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A sign-changing global minimizer on a convex domain

Introduction: Recently one has established the existence of stable sign-changing solutions for the elliptic problem

\[ \begin{align*}
-\Delta u &= f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*} \]

In [5] there is an example of a sign changing stable solution on a convex domain with \( f(0) \neq 0 \). Matano [2] shows the existence of a sign-changing stable solution even with \( f(0) = 0 \). A next question will be: does a global minimizer have a fixed sign? It has been guessed that the answer is positive if the domain is convex.

In this note we will recall a proof for the ball and give a counterexample for a triangle.

We will assume that \( f \in C^{0,1} \), \( \Omega \) is bounded with \( \partial \Omega \in C^{0,1} \) and that \( (1) \) has a solution \( u \) that minimizes the energy functional \( J \). This functional \( J : H^1_0(\Omega) \to \mathbb{R} \) is defined by

\[ J(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} F(v) \, dx, \]

where \( F(s) = \int_0^s f(t) \, dt \). It is classical that if \( J(u) = \min \{ J(v) ; v \in H^1_0(\Omega) \} \), then \( u \) is a \( C^2(\Omega) \cap C(\Omega) \) solution,

\[ J'(u)v := \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\Omega} f(u) v \, dx = 0 \quad \text{for all } v \in H^1_0(\Omega) \]

and

\[ J''(u)(v) := \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} f'(u)v^2 \, dx \geq 0 \quad \text{for all } v \in H^1_0(\Omega). \]

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Proposition 1: If $f$ is antisymmetric or $\Omega$ is a ball in $\mathbb{R}^n$, then the global minimizer $u$ has a fixed sign.

Remark 1: The result that a (local) minimizer cannot change sign on a ball is due to Lin and Ni, [1]. In their unpublished preprint they also prove the result for $\Omega$ being the difference of two balls with the same center. We will sketch their proof.

Proof. i) If $f$ is antisymmetric, $f(-s) = -f(s)$, then $J(|u|) = J(u)$. Hence $|u|$ is a minimizing solution and $|u| \in C^2(\Omega)$. It follows from $x \in \Omega$ and $u(x) = 0$, that $\nu u(x) = 0$. Then the strong maximum principle shows that $u \equiv 0$. Hence $u$ has a fixed sign.

ii) Suppose $\Omega$ is a ball with center 0. Then differentiate the solution $u$ in a tangential direction, that is, apply $\frac{d}{d\theta} = x_1 \frac{d}{dx_1} - x_j \frac{d}{dx_j}$. Since $\frac{d}{d\theta}$ and $\Delta$ commute, the function $\varphi = \frac{d}{d\theta} u$ satisfies $-\Delta \varphi = f'(u) \varphi$ in $\Omega$. Moreover $\varphi = 0$ on $\partial \Omega$. Then either $\varphi = 0$ or $\varphi$ is an eigenfunction (with eigenvalue 0) of

$$
\begin{bmatrix}
-\Delta v - f'(u)v - \lambda v \\
\nu \varphi = 0
\end{bmatrix}
in \Omega,
\varphi = 0
\text{ on } \partial \Omega.
$$

From (4) one finds that all eigenvalues, except maybe the first, are strictly positive. Hence $\varphi$ is a multiple of the first eigenfunction. If $\varphi$ is nonzero this shows $\varphi$ has a fixed sign, which contradicts $\int_0^{2\pi} \varphi \, d\theta = 0$.

Since this holds for all $i$ and $j$, $u$ is radially symmetric. Now suppose $u = u(r)$ changes sign; then there is a positive number $r_0$ such that $u_+(r_0) = 0$. Set

$$
\nu(r) = \begin{cases}
u_+(r) & \text{for } r < r_0, \\
0 & \text{for } r \geq r_0.
\end{cases}
$$

Then $\nu \in H^1_0(\Omega)$ and

$$
0 \leq J'(u)(\nu) = \int_{|x| < r_0} \left( |\nu u_+|^2 - f'(u) u_+^2 \right) dx = \int_{|x| < r_0} u_+(-\Delta u_+ - f'(u) u_+) dx = -(n-1) \int_{|x| < r_0} r^{-2} u_+^2 dx,
$$

which gives a contradiction for nonconstant $u$. $\Box$

232
Proposition 2: There is $f \in C^{0,1}(\mathbb{R})$, with $f(0) = 0$, and $\Omega \subset \mathbb{R}^2$, bounded and convex, such that the global minimizer changes sign.

Remark 2: In this note we will construct just one example. A forthcoming paper of Matano will certainly have a more rigorous approach to sign-changing stable solutions. However, it is not clear if this considers global minimizers.

Remark 3: Without the condition $f(0) = 0$ one can modify the example in [5] to obtain the result of proposition 2.

Proof: Set $\Omega = \{(x_1, x_2) \in \mathbb{R}^2; 2|x_2| < x_1 < 1\}$ and define the Lipschitz-continuous functions $f_\varepsilon$ for $\varepsilon > 0$ by

\begin{align*}
  f_\varepsilon(s) &= 0 \quad \text{on } (-\infty, -2\varepsilon], \\
  f_\varepsilon(s) &= -\varepsilon^{-2}(s+2\varepsilon) \quad \text{on } (-2\varepsilon, -\varepsilon], \\
  f_\varepsilon(s) &= \varepsilon^{-2}s \quad \text{on } (-\varepsilon, 0], \\
  f_\varepsilon(s) &= s \quad \text{on } (0, 2], \\
  f_\varepsilon(s) &= 4 - s \quad \text{on } (2, 4], \\
  f_\varepsilon(s) &= 0 \quad \text{on } (4, \infty].
\end{align*}

Note that $f(s) = -2\varepsilon f(-\varepsilon s)$ for $s > 0$. 

\begin{center}
\begin{tikzpicture}
  \draw[->] (-2.5,0) -- (2.5,0);
  \draw[->] (0,-2) -- (0,2);
  \draw (0,0) -- (2,2) -- (-2,2) -- cycle;
  \node at (2,2) {$2$};
  \node at (-2,2) {$-2\varepsilon$};
  \node at (0,2) {$-\varepsilon$};
  \node at (0,0) {$0$};
  \node at (4,2) {$2$};
  \node at (4,0) {$4$};
\end{tikzpicture}
\end{center}
Let $\lambda_0$ denote the first eigenvalue of

$$
\begin{align*}
-\Delta \varphi &= \lambda \varphi & \text{in } \Omega, \\
\varphi &= 0 & \text{on } \partial \Omega,
\end{align*}
$$

then the bifurcation picture for solutions with fixed sign of

$$
\begin{align*}
-\Delta u &= \lambda f_\varepsilon(u) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
$$

looks as follows.

Since $s^{-1} f_\varepsilon(s)$ is decreasing on $[0,4]$ (and strictly on $[2,4]$) it is known that there is a unique positive solution for every $\lambda > \lambda_0$. See [4].

There is no positive solution for $\lambda < \lambda_0$. Similar arguments hold for negative solutions. Let $U_\lambda$ and $V_\lambda^\varepsilon$ denote the positive, respectively the negative solution of (7) for $\lambda > \lambda_0$.

Let $J_\varepsilon(\lambda,u)$ denote the energy functional for (7), that is

$$J_\varepsilon(\lambda,u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \lambda \int_0^u f_\varepsilon(s) \, ds \right) \, dx.$$

**Lemma 3:** $\lambda^{-1} J_\varepsilon(\lambda,U_\lambda) = -4|\Omega|$ and $\lambda^{-1} J_\varepsilon(\lambda,V_\lambda^\varepsilon) = -|\Omega|$, uniformly for $\varepsilon \in [0,1]$, where $|\Omega|$ is the Lebesgue measure of $\Omega$.

**Proof:** We will show the second statement. Since $V_\lambda^\varepsilon$ is the only stable solution of

$$
\begin{align*}
-\Delta u &= \lambda \min(f_\varepsilon(u),0) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
$$

the function minimises

$$J_\varepsilon(\lambda,u) - \int_\Omega \left( \frac{1}{2} |\nabla u|^2 - \lambda \int_0^u \min(f_\varepsilon(s),0) \, ds \right) \, dx \quad \text{for } \lambda > \lambda_0.$$
Since we can estimate \( J_{\varepsilon}(\lambda, u) \) from below by \(-\lambda|\Omega| \):
\[
J_{\varepsilon}(\lambda, u) \geq -\lambda \int_{\Omega} \min(f_{\varepsilon}(s), 0) \, ds \, dx \geq -\lambda \int_{\Omega} 1 \, dx ;
\]
it is sufficient to show that for all \( \sigma > 0 \) there is \( \varphi_{\varepsilon} \in H^1_0(\Omega) \) such that uniformly for \( \varepsilon \in [0,1] \)
\[
\lim_{\lambda \to \infty} \lambda^{-1} J_{\varepsilon}(\lambda, \varphi_{\varepsilon}) \leq -|\Omega| + \sigma.
\]

Take \( \varphi \in C^0_0(\Omega) \) with \( \varphi = -2 \) in a closed subset of \( \Omega \) with measure larger than \( |\Omega| - \frac{1}{4} \sigma \). The result follows for \( \lambda \) large since
\[
\lambda^{-1} J_{\varepsilon}(\lambda, \varepsilon \varphi) < \lambda^{-1} \varepsilon^2 \int_{\Omega} \frac{1}{2} |\varphi|^2 \, dx - |\Omega| + \frac{1}{4} \sigma \leq \lambda^{-1} \int_{\Omega} \frac{1}{2} |\varphi|^2 \, dx - |\Omega| + \frac{1}{4} \sigma. \tag{\ast}
\]

Because of Lemma 3 there is \( \lambda_1 > \lambda_0 \) such that
\[
J_{\varepsilon}(\lambda, U_{\lambda}) < J_{\varepsilon}(\lambda, V_{\lambda}^\varepsilon) < -\frac{1}{2} |\Omega| \quad \text{for all } \lambda \geq \lambda_1 \text{ and } \varepsilon \in [0,1].
\]

**Lemma 4:** \( U_{\lambda_1}(x_1, x_2) < \frac{1}{3} \lambda_1 \left( x_1^2 - 4x_2^2 \right) \).

**Proof:** For \( t \) large enough
\[
U_{\lambda_1}(x_1, x_2) < \frac{1}{3} \lambda_1 ((x_1 + t)^2 - 4x_2^2) \quad \text{in } \Omega.
\]
Since \(-\lambda_1 ((x_1 + t)^2 - 4x_2^2) = 6\lambda_1 > \lambda_1 \max f_{\varepsilon} \) and since \( \lambda_1 ((x_1 + t)^2 - 4x_2^2) > 0 \) in \( \Omega \) for \( t > 0 \), this function is a supersolution for \( t \geq 0 \).

By the Sweeping Principle [3, Theorem 9] one finds (8) for all \( t \geq 0 \). \( \Box \)

Finally we will show, for \( \epsilon > 0 \) but small enough, that \( U_{\lambda_1} \) does not minimize \( J_{\varepsilon}(\lambda_1, \cdot) \). We will modify \( U_{\lambda_1} \) near \((0,0)\) to obtain a \( H^1_0(\Omega) \)-function with lower
energy. Hence the solution of (7) for \( \lambda = \lambda_1 \) that minimizes \( J_c(\lambda_1, \cdot) \) is not \( U_{\lambda_1} \) or \( \nu_{\lambda_1} \), which are the only stable solutions with fixed sign.

Set

\[
\Omega_0^\delta = \{ (x_1, x_2) \in \Omega ; x_1 < \delta \},
\]

\[
U_0^\delta = \{ (x_1, x_2) \in U ; \delta < x_1 < 2\delta \}
\]

and

\[
\Omega_1^\delta = \{ (x_1, x_2) \in \Omega ; x_1 < 2\delta \}.
\]

Then \( |\Omega_1^\delta| = 2\delta^2 \).

Moreover define \( z \in C^0,1(\mathbb{R}) \) by

\[
\begin{cases}
  z(s) = 0 & \text{for } s \leq 1, \\
  z(s) = s-1 & \text{for } 1 < s \leq 2, \\
  z(s) = 1 & \text{for } 2 < s,
\end{cases}
\]

and set

\[
u_0(x_1, x_2) = z(\delta^{-1}x_1) U_{\lambda_1}(x_1, x_2).
\]

Then \( u_0 \in H^1_0(\Omega) \) and

\[
\nu u_0(x_1, x_2) = \delta^{-1} U_{\lambda_1}(x_1, x_2)(1,0) + z(\delta^{-1}x_1) \nu U_{\lambda_1}(x_1, x_2) \quad \text{in } \Omega_1^\delta.
\]

By using lemma 4 we can estimate the difference in energy as follows:

\[
J_c(\lambda_1 u_0) - J_c(\lambda_1 U_{\lambda_1}) \leq \frac{1}{2} \int_{\Omega_1^\delta} \left( |\nabla u_0|^2 - |\nabla U_{\lambda_1}|^2 \right) dx + \lambda_1 \int_{\Omega_1^\delta} \frac{1}{2} u_0^2 dx \leq \frac{1}{2} \int_{\Omega_1^\delta} \left( \frac{1}{2} \delta^{-2} U_{\lambda_1}^2 dx + \delta^{-1} U_{\lambda_1} z(\delta^{-1}x_1) \frac{d}{dx_1} U_{\lambda_1} \right) dx + \lambda_1 \int_{\Omega_1^\delta} \frac{1}{2} U_{\lambda_1}^2 dx \leq \frac{1}{2} \int_{\Omega_1^\delta} \left( \frac{1}{2} \delta^{-2} \left( \frac{1}{2} \lambda_1 \delta^2 \right)^2 + \delta^{-1} \left( \frac{1}{3} \lambda_1 \delta^2 \right) \| U_{\lambda_1} \|_{\infty} + \frac{1}{2} \lambda_1 \left( \frac{1}{3} \lambda_1 \delta^2 \right)^2 \right) dx \leq 2\delta^2 \left( \frac{8}{9} \lambda_1^2 \delta^2 + \frac{4}{3} \lambda_1 \delta \| U_{\lambda_1} \|_{\infty} + \frac{8}{9} \lambda_1^3 \delta^4 \right) \leq C(\lambda_1) \delta^3 \quad \text{for } \delta < 1.
\]

The function \( \nu_0 \) defined by

\[
\nu_0(x_1, x_2) = -\frac{1}{2} \delta U_{\lambda_1}(\delta^{-1}x_1, \delta^{-1}x_2)
\]

256
satisfies:

\[-\lambda \delta^ 6 (x) - \frac{1}{2} \delta^2 (a U_{\lambda_1})(\delta^6 \lambda_1) - \]
\[- \frac{1}{2} \delta^2 \lambda_1 f_0 (U_{\lambda_1}(\delta^6 \lambda_1)) - \]
\[= \frac{1}{2} \delta^2 \lambda_1 2 \delta f_0 \left( - \frac{1}{2} \delta U_{\lambda_1}(\delta^6 \lambda_1) \right) = \]
\[= \lambda_1 f_0 (v_0^6 (x)). \]

Hence \( \delta \) is a solution of (2) with \( \epsilon = 6 \) and \( \Omega \) replaced by \( \Omega_0^6 \).

After extending \( \delta \) by 0 outside of \( \Omega_0^6 \) we obtain:

\[(10) \quad J_6 (\lambda_1, \delta_0) = \int_{\Omega_0^6} \left( \frac{1}{2} |v_0^6| \right)^2 - \lambda_1 \int_0^{v_0^6} f_0 (s) \, ds \, dx = \]
\[= \frac{1}{2} \int_{\Omega_0^6} \left( \frac{1}{2} |v_0^6| \right)^2 - \lambda_1 \int_0^{v_0^6} f_0 (s) \, ds \, dx = \]
\[= \frac{1}{4} \delta^2 J_6 (\lambda_1, U_{\lambda_1}). \]

Finally, we set \( w_0 = U_0 + \delta_0 \) and we find, since \( \text{supp} \, u_0 \in \Omega \setminus \Omega_0^6 \) and \( \text{supp} \, \delta_0 \in \Omega_0^6 \) that by (9) and (10) for \( \delta \) sufficiently small:

\[J_6 (\lambda_1, w_0) = J_6 (\lambda_1, u_0) + J_6 (\lambda_1, \delta_0) \leq \]
\[\leq (1 + \frac{1}{2} \delta^2) J_6 (\lambda_1, U_{\lambda_1}) + C(\lambda_1) \delta^2 < J_6 (\lambda_1, U_{\lambda_1}). \]

The example uses a triangle for a domain and a piecewise linear right hand side. One can modify both \( \Omega \) and \( f \) to have the same result on a smooth, strictly convex domain with a C^2-function \( f \).

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References:


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