

# On the compactification of hyperconcave ends and the theorems of Siu-Yau and Nadel

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**Abstract.** We show that the ‘pseudoconcave holes’ of some naturally arising class of manifolds, called hyperconcave ends, can be filled in, including the case of complex dimension two. As a consequence we obtain a stronger version of the compactification theorem of Siu-Yau and extend Nadel’s theorems to dimension two.

## 1. Introduction

We will be concerned with the following class of manifolds.

**Definition 1.1.** A complex manifold  $X$  with  $\dim X \geq 2$  is said to be a *strongly pseudoconcave end* if there exist  $a \in \mathbb{R} \cup \{+\infty\}$  and  $c \in \mathbb{R} \cup \{-\infty\}$  and a proper, smooth function  $\varphi : X \rightarrow (c, a)$ , which is strictly plurisubharmonic on a set of the form  $\{\varphi < b\}$ , for some  $b \leq a$ . If  $c = -\infty$ ,  $X$  is called a *hyperconcave end*. For  $e < d < a$  we set  $X_d = \{\varphi < d\}$  and  $X_e^d = \{e < \varphi < d\}$ . We call  $\varphi$  *exhaustion function*.

We say that a strongly pseudoconcave end can be *compactified* or *filled in* if there exists a complex space  $\widehat{X}$  such that  $X$  is (biholomorphic to) an open set in  $\widehat{X}$  and for any  $d < a$ ,  $(\widehat{X} \setminus X) \cup \{\varphi \leq d\}$  is a compact set. We will call  $\widehat{X}$  the *completion* of  $X$ .

By a theorem of Rossi [30, Th. 3, p. 245] and Andreotti-Siu [2, Prop. 3.2] any strongly pseudoconcave end  $X$  can be compactified, provided

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$\dim X \geq 3$ . This is no longer true if  $\dim X = 2$ , as shown in a counter-example of Grauert, Andreotti-Siu and Rossi [18, 2, 30].

Our goal is to compactify the hyperconcave ends also in dimension two. Let us mention some examples. The regular part of a variety with isolated singularities is a hyperconcave end. The same is true for the complement of a compact completely pluripolar set (the set where a strongly plurisubharmonic function equals  $-\infty$ ) in a complex manifold. The first step in the proof of the Siu-Yau compactification theorem [35] is to show that a complete Kähler manifold  $X$  of finite volume and sectional curvature pinched between two negative constants has hyperconcave ends. One can also check that the examples of Grauert, Andreotti-Siu and Rossi are not hyperconcave ends.

**Theorem 1.2.** *Any hyperconcave end  $X$  can be compactified. Moreover, if  $\varphi$  is strictly plurisubharmonic on the whole  $X$ , the completion  $\widehat{X}$  can be chosen a normal Stein space with at worst isolated singularities.*

The motivation for the study of the compactification of hyperconcave ends comes from the theory of complex-analytic compactification of quotients  $X = \mathbb{B}^n/\Gamma$  of the unit ball in  $\mathbb{C}^n$ ,  $n \geq 2$  by arithmetic groups  $\Gamma$ . The Satake-Baily-Borel compactification  $\widehat{X}$  of  $X = \mathbb{B}^n/\Gamma$  is obtained by adding a finite set of points which are isolated singularities. The Siu-Yau theorem gives a differential geometric proof of this fact, first by proving that  $X$  has hyperconcave ends and then showing it can be compactified by adding finitely many points. In this context, our next goal is to find necessary conditions for a manifold with hyperconcave ends to be analytically compactified by adding one point at each end. This also yields a complex analytic proof of the second step of Siu-Yau's theorem (see Corollary 5.1). For more results on the compactification of complete Kähler-Einstein manifolds of finite volume and bounded curvature we refer to Mok [24] and the references therein.

**Theorem 1.3.** *Let  $X$  be a hyperconcave end and let  $\widehat{X}$  be a smooth completion of  $X$ . Assume that  $X$  can be covered by Zariski-open sets which are uniformized by Stein manifolds. Then  $\widehat{X} \setminus X$  is the union of a finite set  $D'$  and an exceptional analytic set which can be blown down to a finite set  $D$ . Each connected component of  $X_c$ , for sufficiently small  $c$ , can be analytically compactified by one point from  $D' \cup D$ . If  $X$  itself has a Stein cover,  $D' = \emptyset$  and  $D$  consists of the singular set of the Remmert reduction of  $\widehat{X}$ .*

Let us note that the result is natural, in the light of a recent result of Colțoiu-Tibăr [8], asserting that the universal cover of a small punctured neighbourhood of an isolated singularity of dimension two is Stein, whenever the fundamental group of the link is infinite.

Theorem 1.3 affords the extension to dimension two of Nadel's theorems [25]. If in Definition 1.1 we may take  $a = +\infty$  and  $\varphi$  bounded from above, the manifold  $X$  is said to be *hyper 1-concave*. In this case  $\{\varphi \geq c\}$  is

compact for every  $c \in \mathbb{R}$ . Hyper 1-concave manifolds are particular cases of strongly 1-concave manifolds in the sense of Andreotti-Grauert [1].

**Corollary 1.4.** *Let  $X$  be a connected manifold of dimension  $n \geq 2$ . The following conditions are necessary and sufficient for  $X$  to be a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many points: (i)  $X$  is hyper 1-concave, (ii)  $X$  admits a line bundle  $E$  such that the ring  $\bigoplus_{k>0} H^0(X, E^k)$  separates points and gives local coordinates, and (iii)  $X$  can be covered by Zariski-open sets which can be uniformized by Stein manifolds. If  $X$  has a Stein cover, one adds only singular points.*

Corollary 1.4 yields, in dimension two, a version of Nadel-Tsuji theorem [26] together with a completely complex-analytic proof of the compactification of arithmetic quotients. It answers also [24, Prob. 1] for the case  $q = 0$ .

In [23] we apply the results of this paper to ball quotients of dimension two having a strongly pseudoconvex boundary, extending results of Burns and Napier-Ramachandran [27]. Another application of the above theorems is the embedding of Sasakian manifolds [23, 29].

The organization of the paper is as follows. In Sect. 2 we construct holomorphic functions on a hyperconcave end and then we prove Theorem 1.2 in Sect. 3. The proof of Theorem 1.3 occupies Sect. 4 and in Sect. 5 we extend Nadel's and Andreotti-Siu theorems to dimension two.

## 2. Existence of holomorphic functions

The idea of proof is to analytically embed small strips  $X_e^{e^*}$ , for  $e < e^*$  in a neighbourhood of minus infinity, into the difference of two concentric polydiscs in the euclidian space. Then we apply the Hartogs extension theorem to extend the image to an analytic set which will provide the compactification. To obtain the embedding we follow the strategy of Grauert and Kohn for the solution of the Levi problem. Namely, we solve the  $L^2$   $\bar{\partial}$ -Neumann for  $(0, 1)$ -forms on domains  $X_d$  with strongly pseudoconvex boundary  $\{\varphi = d\}$  endowed with a complete metric at minus infinity. This entails the finiteness of the  $L^2$  Dolbeault cohomology  $H_{(2)}^{0,1}(X_d)$  which in turn implies the existence of peak holomorphic functions at each point of the boundary  $\{\varphi = d\}$ . Note that, as a consequence of the Andreotti-Grauert theory [1], in dimension two, the sheaf cohomology group  $H^1(X_e^d, \mathcal{F})$  of a strip  $X_e^d$ , where  $e < d < b$ , is infinite dimensional for any coherent analytic sheaf  $\mathcal{F}$ . This makes possible the counterexamples of Grauert-Andreotti-Rossi. On the other hand, the finiteness of this group in dimension greater than three implies the Rossi and Andreotti-Siu theorems [30, 2].

We shall suppose henceforth without loss of generality that  $b > 0$ . The function  $\chi = -\log(-\varphi)$  is smooth and strictly plurisubharmonic on  $X_0$ . We set

$$(2.1) \quad \omega = \sqrt{-1}\partial\bar{\partial}\chi = -\sqrt{-1}\partial\bar{\partial}\log(-\varphi).$$

Note that  $\partial\bar{\partial}\chi = \partial\bar{\partial}\varphi/(-\varphi) + (\partial\varphi \wedge \bar{\partial}\varphi)/\varphi^2$  and  $(\partial\varphi \wedge \bar{\partial}\varphi)/\varphi^2 = \partial\chi \wedge \bar{\partial}\chi$ . Since  $\sqrt{-1}\partial\bar{\partial}\varphi/(-\varphi)$  represents a metric on  $X_0$ , we get the Donnelly-Fefferman condition:

$$(2.2) \quad |\partial\chi|_\omega \leq 1.$$

Since  $\chi : X_0 \rightarrow \mathbb{R}$  is proper, (2.2) also ensures that  $\omega$  is complete. Indeed, (2.2) entails that  $\chi$  is Lipschitz with respect to the geodesic distance induced by  $\omega$ , so any geodesic ball must be relatively compact.

We fix in the sequel a regular value  $d \in (-1, 0)$  of  $\varphi$ . The metric  $\omega$  is complete at the pseudoconcave end of  $X_d$  and extends smoothly over the boundary  $bX_d$ .

We wish to derive the Poincaré inequality for  $(0, 1)$ -forms on  $X_d$ . For this goal we look first at the minus infinity end and use the Berndtsson-Siu trick [5, 34]. Roughly speaking, it uses the negativity of the trivial line bundle, thus avoiding the problems raised by the control of the Ricci curvature of  $\omega$  at  $-\infty$ . Let us denote by  $\mathcal{C}_0^{0,q}(X_d)$  the space of smooth  $(0, q)$ -forms with compact support in  $X_d$ . Let  $\vartheta = - * \partial *$  be the formal adjoint of  $\bar{\partial}$  with respect to the scalar product  $(u, v) = \int_{X_d} \langle u, v \rangle dV_\omega$ , where  $\langle u, v \rangle = \langle u, v \rangle_\omega$  and  $dV_\omega = \omega^n/n!$ .

**Lemma 2.1.** *For any  $v \in \mathcal{C}_0^{0,1}(X_d)$  we have  $\|v\|^2 \leq 8(\|\bar{\partial}v\|^2 + \|\vartheta v\|^2)$ .*

*Proof.* On the trivial bundle  $E = X_d \times \mathbb{C}$  we introduce the auxiliary hermitian metric  $e^{\chi/2}$ . The curvature of  $E$  is then  $\Theta(E) = \partial\bar{\partial}(-\chi/2)$ . Let  $\vartheta_\chi$  be the formal adjoint of  $\bar{\partial}$  with respect to the scalar product  $(u, v)_\chi = \int_{X_d} \langle u, v \rangle e^{\chi/2} dV_\omega$ . Then  $\vartheta_\chi = e^{-\chi/2} \vartheta e^{\chi/2}$ . We apply the Bochner-Kodaira-Nakano formula for  $u \in \mathcal{C}_0^{0,1}(X_d)$ :

$$(2.3) \quad \begin{aligned} & \int_{X_d} \langle [\sqrt{-1} \partial\bar{\partial}(-\chi/2), \Lambda_\omega] u, u \rangle e^{\chi/2} dV_\omega \\ & \leq \int_{X_d} (|\bar{\partial}u|^2 + |\vartheta_\chi u|^2) e^{\chi/2} dV_\omega, \end{aligned}$$

where  $\Lambda_\omega$  represents the contraction with  $\omega$  and  $[A, B] = AB - (-1)^{\deg A \cdot \deg B} BA$  is the graded commutator of the operators  $A, B$ . The idea is to substitute  $v = u e^{\chi/4}$ . It is readily seen that

$$(2.4) \quad \begin{aligned} |\bar{\partial}u|^2 e^{\chi/2} & \leq 2|\bar{\partial}v|^2 + \frac{1}{8}|\bar{\partial}\chi|^2|v|^2, \\ |\vartheta_\chi u|^2 e^{\chi/2} & \leq 2|\vartheta v|^2 + \frac{1}{8}|\partial\chi|^2|v|^2. \end{aligned}$$

Moreover  $\langle [\sqrt{-1} \partial\bar{\partial}(-\chi/2), \Lambda_\omega] u, u \rangle e^{\chi/2} = \langle [\sqrt{-1} \partial\bar{\partial}(-\chi/2), \Lambda_\omega] v, v \rangle$ . In general, for a  $(p, q)$ -form  $\alpha$  we have the identity  $\langle [\omega, \Lambda_\omega]\alpha, \alpha \rangle = (p + q - n)|\alpha|^2$ , where  $n = \dim X$ . Taking into account that  $\omega = \sqrt{-1} \partial\bar{\partial}\chi$

and that  $v$  is a  $(0, 1)$ -form, we obtain

$$(2.5) \quad \langle [\sqrt{-1} \partial \bar{\partial}(-\chi/2), \Lambda_\omega] u, u \rangle e^{\chi/2} = \frac{n-1}{2} |v|^2 \geq \frac{1}{2} |v|^2.$$

By (2.3), (2.4), (2.2),

$$(2.6) \quad \frac{1}{2} \int_{X_d} |v|^2 dV_\omega \leq 2 \int_{X_d} (|\bar{\partial}v|^2 + |\vartheta v|^2) dV_\omega + \frac{1}{4} \int_{X_d} |v|^2 dV_\omega.$$

This immediately implies Lemma 2.1 for elements  $v \in \mathcal{C}_0^{0,1}(X_d)$ .  $\square$

Let  $\eta : (-\infty, 0) \rightarrow \mathbb{R}$  be a smooth function such that  $\eta(t) = 0$  on  $(-\infty, -2]$ ,  $\eta'(t) > 0$ ,  $\eta''(t) > 0$  on  $(-2, 0)$ . Let us introduce the scalar product

$$(2.7) \quad (u, v)_{\eta(\varphi)} = \int_{X_d} \langle u, v \rangle e^{-\eta(\varphi)} dV_\omega,$$

the corresponding norm  $\|\cdot\|_{\eta(\varphi)}$  and  $L^2$  spaces, denoted  $L_2^{0,q}(X_d, \eta(\varphi))$ . Let  $\mathcal{C}_0^{0,q}(\bar{X}_d)$  be the space of smooth  $(0, q)$ -forms with compact support in  $\bar{X}_d$ .

Consider the maximal closed extension of  $\bar{\partial}$  to  $L_2^{0,q}(X_d, \eta(\varphi))$  and let  $\bar{\partial}_{\eta(\varphi)}^*$  and  $\vartheta_{\eta(\varphi)}$  be its Hilbert-space and formal adjoints, respectively. Then  $\vartheta_{\eta(\varphi)} = \vartheta + i(\partial\eta(\varphi))$ , where  $i(\cdot)$  represents the interior product.

We denote by  $\sigma(P, df)$  the symbol of a differential operator of order one, calculated on the cotangent vector  $df$ . Then  $\sigma(\vartheta, df) = *df \wedge *$  and it is clear that  $\sigma(\vartheta_{\eta(\varphi)}, df) = \sigma(\vartheta, df)$  does not depend on  $\eta$ . Set  $B^{0,q} = \{\alpha \in \mathcal{C}_0^{0,q}(\bar{X}_d) : \sigma(\vartheta, d\varphi)\alpha = 0 \text{ on } bX_d\}$ . Integration by parts [15, Prop. 1.3.1–2] yields  $\text{Dom } \bar{\partial}_{\eta(\varphi)}^* \cap \mathcal{C}_0^{0,q}(\bar{X}_d) = B^{0,q}$ ,  $\bar{\partial}_{\eta(\varphi)}^* = \vartheta_{\eta(\varphi)}$  on  $B^{0,q}$ .

**Lemma 2.2.** *The space  $B^{0,q}$  is dense in  $\text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}_{\eta(\varphi)}^*$  in the graph norm*

$$u \mapsto \left( \|u\|_{\eta(\varphi)}^2 + \|\bar{\partial}u\|_{\eta(\varphi)}^2 + \|\bar{\partial}_{\eta(\varphi)}^* u\|_{\eta(\varphi)}^2 \right)^{1/2}.$$

*Proof.* We use first the idea from [4, Lemma 4, p. 92–3] in order to reduce the proof to the case of a compactly supported form  $u \in \mathcal{C}_0^{0,q}(\bar{X}_d)$ . But then the approximation in the graph norm follows from the Friedrichs theorem on the identity of weak and strong derivatives, cf. [22, Prop. 1.2.4].  $\square$

We confine next our attention to the fundamental estimate on  $X_d$ .

**Lemma 2.3.** *If  $\eta$  grows sufficiently fast, there exists a constant  $C > 0$  such that*

$$(2.8) \quad \|u\|_{\eta(\varphi)}^2 \leq C \left( \|\bar{\partial}u\|_{\eta(\varphi)}^2 + \|\bar{\partial}_{\eta(\varphi)}^* u\|_{\eta(\varphi)}^2 + \int_K |u|^2 e^{-\eta(\varphi)} dV_\omega \right),$$

for any  $u \in \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}_{\eta(\varphi)}^* \subset L_2^{0,1}(X_d, \eta(\varphi))$ , where  $K = \{-3 \leq \varphi \leq -3/2\}$ .

*Proof.* The Morrey-Kohn-Hörmander estimate for  $(0, 1)$ -forms shows that there exists  $R > 0$  such that for sufficiently growing  $\eta$  the following estimate holds:

$$(2.9) \quad \|u\|_{\eta(\varphi)}^2 \leq R \left( \|\bar{\partial}u\|_{\eta(\varphi)}^2 + \|\bar{\partial}_{\eta(\varphi)}^* u\|_{\eta(\varphi)}^2 + \int_{\{-3 \leq \varphi \leq -3/2\}} |u|^2 e^{-\eta(\varphi)} dV_\omega \right),$$

for  $u \in \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}_{\eta(\varphi)}^* \subset L_2^{0,1}(X_d, \eta(\varphi))$ ,  $\text{supp } u \subset \{-3 \leq \varphi\}$ . Let  $u \in \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}_{\eta(\varphi)}^* \subset L_2^{0,1}(X_d, \eta(\varphi))$ . The density Lemma 2.2 shows that to prove (2.8) it suffices to consider smooth elements  $u$  compactly supported in  $\bar{X}_d$ . We choose a cut-off function  $\rho_1 \in \mathcal{C}^\infty(\bar{X}_d)$  such that  $\text{supp } \rho_1 = \{-3 \leq \varphi\}$ ,  $\rho_1 = 1$  on  $\{-2 \leq \varphi\}$ . Set  $\rho_2 = 1 - \rho_1$ . On  $\text{supp } \rho_2$ ,  $\eta(\varphi)$  vanishes, therefore  $\bar{\partial}_{\eta(\varphi)}^*(\rho_2 u) = \vartheta(\rho_2 u)$ . Upon applying Lemma 2.1 for  $\rho_2 u$  we get  $\|\rho_2 u\|_{\eta(\varphi)}^2 \leq 8(\|\bar{\partial}(\rho_2 u)\|_{\eta(\varphi)}^2 + \|\bar{\partial}_{\eta(\varphi)}^*(\rho_2 u)\|_{\eta(\varphi)}^2)$ . The latter estimate and estimate (2.9) for  $\rho_1 u$  together with standard inequalities deliver (2.8).  $\square$

In the sequel we fix a function  $\eta$  as in Lemma 2.3. Then the fundamental estimate (2.8) implies the solution of the  $L^2$   $\bar{\partial}$ -Neumann problem. Consider the complex of closed, densely defined operators

$$L_2^{0,0}(X_d, \eta(\varphi)) \xrightarrow{T=\bar{\partial}} L_2^{0,1}(X_d, \eta(\varphi)) \xrightarrow{S=\bar{\partial}} L_2^{0,2}(X_d, \eta(\varphi)),$$

and the closed, densely defined operator

$$\text{Dom } \Delta'' = \{u \in \text{Dom } S \cap \text{Dom } T^* : Su \in \text{Dom } S^*, T^*u \in \text{Dom } T\},$$

$$\Delta''u = S^*Su + T T^*u \quad \text{for } u \in \text{Dom } \Delta''.$$

We know from a theorem of Gaffney [15, Prop. 1.3.8], that  $\Delta''$  is self-adjoint. We denote in the sequel  $\mathcal{H}^{0,1} = \text{Ker } S \cap \text{Ker } T^*$ .

**Theorem 2.4.** *The following assertions hold true:*

- (i) *The operators  $T$  and  $\Delta''$  have closed range,  $\mathcal{H}^{0,1}$  is finite dimensional, and we have the strong Hodge decomposition*

$$L_2^{0,1}(X_d, \eta(\varphi)) = \text{Range}(T T^*) \oplus \text{Range}(S^* S) \oplus \mathcal{H}^{0,1}.$$

- (ii) *There exists a bounded operator  $N$  on  $L_2^{0,1}(X_d, \eta(\varphi))$  such that  $\Delta''N = N\Delta'' = \text{Id} - P_h$ ,  $P_h N = N P_h = 0$ , where  $P_h$  is the orthogonal projection on  $\mathcal{H}^{0,1}$ .*
- (iii) *If  $f \in \text{Range } T$ , the unique solution  $u \perp \text{Ker } T$  of the equation  $Tu = f$  is given by  $u = T^* N f$ .*
- (iv) *The operator  $N$  maps  $L_2^{0,1}(X_d, \eta(\varphi)) \cap \mathcal{C}^{0,1}(X_d)$  into itself.*

*Proof.* The fundamental estimate (2.8) implies as in [28, Prop. 1.2] that for any bounded sequence  $u_k \in \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}_{\eta(\varphi)}^* \subset L_2^{0,1}(X_d, \eta(\varphi))$  with  $\|\bar{\partial} u_k\|_{\eta(\varphi)} \rightarrow 0$ ,  $\|\bar{\partial}^* u_k\|_{\eta(\varphi)} \rightarrow 0$  one can select a strongly convergent subsequence. From this follow assertions (i)–(iii) (see e.g. [22, Prop. 1.1.3]). Since  $\Delta''$  is an extension of  $\bar{\partial}\vartheta_{\eta(\varphi)} + \vartheta_{\eta(\varphi)}\bar{\partial}$ , assertion (iv) follows from the interior regularity for elliptic operators (see e.g. [15, Th. 2.2.9]).  $\square$

*Remark 2.5.* By using the estimates in local Sobolev norms near the boundary points, we can prove as in Folland-Kohn [15] that  $N$  maps  $L_2^{0,1}(X_d, \eta(\varphi)) \cap \mathcal{C}^{0,1}(\bar{X}_d)$  into itself. We could repeat then the solution of the Levi problem as given in [15, Th. 4.2.1], in order to find holomorphic peak functions, for each boundary point. However, we propose in Corollary 2.6, a simpler proof for the existence of peak functions, which doesn't involve the regularity up to the boundary of the  $\bar{\partial}$ -Neumann problem.

**Corollary 2.6.** *Let  $p \in bX_d$  and  $f$  be a holomorphic function on a neighbourhood of  $p$  such that  $\{f = 0\} \cap \bar{X}_d = \{p\}$ . Then for every  $m$  big enough, there is a function  $g \in \mathcal{O}(X_d) \cap \mathcal{C}^\infty(\bar{X}_d \setminus \{p\})$ , a smooth function  $\Phi$  on a neighbourhood  $V$  of  $p$  and constants  $a_1, \dots, a_{m-1}$  such that  $g = f^{-m}(1 + a_{m-1}f + \dots + a_1f^{m-1}) + \Phi$  on  $V \cap X_d$ . In particular, we have  $\lim_{z \rightarrow p} |g(z)| = \infty$ .*

*Proof.* Let  $U$  be a small neighbourhood of  $p$  where  $f$  is defined. Pick  $\psi \in \mathcal{C}_0^\infty(U)$  such that  $\psi = 1$  on a neighbourhood  $V'$  of  $p$ . Set  $h_m = \psi/f^m$  on  $U$ ,  $h_m = 0$  on  $X \setminus \text{supp } \psi$ , and  $v_m = \bar{\partial}h_m$  on  $X$ . Observe that  $v_m$  belongs to  $\mathcal{C}_0^{0,1}(\bar{X}_{d+\delta})$  for  $\delta > 0$  small enough and  $v_m = 0$  on  $V'$ . Moreover, we have  $\bar{\partial}v_m = 0$  on  $X_{d+\delta}$ . Fix such a  $\delta$  and apply Theorem 2.4 for  $X_{d+\delta}$ . By this theorem, the codimension of  $\text{Range } T$  in  $\text{Ker } S$  is finite. For every  $m$  big enough, there are constants  $a_1, \dots, a_{m-1}$  such that  $v = v_m + a_{m-1}v_{m-1} + \dots + a_1v_1$  belongs to  $\text{Range } T$ . Then there is  $\Phi \in \mathcal{C}^{0,0}(X_{d+\delta})$  such that  $\bar{\partial}\Phi = -v$ . Set  $h = h_m + a_{m-1}h_{m-1} + \dots + a_1h_1$  and  $g = h + \Phi$ . We have  $\bar{\partial}g = 0$  on  $X_{d+\delta} \setminus \{f = 0\}$ . Then  $g \in \mathcal{O}(X_d) \cap \mathcal{C}^\infty(\bar{X}_d \setminus \{p\})$ . The function  $\Phi$  in the corollary is equal to  $\Phi$  on  $V'$ . Thus it is smooth on  $V := V' \cap X_{d+\delta}$ . The proof is completed.  $\square$

### 3. The compactification

In this section we prove Theorem 1.2, using the results of Sect. 2 and the method of [2].

**Proposition 3.1.** *Let  $d$  be a regular value of  $\varphi$ . Then for  $\delta > 0$  small enough we have:*

- (a) *the holomorphic functions on  $X_d$  separate points on  $X_{d-\delta}^d$ ,*
- (b) *the holomorphic functions on  $X_d$  give local coordinates on  $X_{d-\delta}^d$ , and*

(c) for any  $e \in (d - \delta, d)$  there exists  $e^* \in (e, d)$ , such that the holomorphically convex hull of  $X_e$ , with respect to the algebra of holomorphic functions on  $X_d$ , is contained in  $X_{e^*}$ .

Proposition 3.1 will be the consequence of the following two lemmas. We can assume that  $bX_{d-\varepsilon}$  is smooth for  $\varepsilon > 0$  small enough. Choose a projection  $\pi$  from a neighbourhood of  $bX_d$  into  $bX_d$ . We will denote by  $(x, \varepsilon)$  the point of  $bX_{d-\varepsilon}$  whose projection is  $x \in bX_d$ .

**Lemma 3.2.** *Let  $x_1, x_2$  be two different points in  $bX_d$ . Then there are two neighbourhoods  $V_1, V_2$  of  $x_1, x_2$  in  $bX_d$  and  $v = v(x_1, x_2) > 0$  such that the holomorphic functions of  $X_d$  separate  $V_1 \times (0, v]$  and  $V_2 \times (0, v]$ .*

*Proof.* This is a direct corollary from the existence of a function holomorphic in  $X_d$ , and  $\mathcal{C}^\infty$  in  $\overline{X}_d \setminus \{x_1\}$  which tends to  $\infty$  at  $x_1$ .  $\square$

**Lemma 3.3.** *Let  $x$  be a point of  $bX_d$ . Then there are a neighbourhood  $V$  of  $x$  in  $bX_d$  and  $\tau = \tau(x) > 0$  such that the holomorphic functions in  $X_d$  give local coordinates for  $V \times (0, \tau]$ .*

*Proof.* Without loss of generality and in order to simplify the notations, we consider the case  $n = 2$ . Choose a local coordinates system such that  $x = 0$  and locally  $X_d \subset \{|z_1 - 1/2|^2 + |z_2|^2 < 1/4\}$ . We now apply Corollary 2.6 for functions  $f_1(z) = z_1$  and  $f_2(z) = z_1(1 - z_2)$ . Denote by  $g_1, g_2$  the holomorphic functions constructed by this corollary for a number  $m$  big enough. We can also construct the analogue functions if we replace  $m$  by  $m + 1$ . Denote by  $g'_1$  and  $g'_2$  these new functions.

Let  $G : X_d \rightarrow \mathbb{C}^4$  given by  $G = (g_1, g_2, g'_1, g'_2)$ . We will prove that  $G$  gives local coordinates. Set  $I(z) = (z_1 z_3^{-1}, 1 - z_2 z_3 z_1^{-1} z_4^{-1})$ . Let  $W$  be a small neighbourhood of 0. By Corollary 2.6, the map  $I \circ G$  is defined on  $W \cap X_d$  and can be extended to a smooth function on  $W$ . Moreover, on  $W$  we have  $I \circ G(z) = (z_1 + O(z_1^2), z_2 + O(z_1))$ . Then  $I \circ G$  gives an immersion of  $W \cap X_d$  in  $\mathbb{C}^2$ , whenever  $W$  is small enough. In consequence,  $G$  gives coordinates on  $W \cap X_d$ .  $\square$

*Proof of Proposition 3.1.* We cover  $bX_d \times bX_d$  by a finite family of open sets of the form  $V_1 \times V_2$  (from Lemma 3.2) and the form  $V \times V$  (from Lemma 3.3). We have a finite family of  $v$ 's and  $\tau$ 's. Then properties (a) and (b) hold for every  $\delta$  smaller than these  $\tau$ 's and  $v$ 's. Property (c) is an immediate consequence of Corollary 2.6.  $\square$

*Proof of Theorem 1.2.* First let us remark that the first assertion is a consequence of the second, so we shall prove only the latter. We assume therefore that the function  $\varphi : X \rightarrow (-\infty, a)$  is strictly plurisubharmonic everywhere and  $a, b > 0$ . The proof of the compactification statement for  $\dim X \geq 3$  in [2, Prop. 3.2] uses only the assertions (a), (b) and (c) of Proposition 3.1, so we just have to follow it. Namely, let  $d, \delta, e$  and  $e^*$  as in



Proposition 3.1 and denote  $P_\varepsilon = \{z \in \mathbb{C}^N : |z_i| < \varepsilon\}$ . Proposition 3.1 implies as in [2, Prop. 3.2] the existence of a holomorphic map  $\alpha : X_d \rightarrow \mathbb{C}^N$  which is an embedding of  $X_e^{e^*}$  and  $\alpha(X_e) \subset P_{1/2}$ ,  $\alpha(\{\varphi = e^*\}) \cap P_1 = \emptyset$ . Set  $H = \alpha^{-1}(P_1 \setminus \overline{P}_{1/2}) \cap X_e^{e^*}$ . Since  $\alpha(H)$  is a complex submanifold of  $P_1 \setminus \overline{P}_{1/2}$  of dimension at least two, it follows from the Hartogs phenomenon [21, Th. VII, D.6] that we can find an  $\varepsilon \in [1/2, 1]$ , such that  $\alpha(H) \cap (P_1 \setminus \overline{P}_\varepsilon)$  can be extended to an analytic subset  $V$  of  $P_1$ . We can glue the topological spaces  $X_d \setminus \alpha^{-1}(\overline{P}_\varepsilon)$  and  $V$  along  $H \setminus \alpha^{-1}(\overline{P}_\varepsilon)$  using the identification given by the holomorphic map  $\alpha$ . Hence, we obtain a complex space  $\widehat{X}_d$ , such that  $X_d \setminus \alpha^{-1}(\overline{P}_\varepsilon)$  and  $V$  are open subsets of  $\widehat{X}_d$ . This turns out to be a Stein space since we can construct a strictly plurisubharmonic exhaustion function, using the function  $\varphi$  and the coordinate functions in  $\mathbb{C}^N$ . The uniqueness of the Stein completion [2, Cor. 3.2] entails that  $\widehat{X}_d$  does not depend on  $d$ , so letting  $d \rightarrow -\infty$  we obtain the desired completion  $\widehat{X}$  of  $X$ .  $\square$

*Remark 3.4.* Our method was to embed small strips  $X_e^{e^*}$  in  $\mathbb{C}^N$  using holomorphic functions and apply the Hartogs phenomenon. One can produce easily holomorphic  $(n, 0)$ -forms on  $X_0$  and an embedding  $\Psi : X_e^{e^*} \rightarrow \mathbb{C}\mathbb{P}^N$ , using the standard  $L^2$  estimates for  $\bar{\partial}$  (cf. [10]). However, it seems that the global Hartogs phenomenon in  $\mathbb{C}\mathbb{P}^N$  is an open question [12, Prob. 1]. Note that by pulling back  $\Psi(X_e^{e^*})$  to  $\mathbb{C}^{N+1} \setminus \{0\}$  we obtain a noncompact manifold, so we cannot apply the known results from the euclidian space (see also [16] where some difficulties of the passing from the local to global Hartogs theorem are exhibited). But we can partly use the projective embedding to show Theorem 1.2. Namely, the existence of a non-constant holomorphic function on  $\Psi(X_e^{e^*}) \subset \mathbb{C}\mathbb{P}^N$  and the arguments from Sarkis [32, Cor. 4.13] show that we can fill in  $X_e^{e^*}$ .

*Remark 3.5 (Generalization of Theorem 1.2).* Theorem 1.2 holds also for normal complex spaces with isolated singularities (which are the only allowed normal singularities in dimension two). Indeed, let  $X$  be a hyperconcave end with isolated normal singularities (Definition 1.1 makes sense also for complex spaces). Let  $\{a_i\}$  denote the singular points and choose functions  $\varphi_i$  with pairwise disjoint compact supports, such that  $\varphi_i$  is strictly plurisubharmonic in a neighbourhood of  $a_i$  and  $\lim_{z \rightarrow a_i} \varphi_i(z) = -\infty$ . Using the function  $\widehat{\varphi} = \varphi + \sum \varepsilon_i \varphi_i$ , with  $\varepsilon_i$  small enough, we see that  $\text{Reg } X$  is a hyperconcave end. By Theorem 1.2 we get a normal Stein completion  $Y$  of  $\text{Reg } X$ . Take  $\{V_i\}$  pairwise disjoint Stein neighbourhoods of  $\{a_i\}$ . Then  $V_i \setminus \{a_i\} \subset \text{Reg } X$  and a normal Stein completion of  $V_i \setminus \{a_i\}$  is  $V_i$ . Using the uniqueness of a normal Stein completion [2, Cor. 3.2] we infer that the  $V_i$  are disjointly embedded in  $Y$ . Therefore,  $Y$  is also a completion of  $X$ . In particular, the singular set of a hyperconcave end with only isolated singularities must be finite.

**Corollary 3.6.** *Let  $V$  be a Stein manifold,  $\dim V \geq 2$ . Let  $K$  be a compact completely pluripolar set,  $K = \varphi^{-1}(-\infty)$  where  $\varphi$  is a strictly plurisubhar-*

monic function defined on a neighbourhood  $U$  of  $K$ , smooth on  $U \setminus K$ . Then any finite non-ramified covering of  $V \setminus K$  can be compactified to a strongly pseudoconvex space.

This follows immediately from Theorem 1.2, since  $V \setminus K$  is a hyperconcave end and any finite non-ramified covering of a hyperconcave end is also a hyperconcave end. Corollary 3.6 is in stark contrast to the examples of non-compactifiable pseudoconcave ends from [18, 2, 30, 14, 11]. They are obtained as finite non-ramified coverings of small neighbourhoods of the boundaries of some Stein manifolds of dimension two. They have ‘big’ holes which cannot be filled, whereas ‘small’, i.e. completely pluripolar holes can always be filled.

*Remark 3.7 (Complex cobordism).* We can recast Theorem 1.2 in the light of the cobordism result of Epstein and Henkin [13]. Namely, if  $Y$  is a compact strongly pseudoconvex CR manifold of real dimension three, strictly complex cobordant to  $-\infty$ , then  $Y$  bounds a strongly pseudoconvex compact manifold. In particular,  $Y$  is embeddable in  $\mathbb{C}^N$ , for some  $N$ . Note that, if  $\dim_{\mathbb{R}} Y > 3$  this is automatic by a theorem of Boutet de Monvel [9, p. 5]. On the other side, the examples of Grauert, Andreotti-Siu, Rossi and also Burns [6] exhibit compact strongly pseudoconvex CR manifolds of dimension three which do not bound a complex manifold and are not embeddable in  $\mathbb{C}^N$ .

#### 4. Compactification by adding finitely many points

The present section is devoted to proving sufficient conditions for the set  $\widehat{X} \setminus X$  to be analytic. In order to prove Theorem 1.3 we consider first the particular case when the completion  $\widehat{X}$  is a Stein space.

We begin with some preparations. Let  $V$  be a complex manifold. We say that  $V$  satisfies the *Kontinuitätssatz* if for any smooth family of closed holomorphic discs  $\overline{\Delta}_t$  in  $V$  indexed by  $t \in [0, 1)$  such that  $\cup b\Delta_t$  lies on a compact subset of  $V$ , then  $\cup \overline{\Delta}_t$  lies on a compact subset of  $V$ . It is clear that every Stein manifold satisfies the *Kontinuitätssatz*, using the strictly plurisubharmonic exhaustion function and the maximum principle. Moreover, if the universal cover of  $V$  is Stein then  $V$  satisfies the *Kontinuitätssatz* since we can lift the family of discs to the universal cover.

Let  $F$  be a closed subset of  $V$ . We say that  $F$  is *pseudoconcave* if  $V \setminus F$  satisfies the local *Kontinuitätssatz* in  $V$ , i.e. for every  $x \in F$  there is a neighbourhood  $W$  of  $x$  such that  $W \setminus F$  satisfies the *Kontinuitätssatz*. Observe that the finite union of pseudoconcave subsets is pseudoconcave and every complex hypersurface is pseudoconcave.

We have the following proposition which implies the Theorem 1.3.

**Proposition 4.1.** *Let  $\widehat{\Omega}$  be a Stein space with isolated singularities  $S$  and  $K$  a completely pluripolar compact subset of  $\widehat{\Omega}$  which contains  $S$ . Assume*

that  $\Omega = \widehat{\Omega} \setminus K$  can be covered by Zariski-open sets which satisfy the local *Kontinuitätssatz* in  $\widehat{\Omega} \setminus S$ . Then  $K$  is a finite set. Moreover, if  $\Omega$  itself satisfies the local *Kontinuitätssatz* in  $\widehat{\Omega} \setminus S$ , we have  $K = S$ .

*Proof.* We can suppose that  $\widehat{\Omega}$  is a subvariety of a complex space  $\mathbb{C}^N$ . Let  $B$  be a ball containing  $K$  such that  $bB \cap \widehat{\Omega}$  is transversal. By hypothesis, we can choose a finite family of Zariski-open sets  $V_1, \dots, V_k$  which satisfy the local *Kontinuitätssatz* in  $\widehat{\Omega} \setminus S$  and  $\cap F_i$  is empty near  $bB$ , where  $F_i = \Omega \setminus V_i$ . Observe that  $F_i$  is an analytic subset of  $\Omega$ ,  $\overline{F_i} \subset F_i \cup K$ . Since  $F_i \cup (K \setminus S)$  is pseudoconcave in  $\widehat{\Omega} \setminus S$ ,  $F_i$  has no component of codimension  $\geq 2$ . By Hartogs theorem, if  $n = \dim X > 2$ , there is a complex subvariety  $\widehat{F_i}$  of  $\widehat{\Omega}$  which contains  $F_i$ . We will prove this property for the case  $n = 2$ . Set  $F = \cup F_i$ .

Observe that  $\Gamma = F \cap bB$  is an analytic real curve. The classical Wermer theorem [36] says that  $\text{hull}(\Gamma) \setminus \Gamma$  is a (possibly void) analytic subset of pure dimension 1 of  $\mathbb{C}^N \setminus \Gamma$ , where  $\text{hull}(\Gamma)$  is the polynomial hull of  $\Gamma$ . By the uniqueness theorem,  $\text{hull}(\Gamma) \subset \widehat{\Omega}$ . Since  $S$  is finite, we have  $\text{hull}(\Gamma \cup S) = \text{hull}(\Gamma) \cup S$ . Set  $F' = (F \cup K) \cap \overline{B}$  and  $F'' = \text{hull}(\Gamma) \cup S$ .

**Lemma 4.2 (in the case  $n = 2$ ).** *We have  $F' \subset F''$ .*

*Proof.* Assume that  $F' \not\subset F''$ . Then there are a point  $p \in F'$  and a polynomial  $h$  such that  $\sup_{F''} |h| < \sup_{F'} |h| = |h(p)|$ . Set  $r = h(p)$ . By the maximum principle, we have  $h^{-1}(r) \cap F' \subset K \setminus S$ . We use here the property that  $K$  is completely pluripolar. In particular,  $p \in K \setminus S$ . Recall that  $F' \setminus S$  is pseudoconcave in  $\widehat{\Omega}' = \widehat{\Omega} \cap B \setminus S$ . We will construct a smooth family of discs which does not satisfy the *Kontinuitätssatz*. This gives a contradiction. The construction is trivial if  $p$  is isolated in  $F'$ . We assume that  $p$  is not isolated. By using a small perturbation of  $h$ , we can suppose that  $h(p)$  is not isolated in  $h(F')$ .

Set  $\Sigma' = h(F')$  and  $\Sigma'' = h(F'')$ . Then  $\Sigma'$  (resp.  $\Sigma''$ ) is included in the closed disk (resp. open disk) of center 0 and of radius  $|r|$ . The holomorphic curves  $\{h = \text{const}\}$  define a holomorphic foliation, possibly singular, of  $\widehat{\Omega}'$ . The difficulty is that the fiber  $\{h = r\}$  can be singular at  $p$ . Denote by  $T$  the set of points  $s$  such that  $h$  is not a submersion on a neighbourhood of  $h^{-1}(s)$  on  $\widehat{\Omega}$ . Then  $T$  is finite.

Denote also  $\Xi$  the unbounded component of  $\mathbb{C} \setminus (\Sigma'' \cup T)$ . It is clear that  $\Sigma'$  meets  $\Xi$ . This property is stable for every small perturbation of the polynomial  $h$ . Since  $K$  is a pluripolar compact set,  $K \cap h^{-1}(a)$  is a polar subset of  $h^{-1}(a)$  for every  $a \in \mathbb{C}$ .

Choose a point  $b \in \Xi$  such that  $0 < \text{dist}(b, \Sigma') < \text{dist}(b, \Sigma'' \cup T)$  and  $a \in \Sigma'$  such that  $\text{dist}(a, b) = \text{dist}(b, \Sigma')$ . We have  $a \notin \Sigma'' \cup T$ . Replacing  $b$  by a point of the interval  $(a, b)$  we can suppose that  $\text{dist}(a, b) < \text{dist}(a', b)$  for every  $a' \in \Sigma' \setminus \{a\}$ . Fix a point  $q \in F'$  such that  $h(q) = a$ . Set  $\delta_1 = |a - b|$ . Since  $a \notin T$ , we can choose a local coordinates system  $(z_1, z_2)$  of an open neighbourhood  $W$  of  $q$  in  $\widehat{\Omega}'$  such that  $z_1 = h(z) - b$ ,

$q = (a - b, 0)$  and  $\{(z_1, z_2), |z_1| < \delta_1 + \delta_2, |z_2| < 2\} \subset W$  with  $\delta_2 > 0$  small enough. We can choose a  $W$  which does not meet  $F''$  and is as small as we want.

Let  $L$  be the complex line  $\{z_1 = a - b\}$ . By the maximum principle,  $K' = F' \cap L$  is equal to  $K \cap L$ . Then  $K'$  is a polar subset of  $L$ . This implies that the length of  $K'$  is equal to 0. Thus, for almost every  $s \in (0, 2)$  the circle  $\{|z_2| = s\} \cap L$  does not meet  $K'$ . Without loss of generality, we can suppose that  $K'$  does not meet  $\{|z_2| = 1\} \cap L$ . Now we define the disk  $\overline{\Delta}_t$  by  $\overline{\Delta}_t = \{z_1 = (a - b)t, |z_2| \leq 1\}$  for  $t \in [0, 1)$ . This smooth family of discs does not verify the local Kontinuitätssatz in  $W \setminus F'$ .  $\square$

Now, denote by  $\widehat{F}_i$  the smallest hypersurface of  $\widehat{\Omega}$  which contains  $F_i$ . Set  $\widehat{F} = \cup \widehat{F}_i$ . If  $n = 2$  we have  $F \cup K \subset \widehat{F}$ . This is also true for  $n > 2$ . It is sufficient to apply the last lemma for linear slices of  $\widehat{\Omega}$ .

**Lemma 4.3.** *Let  $L$  be a pseudoconcave subset of a complex manifold  $V$ . If  $L$  is included in a hypersurface  $L'$  of  $V$  then  $L$  is itself a hypersurface of  $V$ .*

*Proof.* Observe that  $L$  is not included in a subvariety of codimension  $\geq 2$  of  $V$ . Assume that  $L$  is not a hypersurface of  $V$ . Then there is a point  $p$  in  $\text{Reg } L'$  which belongs to the boundary of  $L$  in  $L'$ . Choose a local coordinates system  $(z_1, \dots, z_n)$  of a neighbourhood  $W$  of  $p$  such that  $W$  contains the unit polydisk  $\Delta^n$ ,  $p \in \Delta^n$  and  $L' \cap W = \{z_1 = 0\} \cap W$ . We can suppose that  $0 \notin L$  and we can choose  $W$  small as we want.

Let  $\pi : \Delta^n \rightarrow \Delta^{n-1}$  be the projection on the last  $n - 1$  coordinates. Let  $q \in L^* = \pi(L \cap \Delta^n)$  such that  $\text{dist}(0, L^*) = \text{dist}(0, q)$ . Consider the smooth family of discs given by  $\overline{\Delta}_t = \{z = (z_1, z'') : |z_1| < 1/2, z'' = tq\}$ . This family does not verify the local Kontinuitätssatz in  $W \setminus L$ .  $\square$

*End of the proof of Proposition 4.1.* We know that  $(F_i \cup K) \setminus S$  is pseudoconcave in  $\widehat{\Omega} \setminus S$  and  $F_i \cup K \subset \widehat{F}$ . By Lemma 4.3,  $(F_i \cup K) \setminus S$  is a hypersurface of  $\widehat{\Omega} \setminus S$ . By Remmert-Stein theorem, any analytic set can be extended through a point, so  $F_i \cup K$  is a hypersurface of  $\widehat{\Omega}$ . We deduce that  $K$  is included in the analytic set  $\cap \widehat{F}_i$  of  $\mathbb{C}^N$  which does not intersect  $bB$ . Therefore  $K$  must be a finite set.

If  $\Omega$  itself satisfies the local Kontinuitätssatz in  $\widehat{\Omega} \setminus S$ , we have only one Zariski-open set and the Kontinuitätssatz shows directly that  $K = S$ .  $\square$

*Remark 4.4.* The Proposition 4.1 holds for  $K$  not pluripolar. For this case, the proof is more complicated. Using another submersion of  $\widehat{\Omega}$  in Lemma 4.2, given by the map  $z \mapsto (h(z), h(z) + \varepsilon z_1, \dots, h(z) + \varepsilon z_N)$ , we can suppose that  $R = \max_{K \setminus S} |z| > \max_{F''} |z|$ . Let  $q \in bB_R \cap (K \setminus S)$  where  $B_R$  is the ball of center 0 and radius  $R$ . Using a small affine change of coordinates, we can suppose that  $bB_R \cap \widehat{\Omega}$  is transversal at  $q$ . We then construct easily a family of discs close to  $T_q(bB_R) \cap \widehat{\Omega}$ , which does not satisfy the Kontinuitätssatz, where  $T_q(bB_R)$  is the complex tangent space of  $bB_R$  at  $q$ .

*Proof of Theorem 1.3.* Let  $X$  be a hyperconcave end such that the exhaustion function  $\varphi$  is overall strictly plurisubharmonic. Let  $\widehat{X}$  be a smooth completion of  $X$ . Then  $\widehat{X} \setminus X$  has a strictly pseudoconvex neighbourhood  $V$ . Based on Remmert's reduction theory, Grauert [17, Satz 3, p. 338] showed that there exists a maximal compact analytic set  $A$  of  $V$ . (Note that, by definition, a maximal analytic set has dimension greater than one at each point.) Moreover, by [17, Satz 5, p. 340] there exist a normal Stein space  $V'$  with at worst isolated singularities, a discrete set  $D \subset V'$  and a proper holomorphic map  $\pi : V \rightarrow V'$ , biholomorphic between  $V \setminus A$  and  $V' \setminus D$  and with  $\pi(A) = D$ . That is,  $A$  can be blown down to the finite set  $D$ . Of course,  $\text{Sing}(V') \subset D$ .

The maximum principle for  $\varphi$  implies  $A \subset \widehat{X} \setminus X$ . Let  $\psi : V' \rightarrow [-\infty, \infty)$  be given by  $\psi = \varphi \circ \pi^{-1}$  on  $V' \setminus D$  and  $\psi = -\infty$  on  $\pi(\widehat{X} \setminus X)$ . Then  $\psi$  is a strictly plurisubharmonic function on  $V'$  by [7] and  $\pi(\widehat{X} \setminus X)$  is its pluripolar set. By Proposition 4.1,  $\pi(\widehat{X} \setminus X)$  is a finite set. Therefore  $\widehat{X} \setminus X$  consists of  $A$  and possibly a finite set  $D'$ . If  $X$  has a Stein cover, it follows from the Kontinuitätssatz that  $\pi(\widehat{X} \setminus X) = \text{Sing}(V')$ . Therefore  $D' = \emptyset$  and  $D = \text{Sing}(V')$ .  $\square$

*Remark 4.5.* If in Theorem 1.3 we suppose only that  $X$  admits a Zariski-open dense set which is uniformized by a Stein manifold, we can prove in the same way, that  $\widehat{X} \setminus X$  is included in a hypersurface of  $\widehat{X}$ , i.e.  $X$  contains a Zariski-open dense set of  $\widehat{X}$ .

## 5. Extension of Nadel's theorems

Our goal is to extend to dimension two Nadel's theorems [25]. The arithmetic quotients are, with a few exceptions, pseudoconcave manifolds carrying a positive line bundle. In this respect, Nadel [25] considered the realization as a quasiprojective manifold of a class of manifolds  $X$  with hyperconcave ends and of dimension greater than three. The method of Nadel is to compactify the manifold by the theorem of Rossi, and then to apply differential geometric methods, like the existence of Kähler-Einstein metric and the Schwarz-Pick lemma of Yau and Mok-Yau. Corollary 1.4 is a generalization of [25, Th. 0.2].

*Proof of Corollary 1.4.* The necessity of conditions (i) and (ii) is obvious, while the necessity of (iii) follows from a theorem of Griffiths [20, Th. I].

For the sufficiency, we need the embedding theorem of Andreotti-Tomassini [3, Th. 2, p. 97], [26, Lemma 2.1] (see also [2, Th. 4.1]). This theorem shows that condition (ii) implies the embedding of  $X$  as an open set in a smooth projective manifold  $\widehat{X}$ . We conclude by Theorem 1.3.  $\square$

Our result pertains to the work of Nadel and Tsuji [26] which generalizes the compactification of arithmetic quotients of any rank, by showing

that certain pseudoconcave manifolds are quasi-projective. In dimension two their condition coincides with hyperconcavity. Corollary 1.4 yields an extension of their theorem in dimension two, together with a completely complex-analytic proof of the compactification of arithmetic quotients.

As a consequence of Corollary 1.4 we also get a slightly stronger form of [35, Main Theorem] (also noted by Nadel in dimension greater than three):

**Corollary 5.1.** *Let  $X$  be a complete Kähler manifold of finite volume and bounded negative sectional curvature. If  $\dim X \geq 2$ ,  $X$  is biholomorphic to a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many singular points.*

*Proof.* Indeed, the same argument as in [35] or [26, §3] shows, with the help of the Busemann function, that  $X$  is hyper 1-concave. Moreover, the negativity of the curvature implies that the canonical bundle  $K_X$  is positive and the universal cover of  $X$  is Stein.

We show that the positivity of  $K_X$  implies the ring  $\bigoplus_{k>0} H^0(X, K_X^k)$  separates points and gives local coordinates everywhere on  $X$ . If  $\varphi$  denotes the exhaustion function of  $X$ ,  $\sqrt{-1}(A\Theta(K_X) + \partial\bar{\partial}(-\log(-\varphi)))$ ,  $A \gg 1$ , is a complete Kähler metric on  $X$ , where  $\Theta(K_X)$  is the curvature of  $K_X$ . By the  $L^2$  estimates with singular weights for positive line bundles (see e.g. Demailly [10]) our contention follows. We can thus apply Corollary 1.4.  $\square$

In the same vein as in Corollary 1.4, we can show that Nadel's main result [25, Th.0.1] holds in dimension two. Namely, let  $X$  be a connected Moishezon, hyper 1-concave manifold of dimension  $n \geq 2$ , which can be covered by Zariski-open sets uniformized by Stein manifolds. Then  $X$  can be compactified by adding finitely many points to a compact Moishezon space.

We close the section with one more result about embedding of hyper 1-concave manifolds. Generalizing the Andreotti-Tomassini theorem, Andreotti-Siu [2, Th. 7.1] show that a strongly 1-concave manifold  $X$  of  $\dim X \geq 3$  can be embedded in the projective space, if it admits a line bundle  $E$  such that  $\bigoplus_{k>0} H^0(X, E^k)$  gives local coordinates on a sufficiently large compact of  $X$ . The proof is based on techniques of extending analytic sheaves. Moreover, they show through an example [2, p. 267–70] that the result breaks down in dimension two. We prove now however, that if we impose the condition of hyperconcavity, the result occurs also in dimension two. Here  $\varphi$  and  $b$  have the same meaning as in Definition 1.1.

**Proposition 5.2.** *Let  $X$  be a hyper 1-concave manifold of dimension  $n \geq 2$ . Let  $c$  be a real number such that  $c < b$ . Assume there is a line bundle  $E$  over  $X' = \{\varphi > c\}$  such that the ring  $\bigoplus_{k>0} H^0(X', E^k)$  gives local coordinates on  $X'$ . Then  $X$  is biholomorphic to an open subset of a projective manifold.*

*Proof.* By Theorem 1.2,  $X$  is an open subset of a variety  $\widehat{X}$  with isolated singularities. Moreover  $\widehat{X} \setminus X'$  is a Stein space. Replacing  $c$  by a  $c'$  such that  $c < c' < b$  we can suppose that there are holomorphic sections  $s_0, \dots, s_m$  of  $H^0(X', E^k)$  which give local coordinates of  $X'$  where  $k$  is big enough. We can define a holomorphic map  $\pi : X' \rightarrow \mathbb{P}^m$  by  $\pi(z) = [s_0(z) : \dots : s_m(z)]$ . Then  $\pi$  gives a local immersion of  $X'$  in  $\mathbb{P}^m$ . Since  $\widehat{X} \setminus X'$  is embeddable in an euclidian space, a theorem of Dolbeault-Henkin-Sarkis [12, 31] implies that  $\pi$  can be extended to a meromorphic map from  $\widehat{X}$  into  $\mathbb{P}^m$ .

Denote by  $Z$  the set consisting of the singular points of  $\widehat{X}$ , the points of indeterminacy of  $\pi$  and the critical points of  $\pi$ . Then  $Z$  is a compact analytic subset of  $\widehat{X} \setminus X'$ . Since  $\widehat{X} \setminus X'$  is Stein space,  $Z$  is a finite set. The map  $\pi$  gives local immersion of  $\widehat{X} \setminus Z$  in  $\mathbb{P}^m$ . Let  $H$  be the hyperplane line bundle of  $\mathbb{P}^m$  and set  $L = \pi^*(H)$ . Then  $L$  is a positive line bundle of  $\widehat{X} \setminus Z$ . In particular  $L$  is positive on  $X \setminus Z$  and by a theorem of Shiffman [33] extends to a positive line bundle on  $X$ . By the argument in the proof of Corollary 5.1, we show that  $\bigoplus_{k>0} H^0(X, L^k \otimes K_X)$  separates points and gives local coordinates on  $X$ . By [26, Lemma 2.1]  $X$  is biholomorphic to an open subset of a projective manifold.  $\square$

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