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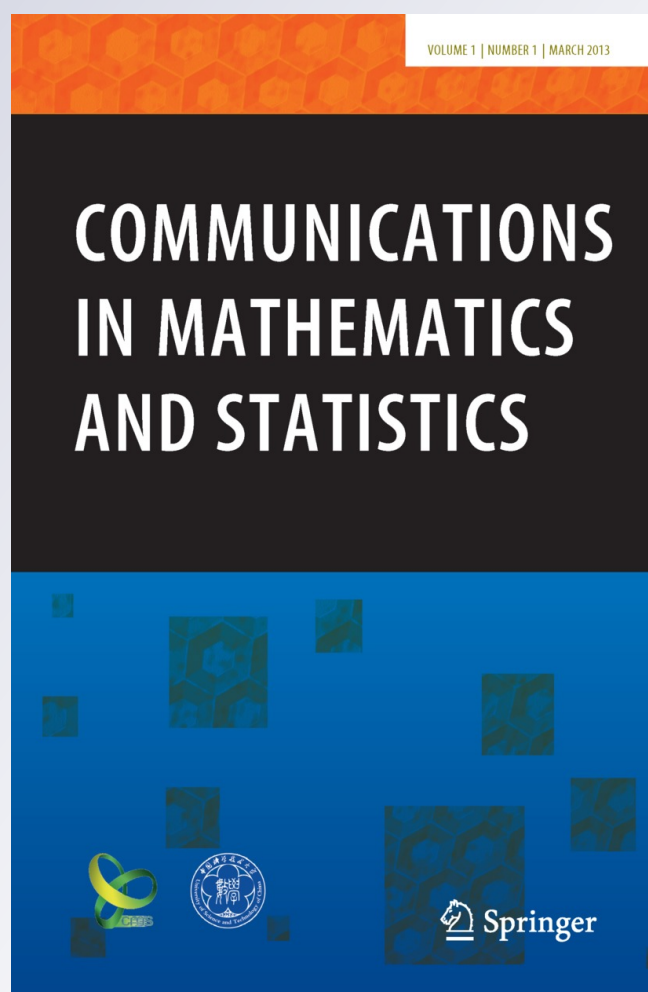
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Remark on the Off-Diagonal Expansion of the Bergman Kernel on Compact Kähler Manifolds

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Abstract In this short note, we compare our previous work on the off-diagonal expansion of the Bergman kernel and the preprint of Lu–Shiffman ([arXiv:1301.2166](https://arxiv.org/abs/1301.2166)). In particular, we note that the vanishing of the coefficient of $p^{-1/2}$ is implicitly contained in Dai–Liu–Ma’s work (*J. Differ. Geom.* 72(1), 1–41, 2006) and was explicitly stated in our book (*Holomorphic Morse inequalities and Bergman kernels. Progress in Math.*, vol. 254, 2007).

Keywords Kähler manifold · Bergman kernel of a positive line bundle

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In this short note we revisit the calculations of some coefficients of the off-diagonal expansion of the Bergman kernel from our previous work [4, 5].

Let (X, ω) be a compact Kähler manifold of $\dim_{\mathbb{C}} X = n$ with Kähler form ω . Let (L, h^L) be a holomorphic Hermitian line bundle on X , and let (E, h^E) be a holomorphic Hermitian vector bundle on X . Let ∇^L, ∇^E be the holomorphic Hermitian connections on $(L, h^L), (E, h^E)$ with curvatures $R^L = (\nabla^L)^2, R^E = (\nabla^E)^2$, respectively. We assume that (L, h^L, ∇^L) is a prequantum line bundle, i.e., $\omega = \frac{\sqrt{-1}}{2\pi} R^L$.

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Let $P_p(x, x')$ be the Bergman kernel of $L^p \otimes E$ with respect to h^L, h^E and the Riemannian volume form $dv_X = \omega^n/n!$. This is the integral kernel of the orthogonal projection from $\mathcal{C}^\infty(X, L^p \otimes E)$ to the space of holomorphic sections $H^0(X, L^p \otimes E)$ (cf. [4, §4.1.1]).

We fix $x_0 \in X$. We identify the ball $B^{T_{x_0}X}(0, \varepsilon)$ in the tangent space $T_{x_0}X$ to the ball $B^X(x_0, \varepsilon)$ in X by the exponential map (cf. [4, §4.1.3]). For $Z \in B^{T_{x_0}X}(0, \varepsilon)$ we identify $(L_Z, h_Z^L), (E_Z, h_Z^E)$ to $(L_{x_0}, h_{x_0}^L), (E_{x_0}, h_{x_0}^E)$ by parallel transport with respect to the connections ∇^L, ∇^E along the curve $\gamma_Z : [0, 1] \ni u \rightarrow \exp_{x_0}^X(uZ)$. Then $P_p(x, x')$ induces a smooth section $(Z, Z') \mapsto P_{p,x_0}(Z, Z')$ of $\pi^* \text{End}(E)$ over $\{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}$, which depends smoothly on x_0 , with $\pi : TX \times_X TX \rightarrow X$ the natural projection. If dv_{TX} is the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$, there exists a smooth positive function $\kappa_{x_0} : T_{x_0}X \rightarrow \mathbb{R}$, defined by

$$dv_X(Z) = \kappa_{x_0}(Z)dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1. \tag{1}$$

For $Z \in T_{x_0}X \cong \mathbb{R}^{2n}$, we denote $z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}$ its complex coordinates, and set

$$\mathcal{P}(Z, Z') = \exp\left(-\frac{\pi}{2} \sum_{i=1}^n (|z_i|^2 + |z'_i|^2 - 2z_i \bar{z}'_i)\right). \tag{2}$$

The near off-diagonal asymptotic expansion of the Bergman kernel in the form established [4, Theorem 4.1.24] is the following.

Theorem 1 *Given $k, m' \in \mathbb{N}, \sigma > 0$, there exists $C > 0$ such that if $p \geq 1, x_0 \in X, Z, Z' \in T_{x_0}X, |Z|, |Z'| \leq \sigma/\sqrt{p}$,*

$$\left| \frac{1}{p^n} P_p(Z, Z') - \sum_{r=0}^k \mathcal{F}_r(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') p^{-\frac{r}{2}} \right|_{\mathcal{C}^{m'}(X)} \leq Cp^{-\frac{k+1}{2}}. \tag{3}$$

where $\mathcal{C}^{m'}(X)$ is the $\mathcal{C}^{m'}$ -norm with respect to the parameter x_0 ,

$$\mathcal{F}_r(Z, Z') = J_r(Z, Z') \mathcal{P}(Z, Z'), \tag{4}$$

$J_r(Z, Z') \in \text{End}(E)_{x_0}$ are polynomials in Z, Z' with the same parity as r and $\text{deg } J_r(Z, Z') \leq 3r$, whose coefficients are polynomials in R^{TX} (the curvature of the Levi-Civita connection on TX), R^E and their derivatives of order $\leq r - 2$.

Remark 2 For the above properties of $J_r(Z, Z')$ see [4, Theorem 4.1.21 and end of §4.1.8]. They are also given in [2, Theorem 4.6, (4.107) and (4.117)]. Moreover, by [4, (1.2.19) and (4.1.28)], κ has a Taylor expansion with coefficients the derivatives of R^{TX} . As in [4, (4.1.101)] or [5, Lemma 3.1 and (3.27)] we have

$$\kappa(Z)^{-1/2} = 1 + \frac{1}{6} \text{Ric}(z, \bar{z}) + \mathcal{O}(|Z|^3) = 1 + \frac{1}{3} R_{\ell\bar{k}k\bar{q}} z_\ell \bar{z}_q + \mathcal{O}(|Z|^3). \tag{5}$$

Note that a more powerful result than the near-off diagonal expansion from Theorem 1 holds. Namely, by [2, Theorem 4.18'] and [4, Theorem 4.2.1], the full off-diagonal expansion of the Bergman kernel holds (even for symplectic manifolds),

i.e., an analogous result to (3) for $|Z|, |Z'| \leq \varepsilon$. This appears naturally in the proof of the diagonal expansion of the Bergman kernel on orbifolds in [2, (5.25)] or [4, (5.4.14), (5.4.23)].

Proposition 3 *The coefficient \mathcal{F}_1 vanishes identically: $\mathcal{F}_1(Z, Z') = 0$ for all Z, Z' . Therefore the coefficient of $p^{-1/2}$ in the expansion of $p^{-n} P_p(p^{-1/2}Z, p^{-1/2}Z')$ vanishes, so the latter converges to $\mathcal{F}_0(Z, Z')$ at rate p^{-1} as $p \rightarrow \infty$.*

Proof This is [4, Remark 4.1.26] or [5, (2.19)], see also [2, (4.107), (4.117), (5.4)]. \square

When $E = \mathbb{C}$ with trivial metric, the vanishing of \mathcal{F}_1 was recently rediscovered in [3, Theorem 2.1] ($b_1(u, v) = 0$ therein). In [3] an equivalent formulation [6] of the expansion (3) is used, based on the analysis of the Szegő kernel from [1]. In [3, Theorem 2.1] further off-diagonal coefficients $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ are calculated in the K -coordinates. From [5, (3.22)], we see that the usual normal coordinates are K -coordinates up to order at least 3. This shows that the vanishing of \mathcal{F}_1 given by Proposition 3 implies the vanishing of b_1 calculated with the help of in K -coordinates. We wish to point out that we calculated in [5] the coefficients $\mathcal{F}_1, \dots, \mathcal{F}_4$ on the diagonal, using the off-diagonal expansion (3) and evaluating \mathcal{F}_r for $Z = Z' = 0$. Thus, off-diagonal formulas for $\mathcal{F}_1, \dots, \mathcal{F}_4$ are implicitly contained in [5]. We show below how the coefficient \mathcal{F}_2 can be calculated in the framework of [5].

We use the notation in [5, (3.6)], then $\mathbf{r} = 8R_{m\bar{q}q\bar{m}}$ is the scalar curvature.

Proposition 4 *The coefficient J_2 in (4) is given by*

$$\begin{aligned}
 J_2(Z, Z') = & -\frac{\pi}{12} R_{k\bar{m}\ell\bar{q}} (z_k z_\ell \bar{z}_m \bar{z}_q + 6z_k z_\ell \bar{z}'_m \bar{z}'_q - 4z_k z_\ell \bar{z}_m \bar{z}'_q \\
 & - 4z_k z'_\ell \bar{z}_m \bar{z}'_q + z'_k z'_\ell \bar{z}'_m \bar{z}'_q) \\
 & - \frac{1}{3} R_{k\bar{m}q\bar{q}} (z_k \bar{z}_m + z'_k \bar{z}'_m) + \frac{1}{8\pi} \mathbf{r} + \frac{1}{\pi} R_{q\bar{q}}^E \\
 & - \frac{1}{2} (z_\ell \bar{z}_q - 2z_\ell \bar{z}'_q + z'_\ell \bar{z}'_q) R_{\ell\bar{q}}^E.
 \end{aligned} \tag{6}$$

Remark 5 Setting $Z = Z' = 0$ in (6) we obtain the coefficient $\mathbf{b}_1(x_0) = J_2(0, 0) = \frac{1}{8\pi} \mathbf{r} + \frac{1}{\pi} R_{q\bar{q}}^E$ of p^{-1} of the (diagonal) expansion of $p^{-n} P_p(x_0, x_0)$, cf. [4, Theorem 4.1.2].

Moreover, in order to obtain the coefficient of p^{-1} in the expansion (3) we multiply $\mathcal{F}_2(\sqrt{p}Z, \sqrt{p}Z')$ to the expansion of $\kappa(Z)^{-1/2} \kappa(Z')^{-1/2}$ with respect to the variable $\sqrt{p}Z$ obtained from (5). If $E = \mathbb{C}$ the result is a polynomial which is the sum of a homogeneous polynomial of order four and a constant, similar to [3].

Proof of Proposition 4 Set

$$\begin{aligned}
 b_i &= -2\frac{\partial}{\partial z_i} + \pi \bar{z}_i, & b_i^+ &= 2\frac{\partial}{\partial \bar{z}_i} + \pi z_i, & \mathcal{L} &= \sum_{i=1}^n b_i b_i^+, \\
 \widetilde{\mathcal{O}}_2 &= \frac{b_m b_q}{48\pi} R_{k\bar{m}\ell\bar{q}} z_k z_l + \frac{b_q}{3\pi} R_{\ell\bar{k}\bar{q}} z^\ell - \frac{b_q}{12} R_{k\bar{m}\ell\bar{q}} z_k z_l \bar{z}'_m.
 \end{aligned}
 \tag{7}$$

By [4, (4.1.107)] or [5, (2.19)], we have

$$\mathcal{F}_{2,x_0} = -\mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_2 \mathcal{P} - \mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P}^\perp.
 \tag{8}$$

By [5, (4.1a), (4.7)] we have

$$(\mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_2 \mathcal{P})(Z, Z') = (\mathcal{L}^{-1} \mathcal{O}_2 \mathcal{P})(Z, Z') = \left\{ \widetilde{\mathcal{O}}_2 + \frac{b_q}{4\pi} R_{\ell\bar{q}}^E z^\ell \right\} \mathcal{P}(Z, Z').
 \tag{9}$$

By the symmetry properties of the curvature [5, Lemma 3.1] we have

$$R_{k\bar{m}\ell\bar{q}} = R_{\ell\bar{m}k\bar{q}} = R_{k\bar{q}\ell\bar{m}} = R_{\ell\bar{q}k\bar{m}}, \quad \overline{R_{k\bar{m}\ell\bar{q}}} = R_{m\bar{k}q\bar{\ell}}, \quad (R_{k\bar{q}}^E)^* = R_{q\bar{k}}^E.
 \tag{10}$$

We use throughout that $[g(z, \bar{z}), b_j] = 2\frac{\partial}{\partial z_j} g(z, \bar{z})$ for any polynomial $g(z, \bar{z})$ (cf. [5, (1.7)]). Hence from (10), we get

$$\begin{aligned}
 b_q R_{k\bar{k}\ell\bar{q}} z^\ell &= R_{k\bar{k}\ell\bar{q}} z^\ell b_q - 2R_{k\bar{k}q\bar{q}}, \\
 b_q R_{k\bar{m}\ell\bar{q}} z_k z_l &= -4R_{k\bar{m}q\bar{q}} z_k + R_{k\bar{m}\ell\bar{q}} z_k z_l b_q, \\
 b_m b_q R_{k\bar{m}\ell\bar{q}} z_k z_l &= R_{k\bar{m}\ell\bar{q}} z_k z_l b_m b_q - 8R_{k\bar{k}\ell\bar{q}} z^\ell b_q + 8R_{m\bar{m}q\bar{q}}.
 \end{aligned}
 \tag{11}$$

Thus from (7) and (11), we get

$$\widetilde{\mathcal{O}}_2 = \frac{1}{48\pi} R_{k\bar{m}\ell\bar{q}} z_k z_l (b_m - 4\pi \bar{z}'_m) b_q + \frac{1}{6\pi} R_{k\bar{k}\ell\bar{q}} z^\ell b_q - \frac{1}{2\pi} R_{m\bar{m}q\bar{q}} + \frac{1}{3} R_{k\bar{m}q\bar{q}} z_k \bar{z}'_m.
 \tag{12}$$

Now, $(b_i \mathcal{P})(Z, Z') = 2\pi(\bar{z}_i - \bar{z}'_i) \mathcal{P}(Z, Z')$, see [4, (4.1.108)] or [5, (4.2)]. Therefore

$$\begin{aligned}
 (\widetilde{\mathcal{O}}_2 \mathcal{P})(Z, Z') &= \left[\frac{\pi}{12} R_{k\bar{m}\ell\bar{q}} z_k z_l (\bar{z}_m - 3\bar{z}'_m) (\bar{z}_q - \bar{z}'_q) + \frac{1}{3} R_{k\bar{k}\ell\bar{q}} z^\ell (\bar{z}_q - \bar{z}'_q) \right. \\
 &\quad \left. - \frac{1}{2\pi} R_{m\bar{m}q\bar{q}} + \frac{1}{3} R_{k\bar{m}q\bar{q}} z_k \bar{z}'_m \right] \mathcal{P}(Z, Z') \\
 &= \left[\frac{\pi}{12} R_{k\bar{m}\ell\bar{q}} z_k z_l (\bar{z}_m - 3\bar{z}'_m) (\bar{z}_q - \bar{z}'_q) + \frac{1}{3} R_{k\bar{m}q\bar{q}} z_k \bar{z}'_m \right. \\
 &\quad \left. - \frac{1}{2\pi} R_{m\bar{m}q\bar{q}} \right] \mathcal{P}(Z, Z').
 \end{aligned}
 \tag{13}$$

We know that for an operator T we have $T^*(Z, Z') = \overline{T(Z', Z)}$, thus

$$\begin{aligned}
 (\widetilde{\mathcal{O}}_2 \mathcal{P})^*(Z, Z') &= \left[\frac{\pi}{12} R_{k\bar{m}\ell\bar{q}} \bar{z}'_m \bar{z}'_q (z'_k - 3z_k)(z'_\ell - z_\ell) + \frac{1}{3} R_{k\bar{m}q\bar{q}} \bar{z}'_m z'_k \right. \\
 &\quad \left. - \frac{1}{2\pi} R_{m\bar{m}q\bar{q}} \right] \mathcal{P}(Z, Z'). \tag{14}
 \end{aligned}$$

We have $(\mathcal{P} \mathcal{O}_2 \mathcal{L}^{-1} \mathcal{P}^\perp)^* = \mathcal{L}^{-1} \mathcal{P}^\perp \mathcal{O}_2 \mathcal{P}$ by [4, Theorem 4.1.8], so from (13) and (14), we obtain the factor of $R_{k\bar{m}\ell\bar{q}}$ in (6).

Let us calculate the contribution of the last term (curvature of E). We have

$$-\left(\frac{b_q}{4\pi} R_{\ell\bar{q}}^E z_\ell \mathcal{P} \right) (Z, Z') = \left(\frac{1}{2\pi} R_{q\bar{q}}^E - \frac{1}{2} z_\ell (\bar{z}_q - \bar{z}'_q) R_{\ell\bar{q}}^E \right) \mathcal{P}(Z, Z') \tag{15}$$

and by (10), we also have

$$-\left(\frac{b_q}{4\pi} R_{\ell\bar{q}}^E z_\ell \mathcal{P} \right)^* (Z, Z') = \left(\frac{1}{2\pi} R_{q\bar{q}}^E - \frac{1}{2} \bar{z}'_\ell (z'_q - z_q) R_{q\bar{q}}^E \right) \mathcal{P}(Z, Z'). \tag{16}$$

The contribution to J_2 of the term on E is thus given by the last two terms in (6). \square

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