CONVERGENCE OF FUBINI-STUDY CURRENTS FOR ORBIFOLD LINE BUNDLES

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ABSTRACT. We discuss positive closed currents and Fubini-Study currents on orbifolds, as well as Bergman kernels of singular Hermitian orbifold line bundles. We prove that the Fubini-Study currents associated to high powers of a semipositive singular line bundle converge weakly to the curvature current on the set where the curvature is strictly positive, generalizing a well-known theorem of Tian. We include applications to the asymptotic distribution of zeros of random holomorphic sections.

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1. INTRODUCTION

Let X be a compact complex manifold and (L,h) be a positive holomorphic line bundle over X, with Chern curvature form $\omega = c_1(L,h)$. By Kodaira's embedding theorem, high powers L^p give rise to embeddings

$$\Phi_p : X \to \mathbb{P}(H^0(X,L^p)^*), \quad x \mapsto \{S \in H^0(X,L^p) : S(x) = 0\}$$

into projective spaces. The Hermitian metric h on L and the volume form $\omega^n/n!$ on X induce an L^2 inner product on $H^0(X,L^p)$, hence an associated Fubini-Study metric ω_{FS} on $\mathbb{P}(H^0(X,L^p)^*)$. This gives an *induced Fubini-Study metric* $\gamma_p = \Phi_p^*\omega_{FS}$ on X. The induced Fubini-Study metrics are in some sense "algebraic" objects used to approximate the "transcendental" metric ω. The relations between these metrics is given by

$$\frac{1}{p}\gamma_p - \omega = \frac{i}{2\pi p}\partial\bar{\partial}\log P_p\,, \tag{1}$$

where P_p is the Bergman kernel function of $H^0(X,L^p)$. Following a suggestion of Yau [Y], Tian [T, Theorem A] proved that

$$\frac{1}{p}\gamma_p \to \omega\,, \quad \text{as } p \to \infty, \text{ in the } \mathscr{C}^2\text{−topology}. \tag{2}$$

Later, Ruan [R] proved the convergence in the $\mathscr{C}^\infty$−topology and improved the estimate of the convergence speed. In view of (1), the proof consists in showing the asymptotics $P_p(x) = p^n + o(p^n)$ in the $\mathscr{C}^k$−topology ($k \geq 2$), which implies

$$\frac{1}{p}\log P_p \to 0\,, \quad \text{as } p \to \infty, \text{ in the } \mathscr{C}^k\text{−topology}. \tag{3}$$

Catlin [Ca], Zelditch [Z], Dai-Liu-Ma [DLM1], and [MM1, MM2, MM3] showed the full asymptotics

$$P_p(x) = \sum_{r=0}^{\infty} b_r(x)p^{n-r} + O(p^{-\infty}), \quad \text{as } p \to \infty, \text{ in the } \mathscr{C}^k\text{−topology}, \tag{4}$$

which obviously implies (3). The asymptotics (2) and (4) play an important role in the Yau-Tian-Donaldson programme of relating the existence of constant scalar curvature Kähler metrics to a suitable notion of stability, see e. g. [D, RT], and also in the equidistribution theorems for zeros of random sections [SZ1, SZ2, DS, SZ3, DMS, CM].

There are several natural generalizations of these results in the presence of singularities. One of them is for a compact orbifold X and a positive orbifold line bundle $L \longrightarrow X$. Then γ_p is degenerate at the points with non-trivial isotropy group, that is at points of X^{orb}_{sing}. However, $\frac{1}{p}\gamma_p$ still approximates the original metric on the regular set X^{orb}_{reg}. Dai-Liu-Ma [DLM1] (see also [MM2, Theorem 5.4.19]) showed that for any $0 \le \alpha < 1$ we have $\frac{1}{p}\log P_p \to 0$ as $p \to \infty$, in the $\mathscr{C}^{1,\alpha}-$topology on local orbifold charts and $\frac{1}{p}\gamma_p \to \omega$ as $p \to \infty$ in the $\mathscr{C}^k-$topology on compact sets of X^{orb}_{reg}. This is based on the orbifold analogue of (4), see [DLM1], [MM2, Theorems 5.4.10–11] and also [DLM2, RT] for results on orbifolds with cyclic stabiliser groups.

Another generalization is to consider a smooth manifold X and a smooth line bundle L, but a *singular* Hermitian metric h on L with strictly positive curvature current $\omega = c_1(L,h)$. In this case $\gamma_p = \Phi_p^*\omega_{FS}$ are positive currents and it was shown in [CM] that the analogue of Tian's result (2) is

$$\frac{1}{p}\gamma_p \to \omega, \quad \text{as } p \to \infty, \text{ weakly in the sense of currents.} \tag{5}$$

This folows via (1) from the fact that $\log P_p$ is locally the difference of two plurisubharmonic functions, thus locally integrable, and

$$\frac{1}{p}\log P_p \to 0, \quad \text{as } p \to \infty, \text{ in } L^1_{loc}(X). \tag{6}$$

It turns out that (5) and (6) are all that is needed to obtain equidistribution results for singular metrics and they are fulfilled in several geometric contexts, see [CM].

In this paper we consider the following setting:

(A) $\mathscr{X} = (X, \mathscr{U})$ is a complex (effective) orbifold of dimension n, $\widetilde{\Omega}$ is a Hermitian form on $\mathscr{X}$ with induced Hermitian form Ω on the orbifold regular locus of X.

(B) $G \subset X$ is an open set with orbifold structure $\mathscr{G} = (G, \mathscr{U}_G)$ induced by $\mathscr{X}$.

(C) $(L, \mathscr{X}, \widetilde{h})$ is an orbifold line bundle on $\mathscr{X}$ endowed with a singular Hermitian metric $\widetilde{h}$ with (semi)positive curvature current $c_1(L, \widetilde{h}) \ge 0$.

We denote by $\widetilde{h}^p$ the metric induced by $\widetilde{h}$ on $L^p := L^{\otimes p}$, and we consider the orbifold canonical bundle $K_{\mathscr{X}}$ endowed with the metric $\widetilde{h}^{K_{\mathscr{X}}}$ induced by the volume form $\widetilde{\Omega}^n$. With this metric data we can define the following Hilbert spaces of L^2-holomorphic sections:

(I) $H^0_{(2)}(\mathscr{X}, L^p \otimes K_{\mathscr{X}})$ is the space of L^2-holomorphic sections of $L^p \otimes K_{\mathscr{X}}$ endowed with the inner product determined by the metric $\widetilde{h}^p \otimes \widetilde{h}^{K_{\mathscr{X}}}$ and the volume form $\widetilde{\Omega}^n$.

(II) $H^0_{(2)}(\mathscr{X}, L^p)$ is the space of L^2-holomorphic sections of L^p endowed with the inner product determined by the metric $\widetilde{h}^p$ and the volume form $\widetilde{\Omega}^n$.

We refer to Section 2 for background about orbifolds and orbifold line bundles. Let us recall that in the above setting X is a normal complex space and let us denote by X_{reg} the set of its regular points, and by $X^{orb}_{reg} \subset X_{reg}$ its orbifold regular locus (see Section 2.1). In Section 3.1 we discuss the notion of orbifold current and show that positive closed orbifold currents of bidegree (1,1) are in one-to-one correspondence to positive closed currents of bidegree (1,1) on X_{reg} (see Propositions 3.1 and 3.3). In view of these, such orbifold currents can be regarded as currents on X_{reg}. The notion of singular metric on an orbifold line bundle is recalled in Section 3.2, and we refer to Section 3.3 for the necessary definitions of spaces of L^2-holomorphic sections, their Fubini-Study currents and Bergman kernel functions.

With these preparations we can state our results.

Theorem 1.1. *Let $(L,\mathscr{X},\widetilde{h})$, $\mathscr{G}$, $\widetilde{\Omega}$ verify assumptions (A)-(C). Suppose that X^{orb}_{reg} carries a complete Kähler metric and there exists a continuous function $\varepsilon : G\cap X^{orb}_{reg} \longrightarrow (0,+\infty)$ so that $c_1(L,\widetilde{h}) > \varepsilon\Omega$ on $G\cap X^{orb}_{reg}$. If γ_p, P_p are the Fubini-Study currents, resp. the Bergman kernel functions, of the spaces $H^0_{(2)}(\mathscr{X},L^p\otimes K_{\mathscr{X}})$, then:*
(i) $\frac{1}{p}\gamma_p \to c_1(L,\widetilde{h})$ weakly as orbifold currents on $\mathscr{G}$ as $p\to\infty$.
(ii) $\frac{1}{p}\log P_p \to 0$ as $p\to\infty$, in $L^1_{loc}(G)$ with respect to the area measure on X.

Note that the metric Ω is not assumed to be complete, we assume just that X^{orb}_{reg} carries *some* complete Kähler metric. The assumption that X^{orb}_{reg} is Kähler complete is known to hold in the following situations, thanks to an argument in [O]:

Proposition 1.2. *X^{orb}_{reg} admits a complete Kähler metric if one of the following three conditions hold:*
(i) $(\mathscr{X},\widetilde{\Omega})$ is a compact Kähler orbifold, or
(ii) X is a Stein space, or
(iii) $\mathscr{X}$ is a complete Kähler manifold.

An immediate consequence of Theorem 1.1 for $L = K_{\mathscr{X}}$ is the following.

Corollary 1.3. *Let $\mathscr{X}$ be an orbifold so that X^{orb}_{reg} carries a complete Kähler metric, $G\subset X$ an open set, and $\widetilde{\Omega}$ a Hermitian metric on $\mathscr{X}$ so that $c_1(K_{\mathscr{X}},\widetilde{h})\geq 0$, where $\widetilde{h}=\widetilde{h}^{K_{\mathscr{X}}}$ is the metric on $K_{\mathscr{X}}$ induced by $\widetilde{\Omega}$. Suppose that there exists a continuous function $\varepsilon : G\cap X^{orb}_{reg} \longrightarrow (0,+\infty)$ so that $c_1(K_{\mathscr{X}},\widetilde{h}) > \varepsilon\Omega$ on $G\cap X^{orb}_{reg}$. Then $\frac{1}{p}\gamma_p \to c_1(K_{\mathscr{X}},\widetilde{h})$ weakly on $\mathscr{G}$, and $\frac{1}{p}\log P_p \to 0$ in $L^1_{loc}(G)$, as $p\to\infty$, where γ_p, P_p are the Fubini-Study currents, resp. the Bergman kernel functions, of $H^0_{(2)}(\mathscr{X},K^p_{\mathscr{X}})$.*

Under stronger hypotheses, we can formulate a version of Theorem 1.1 for L^2-holomorphic sections of L^p rather than L^p-valued holomorphic n-forms:

Theorem 1.4. *Let $(L,\mathscr{X},\widetilde{h})$, $\mathscr{G}$, $\widetilde{\Omega}$ verify assumptions (A)-(C) such that X^{orb}_{reg} carries a complete Kähler metric and $\widetilde{\Omega}$ is a Kähler form with semipositive Ricci form $Ric_{\widetilde{\Omega}}\geq 0$. If there exists a continuous function $\varepsilon : G\cap X^{orb}_{reg} \longrightarrow (0,+\infty)$ so that $c_1(L,\widetilde{h}) > \varepsilon\Omega$ on $G\cap X^{orb}_{reg}$, then the conclusions (i)-(ii) of Theorem 1.1 hold for the Fubini-Study currents γ_p and the Bergman kernel functions P_p of the spaces $H^0_{(2)}(\mathscr{X},L^p)$.*

For overall positive holomorphic orbifold line bundles with smooth metrics we have the following result.

Theorem 1.5. *Let $(\mathscr{X},\widetilde{\Omega})$ be a compact Hermitian orbifold and $(L,\mathscr{X},\widetilde{h})$ be a positive orbifold line bundle endowed with a smooth positively curved metric $\widetilde{h}$. Then $\frac{1}{p}\gamma_p \to c_1(L,\widetilde{h})$ weakly on $\mathscr{X}$ as $p\to\infty$, where γ_p are the Fubini-Study currents of $H^0_{(2)}(\mathscr{X},L^p)$.*

An important application of Theorems 1.1, 1.4 and 1.5 is to the study of the asymptotic distribution of zeros of random holomorphic sections. This is done using a general method due to Shiffman and Zelditch, who describe the asymptotic distribution of zeros of random holomorphic sections of a positive line bundle over a projective manifold endowed with a smooth positively curved metric [SZ1, SZ2, SZ3].

Suppose that we are in either one of the settings of Theorem 1.1, or Corollary 1.3, or Theorem 1.4, or Theorem 1.5. We assume in addition that $\mathscr{X}$ is compact and $\widetilde{\Omega}$ is a Kähler form and we denote by $\mathscr{V}^p$ the corresponding spaces of L^2-holomorphic sections in each of the above settings.

Following the framework in [SZ1], we let λ_p be the normalized surface measure on the unit sphere $\mathscr{S}^p$ of $\mathscr{V}^p$, defined in the natural way by using an orthonormal basis of $\mathscr{V}^p$ (see Section 5). Consider the probability space $\mathscr{S}_\infty=\prod_{p=1}^\infty \mathscr{S}^p$ endowed with the probability measure $\lambda_\infty=\prod_{p=1}^\infty \lambda_p$. We have the following theorem:

Theorem 1.6. *In either one of the settings of Theorem 1.1, 1.4, 1.5, or Corollary 1.3, assume in addition that $\mathscr{X}$ is compact and $\widetilde{\Omega}$ is a Kähler form. Then:*

(i) The Fubini-Study current of $\mathscr{V}^p$ is the expectation $E_p[S=0]$ of the current-valued random variable $S\in\mathscr{S}^p\to[S=0]$, given by

$$\langle E_p[S=0],\widetilde{\theta}\,\rangle=\int_{\mathscr{S}^p}\langle [S=0],\widetilde{\theta}\,\rangle\, d\lambda_p(S),$$

where $\widetilde{\theta}$ is a test form on $\mathscr{X}$. We have that $\frac{1}{p}E_p[S=0]\to c_1(L,\widetilde{h})$ as $p\to\infty$, weakly as orbifold currents on $\mathscr{G}$ (resp. on $\mathscr{X}$, in the case of Theorem 1.5).

(ii) For λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq 1}\in\mathscr{S}_\infty$, we have that $\frac{1}{p}[\sigma_p=0]\to c_1(L,\widetilde{h})$ as $p\to\infty$, weakly as orbifold currents on $\mathscr{G}$ (resp. on $\mathscr{X}$, in the case of Theorem 1.5).

Here $[S=0]$ denotes the current of integration (with multiplicities) over the zero set of a nontrivial section $S\in\mathscr{V}^p$. The arguments of Shiffman and Zelditch [SZ1] needed for the proof of Theorem 1.6 are recalled in Section 5.

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2. Orbifolds and orbifold line bundles

We recall here some necessary notions about (complex effective) orbifolds and (holomorphic) orbifold line bundles, following [BG] (see also [ALR, BGK, GK, MM2]).

2.1. **Orbifolds.** Let X be a (second countable) complex space of dimension n. An orbifold chart on X is a triple $(\widetilde{U},\Gamma,\phi)$ where $\widetilde{U}$ is a domain in $\mathbb{C}^n$, Γ is a finite group acting effectively as automorphisms of $\widetilde{U}$, and $\phi:\widetilde{U}\longrightarrow U$ is an analytic cover (i.e. proper and finite holomorphic map) onto an open set $U\subset X$ such that $\phi\circ\gamma=\phi$ for every $\gamma\in\Gamma$ and the induced natural map $\widetilde{U}/\Gamma\longrightarrow U$ is a homeomorphism. An injection between two charts $(\widetilde{U},\Gamma,\phi)$, $(\widetilde{U}',\Gamma',\phi')$ is a holomorphic embedding $\lambda:\widetilde{U}\longrightarrow\widetilde{U}'$ so that $\phi'\circ\lambda=\phi$. An orbifold atlas on X is a family $\mathscr{U}=\{(\widetilde{U}_i,\Gamma_i,\phi_i)\}$ of orbifold charts such that $X=\bigcup U_i$, where $U_i:=\phi_i(\widetilde{U}_i)$, and, given two charts $(\widetilde{U}_i,\Gamma_i,\phi_i)$, $(\widetilde{U}_j,\Gamma_j,\phi_j)$ and $x\in U_i\cap U_j$, there exist a chart $(\widetilde{U}_k,\Gamma_k,\phi_k)$ with $x\in U_k$ and injections $\lambda_{ik}:(\widetilde{U}_k,\Gamma_k,\phi_k)\longrightarrow(\widetilde{U}_i,\Gamma_i,\phi_i)$, $\lambda_{jk}:(\widetilde{U}_k,\Gamma_k,\phi_k)\longrightarrow(\widetilde{U}_j,\Gamma_j,\phi_j)$. An atlas $\mathscr{U}$ is said to be a refinement of an atlas $\mathscr{V}$ if there exists an injection of every chart of $\mathscr{U}$ into some chart of $\mathscr{V}$. An orbifold $\mathscr{X}=(X,\mathscr{U})$ is a complex space X with a (maximal) orbifold atlas $\mathscr{U}$. It follows that the underlying space X is a reduced normal complex space with at most quotient singularities (see e.g. [BG, Sec. 4.4]).

Given an injection $\lambda:\widetilde{U}\longrightarrow\widetilde{U}'$ and $\gamma\in\Gamma$, one has that there exists a unique $\gamma'\in\Gamma'$ with $\gamma'\circ\lambda=\lambda\circ\gamma$ [BG, Lemma 4.1.2]. Thus we get an injective group homomorphism, denoted still by $\lambda:\Gamma\longrightarrow\Gamma'$, defined by $\lambda(\gamma)=\gamma'$. Moreover, if $\gamma'\lambda(\widetilde{U})\cap\lambda(\widetilde{U})\neq\emptyset$ then $\gamma'\in\lambda(\Gamma)$, and so $\gamma'\lambda(\widetilde{U})=\lambda(\widetilde{U})$ [ALR, p. 3]. This implies that the set $\gamma'\lambda(\widetilde{U})$ depends only on the coset $[\gamma']=\gamma'\lambda(\Gamma)$ and

$$(\phi')^{-1}(U)=\bigcup_{[\gamma']}\gamma'\lambda(\widetilde{U}), \tag{7}$$

where the union is disjoint.

We write

$$X=X_{reg}\cup X_{sing},$$

where X_{reg} (resp. X_{sing}) is the set of regular (resp. singular) points of X. Since X is normal, we have that X_{reg} is a connected complex manifold and X_{sing} is a closed reduced complex subspace of X with $\operatorname{codim}X_{sing}\geq 2$. Given an orbifold chart $(\widetilde{U},\Gamma,\phi)$, the isotropy group Γ_x of $x\in U$ is defined as the isotropy (stabilizer) group Γ_y of any $y\in\phi^{-1}(x)$, which is unique up to conjugacy. The sets of orbifold regular, resp. orbifold singular, points are defined by

$$X^{orb}_{reg}=\{x\in X:|\Gamma_x|=1\},\, X^{orb}_{sing}=\{x\in X:|\Gamma_x|>1\}.$$

Then X^{orb}_{sing} is a closed complex subspace of X and one has that $X^{orb}_{reg}\subset X_{reg}$ and $X_{sing}\subset X^{orb}_{sing}$.

An orbifold $\mathscr{X} = (X, \mathscr{U})$ can be identified with the log pair (X, Δ), where Δ is called the branch divisor and is the $\mathbb{Q}$-divisor defined by

$$\Delta = \sum \left(1 - m^{-1}\right) D\,.$$

Here the sum is taken over all Weil divisors $D \subset X^{orb}_{sing}$ and m is the ramification index over D (see [BGK, GK], [BG, Sec. 4.4]).

2.2. **Orbifold line bundles.** We now recall the notion of a (proper) orbifold line bundle on $\mathscr{X} = (X, \mathscr{U})$ (see [BG]). This is a collection $\{L_{\widetilde{U}_i}\}$ of Γ_i-equivariant holomorphic line bundles $\widetilde{\pi}_i : L_{\widetilde{U}_i} \longrightarrow \widetilde{U}_i$ which satisfy a gluing condition. Equivariance means that Γ_i acts effectively on $L_{\widetilde{U}_i}$ as bundle maps which are isomorphisms along fibers and the following diagram is commutative,

$$\begin{array}{ccc} L_{\widetilde{U}_i} & \xrightarrow{\widetilde{\gamma}} & L_{\widetilde{U}_i} \\ \big\downarrow \widetilde{\pi}_i & & \big\downarrow \widetilde{\pi}_i \\ \widetilde{U}_i & \xrightarrow{\gamma} & \widetilde{U}_i \end{array},$$

where $\widetilde{\gamma}$ is the bundle map corresponding to $\gamma \in \Gamma_i$. The gluing condition is as follows: any injection $\lambda_{ji} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ induces a bundle map $\widetilde{\lambda}_{ji} : L_{\widetilde{U}_j}|_{\lambda_{ji}(\widetilde{U}_i)} \longrightarrow L_{\widetilde{U}_i}$ which is an isomorphism along fibers, so that if $\gamma \in \Gamma_i$ and $\gamma' = \lambda_{ji}(\gamma) \in \Gamma_j$ (i.e. $\gamma' \circ \lambda_{ji} = \lambda_{ji} \circ \gamma$) then $\widetilde{\lambda}_{ji} \circ \widetilde{\gamma}' = \widetilde{\gamma} \circ \widetilde{\lambda}_{ji}$. Moreover, the $\widetilde{\lambda}_{ji}$ are functorial: if $\lambda_{kj} : \widetilde{U}_j \longrightarrow \widetilde{U}_k$ is another injection then $\widetilde{\lambda_{kj} \circ \lambda_{ji}} = \widetilde{\lambda}_{ji} \circ \widetilde{\lambda}_{kj}$.

The total space L of the orbifold line bundle $\{L_{\widetilde{U}_i}\}$ is constructed by gluing the sets $L_{\widetilde{U}_i}/\Gamma_i$ via the maps $\widetilde{\lambda}_{ji}$ in the usual way. Let $\widetilde{\phi}_i : L_{\widetilde{U}_i} \longrightarrow L_{\widetilde{U}_i}/\Gamma_i$ be the natural map. Then the atlas $\{(L_{\widetilde{U}_i}, \Gamma_i, \widetilde{\phi}_i)\}$ gives L the structure of an (effective) orbifold, hence L is a normal complex space. Since

$$\phi_i \circ \widetilde{\pi}_i \circ \widetilde{\gamma} = \phi_i \circ \gamma \circ \widetilde{\pi}_i = \phi_i \circ \widetilde{\pi}_i\,,\ \forall\, \gamma \in \Gamma_i\,,$$

there exists a continuous map π_i which makes the following diagram commutative:

$$\begin{array}{ccc} L_{\widetilde{U}_i} & \xrightarrow{\widetilde{\phi}_i} & L_{\widetilde{U}_i}/\Gamma_i \\ \big\downarrow \widetilde{\pi}_i & & \big\downarrow \pi_i \\ \widetilde{U}_i & \xrightarrow{\phi_i} & U_i \end{array}.$$

Note that π_i is surjective and is holomorphic on $\pi_i^{-1}(U_i \cap X^{orb}_{reg})$, hence it is holomorphic. The maps π_i glue to a surjective holomorphic map $\pi : L \longrightarrow X$.

For brevity, we will denote the orbifold line bundle $\{L_{\widetilde{U}_i}\}$ on $\mathscr{X} = (X, \mathscr{U})$ by $(L, \mathscr{X})$. If $L|_{X^{orb}_{reg}} := \pi^{-1}(X^{orb}_{reg})$, then $\pi : L|_{X^{orb}_{reg}} \longrightarrow X^{orb}_{reg}$ is a holomorphic line bundle. Suppose now that $x \in X$ has non-trivial isotropy group. If $y \in \phi^{-1}(x)$ for some orbifold chart $(\widetilde{U}, \Gamma, \phi)$ near x, then each $\gamma \in \Gamma_y$ induces an isomorphism $\widetilde{\gamma} : L_{\widetilde{U}}|_y \longrightarrow L_{\widetilde{U}}|_y$. Hence

$L|_x := \pi^{-1}(x) \cong (L_{\widetilde{U}}|_y)/\Gamma_y$, and $\pi : L \longrightarrow X$ is in general not a holomorphic line bundle in the usual sense. The latter are sometimes called absolute orbifold line bundles, see [BG]. However if X is compact then there exists $m \in \mathbb{N}$ so that, for every orbifold line bundle L on $\mathscr{X}$, $L^{\otimes m}$ is absolute.

A (holomorphic) section of $(L, \mathscr{X})$ is a collection of sections $\widetilde{S}_i : \widetilde{U}_i \longrightarrow L_{\widetilde{U}_i}$ for each orbifold chart $(\widetilde{U}_i, \Gamma_i, \phi_i)$ so that $\widetilde{S}_i$ is Γ_i-equivariant, i.e. $\widetilde{\gamma} \circ \widetilde{S}_i = \widetilde{S}_i \circ \gamma$ for all $\gamma \in \Gamma_i$, and for every injection $\lambda_{ji} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ one has $\widetilde{\lambda}_{ji} \circ \widetilde{S}_j \circ \lambda_{ji} = \widetilde{S}_i$.

Since $\widetilde{\phi}_i \circ \widetilde{S}_i \circ \gamma = \widetilde{\phi}_i \circ \widetilde{\gamma} \circ \widetilde{S}_i = \widetilde{\phi}_i \circ \widetilde{S}_i$ for each $\gamma \in \Gamma_i$, there exists a continuous map S_i which makes the following diagram commutative:

$$\begin{array}{ccc} L_{\widetilde{U}_i} & \xrightarrow{\widetilde{\phi}_i} & L|_{U_i} \\ \uparrow{\scriptstyle \widetilde{S}_i} & & \uparrow{\scriptstyle S_i} \\ \widetilde{U}_i & \xrightarrow{\phi_i} & U_i \end{array} .$$

Note that S_i is in fact holomorphic on U_i, since it is holomorphic on $U_i \cap X^{orb}_{reg}$. The local sections S_i glue to an injective holomorphic map $S : X \longrightarrow L$ which verifies $\pi \circ S = id_X$. Its restriction to X^{orb}_{reg} is a section of the line bundle $L|_{X^{orb}_{reg}}$.

We denote by $H^0(\mathscr{X}, L)$ the vector space of holomorphic sections of $(L, \mathscr{X})$.

2.3. **Differential forms on orbifolds.** A (p,q) form on an orbifold $\mathscr{X} = (X, \mathscr{U})$ is a collection $\widetilde{\psi} = \{\widetilde{\psi}_i\}$ of smooth (p,q) forms $\widetilde{\psi}_i$ on $\widetilde{U}_i$, for each orbifold chart $(\widetilde{U}_i, \Gamma_i, \phi_i)$, so that $\gamma^\star \widetilde{\psi}_i = \widetilde{\psi}_i$ for each $\gamma \in \Gamma_i$, and $\lambda^\star_{ji} \widetilde{\psi}_j = \widetilde{\psi}_i$ for each injection $\lambda_{ji} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$. Equivalently, a (p,q) form on $\mathscr{X}$ is a smooth (p,q) form ψ on X^{orb}_{reg} so that for each chart $(\widetilde{U}_i, \Gamma_i, \phi_i)$, $\phi^\star_i(\psi|_{U_i \cap X^{orb}_{reg}})$ extends to a smooth form on $\widetilde{U}_i$ [BGK].

A (p,p) form $\widetilde{\psi} = \{\widetilde{\psi}_i\}$ is positive (resp. closed) if each form $\widetilde{\psi}_i$ is positive (resp. closed). A $(1,1)$ form $\widetilde{\Omega} = \{\widetilde{\Omega}_i\}$ is Kähler, resp. Hermitian, if each form $\widetilde{\Omega}_i$ is Kähler, resp. Hermitian (i.e. positive).

One can define the orbifold tangent and cotangent bundles of $\mathscr{X}$ and view orbifold differential forms as sections of such. An important example that we will need is the orbifold canonical bundle $K_{\mathscr{X}}$, defined as the collection of canonical bundles $\{K_{\widetilde{U}_i}\}$, for all charts $(\widetilde{U}_i, \Gamma_i, \phi_i) \in \mathscr{U}$. Note that the equivariance and gluing are given by the pull-back operators, $\widetilde{\gamma} := (\gamma^{-1})^\star$, $\widetilde{\lambda}_{ji} := \lambda^\star_{ji}$.

3. Currents and singular metrics

We collect here a few facts about currents on orbifolds, being especially interested in positive closed currents of bidegree (1,1). We also recall the notion of singular metric on an orbifold line bundle and we introduce the Bergman kernel function and the Fubini-Study currents for subspaces of L^2-holomorphic sections. Throughout this section, we use the notations introduced in Section 2.

3.1. **Currents on orbifolds.** A current of bidegree (p,q) on an orbifold $\mathscr{X}=(X,\mathscr{U})$ is a collection $\widetilde{T}=\{\widetilde{T}_i\}$ of currents $\widetilde{T}_i$ of bidegree (p,q) on $\widetilde{U}_i$, for each orbifold chart $(\widetilde{U}_i,\Gamma_i,\phi_i)$, so that $\gamma_\star\widetilde{T}_i=\widetilde{T}_i$ for each $\gamma\in\Gamma_i$, and $(\lambda_{ji})_\star\widetilde{T}_i=\widetilde{T}_j\,|_{\lambda_{ji}(\widetilde{U}_i)}$ for each injection $\lambda_{ji}:\widetilde{U}_i\longrightarrow\widetilde{U}_j$. A current $\widetilde{T}$ of bidegree (p,p) is positive (resp. closed) if each current $\widetilde{T}_i$ is positive (resp. closed). A sequence of currents $\widetilde{T}^k=\{\widetilde{T}^k_i\}$, $k\geq 1$, converges weakly to a current $\widetilde{T}=\{\widetilde{T}_i\}$ if, for each i, the sequence of current $\widetilde{T}^k_i$ converges weakly on $\widetilde{U}_i$ to $\widetilde{T}_i$ as $k\to\infty$.

Let us define the action of an orbifold current $\widetilde{T}=\{\widetilde{T}_i\}$ of bidegree (p,q) on a $(n-p,n-q)$ test form $\widetilde{\theta}=\{\widetilde{\theta}_i\}$ (where $\operatorname{supp}\widetilde{\theta}\Subset X$). Fix a partition of unity $\{\chi_l\}_{l\geq1}$ on X so that χ_l has compact support contained in U_{i_l} and set

$$\langle\widetilde{T},\widetilde{\theta}\rangle=\sum_{l=1}^{\infty}\frac{1}{m_{i_l}}\langle\widetilde{T}_{i_l},(\chi_l\circ\phi_{i_l})\widetilde{\theta}_{i_l}\rangle\,,$$

where $m_i=|\Gamma_i|$. Note that the standard calculus with currents works as usual. For instance, one checks that the current $d\widetilde{T}:=\{d\widetilde{T}_i\}$ verifies $\langle d\widetilde{T},\widetilde{\theta}\rangle=(-1)^{p+q+1}\langle\widetilde{T},d\widetilde{\theta}\rangle$.

If $U_i\subset X_{reg}$ the current $(\phi_i)_\star\widetilde{T}_i$ is well defined on U_i, since ϕ_i is proper. We show that these currents glue to a global current on X_{reg}:

Proposition 3.1. *Let $\widetilde{T}=\{\widetilde{T}_i\}$ be a current of bidegree (p,q) on an orbifold $\mathscr{X}=(X,\mathscr{U})$. There exists a current T of bidegree (p,q) on X_{reg} so that for every chart $(\widetilde{U}_i,\Gamma_i,\phi_i)$ with $U_i\subset X_{reg}$ we have $T\,|_{U_i}=\frac{1}{m_i}(\phi_i)_\star\widetilde{T}_i$, where $m_i=|\Gamma_i|$.*

Proof. Let $T_i:=\frac{1}{m_i}(\phi_i)_\star\widetilde{T}_i$. If $\lambda_{ji}:\widetilde{U}_i\longrightarrow\widetilde{U}_j$ is an injection with $U_i\subset U_j\subset X_{reg}$ we have to show that $T_j\,|_{U_i}=T_i$. If $V:=\lambda_{ji}(\widetilde{U}_i)$ we have by (7) that

$$\phi_j^{-1}(U_i)=\bigcup_{[\gamma']}\gamma' V\,,$$

where the union is disjoint and $[\gamma']=\gamma'\lambda_{ji}(\Gamma_i)$ denotes the coset of γ' in $\Gamma_j/\lambda_{ji}(\Gamma_i)$. This implies that the map $\hat{\phi}_j:=\phi_j\,|_V\colon V\longrightarrow U_i$ is an analytic cover with topological degree $m_i=|\lambda_{ji}(\Gamma_i)|$ and

$$T_i=\frac{1}{m_i}(\phi_i)_\star\widetilde{T}_i=\frac{1}{m_i}(\hat{\phi}_j)_\star\circ(\lambda_{ji})_\star\widetilde{T}_i=\frac{1}{m_i}(\hat{\phi}_j)_\star(\widetilde{T}_j\,|_V)\,.$$

If θ is a $(n-p,n-q)$ test form supported in U_i then

$$\begin{aligned}\langle T_j\,|_{U_i},\theta\rangle &= \frac{1}{m_j}\langle\widetilde{T}_j,\phi_j^\star\theta\rangle=\frac{1}{m_j}\sum_{[\gamma']}\langle\widetilde{T}_j\,|_{\gamma'V},(\phi_j^\star\theta)\,|_{\gamma'V}\rangle\\ &= \frac{1}{m_j}\sum_{[\gamma']}\langle\gamma'_\star(\widetilde{T}_j\,|_V),(\phi_j^\star\theta)\,|_{\gamma'V}\rangle=\frac{1}{m_j}\sum_{[\gamma']}\langle\widetilde{T}_j\,|_V,\hat{\phi}_j^\star\theta\rangle=\langle T_i,\theta\rangle\,.\end{aligned}$$

□

Let $\widetilde{\mathscr{T}}$, resp. $\mathscr{T}$, denote the set of positive closed currents of bidegree $(1,1)$ on $\mathscr{X}$, resp. on X_{reg}. Proposition 3.1 provides a map $\mathbf{F}:\widetilde{\mathscr{T}}\longrightarrow\mathscr{T}$, $\mathbf{F}(\widetilde{T})=T$, which is continuous with respect to weak* convergence of currents. We will prove that $\mathbf{F}$ is in fact bijective and has continuous inverse. For this, we show first the following:

Proposition 3.2. *Let $\mathscr{X}=(X,\mathscr{U})$ be an orbifold. Then each point $x\in X$ has a neighborhood $U\subset X$ such that for every current $T\in\mathscr{T}$ there is a psh function v on U with $dd^c v=T$ on $U\cap X_{reg}$.*

Proof. The conclusion is clear if $x\in X_{reg}$, so we assume $x\in X_{sing}$. Fix an orbifold chart $(\widetilde{U}_i,\Gamma_i,\phi_i)$ with $x\in U_i=\phi_i(\widetilde{U}_i)$ and set $m_i=|\Gamma_i|$, $\Sigma=U_i\cap X_{sing}$. Let $V\subset\widetilde{U}_i$ be an open set so that all its connected components are simply connected and $\phi_i^{-1}(x)\subset V$. Since ϕ_i is proper and X is locally irreducible, there exists a neighborhood $U\subset U_i$ of x so that $U\setminus\Sigma$ is connected and $\phi_i^{-1}(U)\subset V$.

Let $T\in\mathscr{T}$. Since $\operatorname{codim}\Sigma\geq 2$ and ϕ_i is finite we have $\operatorname{codim}\phi_i^{-1}(\Sigma)\geq 2$. Hence the positive closed $(1,1)$ current $R=\phi_i^\star(T\,|_{U_i\setminus\Sigma})$ on $\widetilde{U}_i\setminus\phi_i^{-1}(\Sigma)$ extends to a positive closed current on $\widetilde{U}_i$. Moreover $\gamma^\star R=R$ on $\widetilde{U}_i$. Indeed, since $\phi_i\circ\gamma=\phi_i$ this holds on the set $\widetilde{U}_i\setminus\phi_i^{-1}(\Sigma)$ which is Γ_i-invariant, hence it holds on $\widetilde{U}_i$. By the assumption on V, we have $R=dd^c u'$ for some psh function u' on V. As $\phi_i^{-1}(U)$ is Γ_i-invariant we can define

$$u=\frac{1}{m_i}\sum_{\gamma\in\Gamma_i}u'\circ\gamma\in PSH(\phi_i^{-1}(U)),\ \text{so}\ dd^c u=\frac{1}{m_i}\sum_{\gamma\in\Gamma_i}\gamma^\star R=R\,.$$

Note that $u\circ\gamma=u$ for all $\gamma\in\Gamma_i$. Thus $u=v\circ\phi_i$ for some upper semicontinuous function v on U. As $\phi_i:\phi_i^{-1}(U\setminus\Sigma)\longrightarrow U\setminus\Sigma$ is proper and $v(y)=u(z)$ for any $z\in\phi_i^{-1}(y)$, it follows that v is psh on $U\setminus\Sigma$. Since v is upper semicontinuous, we have $v\in PSH(U)$ [D2, Theorem 1.7] .

If S is a positive closed (1,1) current on $U\setminus\Sigma$ then $(\phi_i)_\star\circ\phi_i^\star S=m_iS$. Indeed, if θ is a test form supported in $U\setminus\Sigma$ we may assume that $S=dd^c\rho$ for a psh function ρ near the support of θ, and

$$\langle(\phi_i)_\star\circ\phi_i^\star S,\theta\rangle=\langle dd^c(\rho\circ\phi_i),\phi_i^\star\theta\rangle=m_i\langle dd^c\rho,\theta\rangle=m_i\langle S,\theta\rangle\,. \tag{8}$$

Then $\phi_i^\star T=R=dd^c(v\circ\phi_i)=\phi_i^\star(dd^c v)$ on $\phi_i^{-1}(U\setminus\Sigma)$ implies $T=dd^c v$ on $U\setminus\Sigma$. □

Proposition 3.3. *The map $\mathbf{G}:\mathscr{T}\longrightarrow\widetilde{\mathscr{T}}$, $\mathbf{G}(T)=\{\phi_i^\star(T\,|_{U_i})\}$, where $(\widetilde{U}_i,\Gamma_i,\phi_i)$ are the orbifold charts of $\mathscr{X}$, is well defined, continuous with respect to weak* convergence, and it is the inverse of* $\mathbf{F}$.

Proof. We can define $\widetilde{T}_i:=\phi_i^\star(T\,|_{U_i})$ as the positive closed (1,1) current on $\widetilde{U}_i$ with local potentials $v\circ\phi_i$, where v are the local potentials of T near each point of U_i

provided by Proposition 3.2. Clearly, $\gamma^\star \widetilde{T}_i = \widetilde{T}_i$ for all $\gamma \in \Gamma_i$. If $\lambda_{ji} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is an injection then, with the notation $\hat{\phi}_j$ from the proof of Proposition 3.1,

$$\lambda_{ji}^\star(\widetilde{T}_j\,|_{\lambda_{ji}(\widetilde{U}_i)}) = \lambda_{ji}^\star((\phi_j^\star T)\,|_{\lambda_{ji}(\widetilde{U}_i)}) = \lambda_{ji}^\star(\hat{\phi}_j^\star(T\,|_{U_i})) = \phi_i^\star(T\,|_{U_i}) = \widetilde{T}_i\,.$$

Hence $\mathbf{G}(T) \in \widetilde{\mathcal{T}}$. By (8), $(\phi_i)_\star \widetilde{T}_i = m_i T\,|_{U_i}$, where $m_i = |\Gamma_i|$, so $\mathbf{F} \circ \mathbf{G}(T) = T$.

We note that $\mathbf{G}$ is surjective. Indeed, if $\widetilde{T} = \{\widetilde{T}_i\} \in \widetilde{\mathcal{T}}$ and $x \in U_i$, we can repeat the argument in the proof of Proposition 3.2 to show that there exists a small neighborhood $U \subset U_i$ of x and $v \in PSH(U)$ so that $\widetilde{T}_i = dd^c(v \circ \phi_i)$ on $\phi^{-1}(U)$ and $\frac{1}{m_i}(\phi_i)_\star \widetilde{T}_i = dd^c v$ on $U \cap X_{reg}$. Setting $T := \mathbf{F}(\widetilde{T})$ we have for $U_i \subset X_{reg}$, $T\,|_U = \frac{1}{m_i}(\phi_i)_\star \widetilde{T}_i\,|_U = dd^c v$, so $\widetilde{T} = \mathbf{G}(T)$.

To prove the continuity of $\mathbf{G}$, assume that $T^j, T \in \mathcal{T}$ and the sequence T^j converges weakly to T. Fix an orbifold chart $(\widetilde{U}_i, \Gamma_i, \phi_i)$ and let $\Sigma = U_i \cap X_{sing}$. Then $\phi_i^\star T^j$ converges weakly to $\phi_i^\star T$ on $\widetilde{U}_i \setminus \phi_i^{-1}(\Sigma)$. Since $\operatorname{codim} \phi_i^{-1}(\Sigma) \geq 2$, Oka's inequality for currents [FS] implies that the sequence of currents $\{\phi_i^\star T^j\}_j$ has locally bounded mass in $\widetilde{U}_i$. As any limit point equals $\phi_i^\star T$ on $\widetilde{U}_i \setminus \phi_i^{-1}(\Sigma)$, we conclude that the sequence $\phi_i^\star T^j$ converges weakly to $\phi_i^\star T$ on $\widetilde{U}_i$. $\square$

Definition 3.4. A Kähler current on X is a positive closed (1,1) current T on X_{reg} with the property that for every $x \in X$ there exist a neighborhood U of x and a strictly psh function v on U so that $T = dd^c v$ on $U \cap X_{reg}$.

We note that if the local potentials of a Kähler current T are C^∞-smooth then X is a called a Kähler space, cf. [O] (see also [EGZ, Sec. 5.2]). Propositions 3.2 and 3.3 yield the following:

Proposition 3.5. *If $\mathcal{X} = (X, \mathcal{U})$ is a Kähler orbifold then X carries a Kähler current whose local potentials are continuous near each $x \in X$ and smooth near each $x \in X_{reg}^{orb}$.*

Proof. Let $\widetilde{\Omega} = \{\widetilde{\Omega}_i\}$ be a Kähler form on $\mathcal{X}$ and let $\Omega = \mathbf{F}(\widetilde{\Omega}) \in \mathcal{T}$. Proposition 3.3 and its proof shows that every $x \in X$ has a neighborhood $U \subset U_i$, for some orbifold chart $(\widetilde{U}_i, \Gamma_i, \phi_i)$, for which there exists a continuous function $v \in PSH(U)$ so that $u = v \circ \phi_i$ is smooth strictly psh on $\phi_i^{-1}(U)$, $\widetilde{\Omega}_i = dd^c u$ on $\phi_i^{-1}(U)$ and $\Omega = dd^c v$ on $U \cap X_{reg}$. Moreover v is smooth if $x \in X_{reg}^{orb}$. To see that v is strictly psh, consider any local embedding $U \hookrightarrow \mathbb{C}^N$ with coordinates $z = (z_1, \ldots, z_N)$, so $\phi_i : \phi_i^{-1}(U) \longrightarrow \mathbb{C}^N$. Since $\widetilde{\Omega}_i$ is Kähler and $dd^c \|\phi_i\|^2$ is a smooth real form, by shrinking U we can find $\varepsilon > 0$ so that $\widetilde{\Omega}_i > \varepsilon dd^c \|\phi_i\|^2$ on $\phi_i^{-1}(U)$. This shows that $v \circ \phi_i - \varepsilon \|\phi_i\|^2$ is psh on $\phi_i^{-1}(U)$, hence the function $\rho(z) = v(z) - \varepsilon \|z\|^2$ is psh on U. As ρ extends to a psh function in the ambient space, it follows that the function v is strictly psh on U. $\square$

We note that the notion of (positive) orbifold current is more restrictive than that of a (positive) current on X_{reg}. Indeed, an orbifold differential form determines a smooth form on X_{reg}^{orb} whose coefficients may blow up at points of $X_{reg} \setminus X_{reg}^{orb}$. For instance, consider the (global) orbifold structure on $\mathbb{C}^2$ given by the analytic cover $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $\phi(z_1, z_2) = (z_1^2, z_2)$, whose orbifold singular locus is the line $\{x_1 = 0\}$. Then $\theta = \frac{i}{|x_1|} dx_1 \wedge d\overline{x}_1$ is a smooth positive (1,1) orbifold form, since $\phi^\star \theta = 4i dz_1 \wedge d\overline{z}_1$ is smooth.

A current T on X_{reg} arising from an orbifold current acts on such forms θ in the following way: assuming that $\operatorname{supp}\theta \subset U \subset X_{reg}$ for some orbifold chart $(\widetilde{U}, \Gamma, \phi)$, we have $T|_U = \frac{1}{m} \phi_\star \widetilde{T}$ for some current $\widetilde{T}$ on $\widetilde{U}$, where $m = |\Gamma|$. Hence we set $\langle T, \theta \rangle := \frac{1}{m} \langle \widetilde{T}, \phi^\star \theta \rangle$. Returning to the previous example, we see that $T = \frac{i}{|x_1|} dx_2 \wedge d\overline{x}_2$ is a positive (1,1) current on $\mathbb{C}^2$ which does not arise from an orbifold current since $\int_K T \wedge \theta = +\infty$, where K is the bidisk in $\mathbb{C}^2$.

3.2. **Singular Hermitian metrics on orbifold line bundles.** We refer to [D3] for the notion of singular Hermitian metric on a holomorphic line bundle over a complex manifold or complex space (see also [MM2, p. 97]).

Let $(L, \mathscr{X})$ be an orbifold line bundle over the orbifold $\mathscr{X} = (X, \mathscr{U})$. A singular Hermitian metric on L is a collection $\widetilde{h} = \{\widetilde{h}_i\}$ of singular Hermitian metrics on the line bundles $(L_{\widetilde{U}_i}, \widetilde{U}_i)$, for every orbifold chart $(\widetilde{U}_i, \Gamma_i, \phi_i)$, such that:

(i) $\widetilde{h}_i$ is Γ_i-invariant: for every $\gamma \in \Gamma_i$ with induced linear map $\widetilde{\gamma} : L_{\widetilde{U}_i}|_x \longrightarrow L_{\widetilde{U}_i}|_{\gamma x}$ we have $\widetilde{h}_i(\widetilde{\gamma}\ell, \widetilde{\gamma}\ell') = \widetilde{h}_i(\ell, \ell')$ for all $\ell, \ell' \in L_{\widetilde{U}_i}|_x$, $x \in \widetilde{U}_i$.

(ii) $\widetilde{h}_i$ satisfy the following gluing condition: if $\lambda_{ji} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is an injection with associated bundle map $\widetilde{\lambda}_{ji} : L_{\widetilde{U}_j}|_{\lambda_{ji}(\widetilde{U}_i)} \longrightarrow L_{\widetilde{U}_i}$ then $\widetilde{h}_i(\widetilde{\lambda}_{ji}\ell, \widetilde{\lambda}_{ji}\ell') = \widetilde{h}_j(\ell, \ell')$ for all $\ell, \ell' \in L_{\widetilde{U}_j}|_{\lambda_{ji}(x)}$ and $x \in \widetilde{U}_i$.

The curvature current

$$c_1(L, \widetilde{h}) := \{c_1(L_{\widetilde{U}_i}, \widetilde{h}_i)\}$$

is a well defined real closed (1,1) orbifold current. Indeed:

(i) $c_1(L_{\widetilde{U}_i}, \widetilde{h}_i)$ is Γ_i-invariant: by shrinking U_i we may assume that $L_{\widetilde{U}_i}$ has a holomorphic frame $\widetilde{e}_i$ on $\widetilde{U}_i$, so $\widetilde{h}_i(\widetilde{e}_i, \widetilde{e}_i) = e^{-2\psi_i}$ for some function $\psi_i \in L^1_{loc}(\widetilde{U}_i)$. As $\widetilde{\gamma}\widetilde{e}_i$ is also a frame on $\widetilde{U}_i$, we have $\widetilde{\gamma}\widetilde{e}_i(x) = f(x)\widetilde{e}_i(\gamma x)$ for some non-vanishing holomorphic function f on $\widetilde{U}_i$ and $|f|^2 e^{-2\psi_i \circ \gamma} = \widetilde{h}_i(\widetilde{\gamma}\widetilde{e}_i, \widetilde{\gamma}\widetilde{e}_i) = \widetilde{h}_i(\widetilde{e}_i, \widetilde{e}_i) = e^{-2\psi_i}$. Hence $dd^c \psi_i \circ \gamma = dd^c \psi_i$, so $\gamma^\star c_1(L_{\widetilde{U}_i}, \widetilde{h}_i) = c_1(L_{\widetilde{U}_i}, \widetilde{h}_i)$.

(ii) If $\lambda_{ji} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is an injection then $\lambda_{ji}^\star(c_1(L_{\widetilde{U}_j}, \widetilde{h}_j)|_{\lambda_{ji}(\widetilde{U}_i)}) = c_1(L_{\widetilde{U}_i}, \widetilde{h}_i)$: if $\widetilde{e}_j$ is a frame of $L_{\widetilde{U}_j}$ over some set $\lambda_{ji}(V)$ where $V \subset \widetilde{U}_i$ is open, then $\widetilde{\lambda}_{ji}\widetilde{e}_j$ is a frame of $L_{\widetilde{U}_i}$ over V, so $\widetilde{\lambda}_{ji}\widetilde{e}_j(\lambda_{ji}(x)) = f(x)\widetilde{e}_i(x)$ for some non-vanishing holomorphic function f on V. Thus

$$|f|^2 e^{-2\psi_i} = \widetilde{h}_i(\widetilde{\lambda}_{ji}\widetilde{e}_j\lambda_{ji}, \widetilde{\lambda}_{ji}\widetilde{e}_j\lambda_{ji}) = \widetilde{h}_j(\widetilde{e}_j\lambda_{ji}, \widetilde{e}_j\lambda_{ji}) = e^{-2\psi_j \circ \lambda_{ji}} \text{ on } V,$$

which shows that $\lambda_{ji}^\star(dd^c \psi_j) = dd^c \psi_i$ on V.

We say that the metric $\widetilde{h}$ is (semi)positively curved if its curvature $c_1(L,\widetilde{h})$ is a positive current.

Lemma 3.6. *Let $(L,\mathscr{X})$ be an orbifold line bundle over the orbifold $\mathscr{X}=(X,\mathscr{U})$ endowed with a singular Hermitian metric $\widetilde{h}=\{\widetilde{h}_i\}$ and let $S=\{\widetilde{S}_i\}$ be a section of L. There exists a function on X, denoted by $|S|^2_{\widetilde{h}}$, so that for every orbifold chart $(\widetilde{U}_i,\Gamma_i,\phi_i)$ we have $|S|^2_{\widetilde{h}}\circ\phi_i=|\widetilde{S}_i|^2_{\widetilde{h}_i}:=\widetilde{h}_i(\widetilde{S}_i,\widetilde{S}_i)$.*

Proof. Note that the function $|\widetilde{S}_i|^2_{\widetilde{h}_i}$ is Γ_i-invariant, as

$$\widetilde{h}_i\big(\widetilde{S}_i(\gamma(x)),\widetilde{S}_i(\gamma(x))\big)=\widetilde{h}_i\big(\widetilde{\gamma}\widetilde{S}_i(x),\widetilde{\gamma}\widetilde{S}_i(x)\big)=\widetilde{h}_i\big(\widetilde{S}_i(x),\widetilde{S}_i(x)\big)\,,\ \forall\,\gamma\in\Gamma_i\,.$$

Hence there exists as function f_i on U_i so that $|\widetilde{S}_i|^2_{\widetilde{h}_i}=f_i\circ\phi_i$. We have to show that if $\lambda_{ji}:\widetilde{U}_i\longrightarrow\widetilde{U}_j$ is an injection then $f_j\,|_{U_i}=f_i$. Indeed,

$$\begin{aligned} f_j\circ\phi_i(x) &= f_j\circ\phi_j\circ\lambda_{ji}(x)=\widetilde{h}_j\big(\widetilde{S}_j(\lambda_{ji}(x)),\widetilde{S}_j(\lambda_{ji}(x))\big)\\ &= \widetilde{h}_i\big(\widetilde{\lambda}_{ji}\widetilde{S}_j(\lambda_{ji}(x)),\widetilde{\lambda}_{ji}\widetilde{S}_j(\lambda_{ji}(x))\big)=\widetilde{h}_i\big(\widetilde{S}_i(x),\widetilde{S}_i(x)\big)=f_i\circ\phi_i(x),\end{aligned}$$

for every $x\in\widetilde{U}_i$. □

Remark 3.7. If $x\in X^{orb}_{reg}$ there exists an orbifold chart $\phi_i:\widetilde{U}_i\longrightarrow U_i$ so that $x\in U_i$ and ϕ_i is biholomorphic. Thus $L\,|_{U_i}\cong(\phi_i^{-1})^\star L_{\widetilde{U}_i}$ is a holomorphic line bundle with a singular Hermitian metric h_i induced by $\widetilde{h}_i$. It follows that the holomorphic line bundle $L\,|_{X^{orb}_{reg}}$ has a singular Hermitian metric h induced by $\widetilde{h}$ and

$$|S|^2_h:=h(S,S)=|S|^2_{\widetilde{h}}\,|_{X^{orb}_{reg}}$$

for every orbifold section $S:X\longrightarrow L$. Moreover, the curvature current

$$c_1(L\,|_{X^{orb}_{reg}},h)=\mathbf{F}(c_1(L,\widetilde{h}))\,|_{X^{orb}_{reg}}\,,$$

where $\mathbf{F}$ is the map constructed in Proposition 3.1.

3.3. **Bergman kernel and Fubini-Study currents.** Let $(L,\mathscr{X})$ be an orbifold line bundle over $\mathscr{X}=(X,\mathscr{U})$ endowed with a singular Hermitian metric $\widetilde{h}=\{\widetilde{h}_i\}$, and let $\widetilde{\Omega}=\{\widetilde{\Omega}_i\}$ be a Hermitian form on $\mathscr{X}$. Here $\mathscr{U}=\{(\widetilde{U}_i,\Gamma_i,\phi_i)\}$, $U_i=\phi_i(\widetilde{U}_i)$, $m_i=|\Gamma_i|$. We denote by h the singular Hermitian metric induced by $\widetilde{h}$ on the holomorphic line bundle $L\,|_{X^{orb}_{reg}}$. By Proposition 3.1, $\widetilde{\Omega}$ induces a positive (1,1) current Ω on X_{reg}, which clearly is a smooth Hermitian form on X^{orb}_{reg} with $\phi_i^\star(\Omega\,|_{U_i})=\widetilde{\Omega}_i$.

The space $H^0_{(2)}(\mathscr{X},L)$ of L^2-holomorphic sections with respect to this metric data is defined as follows. Fix a partition of unity $\{\chi_l\}_{l\geq1}$ on X so that χ_l has compact support contained in U_{i_l} and set

$$\|S\|^2=\sum_{l=1}^{\infty}\frac{1}{m_{i_l}}\int_{\widetilde{U}_{i_l}}(\chi_l\circ\phi_{i_l})|\widetilde{S}_{i_l}|^2_{\widetilde{h}_{i_l}}\,\widetilde{\Omega}^n_{i_l}\,,\ \text{where } S=\{\widetilde{S}_i\}\in H^0(\mathscr{X},L).$$

Define

$$H^0_{(2)}(\mathscr{X},L) = \{S \in H^0(\mathscr{X},L) : \|S\|^2 < +\infty\},$$

endowed with the obvious inner product. If $A_i = U_i \cap X^{orb}_{sing}$ and $\widetilde{A}_i = \phi_i^{-1}(A_i)$, we note that

$$\frac{1}{m_{i_l}} \int_{\widetilde{U}_{i_l} \setminus \widetilde{A}_{i_l}} (\chi_l \circ \phi_{i_l}) |\widetilde{S}_{i_l}|^2_{\widetilde{h}_{i_l}} \, \widetilde{\Omega}^n_{i_l} = \int_{U_{i_l} \setminus A_{i_l}} \chi_l \, |S|^2_{\widetilde{h}} \, \Omega^n .$$

It follows that

$$\|S\|^2 = \int_{X^{orb}_{reg}} |S|^2_{\widetilde{h}} \, \Omega^n = \int_{X^{orb}_{reg}} |S|^2_h \, \Omega^n ,$$

where $|S|^2_{\widetilde{h}}$ is the function from Lemma 3.6, and the same holds for the inner product.

Since $H^0_{(2)}(\mathscr{X},L)$ is separable, let $\{S_j\}_{j\geq 1}$ be an orthonormal basis and denote by P the function defined on X by

$$P = \sum_{j=1}^{\infty} |S_j|^2_{\widetilde{h}} . \tag{9}$$

This function is independent of the choice of basis (see Lemma 3.8) and it is called the *Bergman kernel function* associated to the space $H^0_{(2)}(\mathscr{X},L)$.

The *orbifold Fubini-Study current* $\widetilde{\alpha} = \{\widetilde{\alpha}_i\}$ is defined as follows: given a chart $(\widetilde{U}_i, \Gamma_i, \phi_i)$ we may assume that $L_{\widetilde{U}_i}$ has a holomorphic frame $\widetilde{e}_i$ on $\widetilde{U}_i$. If $S_j = \{\widetilde{S}_{j,i}\}$ we write $\widetilde{S}_{j,i} = s_{j,i}\widetilde{e}_i$ for some holomorphic functions $s_{j,i}$ on $\widetilde{U}_i$, and we set

$$\widetilde{\alpha}_i = \frac{1}{2} dd^c \log\left(\sum_{j=1}^{\infty} |s_{j,i}|^2 \right), \tag{10}$$

where $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. To explain the terminology, let us assume that $H^0_{(2)}(\mathscr{X},L)$ is finite dimensional and non-trivial. Denote by ω_{FS} the Fubini-Study metric on $\mathbb{P}(H^0_{(2)}(\mathscr{X},L)^*)$ induced by the L^2 inner product on $H^0_{(2)}(\mathscr{X},L)$. Consider the Kodaira map

$$\Phi : X \dashrightarrow \mathbb{P}(H^0_{(2)}(\mathscr{X},L)^*), \quad x \mapsto \left\{S \in H^0_{(2)}(\mathscr{X},L) : S(x) = 0\right\}.$$

Then $\widetilde{\alpha} = \{\widetilde{\alpha}_i\}$, $\widetilde{\alpha}_i = (\Phi \circ \phi_i)^* \omega_{FS}$, is the orbifold Fubini-Study current.

For the convenience of the reader, we include a proof of some properties of these notions in our setting.

Lemma 3.8. *If the singular metrics $\widetilde{h}_i$ have locally upper bounded weights, then:*

(i) $\widetilde{\alpha}$ is a well defined positive closed current of bidegree (1,1) on $\mathscr{X}$.

(ii) The function P is independent of the choice of basis $\{S_j\}_{j\geq 1}$ and

$$P(x) = \max\{|S|^2_{\widetilde{h}}(x) : S \in H^0_{(2)}(\mathscr{X},L), \ \|S\| = 1\}, \ \ \forall x \in X. \tag{11}$$

(iii) *On each chart,* $\log P \circ \phi_i \in L^1_{loc}(\widetilde{U}_i, \widetilde{\Omega}^n_i)$ *and*

$$2\widetilde{\alpha}_i = 2c_1(L\,|_{\widetilde{U}_i}, \widetilde{h}_i) + dd^c \log P \circ \phi_i ,$$

so $\widetilde{\alpha}$ is independent of the choice of basis $\{S_j\}_{j\geq 1}$.

Proof. By the Riesz-Fischer theorem we have that $S \in H^0_{(2)}(\mathscr{X}, L)$ if and only if there exists a sequence $a = \{a_j\} \in l^2$ so that $S = S_a$, where $S_a = \sum_{j=1}^{\infty} a_j S_j$ and $\|S_a\| = \|a\|_2$.

Given a chart $U_i \subset X$ so that $L_{\widetilde{U}_i}$ has a holomorphic frame $\widetilde{e}_i$ on $\widetilde{U}_i$, we write $S_j = \{\widetilde{S}_{j,i}\}$, $S_a = \{\widetilde{S}_{a,i}\}$, $\widetilde{S}_{j,i} = s_{j,i}\widetilde{e}_i$, $\widetilde{S}_{a,i} = s_{a,i}\widetilde{e}_i$, with holomorphic functions $s_{j,i}, s_{a,i}$ on $\widetilde{U}_i$, and $|\widetilde{e}_i|^2_{\widetilde{h}_i} = e^{-2\psi_i}$. It follows that $s_{a,i} = \sum_{j=1}^{\infty} a_j s_{j,i}$ and the series converges locally uniformly on $\widetilde{U}_i$. Indeed, if $K_1 \Subset K_2 \subset \widetilde{U}_i$ are compact sets then, since ψ_i is locally upper bounded, we have

$$\begin{aligned}
\max_{K_1} \Big| \sum_{j=N}^{M} a_j s_{j,i} \Big|^2 &\leq C_1 \int_{K_2} \Big| \sum_{j=N}^{M} a_j s_{j,i} \Big|^2 \widetilde{\Omega}_i^n \leq C_2 \int_{K_2} \Big| \sum_{j=N}^{M} a_j \widetilde{S}_{j,i} \Big|^2_{\widetilde{h}_i} \widetilde{\Omega}_i^n \\
&\leq C_2 m_i \int_{\phi_i(K_2)\cap X^{orb}_{reg}} \Big| \sum_{j=N}^{M} a_j S_j \Big|^2_{\widetilde{h}} \Omega^n \\
&\leq C_2 m_i \Big\| \sum_{j=N}^{M} a_j S_j \Big\|^2 = C_2 m_i \sum_{j=N}^{M} |a_j|^2 .
\end{aligned}$$

As this holds for every sequence $a \in l^2$ we see that $\{s_{j,i}(z)\}_{j\geq 1} \in l^2$ for all $z \in \widetilde{U}_i$. Using this and the same argument from the proof of [CM, Lemma 3.1] we show that the series $\sum_{j=1}^{\infty} |s_{j,i}|^2$ converges locally uniformly on $\widetilde{U}_i$, so its logarithm is a psh function and the current $\widetilde{\alpha}_i$ is a positive closed current on $\widetilde{U}_i$. The proof that $\widetilde{\alpha} \in \widetilde{\mathscr{T}}$ is similar to the one showing that $c_1(L, \widetilde{h})$ is a well-defined orbifold current.

By Lemma 3.6 we see that

$$P \circ \phi_i = \sum_{j=1}^{\infty} |\widetilde{S}_{j,i}|^2_{\widetilde{h}_i} = e^{-2\psi_i} \sum_{j=1}^{\infty} |s_{j,i}|^2,$$

which implies (iii). To prove (ii) we let $x \in U_i$, for U_i as above. If $a \in l^2$, $\|a\|_2 = 1$, and $S_a = \sum_{j=1}^{\infty} a_j S_j$ then on $\widetilde{U}_i$, by the Cauchy-Schwarz inequality,

$$|S_a|^2_{\widetilde{h}} \circ \phi_i = |\widetilde{S}_{a,i}|^2_{\widetilde{h}_i} \leq \sum_{j=1}^{\infty} |\widetilde{S}_{j,i}|^2_{\widetilde{h}_i} = P \circ \phi_i .$$

Moreover, if $y \in \phi_i^{-1}(x)$ set

$$a = \left\{ c^{-1} \overline{s_{j,i}(y)} \right\}_{j\geq 1}, \; c := \left(\sum_{j=1}^{\infty} |s_{j,i}(y)|^2 \right)^{1/2} .$$

Then $\|a\|_2 = 1$, $\widetilde{S}_{a,i}(y) = c\widetilde{e}_i$, so $|S_a|^2_{\widetilde{h}}(x) = |\widetilde{S}_{a,i}(y)|^2_{\widetilde{h}_i} = |c|^2 e^{-2\psi_i(y)} = P(x)$. ☐

4. Proofs of Theorems 1.1, 1.4 and 1.5

Now we will prove the main results. In Section 4.1 we examine the extension of holomorphic square integrable sections defined on the orbifold regular locus. This yields the fact that the logarithm of the Bergman kernel is locally the difference of two psh functions. In Section 4.2 we recall the L^2 estimates for $\overline{\partial}$ in the form we use them. In Section 4.3 we prove the weak asymptotics of the Bergman kernel, $\frac{1}{p}\log P_p \to 0$ as $p \to \infty$, in L^1_{loc}, and deduce Theorems 1.1, 1.4 and 1.5.

4.1. **Extension of holomorphic sections.** We begin with two lemmas, which are formulated in a general context. Let $(L, \mathscr{X})$ be an orbifold line bundle over $\mathscr{X} = (X, \mathscr{U})$ endowed with a singular metric $\widetilde{h} = \{\widetilde{h}_i\}$, and let $\widetilde{\Omega} = \{\widetilde{\Omega}_i\}$ be a Hermitian form on $\mathscr{X}$. As before, we set $\mathscr{U} = \{(\widetilde{U}_i, \Gamma_i, \phi_i)\}$, $U_i = \phi_i(\widetilde{U}_i)$, $m_i = |\Gamma_i|$. Let h be the singular metric induced by $\widetilde{h}$ on the line bundle $L\mid_{X^{orb}_{reg}}$, and Ω be the Hermitian form induced by $\widetilde{\Omega}$ on X^{orb}_{reg}.

Lemma 4.1. *Assume that the singular metrics $\widetilde{h}_i$ have weights that are locally upper bounded and that*

$$S \in H^0(X^{orb}_{reg}, L\mid_{X^{orb}_{reg}}), \ \ \|S\|^2 = \int_{X^{orb}_{reg}} |S|^2_h \, \Omega^n < +\infty.$$

Then S extends to a holomorphic section of L over $\mathscr{X}$ and $S \in H^0_{(2)}(\mathscr{X}, L)$.

Proof. Without loss of generality, we can consider an orbifold chart so that $L_{\widetilde{U}_i}$ has a holomorphic frame $\widetilde{e}_i$ on $\widetilde{U}_i$ and we let $|\widetilde{e}_i|^2_{\widetilde{h}_i} = e^{-2\psi_i}$. Set $A_i = U_i \cap X^{orb}_{sing}$ and $\widetilde{A}_i = \phi_i^{-1}(A_i)$. The action of $\gamma \in \Gamma_i$ on $L_{\widetilde{U}_i}$ is defined by $\widetilde{\gamma}(\widetilde{e}_i(y)) = h_i(\gamma)\widetilde{e}_i(\gamma y)$, where h_i is a group homomorphism of Γ_i to the group of roots of order m_i of unity.

Using the notation from Section 2.2, it follows that if $y \in \widetilde{U}_i \setminus \widetilde{A}_i$ we can define $\widetilde{S}_i(y)$ as the unique element of the set $\widetilde{\phi}_i^{-1}(S(\phi_i(y)))$ that lies in the fiber $L_{\widetilde{U}_i}\mid_y$, and $\widetilde{S}_i$ is a holomorphic section of $L_{\widetilde{U}_i}$ over $\widetilde{U}_i \setminus \widetilde{A}_i$. Writing $\widetilde{S}_i = f\widetilde{e}_i$, we see that

$$\widetilde{\phi}_i^{-1}(S(\phi_i(y))) = \{f(y)h_i(\gamma)\widetilde{e}_i(\gamma y) : \gamma \in \Gamma_i\},$$

so $\widetilde{\gamma}(\widetilde{S}_i(y)) = f(y)h_i(\gamma)\widetilde{e}_i(\gamma y) = \widetilde{S}_i(\gamma(y))$. Since ψ_i is locally upper bounded we have for any compact $K \subset \widetilde{U}_i$,

$$\int_{K\setminus\widetilde{A}_i} |f|^2 \, \widetilde{\Omega}_i^n \leq C_K \int_{K\setminus\widetilde{A}_i} |\widetilde{S}_i|^2_{\widetilde{h}_i} \, \widetilde{\Omega}_i^n \leq C_K m_i \int_{\phi_i(K)\setminus A_i} |S|^2_h \, \Omega^n < +\infty.$$

By Skoda's lemma [MM2, Lemma 2.3.22], we conclude that $\widetilde{S}_i$ extends to an equivariant holomorphic section of $L_{\widetilde{U}_i}$ over $\widetilde{U}_i$. If $\lambda_{ji} : \widetilde{U}_i \longrightarrow \widetilde{U}_j$ is an injection we have $\lambda_{ji}(\widetilde{U}_i \setminus \widetilde{A}_i) = \lambda_{ji}(\widetilde{U}_i) \setminus \widetilde{A}_j$. Indeed, for $y \in \widetilde{U}_i$ we have that the stabilizer of $\lambda_{ji}(y)$ is $(\Gamma_j)_{\lambda_{ji}(y)} = \lambda_{ji}((\Gamma_i)_y)$ (see (7)). Using this we verify that the gluing condition $\widetilde{\lambda}_{ji} \circ \widetilde{S}_j \circ \lambda_{ji} = \widetilde{S}_i$ holds on $\widetilde{U}_i \setminus \widetilde{A}_i$, hence on $\widetilde{U}_i$. $\square$

Lemma 4.2. *Assume that the current $c_1(L,\widetilde{h}) \geq 0$ and let $\widetilde{\alpha} = \{\widetilde{\alpha}_i\}$ be the Fubini-Study current associated to $H^0_{(2)}(\mathscr{X},L)$. Then every point $x \in X$ has a neighborhood U contained in some chart U_i on which there exist psh functions $v = v_x(\widetilde{\alpha})$, $u = u_x(c_1(L,\widetilde{h}))$, so that $\widetilde{\alpha}_i = dd^c(v \circ \phi_i)$, $c_1(L\,|_{\widetilde{U}_i}, \widetilde{h}_i) = dd^c(u \circ \phi_i)$ hold on $\phi_i^{-1}(U)$, and $2v - 2u = \log P$ holds on U.*

Corollary 4.3. *The function $\log P$ is locally the difference of two psh functions near each point of X.*

Proof of Lemma 4.2. Let $x \in U_i$. Assuming that $\widetilde{e}_i$ is a frame for $L_{\widetilde{U}_i}$ we write $|\widetilde{e}_i|^2_{\widetilde{h}_i} = e^{-2\psi_i}$, where $\psi_i \in PSH(\widetilde{U}_i)$. By (10) and Lemma 3.8 we have

$$\widetilde{\alpha}_i = dd^c v_i\,,\ v_i = \frac{1}{2}\log\left(\sum_{j=1}^{\infty}|s_{j,i}|^2\right) \in PSH(\widetilde{U}_i),\, 2v_i - 2\psi_i = \log P \circ \phi_i\,.$$

Letting $\widetilde{v} = m_i^{-1}\sum_{\gamma\in\Gamma_i} v_i \circ \gamma$, $\psi = m_i^{-1}\sum_{\gamma\in\Gamma_i}\psi_i\circ\gamma$, we obtain Γ_i-invariant psh functions so that $\widetilde{\alpha}_i = dd^c\widetilde{v}$, $c_1(L\,|_{\widetilde{U}_i}, \widetilde{h}_i) = dd^c\psi$. Hence $\widetilde{v} = v\circ\phi_i$, $\psi = u\circ\phi_i$, for psh functions v, u on U_i. Since $\phi_i\circ\gamma = \phi_i$ we have $2v_i\circ\gamma - 2\psi_i\circ\gamma = \log P\circ\phi_i$, so $2\widetilde{v} - 2\psi = \log P\circ\phi_i$. $\square$

4.2. **Demailly's estimates for $\overline{\partial}$.** In order to prove our theorems we need the following variants of the existence theorem for $\overline{\partial}$ in the case of singular Hermitian metrics due to Demailly [D1]. Let (M,Ω) be a Hermitian manifold of dimension n, (L,h) be a singular Hermitian holomorphic line bundle. Let $\theta \geq 0$ be a $(1,1)$-current and let θ_{abs} be the absolute continuous component in the Lebesgue decomposition of θ. For any $\alpha \in \Lambda^{n,1}T^*_xX \otimes L_x$, $x \in M$, we define $|\alpha|_{\Omega,\theta} \in [0,\infty]$ to be the smallest number satisfying

$$\big|\langle\alpha,\beta\rangle\big|^2 \leq \big|\alpha\big|^2_{\Omega,\theta}\,\langle\theta_{abs}\wedge\Lambda_\Omega\beta,\beta\rangle, \quad \text{for all } \beta \in \Lambda^{n,1}T^*_xX\otimes L_x,$$

where Λ_Ω is the interior product with Ω. This number is independent of the metric: if Ω_1 is another metric on M, then $|\alpha|^2_{\Omega_1,\theta}\,\Omega_1^n = |\alpha|^2_{\Omega,\theta}\,\Omega^n$, cf. [D1, Lemme 3.2 (3.2)]. Note also that for any $\alpha \in \Lambda^{n,0}T^*_xX\otimes L_x$, $x \in M$, we have $|\alpha|^2_{\Omega_1}\,\Omega_1^n = |\alpha|^2_\Omega\,\Omega^n$, cf. [D1, Remarque 4.5]. Hence for $(n,1)$-forms the estimate in the following theorem is independent of the Kähler metric.

Theorem 4.4 ([D1, Théorème 5.1])**.** *Let (M,Ω) be a Kähler manifold of dimension n which admits a complete Kähler metric. Let (L,h) a singular Hermitian holomorphic line bundle such that $\theta = c_1(L,h) \geq 0$. Then for any form $g \in L^2_{n,1}(M,L,loc)$ satisfying*

$$\overline{\partial} g = 0\,, \quad \int_M |g|^2_{\Omega,\theta}\,\Omega^n < +\infty$$

there exists $u \in L^2_{n,0}(M,L,loc)$ *with* $\overline{\partial}u = g$ *and*

$$\int_M |u|^2_{\Omega}\,\Omega^n \le \int_M |g|^2_{\Omega,\theta}\,\Omega^n .$$

Corollary 4.5. *Let* (M,Ω) *be a Kähler manifold of dimension n which admits a complete Kähler metric. Let* (L,h) *a singular Hermitian holomorphic line bundle and let* $\lambda : M \to [0,+\infty)$ *be a continuous function such that* $c_1(L,h) \ge \lambda\Omega$. *Then for any form* $g \in L^2_{n,1}(M,L,loc)$ *satisfying*

$$\overline{\partial}g = 0\,, \quad \int_M \lambda^{-1}|g|^2\,\Omega^n < +\infty$$

there exists $u \in L^2_{n,0}(M,L,loc)$ *with* $\overline{\partial}u = g$ *and*

$$\int_M |u|^2\,\Omega^n \le \int_M \lambda^{-1}|g|^2\,\Omega^n .$$

Proof. We have $c_1(L,h)_{abs} \ge \lambda\Omega$, so by [D1, Remarque 4.2], $|\alpha|^2_{\Omega,\theta} \le \lambda(x)^{-1}|\alpha|^2$, for any $\alpha \in \Lambda^{n,1}T^*_xX \otimes L_x$ (with the conventions $\frac{1}{0} = +\infty$, $0\cdot\infty = 0$). Hence the conclusion follows immediately from Theorem 4.4. □

Corollary 4.6. *Let* (M,Ω) *be a Kähler manifold such that* $Ric_\Omega \ge 0$. *Assume that M carries a complete Kähler metric. Let* (L,h) *a singular Hermitian holomorphic line bundle and let* $\lambda : M \to [0,+\infty)$ *be a continuous function such that* $c_1(L,h) \ge \lambda\Omega$. *Then for any form* $g \in L^2_{0,1}(M,L,loc)$ *satisfying*

$$\overline{\partial}g = 0\,, \quad \int_M \lambda^{-1}|g|^2\,\Omega^n < +\infty$$

there exists $u \in L^2_{0,0}(M,L,loc)$ *with* $\overline{\partial}u = g$ *and*

$$\int_M |u|^2\,\Omega^n \le \int_M \lambda^{-1}|g|^2\,\Omega^n .$$

Proof. We apply Corollary 4.5 to the line bundle $(F,h^F) = (L \otimes K^*_X, h \otimes h^{K^*_X})$, where $h^{K^*_X}$ is the metric induced by Ω. Obviously, $c_1(F,h^F) \ge \lambda\Omega$. There exists a natural isometry

$$\Psi = \sim : \Lambda^{0,q}(T^*X) \otimes L \longrightarrow \Lambda^{n,q}(T^*X) \otimes F,$$
$$\Psi\, s = \widetilde{s} = (w^1 \wedge \ldots \wedge w^n \wedge s) \otimes (w_1 \wedge \ldots \wedge w_n),$$

where $\{w_j\}_{j=1}^n$ is a local holomorphic frame of $T^{(1,0)}X$ and $\{w^j\}_{j=1}^n$ is the dual frame. The operator Ψ commutes with the action of $\overline{\partial}$. For a form $g \in L^2_{0,1}(M,L,loc)$ with $\overline{\partial}g = 0$, $\int_M \lambda^{-1}|g|^2\,\Omega^n < +\infty$ we have $\widetilde{g} \in L^2_{n,1}(M,L,loc)$, $\overline{\partial}\widetilde{g} = 0$ and $\int_M \lambda^{-1}|\widetilde{g}|^2\,\Omega^n < +\infty$. By Corollary 4.5, there exists $\widetilde{u} \in L^2_{n,0}(M,L,loc)$ with $\overline{\partial}\widetilde{u} = \widetilde{g}$ and $\int_M |\widetilde{u}|^2\,\Omega^n \le \int_M \lambda^{-1}|\widetilde{g}|^2\,\Omega^n$. Then $u = \Psi^{-1}\widetilde{u}$ satisfies the conclusion. □

4.3. **Proofs of Theorems 1.1, 1.4, 1.5.** Theorem 1.1 will follow from Lemmas 4.1, 4.2, and from:

Theorem 4.7. *In the setting of Theorem 1.1, we have that* $\frac{1}{p}\log P_p \to 0$ *as* $p \to \infty$*, in* $L^1_{loc}(G \cap X^{orb}_{reg}, \Omega^n)$.

Proof. Let $X' = X^{orb}_{reg}$ and recall that $L|_{X'}$ is a holomorphic line bundle with a metric h induced by $\widetilde{h}$. We denote by h_p the metric induced by h and $h^{K_{X'}}$ on $(L|_{X'})^p \otimes K_{X'}$. By Lemma 4.1, P_p is the Bergman kernel function of the space $H^0_{(2)}(X', (L|_{X'})^p \otimes K_{X'}) = H^0_{(2)}(\mathscr{X}, L^p \otimes K_{\mathscr{X}})$, where the norm is denoted by $\|\cdot\|$.

We let $x \in G \cap X'$ and $U_\alpha \subset G \cap X'$ be a coordinate neighborhood of x on which there exists a holomorphic frame e_α of $L|_{X'}$ and e'_α of $K_{X'}$. Let ψ_α be a psh weight of h and ρ_α be a smooth weight of $h^{K_{X'}}$ on U_α. Fix $r_0 > 0$ so that the ball $V := B(x, 2r_0) \Subset U_\alpha$ and let $U := B(x, r_0)$.

Following the arguments of [D4, CM] we will show that there exist constants $C > 0$, $p_0 \in \mathbb{N}$ so that

$$-\frac{\log C}{p} \le \frac{1}{p}\log P_p(z) \le \frac{\log(Cr^{-2n})}{p} + 2\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right) \tag{12}$$

holds for all $p > p_0$, $0 < r < r_0$ and $z \in U$ with $\psi_\alpha(z) > -\infty$. By (12) it follows that $\frac{1}{p}\log P_p \to 0$ in $L^1(U, \Omega^n)$, as in [CM, Theorem 5.1].

For the upper estimate, fix $z \in U$ with $\psi_\alpha(z) > -\infty$ and $r < r_0$. Let $S \in H^0_{(2)}(X', (L|_{X'})^p \otimes K_{X'})$ with $\|S\| = 1$ and write $S = s\, e_\alpha^{\otimes p} \otimes e'_\alpha$. Then

$$\begin{aligned}
|S(z)|^2_{h_p} &= |s(z)|^2 e^{-2p\psi_\alpha(z) - 2\rho_\alpha(z)} \le e^{-2p\psi_\alpha(z)} \frac{C_1}{r^{2n}} \int_{B(z,r)} |s|^2 \Omega^n \\
&\le \frac{C_2}{r^{2n}} \exp\left(2p\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right)\right) \int_{B(z,r)} |s|^2 e^{-2p\psi_\alpha - 2\rho_\alpha} \Omega^n \\
&\le \frac{C_2}{r^{2n}} \exp\left(2p\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right)\right),
\end{aligned}$$

where C_2 is a constant that depends only on x. Hence by Lemma 3.8

$$\frac{1}{p}\log P_p(z) = \frac{1}{p}\max_{\|S\|=1} \log |S(z)|^2_{h_p} \le \frac{\log(C_2 r^{-2n})}{p} + 2\left(\max_{B(z,r)} \psi_\alpha - \psi_\alpha(z)\right).$$

We prove next the lower estimate from (12). We proceed like in the proof of [CM, Theorem 5.1] by using an argument of [D5, Section 9] to show that there exist a constant $C_1 > 0$ and $p_0 \in \mathbb{N}$ such that for all $p > p_0$ and all $z \in U$ with $\psi_\alpha(z) > -\infty$ there is a section $S_{z,p} \in H^0_{(2)}(X', (L|_{X'})^p \otimes K_{X'}) = H^0_{(2)}(\mathscr{X}, L^p \otimes K_{\mathscr{X}})$ with $S_{z,p}(z) \neq 0$ and

$$\|S_{z,p}\|^2 \le C_1 |S_{z,p}(z)|^2_{h_p}. \tag{13}$$

Observe that (11) and (13) yield the desired lower estimate

$$\frac{1}{p}\log P_p(z) = \frac{1}{p}\max_{\|S\|=1}\log|S(z)|^2_{h_p} \geq -\frac{\log C_1}{p}.$$

Let us prove the existence of $S_{z,p}$ as above. By the Ohsawa-Takegoshi extension theorem [OT] there exists a constant $C' > 0$ (depending only on x) such that for any $z \in U$ and any p there exists a function $v_{z,p} \in \mathscr{O}(V)$ with $v_{z,p}(z) \neq 0$ and

$$\int_V |v_{z,p}|^2 e^{-2p\psi_\alpha}\Omega^n \leq C'|v_{z,p}(z)|^2 e^{-2p\psi_\alpha(z)}. \tag{14}$$

We shall now solve the $\overline{\partial}$-equation with L^2-estimates in order to extend $v_{z,p}$ to a section of $(L|_{X'})^p \otimes K_{X'}$ over X'. Let $\theta \in \mathscr{C}^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \theta \leq 1$, $\theta(t) = 1$ for $|t| \leq \frac{1}{2}$, $\theta(t) = 0$ for $|t| \geq 1$. Define the quasi-psh function φ_z on X' by

$$\varphi_z(y) = \begin{cases} 2n\theta\big(\frac{|y-z|}{r_0}\big)\log\frac{|y-z|}{r_0}, & \text{for } y \in U_\alpha, \\ 0, \quad \text{for } y \in X' \setminus B(z,r_0). \end{cases} \tag{15}$$

Then there exists $C > 0$ such that $dd^c\varphi_z \geq -C\Omega$ on X' for all $z \in U$. Since the function $\varepsilon > 0$ is continuous, there exists a constant $a' > 0$ such that

$$c_1(L|_{X'}, h) \geq a'\Omega \quad \text{on a neighborhoof of } \overline{V}.$$

Therefore there exist $a > 0$, $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$ and all $z \in U$

$$\begin{aligned} &c_1((L|_{X'})^p, h^p e^{-\varphi_z}) \geq 0 \text{ on } X', \\ &c_1((L|_{X'})^p, h^p e^{-\varphi_z}) \geq ap\,\Omega \text{ on a neighborhoof of } \overline{V}. \end{aligned}$$

Let $\lambda : X' \to [0,+\infty)$ be a continuous function such that $\lambda = ap$ on $\overline{V}$ and

$$c_1((L|_{X'})^p, h^p e^{-\varphi_z}) \geq \lambda\Omega.$$

Consider the form

$$g_{z,p} \in L^2_{n,1}(X', (L|_{X'})^p),\ g_{z,p} = \overline{\partial}\big(v_{z,p}\,\theta\big(\tfrac{|y-z|}{r_0}\big)e_\alpha^{\otimes p} \otimes e'_\alpha\big).$$

For simplicity, let h_p also denote the metric induced on $(L|_{X'})^p \otimes \Lambda^{n,1}(T^*X')$ by h and Ω. Then

$$\int_{X'}\frac{1}{\lambda}|g_{z,p}|^2_{h_p}e^{-\varphi_z}\Omega^n = \int_V \frac{1}{\lambda}|g_{z,p}|^2_{h_p}e^{-\varphi_z}\Omega^n = \frac{1}{ap}\int_V |g_{z,p}|^2_{h_p}e^{-\varphi_z}\Omega^n < +\infty.$$

Note that the integral at the right is finite by (14), since $\psi_\alpha(z) > -\infty$ and

$$\int_V |g_{z,p}|^2_{h_p}e^{-\varphi_z}\Omega^n \leq C'''\int_V |v_{z,p}|^2|\overline{\partial}\theta(\tfrac{|y-z|}{r_0})|^2 e^{-2p\psi_\alpha}e^{-\varphi_z}\Omega^n \leq C''\int_V |v_{z,p}|^2 e^{-2p\psi_\alpha}\Omega^n,$$

where $C''', C'' > 0$ are constants that depend only on x.

By the hypotheses of Theorem 1.1, X' carries a complete Kähler metric. Thus, the hypotheses of Corollary 4.5 are satisfied for the Kähler manifold (X',Ω), the semi-positive line bundle $((L\,|_{X'})^p, h^p e^{-\varphi_z})$ and the form $g_{z,p}$, for all $p \geq p_0$ and $z \in U$. So there exists $u_{z,p} \in L^2_{n,0}(X',(L\,|_{X'})^p)$ such that $\overline{\partial} u_{z,p} = g_{z,p}$ and

$$\int_{X'} |u_{z,p}|^2_{h_p} e^{-\varphi_z}\,\Omega^n \leq \int_{X'} \frac{1}{\lambda} |g_{z,p}|^2_{h_p} e^{-\varphi_z}\Omega^n \leq \frac{C''}{ap}\int_V |v_{z,p}|^2 e^{-2p\psi_\alpha}\Omega^n . \tag{16}$$

Near z, $e^{-\varphi_z(y)} = r_0^{2n}|y-z|^{-2n}$ is not integrable, thus (16) implies that $u_{z,p}(z) = 0$. Define

$$S_{z,p} := v_{z,p}\,\theta\big(\tfrac{|y-z|}{r_0}\big) e_\alpha^{\otimes p} \otimes e'_\alpha - u_{z,p} .$$

Then $\overline{\partial} S_{z,p} = 0$, $S_{z,p}(z) = v_{z,p}(z) e_\alpha^{\otimes p} \otimes e'_\alpha(z) \neq 0$, $S_{z,p} \in H^0_{(2)}(X',(L\,|_{X'})^p \otimes K_{X'})$. Since $\varphi_z \leq 0$ on X', we have by (16) and (14)

$$\begin{aligned}\|S_{z,p}\|^2 = \int_{X'} |S_{z,p}|^2_{h_p}\,\Omega^n \;&\leq\; 2\left(C'''\int_V |v_{z,p}|^2 e^{-2p\psi_\alpha}\Omega^n + \int_{X'} |u_{z,p}|^2_{h_p} e^{-\varphi_z}\,\Omega^n\right)\\ &\leq\; 2C'\left(C''' + \frac{C''}{ap}\right)|v_{z,p}(z)|^2 e^{-2p\psi_\alpha(z)} \leq C_1 |S_{z,p}(z)|^2_{h_p},\end{aligned}$$

with a constant $C_1 > 0$ that depends only on x. This concludes the proof of (13). □

Proof of Theorem 1.1. Let us write $\gamma_p = \{\widetilde{\gamma}_{p,i}\}$, where $\widetilde{\gamma}_{p,i}$ is the corresponding Fubini-Study current on $\widetilde{U}_i$ defined as in (10). We fix $x \in G$ and let $(\widetilde{U}_i,\Gamma_i,\phi_i)$ be an orbifold chart so that $x \in U_i \subset G$ and $L_{\widetilde{U}_i}$, $K_{\widetilde{U}_i}$ have holomorphic frames on $\widetilde{U}_i$. Set $A_i = U_i \cap X^{orb}_{sing}$ and $\widetilde{A}_i = \phi_i^{-1}(A_i)$.

(i) Lemma 4.2 (and its proof) shows that there exist psh functions v_p, u and a continuous function ρ on U_i so that $\rho\circ\phi_i$ is smooth,

$$\widetilde{\gamma}_{p,i} = dd^c(v_p\circ\phi_i),\;\; c_1(L_{\widetilde{U}_i},\widetilde{h}_i) = dd^c(u\circ\phi_i),\;\; c_1(K_{\widetilde{U}_i},\widetilde{h}_i^{K_{\widetilde{U}_i}}) = dd^c(\rho\circ\phi_i),$$

and $2v_p - 2(pu+\rho) = \log P_p$. Hence on $\widetilde{U}_i$,

$$\frac{1}{p}\,v_p\circ\phi_i - u\circ\phi_i = \frac{1}{2p}\,\log P_p\circ\phi_i + \frac{1}{p}\,\rho\circ\phi_i .$$

By Theorem 4.7 we have $\frac{1}{p}\log P_p\circ\phi_i \to 0$, so $\frac{1}{p}v_p\circ\phi_i \to u\circ\phi_i$, in $L^1_{loc}(\widetilde{U}_i\setminus\widetilde{A}_i)$. It follows that the sequence $\{\frac{1}{p}v_p\circ\phi_i\}$ is locally uniformly upper bounded in $\widetilde{U}_i\setminus\widetilde{A}_i$.

If $y \in \widetilde{A}_i$ we may assume that there exist coordinates z on some neighborhood $V \subset \widetilde{U}_i$ of $y = 0$ so that $V\cap\widetilde{A}_i$ is contained in the cone $\{|z_n| \leq \max(|z_1|,\dots,|z_{n-1}|)\}$. Applying the maximum principle on complex lines parallel to the z_n axis, we see that there exist a neighborhood $V_1 \subset V$ of y and a compact set $K \subset V\setminus\widetilde{A}_i$ so that $\sup_{V_1} v_p\circ\phi_i \leq \sup_K v_p\circ\phi_i$. Hence $\{\frac{1}{p}v_p\circ\phi_i\}$ is uniformly upper bounded on V_1.

We conclude that the sequence $\{\frac{1}{p}v_p\circ\phi_i\}$ is locally uniformly upper bounded in $\widetilde{U}_i$. Hence it is relatively compact in $L^1_{loc}(\widetilde{U}_i)$ and it converges to $u\circ\phi_i$ in $L^1_{loc}(\widetilde{U}_i)$, since it does so outside $\widetilde{A}_i$ (see [Ho, Theorem 3.2.12]). This implies that $\frac{1}{p}\widetilde{\gamma}_{p,i}\to c_1(L_{\widetilde{U}_i},\widetilde{h}_i)$ weakly on $\widetilde{U}_i$, which yields (i).

(ii) The proof of (i) implies that $\frac{1}{p}\log P_p\circ\phi_i\to 0$ in $L^1_{loc}(\widetilde{U}_i)$. We may assume that there is an embedding $U_i\hookrightarrow\mathbb{C}^N$. If $K\Subset U_i$ and z denote the coordinates on $\mathbb{C}^N$ then

$$\int_K|\log P_p|(dd^c\|z\|^2)^n=\frac{1}{m_i}\int_{\phi_i^{-1}(K)}|\log P_p\circ\phi_i|(dd^c\|\phi_i\|^2)^n,$$

where $m_i=|\Gamma_i|$. Since $(dd^c\|\phi_i\|^2)^n$ is smooth, it follows that $\frac{1}{p}\log P_p\to 0$ in $L^1_{loc}(U_i)$ with respect to the area measure of X. ☐

To prove Theorem 1.4, we will need the following theorem:

Theorem 4.8. *In the setting of Theorem 1.4, we have that* $\frac{1}{p}\log P_p\to 0$ *as* $p\to\infty$*, in* $L^1_{loc}(G\cap X^{orb}_{reg},\Omega^n)$.

Proof. By Lemma 4.1, P_p is the Bergman kernel function of $H^0_{(2)}(X',(L\,|_{X'})^p)=H^0_{(2)}(\mathscr{X},L^p)$, where the norm is denoted by $\|\cdot\|$. We repeat the proof of Theorem 4.7 by using the same notations and replacing (n,q)-forms with $(0,q)$-forms. Hence the only non-formal difference is the proof of the analogue of lower estimate from (12). More precisely, we have to show that there exist a constant $C_1>0$ and $p_0\in\mathbb{N}$ such that for all $p>p_0$ and all $z\in U$ with $\psi_\alpha(z)>-\infty$ there is a section $S_{z,p}\in H^0_{(2)}(X',(L\,|_{X'})^p)$ with $S_{z,p}(z)\neq 0$ and satisfying (13).

Let $v_{z,p}\in\mathscr{O}(V)$ with $v_{z,p}(z)\neq 0$ satisfying (14), given by the Ohsawa-Takegoshi theorem, and define the quasi-psh function φ_z as in (15). We solve the $\overline{\partial}$-equation with L^2-estimates in order to extend $v_{z,p}(z)e_\alpha^{\otimes p}(z)$ to the desired section of $(L\,|_{X'})^p$ over X'. To this end, we proceed exactly as in the proof of Theorem 4.7, and use Corollary 4.6 instead of Corollary 4.5. ☐

Proof of Theorem 1.4. We follow the proof of Theorem 1.1 and apply Theorem 4.8 instead of Theorem 4.7. ☐

Proof of Theorem 1.5. Since $(L,\mathscr{X},\widetilde{h})$ is a positive orbifold line bundle, [MM2, Theorem 5.4.19] yields $\frac{1}{p}\log P_p\to 0$ in $L^1_{loc}(X^{orb}_{reg},\Omega^n)$ as $p\to\infty$, so the proof of Theorem 1.1 applies again. ☐

In the case of Theorem 1.5 the convergence of the induced Fubini-Study metric to the initial metric is quite explicit. Namely, it is shown in [MM2, Theorem 5.4.19] that there exists $c>0$ such that for any $\ell\in\mathbb{N}$, there exists $C_\ell>0$ with

$$\Big|\frac{1}{p}(\Phi_p^*\omega_{FS})(x)-\omega(x)\Big|_{\mathscr{C}^\ell}\leqslant C_\ell\Big(\frac{1}{p}+p^{\ell/2}\exp\big(-c\sqrt{p}\,d(x,X^{orb}_{sing})\big)\Big),\ p\gg 1.$$

We end the section with two lemmas which imply Proposition 1.2.

Lemma 4.9. *Let $\mathscr{X} = (X, \mathscr{U})$ be a compact Kähler orbifold. Then X^{orb}_{reg} carries a complete Kähler metric.*

Proof. We will adapt the arguments of [O] to our context. Let ω be a Kähler current on X whose local potentials are continuous near each $x \in X$ and smooth near each $x \in X^{orb}_{reg}$ (cf. Proposition 3.5). Let us consider a Hermitian metric η on the complex space X (see e. g. [MM2, Definition 3.4.12]). Now, $X^{orb}_{reg} = X \setminus X^{orb}_{sing}$ and X^{orb}_{sing} is an analytic set containing X_{sing}. Repeating the proof from [O, Proposition 1.1] we find a smooth proper function $\psi : X^{orb}_{reg} \to (-\infty, 0]$ and a constant $A_0 > 0$ such that for all $A > A_0$, $\eta_A = i\partial\bar{\partial}\psi + A\eta$ is a Hermitian metric on X^{orb}_{reg} and the gradient of ψ with respect to η_A is bounded. Since X is compact, there exists $\alpha > 0$ such that $\omega \geq \alpha\eta$ on X_{reg}. Set $\omega_A = i\partial\bar{\partial}\psi + (A/\alpha)\omega$, so $\omega_A \geq \eta_A$. Hence, for A sufficiently large, ω_A is a Kähler metric and the gradient of ψ with respect to ω_A is bounded, thus ω_A is complete. □

Lemma 4.10. *Let X be a reduced Stein space of dimension n and let Y be a (closed) analytic subset of X containing the singular locus X_{sing}. Then $X \setminus Y$ carries a complete Kähler metric.*

Proof. A result of Narasimhan [N, Theorem 5] states that a reduced Stein space X of dimension n admits a holomorphic mapping $\iota : X \to \mathbb{C}^{2n+1}$, which is one-to-one, proper and regular on X_{reg}. Hence $X \setminus Y$ is biholomorphic to $\iota(X) \setminus \iota(Y)$. Since ι is proper, $\iota(Y)$ is an analytic subset of $\mathbb{C}^{2n+1}$. By a theorem of Grauert [G, Satz A, p. 51], $\mathbb{C}^{2n+1} \setminus \iota(Y)$ admits a complete Kähler metric. Its restriction to the closed submanifold $\iota(X) \setminus \iota(Y) \cong X \setminus Y$ is a complete Kähler metric. □

Example 4.11. Let us state some interesting particular cases of our results.

(i) Let X be a Stein manifold of dimension n. Let φ be a psh function on X which is strictly psh on an open set G. Consider the Hilbert spaces

$$H^{n,0}_{(2)}(X, p\varphi) = \left\{ s \text{ holomorphic } n\text{-form} : \int_X i^{n^2} s \wedge \bar{s}\, e^{-2p\varphi} < \infty \right\}.$$

Thus $H^{n,0}_{(2)}(X, p\varphi) = H^{n,0}_{(2)}(X, L^p)$, where (L, h) is the trivial line bundle endowed with the metric $h = e^{-\varphi}$. Let $\{S^p_j\}$ be an orthonormal basis of $H^{n,0}_{(2)}(X, p\varphi)$. Let Ω^n be a volume element on X and define $f^p_j = i^{n^2} S^p_j \wedge \overline{S^p_j}/\Omega^n$. The Fubini-Study current associated to $H^{n,0}_{(2)}(X, p\varphi)$ is $\gamma_p = \frac{1}{2} dd^c \log(\sum_{j=1}^{\infty} |f^p_j|^2)$. By Theorem 1.1 and Proposition 1.2 (ii), we have $\frac{1}{p}\gamma_p \to dd^c\varphi$ weakly as currents on G as $p \to \infty$.

(ii) Let X be a Stein space of pure dimension n. Assume that X has only orbifold singularities, i. e. X admits an orbifold structure such that $X^{orb}_{sing} = X_{sing}$. Let φ be a psh function on X which is strictly psh on an open set G. Consider the Hilbert spaces

$$H^{n,0}_{(2)}(X,p\varphi)=\Big\{s \text{ holomorphic } n\text{-form on } X_{reg}:\int_{X_{reg}} i^{n^2}s\wedge\overline{s}\,e^{-2p\varphi}<\infty\Big\}.$$

By defining the Fubini-Study currents as above with the help of a volume element Ω^n on X_{reg} we obtain that $\frac{1}{p}\gamma_p\to dd^c\varphi$ weakly as currents on $G_{reg}=G\cap X_{reg}$ as $p\to\infty$.

(iii) Let X be a complete Kähler manifold, $(L,h)\longrightarrow X$ a holomorphic line bundle endowed with a singular Hermitian metric h with $c_1(L,h)\geq 0$ in the sense of currents. Assume that $c_1(L,h)$ is strictly positive on an open set G. Denote by γ_p the Fubini-Study current associated to $H^{n,0}_{(2)}(X,L^p)$. Then, as $p\to\infty$, $\frac{1}{p}\log P_p\to 0$ in $L^1_{loc}(G)$ and $\frac{1}{p}\gamma_p\to c_1(L,h)$ weakly as currents on G. Note that actually the full asymptotic expansion of the Bergman kernel (4) on G was obtained in [HsM], in the case that h is smooth.

5. Distribution of zeros of random sections

In this section we prove Theorem 1.6. For the convenience of the reader, we recall here the results of Shiffman and Zelditch [SZ1, SZ3] that are needed to prove Theorem 1.6. As noted in [SZ3], their calculations hold in a general setting.

Let $(L,\mathscr{X},\widetilde{h})$ be an orbifold line bundle over the n-dimensional orbifold $\mathscr{X}$, endowed with a singular Hermitian metric $\widetilde{h}$, and let $\widetilde{\Omega}$ be a Hermitian form on $\mathscr{X}$. No further assumptions on $\mathscr{X},\widetilde{\Omega},\widetilde{h}$ are made at this time.

If $S=\{\widetilde{S}_i\}\in H^0(\mathscr{X},L)$, $S\neq 0$, we denote by $[\widetilde{S}_i=0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{\widetilde{S}_i=0\}\subset\widetilde{U}_i$, and we set $[S=0]=\{[\widetilde{S}_i=0]\}\in\widetilde{\mathscr{T}}$.

5.1. **Expectation and variance estimate.** Suppose $\mathscr{V}\subset H^0_{(2)}(\mathscr{X},L)$ is a *finite dimensional* vector subspace of L^2-holomorphic sections, and let $\{S_1,\dots,S_d\}$ be an orthonormal basis of $\mathscr{V}$, where $d=\dim\mathscr{V}$. The Bergman kernel function P and Fubini-Study current $\widetilde{\alpha}$ are defined as in Section 3.3 (see (9) and (10)). Note that, for every orbifold chart $(\widetilde{U}_i,\Gamma_i,\phi_i)$, $\log P\circ\phi_i\in L^1_{loc}(\widetilde{U}_i,\widetilde{\Omega}^n_i)$ since it is locally the difference of a psh function and the weight of the metric $\widetilde{h}_i$.

Following the framework in [SZ1], we identify the unit sphere $\mathscr{S}$ of $\mathscr{V}$ to the unit sphere $\mathbf{S}^{2d-1}$ in $\mathbb{C}^d$ by

$$a=(a_1,\dots,a_d)\in\mathbf{S}^{2d-1}\longrightarrow S_a=\sum_{j=1}^d a_jS_j\in\mathscr{S},$$

and we let λ be the probability measure on $\mathscr{S}$ induced by the normalized surface measure on $\mathbf{S}^{2d-1}$, denoted also by λ (i.e. $\lambda(\mathbf{S}^{2d-1})=1$).

Consider the function

$$Y(a) = \langle [S_a = 0] - \widetilde{\alpha}, \widetilde{\theta}\,\rangle\,,\ a \in \mathbf{S}^{2d-1}\,,$$

where $\widetilde{\theta}$ is a $(n-1,n-1)$ test form on $\mathscr{X}$. The results in [SZ1, Sec. 3.1, 3.2] are the following:

$$\int_{\mathbf{S}^{2d-1}} Y(a)\,d\lambda(a) = 0\,, \tag{17}$$

$$\int_{\mathbf{S}^{2d-1}} |Y(a)|^2\,d\lambda(a) \le A\,C_{\widetilde{\theta}}\,,\ \text{where } A = \frac{1}{\pi^2}\int_{\mathbb{C}^2} (\log|z_1|)^2 e^{-|z_1|^2-|z_2|^2}\,dz\,, \tag{18}$$

dz is the Lebesgue measure on $\mathbb{C}^2$, and $C_{\widetilde{\theta}}$ is a constant depending only on $\widetilde{\theta}$. Equation (17) says that the expectation of the current-valued random variable $a \to [S_a = 0]$ is the Fubini-Study current $\widetilde{\alpha}$.

5.2. **Almost everywhere convergence.** We assume now that $(\mathscr{X}, \widetilde{\Omega})$ is a *compact Kähler* orbifold and $(E_p, \mathscr{X}, \widetilde{h}_p)$ are singular Hermitian orbifold line bundles with the following property: there exists a constant $C > 0$ so that $\langle c_1(E_p, \widetilde{h}_p), \widetilde{\Omega}^{n-1}\rangle \le Cp$, for all $p > 0$.

Let $\mathscr{V}^p \subset H^0_{(2)}(\mathscr{X}, E_p)$ be vector subspaces of L^2-holomorphic sections with corresponding Fubini-Study currents $\widetilde{\alpha}_p$. Consider the unit sphere $\mathscr{S}^p \subset \mathscr{V}^p$ with the probability measure λ_p induced as above by the normalized area measure on the unit sphere $\mathbf{S}^{2d_p-1}$ in $\mathbb{C}^{d_p}$ via the identification $\mathscr{S}^p \equiv \mathbf{S}^{2d_p-1}$, where $d_p = \dim \mathscr{V}^p$. Finally, consider the probability space $\mathscr{S}_\infty = \prod_{p=1}^\infty \mathscr{S}^p$ endowed with the probability measure $\lambda_\infty = \prod_{p=1}^\infty \lambda_p$.

The result of [SZ1, Sec. 3.3] is that, for λ_∞-a.e. sequence $\{\sigma_p\}_{p\ge1} \in \mathscr{S}_\infty$,

$$\lim_{p\to\infty} \frac{1}{p}\big([\sigma_p = 0] - \widetilde{\alpha}_p\big) = 0\,, \tag{19}$$

weakly in the sense of currents on $\mathscr{X}$. Their argument is as follows. Since

$$\langle [\sigma_p = 0], \widetilde{\Omega}^{n-1}\rangle = \langle \widetilde{\alpha}_p, \widetilde{\Omega}^{n-1}\rangle = \langle c_1(E_p, \widetilde{h}_p), \widetilde{\Omega}^{n-1}\rangle \le Cp\,,$$

it suffices to show that, for a fixed test form $\widetilde{\theta}$, one has

$$\lim_{p\to\infty} \frac{1}{p}\langle [\sigma_p = 0] - \widetilde{\alpha}_p, \widetilde{\theta}\,\rangle = 0\,,$$

for λ_∞-a.e. $\sigma = \{\sigma_p\}_{p\ge1} \in \mathscr{S}_\infty$. Setting

$$Y_p(\sigma) = \frac{1}{p}\langle [\sigma_p = 0] - \widetilde{\alpha}_p, \widetilde{\theta}\,\rangle\,,$$

it follows by (18) that

$$\int_{\mathscr{S}_\infty}\left(\sum_{p=1}^{\infty}|Y_p|^2\right)d\lambda_\infty=\sum_{p=1}^{\infty}\int_{\mathscr{S}^p}|Y_p|^2\,d\lambda_p\le A\,C_{\widetilde{\theta}}\sum_{p=1}^{\infty}\frac{1}{p^2}<+\infty,$$

which yields the desired conclusion.

Proof of Theorem 1.6. This follows from (19) and from Theorems 1.1, 1.4, 1.5, and Corollary 1.3. □

References

[ALR] A. Adem, J. Leida and Y. Ruan, *Orbifolds and Stringy Topology*, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, 2007, 149 p.

[BG] C. Boyer and K. Galicki, *Sasakian Geometry*, Oxford Mathematical Monographs, Oxford University Press, 2008, 631 p.

[BGK] C. Boyer, K. Galicki and J. Kollár, *Einstein metrics on spheres*, Ann. of Math. (2) **162** (2005), 557–580.

[Ca] D. Catlin, *The Bergman kernel and a theorem of Tian*, in *Analysis and geometry in several complex variables (Katata, 1997)*, 1–23, Trends Math., Birkhäuser, Boston, 1999.

[CM] D. Coman and G. Marinescu, *Equidistribution results for singular metrics on line bundles*, preprint, 2011.

[DLM1] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), 1–41; announced in C. R. Math. Acad. Sci. Paris **339** (2004), 193–198.

[DLM2] X. Dai, K. Liu, and X. Ma, *A remark on weighted Bergman kernels on orbifolds*, Math. Res. Lett. **19** (2012), 143–148.

[D1] J. P. Demailly, *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré holomorphe semipositif au–dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. **15** (1982), 457–511.

[D2] J. P. Demailly, *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France (N.S.) No. 19 (1985), 1–125.

[D3] J. P. Demailly, *Singular Hermitian metrics on positive line bundles*, in *Complex algebraic varieties (Bayreuth, 1990)*, Lecture Notes in Math. 1507, Springer, Berlin, 1992, 87–104.

[D4] J. P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), 361–409.

[D5] J. P. Demailly, *A numerical criterion for very ample line bundles*, J. Differential Geom. **37** (1993), 323–374.

[DMS] T. C. Dinh, G. Marinescu and V. Schmidt, *Asymptotic distribution of zeros of holomorphic sections in the non compact setting*, J. Stat. Phys. **148** (2012), no. 1, 113-136.

[DS] T. C. Dinh and N. Sibony, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv. **81** (2006), 221–258.

[D] S. K. Donaldson, *Scalar curvature and projective embeddings. I.*, J. Differential Geom. **59** (2001), no. 3, 479–522.

[EGZ] P. Eyssidieux, V. Guedj and A. Zeriahi, *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), 607–639.

[FS] J. E. Fornæss and N. Sibony, *Oka's inequality for currents and applications*, Math. Ann. **301** (1995), 399–419.

[GK] A. Ghigi and J. Kollár, *Kähler-Einstein metrics on orbifolds and Einstein metrics on spheres*, Comment. Math. Helv. **82** (2007), 877–902.

[G] H. Grauert, *Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik*, Math. Ann. **131** (1956), 38–75.

[Ho] L. Hörmander, *Notions of Convexity*, Birkhäuser, 1994.

[HsM] C. Y. Hsiao and G. Marinescu, *The asymptotics for Bergman kernels for lower energy forms and the multiplier ideal Bergman kernel asymptotics*, preprint, 2011.

[MM1] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), 493–498.

[MM2] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Math., vol. 254, Birkhäuser, Basel, 2007, xiii, 422 p.

[MM3] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, Adv. Math. **217** (2008), 1756–1815.

[N] R. Narasimhan, *Imbedding of holomorphically complete complex spaces*, Amer. J. Math. **82** (1960), 917–934.

[O] T. Ohsawa, *Hodge spectral sequence and symmetry on compact Kähler spaces*, Publ. Res. Inst. Math. Sci. **23** (1987), 613–625.

[OT] T. Ohsawa and K. Takegoshi, *On the extension of L^2 holomorphic functions*, Math. Z. **195** (1987), 197–204.

[RT] J. Ross, and R. Thomas, *Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics*, J. Differential Geom. **88** (2011), no. 1, 109–159.

[R] W. D. Ruan, *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom. **6** (1998), 589–631.

[SZ1] B. Shiffman and S. Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Comm. Math. Phys. **200** (1999), 661–683.

[SZ2] B. Shiffman and S. Zelditch, *Random polynomials with prescribed Newton polytope*, J. Amer. Math. Soc. **17** (2004), 49–108.

[SZ3] B. Shiffman and S. Zelditch, *Number variance of random zeros on complex manifolds*, Geom. Funct. Anal. **18** (2008), 1422–1475.

[T] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), 99–130.

[Z] S. Zelditch, *Szegö kernels and a theorem of Tian*, Internat. Math. Res. Notices **1998**, no. 6, 317–331.

[Y] S. T. Yau, *Nonlinear analysis in geometry*, Enseign. Math. (2) **33** (1987), no. 1-2, 109–158.

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