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Point interactions on bounded domains

Wim Caspers and **Guido Sweers**
Department of Pure Mathematics, Delft University of Technology,
P.O. Box 5031, 2600 GA Delft, The Netherlands

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The Laplacian operator Δ on a bounded domain Ω in $\mathbb{R}^n$ containing 0, with Dirichlet boundary condition, is perturbed by a pseudopotential δ, the Dirac measure at 0. Such a perturbation will be defined in $L_p(\Omega)$ for $n=2$, $1<p<\infty$, and for $n=3$, $\frac{3}{2}<p<3$, and is shown to be the generator of an analytic semigroup. Thus solutions of the corresponding evolutionary system are well defined. The necessary estimates involve the Gagliardo–Nirenberg inequality and the Kato inequality.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N>1)$ that contains 0, and has a C^2-boundary. Consider the evolution type system

$$\begin{cases} \left(\dfrac{\partial}{\partial t}-\Delta+\delta\right)u(t,\cdot)=0 & \text{in } \Omega, \quad t>0, \\ u(t,\cdot)=0 & \text{on } \partial\Omega, \quad t>0, \\ u(0,\cdot)=u_0(\cdot) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\delta(u)(t,\cdot)=u(t,0)\,\delta_0$ and δ_0 is the Dirac measure at $0\in\Omega$. The perturbation of $-\Delta$ by the pseudopotential δ is called a *point interaction*. We show that, at least for some N and p, one can solve (1.1) by

$$u(t)=S(t)u_0,$$

where $\{S(t)\}_{t\geqq 0}$ is an analytic semigroup on $L^p(\Omega)$.

It will be sufficient to give an appropriate version of $-\Delta+\delta$ with domain D, corresponding to the Dirichlet boundary condition, such that for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda$ large

$$\lambda-\Delta+\delta : D\subseteq L^p(\Omega)\to L^p(\Omega)$$

has an inverse and, with some constant M,

$$\|(\lambda-\Delta+\delta)^{-1}\|_{\mathscr{L}(L^p(\Omega))}\leqq M\frac{1}{|\lambda|}.$$

Then, see [**14**, Theorem 2.5.2], $-\Delta+\delta$ will be the negative generator of an analytic semigroup on $L^p(\Omega)$.

For our approach to work we need $W^{2,p}(\Omega)$ and $W^{2,q}(\Omega)$, with q defined by $(1/p)+(1/q)=1$, to be both continuously embedded in $C^0(\bar{\Omega})$, that is $2-(N/p)>0$

and $2-(N/q)>0$. Hence the restriction will be

$$\frac{N}{2}<p<\frac{N}{N-2},$$

which implies $N=2$ and $p\in(1,\infty)$, or $N=3$ and $\frac{3}{2}<p<3$.

The paper is organised as follows. In Section 2 we summarise some facts on point interactions in $L^p(\mathbb{R}^N)$. In Section 3, respectively 4, we construct the point interaction and its resolvent operator. In Section 5 we show pointwise estimates for the Green function $\varphi_\lambda = G_{\lambda,\Omega}(0,\cdot)$ at zero. The L^p norm of φ_λ will be estimated from above using the Gagliardo–Nirenberg inequality. The main result is proved in the last section.

Application of point interactions to the study of singular solutions of the semilinear problem $-\Delta u+|u|^{s-1}u=-\sigma u$ with $\sigma>0$ will be treated in a forthcoming paper. For results on point interactions on bounded domains with periodic boundary conditions, see [**3**].

2. Preliminaries

In this section we summarise some facts on point interactions in $L^p(\mathbb{R}^3)$ taken from [**2, 5, 6**].

Point interactions in $L^2(\mathbb{R}^3)$

Consider the Laplacian operator in $L^2(\mathbb{R}^3)$

$$-\Delta: W^{2,2}(\mathbb{R}^3)\subseteq L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3).$$

The point interaction in $L^2(\mathbb{R}^3)$ can be defined as a selfadjoint extension of the restricted Laplacian operator A_0 in $L^2(\mathbb{R}^3)$ with

$$D(A_0)=\{u_0\in W^{2,2}(\mathbb{R}^3): u_0(0)=0\},\quad A_0u_0=-\Delta u_0$$

(see [**2**]). Note that by the Sobolev embedding theorem (see e.g. [**1**]) $W^{2,2}(\mathbb{R}^3)$ can be embedded into the space of continuous functions. So for $u\in W^{2,2}(\mathbb{R}^3)$ the value $u(0)$ is well defined. If $N\geqq 4$ this is no longer the case and this approach fails. (For point interactions in higher dimensions, see [**7, 16**]) The adjoint operator A_0^* can be described as follows. For $\lambda\in\rho(-\Delta)$, let $G_{\lambda,\mathbb{R}^3}(x,y)$ denote the Green function corresponding to $(\lambda-\Delta)^{-1}$, that is

$$G_{\lambda,\mathbb{R}^3}(x,y)=\frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|}\quad\text{for } x\neq y,$$

and define $\varphi_{\lambda,\mathbb{R}^3}(x)=G_{\lambda,\mathbb{R}^3}(0,x)$ for $x\neq 0$. Then (see e.g. [**15**])

$$D(A_0^*)=\{u+c(\varphi_{-i}+\varphi_i): u\in W^{2,2}(\mathbb{R}^3),\ c\in\mathbb{C}\},$$

$$A_0^*(u+c(\varphi_{-i}+\varphi_i))=-\Delta u+c(i\varphi_{-i}-i\varphi_i).$$

Note that every selfadjoint extension of A_0 is a symmetric restriction of A_0^*. All selfadjoint extensions are given by the one-parameter family of operators

$\{-\Delta_\tau\}_{\tau\in(-\pi,\pi]}$ (see [2, I.1.1]), with

$$D(-\Delta_\tau)=\{u_0+c(e^{-\frac{1}{2}i\tau}\varphi_{-i}+e^{\frac{1}{2}i\tau}\varphi_i):u_0\in W^{2,2}(\mathbb{R}^3),\ u_0(0)=0,\ c\in\mathbb{C}\},$$

$$-\Delta_\tau(u_0+c(e^{-\frac{1}{2}i\tau}\varphi_{-i}+e^{\frac{1}{2}i\tau}\varphi_i))=-\Delta u_0+c(ie^{-\frac{1}{2}i\tau}\varphi_{-i}-ie^{\frac{1}{2}i\tau}\varphi_i).$$

This family of selfadjoint extensions defines a family of point interactions in the following way.

The domain of the point interaction is contained in $D(A_0^*)$. We extend the pseudopotential, the Dirac measure, such that it is defined on φ_{-i} and φ_i as well. The Laplacian operator will also be extended. The sum of this extended Laplacian operator and a multiple of the extended pseudopotential is an operator in $D(A_0^*)$. For suitable $\tau\in(-\pi,\pi]$ the operator $-\Delta_\tau$ is the restriction of this sum to $D(-\Delta_\tau)$.

Let δ_0 denote the Dirac measure at zero. Using the Fermi pseudopotential $\delta(\partial/\partial r)r$ [2, 8] the operator $4\pi\delta: W^{2,2}(\mathbb{R}^3)\to(W^{2,2}(\mathbb{R}^3))'$ with $\delta(u)=4\pi u(0)\delta_0$ can be extended to an operator $4\pi\delta(\partial/\partial r)r: D(A_0^*)\to(W^{2,2}(\mathbb{R}^3))'$, where for $u\in W^{2,2}(\mathbb{R}^3)$ and $c\in\mathbb{C}$

$$4\pi\delta\frac{\partial}{\partial r}r(u+c(\varphi_{-i}+\varphi_i))=\left[\frac{\partial}{\partial r}(4\pi ru+c(e^{-\sqrt{-i}r}+e^{-\sqrt{i}r}))\right](0)\delta_0=[4\pi u(0)-\sqrt{2}c]\delta_0.$$

Next one can extend the Laplacian operator to the operator

$$-\Delta_{-2,2}:L^2(\mathbb{R}^3)\subseteq(W^{2,2}(\mathbb{R}^3))'\to(W^{2,2}(\mathbb{R}^3))'$$

by taking the closure of the Laplacian operator in $(W^{2,2}(\mathbb{R}^3))'$. (Compare [4, Section 2].) Fix $\tau\in(-\pi,\pi]$. Then for $u_0+c(e^{-\frac{1}{2}i\tau}\varphi_{-i}+e^{\frac{1}{2}i\tau}\varphi_i)\in D(-\Delta_\tau)$ and $v\in W^{2,2}(\mathbb{R}^3)$

$$\begin{aligned}&\left[\left(-\Delta_{-2,2}+4\pi a_\tau\delta\frac{\partial}{\partial r}r\right)(u_0+c(e^{-\frac{1}{2}i\tau}\varphi_{-i}+e^{\frac{1}{2}i\tau}\varphi_i))\right](v)\\&\quad=\int_{\mathbb{R}^3}-\Delta u_0 v\,dx+\int_{\mathbb{R}^3}c(ie^{-\frac{1}{2}i\tau}\varphi_{-i}-ie^{\frac{1}{2}i\tau}\varphi_i)v\,dx+c(e^{-\frac{1}{2}i\tau}+e^{\frac{1}{2}i\tau})v(0)\\&\qquad+c(e^{-\frac{1}{2}i\tau}e^{3/4\pi i}+e^{\frac{1}{2}i\tau}e^{-3/4\pi i})\cdot v(0)\\&\quad=\int_{\mathbb{R}^3}(-\Delta u_0+c(ie^{-\frac{1}{2}i\tau}\varphi_{-i}-ie^{\frac{1}{2}i\tau}\varphi_i))v\,dx\end{aligned}$$

for suitable a_τ, assuming $\cos\frac{3}{4}\pi-\frac{1}{2}\tau\neq 0$, that is $\tau\neq\frac{1}{2}\pi$. For this a_τ the operator $-\Delta_\tau$ is the part in $L^2(\mathbb{R}^3)$ of $-\Delta_{-2,2}+4\pi a_\tau\delta(\partial/\partial r)r: D(A_0^*)\subseteq(W^{2,2}(\mathbb{R}^3))'\to(W^{2,2}(\mathbb{R}^3))'$.

Point interactions in $L^p(\mathbb{R}^3)$

For $1<p<\infty$, consider the Laplacian operator in $L^p(\mathbb{R}^3)$

$$-\Delta: W^{2,p}(\mathbb{R}^3)\subseteq L^p(\mathbb{R}^3)\to L^p(\mathbb{R}^3).$$

We define $-\Delta_{-2,p}: L^p(\mathbb{R}^3)\subseteq(W^{2,q}(\mathbb{R}^3))'\to(W^{2,q}(\mathbb{R}^3))'$ as the closure of $-\Delta$ in $(W^{2,q}(\mathbb{R}^3))'$. (Compare [4].) Here $(1/p)+(1/q)=1$. Under the restriction $\frac{3}{2}<p<3$, it follows from the Sobolev embedding theorem that $W^{2,p}(\mathbb{R}^3)$ and $W^{2,q}(\mathbb{R}^3)$ can be embedded continuously into the space of bounded continuous functions. Therefore the value $u(0)$ is well defined for $u\in W^{2,p}(\mathbb{R}^3)$ and $\delta_0\in(W^{2,q}(\mathbb{R}^3))'$. Define

$4\pi\delta(\partial/\partial r)r: W^{2,p}(\mathbb{R}^3)\oplus[\varphi_{1,\mathbb{R}^3}]\to(W^{2,q}(\mathbb{R}^3))'$ by

$$4\pi\delta\frac{\partial}{\partial r}r(u+c\varphi)=(4\pi u(0)-c)\delta_0.$$

For notational convenience, we make use of $\varphi_{1,\mathbb{R}^3}$ instead of $\varphi_{\pm i}$. Note that $\varphi_{1,\mathbb{R}^3}-\frac{1}{2}(\varphi_{-i}+\varphi_i)\in H^{2,p}(\mathbb{R}^3)$. The point interaction

$$-\Delta+4\pi\delta\frac{\partial}{\partial r}r$$

in $L^p(\mathbb{R}^3)$ is by definition the part in $L^p(\mathbb{R}^3)$ of $-\Delta_{-2,p}+4\pi\delta(\partial/\partial r)r$. Explicit formulae for the resolvents of point interactions can be derived using [**11**], see [**5**, **6**]. In order to prove that $-\Delta+4\pi\delta(\partial/\partial r)r$ is the negative generator of an analytic semigroup on $L^p(\mathbb{R}^3)$, estimates on the function $\varphi_{\lambda,\mathbb{R}^3}(x)=G_{\lambda,\mathbb{R}^3}(0,x)$ are needed. The L^p-norm of $\varphi_{\lambda,\mathbb{R}^3}$ has to be estimated from above, whereas estimates on $[\varphi_{\lambda,\mathbb{R}^3}-\varphi_{1,\mathbb{R}^3}](0)$ are needed from below. These estimates can be obtained from the explicit expression $\varphi_{\lambda,\mathbb{R}^3}(x)=e^{-\sqrt{\lambda}|x|}/4\pi|x|$. Similar results hold for point interactions in $L^p(\mathbb{R}^2)$ with $1<p<\infty$. See [**5**].

3. The construction of point interactions in $L^p(\Omega)$

Let Ω be a bounded open subset of $\mathbb{R}^N$ with $N=2$ or 3. Assume that $\partial\Omega$ is C^2 and $0\in\Omega$. Let A_p be the Laplacian operator

$$A_p: W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\subseteq L^p(\Omega)\to L^p(\Omega),$$

$$A_p u=-\Delta u.$$

(For the definitions of $W^{2,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, see e.g. [**1**].) This operator A_p is the negative generator of an analytic semigroup on $L^p(\Omega)$. Assume $N=2$ and $1<p<\infty$, or $N=3$ and $\frac{3}{2}<p<3$. Then the spaces $D(A_p)$ and $D(A_q)$, supplied with the graph norm (where $(1/p)+(1/q)=1$) can be embedded continuously into the space of bounded continuous functions supplied with the supremum norm (see e.g. [**1**]). Therefore the operator $\delta: D(A_p)\to(D(A_q))'$ with

$$\delta(u):=u(0)\delta_0$$

for $u\in W_0^{2,p}(\Omega)$, is well defined. Here δ_0 denotes the Dirac measure centred at zero. We want to define point interactions, that is the perturbation of A by a complex number of times δ, as an operator in $L^p(\Omega)$. For that we extend the operators A_p and δ and define the point interaction as the part in $L^p(\Omega)$ of the sum of these extended operators. Define

$$A_{-1,p}: L^p(\Omega)\subseteq(D(A_q))'\to(D(A_q))'$$

as the closure of A_p in $(D(A_q))'$ under the usual identification $L^p(\Omega)=(L^q(\Omega))'$. (See [**4**, Section 2].) Moreover, extend the operator δ to the operator $\boldsymbol{d}_N$ with

$$D(\boldsymbol{d}_N):=\{u+c\varphi: u\in D(A_p),\ c\in\mathbb{C}\},$$

$$\boldsymbol{d}_N(u+c\varphi):=(u(0)-c)\delta_0,$$

where

$$\varphi := (A_{-1,p})^{-1}\delta_0. \tag{3.1}$$

We use (3.1) throughout the paper. Observe that $\varphi \in C(\bar{\Omega}\backslash\{0\})$ and $\lim_{|x|\to 0}\varphi(|x|)/\mu_N(|x|)$ exits and is positive, with $\mu_2(|x|) = \ln|x|$ and $\mu_3(|x|) = |x|^{-1}$. It follows from our assumption on N and p that $\varphi \in L^p(\Omega)$ and consequently $D(\boldsymbol{d}_N) \subseteq L^p(\Omega)$. For $u + c\varphi \in D(\boldsymbol{d}_N)$,

$$\begin{aligned}(A_{-1,p} + \boldsymbol{d}_N)(u + c\varphi) &= -\Delta u + c\delta_0 + (u(0) - c)\delta_0\\ &= -\Delta u + u(0)\delta_0.\end{aligned}$$

DEFINITION 3.1. The point interaction $-\Delta + \boldsymbol{d}_N$ in $L^p(\Omega)$ is the part of $A_{-1,p} + \boldsymbol{d}_N$ in $L^p(\Omega)$. In other words,

$$\begin{aligned}D(-\Delta + \boldsymbol{d}_N) &:= \{u_0 + c\varphi : u_0 \in D(A_p),\, u_0(0) = 0,\, c \in \mathbb{C}\},\\ (-\Delta + \boldsymbol{d}_N)(u_0 + c\varphi) &:= -\Delta u_0.\end{aligned}$$

For $\alpha \in \mathbb{C}\backslash\{0\}$ the point interaction $-\Delta + \alpha\boldsymbol{d}_N$ in $L^p(\Omega)$ is the part of $A_{-1,p} + \alpha\boldsymbol{d}_N$ in $L^p(\Omega)$. In other words,

$$\begin{aligned}D(-\Delta + \alpha\boldsymbol{d}_N) &:= \{u_0 + c(\varphi + u_\alpha) : u_0 \in D(A_p),\, u_0(0) = 0,\, c \in \mathbb{C}\},\\ (-\Delta + \alpha\boldsymbol{d}_N)(u_0 + c(\varphi + u_\alpha)) &:= -\Delta u_0 - c\Delta u_\alpha,\end{aligned}$$

where $u_\alpha \in D(A_p)$ such that $u_\alpha(0) = (\alpha - 1)/\alpha$.

REMARK 3.2. This second characterisation is independent of the choice for u_α.

4. The resolvent of the point interaction

Let $\alpha \in \mathbb{C}\backslash\{0\}$ and $u_\alpha \in D(A_p)$ such that $u_\alpha(0) = (\alpha - 1)/\alpha$ be fixed. In order to describe the resolvent of the point interaction $-\Delta + \alpha\boldsymbol{d}_N$, auxiliary operators are needed. For $\lambda \in \rho(-A_p)$, define

$$\begin{aligned}L_\lambda &: \mathbb{C} \to L^p(\Omega),\\ L_\lambda(c) &:= c\varphi_\lambda,\end{aligned}$$

for $c \in \mathbb{C}$, where $\varphi_\lambda := (\lambda + A_{-1,p})^{-1}\delta_0$. Note that for $\lambda \in \rho(-A_p)$ we have $\varphi_\lambda \in L^p(\Omega)$ and $\varphi_\lambda - \varphi \in D(A_p)$. So due to the Sobolev embedding theorem the expression $(\varphi_\lambda - \varphi)(0)$ makes sense. We make use of this observation in the sequel. The other auxiliary operator we need is $\Psi : D(\Psi) \to \mathbb{C}$ with

$$\begin{aligned}D(\Psi) &:= \{u + c(\varphi + u_\alpha) : u \in D(A_p),\, c \in \mathbb{C}\},\\ \Psi(u + c(\varphi + u_\alpha)) &:= u(0) - c.\end{aligned}$$

Using these operators we can prove the following theorem (see also [**5**] or [**10, 11**]).

THEOREM 4.1. *Let $\lambda \in \rho(-A_p)$. Then $\lambda \in \rho(\Delta - \alpha\boldsymbol{d}_N)$ if and only if*

$$(\varphi_\lambda - \varphi)(0) \neq \frac{\alpha - 1}{\alpha}.$$

Moreover if $\lambda \in \rho(\Delta - \alpha \boldsymbol{d}_N)$ *then for* $f \in L^p(\Omega)$

$$(\lambda - \Delta + \alpha \boldsymbol{d}_N)^{-1} f = (\lambda + A_p)^{-1} f + \frac{\delta_0[(\lambda + A_p)^{-1} f]}{\delta_0[\varphi - \varphi_\lambda] + \dfrac{\alpha - 1}{\alpha}} \varphi_\lambda.$$

Proof. From the resolvent identity it follows that

$$\begin{aligned} \varphi_\lambda &= (\lambda + A_{-1,p})^{-1} \delta_0 \\ &= -\lambda(\lambda + A_p)^{-1}(A_{-1,p})^{-1}\delta_0 + (A_{-1,p})^{-1}\delta_0 \\ &= \{-\lambda(\lambda + A_{-1,p})^{-1}\varphi - u_\alpha\} + \{\varphi + u_\alpha\} \\ &\in D(\Psi). \end{aligned} \tag{4.1}$$

Therefore the operator

$$(I + L_\lambda \Psi): D(\Psi) \to D(\Psi)$$

is well defined. Moreover, for $u + c(\varphi + u_\alpha) \in D(\Psi)$,

$$\begin{aligned} &(I + L_\lambda \Psi)(u + c(\varphi + u_\alpha)) \\ &\qquad = u + c(\varphi + u_\alpha) + (u(0) - c)(\{-\lambda(\lambda + A_{-1,p})^{-1}\varphi - u_\alpha\} + \{\varphi + u_\alpha\}) \\ &\qquad = u + (u(0) - c)\{-\lambda(\lambda + A_{-1,p})^{-1}\varphi - u_\alpha\} + u(0)(\varphi + u_\alpha). \end{aligned}$$

So $(I + L_\lambda \Psi)(u + c(\varphi + u_\alpha)) \in D(A_p)$ if and only if $u(0) = 0$ or equivalently, if and only if $u + c(\varphi + u_\alpha) \in D(-\Delta + \alpha \boldsymbol{d}_N)$. Furthermore, for $u + c(\varphi + u_\alpha) \in D(-\Delta + \alpha \boldsymbol{d}_N)$

$$\begin{aligned} (\lambda + A_p)(I + L_\lambda \Psi)(u + c(\varphi + u_\alpha)) &= (\lambda + A_p)(u - c\{-\lambda(\lambda + A_{-1,p})^{-1}\varphi - u_\alpha\}) \\ &= (\lambda u - \Delta u + c\lambda\varphi + c\lambda u_\alpha - c\Delta u_\alpha) \\ &= (\lambda(u + c(\varphi + u_\alpha)) - \Delta u - c\Delta u_\alpha) \\ &= (\lambda - \Delta + \alpha \boldsymbol{d}_N)(u + c(\varphi + u_\alpha)). \end{aligned}$$

So $\lambda \in \rho(\Delta - \alpha \boldsymbol{d}_N)$ if and only if $I + L_\lambda \Psi: D(\Psi) \to D(\Psi)$ is invertible and in that case

$$(\lambda - \Delta + \alpha \boldsymbol{d}_N)^{-1} f = (I + L_\lambda \Psi)^{-1}(\lambda + A_p)^{-1} f. \tag{4.2}$$

It is not difficult to verify that $I + L_\lambda \Psi$ is invertible if and only if $I + \Psi L_\lambda : \mathbb{C} \to \mathbb{C}$ is invertible, and that, if those operators are invertible,

$$(I + L_\lambda \Psi)^{-1} = I - L_\lambda(I + \Psi L_\lambda)^{-1}\Psi. \tag{4.3}$$

Finally we remark that for $c \in \mathbb{C}$ (see (4.1))

$$\begin{aligned} (I + \Psi L_\lambda)(c) &= c + \Psi[(-\lambda)(\lambda + A_{-1,p})^{-1}\varphi - u_\alpha + (\varphi + u_\alpha)]c \\ &= c + \left(\Psi[\varphi_\lambda - \varphi] - \frac{\alpha - 1}{\alpha} - 1\right)c \\ &= \left(\delta_0[\varphi_\lambda - \varphi] - \frac{\alpha - 1}{\alpha}\right)c. \end{aligned}$$

The statement of the theorem now follows from (4.2) and (4.3). $\square$

5. Estimates on the Green function at $(0, x)$

In order to prove that point interactions are negative generators of analytic semigroups on $L^p(\Omega)$, we need estimates on the function φ_λ for $\operatorname{Re}\lambda \geqq 0$. Let $\lambda \in \rho(-A_p)$. When $G_{\lambda,\Omega}(x, y)$ denotes the Green function corresponding to $(\lambda + A_p)^{-1}$, then $\varphi_\lambda(x) = G_{\lambda,\Omega}(0, x)$. That is, for $u \in D(A_p)$,

$$\int_\Omega \varphi_\lambda(\lambda - \Delta)u\, dx = u(0).$$

We write $\varphi_{\lambda,\mathbb{R}^N}(x) = G_{\lambda,\mathbb{R}^N}(0, x)$. The relation between φ_λ and $\varphi_{\lambda,\mathbb{R}^N}$ is given by

$$\varphi_\lambda = \varphi_{\lambda,\mathbb{R}^N} - v_\lambda \tag{5.1}$$

on Ω, where v_λ satisfies

$$\begin{cases} (\lambda - \Delta)v_\lambda = 0 & \Omega, \\ v_\lambda = \varphi_{\lambda,\mathbb{R}^N} & \partial\Omega. \end{cases}$$

LEMMA 5.1. *Let $\lambda \in \rho(A_p)$ and $1 < p < \infty$. Then*

$$\|\varphi_\lambda\|_p = \|\delta_0(\lambda + A_q)^{-1}\|_{\mathscr{L}(L^q(\Omega),\mathbb{C})}.$$

Proof. For $f \in L^q(\Omega)$,

$$|\delta_0(\lambda + A_q)^{-1}| = \left|\int_\Omega \varphi_\lambda(x)f(x)\, dx\right| \leqq \|\varphi_\lambda\|_p \|f\|_q.$$

So $\|\delta_0(\lambda + A_q)^{-1}\|_{\mathscr{L}(L^q,\mathbb{C})} \leqq \|\varphi_\lambda\|_p$. Moreover, defining

$$\varphi_\lambda^* := \|\varphi_\lambda\|_p^{2-p}\bar{\varphi}_\lambda|\varphi_\lambda|^{p-2} \in L^q(\Omega),$$

we have $\|\varphi_\lambda^*\|_q = \|\varphi_\lambda\|_p$ and

$$|\delta_0(\lambda + A_q)^{-1}\varphi_\lambda^*| = \left|\int_\Omega \varphi_\lambda\varphi_\lambda^*\, dx\right| = \|\varphi_\lambda\|_p^2 = \|\varphi_\lambda\|_p\|\varphi_\lambda^*\|_q.$$

So $\|\delta_0(\lambda + A_q)^{-1}\|_{\mathscr{L}(L^q,\mathbb{C})} \geqq \|\varphi_\lambda\|_p$. □

PROPOSITION 5.2. *Let $1 < p < \infty$. For every $\theta \in (0, \pi/2)$, there is an $M_\theta \geqq 0$ such that for $\lambda \in \Sigma_{(\pi/2)+\theta}$*

$$\|\varphi_\lambda\|_p \leqq \frac{M_\theta}{|\lambda|^{1-(N/2q)}} \quad \textit{and} \quad \|\delta_0(\lambda + A_p)^{-1}\|_{\mathscr{L}(L^p(\Omega),\mathbb{C})} \leqq \frac{M_\theta}{|\lambda|^{1-(N/2p)}}$$

(Here $\Sigma_{(\pi/2)+\theta} = \{z \in \mathbb{C} : |\arg z| \leqq (\pi/2) + \theta\}$.)

Proof. Let $\theta \in (0, 2\pi)$ and choose $D_\theta \geqq 0$ such that, for every $\lambda \in \Sigma_{(\pi/2)+\theta}$,

$$\|(\lambda + A_q)^{-1}\|_{\mathscr{L}(L^q)} \leqq \frac{D_\theta}{|\lambda|}.$$

Such a D_θ exists as A_q is the negative generator of an analytic semigroup. Let $\lambda \in \Sigma_{(\pi/2)+\theta}$ and $f \in L^q(\Omega)$ and $u = (\lambda + A_q)^{-1}f$. According to the Gagliardo–Nirenberg inequality (see e.g. [**9**])

$$\|u\|_\infty \leqq \|u\|_{2,q}^{1-s}\|u\|_q^s$$

with $s=1-(N/2q)$. We have for some constant $C_\theta \geqq 0$, independent of λ,

$$\|u\|_{2,q} \leqq C_\theta \|f\|_q$$

(see e.g. [**14**, Chapter 5]) and

$$\|u\|_q \leqq \frac{D_\theta}{|\lambda|} \|f\|_q.$$

Therefore for some constant $M_\theta \geqq 0$, independent of λ,

$$\|u\|_\infty \leqq \frac{M_\theta}{|\lambda|^{1-(N/2q)}} \|f\|_q$$

and consequently

$$\|\delta_0(\lambda + A_q)^{-1}\|_{\mathscr{L}(L^q,\mathbb{C})} \leqq \frac{M_\theta}{|\lambda|^{1-(N/2q)}}.$$

The statement follows from Lemma 5.1. $\square$

PROPOSITION 5.3. *Let* $\operatorname{Re}\lambda \geqq 0$. *Then*

$$\|v_\lambda\|_\infty \to 0$$

as $|\lambda| \to \infty$.

Proof. Due to Kato's inequality [**12**, Lemma A], v_λ satisfies

$$\begin{cases} \Delta|v_\lambda| \geqq \operatorname{Re}\lambda |v_\lambda| & \Omega, \\ |v_\lambda| = |\varphi_{\lambda,\mathbb{R}^N}| & \partial\Omega. \end{cases}$$

As $\operatorname{Re}\lambda \geqq 0$, it follows by the maximum principle that

$$\sup_\Omega |v_\lambda| \leqq \sup_{\partial\Omega} |v_\lambda|. \tag{5.2}$$

The statement of this proposition follows from the fact that $|\varphi_{\lambda,\mathbb{R}^N}| \to 0$ uniformly on $\partial\Omega$ as $|\lambda| \to \infty$. $\square$

REMARK 5.4. Inequality (5.2) also follows from [**13**, Corollary 1.3], with $L = (-(\partial/\partial t) + \Delta)$ and $F(t,x) = e^{i\lambda t} v_\lambda(x)$.

6. The point interaction is the negative generator of an analytic semigroup on $L^p(\Omega)$

THEOREM 6.1. *Let* $N=2$ *and* $1<p<\infty$ *or* $N=3$ *and* $\frac{3}{2}<p<3$. *Then, for* $\alpha \in \mathbb{C}\backslash\{0\}$,

$$-\Delta + \alpha \boldsymbol{d}_N$$

is the negative generator of an analytic semigroup on $L^p(\Omega)$.

Proof. Let $\alpha \in \mathbb{C}\backslash\{0\}$. We shall show that for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqq 0$ and $|\lambda|$ large enough: $\lambda \in \rho(\Delta - \alpha \boldsymbol{d}_N)$ and

$$\|(\lambda - \Delta + \alpha \boldsymbol{d}_N)^{-1}\|_{\mathscr{L}(L^p(\Omega))} = O(|\lambda|^{-1}),$$

as $|\lambda| \to \infty$ and $\operatorname{Re}\lambda \geqq 0$.

For $\lambda \in \rho(-A_p)$ we have with (5.1)

$$\begin{aligned}\delta_0[\varphi_\lambda - \varphi] &= \delta_0[\varphi_\lambda - \varphi_1 + \varphi_1 - \varphi] \\ &= \delta_0[\varphi_{\lambda,\mathbb{R}^N} - \varphi_{1,\mathbb{R}^N}] - \delta_0[v_\lambda - v_1] + \delta_0[\varphi_1 - \varphi].\end{aligned}$$

It follows from Proposition 5.3 that $\delta_0[v_\lambda] \to 0$ as $|\lambda| \to \infty$. Moreover, for $|\lambda|$ large enough, $\delta_0[\varphi_{\lambda,\mathbb{R}^N} - \varphi_{1,\mathbb{R}^N}]$ behaves like $\ln \sqrt{\lambda}$ when $N=2$, and like $|\lambda|^{\frac{1}{2}}$ when $N=3$. Indeed

$$\varphi_{\lambda,\mathbb{R}^2}(x) = \frac{1}{2\pi} K_0(\sqrt{\lambda}|x|)$$

(see [**17**, p. 78]) and

$$\varphi_{\lambda,\mathbb{R}^3}(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|\pi|}.$$

By [**17**, formula (14), p. 80] one finds that $K_0(z) = C - \ln z + o(z)$ for $|z| \to 0$ and $z \in \mathbb{C}\backslash(-\infty, 0]$. Consequently for $|\lambda|$ large enough and $\operatorname{Re}\lambda \geqq 0$ we have $\delta_0[\varphi_\lambda - \varphi] \neq (a-1)/\alpha$ and therefore, by Theorem 4.1, $\lambda \in \rho(\Delta - \alpha \boldsymbol{d}_N)$. It also follows, using Lemma 5.1 and Proposition 5.2 and the fact that A_p is the negative generator of an analytic semigroup on $L^p(\Omega)$, that

$$\begin{aligned}&\|(\lambda - \Delta + \alpha \boldsymbol{d}_N)^{-1}\|_{\mathscr{L}(L^p(\Omega))} \\ &\leqq \|(\lambda + A_p)\|_{\mathscr{L}(L^p(\Omega))} + \frac{\|\delta_0[(\lambda + A_p)^{-1}\cdot]\|_{\mathscr{L}(L^q(\Omega),\mathbb{C})}}{\left|\delta_0[\varphi - \varphi_\lambda] + \dfrac{\alpha - 1}{\alpha}\right|} \|\varphi_\lambda\|_p \\ &= \begin{cases} O(|\lambda|^{-1}) + O(|\lambda|^{-1+(1/p)})O(1)O(|\lambda|^{-1+(1/q)}) & \text{when } N=2 \\ O(|\lambda|^{-1}) + O(|\lambda|^{-1+(3/2p)})O(|\lambda|^{-\frac{1}{2}})O(|\lambda|^{-1+(3/2q)}) & \text{when } N=3 \end{cases} \\ &= O(|\lambda|^{-1})\end{aligned}$$

as $|\lambda| \to \infty$ and $\operatorname{Re}\lambda \geqq 0$. □

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References

1 R. A. Adams. *Sobolev Spaces* (New York: Academic Press, 1975).
2 S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden. *Solvable Models in Quantum Mechanics* (New York: Springer, 1988).
3 S. Albeverio and P. Seba. Wave chaos in quantum systems with point interaction. *J. Statist. Phys.* **64** (1991), 369–383.
4 H. Amann. On abstract parabolic fundamental solutions. *J. Math. Soc. Japan* **39** (1987), 93–116.
5 W. Caspers. On point interactions (Thesis, Delft University of Technology, 1992).
6 W. Caspers and Ph. Clément. Point interactions in L^p. *Semigroup Forum* **46** (1993), 253–265.
7 J. F. van Diejen and A. Tip. Scattering from generalized point interactions using self-adjoint extensions in Pontryagin spaces. *J. Math. Phys.* **32** (1991), 630–641.

8 E. Fermi. Sul moto dei neutroni nelle sostanze idrogenate. *Ricerca Sci.* 7 (1936), 13–52; English translation in *E. Fermi Collected papers*, Vol. I, *Italy, 1921–1938,* 980–1016 (Chicago: University of Chicago Press, 1962).
9 A. Friedman. *Partial Differential Equations of Parabolic Type.* (Englewood Cliffs, N.J.: Prentice-Hall, 1964).
10 G. Greiner. Perturbing the boundary conditions of a generator. *Houston J. Math.* **13** (1987), 213–229.
11 G. Greiner and K. G. Kuhn. Linear and semilinear boundary conditions: The analytic case. In *Semigroup Theory and Evolution Equations: The Second International Conference*, eds. Ph. Clément, E. Mitidieri and B. de Pagter (New York: Marcel Dekker, 1991).
12 T. Kato. Schrödinger operators with singular potentials. *Israel J. Math.* **13** (1972), 135–148.
13 G. Lumer. Principes du maximum paraboliques pour des domaines (x, t) non-cylindriques. *Séminaire de Théorie du Potentiel, no. 8*, Lecture Notes in Mathematics 1235, 105–113 (Berlin: Springer, 1987).
14 A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Berlin: Springer, 1983).
15 M. Reed and B. Simon. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness* (New York: Academic Press, 1972).
16 Yu. G. Shondin. Quantum mechanical models in $\mathbb{R}^N$ related with the extensions of energy operators in Pontrjagin spaces. *Theoretical and Mathematical Physics* **74-3** (1988), 331–344 (in Russian).
17 G. N. Watson. *A Treatise on the Theory of Bessel Functions*, 2nd edn (Cambridge: Cambridge University Press, 1962).

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