), (2.19), (2.20) and $\Big\langle[R^{TX}(\overline{z},z),\mathcal{J}]z,\tfrac{\partial}{\partial z_i}\Big\rangle=0$, this term is

$$
\text{(2.36)} \quad I_3=\frac{1}{4}b_i\Big[\Big\langle(\nabla^X\nabla^X\mathcal{J})_{(z,z)}\overline{z}+(\nabla^X\nabla^X\mathcal{J})_{(z,\overline{z})}z+(\nabla^X\nabla^X\mathcal{J})_{(\overline{z},z)}z,\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle
$$

$$
+\frac{1}{3}\Big\langle R^{TX}(\overline{z},\mathcal{J}z)z+R^{TX}(z,\mathcal{J}\overline{z})z,\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle\Big]
$$

$$
=-\frac{\pi}{4}b_i\Big[\Big\langle(\nabla_z^X J)z,3(\nabla_{\overline{z}}^X J)\tfrac{\partial}{\partial \overline{z}_i}-(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\overline{z}\Big\rangle+\frac{4}{3}\Big\langle R^{TX}(z,\overline{z})z,\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle\Big].
$$

Therefore, from (1.101), (2.8), (2.17), (2.21), (2.33) and (2.36), we get

$$
\text{(2.37)} \quad -\Big(\mathscr{L}_0^{-1}P^{N\perp}I_3 P^N\Big)(0,0)=\frac{\pi}{4}\bigg\{\mathscr{L}_0^{-1}b_i\Big[\frac{4}{3}\Big\langle R^{TX}(z,\tfrac{\partial}{\partial \overline{z}_j})z,\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle
$$

$$
+\Big\langle(\nabla_z^X J)z,3(\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\tfrac{\partial}{\partial \overline{z}_i}-(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j}\Big\rangle\Big]\frac{b_j}{2\pi}P^N\bigg\}(0,0)
$$

$$
=-\frac{1}{16\pi}\Big[\frac{4}{3}\Big\langle R^{TX}(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \overline{z}_j})\tfrac{\partial}{\partial z_i}+R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \overline{z}_j})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle
$$

$$
+\Big\langle(\nabla_{\frac{\partial}{\partial z_i}}^X J)\tfrac{\partial}{\partial z_j}+(\nabla_{\frac{\partial}{\partial z_j}}^X J)\tfrac{\partial}{\partial z_i},3(\nabla_{\frac{\partial}{\partial \overline{z}_i}}^X J)\tfrac{\partial}{\partial \overline{z}_j}-(\nabla_{\frac{\partial}{\partial \overline{z}_j}}^X J)\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle\Big]
$$

$$
=-\frac{5}{192\pi}|\nabla^X J|^2-\frac{1}{6\pi}\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \overline{z}_j})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \overline{z}_i}\Big\rangle.
$$

Thanks to (1.101), (2.21) and (2.33) we have

$$\boxed{\text{g59}}\quad (2.38)\quad \frac{1}{3}\Big(P^{N^\perp}\Big\langle R^{TX}(\mathcal{R},\tfrac{\partial}{\partial z_i})\mathcal{R},\tfrac{\partial}{\partial \bar z_i}\Big\rangle P^N\Big)(0,0)$$

$$= \frac{1}{3}\Big(P^{N^\perp}\Big\langle R^{TX}(z,\tfrac{\partial}{\partial z_i})\tfrac{\partial}{\partial z_j} + R^{TX}(\tfrac{\partial}{\partial \bar z_j},\tfrac{\partial}{\partial z_i})z,\tfrac{\partial}{\partial \bar z_i}\Big\rangle \frac{b_j}{2\pi}P^N\Big)(0,0)$$

$$= -\frac{1}{3\pi}\Big[\Big\langle R^{TX}(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial z_i})\tfrac{\partial}{\partial \bar z_j} + R^{TX}(\tfrac{\partial}{\partial \bar z_j},\tfrac{\partial}{\partial z_i})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \bar z_i}\Big\rangle\Big]$$

$$= -\frac{1}{96\pi}|\nabla^X J|^2 + \frac{1}{3\pi}\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_j})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \bar z_i}\Big\rangle.$$

By (2.13), (2.33), (2.34), (2.35), (2.37), (2.38), and the discussion above (2.36),

$$\boxed{\text{g60}}\quad (2.39)\quad -\Big(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_2 P^N\Big)(0,0) = -\Big\{\Big[\frac{b_i b_j}{24\pi}\Big\langle R^{TX}(z,\tfrac{\partial}{\partial \bar z_i})z,\tfrac{\partial}{\partial \bar z_j}\Big\rangle + \frac{b_i}{4\pi}R^E(z,\tfrac{\partial}{\partial \bar z_i})$$

$$+ \frac{b_j}{3\pi}\Big(\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_i})z,\tfrac{\partial}{\partial \bar z_j}\Big\rangle - \Big\langle R^{TX}(z,\tfrac{\partial}{\partial z_i})\tfrac{\partial}{\partial \bar z_j},\tfrac{\partial}{\partial \bar z_j}\Big\rangle\Big)\Big]P^N\Big\}(0,0)$$

$$+ \frac{1}{3}\Big(P^{N^\perp}\Big\langle R^{TX}(\mathcal{R},\tfrac{\partial}{\partial z_i})\mathcal{R},\tfrac{\partial}{\partial \bar z_i}\Big\rangle P^N\Big)(0,0) - \Big(\mathscr{L}_0^{-1}P^{N^\perp}I_3 P^N\Big)(0,0)$$

$$= -\frac{1}{6\pi}\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_i})\tfrac{\partial}{\partial z_j} + R^{TX}(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \bar z_i})\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_j}\Big\rangle + \frac{1}{2\pi}R^E(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_i})$$

$$+ \frac{2}{3\pi}\Big[\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_i})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \bar z_j}\Big\rangle - \Big\langle R^{TX}(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial z_i})\tfrac{\partial}{\partial \bar z_i},\tfrac{\partial}{\partial \bar z_j}\Big\rangle\Big]$$

$$- \frac{7}{192\pi}|\nabla^X J|^2 + \frac{1}{6\pi}\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_j})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \bar z_i}\Big\rangle$$

$$= \frac{1}{2\pi}\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_j})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \bar z_i}\Big\rangle + \frac{1}{2\pi}R^E(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial \bar z_i}).$$

Formulas (2.14), (2.39) and the discussion at the beginning of Section 2.3 yield finally

$$\boxed{\text{g61}}\quad (2.40)\quad b_{0,1}(x_0) = F_{0,2}(0,0) = -\Big(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_2 P^N\Big)(0,0) - \Big(\Big(\mathscr{L}_0^{-1}P^{N^\perp}\mathcal{O}_2 P^N\Big)(0,0)\Big)^*$$

$$= \frac{1}{8\pi}\Big[r_{x_0}^X + \frac{1}{4}|\nabla^X J|_{x_0}^2 + 2\sqrt{-1}R_{x_0}^E(e_j, Je_j)\Big].$$

The proof of Theorem 0.2 is complete.

Remark 2.3. In the Kähler case, i.e. J is integrable and L, E are holomorphic, then $\mathcal{O}_1 = 0$, and the above computation simplifies a lot.

3. APPLICATIONS

$\boxed{\text{s5}}$

In this Section, we discuss various applications of our results. In Section 3.1, we study the density of states function of $\Delta_{p,\Phi}$. In Section 3.2, we explain how to handle the first-order pseudo-differential operator D_b of Boutet de Monvel and Guillemin [13] which was studied extensively by Shiffman and Zelditch [42]. In Section 3.3, we prove a symplectic version of the convergence of the Fubini-Study metric of an ample line bundle [46]. In Section 3.4, we show how to handle the operator $\overline{\partial} + \overline{\partial}^*$ when X is Kähler but $\mathbf{J} \neq J$. Finally, in Sections 3.5, 3.6, we establish some generalizations for non-compact or singular manifolds.

s5.1

3.1. **Density of states function.** Let (X, ω) be a compact symplectic manifold of real dimension $2n$ and (L, ∇^L, h^L) is a pre-quantum line bundle as in Section 0. Assume that E is the trivial bundle $\mathbb{C}$, $\Phi = 0$ and $\mathbf{J} = J$. The latter means, by (0.1), that g^{TX} is the Riemannian metric associated to ω and J. We denote by $\mathrm{vol}(X) = \int_X \frac{\omega^n}{n!}$ the Riemannian volume of (X, g^{TX}). Recall that d_p is defined in (0.6).

Our aim is to describe the asymptotic distribution of the energies of the bound states as p tends to infinity. We define the spectrum counting function of $\Delta_p := \Delta_{p,0}$ by $N_p(\lambda) = \# \{i : \lambda_{i,p} \leqslant \lambda\}$ and the spectral density measure on $[-C_L, C_L]$ by

f2   (3.1)
$$\nu_p = \frac{1}{d_p} \frac{d}{d\lambda} N_p(\lambda), \quad \lambda \in [-C_L, C_L].$$

Clearly, ν_p is a sum of Dirac measures supported on $\mathrm{Spec}\, \Delta_p \cap [-C_L, C_L]$. Set

f3   (3.2)
$$\varrho : X \longrightarrow \mathbb{R}, \quad \varrho(x) = \frac{1}{24} |\nabla^X J|^2.$$

l-density   **Theorem 3.1.** *The weak limit of the sequence $\{\nu_p\}_{p \geqslant 1}$ is the direct image measure* $\varrho_* \left(\frac{1}{\mathrm{vol}(X)} \frac{\omega^n}{n!} \right)$, *that is, for any continuous function $f \in \mathscr{C}([-C_L, C_L])$, we have*

f4   (3.3)
$$\lim_{p \to \infty} \int_{-C_L}^{C_L} f \, d\nu_p = \frac{1}{\mathrm{vol}(X)} \int_X (f \circ \varrho) \frac{\omega^n}{n!}.$$

Proof. By (0.7), we have for $q \geqslant 1$ (now E is trivial): $B_{q,p}(x) = \sum_{i=1}^{d_p} \lambda_{i,p}^q |S_i^p(x)|^2$, which yields by integration over X,

f5   (3.4)
$$\frac{1}{d_p} \int_X B_{q,p} \, dv_X = \frac{1}{d_p} \sum_{i=1}^{d_p} \lambda_{i,p}^q = \int_{-C_L}^{C_L} \lambda^q \, d\nu_p(\lambda),$$

since S_i^p have unit L^2 norm. On the other hand, (0.6), (0.9) entail for $p \to \infty$,

f6   (3.5)
$$\frac{1}{d_p} \int_X B_{q,p} \, dv_X = \frac{p^n}{d_p} \int_X b_{q,0} \, dv_X + \frac{\mathscr{O}(p^{n-1})}{d_p}$$
$$= \frac{1}{\mathrm{vol}(X)} \int_X \varrho^q \, dv_X + \mathscr{O}(p^{-1}).$$

We infer from (3.4) and (3.5) that (3.3) holds for $f(\lambda) = \lambda^q$, $q \geqslant 1$. Since this is obviously true for $f(\lambda) \equiv 1$, too, we deduce it holds for all polynomials. Upon invoking the Weierstrass approximation theorem, we get (3.3) for all continuous functions on $[-C_L, C_L]$. This achieves the proof. $\square$

t5.2   *Remark* 3.2. A function ϱ satisfying (3.3) is called spectral density function. Its existence and uniqueness were demonstrated by Guillemin-Uribe [29]. As for the explicit formula of ϱ, the paper [11] is dedicated to its computation. Our formula (3.2) is different from [11, Theorem 1.2] [3].

_____

[3]In [11, (3.7)], the leading term of G_{0j} should be $\kappa^{-1/2} b_j^{(1)}$ which was missed therein, as the principal terms of $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial y^j}$ are ∂_0, $T_j^l \partial_l$ by [11, equation after (3.11)]. Now, from [11, (3.5)], $b_j^{(1)}$ is $\frac{1}{2} \langle Jz, T_j^l \partial_l \rangle$. Thus $\mathcal{L}_0$ in [11, (3.8)] is incorrect. [11, Theorem 1.2] is $\varrho(x) = -\frac{5}{24} |\nabla^X J|^2$.

An interesting corollary of (3.2) and (3.3) is the following result which was first stated in [11, Cor. 1.3].

Corollary 3.3. *The spectral density function is identically zero if and only if* (X, ω, J) *is Kähler.*

Remark 3.4. Theorem 3.1 can be slightly generalized. Assume namely that $\mathbf{J} = J$ and E is a Hermitian vector bundle as in Section 0 such that $R^E = \eta \otimes \mathrm{Id}_E$, $\Phi = \varphi \, \mathrm{Id}_E$, where η is a 2-form and φ a real function on X. Then there exists a spectrum density function satisfying (3.3) given by

(3.6) $$ \varrho : X \longrightarrow \mathbb{R}, \quad \varrho(x) = \frac{1}{24}|\nabla^X J|^2 + \frac{\sqrt{-1}}{2}\eta(e_j, Je_j) + \varphi. $$

The proof is similar to the previous one, as $\mathrm{Tr}_{E_x} B_{q,p}(x) = \sum_{i=1}^{d_p} \lambda_{i,p}^q |S_i^p(x)|^2$.

3.2. Almost-holomorphic Szegö kernels.

We use the notations and assumptions from Section 3.1, especially, $\mathbf{J} = J$. Then $\tau = 2\pi n$.

Let $Y = \{u \in L^*, |u|_{h^{L^*}} = 1\}$ be the unit circle bundle in L^*. Then the smooth sections of L^p can be identified to the smooth functions $\mathscr{C}^\infty(Y)_p = \{f \in \mathscr{C}^\infty(Y, \mathbb{C}); f(ye^{i\theta}) = e^{ip\theta}f(y) \text{ for } e^{i\theta} \in S^1, y \in Y\}$, here $ye^{i\theta}$ is the S^1 action on Y.

The connection ∇^L on L induces a connection on the S^1-principal bundle $\pi : Y \to X$, and let $T^H Y \subset TY$ be the corresponding horizontal bundle. Let $g^{TY} = \pi^* g^{TX} \oplus d\theta^2$ be the metric on $TY = T^H Y \oplus TS^1$, with $d\theta^2$ the standard metric on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let Δ_Y be the Bochner-Laplacian on (Y, g^{TY}), then by construction, it commutes with the generator ∂_θ of the circle action, and so it commutes with the horizontal Laplacian

(3.7) $$ \Delta_h = \Delta_Y + \partial_\theta^2, $$

then Δ_h on $\mathscr{C}^\infty(Y)_p$ is identical with Δ^{L^p} on $\mathscr{C}^\infty(X, L^p)$ (cf. [10, §2.1]).

In [13, Lemma 14.11, Theorem A 5.9], [14], [29, (3.13)], they construct a self-adjoint second-order pseudodifferential operator Q on Y such that

(3.8) $$ V = \Delta_h + \sqrt{-1}\tau\partial_\theta - Q $$

is a self-adjoint pseudodifferential operator of order zero on Y, and V, Q commute with the S^1-action. The orthogonal projection Π onto the kernel of Q is called the *Szegö projector* associated with the almost CR manifold Y. In fact, the Szegö projector is not unique or canonically defined, but the above construction defines a canonical choice of Π modulo smoothing operators. In the complex case, the construction produces the usual Szegö projector Π.

We denote the operators on $\mathscr{C}^\infty(X, L^p)$ corresponding to Q, V, Π by Q_p, V_p, Π_p, especially, $V_p(x, y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ip\theta} V(xe^{i\theta}, y)d\theta$. Then by (3.8),

(3.9) $$ Q_p = \Delta^{L^p} - p\tau - V_p. $$

By [29, §4], there exists $\mu_1 > 0$ such that for p large,

(3.10) $$ \mathrm{Spec}\, Q_p \subset \{0\} \cup [\mu_1 p, +\infty[. $$

Since the operator V_p is uniformly bounded in p, naturally, from (0.5), (0.6), we get

f24   (3.11) $$\dim \operatorname{Ker} Q_p = d_p = \int_X \operatorname{Td}(TX)\operatorname{ch}(L^p).$$

Now we explain how to study the Szegö projector Π_p [4]. This can be done from our point of view. Recall $\widetilde{F}$ is the function defined after (1.4). Let $\Pi_p(x, x')$, $\widetilde{F}(Q_p)(x, x')$ be the smooth kernels of Π_p, $\widetilde{F}(Q_p)$ with respect to the volume form $dv_X(x')$.

Note that V_p is a 0-order pseudodifferential operator on X induced from a 0-order pseudodifferential operator on Y. Thus from (3.9), (3.10), we have the analogue of [20, Proposition 3.1] (cf. Proposition 1.2): for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $p \geqslant 1$,

f25   (3.12) $$|\widetilde{F}(Q_p)(x, x') - \Pi_p(x, x')|_{\mathscr{C}^m(X \times X)} \leqslant C_{l,m} p^{-l}.$$

By finite propagation speed [45, §4.4], we know that $\widetilde{F}(Q_p)(x, x')$ only depends on the restriction of Q_p to $B^X(x, \varepsilon)$, and is zero if $d(x, x') \geqslant \varepsilon$. It transpires that the asymptotic of $\Pi_p(x, x')$ as $p \to \infty$ is localized on a neighborhood of x. Thus we can translate our analysis from X to the manifold $\mathbb{R}^{2n} \simeq T_{x_0}X =: X_0$ as in Section 1.2, especially, we extend ∇^L to a Hermitian connection ∇^{L_0} on $(L_0, h^{L_0}) = (X_0 \times L_{x_0}, h^{L_{x_0}})$ on $T_{x_0}X$ in such a way so that we still have positive curvature R^{L_0}; in addition $R^{L_0} = R^L_{x_0}$ outside a compact set.

Now, by using a micro-local partition of unity, one can still construct the operator Q^{X_0} as in [13, Lemma 14.11, Theorem A 5.9], [14], [29, (3.13)], such that V^{X_0} differs from V by a smooth operator in a neighborhood of 0. On X_0, and Q^{X_0} still verifies (3.10). Thus we can work on $\mathscr{C}^\infty(X_0, \mathbb{C})$ as in Section 1.3. We rescale then the coordinates as in (1.27) and use the norm (1.38). The $V_p^{X_0}$ is a 0-order pseudodifferential operator on X_0 induced from a 0-order pseudodifferential operator on Y_0. This guarantees that the operator rescaled from $V_p^{X_0}$ will have the similar expansion as (1.29) with leading term $t^2 R_2$ in the sense of pseudo-differential operators.

From (3.11) and [20, (3.89)], similar to the argument in [20, Theorem 3.18], we can also get the full off diagonal expansion for Π_p, which is an extension of [42, Theorem 1], where the authors obtain (3.13) for $|Z|, |Z'| \leqslant C/\sqrt{p}$ with $C > 0$ fixed. More precisely, recalling that P^N is the Bergman kernel of $\mathscr{L}_0$ as in (1.81), (1.94) we have:

tue17   **Theorem 3.5.** *There exist polynomials* $\mathbf{j}_r(Z, Z')$ $(r \geqslant 0)$ *of* Z, Z' *with the same parity with* r, *and* $\mathbf{j}_0 = 1$, $C'' > 0$ *such that for any* $k, m, m' \in \mathbb{N}$, *there exist* $N \in \mathbb{N}, C > 0$

---

[4] As Professor Sjöstrand pointed out to us, in general, $\Pi_p - P_{0,p}$ is not $\mathscr{O}(p^{-\infty})$ as $p \to \infty$, where $P_{0,p}$ is the smooth kernel of the operator $\Delta_{0,p}$ (Definition 1.1). This can also be seen from the presence of a contribution coming from Φ in the expression (0.9) of the coefficient $b_{0,2}$.

such that for $\alpha, \alpha' \in \mathbb{Z}^{2n}$, $|\alpha| + |\alpha'| \leqslant m$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| \leqslant \varepsilon$, $x_0 \in X$, $p > 1$,

f26

(3.13)
$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^n} \Pi_p(Z, Z') - \sum_{r=0}^k (\mathbf{j}_r P^N)(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') p^{-r/2} \right) \right|_{\mathscr{C}^{m'}(X)}$$
$$\leqslant C p^{-(k+1-m)/2} (1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-\sqrt{C''\mu_1}\sqrt{p}|Z - Z'|) + \mathscr{O}(p^{-\infty}).$$

The term $\kappa^{-\frac{1}{2}}$ in (3.13) comes from the conjugation of the operators as in (1.116), $\mathscr{C}^{m'}(X)$ is the $\mathscr{C}^{m'}$-norm for the parameter $x_0 \in X$, and we use the trivializations from Section 1.2, the term $\mathscr{O}(p^{-\infty})$ means that for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that its $\mathscr{C}^{l_1}$-norm is dominated by $C_{l,l_1} p^{-l}$. We leave the details to the interested reader.

s5.31

3.3. **Symplectic version of Kodaira Embedding Theorem.** Let (X, ω) be a compact symplectic manifold of real dimension $2n$ and let (L, ∇^L, h^L) be a pre-quantum line bundle and let g^{TX} be a Riemannian metric on X as in Section 0.

Recall that $\mathcal{H}_p \subset \mathscr{C}^\infty(X, L^p)$ is the span of those eigensections of $\Delta_p = \Delta^{L^p} - \tau p$ corresponding to eigenvalues from $[-C_L, C_L]$. We denote by $\mathbb{P}\mathcal{H}_p^*$ the projective space associated to the dual of $\mathcal{H}_p$ and we identify $\mathbb{P}\mathcal{H}_p^*$ with the Grassmannian of hyperplanes in $\mathcal{H}_p$. The *base locus* of $\mathcal{H}_p$ is the set $\mathrm{Bl}(\mathcal{H}_p) = \{x \in X : s(x) = 0 \text{ for all } s \in \mathcal{H}_p\}$. As in algebraic geometry, we define the Kodaira map

sz0   (3.14)
$$\Phi_p : X \smallsetminus \mathrm{Bl}(\mathcal{H}_p) \longrightarrow \mathbb{P}\mathcal{H}_p^*$$
$$\Phi_p(x) = \{s \in \mathcal{H}_p : s(x) = 0\}$$

which sends $x \in X \smallsetminus \mathrm{Bl}(\mathcal{H}_p)$ to the hyperplane of sections vanishing at x. Note that $\mathcal{H}_p$ is endowed with the induced L^2 product (1.1) so there is a well–defined Fubini–Study metric g_{FS} on $\mathbb{P}\mathcal{H}_p^*$ with the associated form ω_{FS}.

m-Kodaira

Theorem 3.6. *Let* (L, ∇^L) *be a pre–quantum line bundle over a compact symplectic manifold* (X, ω). *The following assertions hold true*:

(i) *For large* p, *the Kodaira maps* $\Phi_p : X \longrightarrow \mathbb{P}\mathcal{H}_p^*$ *are well defined.*

(ii) *The induced Fubini–Study metric* $\frac{1}{p}\Phi_p^*(\omega_{FS})$ *converges in the* $\mathscr{C}^\infty$ *topology to* ω; *for any* $l \geqslant 0$ *there exists* $C_l > 0$ *such that*

sz1   (3.15)
$$\left| \frac{1}{p} \, \Phi_p^*(\omega_{FS}) - \omega \right|_{\mathscr{C}^l} \leqslant \frac{C_l}{p}.$$

(iii) *For large* p *the Kodaira maps* Φ_p *are embeddings.*

Remark 3.7. 1) Assume that X is Kähler and L is a holomorphic bundle. Then Δ_p is the twice the Kodaira-Laplacian and $\mathcal{H}_p$ coincides with the space $H^0(X, L^p)$ of holomorphic sections of L^p. Then (i) and (iii) are simply the Kodaira embedding theorem. Assertion (ii) is due to Tian [46, Theorem A] as an answer to a conjecture of Yau. In [46] the case $l = 2$ is considered and the left–hand side of (3.15) is estimated by $C_l/\sqrt{p}$. Ruan [41] proved the $\mathscr{C}^\infty$ convergence and improved the bound to C_l/p. Both papers use the peak section method, based on L^2–estimates for $\bar{\partial}$. Finally, Zelditch deduced (ii) from the asymptotic expansion of the Szegö

kernel [50]. Bouche [12] proves that the induced Fubini-Study metric $(\Phi_p^* h^{\mathscr{O}(1)})^{1/p}$ on L converges in the $\mathscr{C}^0$ topology to the initial metric h^L.

2) Borthwick and Uribe [10, Theorem 1.1], Shiffman and Zelditch [42, Theorems 2, 3] prove a different symplectic version of [46, Theorem A] when $\mathbf{J} = J$. Instead of $\mathcal{H}_p$, they use the space $H_J^0(X, L^p) := \mathrm{Im}(\Pi_p)$ (cf. [10, p.601], [42, §2.3], §3.2) of 'almost holomorphic sections' proposed by Boutet de Monvel and Guillemin [13, 14].

Proof. Let us first give an alternate description of the map Φ_p which relates it to the Bergman kernel. Let $\{S_i^p\}_{i=1}^{d_p}$ be any orthonormal basis of $\mathcal{H}_p$ with respect to the inner product (1.1). Once we have fixed a basis, we obtain an identification $\mathcal{H}_p \cong \mathcal{H}_p^* \cong \mathbb{C}^{d_p}$ and $\mathbb{P}\mathcal{H}_p^* \cong \mathbb{CP}^{d_p-1}$. Consider the commutative diagram.

(3.16)
$$
\begin{array}{ccc}
X \smallsetminus \mathrm{Bl}(\mathcal{H}_p) & \xrightarrow{\ \Phi_p\ } & \mathbb{P}\mathcal{H}_p^* \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \cong} \\
X \smallsetminus \mathrm{Bl}(\mathcal{H}_p) & \xrightarrow{\ \widetilde{\Phi}_p\ } & \mathbb{CP}^{d_p-1}
\end{array}
$$

Then

(3.17)
$$
\Phi_p^*(\omega_{FS}) = \widetilde{\Phi}_p^* \Big(\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log \sum_{j=1}^{d_p} |w_j|^2 \Big),
$$

where $[w_1, \ldots, w_{d_p}]$ are homogeneous coordinates in $\mathbb{CP}^{d_p-1}$. To describe $\widetilde{\Phi}_p$ in a neighborhood of a point $x_0 \in X \smallsetminus \mathrm{Bl}(\mathcal{H}_p)$, we choose a local unity frame e_L of L and write $S_i^p = f_i^p e_L^{\otimes p}$ for some smooth functions f_i^p. Then

(3.18)
$$
\widetilde{\Phi}_p(x) = [f_1^p(x); \ldots; f_{d_p}^p(x)],
$$

and this does not depend on the choice of the frame e_L.

(i) Let us choose an unit frame e_L of L. Then $|S_i^p|^2 = |f_i^p|^2 |e_L|^{2p} = |f_i^p|^2$, hence

$$
B_{0,p} = \sum_{i=1}^{d_p} |S_i^p|^2 = \sum_{i=1}^{d_p} |f_i^p|^2.
$$

Since $b_{0,0} > 0$, the asymptotic expansion (0.9) shows that $B_{0,p}$ does not vanish on X for p large enough, so the sections $\{S_i^p\}_{i=1}^{d_p}$ have no common zeroes. Therefore Φ_p and $\widetilde{\Phi}_p$ are defined on all X.

(ii) Let us fix $x_0 \in X$. We identify a small geodesic ball $B^X(x_0, \varepsilon)$ to $B^{T_{x_0}X}(0, \varepsilon)$ by means of the exponential map and consider a trivialization of L as in Section 1.2, i.e. we trivialize L by using an unit frame $e_L(Z)$ which is parallel with respect to ∇^L along $[0,1] \ni u \to uZ$ for $Z \in B^{T_{x_0}X}(0, \varepsilon)$. We can express the Fubini–Study metric as

$$
\frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log \Big(\sum_{j=1}^{d_p} |w_j|^2 \Big) = \frac{\sqrt{-1}}{2\pi} \left[\frac{1}{|w|^2} \sum_{j=1}^{d_p} dw_j \wedge d\overline{w}_j - \frac{1}{|w|^4} \sum_{j,k=1}^{d_p} \overline{w}_j w_k \, dw_j \wedge d\overline{w}_k \right],
$$

and therefore, from (3.18),

(3.19) $\quad \Phi_p^*(\omega_{FS})(x_0) = \dfrac{\sqrt{-1}}{2\pi}\left[\dfrac{1}{|f^p|^2}\sum_{j=1}^{d_p} df_j^p \wedge d\overline{f_j^p} - \dfrac{1}{|f^p|^4}\sum_{j,k=1}^{d_p}\overline{f_j^p}f_k^p\, df_j^p \wedge d\overline{f_k^p}\right](x_0)$

$\qquad\qquad = \dfrac{\sqrt{-1}}{2\pi}\left[f^p(x_0,x_0)^{-1}d_xd_yf^p(x,y) - f^p(x_0,x_0)^{-2}d_xf^p(x,y) \wedge d_yf^p(x,y)\right]\big|_{x=y=x_0},$

where $f^p(x,y) = \sum_{i=1}^{d_p} f_i^p(x)\overline{f_i^p}(y)$ and $|f^p(x)|^2 = f^p(x,x)$. Since

(3.20) $\qquad\qquad\qquad P_{0,p}(x,y) = f^p(x,y)e_L^p(x) \otimes e_L^p(y)^*,$

thus $P_{0,p}(x,y)$ is $f^p(x,y)$ under our trivialization of L. By (1.31), Theorem (1.18), and (1.116), we obtain

(3.21) $\quad \dfrac{1}{p}\,\Phi_p^*(\omega_{FS})(x_0) = \dfrac{\sqrt{-1}}{2\pi}\left[\dfrac{1}{F_{0,0}}d_xd_yF_{0,0} - \dfrac{1}{F_{0,0}^2}d_xF_{0,0} \wedge d_yF_{0,0}\right](0,0)$

$\qquad\qquad - \dfrac{\sqrt{-1}}{2\pi}\dfrac{1}{\sqrt{p}}\left[\dfrac{1}{F_{0,0}^2}(d_xF_{0,1} \wedge d_yF_{0,0} + d_xF_{0,0} \wedge d_yF_{0,1})\right](0,0) + \mathscr{O}\big(1/p\big).$

Using again (1.94), (1.112), we obtain

(3.22) $\qquad \dfrac{1}{p}\,\Phi_p^*(\omega_{FS})(x_0) = \dfrac{\sqrt{-1}}{4\pi}\sum_{j=1}^{n} a_j dz_j \wedge d\bar{z}_j\big|_{x_0} + \mathscr{O}\big(1/p\big) = \omega(x_0) + \mathscr{O}\big(1/p\big),$

and the convergence takes place in the $\mathscr{C}^\infty$ topology with respect to $x_0 \in X$.

(iii) Since X is compact, we have to prove two things for p sufficiently large: (a) Φ_p are immersions and (b) Φ_p are injective. We note that (a) follows immediately from (3.15).

To prove (b) let us assume the contrary, namely that there exists a sequence of distinct points $x_p \neq y_p$ such that $\Phi_p(x_p) = \Phi_p(y_p)$. Relation (3.16) implies that $\widetilde{\Phi}_p(x_p) = \widetilde{\Phi}_p(y_p)$, where $\widetilde{\Phi}_p$ is defined by any particular choice of basis.

The key observation is that Theorem 1.19 ensures the existence of a sequence of *peak sections* at each point of X. The construction goes like follows. Let $x_0 \in X$ be fixed. Since Φ_p is point base free for large p, we can consider the hyperplane $\Phi_p(x_0)$ of all sections of $\mathcal{H}_p$ vanishing at x_0. We construct then an orthonormal basis $\{S_i^p\}_{i=1}^{d_p}$ of $\mathcal{H}_p$ such that the first $d_p - 1$ elements belong to $\Phi_p(x_0)$. Then $S_{d_p}^p$ is a unit norm generator of the orthogonal complement of $\Phi_p(x_0)$, and will be denoted by $S_{x_0}^p$. This is a peak section at x_0. We note first that $|S_{x_0}^p(x_0)|^2 = B_{0,p}(x_0)$ and $P_{0,p}(x,x_0) = S_{x_0}^p(x) \otimes S_{x_0}^p(x_0)^*$ and therefore

(3.23) $\qquad\qquad\qquad S_{x_0}^p(x) = \dfrac{1}{B_{0,p}(x_0)}P_{0,p}(x,x_0) \cdot S_{x_0}^p(x_0).$

From (1.116) we deduce that for a sequence $\{r_p\}$ with $r_p \to 0$ and $r_p\sqrt{p} \to \infty$,

(3.24) $\qquad\qquad \displaystyle\int_{B(x_0,r_p)} |S_{x_0}^p(x)|^2\, dv_X(x) = 1 - \mathscr{O}(1/p), \quad \text{for } p \to \infty.$

Relation (3.24) explains the term 'peak section': when p grows, the mass of $S_{x_0}^p$ concentrates near x_0. Since $\Phi_p(x_p) = \Phi_p(y_p)$ we can construct as before the peak

section $S_{x_p}^p = S_{y_p}^p$ as the unit norm generator of the orthogonal complement of $\Phi_p(x_p) = \Phi_p(y_p)$. We fix in the sequel such a section which peaks at both x_p and y_p.

We consider the distance $d(x_p, y_p)$ between the two points x_p and y_p. By passing to a subsequence we have two possibilities: either $\sqrt{p}\, d(x_p, y_p) \to \infty$ as $p \to \infty$ or there exists a constant $C > 0$ such that $d(x_p, y_p) \leqslant C/\sqrt{p}$ for all p.

Assume that the first possibility is true. For large p, we learn from relation (3.24) that the mass of $S_{x_p}^p = S_{y_p}^p$ (which is 1) concentrates both in neighborhoods $B(x_p, r_p)$ and $B(y_p, r_p)$ with $r_p = d(x_p, y_p)/2$ and approaches therefore 2 if $p \to \infty$. This is a contradiction which rules out the first possibility.

To exclude the second possibility we follow [42]. We identify as usual $B^X(x_p, \varepsilon)$ to $B^{T_{x_p}X}(0, \varepsilon)$ so the point y_p gets identified to $Z_p/\sqrt{p}$ where $Z_p \in B^{T_{x_p}X}(0, C)$. We define then

$$\text{(3.25)} \qquad f_p : [0,1] \longrightarrow \mathbb{R}\,, \quad f_p(t) = \frac{|S_{x_p}^p(tZ_p/\sqrt{p})|^2}{B_{0,p}(tZ_p/\sqrt{p})}\,.$$

We have $f_p(0) = f_p(1) = 1$ (again because $S_{x_p}^p = S_{y_p}^p$) and $f_p(t) \leqslant 1$ by the definition of the generalized Bergman kernel. We deduce the existence of a point $t_p \in (0,1)$ such that $f_p''(t_p) = 0$. Equations (1.116), (3.23), (3.25) imply the estimate

$$\text{(3.26)} \qquad f_p(t) = e^{-\frac{t^2}{4}\sum_j a_j |z_{p,j}|^2}\big(1 + g_p(tZ_p)/\sqrt{p}\big)$$

and the $\mathscr{C}^2$ norm of g_p over $B^{T_{x_p}X}(0, C)$ is uniformly bounded in p. From (3.26), we infer that $|Z_p|_0^2 := \frac{1}{4}\sum_j a_j |z_{p,j}|_0^2 = \mathscr{O}(1/\sqrt{p})$. Using a limited expansion $e^x = 1 + x + x^2\varphi(x)$ for $x = t^2|Z_p|_0^2$ in (3.26) and taking derivatives, we obtain $f_p''(t) = -2|Z_p|_0^2 + \mathscr{O}(|Z_p|_0^4) + \mathscr{O}(|Z_p|_0^2/\sqrt{p}) = (-2 + \mathscr{O}(1/\sqrt{p}))|Z_p|_0^2$. Evaluating at t_p we get $0 = f_p''(t_p) = (-2 + \mathscr{O}(1/\sqrt{p}))|Z_p|_0^2$, which is a contradiction since by assumption $Z_p \neq 0$. This finishes the proof of (iii). $\qquad\square$

Remark 3.8. Let us point out complementary results which are analogues of [10, (1.3)–(1.5)] for the spaces $\mathcal{H}_p$. Computing as in (3.19) the pull-back $\Phi_p^* h_{FS}$ of the Hermitian metric $h_{FS} = g_{FS} - \sqrt{-1}\,\omega_{FS}$ on $\mathbb{P}\mathcal{H}_p^*$, we get the similar inequality to (3.15) for g_{FS} and $\omega(\cdot, J\cdot)$. Thus, Φ_p are asymptotically symplectic and isometric. Moreover, arguing as in [10, Proposition 4.4] we can show that Φ_p are 'nearly holomorphic':

$$\text{(3.27)} \qquad \frac{1}{p}\|\partial\Phi_p\| \geqslant C\,, \quad \frac{1}{p}\|\overline{\partial}\Phi_p\| = \mathscr{O}(1/p)\,, \quad \text{for some } C > 0\,,$$

uniformly on X, where $\|\cdot\|$ is the operator norm.

3.4. Holomorphic case revisited.
In this Section, we assume that (X, ω, J) is Kähler and the vector bundles E, L are holomorphic on X, and ∇^E, ∇^L are the holomorphic Hermitian connections on (E, h^E), (L, h^L). As usual, $\frac{\sqrt{-1}}{2\pi}R^L = \omega$.

But we will work with an arbitrary (*non-Kähler*) Riemannian metric g^{TX} on TX compatible with J. That is, in general $\mathbf{J} \neq J$ in (0.1). The use of non-Kähler metrics is useful for example in Section 3.6. Set

$$\text{(3.28)} \qquad \Theta(X, Y) = g^{TX}(JX, Y).$$

Then the 2-form Θ need not to be closed (the convention here is different to [2, (2.1)] by a factor -1). We denote by $T^{(1,0)}X$, $T^{(0,1)}X$ the holomorphic and anti-holomorphic tangent bundles as in Section 1.4. Let $\{e_i\}$ be an orthonormal frame of (TX, g^{TX}).

Let $g_\omega^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$ be the metric on TX induced by ω, J. We will use a subscript ω to indicate the objects corresponding to g_ω^{TX}, especially r_ω^X is the scalar curvature of (TX, g_ω^{TX}), and Δ_ω is the Bochner-Laplace operator as in (1.2) associated to g_ω^{TX}.

Let $\overline{\partial}^{L^p \otimes E, *}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p \otimes E}$ on the Dolbeault complex $\Omega^{0, \bullet}(X, L^p \otimes E)$ with the scalar product induced by g^{TX}, h^L, h^E as in (1.1). Set $D_p = \sqrt{2}(\overline{\partial}^{L^p \otimes E} + \overline{\partial}^{L^p \otimes E, *})$. Then $D_p^2 = 2(\overline{\partial}^{L^p \otimes E}\overline{\partial}^{L^p \otimes E, *} + \overline{\partial}^{L^p \otimes E, *}\overline{\partial}^{L^p \otimes E})$ preserves the $\mathbb{Z}$-grading of $\Omega^{0, \bullet}(X, L^p \otimes E)$. Then for p large enough,

$$\text{(3.29)} \qquad \operatorname{Ker} D_p = \operatorname{Ker} D_p^2 = H^0(X, L^p \otimes E).$$

Here D_p is not a spin^c Dirac operator on $\Omega^{0, \bullet}(X, L^p \otimes E)$, and D_p^2 is not a renormalized Bochner–Laplacian as in (0.4).

Let $P_p(x, x')$ $(x, x' \in X)$ be the smooth kernel of the orthogonal projection from $\mathscr{C}^\infty(X, L^p \otimes E)$ on $\operatorname{Ker} D_p^2$ with respect to the Riemannian volume form $dv_X(x')$ for p large enough. Recall that we denote by $\det_\mathbb{C}$ the determinant function on the complex bundle $T^{(1,0)}X$. We denote by $|\mathbf{J}| = (-\mathbf{J}^2)^{-1/2}$, then $\det_\mathbb{C}|\mathbf{J}| = (2\pi)^{-n}\Pi_i \, a_i$ under the notation in (1.82). Now we explain how to put it in the frame of our work.

Theorem 3.9. *The smooth kernel $P_p(x, x')$ has a full off–diagonal asymptotic expansion analogous to (3.13) with $\mathbf{j}_0 = \det_\mathbb{C}|\mathbf{J}|$ as $p \to \infty$. The corresponding term $b_{0,1}$ in the expansion (0.9) of $B_{0,p}(x) := P_p(x, x)$ is given by*

$$\text{(3.30)} \qquad b_{0,1} = \frac{\det_\mathbb{C}|\mathbf{J}|}{8\pi}\Big[r_\omega^X - 2\Delta_\omega\Big(\log(\det_\mathbb{C}|\mathbf{J}|)\Big) + 4R^E(w_{\omega,j}, \overline{w}_{\omega,j})\Big].$$

here $\{w_{\omega,j}\}$ is an orthonormal basis of $(T^{(1,0)}X, g_\omega^{TX})$.

Proof. As pointed out in [33, Remark 3.1], by [5, Theorem 1], there exist $\mu_0, C_L > 0$ such that for any $p \in \mathbb{N}$ and any $s \in \Omega^{>0}(X, L^p \otimes E) := \bigoplus_{q \geqslant 1} \Omega^{0,q}(X, L^p \otimes E)$,

$$\text{(3.31)} \qquad \|D_p s\|_{L^2}^2 \geqslant (2p\mu_0 - C_L)\|s\|_{L^2}^2.$$

Moreover $\operatorname{Spec} D_p^2 \subset \{0\} \cup [2p\mu_0 - C_L, +\infty[$.

Let S^{-B} denote the 1-form with values in antisymmetric elements of $\operatorname{End}(TX)$ which is such that if $U, V, W \in TX$,

$$\text{(3.32)} \qquad \langle S^{-B}(U)V, W \rangle = -\frac{\sqrt{-1}}{2}\Big((\partial - \overline{\partial})\Theta\Big)(U, V, W).$$

The Bismut connection ∇^{-B} on TX is defined by

$$\text{(3.33)} \qquad \nabla^{-B} = \nabla^{TX} + S^{-B}.$$

Then by [3, Prop. 2.5], ∇^{-B} preserves the metric g^{TX} and the complex structure of TX. Let $\nabla^{\det}$ be the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$ with its curvature $R^{\det}$. Then these two connections induce naturally an unique connection

on $\Lambda(T^{*(0,1)}X)$ which preserves its $\mathbb{Z}$-grading, and with the connections ∇^L, ∇^E, we get a connection ∇^{-B,E_p} on $\Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$. Let Δ^{-B,E_p} be the Laplacian on $\Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$ induced by ∇^{-B,E_p} as in (1.2). For any $v \in TX$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $\overline{v}^*_{1,0} \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then $c(v) = \sqrt{2}(\overline{v}^*_{1,0} \wedge -i_{v_{0,1}})$ defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior product respectively. We define a map $^c : \Lambda(T^*X) \to C(TX)$, the Clifford bundle of TX, by sending $e^{i_1} \wedge \cdots \wedge e^{i_j}$ to $c(e_{i_1}) \cdots c(e_{i_j})$ for $i_1 < \cdots < i_j$. For $B \in \Lambda^3(T^*X)$, set $|B|^2 = \sum_{i<j<k} |B(e_i, e_j, e_k)|^2$. Then we can formulate [3, Theorem 2.3] as following,

$$D_p^2 = \Delta^{-B,E_p} + \frac{r^X}{4} + {}^c(R^E + pR^L + \frac{1}{2}R^{\det}) + \frac{\sqrt{-1}}{2}{}^c(\overline{\partial}\partial\Theta) - \frac{1}{8}|(\partial - \overline{\partial})\Theta|^2. \tag{3.34}$$

We use now the connection ∇^{-B,E_p} instead of ∇^{E_p} in [20, §2]. Then by (3.31), (3.34), everything goes through perfectly well and as in [20, Theorem 3.18], so we can directly apply the result in [20] to get the *full off-diagonal* asymptotic expansion of the Bergman kernel. As the above construction preserves the $\mathbb{Z}$-grading on $\Omega^{0,\bullet}(X, L^p \otimes E)$, we can also directly work on $\mathscr{C}^\infty(X, L^p \otimes E)$.

Now, we need to compute the corresponding $b_{0,1}$. We endow E with the metric $h_\omega^E := (\det_{\mathbb{C}}|\mathbf{J}|)^{-1}h^E$ and let R_ω^E be the curvature associated to the holomorphic Hermitian connection of (E, h_ω^E). Then

$$R_\omega^E = R^E - \overline{\partial}\partial \log(\det_{\mathbb{C}}|\mathbf{J}|). \tag{3.35}$$

Thus

$$\sqrt{-1}R_\omega^E(e_{\omega,j}, Je_{\omega,j}) = 2R_\omega^E(w_{\omega,j}, \overline{w}_{\omega,j}) = \sqrt{-1}R^E(e_{\omega,j}, Je_{\omega,j}) - \Delta_\omega \log(\det_{\mathbb{C}}|\mathbf{J}|). \tag{3.36}$$

Let $\langle \cdot, \cdot \rangle_\omega$ be the Hermitian product on $\mathscr{C}^\infty(X, L^p \otimes E)$ induced by $g_\omega^{TX}, h^L, h_\omega^E$. Then

$$(\mathscr{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega) = (\mathscr{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle), \quad dv_{X,\omega} = (\det_{\mathbb{C}}|\mathbf{J}|)dv_X. \tag{3.37}$$

Observe that $H^0(X, L^p \otimes E)$ does not depend on g^{TX}, h^L or h^E. If $P_{\omega,p}(x, x')$, $(x, x' \in X)$ denotes the smooth kernel of the orthogonal projection from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle \cdot, \cdot \rangle_\omega)$ onto $H^0(X, L^p \otimes E)$ with respect to $dv_{X,\omega}(x)$, we have

$$P_p(x, x') = (\det_{\mathbb{C}}|\mathbf{J}|(x'))P_{\omega,p}(x, x'). \tag{3.38}$$

Now for the kernel $P_{\omega,p}(x, x')$, we can apply Theorem 0.1 (or [20, Theorem 1.3]) since $g_\omega^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a Kähler metric on TX, and (3.30) follows from (0.8) and (3.36). $\qquad\square$

Remark 3.10. Certainly, the argument in this Subsection goes through the orbifold case as in [20, §4.2].

3.5. **Generalizations to non-compact manifolds.** As in Section 3.4, we consider a complex Hermitian manifold (X, J, Θ) of dimension n, where J is the complex structure and Θ is the $(1,1)$ form associated to a riemannian metric g^{TX} compatible with J as in (3.28). The torsion of Θ is $T = [i(\Theta), \partial\Theta]$, where $i(\Theta) = (\Theta \wedge \cdot)^*$ is the interior multiplication with Θ. Let (L, h^L) and (E, h^E) be holomorphic Hermitian vector bundles over X, with $\mathrm{rk}\, L = 1$. We denote by R^L, R^E and $R^{\det}$ the

curvatures of the holomorphic Hermitian connections ∇^L, ∇^E and $\nabla^{\det}$ on L, E and $\det(T^{(1,0)}X)$. Let $J^L \in \mathrm{End}(TX)$ be the endomorphism satisfying $\frac{\sqrt{-1}}{2\pi}R^L(\cdot,\cdot) = \Theta(J^L\cdot,\cdot)$. The line bundle L is supposed to be positive and we set $\omega = \frac{\sqrt{-1}}{2\pi}R^L$. We also keep the notations g_ω^{TX}, Δ_ω and r_ω^X when we refer to Section 3.4.

The space of holomorphic sections of $L^p \otimes E$ which are L^2 with respect to the norm given by (1.1) is denoted by $H_{(2)}^0(X, L^p \otimes E)$. Let $P_p(x,x')$, $(x,x' \in X)$ be the Schwartz kernel of the orthogonal projection P_p from the L^2 section of $L^p \otimes E$ onto $H_{(2)}^0(X, L^p \otimes E)$ with respect to the Riemannian volume form $dv_X(x')$ associated to (X,Θ). Then by the ellipticity of the Kodaira-Laplacian and Schwartz kernel theorem, we know $P_p(x,x')$ is $\mathscr{C}^\infty$. We set $B_p(x) := P_p(x,x) \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. We have the following generalization of Theorem 0.1.

Theorem 3.11. *Assume that (X,Θ) is a complete Hermitian manifold. Suppose that there exist $\varepsilon > 0$, $C > 0$ such that :*

(3.39) $\quad \sqrt{-1}R^L \geqslant \varepsilon\Theta, \quad \sqrt{-1}R^{\det} \geqslant -C\Theta, \quad \sqrt{-1}R^E \geqslant -C\Theta, \qquad |T|_\Theta \leqslant C\Theta.$

The kernel $P_p(x,x')$ has a full off–diagonal asymptotic expansion analogous to (3.13) with $\mathbf{j}_0 = \det_{\mathbb{C}}|J^L|$ as $p \to \infty$, uniformly for any $x,x' \in K$, a compact set of X. Especially there exist coefficients $b_r \in \mathscr{C}^\infty(X)$, $r \in \mathbb{N}$, such that for any compact set $K \subset X$, any $k,l \in \mathbb{N}$, there exists $C_{k,l,K} > 0$ such that for $p \in \mathbb{N}$,

(3.40)
$$\left| \frac{1}{p^n} B_p(x) - \sum_{r=0}^{k} b_r(x)p^{-r} \right|_{\mathscr{C}^l(K)} \leqslant C_{k,l,K}\, p^{-k-1}.$$

Moreover, $b_0 = \det_{\mathbb{C}}|J^L|$ and b_1 equals $b_{0,1}$ from (3.30).

Let us remark that if $L = \det(T^{*(1,0)}X)$ is the canonical bundle, the first two conditions in (3.39) are to be replaced by

(3.41) $\qquad\qquad h^L$ is induced by Θ and $\sqrt{-1}R^{\det} < -\varepsilon\Theta.$

Moreover, if (X,Θ) is Kähler, the condition on the torsion T is trivially satisfied.

Proof. By the argument in Section 1.1, if the Kodaira–Laplacian $\square^{L^p \otimes E} = \frac{1}{2}\Delta_p := \frac{1}{2}\Delta_{p,0}$ acting on sections of $L^p \otimes E$ has a spectral gap as in (0.5), then we can localize the problem, and we get directly (3.40) from Section 1.3. Observe that $D_p^2|_{\Omega^{0,\bullet}} = \Delta_p$. In general, on a non-compact manifold, we define a self-adjoint extension of D_p^2 by

$$\mathrm{Dom}\, D_p^2 = \left\{ u \in \mathrm{Dom}\, \overline{\partial}^{E_p} \cap \mathrm{Dom}\, \overline{\partial}^{E_p *} : \overline{\partial}^{E_p} u \in \mathrm{Dom}\, \overline{\partial}^{E_p *}, \; \overline{\partial}^{E_p *} u \in \mathrm{Dom}\, \overline{\partial}^{E_p} \right\},$$

$$D_p^2 u = 2(\overline{\partial}^{E_p}\overline{\partial}^{E_p *} + \overline{\partial}^{E_p *}\overline{\partial}^{E_p})u, \quad \text{for } u \in \mathrm{Dom}\, D_p^2.$$

where we set $\overline{\partial}^{E_p} := \overline{\partial}^{L^p \otimes E}$. The quadratic form associated to D_p^2 is the form Q_p given by

(3.42)
$$\mathrm{Dom}\, Q_p := \mathrm{Dom}\, \overline{\partial}^{E_p} \cap \mathrm{Dom}\, \overline{\partial}^{E_p *}$$
$$Q_p(u,v) = 2\langle \overline{\partial}^{E_p} u, \overline{\partial}^{E_p} v \rangle + 2\langle \overline{\partial}^{E_p *} u, \overline{\partial}^{E_p *} v \rangle, \quad u,v \in \mathrm{Dom}\, Q_p.$$

In the previous formulas $\overline{\partial}^{E_p}$ is the weak maximal extension of $\overline{\partial}^{E_p}$ to L^2 forms and $\overline{\partial}^{E_p *}$ is its Hilbert space adjoint. We denote by $\Omega_0^{0,\bullet}(X, L^p \otimes E)$ the space of smooth compactly supported forms and by $L_2^{0,\bullet}(X, L^p \otimes E)$ the corresponding L^2-completion.

Under hypothesis (3.39) there exists $\mu > 0$ such that for p large enough

$$\boxed{\text{ell4}} \quad (3.43) \qquad Q_p(u) \geqslant \mu p \, \|u\|^2, \quad u \in \mathrm{Dom}\, Q_p \cap L_2^{0,q}(X, L^p \otimes E) \text{ for } q > 0.$$

Indeed, the estimate holds for $u \in \Omega_0^{0,q}(X, L^p \otimes E)$ since the Bochner-Kodaira-Nakano formula of Demailly [21, Th. 0.3], [40, Th. 1.5], [5, (8)] with torsion term delivers

$$\boxed{\text{ell4,1}} \quad (3.44) \quad Q_p(u) \geqslant \frac{4}{3} \big\langle (pR^L + R^E + R^{\det})(w_i, \overline{w}_j) \, \overline{w}^j \wedge i_{\overline{w}_i} u \,, u \big\rangle$$
$$- \frac{2}{3}(\|Tu\|^2 + \|\overline{T}u\|^2 + \|T^*u\|^2 + \|\overline{T}^*u\|^2)$$

for $u \in \Omega_0^{0,q}(X, L^p \otimes E)$, where $\{w_i\}$ is an orthonormal frame of $T^{(1,0)}X$. Relations (3.44) and (3.39) imply (3.43) for $u \in \Omega_0^{0,q}(X, L^p \otimes E)$. Since $\Omega_0^{0,\bullet}(X, L^p \otimes E)$ is dense in $\mathrm{Dom}\, Q_p$ with respect to the graph norm (due to the completeness of the metric g^{TX}), (3.43) holds in general.

Next, consider $f \in \mathrm{Dom}\, \Delta_p \cap L_2^{0,0}(X, L^p \otimes E)$ and set $u = \overline{\partial}^{E_p} f$. It follows from the definition of the Laplacian and (3.43) that

$$\boxed{\text{ell5}} \quad (3.45) \qquad \|\Delta_p f\|^2 = 2\big\langle \overline{\partial}^{E_p *} u \,, \overline{\partial}^{E_p *} u \big\rangle = Q_p(u) \geqslant \mu p \|u\|^2 = \mu p \big\langle \Delta_p f \,, f \big\rangle.$$

This clearly implies $\mathrm{Spec}(\Delta_p) \subset \{0\} \cup [p\mu, \infty[$ for large p.

As for b_1, the argument leading to (3.35)-(3.38) still holds locally, thus we get b_1 from (3.30). $\square$

Theorem 3.11 permits an immediate generalization of Tian's convergence theorem. Tian [46, Theorem 4.1] already proved a non–compact version for convergence in the $\mathscr{C}^2$ topology and convergence rate $1/\sqrt{p}$. Another easy consequence are holomorphic Morse inequalities for the space $H^0_{(2)}(X, L^p)$.

For simplicity we consider now $\mathrm{rk}\, E = 1$, with the important case $E = K_X = \det(T^{*(1,0)}X)$ in mind. Choose an orthonormal basis $(S_i^p)_{i \geqslant 1}$ of $H^0_{(2)}(X, L^p \otimes E)$. For each local holomorphic frames e_L and e_E of L and E we have

$$\boxed{\text{ell0,1}} \quad (3.46) \qquad\qquad S_i^p = f_i^p e_L^{\otimes p} \otimes e_E$$

for some local holomorphic functions f_i^p. Then $B_p(x) = P_p(x, x) = \sum_{i \geqslant 1} |S_i^p(x)|^2 = \sum_{i \geqslant 1} |f_i^p(x)|^2 |e_L^{\otimes p}|_{h^L}^2 |e_E|_{h^E}^2$ is a smooth function. Observe that the quantity $\sum_{i \geqslant 1} |f_i^p(x)|^2$ is not globally defined, but the current

$$\boxed{\text{ell0}} \quad (3.47) \qquad\qquad \omega_p = \frac{\sqrt{-1}}{2\pi} \, \partial\overline{\partial} \log \Big(\sum_{i \geqslant 1} |f_i^p(x)|^2 \Big)$$

is well defined globally on X. Indeed, since $R^L = -\partial\overline{\partial} \log |e_L|_{h^L}^2$ and $R^E = -\partial\overline{\partial} \log |e_E|_{h^E}^2$ we have

$$\boxed{\text{ell0.1}} \quad (3.48) \qquad\qquad \frac{1}{p}\omega_p - \frac{\sqrt{-1}}{2\pi} R^L = \frac{\sqrt{-1}}{2\pi p} \, \partial\overline{\partial} \log B_p + \frac{\sqrt{-1}}{2\pi p} R^E \,.$$

If E is trivial of rank one and $\dim H^0_{(2)}(X, L^p) < \infty$ we have by (3.14) that $\omega_p = \Phi^*_p(\omega_{FS})$ where Φ_p is defined as in (3.14) with $\mathscr{H}_p$ replaced by $H^0_{(2)}(X, L^p)$.

Corollary 3.12. *Assume that* $\mathrm{rk}\, E = 1$ *and* (3.39) *holds true. Then*:

(a) *for any compact set* $K \subset X$ *the restriction* $\omega_p|_K$ *is a smooth* $(1,1)$-*form for sufficiently large* p; *moreover, for any* $l \in \mathbb{N}$ *there exists a constant* $C_{l,K}$ *such that*

$$\left| \frac{1}{p}\omega_p - \frac{\sqrt{-1}}{2\pi}R^L \right|_{\mathscr{C}^l(K)} \leqslant \frac{C_{l,K}}{p} \, ;$$

(b) *the Morse inequalities hold in bidegree* $(0,0)$:

(3.49)
$$\liminf_{p \longrightarrow \infty} p^{-n} \dim H^0_{(2)}(X, L^p \otimes E) \geqslant \frac{1}{n!} \int_X \left(\frac{\sqrt{-1}}{2\pi}R^L \right)^n.$$

In particular, if $\dim H^0_{(2)}(X, L^p \otimes E) < \infty$, *the manifold* (X, Θ) *has finite volume.*

Proof. Due to (3.40), B_p doesn't vanish on any given compact set K for p sufficiently large. Thus, (a) is a consequence of (3.40) and (3.48).

Part (b) follows from Fatou's lemma, applied on X with the measure $\Theta^n/n!$ to the sequence $p^{-n}B_p$ which converges pointwise to $(\det J^L)^{1/2} = \left(\frac{\sqrt{-1}}{2\pi}R^L \right)^n / \Theta^n$ on X. $\square$

Remark 3.13. Under the hypothesis (3.41), the inequality (3.49) (with E trivial) is [38, Theorem 1.1] of Nadel–Tsuji, where Demailly's holomorphic inequalities on compact sets $K \subset X$ were used. The volume estimate is essential in their compactification theorem of complete Kähler manifolds with negative Ricci curvature (a generalization of the fact that arithmetic varieties can be complex–analytically compactified). The Morse inequalities (3.49) were also used by Napier–Ramachandran [39] to show that some quotients of the unit ball in $\mathbb{C}^n$ ($n > 2$) having a strongly pseudoconvex end have finite topological type (for the compactification of such quotients see also [37]).

Another generalization is a version of Theorem 0.1 for covering manifolds. Let $\widetilde{X}$ be a paracompact smooth manifold, such that there is a discrete group Γ acting freely on $\widetilde{X}$ with a compact quotient $X = \widetilde{X}/\Gamma$. Let $\pi_\Gamma : \widetilde{X} \longrightarrow X$ be the projection. Assume that there exists a Γ–invariant pre–quantum line bundle $\widetilde{L}$ on $\widetilde{X}$ and a Γ–invariant connection $\nabla^{\widetilde{L}}$ such that $\widetilde{\omega} = \frac{\sqrt{-1}}{2\pi}(\nabla^{\widetilde{L}})^2$ is non–degenerate. We endow $\widetilde{X}$ with a Γ–invariant Riemannian metric $g^{T\widetilde{X}}$. Let $\widetilde{J}$ be an Γ-invariant almost complex structure on $T\widetilde{X}$ which is separately compatible with $\widetilde{\omega}$ and $g^{T\widetilde{X}}$. Then $\mathbf{J}$, $g^{T\widetilde{X}}$, $\widetilde{\omega}$, $\widetilde{J}$, $\widetilde{L}$, $\widetilde{E}$ are the pull-back of the corresponding objects in Section 0 by the projection $\pi_\Gamma : \widetilde{X} \to X$. Let Φ be a smooth Hermitian section of $\mathrm{End}(E)$, and $\widetilde{\Phi} = \Phi \circ \pi_\Gamma$. Then the renormalized Bochner-Laplacian $\widetilde{\Delta}_{p,\widetilde{\Phi}}$ is

$$\widetilde{\Delta}_{p,\widetilde{\Phi}} = \Delta^{\widetilde{L}^p \otimes \widetilde{E}} - p\,(\tau \circ \pi_\Gamma) + \widetilde{\Phi}$$

which is an essentially self–adjoint operator. It is shown in [33, Corollary 4.7] that

(3.50)
$$\mathrm{Spec}\,\widetilde{\Delta}_{p,\widetilde{\Phi}} \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty[,$$

where C_L is the same constant as in Section 0 and μ_0 is introduced in (0.3). Let $\widetilde{\mathcal{H}}_p$ be the eigenspace of $\widetilde{\Delta}_{p,\widetilde{\Phi}}$ with the eigenvalues in $[-C_L, C_L]$:

$$(3.51) \qquad \widetilde{\mathcal{H}}_p = \operatorname{Range} E\big([-C_L, C_L], \widetilde{\Delta}_{p,\widetilde{\Phi}}\big),$$

where $E(\,\cdot\,, \widetilde{\Delta}_{p,\widetilde{\Phi}})$ is the spectral measure of $\widetilde{\Delta}_{p,\widetilde{\Phi}}$. From [33, Corollary 4.7], the von Neumann dimension of $\widetilde{\mathcal{H}}_p$ equals $d_p = \dim \mathcal{H}_p$. Finally, we define the generalized Bergman kernel $\widetilde{P}_{q,p}$ of $\widetilde{\Delta}_{p,\widetilde{\Phi}}$ as in Definition 1.1. Unlike most of the objects on $\widetilde{X}$, $\widetilde{P}_{q,p}$ is not Γ–invariant.

By (3.50) and the proof of Proposition 1.2, the analogue of (1.7) still holds on any compact set $K \subset \widetilde{X}$. By the finite propagation speed as the end of Section 1.1, we have:

Theorem 3.14. *We fix $0 < \varepsilon_0 < \inf_{x \in X}\{$injectivity radius of $x\}$. For any compact set $K \subset \widetilde{X}$ and $k, l \in \mathbb{N}$, there exists $C_{k,l,K} > 0$ such that for $x, x' \in K$, $p \in \mathbb{N}$,*

$$(3.52) \qquad \begin{aligned} \left|\widetilde{P}_{q,p}(x,x') - P_{q,p}(\pi_\Gamma(x), \pi_\Gamma(x'))\right|_{\mathscr{C}^l(K \times K)} &\leqslant C_{k,l,K}\, p^{-k-1}, \quad \text{if } d(x,x') < \varepsilon_0, \\ \left|\widetilde{P}_{q,p}(x,x')\right|_{\mathscr{C}^l(K \times K)} &\leqslant C_{k,l,K}\, p^{-k-1}, \quad \text{if } d(x,x') \geqslant \varepsilon_0. \end{aligned}$$

Especially, $\widetilde{P}_{q,p}(x,x)$ has the same asymptotic expansion as $B_{q,p}(\pi_\Gamma(x))$ in Theorem 0.1 on any compact set $K \subset \widetilde{X}$.

Remark 3.15. Theorem 3.14 works well for coverings of non-compact manifolds. Let (X, Θ) be a complete Kähler manifold, (L, h^L) be a holomorphic line bundle on X and let $\pi_\Gamma : \widetilde{X} \to X$ be a Galois covering of $X = \widetilde{X}/\Gamma$. Let $\widetilde{\Theta}$ and $(\widetilde{L}, h^{\widetilde{L}})$ be the inverse images of Θ and (L, h^L) through π_Γ. If (X, Θ) and (L, h^L) satisfy one of the conditions (3.39) or (3.41), $(\widetilde{X}, \widetilde{\Theta})$ and $(\widetilde{L}, h^{\widetilde{L}})$ have the same properties. We obtain therefore as in (3.49) (by integrating over a fundamental domain):

$$(3.53) \qquad \liminf_{p \longrightarrow \infty} p^{-n} \dim_\Gamma H^0_{(2)}(\widetilde{X}, \widetilde{L}^p) \geqslant \frac{1}{n!} \int_X \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n.$$

where $\dim_\Gamma$ is the von Neumann dimension of the Γ–module $H^0_{(2)}(X, L^p)$. Such type of inequalities imply as in [47] weak Lefschetz theorems à la Nori.

The example of non-compact manifolds emphasizes very well our philosophy about the existence of the asymptotic expansion of the Bergman kernel of Laplacian type operators when the power of the line bundle tends to ∞. In fact, from the argument in Section 1.1, the spectral gap property allows to localize our problem whenever the manifold X is compact or not. Thus from the argument in Section 1.3 or [20, §4.4], it implies the existence of the asymptotic expansion. Moreover, the formal power series trick in Section 1.5 gives a general way to compute the coefficients. As an example, we state the following result which is an extension of [20, Theorem 4.18] to non-compact case and we use the notation therein. Let (X, g^{TX}) be a Riemannian manifold with almost complex structure J which is compatible with g^{TX}, and let (L, h^L, ∇^L) and (E, h^E, ∇^E) be Hermitian bundles as in Introduction. We consider the associated spin^c Dirac operator D_p. Let $R^{T^{(1,0)}X}$ be

the curvature of the connection on $T^{(1,0)}X$ induced by ∇^{TX} by projection. We denote by $I_{\mathbb{C}\otimes E}$ the projection from $\Lambda(T^{*(0,1)}X)\otimes E$ onto $\mathbb{C}\otimes E$ under the decomposition $\Lambda(T^{*(0,1)}X) = \mathbb{C}\oplus\Lambda^{>0}(T^{*(0,1)}X)$.

Theorem 3.16. *We suppose that (X, g^{TX}) is complete and the scalar curvature r^X of (X, g^{TX}), R^E and $\mathrm{Tr}[R^{T^{(1,0)}X}]$ are uniformly bounded on (X, g^{TX}). Moreover, there exists $\varepsilon > 0$ such that on X,*

(3.54) $$\sqrt{-1}R^L(\cdot, J\cdot) > \varepsilon g^{TX}(\cdot, \cdot).$$

Then the smooth kernel $P_p(x, x')$ of the orthogonal projection from $\Omega^{0,\bullet}(X, L^p\otimes E)$ onto $\ker D_p$ with respect to $dv_X(x')$, has a full off-diagonal expansion analogous to Theorem 3.11 with $\mathbf{j}_0 = \det_{\mathbb{C}}|J^L|I_{\mathbb{C}\otimes E}$ as $p\to\infty$, uniformly on compact sets of X.

Proof. By the proof of [33, Theorem 2.5], we know that the spectral gap properties $\mathrm{Spec}D_p^2 \subset \{0\}\cup[2\mu_0 p - C_L, \infty[$ still holds under our condition. Thus we get Theorem 3.16. $\qquad\square$

3.6. **Singular polarizations.** Let X be a compact complex manifold. A *singular Kähler metric* on X is a closed, strictly positive $(1,1)$-current ω. This means there exist locally strictly plurisubharmonic functions $\varphi \in L^1_{loc}$ such that $\sqrt{-1}\partial\overline{\partial}\varphi = \omega$.

If the cohomology class of ω in $H^2(X, \mathbb{R})$ is integral, there exists a holomorphic line bundle (L, h^L), endowed with a singular Hermitian metric, such that $\frac{\sqrt{-1}}{2\pi}R^L = \omega$ in the sense of currents. We call (L, h^L) a *singular polarization* of ω. If we change the metric h^L, the curvature of the new metric will be in the same cohomology class as ω. In this case we speak of a polarization of $[\omega] \in H^2(X, \mathbb{R})$. Our purpose is to define an appropriate notion of polarized section of L^p, possibly by changing the metric of L, and study the associated Bergman kernel.

First recall that a Hermitian metric h^L is called *singular* if it is given in local trivialization by functions $e^{-\varphi}$ with $\varphi \in L^1_{\mathrm{loc}}$. The curvature current R^L of h^L is well defined and given locally by the currents $\partial\overline{\partial}\varphi$.

By the approximation theorem of Demailly [23, Theorem 1.1], we can assume that h^L is smooth outside a proper analytic set $\Sigma \subset X$. Using this fundamental fact, we will introduce in the sequel the *generalized Poincaré metric* on $X\smallsetminus\Sigma$. Let $\pi : \widetilde{X} \longrightarrow X$ be a resolution of singularities such that $\pi : \widetilde{X}\smallsetminus\pi^{-1}(\Sigma) \longrightarrow X\smallsetminus\Sigma$ is biholomorphic and $\pi^{-1}(\Sigma)$ is a divisor with only simple normal crossings. Let $g_0^{T\widetilde{X}}$ be an arbitrary smooth J-invariant metric on $\widetilde{X}$ and $\Theta'(\cdot, \cdot) = g_0^{T\widetilde{X}}(J\cdot, \cdot)$ the corresponding $(1,1)$-from. The generalized Poincaré metric on $X\smallsetminus\Sigma = \widetilde{X}\smallsetminus\pi^{-1}(\Sigma)$ is defined by (cf. [16, §2], [18, §6])

(3.55) $$\Theta_{\varepsilon_0} = \Theta' - \varepsilon_0\sqrt{-1}\sum_i \partial\overline{\partial}\log\left((-\log\|\sigma_i\|_i^2)^2\right), \quad 0 < \varepsilon_0 \ll 1 \text{ fixed},$$

where $\pi^{-1}(\Sigma) = \cup_i\Sigma_i$ is the decomposition into irreducible components Σ_i of $\pi^{-1}(\Sigma)$ and each Σ_i is non-singular; σ_i are sections of the associated holomorphic line bundle $[\Sigma_i]$ which vanish to first order on Σ_i, and $\|\sigma_i\|_i$ is the norm for a smooth Hermitian metric on $[\Sigma_i]$ such that $\|\sigma_i\|_i < 1$. The fist part of the following lemma generalizes previous work on the generalized Poincaré metric [16, 18, 51].

$\boxed{\text{lem-poin}}$ **Lemma 3.17.** (i) *The generalized Poincaré metric* $\overset{\text{poin}}{(3.55)}$ *is a complete Hermitian metric of finite volume. It has bounded torsion and the curvature* $R^{\det}$ *is also bounded.*

(ii) *If* (E, h^E) *is a holomorphic bundle over* X *with smooth Hermitian metric and* $H^0_{(2)}(X \smallsetminus \Sigma, E) = \left\{ u \in L^{0,0}_2(X \smallsetminus \Sigma, E, \Theta_{\varepsilon_0}, h^E) : \overline{\partial}^E u = 0 \right\}$ *then* $H^0_{(2)}(X \smallsetminus \Sigma, E) = H^0(X, E)$.

Proof. To describe the metric more precisely we denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and by $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$. On the product $(\mathbb{D}^*)^l \times \mathbb{D}^{n-l}$ we introduce the metric

$\boxed{\text{compl12,16}}$ (3.56) $$\omega_P = \frac{\sqrt{-1}}{2} \sum_{k=1}^{l} \frac{dz_k \wedge d\overline{z}_k}{|z_k|^2 (\log |z_k|^2)^2} + \frac{\sqrt{-1}}{2} \sum_{k=l+1}^{n} dz_k \wedge d\overline{z}_k.$$

For any point $p \in \pi^{-1}(\Sigma)$ there exists a coordinate neighbourhood U of p isomorphic to $\mathbb{D}^n$ in which $(X \smallsetminus \pi^{-1}(\Sigma)) \cap U = \{ z = (z_1, \ldots, z_n) : z_1 \neq 0, \ldots, z_l \neq 0 \}$. Such coordinates are called special. We endow $(X \smallsetminus \pi^{-1}(\Sigma)) \cap U \cong (\mathbb{D}^*)^l \times \mathbb{D}^{n-l}$ with the metric $\overset{\text{compl12,16}}{(3.56)}$.

We have

$\boxed{\text{zar-ell10}}$ (3.57) $\quad -\sqrt{-1}\partial \overline{\partial} \log \left((-\log \|\sigma_i\|_i^2)^2 \right) = 2\sqrt{-1} \left(\frac{R^{[\Sigma_i]}}{\log \|\sigma_i\|_i^2} + \frac{\partial \log \|\sigma_i\|_i^2 \wedge \overline{\partial} \log \|\sigma_i\|_i^2}{(\log \|\sigma_i\|_i^2)^2} \right).$

Since the terms $R^{[\Sigma_i]} / \log \|\sigma_i\|_i^2$ tend to zero as we approach Σ,

$\boxed{\text{zar10}}$ (3.58) $$\Theta' + 2\sqrt{-1}\varepsilon_0 \sum_i \frac{R^{[\Sigma_i]}}{\log \|\sigma_i\|_i^2} > 0,$$

for ε_0 small enough. For a $(1,1)$-form Ω, we write $\Omega > 0$ (resp. $\geqslant 0$) if $\Omega(\cdot, J\cdot) > 0$ (resp. $\geqslant 0$). The last term in $\overset{\text{zar-ell10}}{(3.57)}$ is $\geqslant 0$, as $\sqrt{-1}\partial g \wedge \overline{\partial} g \geqslant 0$ for any real function g on $\widetilde{X}$. Thus Θ_{ε_0} is positive for ε_0 small enough.

We choose special coordinates in a neighborhood U of x_0 in which Σ_j has the equation $z_j = 0$ for $j = 1, \ldots, k$ and Σ_j, $j > k$, do not meet U. Then for $1 \leqslant i \leqslant k$, $\|\sigma_i\|_i^2 = u_i |z_i|^2$ for some positive smooth function u_i on U and

$\boxed{\text{zar-ell11}}$ (3.59) $$\frac{\partial \log \|\sigma_i\|_i^2 \wedge \overline{\partial} \log \|\sigma_i\|_i^2}{(\log \|\sigma_i\|_i^2)^2} = \frac{dz_i \wedge d\overline{z}_i + v_i}{|z_i|^2 (\log \|\sigma_i\|_i^2)^2}$$

where v_i is a smooth $(1,1)$-form on U such that $v_i|_{z_i=0} = 0$.

As in $\overset{\text{Zu:79}}{[51, \text{Prop. 3.4}]}$, we show using $\overset{\text{zar-ell10}}{(3.57)}$ and $\overset{\text{zar-ell11}}{(3.59)}$ that the metrics $\overset{\text{poin}}{(3.55)}$ and $\overset{\text{compl12,16}}{(3.56)}$ are equivalent for $|z_i|$ small. From this the first assertion of (i) follows.

Recall that $R^{\det}$ is the curvature of the holomorphic Hermitian connection on $\det(T^{(1,0)}X)$ with respect to the Hermitian metric induced by Θ_{ε_0}. We wish to show that there exist a constant $C > 0$ such that

$\boxed{\text{zar-ell9}}$ (3.60) $$-C\Theta_{\varepsilon_0} < \sqrt{-1} R^{\det} < C\Theta_{\varepsilon_0}, \quad |T_{\varepsilon_0}|_{\Theta_{\varepsilon_0}} < C.$$

where $T_{\varepsilon_0} = [i(\Theta_{\varepsilon_0}), \partial \Theta_{\varepsilon_0}]$ is the torsion operator of Θ_{ε_0} and $|T_{\varepsilon_0}|_{\Theta_{\varepsilon_0}}$ is its norm with respect to Θ_{ε_0}. Since $\partial \Theta_{\varepsilon_0} = \partial \Theta'$ by $\overset{\text{poin}}{(3.55)}$, $\partial \Theta_{\varepsilon_0}$ extends smoothly over $\widetilde{X}$, and thus we get the second relation of $\overset{\text{zar-ell9}}{(3.60)}$.

We turn now to the first condition of (3.60). By (3.55), (3.57) and (3.59), we know that

(3.61) $$\Theta_{\varepsilon_0}^n = \frac{2^k \varepsilon_0^k + \beta(z)}{\prod_{i=1}^k |z_i|^2 (\log \|\sigma_i\|_i^2)^2} \prod_{j=1}^n (\sqrt{-1} dz_j \wedge d\overline{z}_j) =: \gamma(z) \prod_{j=1}^n (\sqrt{-1} dz_j \wedge d\overline{z}_j).$$

Here $\beta(z)$ is a polynomial in the functions $a_{i\alpha}(z) |z_i|^2 (\log \|\sigma_i\|_i^2)^2$, $b_{i\alpha}(z) |z_i|^2 \log \|\sigma_i\|_i^2$ and $c_{i\alpha}(z)$, $(1 \leqslant i \leqslant k)$, with $a_{i\alpha}, b_{i\alpha}$ smooth functions on U and $c_{i\alpha}$ smooth functions on U such that $c_{i\alpha}(z)|_{z_i=0} = 0$. Moreover, $2^k \varepsilon_0^k + \beta(z)$ is positive on U as Θ_{ε_0} is positive. Since

(3.62) $$|\tfrac{\partial}{\partial z_1} \wedge \cdots \wedge \tfrac{\partial}{\partial z_n}|_{\Theta_{\varepsilon_0}}^2 \prod_{j=1}^n (\sqrt{-1} dz_j \wedge d\overline{z}_j) = \Theta_{\varepsilon_0}^n.$$

we get from (3.61) and (3.62)

(3.63) $$R^{\det} = -\partial\overline{\partial} \log \gamma(z) = -\partial\overline{\partial} \log \left(2^k \varepsilon_0^k + \beta(z) \right) + \sum_{i=1}^k \partial\overline{\partial} \log \left((\log \|\sigma_i\|_i^2)^2 \right).$$

By (3.57), the last term of (3.63) it is bounded with respect to Θ_{ε_0}. To examine the first term of the sum we write

(3.64) $$\partial\overline{\partial} \log(2^k \varepsilon_0^k + \beta(z)) = \frac{\partial\overline{\partial}\beta(z)}{2^k \varepsilon_0^k + \beta(z)} - \frac{\partial\beta(z) \wedge \overline{\partial}\beta(z)}{(2^k \varepsilon_0^k + \beta(z))^2}.$$

Now we observe that for $W_i(z) = |z_i|^2 (\log \|\sigma_i\|_i^2)^2$ or $|z_i|^2 \log \|\sigma_i\|_i^2$, the terms $\partial\overline{\partial} W_i(z)$, $\partial W_i(z)$, $\overline{\partial} W_i(z)$ are bounded with respect to the Poincaré metric (3.56), thus with respect to Θ_{ε_0}. Combining with the form of β given after (3.61), this achieves the proof of (3.60).

Let us prove (ii). First observe that Θ_{ε_0} dominates the euclidian metric in special coordinates near $\pi^{-1}(\Sigma)$, being equivalent with (3.56). Therefore it dominates some positive multiple of any smooth Hermitian metric on $\widetilde{X}$. We deduce that, given a smooth Hermitian metric Θ'' on X, there exists a constant $c > 0$ such that $\Theta_{\varepsilon_0} \geqslant c\Theta''$ on $X \smallsetminus \Sigma$. It follows that elements of $H_{(2)}^0(X \smallsetminus \Sigma, E)$ are L^2 integrable with respect to the smooth metrics Θ'' and h^E over X, which entails they extend holomorphically to sections of $H^0(X, E)$ by [22, Lemme 6.9]. We have therefore $H_{(2)}^0(X \smallsetminus \Sigma, E) \subset H^0(X, E)$. The reverse inclusion follows from the finiteness of the volume of $X \smallsetminus \Sigma$ in the Poincaré metric. $\square$

We can construct as in [43, §4] a singular Hermitian line bundle $(\widetilde{L}, h^{\widetilde{L}})$ on $\widetilde{X}$ which is strictly positive and $\widetilde{L}|_{\widetilde{X} \smallsetminus \pi^{-1}(\Sigma)} \cong \pi^*(L^{p_0})$, for some $p_0 \in \mathbb{N}$. We introduce on $L|_{X \smallsetminus \Sigma}$ the metric $(h^{\widetilde{L}})^{1/p_0}$ whose curvature extends to a strictly positive $(1,1)$-current on $\widetilde{X}$. Set

(3.65a) $$h_\varepsilon^L = (h^{\widetilde{L}})^{1/p_0} \prod_i (-\log \|\sigma_i\|_i^2)^\varepsilon, \quad 0 < \varepsilon \ll 1,$$

(3.65b) $$H_{(2)}^0(X \smallsetminus \Sigma, L^p) = \left\{ u \in L_2^{0,0}(X \smallsetminus \Sigma, L^p, \Theta_{\varepsilon_0}, h_\varepsilon^L) : \overline{\partial}^{L^p} u = 0 \right\}.$$

The space $H_{(2)}^0(X \smallsetminus \Sigma, L^p)$ is the space of L^2-holomorphic sections relative to the metrics Θ_{ε_0} on $X \smallsetminus \Sigma$ and h_ε^L on $L|_{X \smallsetminus \Sigma}$. Since $(h^{\widetilde{L}})^{1/p_0}$ is bounded away from zero

(having plurisubharmonic weights), the elements of this space are L^2 integrable with respect to the Poincaré metric and a smooth metric h_*^L of L over whole X. By Lemma 3.17 (ii) we have $H^0_{(2)}(X \smallsetminus \Sigma, L^p) \subset H^0(X, L^p)$. (Here we cannot infer the other inclusion since h^L might be infinity on Σ.) The space $H^0_{(2)}(X \smallsetminus \Sigma, L^p)$ is our space of polarized sections of L^p.

Corollary 3.18. *Let (X, ω) be a compact complex manifold with a singular Kähler metric with integral cohomology class. Let (L, h^L) be a singular polarization of $[\omega]$ with strictly positive curvature current having singular support along a proper analytic set Σ. Let (E, h^E) be a holomorphic Hermitian bundle on X. Then the Bergman kernel of the space of polarized sections (3.65b) has the asymptotic expansion as in Theorem 3.11 for $X \smallsetminus \Sigma$.*

Proof. We will apply Theorem 3.11 to the non–Kähler Hermitian manifold $(X \smallsetminus \Sigma, \Theta_{\varepsilon_0})$ equipped with the Hermitian bundle $(L|_{X \smallsetminus \Sigma}, h_\varepsilon^L)$ and (E, h^E). Certainly, R^E is bounded. Thus we have to show that there exist constants $\eta > 0$, $C > 0$ such that

(3.66) $$\sqrt{-1} R^{(L|_{X \smallsetminus \Sigma}, h_\varepsilon^L)} > \eta \Theta_{\varepsilon_0} \, , \quad \sqrt{-1} R^{\det} > -C \Theta_{\varepsilon_0} \, , \quad |T_{\varepsilon_0}|_{\Theta_{\varepsilon_0}} < C \, .$$

where $T_{\varepsilon_0} = [i(\Theta_{\varepsilon_0}), \partial \Theta_{\varepsilon_0}]$ is the torsion operator of Θ_{ε_0} and $|T_{\varepsilon_0}|_{\Theta_{\varepsilon_0}}$ is its norm with respect to Θ_{ε_0}. The first one results for all ε small enough from (3.55), (3.65a) and the fact that the curvature of $(h^{\widetilde{L}})^{1/p_0}$ extends to a strictly positive $(1,1)$–current on $\widetilde{X}$ (dominating a small positive multiple of Θ' on $\widetilde{X}$). The second and third relations were proved in (3.60). This achieves the proof of Corollary 3.18. $\qquad\square$

Remark 3.19. (a) Corollary 3.18 gives an alternative proof of the characterization of Moishezon manifolds given by Ji–Shiffman [30], Bonavero [8] and Takayama [43]. Indeed, any Moishezon manifold possesses a strictly positive singular polarization (L, h^L). Conversely, suppose X has such a polarization. Then as in (3.49), we have $\dim H^0_{(2)}(X \smallsetminus \Sigma, L^p) \geqslant C p^n$ for some $C > 0$ and p large enough. Since $H^0_{(2)}(X \smallsetminus \Sigma, L^p) \subset H^0(X, L^p)$, it follows that L is big and X is Moishezon.
   (b) By [24, Proposition 6.6. (f)], or [48], any big line bundle L on a projective manifold carries a singular Hermitian metric having strictly positive curvature current with singularities along a proper analytic set.

   (c) The results of this section hold also for reduced compact complex spaces X possessing a holomorphic line bundle L with singular Hermitian metric h^L having positive curvature current (see [43] for definitions). This is just a matter of desingularizing X. As space of polarized sections we obtain $H^0_{(2)}(X \smallsetminus \Sigma, L^p)$ where Σ is an analytic set containing the singular set of X.
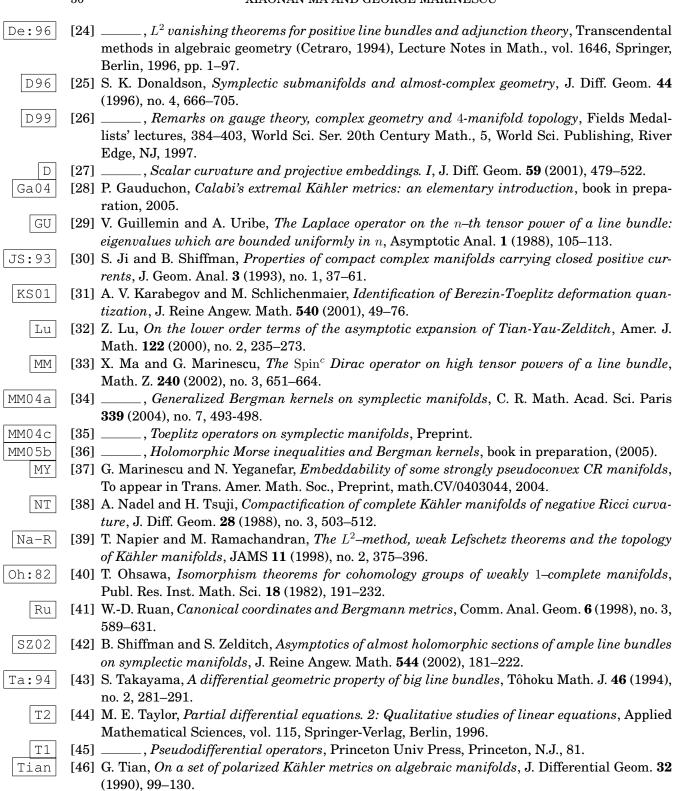
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