ON THE APPROXIMATION OF POSITIVE CLOSED CURRENTS ON COMPACT KÄHLER MANIFOLDS

DAN COMAN AND GEORGE MARINESCU

ABSTRACT. Let L be a holomorphic line bundle over a compact Kähler manifold X endowed with a singular Hermitian metric h with curvature current $c_1(L,h) \geq 0$. In certain cases when the wedge product $c_1(L,h)^k$ is a well defined current for some positive integer $k \leq \dim X$, we prove that $c_1(L,h)^k$ can be approximated by averages of currents of integration over the common zero sets of k-tuples of holomorphic sections over X of the high powers $L^p := L^{\otimes p}$.

In the second part of the paper we study the convergence of the Fubini-Study currents and the equidistribution of zeros of L^2-holomorphic sections of the adjoint bundles $L^p \otimes K_X$, where L is a holomorphic line bundle over a complex manifold X endowed with a singular Hermitian metric h with positive curvature current. As an application, we obtain an approximation theorem for the current $c_1(L,h)^k$ using currents of integration over the common zero sets of k-tuples of sections of $L^p \otimes K_X$.

1. INTRODUCTION

Let L be a holomorphic line bundle over the compact Kähler manifold (X, Ω) and let $n = \dim X$. Given a holomorphic section $\sigma \in H^0(X, L^p)$, where $L^p := L^{\otimes p}$, we denote by $[\sigma = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{\sigma = 0\} \subset X$. We define $\mathcal{A}_k(L^p)$, $k \leq n$, to be the space of positive closed currents R of bidegree (k,k) on X of the form

$$R = \frac{1}{p^k N} \sum_{\ell=1}^{N} [\sigma_{\ell,1} = 0] \wedge \ldots \wedge [\sigma_{\ell,k} = 0] \,. \tag{1}$$

Here $N \in \mathbb{N}$ and $\sigma_{\ell,j} \in H^0(X, L^p)$ are such that $[\sigma_{\ell,1} = 0] \wedge \ldots \wedge [\sigma_{\ell,k} = 0]$ is a well defined positive closed current on X in the sense of [D4, FS]. Recall that in this case the set $\{\sigma_{\ell,i_1} = 0\} \cap \ldots \cap \{\sigma_{\ell,i_j} = 0\}$ has pure dimension $n - j$ for every $i_1 < \ldots < i_j$ in $\{1, \ldots, k\}$ and the current $[\sigma_{\ell,1} = 0] \wedge \ldots \wedge [\sigma_{\ell,k} = 0]$ is equal to the current of integration with multiplicities along the analytic set $\{\sigma_{\ell,1} = 0\} \cap \ldots \cap \{\sigma_{\ell,k} = 0\}$ [D4, Corollary 2.11, Proposition 2.12]. Building on our previous work [CM1] we will prove the following approximation result for wedge products of integral positive closed currents by currents of integration along the common zero sets of k-tuples of holomorphic sections of high powers of L.

Date: January 24, 2013.

2010 *Mathematics Subject Classification.* Primary 32L10; Secondary 32U40, 32W20, 53C55.

G. Marinescu is partially supported by SFB TR 12.

Theorem 1.1. *Let (X,Ω) be a compact Kähler manifold and (L,h_0) be a holomorphic line bundle on X endowed with a singular Hermitian metric h_0 with strictly positive curvature current, i.e. $c_1(L,h_0)\geq c\,\Omega$ on X, for some constant $c>0$. Assume that h_0 is continuous on $X\setminus\Sigma$, where Σ is an analytic subset of X. If $k\leq\operatorname{codim}\Sigma$ and h is a singular Hermitian metric on L with curvature current $c_1(L,h)\geq 0$ on X and with locally bounded weights on $X\setminus\Sigma$, then there exists a sequence of currents $R_j\in\mathcal{A}_k(L^{p_j})$, where $p_j\nearrow\infty$, such that R_j converges weakly on X to $c_1(L,h)^k$.*

The hypothesis on h means the following: if U_α is an open set of X on which there exists a local holomorphic frame e_α of L and $|e_\alpha|_h=e^{-\psi_\alpha}$, then ψ_α is plurisubharmonic (psh) on U_α and locally bounded on $U_\alpha\setminus\Sigma$. Note that in Theorem 1.1 the line bundle L is big and X is Moishezon [JS]. Hence X is a projective manifold, since it is assumed to be Kähler (see e. g. [MM1, Theorem 2.2.26]).

The proof relies on the convergence of Fubini-Study currents obtained in [CM1, Theorem 5.4] . This follows in turn from the weak asymptotic behavior of the Bergman kernel, $\frac{1}{p}\log P_p\to 0$ in L^1_{loc}, see [CM1, Theorems 1.1, 1.2]. For equidistribution results concerning some non-compact manifolds see [DMS], where the full asymptotic expansion of the Bergman kernel [MM1, Theorem 6.1.1] [MM2, Theorem 3.11] is used.

In the case that the line bundle L is *positive* (i. e. it admits a smooth Hermitian metric with positive curvature), one has the following theorem in which the approximation is achieved by using currents of integration over common zero sets of k-tuples of sections rather than averages of such.

Theorem 1.2. *Let X be a compact Kähler manifold of dimension n and L be a positive holomorphic line bundle on X. Assume that h is a singular Hermitian metric on L with positive curvature current $c_1(L,h)\geq 0$ such that the current $c_1(L,h)^k$ is well defined for some $k\leq n$. Then there exist a sequence of integers $p_j\nearrow\infty$ and sections $\sigma_{j,1},\dots,\sigma_{j,k}\in H^0(X,L^{p_j})$ such that $T_j:=[\sigma_{j,1}=0]\wedge\ldots\wedge[\sigma_{j,k}=0]$ are well defined positive closed currents of bidegree (k,k) and $p_j^{-k}\,T_j\to c_1(L,h)^k$ weakly on X.*

Theorem 1.2 is a consequence of the equidistribution results of common zeros of k-tuples of holomorphic sections [DS, SZ1, SZ2] and the regularization results of [D3, GZ1, BK]. The hypothesis on the wedge product current $c_1(L,h)^k$ is that it is well defined locally, in the sense of Bedford and Taylor [BT1, BT2]. We recall briefly its definition at the beginning of Section 2.

In the situation of Theorem 1.2 for $k=1$, Demailly [D1, Théorème 1.9] proved that $c_1(L,h)$ can be approximated by currents of integration over irreducible hypersurfaces in X. Theorem 1.2 shows that these currents can actually be chosen of type $[\sigma_j=0]$, $\sigma_j\in H^0(X,L^{p_j})$. In the case when X is a projective homogeneous manifold or when $X\setminus\operatorname{supp}c_1(L,h)$ satisfies a certain convexity property it was shown in [G, Theorems 0.5, 1.6] that the approximating sections σ_j can be chosen is such a way that $\{\sigma_j=0\}$ converge in the Hausdorff metric to the support of $c_1(L,h)$. Such an approximation theorem with control on the supports was first proved in [DuS] for positive closed currents of bidegree (1,1) on pseudoconvex domains in $\mathbb{C}^n$.

When $k=1$, Theorem 1.2 holds in fact in the more general case of a *big* line bundle with an arbitrary positively curved singular Hermitian metric.

Theorem 1.3. *Let L be a big line bundle over the compact Kähler manifold X and h be a singular Hermitian metric on L with positive curvature current $c_1(L,h) \geq 0$. Then there exists a sequence of sections $\sigma_j \in H^0(X, L^{p_j})$, where $p_j \nearrow \infty$, such that $p_j^{-1}\,[\sigma_j = 0] \to c_1(L,h)$ weakly on X as $j \to \infty$.*

Theorems 1.1, 1.2 and 1.3 are proved in Section 2. In Section 3 we consider the general framework as in [CM1] of a holomorphic line bundle L over a complex manifold X endowed with a positively curved singular Hermitian metric h which is continuous outside a compact analytic set $\Sigma \subset X$, and we work instead with the spaces of L^2-holomorphic sections of the adjoint bundles $L^p \otimes K_X$ relative to the metrics induced by h and some positive $(1,1)$ form on X. If γ_p are the corresponding Fubini-Study currents and $k \leq \operatorname{codim}\Sigma$ we prove that $\frac{1}{p^k}\,\gamma_p^k$ converge weakly on X to $c_1(L,h)^k$. We also study the equidistribution of common zeros of k-tuples of L^2-holomorphic sections of $L^p \otimes K_X$. We conclude Section 3 by noting that working with sections of adjoint bundles one can obtain a more general approximation theorem than Theorem 1.1, in the sense that the metric h_0 is assumed to verify a weaker positivity condition. Let $\mathcal{A}_k(L^p \otimes K_X)$ be the spaces of positive closed currents of bidegree (k,k) on X defined as in (1) using sections $\sigma_{\ell,j} \in H^0(X, L^p \otimes K_X)$. Namely, we prove the following:

Theorem 1.4. *Let (X,Ω) be a compact Kähler manifold and (L,h_0) be a holomorphic line bundle on X endowed with a singular Hermitian metric h_0 with positive curvature current $c_1(L,h_0) \geq 0$. Assume that there exists an analytic subset Σ of X such that h_0 is continuous on $X \setminus \Sigma$ and $c_1(L,h_0)$ is strictly positive on $X \setminus \Sigma$, i.e. $c_1(L,h_0) \geq \varepsilon\,\Omega$ for some continuous function $\varepsilon : X \setminus \Sigma \longrightarrow (0,+\infty)$. If $k \leq \operatorname{codim}\Sigma$ and h is a singular Hermitian metric on L with curvature current $c_1(L,h) \geq 0$ on X and with locally bounded weights on $X \setminus \Sigma$, then there exists a sequence of currents $R_j \in \mathcal{A}_k(L^{p_j} \otimes K_X)$, where $p_j \nearrow \infty$, such that R_j converges weakly on X to $c_1(L,h)^k$.*

2. Proofs of Theorems 1.1, 1.2 and 1.3

We recall first the inductive definition of complex Monge-Ampère operators due to Bedford and Taylor. Let T be a positive closed current of bidimension (ℓ,ℓ), $\ell > 0$, on an open set U in $\mathbb{C}^n$ and let $|T|$ denote its trace measure. If u is a psh function on U so that $u \in L^1_{loc}(U,|T|)$ we say that the wedge product $dd^c u \wedge T$ is well defined. This is the positive closed current of bidimension $(\ell-1,\ell-1)$ defined by $dd^c u \wedge T = dd^c(uT)$.

If $u_1,\dots,u_k$ are psh functions on U we say that $dd^c u_1 \wedge \ldots \wedge dd^c u_k$ is well defined if one can define inductively as above all intermediate currents

$$dd^c u_{j_1} \wedge \ldots \wedge dd^c u_{j_\ell} = dd^c(u_{j_1} dd^c u_{j_2} \wedge \ldots \wedge dd^c u_{j_\ell}),\ 1 \leq j_1 < \ldots < j_\ell \leq k.$$

The wedge product is well defined for locally bounded psh functions [BT1, BT2], for psh functions that are locally bounded outside a compact subset of a pseudoconvex open set U, or when the mutual intersection of their unbounded loci is small in a certain sense [S, D4, FS]. If T_j, $1 \leq j \leq k$, are positive closed currents of bidegree $(1,1)$ on a complex manifold then one defines $T_1 \wedge \ldots \wedge T_k$ by working with their local psh potentials.

We will need the following lemma:

Lemma 2.1 ([KN, Lemma 2.1, p. 182]). *Let μ be a positive Radon measure on a compact metric space K with $\mu(K)=1$, let $\mathcal{B}$ be the σ-algebra of Borel sets in K, and let $f:X\longrightarrow\mathbb{C}$ be a given bounded Borel measurable function. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\ell=1}^{N}f(x_\ell)=\int_X f\,d\mu$$

for almost all sequences $(x_\ell)\in\prod_{\ell=1}^{\infty}(K,\mathcal{B},\mu)$.

Proof of Theorem 1.1. Since $\dim\Sigma\leq n-k$ and h has locally bounded weights on $X\setminus\Sigma$, $c_1(L,h)^k$ is a well defined positive closed current on X by [D4, Corollary 2.11]. The proof of Theorem 1.1 is divided in three steps.

Step 1. We introduce here the spaces of L^2-holomorphic section of L^p and their Fubini-Study currents that will be used in the proof. Assume for the moment that h is any singular metric on L with positive curvature current $c_1(L,h)\geq 0$. Let $H^0_{(2)}(X,L^p)$ be the space of L^2-holomorphic sections of L^p relative to the metric h_p induced by h and the volume form Ω^n on X,

$$H^0_{(2)}(X,L^p)=\left\{S\in H^0(X,L^p):\,\|S\|_p^2:=\int_X|S|^2_{h_p}\,\Omega^n<\infty\right\},$$

endowed with the obvious inner product. Let $d_p=\dim H^0_{(2)}(X,L^p)$ and fix an orthonormal basis $\{S_i^p\}_{1\leq i\leq d_p}$ of $H^0_{(2)}(X,L^p)$. We denote by γ_p the Fubini-Study current of the space $H^0_{(2)}(X,L^p)$, defined by

$$\gamma_p\mid_{U_\alpha}=\frac{1}{2}\,dd^c\log\left(\sum_{i=1}^{d_p}|s_i^p|^2\right),\;U_\alpha\subset X\;\text{ open}\,, \tag{2}$$

where $d^c=\frac{1}{2\pi i}(\partial-\overline{\partial})$, $S_i^p=s_i^pe_\alpha^{\otimes p}$, and e_α is a local holomorphic frame for L on U_α. Recall that the currents γ_p are independent of the choice of basis $\{S_i^p\}$ and we have

$$\frac{1}{p}\,\gamma_p=c_1(L,h)+\frac{1}{2p}\,dd^c\log P_p\,,$$

where $P_p=\sum_{i=1}^{d_p}|S_i^p|^2_{h_p}$ is the Bergman kernel function of the space $H^0_{(2)}(X,L^p)$ (cf. [CM1, Lemma 3.2] or [CM2, Lemma 3.8]). If $S\in H^0(X,L^p)$ the Lelong-Poincaré equation [MM1, Theorem 2.3.3] takes the form

$$\frac{1}{p}\,[S=0]=c_1(L,h)+\frac{1}{2p}\,dd^c\log|S|^2_{h_p}\,.$$

Let θ be a closed smooth (1,1) form representing the first Chern class of L. Then any positive closed current T of bidegree (1,1) in the first Chern class of L can be written at $T=\theta+dd^c\varphi$ where φ is a θ-psh function on X. These are quasiplurisubharmonic (qpsh) functions φ on X, i.e. functions that can be written locally as a sum of a psh function and a smooth one, such that the current $\theta+dd^c\varphi\geq 0$. It follows by the inductive (local) definition of the complex Monge-Ampère operator that if, for some $k\leq n$, the wedge product currents $c_1(L,h)^k$, γ_p^k, $[\sigma_1=0]\wedge\ldots\wedge[\sigma_k=0]$ are well defined, where $\sigma_i\in H^0(X,L^p)$, then

$$\frac{1}{p^k}\int_X\gamma_p^k\wedge\Omega^{n-k}=\frac{1}{p^k}\int_X[\sigma_1=0]\wedge\ldots\wedge[\sigma_k=0]\wedge\Omega^{n-k}=\int_X c_1(L,h)^k\wedge\Omega^{n-k}\,, \tag{3}$$

all being equal to $\int_X\theta^k\wedge\Omega^{n-k}$ (cf. [GZ1]).

We identify the unit sphere $\mathcal{S}^p$ of $(H^0_{(2)}(X,L^p),\|\cdot\|_{L^2})$ to the unit sphere $\mathbf{S}^{2d_p-1}$ in $\mathbb{C}^{d_p}$ by

$$\mathbf{S}^{2d_p-1}\ni a=(a_1,\ldots,a_{d_p})\longmapsto S_a=\sum_{i=1}^{d_p}a_iS_i^p\in\mathcal{S}^p,$$

and we let λ_p be the probability measure on $\mathcal{S}^p$ induced by the normalized surface measure on $\mathbf{S}^{2d_p-1}$, denoted also by λ_p (i.e. $\lambda_p(\mathbf{S}^{2d_p-1})=1$). If $k\geq 2$, we let λ_p^k denote the product measure on $(\mathcal{S}^p)^k$ determined by λ_p.

Step 2. We consider now the case when the metric h has strictly positive curvature current and is continuous on $X\setminus\Sigma$. By [CM1, Theorem 5.4] (see also [CM1, Theorems 1.1, 1.2]), there exists p_0 so that the currents $c_1(L,h)^k$ and γ_p^k, for $p\geq p_0$, are well defined and $\frac{1}{p^k}\,\gamma_p^k\to c_1(L,h)^k$ weakly on X as $p\to\infty$. Moreover, for each $p\geq p_0$ there exists a set $\mathcal{N}_p\subset(\mathcal{S}^p)^k$ with $\lambda_p^k(\mathcal{N}_p)=0$ such that

$$[\sigma=0]:=[\sigma_1=0]\wedge\ldots\wedge[\sigma_k=0]\,,\ \text{ where } \sigma=(\sigma_1,\ldots,\sigma_k)\in(\mathcal{S}^p)^k\setminus\mathcal{N}_p\,,$$

is a well defined positive closed current of bidegree (k,k) on X and

$$\int_{(\mathcal{S}^p)^k}\big\langle[\sigma=0],\theta\big\rangle\,d\lambda_p^k=\langle\gamma_p^k,\theta\rangle\,,$$

for every test form θ on X.

Let $\{\theta_m\}_{m\geq1}$ be a dense set of smooth $(n-k,n-k)$ forms on X and let $p\geq p_0$. We define the function

$$f_m:(\mathcal{S}^p)^k\to\mathbb{C},\ \ f_m(\sigma)=\begin{cases}\langle[\sigma=0],\theta_m\rangle\,, & \text{if } \sigma\in(\mathcal{S}^p)^k\setminus\mathcal{N}_p\,,\\ 0\,, & \text{if } \sigma\in\mathcal{N}_p\,.\end{cases}$$

Then f_m is Borel measurable and bounded in view of (3). Since $\lambda_p^k(\mathcal{N}_p)=0$, Lemma 2.1, applied to the finite collection of functions f_m, $1\leq m\leq p$, implies that for almost all sequences $(\sigma_\ell)\in\prod_{\ell=1}^\infty((\mathcal{S}^p)^k\setminus\mathcal{N}_p,\lambda_p^k)$ we have

$$\frac{1}{N}\sum_{\ell=1}^N f_m(\sigma_\ell)=\frac{1}{N}\sum_{\ell=1}^N\langle[\sigma_\ell=0],\theta_m\rangle\to\int_{(\mathcal{S}^p)^k}f_m\,d\lambda_p^k=\langle\gamma_p^k,\theta_m\rangle\,,$$

as $N\to\infty$, for all m, $1\leq m\leq p$.

It folows that for each $p\geq p_0$ there exists a current $T_p=\frac{1}{N_p}\sum_{\ell=1}^{N_p}[\sigma_\ell^p=0]$, where $N_p\in\mathbb{N}$, $\sigma_\ell^p\in(\mathcal{S}^p)^k\setminus\mathcal{N}_p$, such that

$$\big|\langle T_p,\theta_m\rangle-\langle\gamma_p^k,\theta_m\rangle\big|<1\,,\ \ 1\leq m\leq p\,. \tag{4}$$

By (3) the currents $\frac{1}{p^k}\,\gamma_p^k$ and $R_p := \frac{1}{p^k}\,T_p \in \mathcal{A}_k(L^p)$ have uniformly bounded mass. Since $\frac{1}{p^k}\,\gamma_p^k \to c_1(L,h)^k$, we conclude by (4) that $R_p \to c_1(L,h)^k$ weakly on X as $p \to \infty$.

Step 3. We treat here the general case when h has locally bounded weights on $X \setminus \Sigma$ and $c_1(L,h) \geq 0$. Fix a smooth Hermitian metric h_s on L and let $\theta = c_1(L,h_s)$. By [D2] (see also [GZ1]) the set of positively curved singular metrics on L is in one-to-one correspondence with the set of θ-psh functions on X. Let φ, resp. φ_0, be the θ-psh function determined by h, resp. h_0.

By hypothesis, φ is locally bounded on $X \setminus \Sigma$, φ_0 is continuous on $X \setminus \Sigma$, and

$$c_1(L,h) = \theta + dd^c\varphi \geq 0\,,\;\; c_1(L,h_0) = \theta + dd^c\varphi_0 \geq c\,\Omega\,.$$

Subtracting a constant we may assume $\varphi < 0$ on X. By [DP, Theorem 3.2] (see also [D3, Proposition 3.7]) there exist a sequence of qpsh functions $\varphi_j < 0$ decreasing to φ on X and a sequence $\varepsilon_j \in (0,c)$, $\varepsilon_j \searrow 0$, such that φ_j are smooth on $X \setminus \Sigma$ and $\theta + dd^c\varphi_j \geq -\varepsilon_j\Omega$ on X. If $\widetilde{\varphi}_j = (1-\varepsilon_j/c)\varphi_j + (\varepsilon_j/c)\varphi_0$ then

$$\theta + dd^c\widetilde{\varphi}_j \geq \theta - (1-\varepsilon_j/c)(\theta + \varepsilon_j\Omega) + (\varepsilon_j/c)(-\theta + c\,\Omega) = (\varepsilon_j^2/c)\,\Omega\,,$$

hence $\widetilde{\varphi}_j$ are θ-psh. Let h_j denote the metric on L determined by $\widetilde{\varphi}_j$, so $c_1(L,h_j) \geq (\varepsilon_j^2/c)\,\Omega$ on X and h_j is continuous on $X \setminus \Sigma$.

We claim that $c_1(L,h_j)^k \to c_1(L,h)^k$ weakly on X as $j \to \infty$. Indeed, let $U \subset X$ be a coordinate ball and write $\theta = dd^c\chi$, $\Omega = dd^c\rho$, where χ, ρ are smooth functions so that $\chi < 0$, $\rho > 0$ on U. The functions

$$\psi_j = (1-\varepsilon_j/c)(\chi + \varphi_j) + \varepsilon_j\rho\,,\;\; \psi_0 = \chi + \varphi_0 - c\rho\,,$$

are continuous on $U \setminus \Sigma$ and psh on U, since $dd^c\psi_j \geq (\varepsilon_j^2/c)\,\Omega$, $dd^c\psi_0 \geq 0$. As $\chi < 0$, $\rho > 0$, $0 > \varphi_j \searrow \varphi$, $\varepsilon_j \searrow 0$, it follows that $\psi_j \searrow \chi + \varphi$ on U. Note that on U

$$c_1(L,h_j) = \theta + dd^c\widetilde{\varphi}_j = dd^c\widetilde{\psi}_j \;\text{ where } \widetilde{\psi}_j = \psi_j + (\varepsilon_j/c)\psi_0\,.$$

Since $k \leq \operatorname{codim}\Sigma$, it follows by [D4, Corollary 2.11] that $(dd^c\psi_j)^{k-\ell} \wedge (dd^c\psi_0)^\ell$, $(dd^c(\chi+\varphi))^{k-\ell} \wedge (dd^c\psi_0)^\ell$, $0 \leq \ell \leq k$, are well defined positive closed currents on U. Moreover, since $\varepsilon_j \to 0$ and by [D4, Proposition 2.9], $(dd^c\psi_j)^{k-\ell} \wedge (dd^c\psi_0)^\ell \to (dd^c(\chi+\varphi))^{k-\ell} \wedge (dd^c\psi_0)^\ell$, we have

$$c_1(L,h_j)^k = (dd^c\widetilde{\psi}_j)^k = \sum_{\ell=0}^{k}\binom{k}{\ell}\frac{\varepsilon_j^\ell}{c^\ell}\,(dd^c\psi_j)^{k-\ell} \wedge (dd^c\psi_0)^\ell \to c_1(L,h)^k\,,$$

weakly on U as $j \to \infty$.

Let $\{\theta_m\}_{m\geq 1}$ be a dense set of smooth $(n-k,n-k)$ forms on X. Applying the result of Step 2 to each metric h_j we obtain a sequence of integers $p_j \nearrow \infty$ and of currents $R_j \in \mathcal{A}_k(L^{p_j})$ such that

$$\big|\big\langle R_j, \theta_m\big\rangle - \big\langle c_1(L,h_j)^k, \theta_m\big\rangle\big| < 1/j\,,\;\; 1 \leq m \leq j\,.$$

Since R_j, $c_1(L,h_j)^k$, have uniformly bounded mass and $c_1(L,h_j)^k \to c_1(L,h)^k$, it follows that $R_j \to c_1(L,h)^k$ weakly on X as $j \to \infty$. The proof of Theorem 1.1 is complete. □

Remark 2.2. The hypothesis of Theorem 1.1 that X carries a Kähler form Ω is essential. In Step 2 of the proof we apply [CM1, Theorem 5.4], which requires X to be a Kähler manifold. Moreover, we use that Ω is a Kähler form in formula (3), which shows that all currents $\frac{1}{p^k}\,\gamma_p^k$, $R_p \in \mathcal{A}_k(L^p)$, have equal mass with respect to Ω^{n-k}.

Proof of Theorem 1.2. We note first that Theorem 1.2 holds if h is a smooth metric with positive curvature $c_1(L,h) > 0$. Indeed, by [DS, Théorème 7.3] (see also [SZ2]) there exist sections $\sigma_{p,1},\ldots,\sigma_{p,k} \in H^0(X,L^p)$ such that $[\sigma_{p,1}=0]\wedge\ldots\wedge[\sigma_{p,k}=0]$ are well defined positive closed currents of bidegree (k,k) and $\frac{1}{p^k}\,[\sigma_{p,1}=0]\wedge\ldots\wedge[\sigma_{p,k}=0] \to c_1(L,h)^k$ weakly on X as $p\to\infty$.

We consider next the general case of a singular metric h with $c_1(L,h)\geq 0$ and such that the current $c_1(L,h)^k$ is well defined for some $k\leq n$. Let h_0 be a smooth metric on L such that $\Omega = c_1(L,h_0)$ is a Kähler form on X and let φ be the Ω-psh function determined by h (see [D2, GZ1]). The regularization theorem [BK, Theorem 1] (see also [D3], [GZ1, Theorem 8.1]) yields a decreasing sequence of smooth Ω-psh functions $\varphi_j \searrow \varphi$ on X. Subtracting a constant, we may assume that $\varphi_1 < 0$ on X. Then $\psi_j := \frac{j}{j+1}\,\varphi_j$ are smooth Ω-psh functions on X and $\psi_j \searrow \varphi$ as $j\to\infty$. If h_j is the metric on L defined by ψ_j then $c_1(L,h_j) = \Omega + dd^c\psi_j \geq \frac{1}{j+1}\,\Omega > 0$. Since the Monge-Ampère operator is continuous on decreasing sequences it follows that $c_1(L,h_j)^k \to c_1(L,h)^k$ weakly on X.

We proceed now as in Step 3 from the proof of Theorem 1.1. Let $\{\theta_m\}_{m\geq 1}$ be a dense set of smooth $(n-k,n-k)$ forms on X. As noted at the beginning of the proof, we can apply the conclusion of Theorem 1.2 for each smooth positively curved metric h_j. Thus we obtain a sequence of integers $p_j \nearrow \infty$ and sections $\sigma_{j,1},\ldots,\sigma_{j,k} \in H^0(X,L^{p_j})$ such that $T_j := [\sigma_{j,1}=0]\wedge\ldots\wedge[\sigma_{j,k}=0]$ are well defined positive closed currents of bidegree (k,k) and

$$\big|\big\langle p_j^{-k}\,T_j,\theta_m\big\rangle - \langle c_1(L,h_j)^k,\theta_m\rangle\big| < 1/j\ ,\ \ 1\leq m\leq j\,.$$

Since the currents $p_j^{-k}\,T_j$, $c_1(L,h_j)^k$, have uniformly bounded mass and $c_1(L,h_j)^k \to c_1(L,h)^k$, it follows that $p_j^{-k}\,T_j \to c_1(L,h)^k$ weakly on X as $j\to\infty$. □

Proof of Theorem 1.3. Since L is big, it carries a singular Hermitian metric h_0 with strictly positive curvature current [JS] (see also [MM1, Theorem 2.3.8]). If h is any such metric, [CM1, Theorem 5.1] shows that there exists a sequence of sections $\sigma_p \in H^0(X,L^p)$ such that $\frac{1}{p}\,[\sigma_p=0]\to c_1(L,h)$ weakly on X as $p\to\infty$.

Assume now that h is a singular Hermitian metric on L with positive curvature current $c_1(L,h)\geq 0$. Let h_s be a fixed smooth metric on L, $\theta = c_1(L,h_s)$, and let φ,φ_0 be the θ-psh functions determined by h, resp. h_0. If h_j is the singular metric on L determined by the θ-psh function $\varphi_j = (j\varphi+\varphi_0)/(j+1)$ then its curvature is a Kähler current, since $c_1(L,h_j)\geq (j+1)^{-1}c_1(L,h_0)$. As $\varphi_j\to\varphi$ in $L^1(X)$, it follows that $c_1(L,h_j)\to c_1(L,h)$ weakly on X. Then we conclude the proof proceeding as at the end of the proof of Theorem 1.2, since Theorem 1.3 holds for each metric h_j. □

Example 2.3. We conclude this section by discussing a fundamental example where Theorem 1.2 applies. Consider the line bundle $L=\mathcal{O}(1)$ over $X=\mathbb{P}^n$. The global holomorphic sections of L^d are given by homogeneous polynomials of degree d on $\mathbb{C}^{n+1}$. If

ω denotes the Fubini-Study Kähler form on $\mathbb{P}^n$ then the set of singular metrics h on L is in one-to-one correspondence to the set of ω-psh functions φ on $\mathbb{P}^n$, and we have $c_1(L,h)=\omega+dd^c\varphi$ ([D2], [GZ1]). Moreover, the latter class is in one-to-one correspondence to the Lelong class $\mathcal{L}(\mathbb{C}^n)$ of entire psh functions with logarithmic growth, where we consider the standard embedding $\mathbb{C}^n\hookrightarrow\mathbb{P}^n$ (cf. [GZ1, Section 2]). Recall that a psh function u on $\mathbb{C}^n$ is said to have logarithmic growth if $u(z)\leq\log^+\|z\|+C$ holds on $\mathbb{C}^n$, with some constant C depending on u.

The domain $DMA(\mathbb{P}^n,\omega)$ of the complex Monge-Ampère operator is defined in [CGZ] to be the set of ω-psh functions φ on $\mathbb{P}^n$ for which there is a positive Radon measure $MA(\varphi)$ with the following property: If $\{\varphi_j\}$ is any sequence of *bounded* ω-psh functions decreasing to φ then $(\omega+dd^c\varphi_j)^n\to MA(\varphi)$, in the weak sense of measures. We set $(\omega+dd^c\varphi)^n:=MA(\varphi)$. This domain includes all the classes in which the operator was defined earlier, either as a consequence of the local theory [BT1, BT2, Ce, Bl1, Bl2], or genuinely in the compact setting (the class $\mathcal{E}$ from [GZ2]).

Theorem 2.4. *For every $\varphi\in DMA(\mathbb{P}^n,\omega)$ there exist a sequence of integers $d_j\nearrow\infty$ and n-tuples $(P_{j,1},\dots,P_{j,n})$ of homogeneous polynomials of degree d_j on $\mathbb{C}^{n+1}$ such that for each j the set $\{P_{j,1}=0\}\cap\ldots\cap\{P_{j,n}=0\}\subset\mathbb{P}^n$ is finite and the measures $d_j^{-n}[P_{j,1}=0]\wedge\ldots\wedge[P_{j,n}=0]$ converge weakly on $\mathbb{P}^n$ to $(\omega+dd^c\varphi)^n$.*

This follows immediately by applying Theorem 1.2 to the singular metric on L determined by φ.

3. Equidistribution for sections of adjoint bundles

We return here to a general framework analogous to [CM1] and consider sections of adjoint bundles. The benefit is that the analysis of the Bergman kernel becomes easier, since its definition doesn't depend in this case on the ground metric and L^2 estimates do not involve the Ricci curvature. This is especially useful when dealing with singular spaces or metrics and was already done for orbifolds in [CM2]. We consider the following setting:

(A) X is a complex manifold of dimension n (not necessarily compact), Σ is a compact analytic subvariety of X, and Ω is a smooth positive $(1,1)$ form on $X\setminus\Sigma$.

(B) (L,h) is a holomorphic line bundle on X with a singular Hermitian metric h with positive curvature current $c_1(L,h)\geq0$ on X.

Consider the space $H^0_{(2)}(X\setminus\Sigma,L^p\otimes K_X)$ of L^2-holomorphic sections of $L^p\otimes K_X\mid_{X\setminus\Sigma}$ relative to the metrics h_p on L^p induced by h, h^{K_X} on $K_X\mid_{X\setminus\Sigma}$ induced by Ω and the volume form Ω^n on $X\setminus\Sigma$, endowed with the inner product

$$(S,S')_p=\int_{X\setminus\Sigma}\langle S,S'\rangle_{h_p,\Omega}\,\Omega^n\,,\quad S,S'\in H^0_{(2)}(X\setminus\Sigma,L^p\otimes K_X).$$

The interesting point is that the space $H^0_{(2)}(X\setminus\Sigma,L^p\otimes K_X)$ does not depend on the choice of the form Ω. Indeed, for any $(n,0)$-form S with values in L^p, and any metrics Ω, Ω_1 on $X\setminus\Sigma$, we have pointwise $|S|^2_{h_p,\Omega}\Omega^n=|S|^2_{h_p,\Omega_1}\Omega_1^n$. Therefore, we can take Ω to be a smooth positive (1,1) form on X. Since $c_1(L,h)\geq0$, the metric h has psh local weights, so it is locally bounded below away from zero. Thus sections $S \in H^0_{(2)}(X \setminus \Sigma, L^p \otimes K_X)$ are locally integrable on X (with respect to the Lebesgue measure). Skoda's lemma [MM1, Lemma 2.3.22] shows that sections in $H^0_{(2)}(X \setminus \Sigma, L^p \otimes K_X)$ extend holomorphically to X, therefore $H^0_{(2)}(X \setminus \Sigma, L^p \otimes K_X) \subset H^0(X, L^p \otimes K_X)$ and

$$\begin{aligned} H^0_{(2)}(X \setminus \Sigma, L^p \otimes K_X) &= H^0_{(2)}(X, L^p \otimes K_X) \\ &= \Big\{ S \in H^0(X, L^p \otimes K_X) : \int_X |S|^2_{h_p, \widetilde{\Omega}} \, \widetilde{\Omega}^n < \infty \Big\}, \end{aligned}$$

where $\widetilde{\Omega}$ is any smooth positive $(1,1)$ form on X.

Let $d_p := \dim H^0_{(2)}(X, L^p \otimes K_X) \in \mathbb{N} \cup \{\infty\}$ and let $\{S^p_j\}_j$ be an orthonormal basis of $H^0_{(2)}(X, L^p \otimes K_X)$. Denote by P_p the Bergman kernel function defined by

$$P_p(x) = \sum_{j=1}^{d_p} |S^p_j(x)|^2_{h_p,\Omega} \,, \quad |S^p_j(x)|^2_{h_p,\Omega} := \langle S^p_j(x), S^p_j(x) \rangle_{h_p,\Omega} \,, \ x \in X \,. \tag{5}$$

We denote by γ_p the Fubini-Study current of the space $H^0_{(2)}(X, L^p \otimes K_X)$, defined by

$$\gamma_p \,|_U = \frac{1}{2} \, dd^c \log \Big(\sum_{j=1}^{d_p} |s^p_j|^2 \Big), \ U \subset X \ \text{open}\,, \tag{6}$$

where $S^p_j = s^p_j \, e^{\otimes p} \otimes e'$, and e, e' are local holomorphic frames for L, K_X on U. As in [CM1, Lemma 3.2] (see also [MM2, (3.48)]) we see that the currents γ_p are independent of the choice of basis $\{S^p_j\}$ and we have

$$\frac{1}{p} \, \gamma_p = c_1(L,h) + \frac{1}{p} \, c_1(K_X, h^{K_X}) + \frac{1}{2p} \, dd^c \log P_p \,.$$

By taking Ω to be smooth on X, we have $\frac{1}{p} \, c_1(K_X, h^{K_X}) \to 0$ locally uniformly as $p \to \infty$.

We denote by $[S = 0]$ the current of integration (with multiplicities) over the analytic hypersurface $\{S = 0\}$ determined by a nontrivial section $S \in H^0(X, L^p \otimes K_X)$. The Lelong-Poincaré equation reads

$$\frac{1}{p} \, [S = 0] = c_1(L,h) + \frac{1}{p} \, c_1(K_X, h^{K_X}) + \frac{1}{2p} \, dd^c \log |S|^2_{h_p,\Omega} \,.$$

When X is compact, let $\mathcal{S}^p$ be the unit sphere of $(H^0_{(2)}(X, L^p \otimes K_X), \|\cdot\|_{L^2})$ and let λ_p be the normalized surface measure on $\mathcal{S}^p$. We denote by λ^k_p the product measure on $(\mathcal{S}^p)^k$. We also consider the probability space $\mathcal{S}_\infty = \prod_{p=1}^\infty \mathcal{S}^p$ endowed with the probability measure $\lambda_\infty = \prod_{p=1}^\infty \lambda_p \,$.

Theorem 3.1. *Let X, Σ, Ω, (L,h) satisfy (A), (B) and assume that there exists an open set $G \subset X$ such that $c_1(L,h)$ is strictly positive on $G \setminus \Sigma$, i.e. $c_1(L,h) \geq \varepsilon \, \Omega$ on $G \setminus \Sigma$, for some continuous function $\varepsilon : G \setminus \Sigma \longrightarrow (0, +\infty)$.*
(i) Assume that $X \setminus \Sigma$ admits a complete Kähler metric. Then $\frac{1}{p} \gamma_p \to c_1(L,h)$ weakly as currents on G and $\frac{1}{p} \log P_p \to 0$ in $L^1_{loc}(G, \Omega^n)$, as $p \to \infty$.

(ii) Assume that X is a compact Kähler manifold. Then $\frac{1}{p}\,[\sigma_p=0]\to c_1(L,h)$ weakly on G as $p\to\infty$, for λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq 1}\in\mathcal{S}_\infty$.

Proof. To prove (i), we repeat the proof of [CM2, Theorem 1.1], by replacing the orbifold regular locus X^{orb}_{reg} in loc. cit. by $X\setminus\Sigma$. The proof of (ii) is analogous to the proof of [CM2, Theorem 1.6]. ☐

Consider now the following condition:

(B′) (L,h) is a holomorphic line bundle on X with a singular Hermitian metric h on X such that h is *continuous* on $X\setminus\Sigma$ and $c_1(L,h)\geq 0$ on X.

Proceeding as in the proof of Theorem [CM1, Theorem 1.1] we obtain the following:

Theorem 3.2. *Let $X,\,\Sigma,\,\Omega,\,(L,h)$ satisfy (A), (B′) and assume that*

$$\lim_{p\to\infty}\frac{1}{p}\,\log P_p(x)=0,\ \textit{locally uniformly on } X\setminus\Sigma\,. \tag{7}$$

Then $\frac{1}{p}\,\gamma_p\to c_1(L,h)$ weakly on X. If, in addition, $\dim\Sigma\leq n-k$ for some $2\leq k\leq n$, then the currents γ^k and γ^k_p are well defined on X, respectively on each relatively compact neighborhood of Σ, for all p sufficiently large. Moreover, $\frac{1}{p^k}\,\gamma^k_p\to\gamma^k$ weakly on X.

By slight modifications of the proofs of [CM1, Theorems 1.2, 4.3] we obtain:

Theorem 3.3. *Let $X,\,\Sigma,\,\Omega,\,(L,h)$ satisfy (A), (B′). Assume that X is compact, $\dim\Sigma\leq n-k$ for some $1\leq k\leq n$, and that (7) holds. Then, for all p sufficiently large:*

(i) $[\sigma=0]:=[\sigma_1=0]\wedge\ldots\wedge[\sigma_k=0]$ is a well defined positive closed current of bidegree (k,k) on X, for λ^k_p-a.e. $\sigma=(\sigma_1,\ldots,\sigma_k)\in(\mathcal{S}^p)^k$.

(ii) The expectation $E^k_p[\sigma=0]$ of the current-valued random variable $\sigma\to[\sigma=0]$, given by $\langle E^k_p[\sigma=0],\varphi\rangle=\int_{(\mathcal{S}^p)^k}\langle[\sigma=0],\varphi\rangle\,d\lambda^k_p$, where φ is a test form on X, is a well defined current and $E^k_p[\sigma=0]=\gamma^k_p$.

(iii) We have $\frac{1}{p^k}\,E^k_p[\sigma=0]\to\gamma^k$ as $p\to\infty$, weakly in the sense of currents on X.

Theorem 3.4. *Let $X,\,\Sigma,\,\Omega,\,(L,h)$ satisfy (A), (B′) such that X is compact and (7) holds. Then $\frac{1}{p}\,[\sigma_p=0]\to c_1(L,h)$ as $p\to\infty$ weakly on X, for λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq 1}\in\mathcal{S}_\infty$.*

Taking advantage of the fact that we work with adjoint bundles, we apply in the next result the analysis already used in [CM2] (especially the resolution of the $\overline{\partial}$-equation on complete Kähler manifolds, cf. [CM2, §4.2-3]).

Theorem 3.5. *Let $X,\,\Sigma,\,\Omega,\,(L,h)$ satisfy (A), (B′) and let $k\leq\operatorname{codim}\Sigma$. Assume that $X\setminus\Sigma$ admits a complete Kähler metric and $c_1(L,h)$ is strictly positive on $X\setminus\Sigma$, i.e. $c_1(L,h)\geq\varepsilon\,\Omega$ on $X\setminus\Sigma$, for some continuous function $\varepsilon:X\setminus\Sigma\longrightarrow(0,+\infty)$. Then (7) holds and the conclusions of Theorem 3.2 hold. If, in addition, X is compact, then $\frac{1}{p}\,[\sigma_p=0]\to c_1(L,h)$ as $p\to\infty$ weakly on X, for λ_∞-a.e. sequence $\{\sigma_p\}_{p\geq 1}\in\mathcal{S}_\infty$. Moreover, the conclusions of Theorem 3.3 hold.*

Proof. The locally uniform convergence (7) follows by showing the estimate [CM2, (12)] as in [CM2, Theorem 4.7], whereby X^{orb}_{reg} in loc. cit. is replaced by $X\setminus\Sigma$. Hence we can apply Theorems 3.3 and 3.4 in this situation. ☐

Proof of Theorem 1.4. We repeat the proof of Theorem 1.1, working now with the spaces $H^0_{(2)}(X, L^p \otimes K_X)$ of L^2-holomorphic sections of adjoint bundles, and using Theorem 3.5 instead of [CM1, Theorem 5.4] in Step 2 of the proof. Note that $X \setminus \Sigma$ admits a complete Kähler metric since X is a compact Kähler manifold (cf. [O]; see also [CM2, Lemma 4.9]). □

Along the same lines as above, there are versions of [CM1, Theorems 6.19, 6.21] for n-forms.

Theorem 3.6. *Let X be a 1-convex manifold and $\Sigma \subset X$ be the exceptional set of X. Let (L, h) be a holomorphic line bundle on X with singular metric h such that $c_1(L, h) \geq 0$ on X, $c_1(L, h)$ is strictly positive on $X \setminus \Sigma$, and $h\mid_{X\setminus\Sigma}$ is continuous. Then* (7) *holds. In particular, the conclusions of Theorem 3.2 hold for the spaces $H^0_{(2)}(X, L^p \otimes K_X)$.*

Proof. Indeed, $X \setminus \Sigma$ admits a complete Kähler metric, so this is an immediate application of Theorem 3.5. □

Remark 3.7. If we suppose only that $c_1(L, h)$ is strictly positive in a neighborhood of Σ, we can modify the metric h by $he^{-A\rho}$, where ρ is the exhaustion function of X and A is an appropriate constant, so that the curvature of the line bundle $(L, he^{-A\rho})$ becomes positive on the whole $X \setminus \Sigma$. Thus $\frac{1}{p}\gamma_p \to c_1(L, he^{-A\rho}) = c_1(L, h) + A\, dd^c\varphi$ as $p \to \infty$ on X.

References

[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*, Invent. Math. **37** (1976), 1–44.

[BT2] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1–40.

[Bl1] Z. Błocki, *On the definition of the Monge-Ampère operator in $\mathbb{C}^2$*, Math. Ann. **328** (2004), 415–423.

[Bl2] Z. Błocki, *The domain of definition of the complex Monge-Ampère operator*, Amer. J. Math. **128** (2006), 519–530.

[BK] Z. Błocki and S. Kołodziej, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc. **135** (2007), 2089–2093.

[Ce] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), 159–179.

[CGZ] D. Coman, V. Guedj, and A. Zeriahi, *Domains of definition of Monge-Ampère operators on compact Kähler manifolds*, Math. Z. **259** (2008), 393–418.

[CM1] D. Coman and G. Marinescu, *Equidistribution results for singular metrics on line bundles*, preprint available at arXiv:1108.5163.

[CM2] D. Coman and G. Marinescu, *Convergence of Fubini-Study currents for orbifold line bundles*, preprint available at arXiv:1210.5604.

[D1] J.-P. Demailly, *Courants positifs extrêmaux et conjecture de Hodge*, Invent. Math. **69** (1982), 347–374.

[D2] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, in *Complex algebraic varieties (Bayreuth, 1990)*, Lecture Notes in Math. 1507, Springer, Berlin, 1992, 87–104.

[D3] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), 361–409.

[D4] J.-P. Demailly, *Monge-Ampère operators, Lelong numbers and intersection theory*, in *Complex analysis and geometry*, Plenum, New York, 1993, 115–193.

[DP] J. P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), 1247–1274.

[DMS] T.-C. Dinh, G. Marinescu and V. Schmidt, *Asymptotic distribution of zeros of holomorphic sections in the non compact setting*, J. Stat. Phys. **148** (2012), 113–136.

[DS] T.-C. Dinh and N. Sibony, *Distribution des valeurs de transformations méromorphes et applications*, Comment. Math. Helv. **81** (2006), 221–258.

[DuS] J. Duval and N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. **79** (1995), 487–513.

[FS] J. E. Fornæss and N. Sibony, *Oka's inequality for currents and applications*, Math. Ann. **301** (1995), 399–419.

[G] V. Guedj, *Approximation of currents on complex manifolds*, Math. Ann. **313** (1999), 437–474.

[GZ1] V. Guedj and A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. **15** (2005), 607–639.

[GZ2] V. Guedj and A. Zeriahi, *The weighted Monge-Ampère energy of quasiplurisubharmonic functions*, J. Func. Anal. **250** (2007), 442–482.

[JS] S. Ji and B. Shiffman, *Properties of compact complex manifolds carrying closed positive currents*, J. Geom. Anal. **3** (1993), 37–61.

[KN] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974, xiv+390 pp.

[MM1] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Math., vol. 254, Birkhäuser, Basel, 2007, xiii, 422 pp.

[MM2] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, Adv. Math. **217** (2008), 1756–1815.

[O] T. Ohsawa, *Hodge spectral sequence and symmetry on compact Kähler spaces*, Publ. Res. Inst. Math. Sci. **23** (1987), 613–625.

[SZ1] B. Shiffman and S. Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Comm. Math. Phys. **200** (1999), 661–683.

[SZ2] B. Shiffman and S. Zelditch, *Number variance of random zeros on complex manifolds*, Geom. Funct. Anal. **18** (2008), 1422–1475.

[S] N. Sibony, *Quelques problèmes de prolongement de courants en analyse complexe*, Duke Math. J. **52** (1985), 157–197.

D. Coman: Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150, USA
E-mail address: dcoman@syr.edu

G. Marinescu: Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany & Institute of Mathematics 'Simion Stoilow', Romanian Academy, Bucharest, Romania
E-mail address: gmarines@math.uni-koeln.de