Table of inequalities in elliptic boundary value problems

C. Bandle, M. Flucher

Version April 30, 1997

Contents

AMS Subject Classification Code: 35J

1 Introduction and notation

This contribution contains a compiled list of inequalities that are frequently used in the calculus of variations and elliptic boundary value problems. The selection reflects the authors personal taste and experience. Purely one dimensional results are omitted. No proofs are given. For those we refer e.g. to [71, 71]. Frequently we refer to textbooks rather than original sources. General references are Pólya and Szegö [88], Morrey [73], Giaquinta [40, 41], Gilbarg and Trudinger [43], Kufner, John and Fucik [61], Ziemer [113].

We hope that this table will be useful to other mathematicians working in these fields and a stimulus to study some of the subjects more deeply.

A postscript file of this text is available at the WWW site `http://www.math.unibas.ch/~flucher/`. It is periodically updated an improved based on suggestions made by users. If your favourite inequality is missing or if you find any unprecise statement, please let us know. Other users will be grateful.

1.1 Notation

Unless otherwise stated Ω is a bounded, connected domain in $\mathbb{R}^n$ with Lipschitz boundary. The exterior unit normal is denoted by ν, the distance of a point from the boundary by

$$d(x) \quad := \quad \inf\{|x-y| \,:\, y \notin \Omega\}.$$

The letter c stands for a generic constants which is independent of the functions involved, ε stands for a positive constant that may be arbitrarily small and $\theta \in (0,1)$ is an interpolation parameter. The positive part of a function is $u^+ := \max(u, 0)$. For a set $A \subset \mathbb{R}^n$ we denote by $|A|$ and $|\partial A|$ its volume and surface area in the sense of Hausdorff measure. B_x^r is a ball in $\mathbb{R}^n$ of radius ρ centered at x. The symmetrized domain Ω^* is a ball centered at the origin having the same volume as Ω. The volume and surface of the unit ball are

$$|B| \;=\; \frac{2\pi^{n/2}}{n\Gamma(n/2)}, \qquad |S^{n-1}| \;=\; \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Let $u : \Omega \to \mathbb{R}$ be a measurable function. The function

$$u_*(a) \quad := \quad \sup\{t \,:\, |\{|u| \geq t\}| \geq a\}$$

is called *decreasing rearrangement* of u. The function

$$u^*(x) \quad := \quad u_*\left(|B|^n\, |x|^n\right)$$

defined on Ω^* is called *Schwarz symmetrization* of the positive function u. It is radially symmetric and $|\{u^* > t\}| = |\{|u| > t\}|$ for every $t \geq 0$. The relative *capacity* of a set $A \subset \Omega$ is defined as

$$\mathrm{cap}_\Omega(A) \quad := \quad \min\left\{\int_\Omega |\nabla u|^2 \,:\, u \in H_0^1,\; u \geq 1 \;\text{ on }\; B,\; A \subset B,\; B \text{ open}\right\}$$

and $\mathrm{cap}(A) := \mathrm{cap}_{\mathbb{R}^n}(A)$. The minimum is attained by the *capacity potential* of A.

1.2 Function spaces

All functions (with a few exceptions) are scalar functions defined on $\overline{\Omega}$. Sequences are denoted by (u_i). Integrals are taken with respect to Lebesgue measure. The mean value of a function is denoted by $u_\Omega := \frac{1}{|\Omega|}\int_\Omega u$. Convergence almost everywhere with respect to Lebesgue measure is abbreviated as a.e. The *convolution* of two functions given on all of $\mathbb{R}^n$ is defined as $(u * v)(x) := \int_{\mathbb{R}^n} dy\, u(x-y)v(y)$. In particular convolution with the *Riesz kernel* $K_\lambda(x) := |x|^{-\lambda}$ is considered. If $\lambda = n-2$ it is used to solve the Dirichlet problem $-\Delta u = f$. The space L^p is endowed with the norm

$$\|u\|_p^p \quad := \quad \int_\Omega |u|^p$$

where $1 \leq p < \infty$. A sequence (u_i) of L^1 functions is said to be *equi-integrable* or *uniformly integrable* [30] if

$$\lim_{|A|\to 0}\,\sup_i \int_A |u_i| \;=\; 0 \quad\text{or}\quad \lim_{\tau\to\infty}\,\sup_i \int_{\{|u_i|\geq\tau\}} |u_i| \;=\; 0.$$

Moreover $u_i \to u$ in measure if

$$|\{x \,:\, |u_i - u| \geq \varepsilon\}| \quad \to \quad 0$$

for every $\varepsilon > 0$. The dual exponent p' of $p \in [1,\infty]$ is defined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. The *Sobolev space* $H^{k,p}$ is given by the norm

$$\|u\|_{k,p}^p \quad := \quad \sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p\,.$$

If $p = 2$ we write $H^k := H^{k,2}$. $D^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm $||\nabla v||_p$. In the case of Orlicz spaces the power function is replaced by a more general *N-function* $A(t) := \int_0^t a$ with a positive, strictly increasing, upper semi-continuous function a with $a(0) = 0$. The dual N-function is defined as $\widetilde{A}(t) := \int_0^t a^{-1}$. If $A(2t) \le c\,A(t)$ for large t, then

$$||u||_A := \inf\left\{c > 0 \,:\, \int_\Omega A \circ \frac{u}{c} \le 1\right\},$$
$$||u||_{k,A} := \sum_{|\alpha| \le k} ||D^\alpha u||_A$$

defines the norms on the *Orlicz space* L_A and the *Sobolev-Orlicz space* $H^{k,A}$ respectively. In particular $u \in L_A$ if and only if $\int_\Omega A \circ u < \infty$. Another important generalization of L^p are the *Lorentz spaces* $L(p,q)$ on $\mathbb{R}^n$ given by

$$||u||_{(p,q)}^q := \int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{q}-1}\int_0^t u_*\right)^q dt,$$
$$||u||_{(p,\infty)} := \sup_{t>0} t^{\frac{1}{p}-1}\int_0^t u_*$$

with u_* as in Section 1.1. If $1 < p, q < \infty$ this norm is equivalent to

$$\int_0^\infty t^{\frac{q}{p}-1} u_*(t)^q\,dt.$$

In particular $L(p,p) = L^p(\mathbb{R}^n)$ [113]. *Campanato spaces* are given by the norm

$$||u||_{L^{p,\lambda}} := ||u||_p + \sup_{B_x^r \subset \Omega} \rho^{-\frac{\lambda}{p}} ||u - u_{B_x^r}||_{L^p(B_x^r)}.$$

The *John-Nirenberg space* of functions of bounded mean oscillation can be defined as BMO $:= L^{p,n}$. On $\mathbb{R}^n$

$$||u||_{\mathrm{BMO}} := \sup_{B_x^r} \frac{1}{|B_x^r|}\int_{B_x^r} |u - u_{B_x^r}|$$

defines a norm if we identify functions whose difference is a constant. Because the smooth functions are not dense in this space it is more convenient to consider the space VMO of functions of vanishing mean oscillations defined as the closure of C^0 with respect to the BMO-norm. A function u is in VMO if the above supremum vanishes in the limit $\rho \to 0$ and $\rho \to \infty$. The function $x \mapsto \log|x|$ is in BMO but not in L^∞ while $x \mapsto \log^\alpha|x|$ is in VMO for $\alpha < 1$, but not for $\alpha = 1$. BMO is a substitute for L^∞ while the *Hardy space* $\mathcal{H}^1$ substitutes for L^1. For $f \in L^1(\mathbb{R}^n)$ define

$$||f||_{\mathcal{H}^1} := \int_{\mathbb{R}^n} dx \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^n} \frac{dy}{\varepsilon^n}\phi\left(\frac{x-y}{\varepsilon}\right) f(y)\right|$$

where $\phi \in C_0^\infty(B_0^1)$ is a mollifying kernel with $\int_{B_0^1}\phi = 1$. Different ϕ lead to equivalent norms. References on Hardy spaces are [32, 97, 92]. We follow [77]. A local Hardy space was introduced by Goldberg [44]. Basic facts on interpolation spaces are summarized in [57].

An embedding of normed spaces, denoted by $X \subset Y$, is a bounded linear injection $j \in \mathcal{L}(X,Y)$. If j is a compact map we write $X \subset\subset Y$.

1.3 Elliptic boundary value problems

In the most general case we consider uniformly elliptic operators of the form

$$Lu := -\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

defined for $u \in H^1$. Several estimates deal with the Dirichlet problem

$$(1)\qquad \begin{aligned} Lu &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

which is the prototype of an elliptic boundary value problem. The natural space for its solutions is H_0^1 where the subscript refers to homogeneous Dirichlet boundary values. The corresponding principal Dirichlet eigenvalue is denoted by λ_1. For simplicity most results are stated for the Laplacian although they carry over to more general elliptic operators. The Dirichlet Green's function G_y is the solution of

$$\begin{aligned} -\Delta G_y &= \delta_y \text{ in } \Omega, \\ G_y &= 0 \text{ on } \partial\Omega \end{aligned}$$

where δ_y is the Dirac distribution with singularity at y.

2 L^p-spaces

Most inequalities of this section are proved in standard books on functional analysis (see e.g. [1, 3]).

2-1 Cauchy-Schwarz's inequality:

$$\int_\Omega uv \le \|u\|_2\|v\|_2.$$

2-2 Hölder's inequality: If $1 \le p \le \infty$ then

$$\int_\Omega uv \le \|u\|_p\|v\|_{p'}.$$

A useful variant in one dimension is the following optimal inequality.

$$\int_0^\infty fg \le \sup_{T>0} \frac{1}{T}\int_0^T f \int_0^\infty g$$

for measurable $f, g \ge 0$ and non-increasing g. Extremal functions are known [33]. With the Orlicz norm as defined in Section 1.2 [1, p. 237] one has

$$\int_\Omega uv \le 2\|u\|_A\|v\|_{\widetilde{A}}.$$

If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ then

$$\|u_1u_2\|_{(p,q)} \le \|u_1\|_{(p_1,q_1)}\|u_2\|_{(p_2,q_2)}.$$

In fact the dual space of the Lorentz space $L(p,q)$ is $L(p',q')$ [113].

2-3 Calderon's lemma: If $p_1 \le p_2$ then

$$\|u\|_{p_1} \le |\Omega|^{\frac{1}{p_1}-\frac{1}{p_2}}\|u\|_{p_2}$$

hence $L^{p_2} \subset L^{p_1}$. If $q_1 \le q_2$ then

$$\|u\|_{(p,q_2)} \le \left(\frac{q_1}{p}\right)^{\frac{1}{q_1}-\frac{1}{q_2}}\|u\|_{(p,q_1)}$$

hence $L(p,q_1) \subset L(p,q_2)$ [113, p. 37].

2-4 Young's inequality: [61, 1, p. 229]. If $1 < p < \infty$ then

$$\int_\Omega uv \leq \frac{1}{p}||u||_p^p + \frac{1}{p'}||v||_{p'}^{p'},$$
$$\int_\Omega uv \leq \frac{\varepsilon^p}{p}||u||_p^p + \frac{\varepsilon^{-p'}}{p'}||v||_{p'}^{p'},$$
$$\int_\Omega uv \leq \int_\Omega A \circ u + \int_\Omega \widetilde{A} \circ v$$

where A is an N-function with dual $\widetilde{A}$ as defined in Section 1.2 and $u, v \geq 0$.

2-5 Bank's inequality: [11, p. 69]. If $u_1, u_2, \phi \in L^2$ with $\int_\Omega u_1 \geq \int_\Omega u_2$ and $0 \leq \phi(x_1) \leq \phi(x_2)$ whenever $u_1(x_1) \leq u_2(x_1)$ and $u_1(x_2) \geq u_2(x_2)$, then

$$\int_\Omega u_1\phi \geq \int_\Omega u_2\phi.$$

2-6 Jensen's inequality: If $\phi \geq 0$ is convex then

$$\phi\left(\frac{1}{|\Omega|}\int_\Omega u\right) \leq \frac{1}{|\Omega|}\int_\Omega \phi \circ u.$$

2-7 Minkowski's inequality:

$$||u+v||_p \leq ||u||_p + ||v||_p,$$
$$\int_\Omega |u+v|^p \leq (1+\varepsilon)^{p-1}\int_\Omega |u|^p + \left(1+\frac{1}{\varepsilon}\right)^{p-1}\int_\Omega |v|^p$$

for $\varepsilon > 0$.

2-8 Clarkson's inequalities: [4, p. 89].

$$||u+v||_p^p + ||u-v||_p^p \leq 2^{p-1}(||u||_p^p + ||v||_p^p), \quad 2 \leq p < \infty,$$
$$||u+v||_p^{p'} + ||u-v||_p^{p'} \geq 2(||u||_p^p + ||v||_p^p)^{p'-1}, \quad 2 \leq p < \infty,$$
$$||u+v||_p^{p'} + ||u-v||_p^{p'} \leq 2(||u||_p^p + ||v||_p^p)^{p'-1}, \quad 1 < p \leq 2,$$
$$||u+v||_p^p + ||u-v||_p^p \geq 2^{p-1}(||u||_p^p + ||v||_p^p), \quad 1 < p \leq 2.$$

2-9 Monotonicity of p-Laplacian: If $p \geq 2$ then

$$(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u - \nabla v) \geq c_p|\nabla u - \nabla v|^2(|\nabla u|^{p-2} + |\nabla v|^{p-2})$$

with $c_p \geq c_2 = 0.5$ and $c_p = 1$ for $p \geq 3$.

2-10 Interpolation inequality: [43, p. 146]. If $p \leq r \leq q$ and $\frac{1}{r} \geq \frac{\theta}{p} + \frac{1-\theta}{q}$ then

$$||u||_r \leq ||u||_p^\theta ||u||_q^{1-\theta}.$$

2-11 Riesz-Thorin theorem: [57]. If a linear operator T satisfies $||Tu||_{q_0} \leq c_0||u||_{p_0}$, $||Tu||_{q_1} \leq c_1||u||_{p_1}$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $0 \leq \theta \leq 1$ then

$$||Tu||_q \leq c_0^{1-\theta}c_1^\theta||u||_p.$$

2-12 Convolution inequality: [4, p.89], [113, p. 96]. If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $1 \leq p,\ q \leq \infty$ then

$$||u * v||_r \leq ||u||_p ||v||_q$$

with $u * v$ as in Section 1.2. If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ then

$$||u_1 * u_2||_{(p,q)} \leq 3p\,||u_1||_{(p_1,q_1)}||u_2||_{(p_2,q_2)}.$$

If one of the factors is the Riesz kernel K_λ then $K_\lambda \in L\left(\frac{n}{\lambda}, \infty\right)$ and

$$||K_\lambda * u||_{\left(\frac{np}{n-(n-\lambda)p}, q\right)} \leq ||K_\lambda||_{\left(\frac{n}{\lambda}, \infty\right)} ||u||_{(p,q)}.$$

2-13 Hardy-Littlewood-Sobolev inequality: [51, 94, 57]. If $0 < \lambda < n$, $1 < p < \frac{n}{n-\lambda}$, and $\frac{1}{p} + \frac{\lambda}{n} = \frac{1}{q} + 1$ then

$$||K_\lambda * u||_{L^q(\mathbb{R}^n)} \leq c||u||_{L^p(\mathbb{R}^n)}.$$

2-14 Hardy-Littlewood maximal function theorem: [98, p.55-58]. The *maximal function* $Mu(x) := \sup_{\rho>0} \frac{1}{|B^\rho|} \int_{B_x^r} |u|$ of $u \in L^1$ satisfies

$$|\{Mu > \tau\}| \leq \frac{c}{\tau}||u||_1,$$
$$||Mu||_p \leq c||u||_p, \quad \text{for } p > 1.$$

2-15 Hardy inequalities in one dimension: [49, 50]. If $u(0) = 0$ then

$$\int_0^1 \left|\frac{u}{x}\right|^2 \leq 4\int_0^1 |u'|^2.$$

More generally, if $\alpha > 2k - 1$ then

$$\int_0^\infty x^{\alpha-2k}|f|^2 \leq \frac{4^k}{(1-\alpha)^2 \ldots (2k-1-\alpha)^2} \int_0^\infty x^\alpha |f^{(k)}|^2.$$

If $\alpha < 1$ and $f^{(i)}(0) = 0$ for $i = 0, \ldots, k-1$ then

$$\int_0^\infty x^{\alpha-2k}|f|^2 \leq \frac{4^k}{(1-\alpha)^2 \ldots (2k-1-\alpha)^2} \int_0^\infty x^\alpha |f^{(k)}|^2.$$

2-16 Hardy inequalities in higher dimensions: [80]. If $p > 1$ then

$$\int_\Omega \left|\frac{u}{d}\right|^p \leq c\int_\Omega |\nabla u|^p$$

for all $u \in H_0^{1,p}(\Omega)$, d=distance from boundary, $c \geq \left(\frac{p}{p-1}\right)^p$. For convex domains $c = \left(\frac{p}{p-1}\right)^p$. In 3 dimensions

$$\int_\Omega \frac{|u|^2}{1+|x|^2} \leq 4\int_\Omega |\nabla u|^2$$

for every $u \in H_0^1(\Omega)$ [18]. If $0 \neq \overline{\Omega}$ then also

$$\int_\Omega \left|\frac{u}{x}\right|^2 \leq 4\int_\Omega |\nabla u|^2.$$

For the exterior domain $\Omega = \mathbb{R}^n \setminus B_0^{2r}$ one has

$$\int_\Omega \frac{|u|^p}{|x|^p} \leq \left(\frac{p}{|n-p|}\right)^p \int_\Omega |\nabla u|^p$$

for $p \neq n$ and $u \in H_0^{1,p}(\Omega)$ and

$$\int_\Omega \frac{|u|^n}{|x|^n \log^n(|x|/r)} \leq \left(\frac{n}{n-1}\right)^n \int_\Omega |\nabla u|^n$$

for $u \in H_0^{1,n}(\Omega)$ [80].

2-17 Hardy inequality: [57, 113, p.35]. If $p > 1$, $r > 0$, $U(x) := \frac{1}{x}\int_0^x u$ for $x > 0$ and $\theta(x) := \sup_{\xi > x} \frac{1}{\xi - x}\int_x^\xi u$ for $x \in \mathbb{R}$ then

$$\begin{aligned}
\int_0^\infty U(t)^p t^{p-r-1}dt &\leq \left(\frac{p}{r}\right)^p \int_0^\infty u(t)^p t^{p-r-1}dt, \\
\|U\|_p &\leq \frac{p}{p-1}\|u\|_p, \\
\|\theta\|_p &\leq \frac{p2^{1/p}}{p-1}\|u\|_p.
\end{aligned}$$

3 Convergence theorems in L^p

3-1 Fatou's lemma: [3]. If $u_i \geq 0$ then

$$\int_\Omega \liminf u_i \leq \liminf \int_\Omega u_i.$$

If $u_i \leq v_i \to v$ in L^1 then also

$$\limsup \int_\Omega u_i \leq \int_\Omega \limsup u_i.$$

3-2 Lebesgue's differentiation theorem: [55]. If $u \in L^1$ then

$$\lim_{\rho \to 0} \frac{1}{|B_x^r|}\int_{B_x^r} u = u(x)$$

for a.e. $x \in \Omega$.

3-3 Equi-integrability theorem: If $u \in L^1$ and $\varepsilon > 0$ then

$$\int_A |u| < \varepsilon$$

for all $A \subset \Omega$ with $|A| < \delta(\varepsilon)$.

3-4 Lusin's continuity theorem: If $u \in L^1$ and $\varepsilon > 0$ then u is uniformly continuous on $\Omega \setminus E$ with $|E| < \varepsilon$.

3-5 Egoroff's theorem: If $u_i \to u$ a.e. (all measurable) and $\varepsilon > 0$ then

$$u_i \to u \text{ uniformly on } \Omega \setminus E$$

with $|E| < \varepsilon$.

3-6 Lebesgue's convergence theorem: If $u_i \to u$ a.e. and $|u_i| \le v_i \to v$ in L^1 then

$$u_i \quad \to \quad u \text{ in } L^1.$$

3-7 Vitali's convergence theorem: If (u_i) is equi-integrable and $u_i \to u$ in measure then

$$u_i \to u \text{ in } L^1.$$

If $u_i \in L^p$, $u_i \to u$ a.e., and (u_i^p) is equi-integrable then

$$u_i \quad \to \quad u \text{ in } L^p.$$

4 Sobolev spaces

Most inequalities of this section can be found in [73, 1, 61, 43, 66, 113]. See Section 1.2 for definitions.

4-1 Poincaré's inequalities:

(a) For every $u \in H_0^1$

$$\int_\Omega |u|^2 \quad \le \quad \frac{1}{\lambda_1} \int_\Omega |\nabla u|^2 .$$

(b) For every $u \in H_0^{1,p}$

$$\|u\|_p \quad \le \quad c\, \|\nabla u\|_p .$$

(c) For every $u \in H^1$ and $B_x^r \subset \Omega$

$$\int_{B_x^r} \left| u - u_{B_x^r} \right|^2 \quad \le \quad c_n \rho^2 \int_{B_x^r} |\nabla u|^2 .$$

(d) [73]. If $0 < \theta < 1$ then

$$\|u\|_2 \quad \le \quad c_\theta \, \|\nabla u\|_2$$

for every $u \in H^1$ with $|\{u = 0\}| \ge \theta\, |\Omega|$.

(e) [38, p. 15]. If u vanishes on a set of non-vanishing capacity (Section 1.1) then

$$\int_\Omega |u|^2 \quad \le \quad \frac{c}{\operatorname{cap}(\{u=0\})} \int_\Omega |\nabla u|^2 .$$

(f) [26]. If $u \in H_0^1(\Omega \times \mathbb{R}^m)$ then

$$\int_{\Omega \times \mathbb{R}^m} |u|^2 \quad \le \quad \frac{1}{\lambda_1} \int_{\Omega \times \mathbb{R}^m} |\nabla u|^2 .$$

(g) If Ω bounded in one direction then

$$\int_\Omega |u|^2 \quad \le \quad c \left(\int_\Omega u \right)^2 + c \int_\Omega |\nabla u|^2$$

for every $u \in H^1$.

(h) A one dimensional version is Wirtinger's inequality: If $u \in H^1(0,2\pi)$ is periodic with vanishing mean value then

$$\int_0^{2\pi} u^2 \quad \leq \quad \int_0^{2\pi} (u')^2.$$

Equality holds if and only if $u(t) = a\cos(t) + b\sin(t)$ [106, 15].

4-2 Gårding's inequality: [41, p. 7-9] If A is a uniformly positive definit matrix and $A \in L^\infty$, $b \in L^n$ and $d \in L^{n/2}$ then there is a constant $c_1 > 0$ such that

$$\int_\Omega \nabla u \cdot A(x)\nabla u + 2u\, b(x) \cdot \nabla u + d(x)uv \quad \geq \quad c_1 \int_\Omega |\nabla u|^2 + c_2 \int_\Omega |u|^2$$

for every $u \in H^1$. The same is true for systems with continuous A satisfying the Legendre Hadamard condition.

4-3 Korn's inequality: [112]. In terms of the symmetric gradient $E := \frac{1}{2}\left(Du + Du^T\right)$

$$\int_\Omega |u|^2 + |Du|^2 \quad \leq \quad c\int_\Omega \mathrm{Tr}\left(\mathrm{E^T E}\right) \quad \text{for} \quad \mathrm{u} \in \mathrm{H}^1_0(\Omega, \mathrm{IR}^n)$$

$$\|Du\|_p \quad \leq \quad c(\|u\|_p + \|E\|_p) \quad \text{for} \quad u \in H^p(\Omega, \mathrm{IR}^n).$$

4-4 Poincaré's inequality for capacity potentials: [35]. Let (u_i) be a sequence of capacity potentials (Section 1.1) with $\mathrm{cap}(A_i) \to 0$ and $p < \frac{2n}{n-2}$. Then

$$\frac{\|u_i\|_p}{\|\nabla u_i\|_p} \quad \to \quad 0.$$

4-5 Gagliardo-Nirenberg's inequality: [39, 79], [4, p. 38].

$$\|u\|_{\frac{n}{n-1}} \quad \leq \quad \frac{1}{2}\|\nabla u\|_1$$

for every $u \in H_0^{1,1}$. This implies: If $\frac{\theta}{p} + \frac{1-\theta}{q} = \theta$ then

$$\|u\|_{\frac{n}{\theta(n-1)}} \quad \leq \quad (2\theta)^{-\theta}\|\nabla u\|_p^\theta \|u\|_q^{1-\theta}$$

for every $u \in H_0^{1,p}$.

4-6 Ehrling-Browder's inequality: [1], [4, p. 94]. If $\frac{k}{k'} \leq \theta \leq 1$ and $\frac{1}{p} = \frac{k}{n} + \theta\left(\frac{1}{p'} - \frac{k'}{n}\right) + \frac{1-\theta}{q}$ then

$$\|\nabla^k u\|_p \quad \leq \quad c\,\|\nabla^{k'} u\|_{p'}^\theta \|u\|_q^{1-\theta}.$$

For $n \geq 3$ Sobolev's inequality follows. Moreover

$$\|\nabla u\|_2 \quad \leq \quad \|\Delta u\|_p \|u\|_{p'}$$

for every $u \in H_0^{2,p}$. If $n = 2$ then

$$\|u\|_p \quad \leq \quad c\,\|\nabla u\|_2^{\frac{p-1}{p}}\, \|u\|_1^{\frac{1}{p}}.$$

4-7 Sobolev's inequalities:

(a) [80, 3]. If $1 \le p < n/k$ then

$$\|u\|_{\frac{np}{n-kp}} \le c\|D^k u\|_p$$

for all $u \in H_0^{1,p}$.

(b) [103], [4, p. 39]. If $1 < p < n$ then

$$\|u\|_{\frac{np}{n-p}} \le \pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{1-1/p} \left(\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(n+1-n/p)}\right)^{1/n} \|\nabla u\|_p$$

for $u \in H_0^{1,p}$. Extremal functions are of the form $u(x) = \left(c + |x - x_0|^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}$.

(c)

$$\|u\|_{\frac{n}{n-1}} \le \frac{\Gamma(1+n/2)^{1/n}}{\sqrt{\pi} n} \|\nabla u\|_1$$

for every $u \in H_0^{1,1}$ [31].

(d) [36]. If $\theta > 0$ then

$$\|u\|_{\frac{2n}{n-2}} \le c_\theta \|\nabla u\|_2$$

for every $u \in H_0^1$ with $|\{u = 0\}| \ge \theta\, |\Omega|$.

(e) If $0 < \alpha < k - \frac{n}{p}$ then

$$\|u\|_{C^\alpha} \le c\,\|u\|_{k,p},$$
$$H^{n,1} \subset C^0.$$

(f) If $\frac{1}{p'} = \frac{1}{p} - \frac{k-k'}{n}$, $k \ge k' \ge 0$ and $1 \le p \le p'$ then

$$\|u\|_{k',p'} \le c\,\|u\|_{k,p}.$$

4-8 Weighted Sobolev inequalities: [66, p. 98]. If $kp < n$, $1 \le p \le q \le \frac{np}{n-kp}$, and $\beta + \frac{n}{q} = \alpha - k + \frac{n}{p} > \frac{n-m}{q}$ (spacial homogeneity) then

$$\| |(x_1 \ldots x_m)|^\beta u\|_q \le c\, \| |(x_1 \ldots x_m)|^\alpha D^k u\|_p$$

for all u with compact support in $\mathbb{R}^n$. If $\partial\Omega$ is Lipschitz and $1 < p < \infty$ then

$$\int_\Omega |u|^p\, d(x)^{\alpha-p} \le c \int_\Omega |\nabla u|^p\, d(x)^\alpha \quad \text{if } \alpha > p-1,$$
$$\int_\Omega |u|^p\, d(x)^{-1+\varepsilon} \le c_\varepsilon \int_\Omega |\nabla u|^p\, d(x)^\alpha \quad \text{if } \alpha \le p-1$$

for every $\varepsilon > 0$ [60].

4-9 Radial Sobolev inequality: [78, 99].

$$|u(x)| \le \left((n-2)\,|S^{n-1}|\right)^{-1/2} |x|^{\frac{2-n}{2}} \|\nabla u\|_2$$

for every $u \in D^{1,2}_{\text{rad}}(\mathbb{R}^n)$.

$$|u(x)| \le c_n\, |x|^{\frac{1-n}{2}} \|u\|_{1,2}$$

for every $u \in H^1(\mathbb{R}^n)$. Moreover $H^1_{\text{rad}}(\mathbb{R}^n) \subset\subset L^p(\mathbb{R}^n)$ for $p < \frac{2n}{n-2}$ [17].

4-10 Generalized Sobolev inequality: [36]. If $0 \le f(\tau) \le c\,|\tau|^{\frac{2n}{n-2}}$ then

$$\int_\Omega f \circ u \quad \le \quad S^f \,\|\nabla u\|_2^{\frac{2n}{n-2}}$$

where $S^f := \sup\left\{\int_{\mathbb{R}^n} f(v) \,:\, v \in C_c^\infty(\mathbb{R}^n),\, \|\nabla v\|_2 \le 1\right\}$. This statement can be localized. For every $\delta > 0$ there is an optimal ratio $k(\delta)$ (explicit) such that

$$\int_{B_x^\rho} f(u) \quad \le \quad S^f\left(\int_{B_x^R} |\nabla u|^2 + \delta \int_{\mathbb{R}^n} |\nabla u|^2\right)^{2^*/2},$$

$$\int_{\mathbb{R}^n \setminus B_x^R} f(u) \quad \le \quad S^f\left(\int_{\mathbb{R}^n \setminus B_x^\rho} |\nabla u|^2 + \delta \int_{\mathbb{R}^n} |\nabla u|^2\right)^{2^*/2}$$

for every $\rho/R \le k(\delta)$, $x \in \mathbb{R}^n$, and $u \in D^{1,2}(\mathbb{R}^n)$.

4-11 Traces: [3, p. 168]. If $1 \le p < \infty$ then

$$\int_{\partial\Omega} |u|^p \quad \le \quad c \int_\Omega |\nabla u|^p\,, \quad \text{i.e.}$$

$$H^{1,p} \quad \subset \quad L^p(\partial\Omega).$$

The embedding is compact for $p < \infty$ and continuous for $p = \infty$. [1, p. 114], [61, p. 328 and 337]. If $\partial\Omega$ is C^k then

$$H^{k,p} \quad \subset \quad L^{\frac{(n-1)p}{n-kp}}(\partial\Omega) \;\text{ if }\; kp < n,$$
$$H^{k,p} \quad \subset \quad L^q(\partial\Omega) \;\text{ for every } q \text{ if }\; kp \ge n,$$
$$H^{1,p} \quad \subset \quad H^{1-\frac{1}{p},p}(\partial\Omega).$$

If M is a m-dimensional submanifold of $\overline{\Omega}$ and $\overline{p} < \frac{mp}{n-(k-\overline{k})p}$ then

$$H^{k,p} \quad \subset \quad H^{\overline{k},\overline{p}}(M \cap \overline{\Omega}).$$

5 Critical Sobolev embeddings

In this section we consider the spaces $H^{k,p}$ with $kp = n$. In this case the measure $\left|D^k u\right|^p dx$ which contributes to the leading term of the norm is conformally invariant. By Sobolev's theorem $H^{k,p} \subset\subset L^q$ for every $q < \infty$ but $H^{k,p} \not\subset L^\infty$. See Section 1.2 for definitions of spaces and norms.

5-1 Orlicz' inequality: [4, p. 63].

$$\int_\Omega \exp(u) \quad \le \quad c \exp\left(\alpha \int_\Omega |u|^n + \beta \int_\Omega |\nabla u|^n\right).$$

5-2 Strichartz's inequality: [1, p. 242]. If $kp = n$ and $A(t) := \exp\left(t^{\frac{p}{p-1}}\right) - 1$ then

$$\|u\|_A \quad \le \quad c\,\|u\|_{k,p}.$$

5-3 Trudinger-Moser's inequality: [4, p. 65].

$$\int_\Omega \exp\left(n\,|S^{n-1}|^{\frac{1}{n-1}}\,|u|^{\frac{n}{n-1}}\right) \leq c\,|\Omega|,$$

$$\int_\Omega \exp\left(4\pi u^2\right) \leq c\,|\Omega| \quad (n=2)$$

for every $u \in H_0^{1,n}$ with $\|\nabla u\|_n \leq 1$.

5-4 Orlicz-Sobolev embedding: [1, p. 252]. If $\int_1^\infty \frac{A(t)}{t^{\frac{n+1}{n}}}dt < \infty$ then

$$\|u\|_\infty \leq c\,\|u\|_{1,A}$$

for every $u \in H^{1,A}$. In fact u is continuous.

5-5 Wente's inequality: [110, 53]. For $f, g, h \in H^1(\mathbb{R}^2)$ one has

$$\int_{\mathbb{R}^2} f \det(\nabla g, \nabla h) \leq c\,\|\nabla f\|_2 \|\nabla g\|_2 \|\nabla h\|_2.$$

5-6 Higher integrability of Jacobians: [28, 68, 76]. If $u \in H^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ then

$$\|\det Du\|_{\mathcal{H}^1} \leq c\,\|Du\|_{L^n}^n.$$

5-7 Poincaré-Sobolev inequality for BMO functions: If $p < \infty$ then

$$\|u\|_p \leq c\,\|u\|_{\mathrm{BMO}} \leq c\,\|u\|_\infty.$$

If $kp = n$ then

$$\|u\|_{\mathrm{BMO}} \leq c\,\|u\|_{k,p}.$$

5-8 Fefferman-Stein duality: [97, 32, 92, 107].

$$\int_{\mathbb{R}^n} fg \leq c\,\|f\|_{\mathcal{H}^1} \|g\|_{\mathrm{BMO}}.$$

In fact BMO is the dual space of $\mathcal{H}^1$ and $\mathcal{H}^1$ is the dual space of VMO.

5-9 John-Nirenberg's inequality: [97, 43]. For $p < \infty$ and $\tau > 0$ one has

$$\frac{1}{|B_x^r|}\int_{B_x^r} |u - u_{B_x^r}|^p \leq c_p \|u\|_{\mathrm{BMO}}^p,$$

$$\left|\left\{y \in B_x^r \,:\, |u(y) - u_{B_x^r}| > \tau\right\}\right| \leq c_1\,|B_x^r| \exp\left(-\frac{c_2 \tau}{\|u\|_{\mathrm{BMO}}}\right).$$

If Ω is convex, $u \in H^{1,1}$, $\int_{B_x^r} |\nabla u| \leq A\rho^{n-1}$ for all balls then

$$\int_\Omega \exp\left(b\,|u - u_\Omega|\right) \leq c\,|\Omega|.$$

6 Maximum and comparison principles

In this section all functions are supposed to be C^2 (except for the weak maximum principle).

6-1 Maximum principle: [89, 43]. If $\Delta u + g(\cdot, u) \geq \Delta v + g(\cdot, v)$ in Ω and $u \geq v$ on $\partial\Omega$.

(a) If $g(x, \cdot)$ is non-increasing for every x, then

$$u \geq v \text{ in } \Omega.$$

(b) If in addition $g(x, \cdot)$ is Lipschitz and $u \neq v$ then

$$u > v \text{ in } \Omega.$$

(c) If Ω satisfies an interior ball condition and if $u(x) = v(x)$ for some $x \in \partial\Omega$ and $u \neq v$, then

$$\frac{\partial u}{\partial \nu}(x) < \frac{\partial v}{\partial \nu}(x)$$

(d) If $(-\Delta - \lambda)u \geq 0$ with $u = 0$ on $\partial\Omega$, $\lambda < \lambda_1$ and $u \neq 0$ then

$$u > 0 \text{ in } \Omega.$$

6-2 Weak maximum principle: [43, p. 179]. If $u \in H^1$ is subharmonic ($\int_\Omega \nabla u \nabla \phi \leq 0$ for all $\phi \in H_0^1$, $\phi \geq 0$) then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+.$$

6-3 Giraud's maximum principle: [70]. If $\partial\Omega$ is Hölder continuous $-\Delta u \leq 0$ and u assumes its maximum at a point $x \in \partial\Omega$, then

$$|u(x) - u(y)| \leq c\,|x - y|$$

for every $y \in \Omega$.

6-4 Bernstein inequality: If $au_{xx} + 2bu_{xy} + cu_{yy} = f$ then

$$u_{xx}^2 + 2u_{xy}^2 u_{yy}^2 + 2A(u_{xx}u_{yy} - u_{xy}^2) \leq Bf^2$$

where $A > \frac{1}{2}\frac{a^2+2b^2+c^2}{ac-b^2}$ and $B \geq \frac{A-1}{2A(ac-b^2)-(a^2+2b^2+c^2)}$.

6-5 Bernstein type inequalities:

(a) If $\Delta u = 0$, then $\Delta |\nabla u|^2 \geq 0$ and the maximum of $|\nabla u|^2$ is attained on $\partial\Omega$.

(b) [90]. If $Lu = 0$ then for some constant c the maximum of $|\nabla u|^2 + c\,|u|^2$ is attained on $\partial\Omega$.

(c) [69, 95]. Let u be the solution of the *torsion problem*

$$\begin{aligned} -\Delta u &= 1 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Then the maximum of $|\nabla u|^2 + 2\,|u|^2$ is attained on $\partial\Omega$.

6-6 Payne-Philippin maximum principle: [85]. Let u be a solution of the elliptic problem

$$\nabla \cdot \left(g\left(|\nabla u|^2\right)\nabla u\right) + \rho\left(|\nabla u|^2\right) f(u) = 0 \text{ in } \Omega$$

with $g(t) + 2tg'(t) > 0$. Define

$$P(x) := \int_0^{|\nabla u(x)|^2} \frac{g(\xi) + 2\xi g'(\xi)}{\rho(\xi)} d\xi + 2\int_0^{u(x)} f(\eta)\, d\eta.$$

Then the maximum of P is attained on $\partial\Omega$ or at a critical point of u.

6-7 Miranda's biharmonic maximum principle: [69]. If Ω is sufficiently smooth and $\Delta^2 u = 0$ then $\max |\nabla u|^2 - u\Delta u$ is attained at the boundary.

6-8 Boundary blow up: [62, 13]. If $\Delta u \geq u^p$ then

$$\begin{aligned} u(x) &\leq c\phi(d(x)) \text{ if } p > 1, \\ u(x) - \phi(d(x)) &\leq c \text{ if } p > 3. \end{aligned}$$

where $\phi(t) := \left(\frac{p-1}{\sqrt{2(p+1)}} t\right)^{-\frac{2}{p-1}}$.

6-9 Whitney's inequality: [91]. There exists a function $\tilde{d} \in C^\infty$ with bounded gradient such that

$$\frac{1}{c} d \leq \tilde{d} \leq c\, d, \quad \left|\Delta \tilde{d}\right| \leq \frac{c}{\tilde{d}}.$$

7 Elliptic regularity theory

We start with the weakest hypothesis on Lu.

7-1 Weinberger's inequalities: [109]. Let $Lu := \nabla \cdot A(x) \nabla u$ be an elliptic operator in divergence form and $\lambda := \inf_\Omega \lambda_1(A)$. Then the Dirichlet Green's function G_y of L satisfies:

$$\|G_y\|_{p'} \leq c_{p,n} \frac{1}{\lambda} |\Omega|^{\frac{2}{n} - \frac{1}{p}}$$

where $c_{p,n} := (n-2)^{-2+\frac{1}{p}} n^{\frac{2}{n} - \frac{1}{p}} \beta^{1-\frac{1}{p}} \left(\frac{2p-1}{p-1}, \frac{2}{n-1} - \frac{1}{p-1}\right) |B|^{-\frac{2}{n}}$ and β is the beta function [22]. For $p > n$

$$\|\nabla G_y\|_{p'} \leq \overline{c}_{p,n} \lambda |\Omega|^{\frac{1}{n} - \frac{1}{p}}$$

where $\overline{c}_{p,n} := |B|^{-\frac{1}{n}} \left(\frac{p-1}{p-n}\right)^{1-\frac{1}{p}} n^{\frac{1}{n} - \frac{1}{p}}$. For the Laplacian equality holds when Ω is a ball centered at x. As a consequence the solution of (1) satisfies

$$\|u\|_\infty \leq c_{p,n} \frac{1}{\lambda} |\Omega|^{\frac{2}{n} - \frac{1}{p}} \|f\|_p \text{ for } p > \frac{n}{2}.$$

If $f = \nabla \cdot v$, then

$$\|u\|_\infty \leq \overline{c}_{p,n} \lambda |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|v\|_p \text{ for } p > n.$$

7-2 Estimates for the Green's function near the boundary: [101]. The Dirichlet Green's function of a second order uniformly elliptic operator L with $C^{1,\alpha}$-coefficients on a $C^{1,\alpha}$ domain satisfies

$$c_1 g(x,y) \leq G_x(y) \leq c_2 g(x,y)$$

with positive constants and

$$g(x,y) = \begin{cases} |x-y|^{2-n} \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right) & (n \geq 3), \\ \log\left(1 + \frac{d(x)d(y)}{|x-y|^2}\right) & (n = 2), \\ \sqrt{d(x)d(y)} \min\left(1, \frac{\sqrt{d(x)d(y)}}{|x-y|}\right) & (n = 1). \end{cases}$$

7-3 Grisvard's inequality: [45, 46, 65]. If $\partial\Omega$ is smooth, $-\Delta u \in L^2$ and $\partial_\nu u \in H^{1/2}$ then

$$u \in H^2.$$

7-4 Hardy-Littlewood-Sobolev inequality: [51, 94, 57]. If $1 < p < \frac{n}{2}$ then the solution $u = K_{n-2} * f$ of $-\Delta u = f$ satisfies

$$||u||_{\frac{np}{n-2p}} \leq c_{n,p}||\Delta u||_p.$$

7-5 Regularity in Lorentz spaces: If $f \in L(p,q)$ with $1 < p < \frac{n}{2}$ and $-\Delta u = f$ in $\mathbb{R}^n$ then $u = u_0 + h$ with $-\Delta h = 0$ and

$$\begin{aligned} ||u_0||_{(\frac{np}{n-2p},q)} &\leq c\,||f||_{(p,q)}, \\ ||u_0||_{L^{\frac{np}{n-2p}}(\mathbb{R}^n)} &\leq c\,||f||_{L^p(\mathbb{R}^n)}. \end{aligned}$$

This follows by convolution with K_{n-2} and the Hardy-Littlewood-Sobolev inequality.

7-6 Regularity in Hardy spaces: [32]. If $f \in \mathcal{H}^1$ and $-\Delta u = f$ in $\mathbb{R}^n$ then $u = u_0 + h$ with $-\Delta h = 0$ and

$$||D^2 u_0||_{\mathcal{H}^1} \leq c\,||f||_{\mathcal{H}^1}.$$

In two dimensions also

$$||\nabla u_0||_{L^2} + ||u_0||_{L^\infty} \leq c\,||f||_{\mathcal{H}^1}.$$

7-7 Riesz operators: The operators

$$-\Delta^{-1}\partial_i\partial_j : L^p \to L^p,\ L^\infty \to \mathrm{BMO},\ \mathrm{BMO} \to \mathrm{BMO}$$

are bounded.

7-8 Calderón-Zygmund's inequality: [43, Lemma 9.17]. If $\partial\Omega$ is C^2, $1 < p < \infty$ and $u \in H^{2,p} \cap H_0^1$ then

$$||u||_{2,p} \leq c\,||\Delta u||_p.$$

7-9 Brézis-Merle inequality: [21]. If $n = 2$ and $u = 0$ on $\partial\Omega$ then

$$\int_\Omega \exp\left(\frac{(4\pi - \varepsilon^2)u}{||\Delta u||_1}\right) \leq \left(\frac{2\pi\,\mathrm{diam}(\Omega)}{\varepsilon}\right)^2.$$

7-10 Meyers' inequality: [67]. If A is bounded and $|p-2|$ small enough then

$$||u||_{1,p} \leq c\,||\nabla(A\nabla u)||_{-1,p}.$$

7-11 Regularity theorem for smooth operators: [4, p. 85]. If L has C^∞ coefficients, $k \geq 0$, $1 < p < \infty$ and $0 < \alpha < 1$ then

$$\begin{aligned} ||u||_{k+2,p} &\leq c\,||Lu||_{k,p}, \\ ||u||_{C^{k+2+\alpha}} &\leq c\,||Lu||_{C^{k+\alpha}}. \end{aligned}$$

7-12 Schauder estimates: [3, 41, p. 48-53]. If $\partial\Omega$ is $C^{2+\alpha}$ then

$$\begin{aligned} \|u\|_{C^{2+\alpha}_{\text{loc}}} &\leq c\left(\|u\|_{L^\infty} + \|\Delta u\|_{C^\alpha}\right), \\ \|\nabla u\|_{C^\alpha} &\leq c\left(\|u\|_{L^2} + \|\Delta u\|_{C^\alpha}\right), \\ \|u\|_{C^{2+\alpha}} &\leq c\left(\|u\|_{C^0} + \|\Delta u\|_{C^\alpha} + \|u\|_{C^{2+\alpha}(\partial\Omega)}\right). \end{aligned}$$

If L has trivial kernel then

$$\|u\|_{2,2} \leq c\,\|Lu\|_{C^\alpha}.$$

7-13 Cordes-Nirenberg's inequality: [4].

$$\|u\|_{C^{2-\epsilon}_{\text{loc}}} \leq c(\|u\|_{L^\infty} + \|\Delta u\|_{L^\infty}).$$

7-14 Hölder regularity near boundary: [48, Lemma 2]. For open sets U and U' satisfying $\partial\Omega \subset U' \subset\subset U$ one has

$$\|u\|_{1,p} + \|u\|_{C^{1,\alpha}(U')} \leq c\left(\|\Delta u\|_1 + \|\Delta u\|_{C^\infty(U)}\right)$$

for $p < \frac{n}{n-1}$ and every function u vanishing on $\partial\Omega$.

7-15 De Giorgi-Nash-Moser regularity theorem: [74, 40, p. 53]. If u is a weak solution of $\nabla \cdot A(x)\nabla u = 0$ with uniformly positive $A \in L^\infty$ and $\Omega' \subset\subset \Omega$ then

$$\|u\|_{C^\alpha(\Omega')} \leq c\|u\|_2$$

for some $\alpha > 0$.

7-16 A priori estimates for nonlinear equations: [42]. Every positve solution of the Dirichlet problem $Lu = f(x,u)$ with subcritical f satisfies

$$\|u\|_\infty \leq c_\Omega.$$

7-17 Campanato's theorem: [61, 40, p. 70-72], [41, p. 41]. If $0 < \alpha < 1$ and $\int_{B_x^r} |u - u_{B_x^r}|^p \leq c\rho^{n+\alpha p}$ then

$$u \in C^\alpha_{loc}.$$

If $n < \lambda \leq n + p$ then

$$\|u\|_{C^{\frac{\lambda-n}{p}}} \leq c\|u\|_{L^{p,\lambda}}.$$

In fact $L^{p,\lambda} = C^{\frac{\lambda-n}{p}}$ are isomorphic.

7-18 Morrey's Dirichlet growth theorem: [40, 41]. If $\alpha > 0$, $u \in H^{1,p}_{\text{loc}}$ and $\int_{B_x^r} |\nabla u|^p \leq c\rho^{n-p+\alpha p}$ for every ball then

$$u \in C^\alpha_{\text{loc}}.$$

8 Further integral inequalities for solutions of elliptic differential equations

8-1 Mean value properties: If $-\Delta u \leq 0$ then

$$u(x) \leq \frac{1}{|\partial B^\rho|} \int_{\partial B_x^r} u$$

whenever $B_x^r \subset \Omega$. If $-\Delta u = 0$ then

$$|D^\alpha u(x)| \leq c_\alpha \frac{1}{\rho^{\frac{n}{2}+|\alpha|}} \|u\|_{L^2(B_x^r)}.$$

If $n = 2$ and $-\Delta u \leq Ke^u$ then

$$\int_{B_x^r} e^u \leq \frac{c_1 \rho^2}{1 + c_2 \rho^2}$$

for ρ small enough. Best constants are known [8].

8-2 Harnack's inequality: [43]. If $K \subset\subset \Omega$ then

$$\sup_K u \leq c \inf_K u$$

for every $u \in C^2$ with $\Delta u = 0$ and $u > 0$ in Ω.

8-3 Weak Harnack inequality: [75, 43, p. 194]. If $u \geq 0$, $1 < p < \frac{n}{n-2}$ and $q > n$ then

$$\frac{1}{c}\rho^{\frac{n}{p}} \left(\sup_{B_x^{\rho/2}} u - \rho^{2-\frac{n}{q}} \|\Delta u\|_q \right) \leq \|u\|_{L^p(B_x^r)} \leq c\rho^{\frac{n}{p}} \left(\inf_{B_x^{2\rho}} u + \rho^{2-\frac{n}{q}} \|\Delta u\|_q \right).$$

8-4 Caccioppoli's inequality: [40, p. 77]. If $-\Delta u = 0$ then

$$\int_{B_x^r} |\nabla u|^2 \leq \frac{c}{(r-\rho)^2} \int_{B_x^r \setminus B_x^\rho} |u|^2.$$

8-5 Reverse Hölder inequality: [40, p. 119, 136]. If $-\Delta u = 0$ then

$$\int_{B_x^r} |\nabla u|^2 \leq \frac{c}{\rho^2} \left(\int_{B_x^{2\rho}} |\nabla u|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}},$$

$$\int_{B_x^r} |\nabla u|^p \leq \frac{c}{\rho^{\frac{n(p-2)}{2}}} \left(\int_{B_x^{2\rho}} |\nabla u|^2 \right)^{\frac{p}{2}}$$

for $p > 2$.

8-6 Monotonicity formula for harmonic maps: [93]. If $u : \Omega \to \mathbb{R}^n$ is a harmonic map (strongly) and $R \geq \rho$ then

$$\rho^{2-n} \int_{B_x^r} |Du|^2 \leq R^{2-n} \int_{B_x^R} |Du|^2$$

8-7 Kato's inequality: [58, Lemma 9]. If $u \in C^2$, $\phi \in C_0^\infty$, $\phi \geq 0$ then

$$\int_\Omega \Delta\phi\, |u| \;\geq\; \int_\Omega \mathrm{sign}\,(\mathrm{u})\, \phi \Delta \mathrm{u}.$$

8-8 Inequality for sub- and supersolutions: [64]. A pointwise maximum (minimum) of subsolutions (supersolutions) of $Lu = f$ is a subsolution (supersolution). The same is true for H^1 functions.

8-9 Pohozaev identity: [100]. If

$$\begin{aligned} -\Delta u &= f'(u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

then

$$\frac{n-2}{2}\int_\Omega |\nabla u|^2 - n\int_\Omega f \circ u + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 x \cdot \nu \;=\; 0.$$

If Ω is starshaped, then

$$\int_\Omega |\nabla u|^2 \;\leq\; \frac{2n}{n-2}\int_\Omega f \circ u.$$

9 Calculus of variations

9-1 Direct method: [100, p. 4]. A weakly lower semicontinuous coercive functional on a reflexive Banach space attains its minimum. I.e. if $F(u_i) \to \inf F$ and $u_i \rightharpoonup u$ weakly then $F(u) \leq \liminf F(u_i) = \inf F$.

9-2 Weak lower semi-continuity of norm: If (u_i) is a bounded sequence in a reflexive Banach space then

$$u_i \;\rightharpoonup\; u \text{ weakly}$$

for a subsequence. If $u_i \rightharpoonup u$ weakly in a Banach space then

$$\|u\| \;\leq\; \liminf \|u_i\|.$$

If $u_i \rightharpoonup u$ weakly and $\|u_i\| \to \|u\|$ in a uniformly convex Banach space then

$$u_i \;\to\; u.$$

9-3 Brézis-Lieb's lemma: [20]. If a bounded sequence (u_i) in L^p converges pointwise a.e. to a function u then

$$\liminf \|u_i - u\|_p^p \;=\; \liminf \|u_i\|_p^p - \|u\|_p^p.$$

9-4 Maximal distance to weak L^p-limits: [34]. If $u_i \rightharpoonup u$ weakly in L^p then

$$\liminf \|u_i - u\|_p^p \;\leq\; c \liminf \|u_i\|_p^p$$

with $c = \max_{0\leq\alpha\leq 1}\left(\alpha^{p-1} + (1-\alpha)^{p-1}\right)\left(\alpha^{\frac{1}{p-1}} + (1-\alpha)^{\frac{1}{p-1}}\right)^{p-1}$.

9-5 Semicontinuity theorem: [40, p. 23 and 25], [41, p. 13]. If $f \in C(\Omega, \mathbb{R}^m, \mathbb{R}^{mn})$ is bounded below and convex and continuous in the last argument, $u_i \rightharpoonup u$ weakly in $H^{k,p}_{\mathrm{loc}}$ or $u_i \to u$ in L^1_{loc} then

$$\int_\Omega f(\cdot, u, Du) \;\leq\; \liminf \int_\Omega f(\cdot, u_i, Du_i).$$

10 Compactness theorems

10-1 Ascoli's compactness theorem: [3]. If (u_i) is a bounded sequence of equi-continuous functions in $C(K)$ with compact K then

$$u_i \ \rightarrow \ u \ \text{in} \ C(K)$$

for a subsequence.

10-2 Dunford-Pettis compactness criterion: [3, p 176]. [30, Theorem 25]. If the sequence (u_i) is bounded and equi-integrable in L^1 then

$$u_i \ \rightharpoonup \ u \ \text{in} \ L^1$$

for a subsequence.

10-3 Fréchet-Kolmogorov compactness theorem: Suppose (u_i) is a bounded sequence in L^p with $p < \infty$. If for every ε a compact set $K \subset\subset \Omega$ exists such that $\sup_i \|u_i\|_{L^p(\Omega\setminus K)} < \varepsilon$ and $\sup_i \|u_i(\cdot - h) - u_i\|_p \rightarrow 0$ as $h \rightarrow 0$ then

$$u_i \ \rightarrow \ u \ \text{in} \ L^p$$

for a subsequence.

10-4 Rellich-Kondrachov compactness theorem: [3].

$$H^{k,p} \ \subset\subset \ L^q \ \text{for} \ q < \frac{np}{n-kp},$$
$$H^{k,p} \ \subset\subset \ C^\alpha \ \text{for} \ \alpha < k - \frac{n}{p}.$$

10-5 Weak compactness in non-reflexive Sobolev spaces: [40, p. 29]. If (u_i) is bounded in $H^{1,1}$ with (∇u_i) uniformly equi-integrable then

$$u_i \ \rightharpoonup \ u \ \text{weakly in} \ H^{1,1}$$

for a subsequence.

10-6 Murat's compactness theorem: [100, p. 30]. If $u_i \rightharpoonup u$ weakly in H_0^1 and (Δu_i) is bounded in L^1 then

$$\nabla u_i \rightarrow \nabla u \ \text{in} \ L^q$$

for every $q < 2$ and a.e.

10-7 Ehrling lemma: [3]. For every triple of nested Banach spaces $X \subset\subset Y \subset Z$ one has

$$\|x\|_Y \ \leq \ \varepsilon\|x\|_X + c_\varepsilon\|x\|_Z.$$

11 Geometric isoperimetric inequalities

The perimeter of a set $A \subset \mathbb{R}^n$ is defined as

$$|\partial A| \;:=\; \sup\left\{\int_A \nabla\cdot v \,:\, v \in C_0^\infty(\mathbb{R}^n,\mathbb{R}^n), |v| \le 1\right\}$$

while the relative perimeter of $A \subset \Omega$ is given by

$$|\partial A|_\Omega \;:=\; \sup\left\{\int_A \nabla\cdot v \,:\, v \in C_0^\infty(\Omega,\mathbb{R}^n),\ |v| \le 1\right\}.$$

For smooth sets $|\partial A|_\Omega = |\partial A \setminus \partial\Omega|$.

11-1 Isoperimetric inequality for perimeter: [81, 24, 47].

$$|\partial A|^{\frac{n}{n-1}} \;\ge\; \left|S^{n-1}\right|^{\frac{n}{n-1}} \frac{|A|}{|B|} \;=\; n\left|S^{n-1}\right|^{\frac{1}{n-1}} |A|$$

with equality for balls. The Fourier analysis proof of Hurwitz and Lebesgue in two dimensions can be found in [106] as well as a variational approach to the general case. A similar inequality holds in spaces of constant curvature [24].

11-2 Bonnesen's inequality: For every set $A \subset \mathbb{R}^2$ one has the following quantitative stability estimate involving the deviation from a disk

$$|\partial A|^2 - 4\pi|A| \;\ge\; \pi^2 h^2$$

where h denotes the minimal width of an annulus containing ∂A. Similar results for higher dimensional convex sets can be found in [82].

11-3 Relative isoperimetric inequality: [24, 83]. If Ω satisfies an interior cone condition, then

$$\min\{|A|, |\Omega\setminus A|\} \;\le\; c\,|\partial A|_\Omega^{\frac{n}{n-1}}.$$

If Ω is a ball equality holds for half balls.

11-4 Relative isoperimetric inequality for planar sets: [7, 9]. Suppose $A \subset \mathbb{R}^2$ is simply connected with $\partial A = \partial A_1 \bigcup \partial A_2$ (disjoint). Denote by κ the curvature of ∂A with respect to the exterior normal. Then

$$|\partial A_2|^2 \;\ge\; 2\left(\pi - \int_{\partial A_1} \kappa^+\right)|A|.$$

Equality holds for sectors.

11-5 Isoperimetric inequality for two-dimensional manifolds: [2, 10, 19, 56]. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain endowed with the conformal metric $p\,|dx|$ of Gaussian curvature K, i.e. $-\Delta\log(p) = Kp^2$. Then

$$|\partial\Omega|_p^2 \;\ge\; 4\pi|\Omega|_p \int_\Omega p^2 - 2|\Omega|_p \int_\Omega K^+$$

where $|\Omega|_p := \int_\Omega p^2$ and $|\partial\Omega|_p^2 := \int_{\partial\Omega} p$. Equality holds for balls in the limit as K tends to a Dirac measure at the center. Moreover

$$L^2 \;\ge\; 4\pi|\Omega|_p - \left(\sup_\Omega K\right)|\Omega|_p^2,$$

Equality holds e.g. if Ω is a ball, K a constant and $p(x) = \frac{1}{1+\frac{K}{4}|x|^2}$.

11-6 Gromov's isoperimetric inequality: [16]. Let M be a compact Riemannian manifold of dimension n and $A \subset M$. If the Ricci curvature of M satisfies $\mathrm{Ric}\,(M) \geq \mathrm{Ric}\,(S^n) = n-1$ then

$$\frac{|\partial A|}{|\partial A^*|} \geq \left(\frac{|M|}{|S^n|}\right)^{\frac{n-1}{n}}$$

where A^* is a gedodesic ball on $S^n = \partial B_0^1 \subset \mathbb{R}^{n+1}$ with $|A^*| = |A|$.

11-7 Isoperimetric inequality of Reilly and Chavel: If $A \subset \mathbb{R}^n$ has smooth boundary then

$$\frac{|\partial A|}{|A|} \geq \frac{n}{\sqrt{n-1}}\sqrt{\mu_2(\partial A)}$$

where μ_2 denotes the first nonzero eigenvalue of the Laplace-Beltrami operator on ∂A [106].

12 Symmetrization

The decreasing rearrangement of a function $u : \Omega \to \mathbb{R}^+$ has been defined in Section 1.1.

12-1 Cavalieri's principle: [88]. The decreasing rearrangement of a positive function satisfies

$$\int_\Omega f \circ u = \int_0^{|\Omega|} f \circ u_*,$$
$$\|u\|_p^p = \int_0^{|\Omega|} {u_*}^p,$$

12-2 Rearrangement inequalities: [52].

$$\int_\Omega uv \leq \int_0^{|\Omega|} u_* v_*,$$
$$\int_\Omega \phi \circ u \leq \int_\Omega \phi \circ v$$

if ϕ is non-decreasing and convex and $\int_0^a u_* \leq \int_0^a v_*$ for every $a \leq |\Omega|$.

12-3 Schwarz symmetrization: [88, 11, 59, 104, 105, 6]. The symmetrized function u^* defined in Section 1.1 satisfies

$$\int_{\Omega^*} \phi\left(|\nabla u^*|\right) \leq \int_\Omega \phi\left(|\nabla u|\right)$$

for every convex, non-decreasing positive function ϕ and every $u \in H_0^1$. In particular

$$\int_{\Omega^*} |\nabla u^*|^p \leq \int_\Omega |\nabla u|^p,$$
$$\int_{\Omega^*} f \circ u^* = \int_\Omega f \circ u$$

for $1 \leq p < \infty$. Equality in the first relation with $p > 1$ implies that $u = u^*$ a.e. up to translation provided that no level set below the top level has positive measure [23].

12-4 Schmidt's inequality: [47]. For every $A \subset B \subset \mathbb{R}^n$ one has

$$\mathrm{dist}\,(\partial A, \partial B) \leq \mathrm{dist}\,(\partial A^*, \partial B^*).$$

12-5 Brunn-Minkowski's inequality: [47, 24]. For $A, B \subset \mathbb{R}^n$ one has

$$|\theta A + (1-\theta)B|^{\frac{1}{n}} \geq \theta\,|A|^{\frac{1}{n}} + (1-\theta)\,|B|^{\frac{1}{n}}$$

where $\theta A + (1-\theta)B := \{\theta a + (1-\theta)b : a \in A, b \in B\}$ and $0 \leq \theta \leq 1$. The same is true for the exterior Lebesgue measure. If A and B are convex and $0 < \theta < 1$ then equality holds if and only if A and B are homothetic.

12-6 Riesz' rearrangement inequality: [52, 25].

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, f(y)g(x-y)h(x) \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, f^*(y)g^*(x-y)h^*(x).$$

12-7 Weinberger-Talenti's inequality: [104]. If

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

and

$$\begin{aligned} -\Delta \tilde{u} &= f^* \text{ in } \Omega^*, \\ \tilde{u} &= 0 \text{ on } \partial\Omega^* \end{aligned}$$

then

$$u^* \leq \tilde{u} \text{ in } \Omega^*.$$

12-8 Harmonic transplantation: [54, 12]. Let $r(x)$ denote the harmonic radius of Ω at x. For radially symmetric $u = \mu \circ G_0 : B_0^{r(x)} \to \mathbb{R}$ define $u_x := \mu \circ G_x$. Then

$$\begin{aligned} \int_\Omega |\nabla u_x|^2 &= \int_{B_0^{r(x)}} |\nabla u|^2, \\ \int_\Omega f \circ u_x &\geq \int_{B_0^{r(x)}} f \circ u \end{aligned}$$

for every $f : \mathbb{R} \to \mathbb{R}^+$. This fact allows to derive upper bounds for eigenvalues and related quantities while symmetrization gives lower bounds.

12-9 Isoperimetric inequality for capacity: [88, 37].

$$\begin{aligned} \frac{\mathrm{cap}_\Omega(A)}{|A|^{\frac{n-2}{n}}} &\geq \frac{\mathrm{cap}(B_0^1)}{|B_0^1|^{\frac{n-2}{n}}} = n(n-2)\,|B_0^1|^{\frac{2}{n}} \quad (n \geq 3), \\ \mathrm{cap}_\Omega(A) \log\frac{|\Omega|}{|A|} &\geq \mathrm{cap}_{B_0^1}(B_0^\rho) \log\frac{|B_0^1|}{|B_0^\rho|} = 4\pi \quad (n = 2) \end{aligned}$$

Equality holds if and only if A is a ball and $\Omega = \mathbb{R}^n$ (in two dimensions if Ω and A are concentric balls).

12-10 Subadditivity of modulus: [84, 54, 12]. If $A \subset B \subset C$ then

$$\frac{1}{\mathrm{cap}_C(A)} \geq \frac{1}{\mathrm{cap}_B(A)} + \frac{1}{\mathrm{cap}_C(B)}.$$

Equality holds if and only if B is a level set of the capacity potential of A with respect to C.

13 Inequalities for eigenvalues

Let $\lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ be the Dirichlet eigenvalues of Ω with corresponding L^2 orthogonal eigenfunctions (ϕ_i) and $E_i := \mathrm{span}(\phi_1, \ldots, \phi_i)$. The Neumann eigenvalues are denoted by $0 = \mu_1 < \mu_2 \le \mu_3 \le \ldots$ A survey on this subject can be found in [84].

13-1 Rayleigh-Ritz characterization of eigenvalues: [29, 7].

$$\begin{aligned} \lambda_i &= \sup_{u \in E_i \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}, \\ \lambda_{i+1} &= \inf_{u \in E_i^\perp \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}. \end{aligned}$$

13-2 Poincaré principle: [87]

$$\lambda_i = \inf_{\substack{E \subset H_0^1 \\ \dim E = i}} \sup_{u \in E \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

and similarly for μ_i with H_0^1 replaced by H^1. This implies:

$$\mu_i \le \lambda_i.$$

13-3 Barta's inequalities: [14, 90]. For every $u \in C^2$, $u > 0$

$$\lambda_1 \ge \inf_{x \in \Omega} \frac{-\Delta u(x)}{u(x)}.$$

If in addition $u = 0$ on $\partial\Omega$ then

$$\lambda_1 \le \sup_{x \in \Omega} \frac{-\Delta u(x)}{u(x)}.$$

In both cases equality holds for the principal eigenfunction.

13-4 Rayleigh-Faber-Krahn's inequality:

$$\lambda_1 \ge \lambda_1(\Omega^*) = \left(\frac{|B|}{|\Omega|}\right)^{\frac{2}{n}} j^2_{\frac{n-2}{2}}$$

where $j^2_{\frac{n-2}{2}}$ is the first zero of the Bessel function $J^2_{\frac{n-2}{2}}$. Equality holds for balls.

13-5 Cheeger-Yau's inequality: [27, 5, 111, 63].

$$\lambda_1 \ge \frac{1}{4} \inf_{A \subset \Omega} \left(\frac{|\partial A|}{|A|}\right)^2 = \frac{1}{4} \inf_{u \in H_0^{1,1}} \left(\frac{\int_\Omega |\nabla u|}{\int_\Omega |u|^2}\right)^2.$$

13-6 Szegö-Weinberger's inequality: [102, 108].

$$\mu_2 \le \mu_2(\Omega^*).$$

13-7 Payne-Weinberger's inequality: [86]. If Ω is convex then

$$\mu_2 \geq \left(\frac{\pi}{\operatorname{diam}(\Omega)}\right)^2$$

13-8 Lichnerowicz-Obata's inequality: [111]. The first nontrivial eigenvalue of a compact Riemannian manifold M is

$$\mu_2(M) \geq \frac{n}{n-1} \inf \operatorname{Ric}(M).$$

Acknowledgment: We thank S. Müller and W. Dörfler for a number of valuable suggestions.

References

[1] ADAMS R.A. *Sobolev spaces*, Academic Press (1975).

[2] ALEXANDROV A.D. *Die innere Geometrie der konvexen Flächen*, Akademie-Verlag, Berlin (1955).

[3] ALT H.W. *Lineare Funktional Analysis*, Springer (1985).

[4] AUBIN T. *Nonlinear Analysis on manifolds, Monge-Ampere equations*, Springer (1982).

[5] A. AVINYÓ, MORA X. *Lower bounds of Cheeger-Osserman type for the first eigenvalue of the n-dimensional membrane problem*, ZAMP 41 (1990), 426-430.

[6] BAERNSTEIN II A. *A uniform approach to symmetrization*, Symposia Math. 35 (1994), 47-91.

[7] BANDLE C. *Extremaleigenschaften von Kreissektoren und Halbkugeln*, Comment. Math. Helv. 46 (1971), 356-380.

[8] BANDLE C. *Mean value theorems for functions satisfying the inequality* $\Delta u + Ke^u \geq 0$, Arch. Rat. Mech. Anal. 51(1973), 70-84.

[9] BANDLE C. *A Geometrical Isoperimetric Inequality and Applications to Problems of Mathematical Physics*, Comment. Math. Helv.49 (1974), 496-511.

[10] BANDLE C. *On a differential inequality and its applications to geometry*, Math. Z. 147 (1976), 253-261.

[11] BANDLE C. *Isoperimetric inequalities and applications*, Pitman Monographs and Studies in Mathematics (1980).

[12] BANDLE C., FLUCHER M. *Harmonic radius and concentration of energy; hyperbolic radius and Liouville's equations* $\Delta U = e^U$ *and* $\Delta U = U^{\frac{n+2}{n-2}}$, SIAM Review 38-2 (1996), 191-238.

[13] BANDLE C., MARCUS M. *Asymptotic behaviour of solutions and their derivatives for semilinear elliptic problems with blowup at the boundary*, Ann. Inst. Henri Poincaré 12 (1995), 155-171.

[14] BARTA J. *Sur la vibration fondamentale d'une membrane*, C.R. Acad. Sc. Paris, 204 (1937), 472-473.

[15] BECKENBACH E.F., BELLMAN R. *Inequalities*, Springer (1983).

[16] BÉNARD P., BESSON G., GALLOT S. *Sur une inégalité isopérimétrique qui généralise celle de Paul-Lévy-Gromov*, Invent. math. 80 (1985), 295-308.

[17] BERESTYCKI H., LIONS P.L. *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. 82.4 (1983), 313-345.

[18] BERNADI C., GIRAULT V., HALPERN L. *Variational formulation for a nonlinear elliptic equation in a three-dimensional exterior domain*, Nonlinear Analysis 15 (1990), 1017–1029.

[19] BOL G. *Isoperimetrische Ungleichung für Bereiche auf Flächen*, Jber. Deutschen Math. Vereinigung 51 (1941) 219-257.

[20] BRÉZIS H., LIEB E. *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88.3 (1983), 486-490.

[21] BRÉZIS H., MERLE F. *Uniform estimate and blow-up behaviour for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Part. Diff. Eq. 16 (1991), 1223-1253.

[22] BRONSTEIN I.N., SEMENDJAJEW K.A. *Taschenbuch der Mathematik*, Harry Deutsch (1978), second edition to appear.

[23] BROTHERS J.E., ZIEMER W.P. *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math. 384 (1988), 153-179.

[24] BURAGO Y.D., ZALGALLER V.A. *Geometric Inequalities*, Springer (1988).

[25] BURCHARD A. *Cases of equality in the Riesz rearrangement inequality*, Annals of Math. 143 (1996), 499-527.

[26] BURTON G.R. *Semilinear elliptic equations on unbounded domains*, Math. Z., 190 (1985), 519-525.

[27] CHEEGER J. *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in Analysis (ed. R. C. Gunning), Princeton Univ. Press (1970), 195-199.

[28] COIFMAN R., LIONS P.L., MEYER Y., SEMMES S. *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. 72(1993), 247-286.

[29] COURANT R., HILBERT D. *Methods of mathematical physics*, Wiley (1989).

[30] DELLACHERIE C., MEYER P.A. *Probabilities and potential*, North-Holland, 1978.

[31] FLEMING W., RISHEL R. *An integral formula for the total gradient variation*, Arch. Math. 11 (1960).

[32] FEFFERMAN C., STEIN E.M. *H^p spaces of several variables*, Acta Math. 129 (1972), 137-193.

[33] FINK A.M., JODEIT M. *Inequalities using maximal averages*, Preprint, 1994.

[34] FLUCHER M. *Best inequalities for the distance of L^p-functions to mean values and weak limits*, Nonlin. Anal., TMA 20 (1993), 1021-1027.

[35] FLUCHER M. *Approximation of Dirichlet eigenvalues on domains with small holes*, J. Math. Analysis & Appl. 193 (1995), 169-199.

[36] FLUCHER M., MÜLLER S. *Concentration of low energy extremals*, Ann. Inst. H. Poincaré - Analyse nonlinéaire, submitted.

[37] FLUCHER M., RUMPF M. *Bernoulli's free-boundary problem, qualitative theory and numerical approximation*, J. Reine Angew. Math. 486 (1997).

[38] FREHSE J. *Capacity methods in the theory of partial differential equations*, Jahresbericht der Deutschen Mathematiker-Vereinigung 84 (1982), 1-44.

[39] GAGLIARDO E. *Proprietà di alcune classi di funzioni in piu variabili*, Ric. Mat. 7 (1958), 102-137.

[40] GIAQUINTA M. *Multiple integrals in the calculus of variations*, Princeton university press (1983).

[41] GIAQUINTA M. *Introduction to regularity theory for nonlinear elliptic systems*, Birkhäuser (1993).

[42] GIDAS B., SPRUCK J. *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Part. Diff. Eqns. 6.8 (1981), 883-901.

[43] GILBARG D., TRUDINGER N.S. *Elliptic partial differential equations of second order*, Springer-Verlag (1983).

[44] GOLDBERG D. *A local version of the Hardy space*, Duke Math. J. 46 (1979), 27-42.

[45] GRISVARD P. *Edge behavior of the solution of an elliptic problem*, Math. Nachrichten 132 (1987), 281-299.

[46] GRISVARD P. *Singular behaviour of elliptic problems in non Hilbertian Sobolev spaces*, Preprint 321, Nice France (1992).

[47] HADWIGER H. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer (1957).

[48] HAN Z-C. *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré - Analyse nonlinéaire 8.2 (1991), 159-174.

[49] HARDY G.H. *Note on a theorem of Hilbert*, Math. Z. 6 (1920), 314-317.

[50] HARDY G.H. *An inequality between integrals*, Messenger of Math. 54 (1925), 15-156.

[51] HARDY G., LITTLEWOOD J. *Some properties of fractional integrals*, Math. Z. 27 (1928), 565-606.

[52] HARDY G.H., LITTLEWOOD J.E., PÓLYA G. *Inequalities*, Cambridge University Press (1952).

[53] HEINZ E. *Elementare Bemerkung zur isoperimetrischen Ungleichung im* $\mathbb{R}^3$, Math. Z. 132 (1973), 319-322.

[54] HERSCH J. *Transplantation harmonique, transplantation par modules, et théorèmes isopérimétriques*, Comment. Math. Helvetici 44.3 (1969), 354-366.

[55] HEWITT E., STROMBERG K. *Real abstract analysis*, Springer (1965).

[56] HUBER A. *On the isoperimetric inequality on surfaces of variable Gaussian curvature*, Ann. of Math. 60, (1954), 237-247.

[57] ITÔ K. (ED.) *Encyclopedic dictionary of mathematics*, MIT Press (1993).

[58] KATO T. *Schrödinger operators with singular potential*, Israel J. Math. 13, (1972), 135-148.

[59] KAWOHL B. *Rearrangements and convexity of level sets in P.D.E*, Springer Lecture Notes 1150 (1985).

[60] KUFNER A. *Weighted Sobolev spaces*, Wiley (1985).

[61] KUFNER A., JOHN O., FUCIK S. *Function spaces*, Noordhoff (1977).

[62] LAZER A.C., MCKENNA P.J. *Asymptotic behaviour of solutions of boundary blow up problems*, Diff. Int. Equ. 7.4 (1994), 1001-1019.

[63] YAU S.T., LI P. *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Symp. Pure Math. 36 (1980), 205-239.

[64] MATANO H. *Asymptotic behaviour and stability of solutions of semilinear diffusion equations*, Publ. RIMS, Kyoto Univ. 15 (1979), 401-454.

[65] MAZ'JA *L^p-estimates of solutions of elliptic boundary value problems in domains with edges*, (1980).

[66] MAZ'JA V.G. *Sobolev spaces*, Springer (1985).

[67] MEYERS N.G. *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Sc. Norm. Sup. Pisa (3) 17 (1963), 717-721.

[68] MILMAN M. *Integrability of the Jacobien of orientation preserving maps: interpolation methods*, C.R.A.S. Paris 317 (1993), 539-543.

[69] MIRANDA C. *Formule di maggiorzione e teorema di esistenza per le funzioni biarmoniche di du variabili*, Giorn. Mat.Battaglini 2(78) (1967), 97-118.

[70] MIRANDA C. *Partial differential equations of elliptic type*, Springer (1970).

[71] MITRINOVIĆ D.S., PEČARIĆ J.E., FINK A.M. *Inequalities involving functions and their integrals and derivatives*, Kluwer, 1991.

[72] MITRINOVIĆ D.S., PEČARIĆ J.E., FINK A.M. *Classical and new inequalities in analysis*, Kluwer, 1993. (Reviewer: Ion Raşa) 26D10 (34A40 45K05)

[73] MORREY B.M. *Multiple integrals in the calculus of variations*, Springer (1966).

[74] MOSER J. *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. 13 (1960), 457-468.

[75] MOSER J. *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. 14 (1961), 577-591.

[76] MÜLLER *A surprising higher integrability property of mappings with nonnegative Jacobien*, Bull. AMS, 21 (1989), 245-248.

[77] MÜLLER S. *Hardy space methods for nonlinear partial differential equations*, Proceedings of EQUADIFF 8, Tatra Mount. Math. Publ. 4 (1994), 159-168.

[78] NI W.M. *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. J. 31 (1988), 801-807.

[79] NIRENBERG L. *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. 13 (1959), 116-162.

[80] OPIC B., KUFNER A. *Hardy-type inequalities*, Pitman Research Notes in Mathematical Series 219 (1990).

[81] OSSERMAN R. *The isoperimetric inequality*, Bull. AMS 84 (1978), 1182-1238.

[82] OSSERMAN R. *A strong form of the isoperimetric inequality in* $\mathbb{R}^n$, Complex variables 9 (1987), 241-249.

[83] PACELLA F., TRICARICO M. *Symmetrization for a class of elliptic equations with mixed boundary conditions*, Atti Sem.Fis. Univ. Modena 34 (1985-86), 75-94.

[84] PAYNE L. E. *Isoperimetric inequalities and their applications*, SIAM Review 9 (1967), 453-488.

[85] PAYNE L.E., PHILIPPIN G.A. *Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillary surfaces and to surfaces of constant mean curvature*, Nonlinear Analysis 3 (1979), 193-211.

[86] PAYNE L.E., WEINBERGER H. *An optimal Poincaré inequality for convex domains*, Arch. Rat. Mech. Anal. 5 (1960), 286-292.

[87] PÓLYA G., SCHIFFER M. *Convexity of functionals by transplantation*, J. d'Anal. Math. 3.2 (1953/54), 245-345.

[88] PÓLYA G., SZEGÖ G. *Isoperimetric inequalities in mathematical physics*, Princeton University Press (1951).

[89] PROTTER M.H., WEINBERGER H. *Maximum principles in differential equations*, Prentice Hall (1967).

[90] PROTTER M.H., WEINBERGER H. *A maximum principle and gradient bounds for linear elliptic equations*, Indiana Univ. Math. J. 23 (1973), 239-249.

[91] REED M., SIMON B. *Methods of mathematical physics. Vol 4 Analysis of operators*, Academic Press (1978).

[92] SEMMES S. *A primer on Hardy spaces and some remarks on a result by Evans and Müller*, Comm. PDE 19 (1994), 277-319.

[93] SIMON L. *Lectures on geometric measure theory*, Australian National University 3 (1983).

[94] SOBOLEV S. *On a theorem of functional analysis*, Amer. Math. Soc. Transl. Ser. 2, 34 (1963), 39-68.

[95] SPERB R. *Maximum principles and their applications*, Academic Press 157 (1981).

[96] STEIN E. *Singular integrals and differentiability properties of functions*, Princeton Univ. Press (1970).

[97] STEIN E. *Harmonic analysis*, Princeton Unversity Press, 1993.

[98] STEIN E., WEISS G. *Fourier analysis on Euclidean spaces*, Princeton Univ. Press (1971).

[99] STRAUSS, W.A. *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. 55 (1977), 149-162.

[100] STRUWE M. *Variational methods*, Springer-Verlag (1990, 1996).

[101] SWEERS G. *Positivity for a strongly coupled elliptic system by Green function estimates*, J. Geometric Analysis 4 (1994), 121-142.

[102] SZEGÖ G. *Inequalities for certain eigenvalues of a membrane of given area*, J. Rat. Mech. Anal. 3 (1954), 343-356.

[103] TALENTI G. *Best constant in Sobolev inequality*, Ann. Math. Pura Appl. 110 (1976), 353-372.

[104] TALENTI G. *Elliptic equations and rearrangements*, Ann. Scuola Norm. Pisa 4.3 (1987), 185-195.

[105] TALENTI G. *On isoperimetric theorems of mathematical physics*, Handbook of convex geometry B (1993), 1131-1147.

[106] TALENTI G. *The standard isoperimetric theorem*, Handbook of convex geometry A (1993), 73-124.

[107] TORCHINSKY A. *Real variable methods in harmonic analysis*, Academic Press (1986).

[108] WEINBERGER H. *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Rat. Mech. Anal., 5 (1959), 533-636.

[109] WEINBERGER H. *Symmetrization in uniformly elliptic problems*, Studies in Mathematical Analysis and Related Topics Stanford University Press (1962).

[110] H.C. WENTE *An existence theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl. 26 (1969), 318-344.

[111] YAU S.T. *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. 8 (1975), 487-507.

[112] ZEIDLER E. *Nonlinear functional analysis and its applications, Vol. IV*, Springer (1988).

[113] ZIEMER, W.P. *Weakly Differentiable Functions, Sobolev Spaces and Functions of Bounded Variation*, Springer (1989).

Catherine Bandle
Universität Basel
Mathematisches Institut
Rheinsprung 21
CH–4051 Basel
SCHWEIZ
bandle@math.unibas.ch

Martin Flucher
Universität Basel
Mathematisches Institut
Rheinsprung 21
CH–4051 Basel
SWITZERLAND
flucher@math.unibas.ch
http://www.math.unibas.ch/~flucher/