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Some Results for a Semilinear Elliptic Problem with a Large Parameter

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ABSTRACT

Consider the following eigenvalue problem

$$(P)\quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n, \text{ bounded,} \\ u = 0 & \text{on } \partial\Omega, \text{ smooth,} \end{cases}$$

where f changes sign. In this note we will show results which can be found by using the so-called sweeping principle of Serrin, 1971. Especially we will give estimates for the boundary layer of positive solutions near a zero of f. For some f a solution u will have a free boundary. We show for such f that $f(u)=0$ except near $\partial\Omega$. Next to this we improve a result for existence of a solution.

1. INTRODUCTION

We are interested in pairs $(\lambda,u) \in \mathbb{R}_+ \times C^2(\Omega)$ satisfying (P) and $u > 0$ in Ω. First, note that a solution satisfies $f(\max u) \geq 0$. If $f \in C^1$, the strong maximum principle even shows $f(\max u) > 0$. Secondly, if ρ is a zero of f then $u \equiv \rho$ satisfies the differential equation for all λ. So one could expect the existence of a solution (λ,u), where λ is large and u is near a zero of f (with $f(\max u) \geq 0$) except for a boundary layer. Results for this problem were presented by Fife,1973 and by Clément et al., 1986. The results here are strongly related to this last paper.

Assume that there are two numbers $0 < \rho_1 < \rho_2$ such that

(F1) $\qquad f(\rho_1) = f(\rho_2) = 0$ and $f > 0$ in (ρ_1,ρ_2), (F2) $\quad f \in C^{\gamma}(-\infty,\rho_2] \cap C^1(-\infty,\rho_2)$ and there is $\delta > 0$ such that $f' \leq 0$ in $(\rho_2-\delta,\rho_2)$.

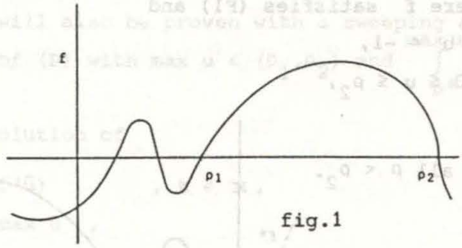


fig.1

In 1981 Hess showed, if $f(0) > 0$, that the following condition is sufficient for existence of a positive solution (λ,u) with $\max u \in (\rho_1,\rho_2)$.

(F3) $\quad J(\rho) := \int_{\rho}^{\rho_2} f(s)ds > 0$ for every $\rho \in [0,\rho_1]$.

In the first theorem, it will be proven that this condition is sufficient and necessary when $f \in C^1[0,\max u]$, even if $f(0) < 0$. In the second theorem we will show that the solutions, which are found in this way, are near ρ_2.

2. THEOREMS AND PROOFS

Before stating the first theorem we will shortly explain the sweeping principle of Serrin, 1971. A formulation can also be found in the paper by Clément et al., 1986.

Fix λ, let u be a solution of (P) and let $\{v(t) \in C(\bar{\Omega});\ t \in [0,1]\}$ be a continuous family of subsolutions, such that $v(0) < u$ in Ω and for all t $v(t) < u$ on $\partial\Omega$ as well as $v(t) < \rho_2$ in Ω. Then $v(t) < u$ in Ω for all $t \in [0,1]$. Since, if there exists $t^* \in [0,1]$ such that $v(t^*) \leq u$ and for some $x^* \in \Omega$ $v(t^*,x^*) = u(x^*)$, the strong maximum principle implies $v(t^*) \equiv u$, a contradiction.

THEOREM 1 :

Let f satisfy (F1)(F2)(F3) and let Ω satisfy a uniform interior sphere condition. Then there exists $c_1 > 0$, $c_2 \in (\rho_1,\rho_2)$ and $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ a positive solution $(\lambda,u(\lambda))$ of (P) exists with

(1) $\quad \min(c_1 . d(x,\partial\Omega) . \lambda^{\frac{1}{2}}, c_2) < u(\lambda) \leq \rho_2$.

Moreover every solution (λ,u) of (P) (not necessarily positive) with $\max u \in (\rho_1,\rho_2)$ satisfies $\int_\rho^{\max u} f(s)ds > 0$ for every $\rho \in [0,\rho_1]$.

PROOF:

Replace f by f^*, where f^* satisfies (F1) and

$$f^*(u) = 1 \quad \text{for } u < -1,$$
$$f^*(u) \le f(u) \text{ for } 0 \le u \le \rho_2,$$
$$f^* \in C^1(\mathbb{R}),$$
$$\int_\rho^{\rho_2} f^*(s)ds > 0 \text{ for all } \rho < \rho_2.$$

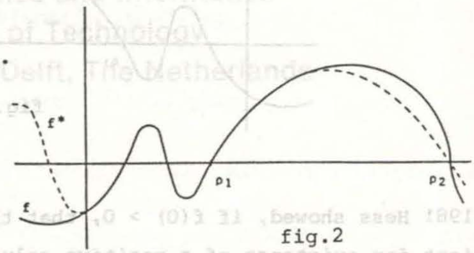


fig.2

Like Hess in 1981, one finds for μ large enough, a minimizer v of $I(v,\mu) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \mu \int_\Omega \int_0^v f^*(s)ds\, dx$ in the cone $\{v \in W^{1,2}(\Omega);\ v > -1 \text{ in } B,\ v = -1 \text{ on } \partial B\}$ with $\max v \in (\rho_1,\rho_2)$. (B denotes the unit ball). Gidas et al. showed in 1979 that v is radially symmetric and $v'(r) < 0$ for $r \in (0,1]$. Let $\theta \in (0,1)$ be the number such that $v(\theta) = 0$. Since Ω satisfies a uniform interior sphere condition, $\Omega = \cup\{B(x,\varepsilon);\ x \in \Omega(\varepsilon)\}$ for all $\varepsilon \in (0,\varepsilon_0)$, where ε_0 is some positive constant and $B(x,\varepsilon) = \{y \in \mathbb{R}^N;\ |x-y| < \varepsilon\}$, $\Omega(\varepsilon) = \{x \in \Omega;\ d(x,\partial\Omega) > \varepsilon\}$. Then $w(\lambda,x) := \sup\{v(\theta.\varepsilon^{-1}.|x-y|);\ y \in \Omega(\varepsilon)\}$, with $\lambda = \mu.(\theta/\varepsilon)^2$, is a subsolution of (P), with f replaced by f^*, for all $\lambda \ge \lambda_0 := \mu.(\theta/\varepsilon_0)^2$. Since $0 < w(\lambda) < \rho_2$ and $f^* \le f$ on $[0,\rho_2]$, $w(\lambda)$ is also a subsolution of the original (P). Note that $W(\lambda) \equiv \rho_2$ is a supersolution of (P) for all λ. By an iteration scheme one shows the existence of a solution in between. By condition (F2) there exist two strictly increasing continuous functions f_1 and f_2 such that $f = f_1 - f_2$ on $[0,\rho_2]$ and $f_2(0) = 0$. Because of (F2) one may assume $f_1 \in C^1[0,\rho_2]$. Define T by $u = T(v)$, where u is the unique solution of

$$\begin{cases} -\Delta u + \lambda f_2(u) = \lambda f_1(v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

See the paper of Brezis et al. from 1973. Define $W_n = T^n(W(\lambda))$ and $w_n = T^n(w(\lambda))$. $\{W_n\}$ and $\{w_n\}$ are sequences of respectively decreasing supersolutions and increasing subsolutions. Since $W_n > w_n$ in Ω the sequences converge to a solution of (P). Standard regularity theory shows that these solutions, or maybe just one solution , are $C^2(\Omega)$. The estimate (1) is valid since the solutions are between $w(\lambda)$ and $W(\lambda)$.

The last part will also be proven with a sweeping argument. Suppose there is a solution of (P) with $\max u \in (\rho_1,\rho_2)$ and $\int_{\rho^*}^{\max u} f(s)ds = 0$ for some $\rho^* \in [0,\rho_1]$.

Let $\bar{u}$ be the solution of

$$
\begin{aligned}
-\bar{u}'' &= \lambda f(\bar{u}) \qquad , t \in \mathbb{R} , \\
\bar{u}(0) &= \max u \quad , \\
\bar{u}'(0) &= 0.
\end{aligned}
$$

Set $U(t,x_1,\dots,x_N) = \bar{u}(x_1-t)$ for $x \in \mathbb{R}^N$.

Note that $\max U = \max u$ and $\inf U \geq \rho^*$. Moreover there exists t^* and $x^* \in \overline{\Omega^*}$, with $\Omega^* = \Omega \cap \{x \in \mathbb{R}^N ; x_1 > t^*\}$, such that

$$U(t^*) \geq u \text{ in } \Omega^*,$$

$$U(t^*,x^*) = u(x^*) \text{ and } \nabla U(t^*,x^*) = \nabla u(x^*).$$

The strong maximum principle shows $U(t^*) \equiv u$, which is a contradiction. For a more detailed proof see the authors paper of 1986. □

THEOREM 2:

Let Ω satisfy an interior sphere condition and let f satisfy (F1) and (F2) with ρ_1 not necessarily positive. If $\rho_1 > 0$ then assume (F3) is also satisfied.

Suppose that $f(u) > c(\rho_2-u)^\alpha$ for $u \in (\rho_2-\delta,\rho_2)$, where $c,\alpha,\delta > 0$. Then there is $C > 0$ such that for any nonnegative $z \in C_0^\infty(\Omega)$, with $\max z \in (\rho_1,\rho_2)$, $\lambda(z) > \lambda_0$ exists for which the following holds.

Let (λ,u) be a solution of (P) with $z \leq u \leq \rho_2$ in Ω and $\lambda > \lambda(z)$.

1) If $0 < \alpha < 1$ then $u(x) \geq \min (C.\lambda^{\frac{1}{2}}.d(x,\partial\Omega),\rho_2)$.

2) If $\alpha = 1$, then $u(x) > \rho_2(1-\exp(-C.\lambda^{\frac{1}{2}}.d(x,\partial\Omega)))$, for $x \in \Omega$.

3) If $1 < \alpha$, then $u(x) > \rho_2(1-(1+C.\lambda^{\frac{1}{2}}.d(x,\partial\Omega))^{-p})$ for $x \in \Omega$, with $p = 2(\alpha-1)^{-1}$.

REMARK 1.

Case 1) shows that a solution near ρ_2 will have a free boundary within a distance of order $\lambda^{-\frac{1}{2}}$ from $\partial\Omega$.

Define $m := \frac{1}{2}(\rho_1 + \max z)$ and $M := v(0)$. Then there exists a ball $B(x^*, r)$, such that $B(x^*, r) \subset \{x \in \Omega;\ z(x) > m\}$, and a constant σ, such that $f(u) > \sigma(u-m)$ for $u \in [m, M]$. By the lemma one finds

$$u(x) > M \text{ for } x \in B(x^*, r-(\sigma.\lambda/\nu)^{-\frac{1}{2}}).$$

When $r-(\sigma.\lambda/\nu)^{-\frac{1}{2}} > \theta.(\lambda/\mu)^{-\frac{1}{2}}$ the first step is finished since

$$u(x) > M \geq v((\lambda/\mu)^{\frac{1}{2}}|x-x^*|) \text{ for } x \in B(x^*, \theta(\lambda/\mu)^{-\frac{1}{2}})$$

Hence set $\lambda(z) = \max(\lambda_0, r^{-2}((\nu/\sigma)^{\frac{1}{2}} + \mu^{\frac{1}{2}})^2)$.

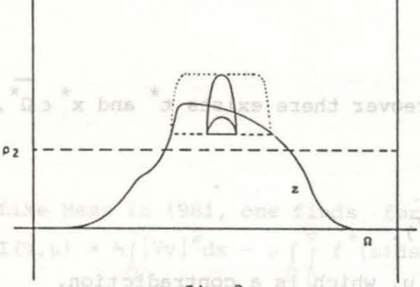


fig.3

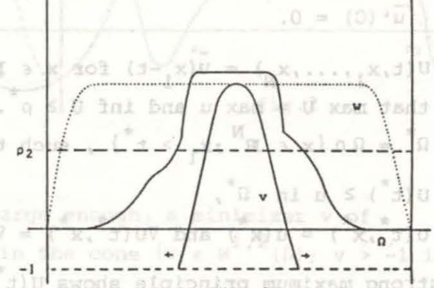


fig.4

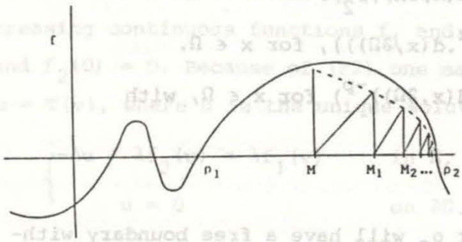
In the second step we prove that a solution (λ, u), with $u \in (w(\lambda), \rho_2]$ and $\lambda > \lambda(z)$, satisfies the statement of the theorem. If $\rho_1 < 0$ set $M = 0$. We may assume that c is such that

$$f(u) > c(\rho_2-u)^\alpha \text{ for } u \in [M, \rho_2].$$

Define $M_k = \rho_2 - 2^{-k}.(\rho_2-M)$

and $\sigma_k = c.2^{-(k+1).(\alpha-1)}.(\rho_2-M)^{\alpha-1}$

Then $f(u) > \sigma_k.(u-M_k)$ for $u \in [M_k, M_{k+1}]$.



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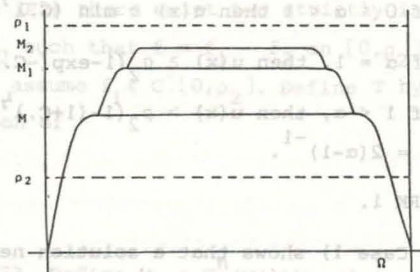

fig.6

REMARK 2.

For the cases ii) and iii) it was proven by Clément et al. in 1986, if $f \in C^{1,\gamma}[0,\rho_2]$ and $\partial\Omega \in C^3$, that there exists a unique solution $u \in [z,\rho_2]$ for every λ large enough.

The key to the proof of theorem 2 will be the following lemma.

LEMMA :

Let (λ,u) be a solution of (P) and let ν be the first eigenvalue of

$$\text{(L)} \quad \begin{cases} -\Delta\psi = \nu.\psi & \text{in } B(0,1), \\ \psi = 0 & \text{on } \partial B(0,1). \end{cases}$$

If $f(u) > \sigma.(u-m)$ for $u \in [m,M]$ and $u > m$ on $B(y,(\sigma.\lambda/\nu)^{-\frac{1}{2}})$ then $u(y) > M$.

PROOF :

Let ψ be the associated eigenfunction of (L) with $\psi(0) = 1$. Define $v(t,x) = m + (t-m).\psi((\sigma.\lambda/\nu)^{\frac{1}{2}}.|x-y|)$ for $x \in B(y,(\sigma.\lambda/\nu)^{-\frac{1}{2}})$.

Then
$$\begin{aligned} -\Delta v(t) &= (t-m).(\sigma.\lambda/\nu).(-\Delta\psi) = \\ &= \lambda.\sigma.(t-m)\psi = \\ &= \lambda.\sigma.(v(t)-m) < \lambda f(v(t)) \qquad \text{for } t \in [m,M]. \end{aligned}$$

Since $v(t,x) = m < u(x)$ for $x \in \partial B(y,(\sigma\lambda/\nu)^{-\frac{1}{2}})$, $v(t)$ is a subsolution of (P) for all $t \in [m,M]$. And since $v(m,x) \equiv m < u(x)$ the sweeping principle shows $v(M,x) < u(x)$ in $B(y,(\sigma\lambda/\nu)^{-\frac{1}{2}})$. Hence $v(M,y) = M < u(y)$. □

PROOF OF THEOREM 2:

In the first step we will show that there exists $\lambda(z)$ such that if (λ,u) is a solution of (P) with $\lambda > \lambda(z)$ and $z < u$ then $u > w(\lambda)$, which is also defined in the proof of theorem 1. If $\rho_1 < 0$ then set $w(\lambda) \equiv 0$. If (F3) is satisfied there exists a radially symmetric solution (μ,v) of

$$\begin{cases} -\Delta v = \mu.f^*(v) & \text{in } B(0,1), \\ v = -1 & \text{on } \partial B(0,1) \end{cases}$$ with $0 \le \max v \in (\rho_1,\rho_2)$. Let σ and ε_0 be as before, and set

$$w(\lambda,x) = \sup\,\{v((\lambda/\mu)^{\frac{1}{2}}|x-y|);\ y \in \Omega(\theta.(\lambda/\mu)^{-\frac{1}{2}})\},$$

which is a positive subsolution of (P) for $\lambda \ge \lambda_0 = \mu.(\theta/\varepsilon_0)^2$. If one can show $u(x) > v((\lambda/\mu)^{\frac{1}{2}}|x-y|)$ for some $y \in \Omega\,(\theta.(\lambda/\mu)^{-\frac{1}{2}})$ then by sweeping and the fact that $\Omega(\theta.(\lambda/\mu)^{-\frac{1}{2}})$ is connected by arc, this inequality holds for all $y \in \Omega(\theta.(\lambda/\mu)^{-\frac{1}{2}})$.

Since $u > w(\lambda)$ one finds that

$$u(x) > M \text{ for } x \in \Omega(\theta.(\lambda/\mu)^{-\frac{1}{2}}).$$

The lemma then yields

$$u(x) > M_1 \text{ for } x \in \Omega((\theta.\mu^{\frac{1}{2}} + (\nu/\sigma_1)^{\frac{1}{2}}).\lambda^{-\frac{1}{2}}).$$

And after applying the lemma n times

$$u(x) > M_n \text{ for } x \in \Omega((\theta.\mu^{\frac{1}{2}} + \nu^{\frac{1}{2}}.\sum_{k=1}^{n}(\sigma_k)^{-\frac{1}{2}}).\lambda^{-\frac{1}{2}}).$$

By the definition of σ_k one finds, if $\alpha \neq 1$, that

$$\sum_{k=1}^{n}(\sigma_k)^{-\frac{1}{2}} = c^{-\frac{1}{2}}.(\tfrac{1}{2}(\rho_2-M))^q.\sum_{k=1}^{n}(\tfrac{1}{2})^{qk} = c^{-\frac{1}{2}}.(\tfrac{1}{4}(\rho_2-M))^q.\frac{1-(\frac{1}{2})^{qn}}{1-(\frac{1}{2})^q}, \text{ with}$$

$$q = \tfrac{1}{2}(1-\alpha).$$

If $\alpha = 1$, then $\sum_{k=1}^{n}(\sigma_k)^{-\frac{1}{2}} = n.c^{-\frac{1}{2}}$.

CASE 1 : $0 < \alpha < 1$.

For every $n \in \mathbb{N}$, $u > M_n$ in $\Omega(C_1.\lambda^{-\frac{1}{2}})$, with $C_1 = \theta.\mu^{\frac{1}{2}} + \nu^{\frac{1}{2}}.c^{-\frac{1}{2}}(\frac{1}{4}(\rho_2-M))^q.$ $.(1-(\frac{1}{2})^q)^{-1}$. Hence $u = \rho_2$ in $\Omega(C_1.\lambda^{-\frac{1}{2}})$, which proves together with $u > w(\lambda)$ the first statement.

CASE 2 : $\alpha = 1$.

For every $n \in \mathbb{N}$, $u > M_n$ in $\Omega((\theta.\mu^{\frac{1}{2}}+n.\nu^{\frac{1}{2}}.c^{-\frac{1}{2}}).\lambda^{-\frac{1}{2}})$. The inequality $u > M_n$ is equivalent with

$$\rho_2 - u < (\rho_2-M).\exp(-n\ln 2).$$

By setting $n = [(c\lambda/\nu)^{\frac{1}{2}}.d(x,\partial\Omega) - \theta.(c\mu/\nu)^{\frac{1}{2}}]$, where $[.]$ denotes the integer function, one finds

$$\rho_2-u < (\rho_2-M)\exp(-\ln 2.((c\lambda/\nu)^{\frac{1}{2}}.d(x,\partial\Omega) - \theta(c\mu/\nu)^{\frac{1}{2}}-1)).$$

Together with $u > w(\lambda)$ this proves the second statement.

CASE 3 : $\alpha > 1$ (hence $q = \frac{1}{2}(1-\alpha) < 0$).

For every $n \in \mathbb{N}$, $u > M_n$ in $\Omega((\theta.\mu^{\frac{1}{2}}+C_1.2^{-qn})\lambda^{-\frac{1}{2}})$ with $C_1 = \nu^{\frac{1}{2}}.c^{-\frac{1}{2}}.$ $.(\frac{1}{4}(\rho_2-M))^q.(2^{-q}-1)^{-1}$. Then $u(x) > \rho_2 - 2(\rho_2-M).(C_1^{-1}.\lambda^{\frac{1}{2}}.d(x,\partial\Omega)-C_1^{-1}.\theta.\mu^{\frac{1}{2}})^{-p}$ with $p = (-q)^{-1} = 2.(\alpha-1)^{-1}$. Together with $u > w(\lambda)$ this proves the third statement. □

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