# On Quasiconvexity, Rank-One Convexity and Symmetry

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*Subject of this note are partial answers to the question whether rank–one convexity implies quasiconvexity. This problem has been an open one for more than 35 years and has resisted many attempts of solving it completely* [8,9,2,5]. *Our results indicate why it is so hard to find a possible counterexample. Moreover, we show some domains with a triangulation for which the condition for quasiconvexity restricted to piecewise linear testfunctions is implied by rank–one convexity. Also we give a simple triangulation for which this implication is not yet known. It might help to find a rank–one convex function that is not quasiconvex.*

## Introduction.

To explain the question under investigation we have to introduce some notation. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a vector–valued function. Our main result will be stated for the case m=n=2. We denote the Jacobian of $\varphi$, if it exists, by

$$\nabla\varphi = \begin{bmatrix} \frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial\varphi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial\varphi_m}{\partial x_1} & \cdots & \frac{\partial\varphi_m}{\partial x_n} \end{bmatrix}.$$

<u>Definitions</u>:

Let $f : \mathbb{R}^{nm} \to \mathbb{R}$ be a continuous map.
The function f is called <u>rank–one convex</u> iff

$$f(\lambda A+(1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B) \tag{1}$$

The function f is called <u>quasiconvex</u> iff

$$f(A) \le \frac{1}{|D|} \int_D f(A+\nabla\varphi(x))dx \tag{2}$$

for every (constant) matrix $A \in \mathbb{R}^{nm}$, for any $\varphi \in [W_0^{1,\infty}(D)]^m$ and for the cube $D = (-1,1)^n$ with volume $|D| = 2^n$.

Remark 1:
In the definition of quasiconvexity one may replace the cube D by any bounded domain. However, the definition which we gave will be convenient for our purposes.

Remark 2:
Quasiconvexity implies rank–one convexity. The converse implication has been shown to be true under certain additional assumptions, e.g. if f is quadratic or if m=1 or n=1. We refer to [2,5] for a detailed discussion of the state of the art. Our contribution will discuss the case n=2.

Remark 3:
If f is of class $C^1$, then rank–one convexity is equivalent to

$$f(B) - f(A) \ge \frac{\partial f}{\partial p^k_\alpha}(A)(B-A)_{k\alpha}$$

for all $A,B \in \mathbb{R}^{nm}$ with rank $\{A-B\} \le 1$.
Here $(\cdot)_{k\alpha}$ denotes the matrix element in row $k \in \{1,\dots,m\}$ and column $\alpha \in \{1,\dots,n\}$. Moreover we use the summation convention.
If f is of class $C^2$, then rank–one convexity is equivalent to the Legendre – Hadamard condition

$$\frac{\partial^2 f}{\partial p^i_\alpha \partial p^j_\beta} \lambda_i \lambda_j \mu_\alpha \mu_\beta \ge 0$$

for any $\lambda \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$.

# Analytical results.

Theorem 1:

Suppose that n=2 and that f is continuous and rank–one convex. Then inequality (2) holds for every function $\varphi \in [W_0^{1,\infty}(D)]^m$ which satisfies

$$\varphi(x_1,x_2) = \tilde{\varphi}(x_1,x_2) := \varphi(-x_1,x_2). \tag{4}$$

Remark 4:

Notice that Theorem 1 does not state that f is quasiconvex. However, Corollary 1 below states that the quasiconvexity can be established under a structural assumption (H) on f. While previous attempts to disprove that rank–one convexity implies quasiconvexity have shown that a counterexample would have to use a complicated matrix function f, our Theorem 1 indicates that a counterexample cannot be constructed with a relatively simple testfunction $\varphi$, that is, a testfunction with some symmetry.

Remark 5:

The restriction to the case n=2 is of a technical nature. It seems possible to prove the result for higher dimensions under additional symmetry assumptions on f. Since an extension to higher dimensions does not provide any additional insight, we restrict ourselves to n=2.

Hypothesis (H):

We say that f satisfies (H) if for every $\Lambda \in \mathbb{R}^{2m}$ and $\varphi \in [W_0^{1,\infty}(B)]^m$ there is some reflection R with R0 = 0, such that

$$\int_B f(\Lambda+\nabla\varphi)dx = \int_B f(\Lambda+\nabla(R\varphi))dx\ , \tag{5}$$

where B is a ball in $\mathbb{R}^2$ with center 0.

Remark 6:

At present it is not clear to us, how strong an assumption (H) is.

If m=2 and if $f(\Lambda)=\langle M\Lambda,\Lambda\rangle$, where M is a real 2×2 matrix and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^4$, then a simple calculation shows that property (H) holds. Therefore Corollary 1 below provides a new proof of the known result, that for such an f quasiconvexity and rank–one convexity are equivalent. On the other hand there are even polyconvex functions f, e.g. f(A)=det A, for which property (H) fails. Polyconvexity implies quasiconvexity. See [5].

Corollary 1:

Suppose that n=2 and that f is rank–one convex and satisfies (H). Then f is quasiconvex.

Proof of Corollary 1:

Let $\varphi \in [W_0^{1,\infty}(D)]^m$. Because of hypothesis (H) there exists a reflexion R such that

$$\int_D f(A+\nabla\varphi)dx = \int_{RD} f(A+\nabla R\varphi)dx .$$

Denote the left half of D by $D_\ell = (-1,0) \times (-1,1)$. Without loss of generality we may assume that $R\varphi = \tilde{\varphi}$. By Remark 1 we may even assume that supp $\varphi \subset D_\ell$.

Set $$\psi(x) := \begin{cases} \varphi(x) & \text{in } D_\ell , \\ \tilde{\varphi}(x) & \text{in } D \setminus D_\ell \end{cases}$$

It follows from Theorem 1 that

$$f(A) \leq \frac{1}{|D|} \int_D f(A+\nabla\psi)dx = \frac{1}{|D_\ell|} \int_{D_\ell} f(A+\nabla\varphi)dx .$$

This implies the quasiconvexity of f. □

We will give two proofs of Theorem 1.

Proof of Theorem 1 by analytical methods:

Using the symmetry of $\varphi$ and replacing $-x_1$ by $x_1$ one finds

$$\int_D f(A+\nabla\varphi)dx = \int_D f(A+\begin{bmatrix} \frac{d}{dx_1}\varphi_1(x_1,x_2) & \frac{d}{dx_2}\varphi_1(x_1,x_2) \\ \frac{d}{dx_1}\varphi_2(x_1,x_2) & \frac{d}{dx_2}\varphi_2(x_1,x_2) \end{bmatrix})dx_1dx_2 =$$

$$= \int_D f(A+\begin{bmatrix} \frac{d}{dx_1}\varphi_1(-x_1,x_2) & \frac{d}{dx_2}\varphi_1(-x_1,x_2) \\ \frac{d}{dx_1}\varphi_2(-x_1,x_2) & \frac{d}{dx_2}\varphi_2(-x_1,x_2) \end{bmatrix})dx_1dx_2 =$$

$$= \int_D f(A+\begin{bmatrix} -\partial_1\varphi_1(-x_1,x_2) & \partial_2\varphi_1(-x_1,x_2) \\ -\partial_1\varphi_2(-x_1,x_2) & \partial_2\varphi_2(-x_1,x_2) \end{bmatrix})dx_1dx_2 =$$

$$= \int_D f(A+\begin{bmatrix} -\partial_1\varphi_1(x_1,x_2) & \partial_2\varphi_1(x_1,x_2) \\ -\partial_1\varphi_2(x_1,x_2) & \partial_2\varphi_2(x_1,x_2) \end{bmatrix})dx_1dx_2 . \tag{6}$$

Hence

$$\int_D f(A+\nabla\varphi)dx = \frac{1}{2}\int_D f(A+\begin{bmatrix} -\partial_1\varphi_1 & \partial_2\varphi_1 \\ -\partial_1\varphi_2 & \partial_2\varphi_2 \end{bmatrix})dx + \frac{1}{2}\int_D f(A+\begin{bmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 \end{bmatrix})dx.$$

By the rank–one convexity inequality one finds

$$\int_D f(A+\nabla\varphi)dx \geq \int_D f(A+\begin{bmatrix} 0 & \partial_2\varphi_1 \\ 0 & \partial_2\varphi_2 \end{bmatrix})dx .$$

Since every pair of matrices $A + (0,\partial_2\varphi(x))$ differ by a rank–one matrix, the rank–one convexity implies

$$\begin{aligned} &\int_D f(A+(0,\partial_2\varphi))dx \geq \\ &\geq |D|\ f(\frac{1}{|D|}\int_D (A+(0,\partial_2\varphi))dx) = \\ &= |D|\ f(A) . \end{aligned} \tag{7}$$

Since f is convex on the hyperplane $\{A+\begin{pmatrix} 0 & \lambda \\ 0 & \mu \end{pmatrix};\ \lambda,\ \mu \in \mathbb{R}\}$ one uses Jensen's inequality in the last step. □

Remark 7:
Notice that is if sufficient for this proof to assume $\varphi(x_1,-1) = \varphi(x_1,1)$ for all $x \in [-1,1]$, instead of assuming $\varphi = 0$ everywhere on the boundary.

# Discrete Results.

Proof of Theorem 1 by using finite elements:

It is known, see [3,7], that a function $V \in W_0^{1,\infty}(D)$ can be approximated in the $\|\cdot\|_{1,\infty}$–norm by functions $V_h \in C_0(D)$ which are linear on each triangle of a mesh of sufficiently small triangles. For the proof it will be convenient to choose a fishbone mesh as in Figure 1 which is symmetric with respect to $\{x_1=0\}$ and which has 4NM congruent triangles. Notice that the mesh–lengths are not essential. Therefore, without loss of generality, we may set the mesh length equal to one. Now if suffices to prove inequality (2) for those functions $\psi \in [C((-N,N)\times(0,M))]^m = [C(D)]^m$ which satisfy

$$\psi \text{ is linear on each triangle,} \tag{8}$$

$$\psi(x_1,0) = \psi(x_1,M), \tag{9}$$

$$\psi(x_1,x_2) = \psi(-x_1,x_2). \tag{4}$$

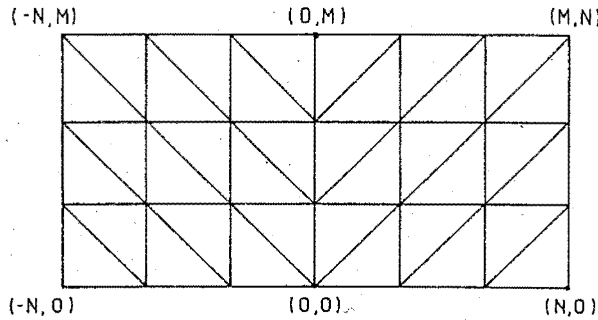



Figure 1.

For $n \in \{1,\dots,N\}$ let $(S_n)$ be the following statement:

$$f(A) \le \frac{1}{2Mn} \int_0^M \int_{-n}^{n} f(A+\nabla\psi(x_1,x_2))dx_1dx_2$$

for all $A \in \mathbb{R}^{2m}$ and $\psi$ satisfying (8)(9)(4).

The validity of $(S_N)$ implies Theorem 1. We intend to verify $(S_N)$ by induction with respect to $n \in \{1,\dots,N\}$. Step 1 will be that property $(S_1)$ holds. This will be shown in Lemma 1 below. For step 2 we suppose that Property $(S_n)$ holds. Then a direct computation which uses $(S_1)$ and $(S_n)$ gives

$$\int_0^M \int_{-n-1}^{n+1} f(A+\nabla\psi(x_1,x_2))dx_1dx_2 =$$

$$= \int_0^M \Big( \int_{-n-1}^{-n} + \int_n^{n+1} \Big) f(A+\nabla\psi(x_1,x_2))dx_1dx_2 + \int_0^M \int_{-n}^{n} f(A+\nabla\psi(x_1,x_2))dx_1dx_2 =$$

$$= \int_0^M \int_{-1}^{1} f(A+\nabla\psi(x_1+(N-1)\mathrm{sign}(x_1),x_2)dx_1dx_2 + \int_0^M \int_{-n}^{n} f(A+\nabla\psi(x_1,x_2))dx_1dx_2 \geq$$

$$\geq 2M\, f(A) + 2nM\, f(A) = 2(n+1)M\, f(A).$$

This proves $(S_{n+1})$ and completes the proof of Theorem 1. □

Lemma 1:
Let $\Omega = (-1,1) \times (0,M)$ and $\Omega_i = (-1,1) \times (i,i+1)$, $i \in \{0,...,M-1\}$.
Suppose that $\psi \in [C(\Omega)]^m$ satisfies (8)(9)(4).
Then $(S_1)$ holds, i.e.

$$f(A) \leq \frac{1}{2M} \int_\Omega f(A+\nabla\psi)dx \qquad \text{for all } A \in \mathbb{R}^{2m}.$$

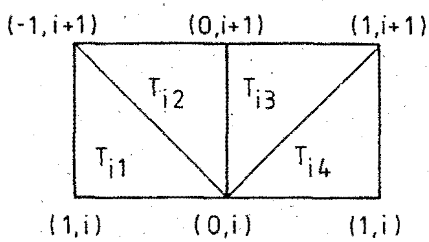



Figure 2.

Proof of Lemma 1:
We introduce some Notation. Let $T_{ij}$ denote the triangles in $\Omega_i$ as depicted in Figure 2, $i \in \{0,...,M\}$, $j \in \{1,...,4\}$. For ease of writing we use the notation $\varphi_i = \psi(0,i)$, $\omega_i = \psi(1,i)$. Then one directly calculates, using (4),

$$\nabla\psi = \begin{cases} (\varphi_i - \omega_i, \omega_{i+1} - \omega_i) & \text{im } T_{i1}, \\ (\varphi_{i+1} - \omega_{i+1}, \varphi_{i+1} - \varphi_i) & \text{im } T_{i2}, \\ (\omega_{i+1} - \varphi_{i+1}, \varphi_{i+1} - \varphi_i) & \text{im } T_{i3}, \\ (\omega_i - \varphi_i, \omega_{i+1} - \omega_i) & \text{im } T_{i4}. \end{cases}$$

We observe that $\nabla\psi$ on $T_{i1}$ (resp. $T_{i2}$) differs by rank one from $\nabla\psi$ on $T_{i4}$ (resp. $T_{i3}$).

Therefore we may use the rank–one convexity of f to obtain

$$\int_{\Omega_i} f(A+\nabla\psi)dx = \sum_{j=1}^{4} \frac{1}{2} f(A+\nabla\psi(T_{ij})) =$$

$$= \frac{1}{2} f(A+\nabla\psi(T_{i1})) + \frac{1}{2} f(A+\nabla\psi(T_{i4})) + \frac{1}{2} f(A+\nabla\psi(T_{i2})) + \frac{1}{2} f(A+\nabla\psi(T_{i3})) \geq$$

$$\geq f(A+(0,\omega_{i+1}-\omega_i)) + f(A+(0,\varphi_{i+1}-\varphi_i)) \geq$$

$$\geq 2f(A+(0,\tfrac{1}{2}(\varphi_{i+1}-\varphi_i+\omega_{i+1}-\omega_i)).$$

A repeated use of the rank–one convexity of f allows us to sum over i:

$$\int_{\Omega} f(A+\nabla\varphi) = 2 \sum_{i=0}^{M-1} f(A+(0,\tfrac{1}{2}(\varphi_{i+1}-\varphi_i+\omega_{i+1}-\omega_i)) \geq$$

$$\geq 2M\ f(A+(0,\tfrac{1}{2M}(\varphi_M-\varphi_0+\omega_M-\omega_0)).$$

Now we observe that $\varphi_M = \varphi_0$ and $\omega_M = \omega_0$ due to (9).

This completes the proof of Lemma 1. □

In the proof of Theorem 1 we showed

a) that condition (H) allows us to work on a rather "narrow" mesh, and

b) that on a narrow mesh rank–one convexity implies quasiconvexity.

In the remainder of this note we shall drop assumption (H) and try to enlarge the mesh. Recall that one has to show

$$f\left(\sum_{i=1}^{N} \lambda_i A_i\right) \leq \sum_{i=1}^{N} \lambda_i f(A_i) \tag{10}$$

for constant matrices $A_i = A + \nabla\psi$ on triangles, where $\psi$ satisfies (8) and $\psi = 0$ on $\partial D$, and where $\lambda_i \geq 0$, $\sum_{i=1}^{N} \lambda_i = 1$.

If (10) were true for arbitrary matrices $A_i \in \mathbb{R}^{nm}$, then f would even be convex. One set of assumptions on $(\lambda_i, A_i)$ which makes sure that rank–one convexity implies the validity of (10) was described by Dacorogna in [4,5]. Here is another result of this type.

Theorem 2:

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be of class $C^1$ and suppose that $(\lambda_1, A_i)$ satisfy $\sum_{j=1}^{N} \lambda_j = 1$ and

$$\operatorname{rank}\{A_i - \sum_{j=1}^{N} \lambda_j A_j\} \le 1 \qquad \text{for } i = 1,\dots,N. \tag{11}$$

Then (10) follows from rank–one convexity.

Remark 8: if $N = 2$ (10) is just the inequality from rank–one–convexity.

Proof: To prove the converse we set $A = \sum_{j=1}^{N} \lambda_j A_j$ and use Remark 3 and (11) to obtain

$$f(A_i) - f(A) \ge \frac{\partial f}{\partial p_\alpha^k}(A)(A_i - A)_{k\alpha}. \tag{12}$$

Now we multiply (12) by $\lambda_i$ and sum over i to obtain (10). □

Remark 9:

Theorem 2 could be used for an alternate proof of Theorem 1 as follows. Set $2A_j$ = sum of $(A+\nabla\psi)$ on two symmetric triangles. Then rank $\{A_j\} = 1$ and rank $\{A\} = 0$. Therefore Theorem 2 applies, provided f is differentiable.

We have attempted to verify inequality (10) for more general meshes, in particular without symmetry assumption (H). The only essential case that we could treat is depicted in Figure 3 below:

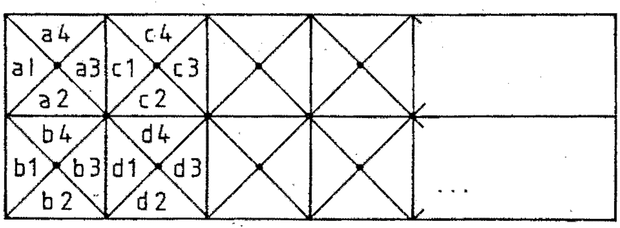



Figure 3.

Notice that the meshes in Figure 4 can be treated as special cases of Figure 3.

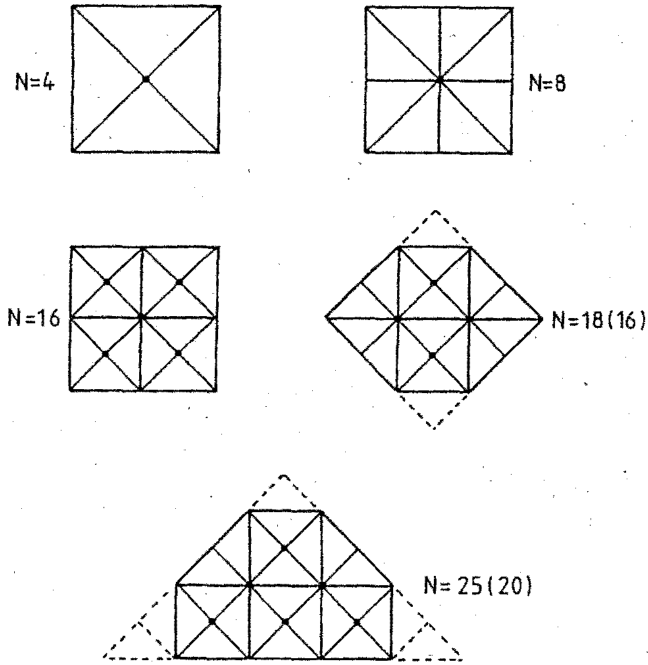



Figure 4.

Ball [1,p.355] and Dacorogna [6] have expressed doubts that (10) holds for N = 16. The configuration of 16 triangles which they had in mind was a triangulation which cannot be considered a special case of Figure 3.
For the meshes in Figure 5, with N = 11 and N = 10, we could not find a direct way to derive (10) from rank–one convexity. Maybe it will be possible to use this mesh in order to find a counterexample.

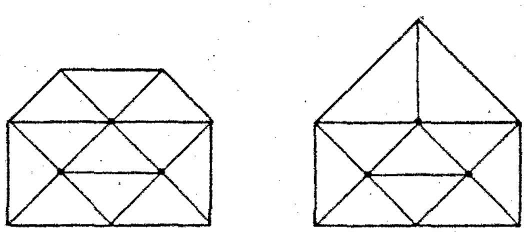


Figure 5.

Theorem 3:

Suppose that a rectangle $D \subset \mathbb{R}^2$ is divided into 8N triangles according to Figure 3 and suppose f is continuous and rank–one convex. Then the quasiconvexity inequality (2) holds for functions $\psi \in [C(D)]^m$ which are linear on each triangle and which vanish on $\partial D$.

For the proof we suppose that $a_i, b_i$ etc denotes $A + \nabla\psi$ in the corresponding triangle. Then a repeated use of rank–one convexity in the right order leads to

$$\int_D f(A+\nabla\psi)dx = \frac{1}{8N}\{f(a_1)+f(a_2)+f(a_3)+f(a_4)+f(b_1)+...\} \geq$$

$$\geq \frac{1}{4N}\{f(\tfrac{1}{2}(a_1+a_2))+f(\tfrac{1}{2}(a_3+a_4))+f(\tfrac{1}{2}(b_1+b_2))+...\} \geq$$

$$\geq \frac{1}{2N}\{f(\tfrac{1}{4}(a_1+...+a_4))+f(\tfrac{1}{4}(b_1+...+b_4))+...\} \geq$$

$$\geq \frac{1}{N}\{f(\tfrac{1}{8}(a_1+...+a_4+b_1+...+b_4))+f(\tfrac{1}{8}(c_1+...+d_4))+...\} \geq$$

$$\geq f(\tfrac{1}{8N}(a_1+...+d_4+...)).$$

In the last step we have used the fact that the mean values over opposing squares differ by a rank–one matrix (with vanishing second column). □

Although the meshes which are considered up till now look quite simple, it is possible to prove inequality (2) on a more complicated mesh without a symmetry assumption.

Lemma 2:

Suppose f is continuous and rank–one convex. Let $D \subset \mathbb{R}^2$ be divided into triangles according to figure 6. Then inequality (2) holds for functions $\psi \in [C(D)]^m$ which are linear on eacht triangle and which vanish on $\partial D$.

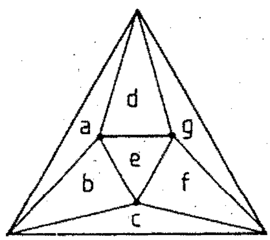
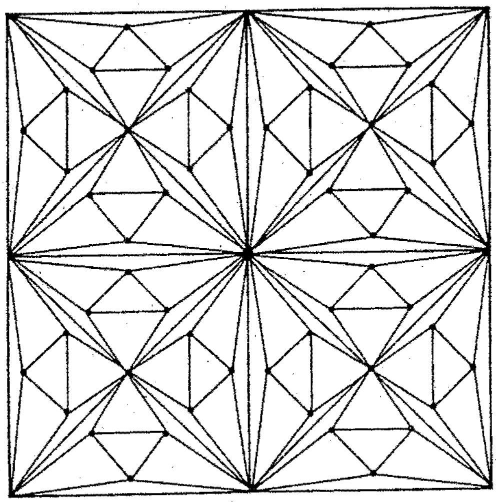



Figure 6.

Proof:
Using the inequality of rank–one convexity in the right order shows inequality (2). Denoted in shorthand:

$$(((a+d)+(b+e))+(c+f)) + g. \qquad \square$$

Finally, by combining the results of Theorem 3 and Lemma 2 one can even prove the statement of Lemma 2 for the meshes in figure 7. However it is not possible to approximate every $W_0^{1,\infty}$–function by refining a mesh in such a way.



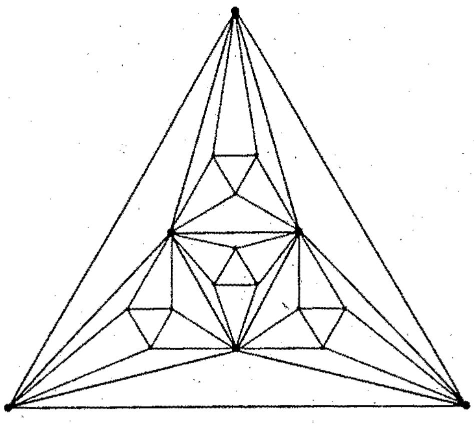


Figure 7.

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