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Minicourse: Elliptic PDEs on domains with corners

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Abstract

When one is young and starts studying boundary value problems for pde, the first assumption that one usually makes, is to take the boundary smooth 'enough'. One hopes that the result one proves for such domains may also hold for less smooth domains, at least if one works hard enough to crack all those nasty technical details. In this minicourse I'll hope to demonstrate through several examples that, although indeed most of the time it is just a lot technical details, sometimes the results are strikingly different in the presence of corners.

- The first example concerns a semilinear Laplace equation. It has been guessed that stable solutions of $-\Delta u = f(u)$ in Ω with $u = 0$ on $\partial\Omega$ should have a fixed sign. This is not true but, when the domain Ω and f are bounded and smooth, and one considers $-\Delta u = \lambda f(u)$ for λ large, then at least a global minimizer seems to be of fixed sign. However, if one allows some corner in $\partial\Omega$, then suddenly the global minimizer can be sign-changing no matter how large λ is.
- Sapondzhyan mentioned in 1975 that the model he used for a hinged thin elastic plate resulted in some inconsistency: Whenever concave corners were present, the energy density blows up near such a corner which, in a physical sense, is pretty bad for the material.
- The famous Babushka paradox describes how that same model does not give continuity in the approximation of the solution on a round plate by the solutions on regular n-polygons.
- When using simple finite elements for numerical approximations, one replaces boundary curves by polygons. Is that always ok? And even if one would be using curvilinear elements, how to deal with corners?
- When putting a moderate weight on a horizontal plate that is supported at the edges, one expects it to bend downwards. For a supported plate that is pushed downwards, corners however result locally in upwards(!) moving plates.

- For the eigenvalue problem $\Delta^2\varphi = \lambda\varphi$ in Ω with $\varphi = \Delta\varphi = 0$ on $\partial\Omega$, which corresponds to the hinged plate problem, there is nothing new on smooth domains when compared with the most well-studied Dirichlet-Laplace eigenvalue problem. The eigenfunctions are indeed the same and only the eigenvalues are squared. But how can it happen, that whenever Ω has a concave corner, one may compute two(!) positive eigenfunctions?

It goes without saying that in order to study these examples we have to address the theory for elliptical problems near corners originating from Williams, Kondrat'ev and Grisvard, which, besides discussing these examples, I plan to explain for the elliptic operators appearing above.

Finally I would like to mention Sergei Nazarov and Serge Nicaise, who helped me pass around the corners.

Some preliminairies with corners

1 A Dirichlet problem on a sector

Let us fix standard sector in $\mathbb{R}^2$ with opening angle $\alpha \in (0, 2\pi)$ as follows:

$$\mathsf{C}_\alpha = \left\{(r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2;\ 0 < r < 1 \text{ and } |\varphi| < \tfrac{1}{2}\alpha\right\}. \tag{1}$$

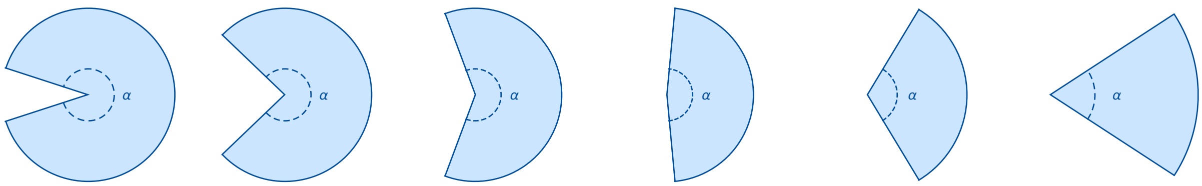
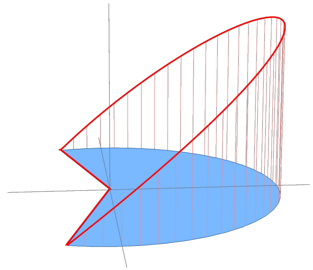


Figure 1: Some sectors from (1)

We consider the Dirichlet problem



$$\begin{cases} -\Delta u = 0 & \text{in } \mathsf{C}_\alpha, \\ u\left(r\cos(\tfrac{1}{2}\alpha), \pm r\sin(\tfrac{1}{2}\alpha)\right) = 0 & \text{for } 0 \le r \le 1, \\ u\left(\cos\varphi, \sin\varphi\right) = \cos\left(\tfrac{\pi}{\alpha}\varphi\right) & \text{for } |\varphi| \le \tfrac{1}{2}\alpha. \end{cases} \tag{2}$$

One may check directly that a solution to (2) is

$$u_\alpha\left(r\cos\varphi, r\sin\varphi\right) = r^{\pi/\alpha}\cos\left(\tfrac{\pi}{\alpha}\varphi\right). \tag{3}$$

Several of these solutions are sketched in Figure 2.

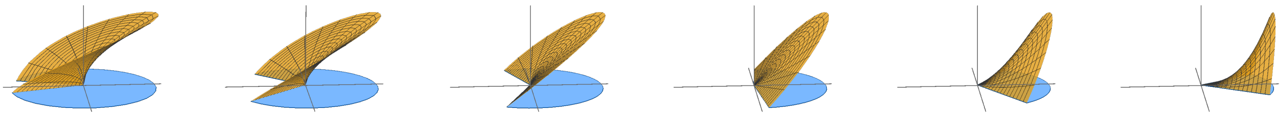


Figure 2: Some solutions from 2

In which function spaces are the functions in (3)? By direct computation one finds:

Lemma 1 *Let $\alpha \in (0, 2\pi)$ and u_α be defined in (3).*

1. *If $\alpha = \frac{\pi}{k}$ for $k \in \mathbb{N}^+$, then $u_\alpha \in C^\infty(\overline{\mathsf{C}_\alpha})$.*
2. *If $\alpha \in (\pi, 2\pi)$, then $u_\alpha \in C^{0,\pi/\alpha}(\overline{\mathsf{C}_\alpha})$.*
3. *If $\alpha \in \left(\frac{\pi}{k+1}, \frac{\pi}{k}\right)$ for some $k \in \mathbb{N}^+$, then $u_\alpha \in C^{k,\pi/\alpha-k}(\overline{\mathsf{C}_\alpha})$.*
4. *If $\left(k - \frac{2}{p}\right)\alpha < \pi$ for some $k \in \mathbb{N}^+$, then $u_\alpha \in W^{k,p}(\mathsf{C}_\alpha)$.*

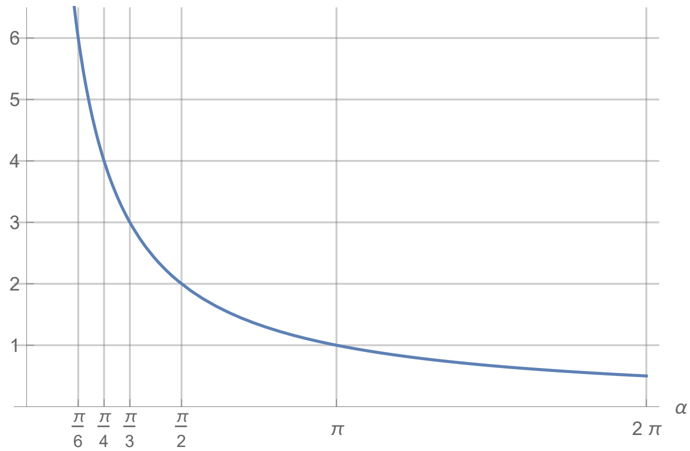


Figure 3: The relation between the opening angle and the regularity of u_α

Remark 2 *A rough conclusion of this lemma would be: smaller angle implies more regularity.*

Remark 3 *The last item implies that for any $\alpha \in (0, 2\pi)$ one finds that $u_\alpha \in W^{1,2}(\mathsf{C}_\alpha)$. Since solutions in $W^{1,2}(\mathsf{C}_\alpha)$ are unique, the function u_α is the unique (weak) solution of (2).*

For the first item one should notice that

$$u_\pi(x) = x_1,\ u_{\pi/2}(x) = x_1 x_2,\ u_{\pi/3}(x) = x_1 \left(x_1^2 - 3x_2^2\right) \text{ etc.}$$

For the Sobolev spaces one finds for noninteger coefficients π/α that $u_\alpha \in W^{k,p}(\mathsf{C}_\alpha)$, if and only if

$$\int_0^1 \left(r^{\pi/\alpha - k}\right)^p \ r\, dr < \infty,$$

which is equivalent to $\left(\frac{\pi}{\alpha} - k\right) p + 1 > -1$.

2 Poisson problems and corners

Without going into details we will compare the solutions u_e and ϕ_1 of

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta \varphi = \lambda_1 \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

for some special domains Ω in $\mathbb{R}^2$. The solution on the left we u_e. On the right is λ_1 the first eigenvalue and φ_1 the first eigenfunction that we may take positive. For a smooth domain such as the disk one finds that there are positive constants c_1 and c_2 such that

$$c_1 u_e(x) \leq \varphi_1(x) \leq c_2 u_e(x) \text{ for all } x \in \Omega. \tag{5}$$

In that case we will write $u_e \sim \varphi_1$ on Ω. For the disk see Figure 4.

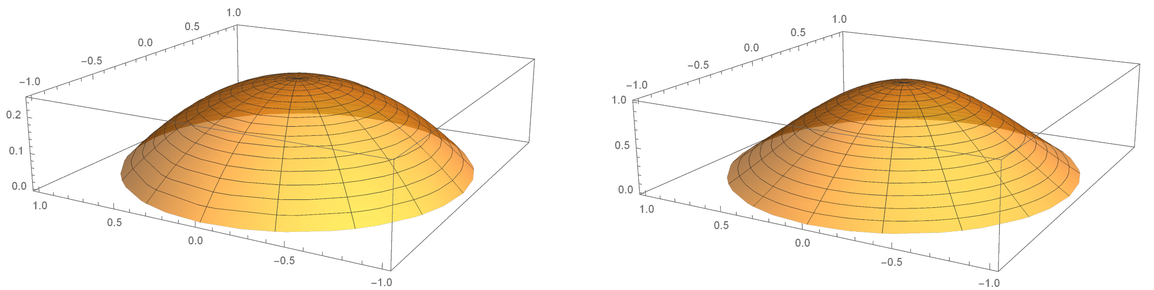


Figure 4: On the left the solution of (4); on the right the first eigenfunction.

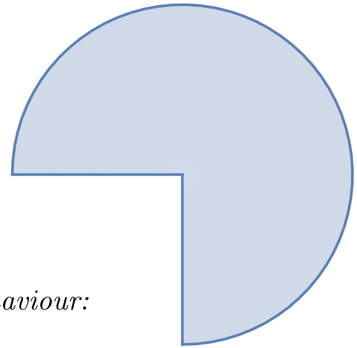
If $\partial\Omega$ is smooth, then Hopf's boundary point Lemma states that for a solution of

$$\begin{cases} -\Delta u = f \gneq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

a constant $c_f > 0$ exists, such that $u(x) \geq c_f d\,(x, \partial\Omega)$ for all $x \in \Omega$. The function $x \mapsto d\,(x, \partial\Omega)$ is the distance to the boundary:

$$d\,(x, \partial\Omega) = \inf\left\{|x - x^*|\,; x^* \in \partial\Omega\right\}.$$

The estimate from above depends on the regularity of the solution. On smooth domains one finds that $u \in C^1(\overline{\Omega})$, whenever $f \in L^p\,(\Omega)$ with $p > n$. See [24].



Example 4 *We take*

$$\Omega = \left\{ (r\cos\theta, r\sin\theta) \,; 0 < \theta < \tfrac{3}{2}\pi \text{ and } 0 < r < 1 \right\}.$$

Near the reentrant corner in 0 *one finds for* u_e *and for* φ_1 *the same behaviour:*

$$u_e\,(r\cos\theta, r\sin\theta) \sim \varphi_1\,(r\cos\theta, r\sin\theta) \sim r^{2/3}\sin\left(\tfrac{2}{3}\theta\right) \text{ for } r < \tfrac{1}{2}.$$

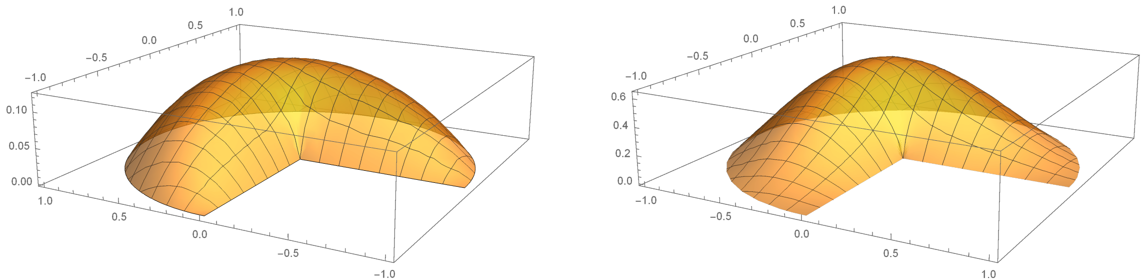


Figure 5: u_e and φ_1 on a 3 quarter disk

Example 5 *For the square* $\Omega = \{(x,y)\,;\ x,y \in (0,1)\}$ *something different happens:*

$$u_e\left(r,r\right) \sim r^2 \log\left(r^{-1}\right) \qquad \textit{and} \qquad \varphi_1\left(r,r\right) \sim r^2.$$

One has different behaviour for constant and linear terms on the right hand side.

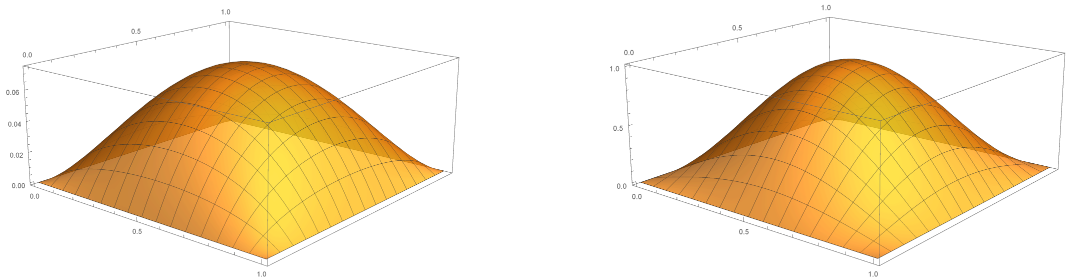
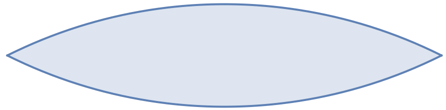


Figure 6: u_e and φ_1 on the square

One may show that in 2d the behaviour of u_e and φ_1 is different if and only if the angle is less than or equal $\pi/2$.

Example 6 *Finally the intersection of two disks:*



$$\Omega = B_{\sqrt{5}}(0,2) \cap B_{\sqrt{5}}(0,-2).$$

The two end points of the intersection are $(\pm 1, 0)$*. One finds near the left end point the following behaviour:*

$$u_e(x,0) \sim (x+1)^2 \qquad \textit{and} \qquad \varphi_1(x,0) \sim (x+1)^{3.3879\ldots}.$$

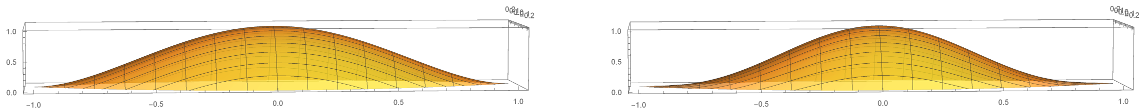


Figure 7: u and φ on the intersection of 2 disks

So, what will be the behaviour of u on domains with corners for $f(u) = c_1 + c_2 u$

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

or even more general f?

3 A second order semilinear elliptic eigenvalue problem and corners

This first example is second order, and being second order it has a huge advantage above higher order, namely one is able to employ the maximum principle, which allows one to transfer pointwise estimates on the source to pointwise estimates of a solution.

Consider for $\Omega \subset \mathbb{R}^n$ a bounded domain the semilinear boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

A domain is an open and connected subset of $\mathbb{R}^n$. The specific f we are interested in, is a bounded sign changing function like the one sketched in Figure 8.

Condition 7 *The semilinear term f in (6) is as follows:*

1. $f \in C^1(\mathbb{R}; \mathbb{R})$;
2. *it has a positive falling zero:* $\exists \rho > 0$ *with* $f(\rho) = 0$ *and such that* $f'(s) < 0$ *for* $s \in (\rho - \varepsilon, \rho)$;
3. $\displaystyle\int_s^\rho f(t)dt > 0$ *for all* $s \in [0, \rho)$.

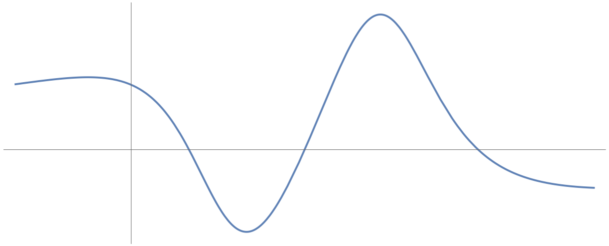


Figure 8: An example for the f in (6)

For the function sketched in Figure 8 one takes $\rho = \rho_3$. For that f Condition 7.3 means that the red area is smaller than the blue area in Figure 9.

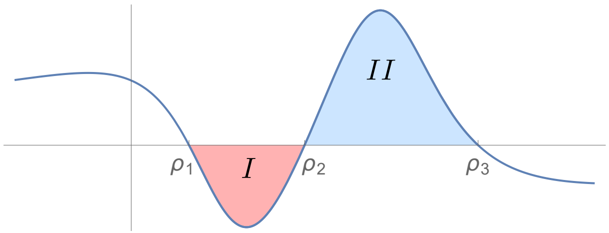


Figure 9: The integral condition in Condition 7 means $I < II$.

Theorem 8 (Clément-S. [10], [50]) *Let f satisfy Condition 7. If $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^{3,\gamma}$ for some $\gamma > 0$, then there is $\varepsilon_0 > 0$ and $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ there is precisely one solution $(\lambda, u_\lambda) \in \mathbb{R}^+ \times C^2(\overline{\Omega})$ satisfying (6),*

$$u_\lambda > 0 \text{ in } \Omega \text{ and } \max_{x\in\Omega} (u_\lambda) \in (\rho_3 - \varepsilon_0, \rho_3)\,.$$

For smooth domains it hardly plays a role if we have $f(0) \geq 0$ or $f(0) < 0$. For the existence proof one starts by constructing a sub- below a supersolution and use the result, that with such preliminaries one may use the maximum principle and a fix-point argument to find that a solution in between exists.

The case $f(0) \geq 0$. Suppose that $B_r(x_0) \subset \Omega$. By ode-methods one may find a radially symmetric solution (λ, v_λ) on the domain $B_r(0)$ f or all $\lambda > \lambda_{B_r(0)}$. One defines a subsolution by

$$\underline{u}(x) = \begin{cases} v_\lambda\,(x - x_0) & \text{for } x \in B_r(x_0), \\ 0 & \text{for } x \notin B_r(x_0). \end{cases} \tag{7}$$

and a supersolution by $\overline{u} = \rho_3$.

Although we call $\underline{u}$ a subsolution, it is not so in the classical sence, since it is just Lipschitz and not C^2. The classical definition of a subsolution is a function satisfying (6) with $\leq$ instead of $=$, that is

$$\begin{cases} -\Delta\underline{u} \leq \lambda f(\underline{u}) & \text{in } \Omega, \\ \underline{u} \leq 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

For a subsolution the inequalities are reversed. One may even take sub- or supersolutions in $C(\overline{\Omega})$ and consider the differential equation in distributional sense.

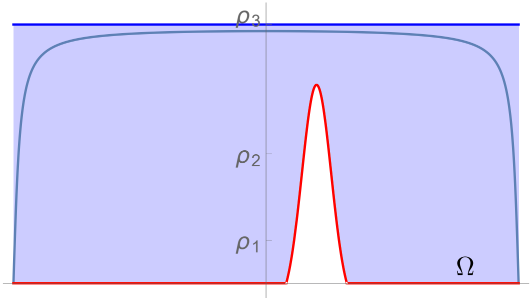


Figure 10: The initial sub- and supersolution for (6) when $f(0) \geq 0$

Definition 9 *$\underline{u}$ is a C-subsolution to (6) if*

$$\left\{ \begin{array}{c} \underline{u} \in C(\overline{\Omega}) \text{ with } \underline{u} \leq 0 \text{ on } \partial\Omega \text{ and} \\ \int_{\Omega} \left(\underline{u}\left(-\Delta\varphi\right) - \lambda f(\underline{u})\varphi\right) dx \leq 0 \text{ for all } 0 \leq \varphi \in C_0^{\infty}(\Omega). \end{array} \right.$$

For C-supersolutions both inequalities involving u are reversed.

Lemma 10 *If $\underline{u}_1$ and $\underline{u}_2$ are C-subsolutions, then $\underline{u}$, defined by $\underline{u}(x) = \max(\underline{u}_1(x), \underline{u}_2(x))$, is a C-subsolution.*

Theorem 11 (See [3, 46, 10]) *Let $\partial\Omega$ be regular in the sense of Perron. Suppose that f is Lipschitz-continuous. Fix λ. If there exists a C-subsolution $\underline{u}$ and a C-supersolution $\overline{u}$ to (6) with $\underline{u} \leq \overline{u}$, then there exists a solution $u \in [\underline{u}; \overline{u}]$.*

The $\underline{u}$ in (7) is such a C-subsolution. Setting $M = \max\{|f'(t)|; t \in [0, \rho_3]\}$ one finishes with a fix-point argument for

$$T_\lambda \cdot \; = (-\Delta + \lambda M)_0^{-1} (\lambda f(\cdot) + \lambda M \cdot) : C(\overline{\Omega}) \to C(\overline{\Omega})$$

on the ordered set $[\underline{u}, \overline{u}] \subset C(\overline{\Omega})$. Indeed

$$u \mapsto \lambda f(u) + \lambda M u \tag{9}$$

is increasing and for $(-\Delta + \lambda M)_0^{-1}$ one uses the maximum principle. Together they imply that

$$u \leq v \implies T_\lambda u \leq T_\lambda v.$$

Remark 12 *For the function in (9) to be increasing on $[0, \rho_3]$ for some large it is sufficient that f is the difference $f_1 - f_2$ of an increasing function f_1 and a Lipschitz-function f_2. For $(-\Delta + \lambda M)_0^{-1}$: $L^\infty(\Omega) \to C(\overline{\Omega})$ to be well-defined $\partial\Omega$ should be regular in the sense of Perron, which holds for example if $\partial\Omega$ is Lipschitz. So when $f(0) \geq 0$ corners are allowed for the existence statement in Theorem 8.*

For a C-subsolution and for a C-supersolution one also finds

$$T_\lambda \underline{u} \geq \underline{u} \text{ and } T_\lambda \overline{u} \leq \overline{u} \text{ respectively.}$$

So

$$\underline{u} \leq T_\lambda \underline{u} \leq T_\lambda^2 \underline{u} \leq T_\lambda^3 \underline{u} \leq \cdots \leq T_\lambda^3 \overline{u} \leq T_\lambda^2 \overline{u} \leq T_\lambda \overline{u} \leq \overline{u} \text{ on } \overline{\Omega}.$$

Both $T_\lambda^k \underline{u}$ and $T_\lambda^k \overline{u}$ converge to functions satisfying $T_\lambda u = u$, that is, a solution u exists with $u \in [\underline{u}, \overline{u}]$. See [3].

The case $f(0) < 0$. If $f(0) < 0$, for example as in Figure 11 on the left, one cannot use $\underline{u} = 0$ for a subsolution since 0 is no subsolution. In order not to violate Condition 7.3, one may need to modify $f(t)$ for $t < 0$. See Figure 11 on the right. If one does find a positive solution, the values of $f(t)$ for $t < 0$ do not matter. If not, then one doesn't have a solution to the original problem.

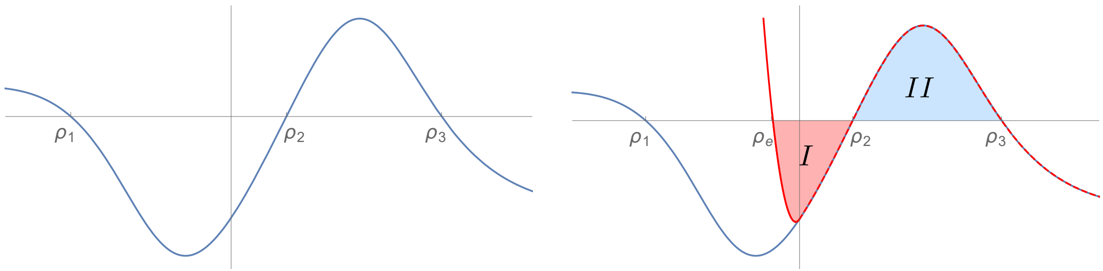


Figure 11: An f with $f(0) < 0$ and a modified f to have $I < II$.

So we consider f as on the right in Figure 11. Let (λ, v_λ) be a radially symmetric solution of

$$\begin{cases} -\Delta v = \lambda f(v) & \text{in } B_r(0), \\ v = \rho_e & \text{on } \partial B_r(0), \end{cases} \tag{10}$$

with $v_\lambda(0) \in (\rho_2, \rho_2)$, which is radially symmetric and can be obtained as the solution of an ode.

We construct the subsolution by

$$\underline{u}_{x_0}(x) = \begin{cases} v_\lambda (x - x_0) & \text{for } x \in B_r(x_0), \\ \rho_e & \text{for } x \notin B_r(x_0). \end{cases} \tag{11}$$

Keeping $\overline{u} = \rho_3$ as a supersolution one finds again a solution in $[\underline{u}_{x_0}, \overline{u}]$. However, such a solution is not a priori positive.

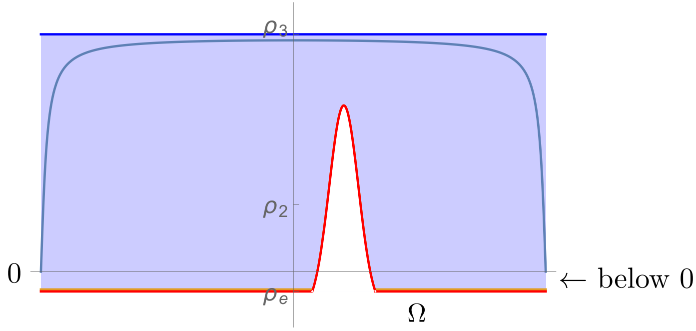


Figure 12: The initial sub- and supersolution for (6) when $f(0) < 0$

What helps us, is a result that combines the maximum principle with a continuity argument:

Theorem 13 (McNabb [35], Serrin's Sweeping Principle [47]) *Suppose that $\overline{u}$ is a supersolution to (6) in the sense of Definition 9 and that $\{\underline{u}_t; t \in [0,1]\}$ is a continuous family in $C(\overline{\Omega})$ of subsolutions to (6). If $\underline{u}_0 \leq \overline{u}$ in Ω, then either*

1. *$\underline{u}_t < \overline{u}$ in Ω for all $t \in [0,1]$, or*

2. *$\underline{u}_t = \overline{u}$ in Ω for some $t \in [0,1]$.*

If $\partial\Omega \in C^{1,1}$ holds, then Ω satisfies a uniform interior sphere condition and hence for some $r > 0$ the domain can be filled with balls of radius r:

$$\Omega = \bigcup_{x \in \Omega_r} B_r(x),$$

where also $\Omega_r = \{x \in \Omega; d(x, \partial\Omega) > \varepsilon\}$ is a connected set. For this r we consider the C-subsolution in (11). Since the solution is radially symmetric one finds $\{x; u(x) > 0\} = B_{r'}(x_0)$ for some $r' < r$. Also $\Omega_{r'}$ is a connected set and one applies Theorem 13 to any curve in $\Omega_{r'}$ starting in x_0. It follows that the solution $u \in [\underline{u}_{x_0}, \overline{u}]$ satisfies

$$u > \underline{u}_{\tilde{x}} \text{ for all } \tilde{x} \in \Omega_{r'},$$

which implies that $u > 0$ in Ω. See Figure 13. Moreover, only if $u \geq 0$ one finds that the solution with the modified f is a solution with the original f.

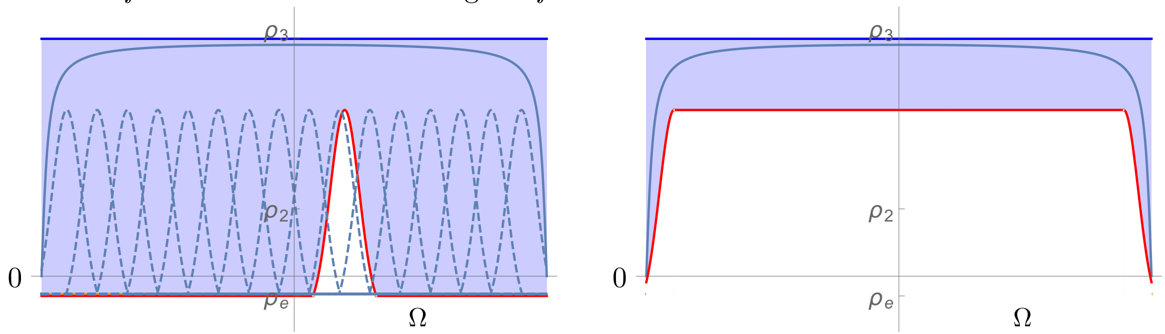


Figure 13: Sliding subsolutions for (6) when $f(0) < 0$

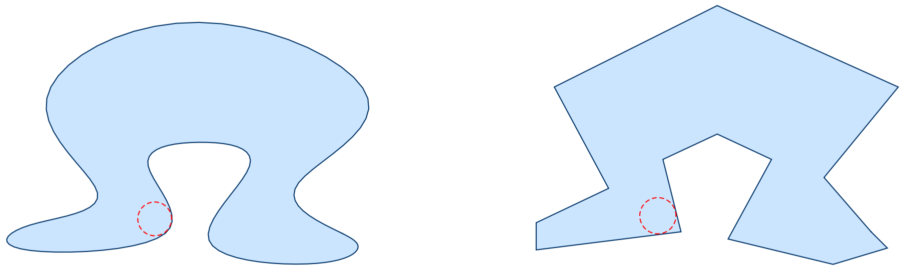


Figure 14: Smooth domains can be filled with uniformly sized balls

Remark 14 *Notice that this argument uses that Ω satisfies a uniform interior sphere condition. Even if one started with another set Q_{x_0} instead of $B_r(x_0)$, the set $Q'_{x_0} := \{x \in Q_{x_0}; v(x) > 0\}$ would be smooth. One cannot fill up a domain Ω with $Q'_{\tilde{x}}$ when $\partial\Omega$ has convex corners.*

As in $\mathbb{R}^2$ we will fix a standard cone in $\mathbb{R}^n$ with opening angle $\alpha \in (0, \pi)$ as follows:

$$\mathsf{C}_\alpha = \left\{ \left(x_1, x'\right) \in \mathbb{R}^+ \times \mathbb{R}^{n-1};\ \left|x'\right| < x_1 \tan\left(\tfrac{1}{2}\alpha\right) \right\}. \tag{12}$$

Theorem 15 (S. [51]) *Let f satisfy Condition 7 and be such that $f(0) < 0$. Suppose that Ω is a bounded domain with $\partial\Omega \in C^{0,1}$. Then there is $\alpha_f \in (0, \pi)$ such that if*

$$\Omega \cap B_\varepsilon(0) = \mathsf{C}_\alpha \cap B_\varepsilon(0)$$

for some $\varepsilon > 0$ and $\alpha < \alpha_f$, (6) does not have a positive solution.

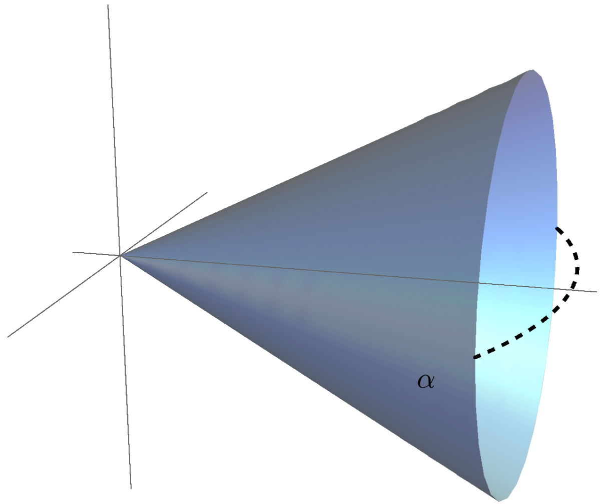


Figure 15: A standard cone C_α

Proof. If u in $C(\overline{\Omega})$, then $f(u) \in C(\overline{\Omega}) \subset L^p(\Omega)$ and since $\partial\Omega \in C^{0,1}$ we find that $u \in W^{2,p}_{loc}(\Omega)$ for all $p \in (1,\infty)$. For p large enough, we then find that $u \in C^1(\Omega)$. So $f(u) \in C^{0,1}(\Omega) \subset W^{1,p}_{loc}(\Omega)$ and $u \in W^{3,p}_{loc}(\Omega)$ for all $p \in (1,\infty)$ and $u \in C^{2,\gamma}(\Omega)$. Note that $C(\overline{\Omega})$ and $L^p(\Omega)$ are normed vectorspaces (even Banach) while $C^{0,1}(\Omega)$ or $W^{3,p}_{loc}(\Omega)$ are just vectorspaces.

Through a scaling argument, $x_{new} := \lambda^{1/2}x$ and $\varepsilon_{new} := \lambda^{1/2}\varepsilon$, we may assume that $\lambda = 1$. We proceed through a series of pointwise estimates. We may take α_f a priori as small as we like and with the square in mind, with $\pi/2$ as the angle where the behaviour of u_e and φ_1 diverged, see Example 5, we take $\alpha_f \leq \frac{1}{2}\pi$.

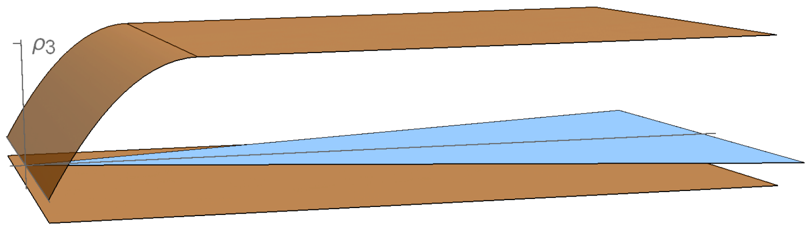


Figure 16: In blue part of the domain with the supersolution $\overline{u}_1$ above and constant subsolution below

1. Set $f_m = \max\{f(u); 0 \leq u \leq \rho_3\}$, $K = \sqrt{2\rho_3 f_m}$ and

$$\overline{u}_1(x) = \begin{cases} Kx_1 - \frac{1}{2}f_m x_1^2 & \text{for } x_1 \leq K/f_m, \\ \rho_3 & \text{for } x_1 > K/f_m. \end{cases}$$

One finds that $-\Delta\overline{u}_1(x) = f_m$ and together with the boundary conditions it implies that $\overline{u}_1$ is a supersolution and by sweeping from above, i.e. considering $\overline{u} + t$ and $t \downarrow 0$ one finds $u \leq \overline{u}_1$.

2. Fix $\varepsilon^* \in (0, K)$ such that $f(s) \leq \frac{1}{2} f(0)$ for $s \in [0, \varepsilon^*]$ and fix the subdomain

$$\Omega^* \subset \left\{ x \in \Omega \cap B_\varepsilon(0) ; x_1 < K^{-1}\varepsilon^* \right\}.$$

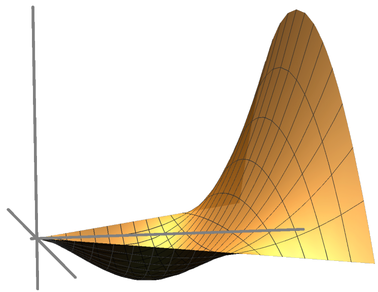


Figure 17: The supersolution v_c

Consider

$$v_c(x) = -\tfrac{1}{8} f(0) \left(x_1^2 - x_2^2 \right) \log(c x_1) \tag{13}$$

to find, independent of $c \in \mathbb{R}^+$, that

$$-\Delta v_c(x) = \left(3 + \left(\frac{x_2}{x_1} \right)^2 \right) \tfrac{1}{8} f(0) \geq \tfrac{1}{2} f(0) \text{ for } x \in \Omega^*.$$

Next we take c such that

$$v_c(x) \geq u(x) \text{ for all } x \in \left\{x \in \partial\Omega^*;\ x_1 = K^{-1}\varepsilon^*\right\}.$$

On the rest of $\partial\Omega^*$ we have $v_c(x) \geq 0 = u(x)$. Note that the function v_c has a negative minimum in

$$\tilde{x} = \left(\frac{1}{c\sqrt{e}}, 0\right).$$

3. Again we use the Sweeping Principle with $t > 0$ and large such that

$$v_c(x) + t > u(x) \text{ for } x \in \Omega^*.$$

In fact, we suppose that $u \geq 0$ and and reduce t until

$$v_c(x) + t \geq u(x) \text{ for all } x \in \Omega^* \text{ and}$$
$$v_c(\tilde{x}) + t = u(\tilde{x}) \text{ for some } \tilde{x} \in \overline{\Omega^*}.$$

If such t is strictly positive, then $\tilde{x} \in \Omega^*$ and by the Strong Maximum Principle $v_c(x) + t = u(x)$ for all $x \in \Omega^*$, which cannot be true at all of the boundary $\partial\Omega^*$ whenever $t > 0$. So $t = 0$ remains and this implies that

$$u(\tilde{x}) \leq v_c(\tilde{x}) < 0.$$

Whenever $f(0) < 0$ and $\alpha \leq \frac{1}{2}\pi$ no positive solution exists. ■

Remark 16 *It is not clear if $\alpha_f = \frac{1}{2}\pi$ is optimal.*

Remark 17 *In higher dimensions one may use similar arguments when replacing v_c in (13) by*

$$v_c(x_1, x') = -\tfrac{1}{8} f(0)\left(x_1^2 - \tfrac{1}{n-1}\left|x'\right|^2\right)\log(cx_1).$$

4 The paradox of Saponzhyan

Saponzhyan interested in the fourth order problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

which is known for $\Omega \subset \mathbb{R}^2$ as the hinged plate problem. The name supported plate, which one also finds for (14), is not appropriate, since it assumes the plate to stick to the supporting edges all around. For Saponzhyan one refers to his book [45] and earlier computations on some special domains in [44].

It is very tempting to write (14) as a system:

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \text{ and } \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Without any condition on Ω besides being bounded, for each $f \in L^2(\Omega)$ the left hand side of (15) has a unique weak solution in $W_0^{1,2}(\Omega)$.

Definition 18 *A weak solution v of (15-left) is a function $v \in W_0^{1,2}(\Omega)$ satisfying*

$$\int_\Omega (\nabla v \cdot \nabla \varphi - f\ \varphi)\, dx = 0 \textit{ for all } \varphi \in W_0^{1,2}(\Omega).$$

Such a solution can even be found for $f \in W^{-1,2}(\Omega) = \left(W_0^{1,2}(\Omega)\right)'$ when we replace $\int_\Omega f\ \varphi dx$ by the duality relation.

The existence of such a weak solution follows from Riesz' Representation Theorem and doesn't need any regularity of the boundary. The only condition that is necessary is $u \mapsto \|\nabla u\|_{L^2(\Omega)}$ being a norm and for that one uses the Poincaré-Friedrichs inequality

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega)}$$

for which it is sufficient that Ω is bounded.

Problem 19 (Sapondzhyan) *Suppose that Ω has a reentrant corner such as for example*

$$\Omega = (-1,1)^2 \setminus (-1,0]^2 \quad \textit{or} \quad \Omega = B_1(0,0) \setminus (-1,0]^2 .$$

The system solution u obtained through iteration of weak solutions in (15) satisfies

$$(u, \Delta u) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega). \tag{16}$$

If $0 \lneq f \in C^\infty(\overline{\Omega})$, then u, v from (15) satisfy $v(x) > 0$ in Ω and $u \sim r^{2/3} \sin\left(\frac{2}{3}\varphi + \frac{1}{3}\pi\right)$. So

$$E(u) \sim \|u\|_{W^{2,2}(\Omega)} \geq c \int_{r=0}^{1/2} \left(r^{2/3-2}\right)^2 r dr = c \int_{r=0}^{1/2} r^{-4/3} dr = \infty.$$

However, simplifying the model somewhat, a solution consistent to the elastic energy model should be a minimizer of

$$J(u) = \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 - f\ u\right) dx \tag{17}$$

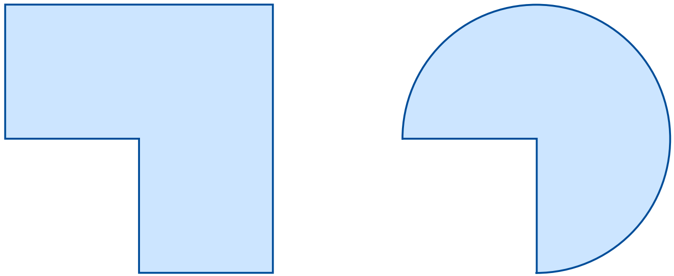


Figure 18: L-shaped domains

and the correct space would be

$$u \in \mathcal{W} := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \tag{18}$$

or written similarly to (16)

$$(u, \Delta u) \in \mathcal{W} \times L^2(\Omega). \tag{19}$$

The first thing that one should notice, is that u from (16) has two, while u from (19) seems to have to satisfy just one boundary condition. Indeed $\Delta u \in W_0^{1,2}(\Omega)$ implies that in the sense of traces $\Delta u = 0$ on the boundary in the sense of traces. See [20]. A minimizer of the functional J in (17) satisfies

$$\int_\Omega (\Delta u \ \Delta\varphi - f \ \varphi)\, dx = 0 \text{ for all } \varphi \in \mathcal{W}$$

and assuming that $u \in W^{4,2}_{loc}(\Omega)$, where the local restriction means inside and on every smooth boundary part, we may test with φ having support in the subdomain where u is $W^{4,2}$, and find after integration by parts for all those $\varphi \in \mathcal{W} \subset W_0^{1,2}(\Omega)$, that

$$\begin{aligned} 0 &= \int_{\partial\Omega} \left(\Delta u \tfrac{\partial}{\partial n}\varphi - \left(\tfrac{\partial}{\partial n}\Delta u\right)\varphi\right) d\sigma_x + \int_\Omega \left(\Delta^2 u - f\right)\varphi dx \\ &= \int_{\partial\Omega} \Delta u \tfrac{\partial}{\partial n}\varphi \; d\sigma_x + \int_\Omega \left(\Delta^2 u - f\right)\varphi dx. \end{aligned}$$

Restricting moreover to $\varphi \in W_0^{2,2}(\Omega)$, it follows that

$$\int_\Omega \left(\Delta^2 u - f\right)\varphi dx = 0.$$

Hence also

$$\int_{\partial\Omega} \Delta u \tfrac{\partial}{\partial n}\varphi \; d\sigma_x = 0$$

and this implies $\Delta u = 0$ on smooth boundary parts. The second boundary condition appears as a natural condition from the variational formulation.

Now for the difference of the solutions in (16) and (19).

Theorem 20 (see [39]) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and suppose that $\partial\Omega$ has one reentrant corner in $(0,0)$. More precisely, we assume for some $\varepsilon > 0$ and $\alpha \in (\pi, 2\pi)$ that*

$$\Omega \cap B_\varepsilon(0,0) = \left\{ r(\cos\theta, \sin\theta); 0 < r < \varepsilon \text{ and } |\theta| < \tfrac{1}{2}\alpha \right\}.$$

Then there exists a function $s \in C\left(\overline{\Omega}\right) \cap C^{\infty}(\Omega)$, with $O = \{(0,0)\}$, satisfying

$$\begin{cases} \Delta^2 s(x) = 0 & \text{for all } x \in \Omega, \\ s(x) = 0 & \text{for all } x \in \partial\Omega, \\ \lim\limits_{\Omega \ni y \to x} \Delta s(y) = 0 & \text{for all } x \in \partial\Omega \setminus \{(0,0)\}, \end{cases}$$

and a constant c, such that with u the solution for (16), one has that

$$u_2 := u + cs \in \mathcal{W} \tag{20}$$

is the minimizer of J from (17) in (19).

Remark 21 *For this function s one finds $s \notin \mathcal{W}$ as well as $\Delta s \notin W_0^{1,2}(\Omega)$.*

Remark 22 *For the sector C_α one finds the explicit formula*

$$s(x) = \frac{\left(1+\frac{\pi}{\alpha}\right) r^{2-\pi/\alpha} - \left(1-\frac{\pi}{\alpha}\right) r^{2+\pi/\alpha} - 2\frac{\pi}{\alpha} r^{\pi/\alpha}}{4\pi\left(1+\frac{\pi}{\alpha}\right)\left(1-\frac{\pi}{\alpha}\right)} \cos\left(\tfrac{\pi}{\alpha}\theta\right), \tag{21}$$

writing $x = (x_1, x_2) = r(\cos\theta, \sin\theta)$ if convenient.

The construction of s is as follows. We start with

$$\zeta_0(x) = \tfrac{1}{\pi} r^{-\pi/\alpha} \cos\left(\tfrac{\pi}{\alpha}\theta\right).$$

One finds $\zeta_0 \in C^\infty\left(\overline{\Omega}\setminus O\right)$ for $O=\{(0,0)\}$ and moreover $\zeta_0=0$ on $(\partial\Omega\setminus O)\cap B_\varepsilon(0,0)$. Notice that $\alpha>\pi$ implies that $\zeta_0\in L^2(\Omega)\setminus W^{1,2}(\Omega)$. Next let χ be a $C^\infty\left(\mathbb{R}^2\right)$ function with

$$\chi(x)=0 \text{ for } |x|<\tfrac{1}{2}\varepsilon \text{ and } \chi(x)=1 \text{ for } |x|>\varepsilon.$$

and set ζ_1 to be the $W_0^{1,2}(\Omega)$ solution of

$$\begin{cases} -\Delta\zeta=-\Delta(\chi\zeta_0) & \text{in } \Omega,\\ \zeta=0 & \text{on } \partial\Omega. \end{cases}$$

Notice that $\Delta(\chi\zeta_0)\in C^\infty\left(\overline{\Omega}\right)$. We set

$$\zeta=(1-\chi)\zeta_0-\zeta_1.$$

Near 0 one finds that $(1-\chi)\zeta_0=\zeta_0$ and hence $(1-\chi)\zeta_0\in L^2(\Omega)\setminus W^{1,2}(\Omega)$. Since $\zeta_1\in W^{1,2}(\Omega)$ the function ζ is nontrivial and satisfies

$$\begin{cases} -\Delta\zeta=0 & \text{in } \Omega,\\ \zeta=0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Remark 23 *Although this might seem a contradiction, it is not. Uniqueness for (22) holds in $W^{1,2}(\Omega)$ and not in $L^2(\Omega)$.*

Finally we define s, the so-called dual singular function, as the solution in $W_0^{1,2}(\Omega)$ of

$$\begin{cases} -\Delta s=\zeta & \text{in } \Omega,\\ s=0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Combining (22) and (23) one finds that for any $c \in \mathbb{R}$ the function $u_2 = u + cs$ satisfies

$$\begin{cases} \Delta^2 u_2 = f & \text{on } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega, \\ \text{“ } \Delta u_2 = 0 \text{ ”} & \text{on } \partial\Omega \setminus \{(0,0)\}. \end{cases}$$

The equations need some explanation. Only for smooth f these are satisfied pointwise inside Ω and on smooth boundary parts.

So how do we need to choose $c \in \mathbb{R}$ such that $u_2 \in \mathcal{W}$?

We will give the formula, but the actual explanation needs the theory initiated by Kondrat'ev, Grisvard. For that formula for u_2 it is convenient to use the notation

$$\mathcal{G} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$$

for the weak solution operator for the Poisson problem, namely,

$$u = \mathcal{G}f \text{ is the weak solution of } \begin{cases} \Delta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 24 *Suppose that Ω is as in Theorem 20, $f \in L^2(\Omega)$ and $u = \mathcal{G}^2 f$. Let ζ and s be as in (22,23). Then*

$$u_2 = \left(\mathcal{G}^2 - \mathcal{G}\mathcal{P}_\zeta\mathcal{G}\right) f \text{ with } \mathcal{P}_\zeta v = \frac{\int_\Omega v\ \zeta\ dx}{\int_\Omega \zeta^2\ dx}\zeta$$

so $\mathcal{P}_\zeta$ is the $L^2(\Omega)$-projection on ζ.

Remark 25 *In other words: for the c in (20), which is called the 'stress intensity factor', one has*

$$c = -\frac{\langle \mathcal{G} f, \zeta \rangle_{L^2(\Omega)}}{\langle \zeta, \zeta \rangle_{L^2(\Omega)}}.$$

The following result concerning solutions for (14) can be found in [39], where also domains with cones in higher dimensions are considered.

Theorem 26 (Nazarov-S. [39]) *If $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, with at least one corner with angle larger than π, then (14) h as a unique system solution u_1 such that $(u_1, \Delta u_1) \in \left(W_0^{1,2}(\Omega)\right)^2$, as well as a unique weak solution u_2 with $(u_2, \Delta u_2) \in \left(\mathcal{W} \times L^2(\Omega)\right)$. Generically $u_1 \neq u_2$.*

When approximating solutions by numerical methods such as finite elements, second order problems can be dealt with by piecewise linear test-functions. Fourth order problems need quadratic test-functions which result in much more complicated computations. So the system solution is much easier to implement than directly trying to approximate the fourth order solution. There are some pitfalls, see [6, 23]. Indeed, the above tells one that whenever nonconvex corners are present, one has to deal with those corners in a special way if one wants to approximate u_2 by second order methods. See [13].

Even for the clamped plate, where the boundary conditions do not decouple nicely with the differential equations, one may use a second order system approach. See [9]. Also here nonconvex corners need a special treatment as has been shown in [12].

5 The paradox of Babushka

The linear 2d Kirchhoff-Love model for a thin plate is derived from a 3d elasticity model with the thickness going to 0 assuming that the deviation and its first and second derivatives are small. For smooth domains that could be fine but corners, as we have seen, might violate the smallness assumption of first or second derivatives.

For a heuristic derivation of the energy model one may also consider the following approach. Let $\Omega \subset \mathbb{R}^2$ be the shape of the plate and let $u : \Omega \to \mathbb{R}$ be the deviation from 0 under a force with density $f : \Omega \to \mathbb{R}$. Using the following approximations of the principal curvatures

$$\kappa_1(x) = \max_{|v|=1} \frac{\partial^2}{\partial v^2} u(x) \text{ and } \kappa_2(x) = \min_{|v|=1} \frac{\partial^2}{\partial v^2} u(x)$$

and with σ the Poisson ratio between the effect of bending and torsion, a simple formula for the elastic energy is

$$J(u) = \int_\Omega \left(\tfrac{1}{2} \left(\kappa_1^2 + \kappa_2^2 \right) + \sigma \kappa_1 \kappa_2 - f \ u \right) dx. \tag{24}$$

The Poisson ratio is a material depending constant $\sigma \in \left(-1, \frac{1}{2}\right)$. For most metals one measures $\sigma \approx 0.3$. Since $(\kappa_1 + \kappa_2)^2 = \Delta u$ and $\kappa_1 \kappa_2 = u_{xy}^2 - u_{xx} u_{yy}$ one may simplify this expression to

$$J(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 + (1 - \sigma) \left(u_{xx} u_{yy} - u_{xy}^2 \right) - f \ u \right) dx. \tag{25}$$

The appropriate space for the functions u is $\mathcal{W} = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ with the 0 taking care of the boundary condition. Through (24) one finds that $|\sigma| < 1$ is sufficitnt for J to be coercive. For a minimizer $u \in \mathcal{W}$ one has

$$\partial J(u; v) = 0 \text{ for all } v \in \mathcal{W}.$$

The middle term in (25) is a so-called nulllagrangian. It doesn't show up in the Euler-Lagrange differential equation but does have its effect on the boundary conditions. We find

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - (1-\sigma)\,\kappa u_\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

Here κ is the signed curvature of the boundary and taken positive on convex parts and u_ν is the outside normal derivative. More details can be found in [22].

On polygonal domains (26) seems to turn into

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Problem 27 (Babuška) *Set $f = 1$ and $\sigma = .3$. Let u_n be the solution of (27) for $\Omega = \Omega_n$, the regular n-gon with corners on ∂D and let $u_{\sigma=.3}$ be the solution of (26) for $\Omega = D$. Then*

$$\lim_{n\to\infty} u_n = u_{\sigma=1} \neq u_{\sigma=.3}.$$

One should notice that at a corner the term κu_ν in (26) is somewhat like $\pm\infty \cdot 0$. On a convex domain, the solution of the Poisson problem with $f \in L^2(\Omega)$ satisfies $u \in W^{2,2}(\Omega)$ which means $\nabla u \in L^q(\Omega)$ but not yet $L^\infty(\Omega)$.

Maz'ya and Nazarov [32, 33] and Davini [15] looked into ways to approximate the J_σ-minimizer on a smooth Ω by polygons.

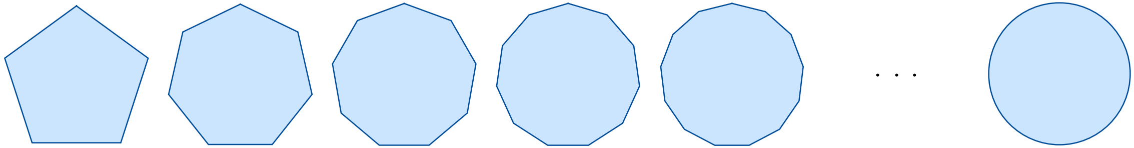


Figure 19: Approximation by n-gons

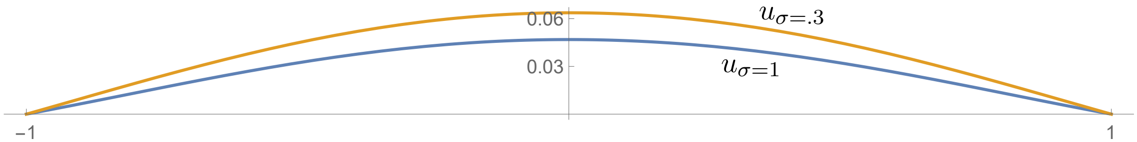


Figure 20: Top: $u_{\sigma=.3}$ and bottom: $u_{\sigma=1} = u_D$

Remark 28 *When there are boundary conditions we are used to think first of Dirichlet and Dirichlet conditions are quite robust. Indeed, for*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

with $f \in L^2(\Omega)$ there is a unique weak solution $u \in W_0^{1,2}(\Omega)$. If we would consider the Babuška-type problem for (28), then, since $W_0^{1,2}(\Omega)$ is the closure in $\|\cdot\|_{W^{1,2}(\Omega)}$ of $C^\infty(\Omega)$-functions with compact support, each function in $W_0^{1,2}(\Omega)$ can be approximated by functions in $W_0^{1,2}(\Omega_n)$ extended by 0 on $\Omega \setminus \Omega_n$. And for the actual case $f = 1$ the pointwise convergence follows directly from $\Omega_n \subset D \subset$

$\left(1+\left(\frac{\pi}{n}\right)^2\right)\Omega_n$, by taking the solutions u_n on Ω_n and $\tilde{u}_n$ on $\left(1+\left(\frac{\pi}{n}\right)^2\right)\Omega_n$ as a direct consequence of the maximum principle. One finds

$$u_n(x) \leq u(x) \text{ for } x \in \Omega_n \quad \text{and} \quad u(x) \leq \tilde{u}_n(x) \text{ for } x \in D.$$

Here we may even use $\tilde{u}_n(x) = \left(1+\left(\frac{\pi}{n}\right)^2\right)^2 u_n\left(\left(1+\left(\frac{\pi}{n}\right)^2\right)^{-1} x\right)$. Hence

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty(\Omega_n)} = 0. \tag{29}$$

For nonconstant f (first of fixed sign) one does not find the explicit scaling between u_n and $\tilde{u}_n$ but nevertheless, since

$$\lim_{n\to\infty} \left\| f\left(\left(1+\left(\frac{\pi}{n}\right)^2\right)^{-1} \cdot\right) - f \right\|_{L^2(\Omega_n)} = 0$$

one may derive that (29) still holds.

Iterating the result of the last remark one finds that the system solutions on $\Omega = \Omega_n$ as in (15) converge to the solution of (14). So indeed, $u_D = u_{\sigma=1}$.

6 Existence, uniqueness and regularity in the classical situation

Suppose that $\Omega \subset \mathbb{R}^n$ is bounded and that $\partial\Omega$ is smooth, say $\partial\Omega \in C^\infty$. Let us consider a $2m^{th}$-order elliptic boundary value problem with m boundary conditions of order m_j for $j = 1, \ldots, m$

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ B_j u = 0 & \text{on } \partial\Omega \text{ for } 1 \leq j \leq m. \end{cases}$$

- One assumes that the B_j are independent boundary operators. It can be formulated as an algebraic condition in each boundary point and involves the elliptic operator. It is usually called complementing condition or named after Shapiro-Lopatinski. Both Dirichlet, i.e.

  $$B_j u = \left(\frac{\partial}{\partial \nu}\right)^{j-1} u$$

  and Navier, i.e.

  $$B_j u = \Delta^{j-1} u$$

  satisfy the complementing condition. See [11].

If B_j is of order m_j with $m_1 < m_2 < \cdots < m_j < 2m$ and

$$B_j u = \left(\frac{\partial}{\partial\nu}\right)^{m_j} + \sum_{i<m_j} \sum_{|\beta|\le m_j - i} a_{i,\beta} \left(\frac{\partial}{\partial\nu}\right)^{i} \left(\frac{\partial}{\partial\tau}\right)^{\beta},$$

then the complementing condition is satisfied.

- The value 0 should not be in the spectrum. For the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega \end{cases}$$

the value 0 is an eigenvalue.

For the existence of a solution there are several approaches:

1. Using a singular integral approach by Calderon-Zygmund [8] one derives the existence of a strong solution in $W^{2m,p}(\Omega)$. One needs a sufficiently smooth boundary.

2. For the Dirichlet problem and using the self-adjointness one may obtain a weak solution by Riesz Representation Theorem. Extending Riesz by Lax-Milgram one can allow some perturbations.

3. Again in a self-adjoint setting one may consider a variational formulation and its minimizer will be the weak solution. In this case higher order boundary conditions may follow as a natural condition for the minimizer.

Uniqueness is a very problem-depending question. For second order problems the maximum principle is very helpful. For higher order one may add a term λu to the equation and show coercivity for that λ. Using regularity result from Agmon, Douglis and Nirenberg [1, 2] the solution operator for that λ will be compact, implying that that there is a solution for all λ except for a most countably may exceptions.

Example 29 *Consider the Navier bilaplace problem*

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

1. *Let $f \in L^p(\Omega)$ for $p \in (1, \infty)$. Then there is a solution $u \in W^{4,p}(\Omega)$ and moreover the regularity results from [1] state that for $f \in W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ one obtains that $u \in W^{4+k,p}(\Omega)$.*

2. *Let $f \in W^{-1,2}(\Omega)$ be the dual of $W_0^{1,2}(\Omega)$. As explained before around (15) there is a unique (weak-weak) solution of (30), meaning $(u, \Delta u) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$. For this approach no regularity condition for the boundary is needed.*

3. *The variational approach for (30) uses the functional*

$$J(u) := \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 + f\ u \right) dx$$

for $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and the minimizer for J will be a W^2-weak solution. The only a priori boundary condition is $u = 0$ following from $u \in W_0^{1,2}(\Omega)$. For the minimizer we find

$$0 = \partial J(u, \varphi) = \int_\Omega (\Delta u\ \Delta\varphi + f\ \varphi)\, dx$$

and when satisfies $\Delta u = 0$ *as a natural condition, meaning that if* $\partial\Omega \in C^3$*, one may prove* $u \in W^{3,2}(\Omega)$ *implying that the trace* $\Delta u_{|\partial\Omega}$ *is defined and one obtains* $\Delta u = 0$ *on* $\partial\Omega$ *as trace. See [20] for a definition of trace. A condition that one needs is that the functional is coercive on* $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$*. By Poincaré-Friedrichs one finds that* $C_1 \in \mathbb{R}$ *exists such that*

$$\|u\|_{W^{1,2}(\Omega)} = \left(\int_\Omega \sum\nolimits_{|\alpha|\le 1} (\partial^\alpha u)^2 \, dx\right)^{1/2} \le C_1 \, \||\nabla u|\|_{L^2(\Omega)} .$$

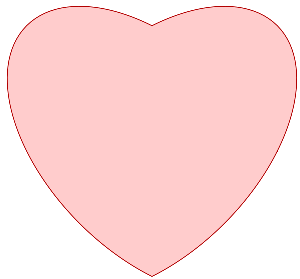
On smooth domains one even finds that $C_2 \in \mathbb{R}$ *exists such that*

$$\|u\|_{W^{2,2}(\Omega)} = \left(\int_\Omega \sum\nolimits_{|\alpha|\le 2} (\partial^\alpha u)^2 \, dx\right)^{1/2} \le C_2 \, \|\Delta u\|_{L^2(\Omega)} . \tag{31}$$

Whenever Ω *has a reentrant corner, the estimate in (31) is not so clear. A reentrant corner has angle in* $(\pi, 2\pi)$ *as on the upper side of the next domain.*

Remark 30 *Take* $\Omega = \mathsf{C}_\alpha \cap B_1(0)$ *for* $\alpha \in (\pi, 2\pi)$ *and the function*

$$v(r,\theta) = \left(r^{\pi/\alpha} - r^2\right) \cos\left(\frac{\pi}{\alpha}\theta\right).$$



One finds that v *is the unique weak solution of*

$$\begin{cases} -\Delta v = \left(4 - (\pi/\alpha)^2\right) \cos\left(\frac{\pi}{\alpha}\theta\right) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\Delta v \in L^2(\Omega)$ *but* $\|v\|_{W^{2,2}(\Omega)} = \infty$*.*

7 Elliptic problems on domains with cones

The first remark that one should make, is that regularity is 'local'. If the coefficients of the operator are sufficiently smooth and the operator is uniformly elliptic of order $2m$, then, away from singular points of the boundary, the regularity of a solution u near a point x depends on the regularity of f near that point and, when x is a boundary point, the regularity of the boundary near x. So for deriving regularity results near a singular point of the boundary, we may zoom it at that singular point and proceed 'locally'. Kondrat'ev [29] focuses on the highest order derivatives and freezes the coefficients to the values at the singular point.

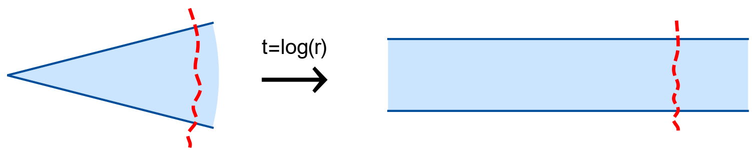


Figure 21: The transformation. We may assume that $f = 0$ on the right of the dashed line.

We assume that the singularity lies in 0 and that Ω nearby is cone shaped: there is $\mathcal{W} \subset \partial B_1(0)$, a domain on the sphere in $\mathbb{R}^n$ such that

$$\Omega \cap B_\varepsilon(0) = \mathsf{C}_{\mathcal{W},\varepsilon} := \{x = r\omega; 0 < r < \varepsilon \text{ and } \omega \in \mathcal{W}\}. \tag{32}$$

As a first step one proceeds by spherical coordinates $x = r\omega$ with $\omega \in \mathcal{W}$ and a second transformation $r = e^t$. One obtains

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + r^{-2}\Delta_\omega = e^{-2t}\left(\frac{\partial^2}{\partial t^2} + (n-2)\frac{\partial}{\partial t} + \Delta_\omega\right).$$

Here Δ_ω denotes the Laplace-Beltrami operator on $\mathcal{W} \subset \partial B_1(0)$. So

$$-\Delta u = f \quad \text{becomes} \quad \left(-\frac{\partial^2}{\partial t^2} - (n-2)\frac{\partial}{\partial t} - \Delta_\omega\right)\tilde{u} = e^{2t}\ \tilde{f}, \tag{33}$$

now defined on the cylinder $\mathbb{R} \times \mathcal{W}$. Note that $e^{2t}\ \tilde{f}(t,\omega) = r^2 f(r,\omega)$.

For the second step one uses that in L^2-type spaces the Fourier transform and its inverse

$$(\mathcal{F}u)(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-it\xi}u(t)\,dt \text{ and } \left(\mathcal{F}^{-1}v\right)(t) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it\xi}v(\xi)\,d\xi$$

is most helpful through the following properties:

1. $\mathcal{F} : L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ is an isomorphism (Plancherel Theorem):

$$\|\mathcal{F}u\|_{L^2(\mathbb{R})} = \|u\|_{L^2(\mathbb{R})} \text{ for all } u \in L^2(\mathbb{R}).$$

2. Differentiation is replaced by multiplication:

$$\left(\mathcal{F}u'\right)(\xi) = i\xi\,(\mathcal{F}u)(\xi).$$

Using the Fourier transform with respect to t in the differential equation on the right in (33) one obtains

$$\left(\xi^2 - i\,(n-2)\,\xi - \Delta_\omega\right)\left(\mathcal{F}_{t\to\xi}\tilde{u}\,(t,\omega)\right) = \mathcal{F}_{t\to\xi}\left(e^{2t}\,\tilde{f}\,(t,\omega)\right), \tag{34}$$

and one finds for

$$\begin{cases} -\Delta u = f & \text{in } \mathsf{C}_{\omega,\infty}, \\ u = 0 & \text{on } \partial\mathsf{C}_{\omega,\infty}. \end{cases} \tag{35}$$

a parameter dependent boundary value problem:

$$\begin{cases} \left(\xi^2 - i\,(n-2)\,\xi - \Delta_\omega\right) v = g & \text{in } \omega, \\ v = 0 & \text{on } \partial\omega. \end{cases} \tag{36}$$

Assuming $\partial\omega$ to be smooth the usual elliptic regularity applies for (36) although we need to find estimates with an optimal dependence on ξ.

For a general $2m$-th order elliptic boundary value problem and Ω satisfying (32) one goes through polar coordinates from

$$\begin{cases} A\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right) u = f & \text{in } \Omega, \\ B\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right) u = \vec{g} & \text{on } \partial\Omega, \end{cases} \quad \text{to} \quad \begin{cases} \tilde{A}\left(\frac{\partial}{\partial r},\frac{\partial}{r\partial\omega_1}\dots,\frac{\partial}{r\partial\omega_{n-1}}\right) u = f & \text{in } \Omega, \\ \tilde{B}\left(\frac{\partial}{r\partial\omega_1}\dots,\frac{\partial}{r\partial\omega_{n-1}}\right) u = \vec{g} & \text{on } \partial\Omega. \end{cases}$$

To study the local behaviour near the vertex of the cone one freezes the coefficients at 0 and consider the highest order derivatives only. Next one replaces Ω by $\mathsf{C}_{\omega,\infty}$ and transform that problem through $r = e^t$ to a boundary value problem on the cylinder

$$\begin{cases} \tilde{A}\left(\frac{\partial}{\partial t},\frac{\partial}{\partial\omega_1}\dots,\frac{\partial}{\partial\omega_{n-1}}\right) u = e^{2mt} f & \text{in } \Omega, \\ \tilde{B}\left(\frac{\partial}{\partial\omega_1}\dots,\frac{\partial}{\partial\omega_{n-1}}\right) u = \left\{e^{2m_k t} g_k\right\} & \text{on } \partial\Omega. \end{cases}$$

Then by a Fourier transform with respect to t one becomes

$$\begin{cases} \tilde{A}\left(\xi, \frac{\partial}{\partial \omega_1} \ldots, \frac{\partial}{\partial \omega_{n-1}}\right) \mathfrak{u} = \mathfrak{f}\left(\cdot - 2mi\right) & \text{in } \omega, \\ \tilde{B}\left(\frac{\partial}{\partial \omega_1} \ldots, \frac{\partial}{\partial \omega_{n-1}}\right) \mathfrak{u} = \left\{\mathfrak{g}_i\left(\cdot - 2m_k i\right)\right\} & \text{on } \partial\omega. \end{cases}$$

Definition 31 *One calls*

$$\mathcal{U}\left(\xi\right) = \begin{pmatrix} \tilde{A}\left(\xi, \frac{\partial}{\partial \omega_1} \ldots, \frac{\partial}{\partial \omega_{n-1}}\right) \\ \tilde{B}\left(\frac{\partial}{\partial \omega_1} \ldots, \frac{\partial}{\partial \omega_{n-1}}\right) \end{pmatrix}$$

an operator pencil. The formal adjoint is

$$\mathcal{U}^*\left(\xi\right) = \begin{pmatrix} \tilde{A}^*\left(\overline{\xi}, \frac{\partial}{\partial \omega_1} \ldots, \frac{\partial}{\partial \omega_{n-1}}\right) \\ \tilde{B}^*\left(\frac{\partial}{\partial \omega_1} \ldots, \frac{\partial}{\partial \omega_{n-1}}\right) \end{pmatrix}.$$

For the Dirichlet-Laplace problem we have

$$\mathcal{U}\left(\xi\right) v = \begin{pmatrix} \left(\xi^2 - i\left(n-2\right)\xi - \Delta_\omega\right) v \\ v_{|\partial\omega} \end{pmatrix} \text{ and } \mathcal{U}^*\left(\xi\right) v = \begin{pmatrix} \left(\overline{\xi}^2 + i\left(n-2\right)\overline{\xi} - \Delta_\omega\right) v \\ v_{|\partial\omega} \end{pmatrix}.$$

Here the problem

$$\mathcal{U}\left(\xi\right) v = \begin{pmatrix} g \\ 0 \end{pmatrix} \tag{37}$$

for $g \in L^2(\mathcal{\omega})$ can be solved in $W^{2,2}(\mathcal{\omega}) \cap W_0^{1,2}(\mathcal{\omega})$, whenever $\xi^2 - i\,(n-2)\,\xi$ is not an eigenvalue of

$$\begin{cases} -\Delta_\omega \varphi = \mu\ \varphi & \text{in } \mathcal{\omega}, \\ \varphi = 0 & \text{on } \partial\mathcal{\omega}. \end{cases} \tag{38}$$

Moreover, if $\{\mu_k\}_{k=1}^{\infty}$ are the eigenvalues for (38) with eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$, then these eigenfunctions form a complete orthonormal system and can also be used to solve (37), since

$$\mathcal{U}\,(\xi)\,\phi_k = \begin{pmatrix} \xi^2 - i\,(n-2)\,\xi + \mu_k \\ 0 \end{pmatrix} \phi_k,$$

results in

$$\mathcal{U}\,(\xi)\,v = \begin{pmatrix} g \\ 0 \end{pmatrix} \Leftrightarrow v = \sum_{k=1}^{\infty} \frac{1}{\xi^2 - i\,(n-2)\,\xi + \mu_k} \langle \phi_k, g \rangle_{L^2(\omega)}\,\phi_k.$$

For the Dirichlet-Laplace we not only have a complete orthonormal system but moreover, although $\mathcal{U}\,(\xi) \neq \mathcal{U}^*\,(\xi)$, we may use the same eigenfunctions.

It is not sufficient to just solve (38) but one will also need an optimal ξ-depending estimate

$$\|v\|_{W^{1,2}(\omega)} \leq C(\xi)\,\|g\|_{W^{-1,2}(\omega)} \text{ or } \|v\|_{W^{2,2}(\omega)} \leq C(\xi)\,\|g\|_{L^2(\omega)}\,.$$

Such an estimate one may get by constructing an approximating Green operator. Or maybe one can proceed by Weil's asymptotics for the distribution of eigenvalues.

Going back from (t,ω) to (r,ω) one arrives in a weighted space, where each radial derivative has an additional factor r:

$$\frac{\partial}{\partial t}u\left(e^t,\omega\right) = r\frac{\partial}{\partial r}u\left(r,\omega\right) \quad \text{and}$$
$$\frac{\partial^2}{\partial t^2}u\left(e^t,\omega\right) = r^2\frac{\partial^2}{\partial r^2}u\left(r,\omega\right) + r\frac{\partial}{\partial r}u\left(r,\omega\right).$$

Instead of the usual Sobolev spaces the space $W^{2,2}(\omega) \cap W_0^{1,2}(\omega)$ would lead us to a space with $r^2 u_{rr}$, $r\nabla_\omega u_r$, $\Delta_\omega u$, ru_r, $\nabla_\omega u$ and u all in a $L^2(\Omega)$-type space. All these functions in $L^2(\Omega)$ is essentially more than $u \in W^{2,2}(\Omega)$.

With the extra factor $e^{2t} = r^2$ in (34) appearing, it seems that one obtains something as follows:

$$x \mapsto |x|^2 f(x) \in L^2(\Omega) \implies x \mapsto \left(|x|^2 \nabla^2 u(x), |x| \nabla u(x), u(x)\right) \in L^2(\Omega).$$

For $f \in L^2(\mathbb{R})$ satisfying $f(t) = 0$ for $t > t_0$ the function $\xi \mapsto \mathcal{F}f(\xi)$ is well defined for $\xi \in \mathbb{C}$ with $\operatorname{Im}\xi \leq 0$. Moreover, setting $u_a(t) = e^{at}u(t)$ one finds for $a > 0$, that

$$(\mathcal{F}u_a)(\xi) = (\mathcal{F}u)(\xi - ai).$$

For appropriate f this formula may also hold for $a < 0$. So we may play with the weight a. Let's make a more precise line-up:

- From $x \mapsto |x|^{2+a} f(x) \in L^2(\Omega)$ one finds $(t,\omega) \to e^{(2+a)t}\ \tilde{f}(t,\omega) \in L^2(\mathbb{R} \times \omega)$.
- With the Fourier transform $\mathcal{F}$ from t to ξ one finds $\mathfrak{f}_a(\cdot,\omega) := \left(\mathcal{F}_{t\to\xi} e^{(2+a)t}\ \tilde{f}(t,\omega)\right) \in L^2(\mathbb{R} \times \omega)$. Or combine both steps by a Mellin transform.

- Next $\left\{\langle \phi_k, \mathfrak{f}_a(\xi,\cdot)\rangle_{L^2(\omega)}\right\}_{k\in\mathbb{N}^+} \in \ell^2$ with its t-dependent ℓ^2-norm as a function of ξ in $L^2(\mathbb{R})$.
- Then a bound for $\left\{\langle \phi_k, \mathfrak{u}_a(\xi,\cdot)\rangle_{L^2(\omega)}\right\}_{k\in\mathbb{N}^+}$ with

$$\langle \phi_k, \mathfrak{u}_a(\xi,\cdot)\rangle_{L^2(\omega)} = \tfrac{1}{(\xi-ai)^2 - i(n-2)(\xi-ai)+\mu_k} \langle \phi_k, \mathfrak{f}_a(\xi,\cdot)\rangle_{L^2(\omega)},$$

whenever

$$(\xi - ai)^2 - i(n-2)(\xi - ai) + \mu_k \neq 0, \tag{39}$$

in a weighted ℓ^2-space with a corresponding ξ-dependent bound. The condition in (39) is satisfied for all $\xi \in \mathbb{R}$ precisely when it is nonzero for $\xi = 0$, that is, when

$$-a^2 - (n-2)a + \mu_k \neq 0,$$

and this expression can be rewritten as

$$a \neq -\frac{n-2}{2} \pm \sqrt{\mu_k + \left(\frac{n-2}{2}\right)^2}. \tag{40}$$

- $\|\mathfrak{u}_a(\xi,\cdot)\|_{L^2(\omega)} \leq \frac{c_a(\xi)}{\xi^2+1} \|\mathfrak{f}_a(\xi,\cdot)\|_{L^2(\omega)}$ and $\left\|\nabla_\omega^2 \mathfrak{u}_a(\xi,\cdot)\right\|_{L^2(\omega)} \leq c_a(\xi) \|\mathfrak{f}_a(\xi,\cdot)\|_{L^2(\omega)}$?
- With the inverse Fourier transform, which is well defined if (39) holds for all $\xi \in \mathbb{R}$, one obtains

$$e^{at}(\tilde{u}_{tt}, \tilde{u}_{t\omega}, \tilde{u}_{\omega\omega}, \tilde{u}_t, \tilde{u}_\omega, \tilde{u}) \in L^2(\mathbb{R} \times \omega).$$

- Returning from (t,ω) to (r,ω) and x one obtains

$$x \mapsto \left(|x|^{2+a} \nabla^2 u(x), |x|^{1+a} \nabla u(x), |x|^a u(x) \right) \in L^2(\Omega).$$

Also for an L^p-type setting one may proceed through these kind of arguments using Mikhlins Multiplier Theorem. One still finds extra weigths $|x|^{2m}$, where $2m$ is the order of the elliptic operator and an extra power of $|x|$ compensating each derivative. So one needs weighted spaces of the following type:

Definition 32 *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}$ and $\beta \in \mathbb{R}$ and set*

$$V_\beta^{\ell,p}(\Omega) = \overline{C_c^\infty(\overline{\Omega} \setminus O)}^{\|\cdot\|} \qquad \textit{with}$$

$$\|u\| = \|u\|_{V_\beta^{\ell,p}(\Omega)} := \left(\sum\nolimits_{j=0}^{\ell} \left\| |x|^{\beta-\ell+j} \left| \nabla_x^j u \right| \right\|_{L^p(\Omega)}^p \right)^{1/p}.$$

In order to define suitable boundary conditions one also needs:

$$V_{\beta,0}^{\ell,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|}.$$

Here $C_c^\infty(\overline{\Omega} \setminus O)$ is the set of all $C^\infty(\overline{\Omega})$-functions which have compact support that does not contain $O = \{0\}$.

Remark 33 *Notice the similarity to the definition of $W_0^{\ell,p}(\Omega)$.*

Let us go back to the Dirichlet-Laplace case and (40). Writing

$$\Lambda_{\pm k} = -\frac{n-2}{2} \pm \sqrt{\mu_k + \left(\frac{n-2}{2}\right)^2}$$

one finds power type functions

$$U_{\pm k}(x) = |x|^{\Lambda_{\pm k}} \ \phi_k\left(\frac{x}{|x|}\right),$$

which satisfy

$$\begin{cases} -\Delta U_{\pm k}(x) = 0 & \text{in } \mathsf{C}_{\omega,\infty}, \\ U_{\pm k}(x) = 0 & \text{on } \partial\mathsf{C}_{\omega,\infty}. \end{cases}$$

and

$$\begin{aligned} U_{\pm k} \in V_\beta^{\ell,2}(\mathsf{C}_{\omega,1}) &\Leftrightarrow 2(\beta - \ell + \Lambda_{\pm k}) + n - 1 > -1 \\ &\Leftrightarrow \Lambda_{\pm k} > \ell - \beta - \tfrac{1}{2}n. \end{aligned}$$

Let us fix the range

$$\Lambda_{-1} < \ell + 1 - \beta - \tfrac{1}{2}n < \Lambda_1.$$

Writing $\Delta_\beta^{\ell,p}$ with $D\left(\Delta_\beta^{\ell,p}\right) = V_\beta^{\ell+1,p}(\Omega) \cap V_{1,0}^{\beta-\ell,p}(\Omega)$ for the corresponding operator to the Laplace-Poisson boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

one obtains the following result:

Proposition 34 *Let $\Omega \subset \mathbb{R}^n$ be as in (32) and set*

$$\Lambda_{\pm 1} = -\frac{n-2}{2} \pm \sqrt{\mu_1 + \left(\frac{n-2}{2}\right)^2}$$

where μ_1 is the first eigenvalue of (38). Then

$$\Delta_\beta^{\ell,p} : V_\beta^{\ell+1,p}(\Omega) \cap V_{\beta-\ell,0}^{1,p}(\Omega) \to V_\beta^{\ell-1,p}(\Omega)$$

is an isomorphism if and only if

$$\Lambda_{-1} < \ell + 1 - \beta - \tfrac{1}{p}n < \Lambda_1.$$

Remark 35 *If $\Lambda_1 < \ell + 1 - \beta - \frac{1}{p}n$ then there are $f \in V_\beta^{\ell-1,p}(\Omega)$ without a solution. For example when $f > 0$ the solution contains a part that behaves near 0 like $c\,|x|^{\Lambda_1}\,\phi_1\left(\frac{x}{|x|}\right)$ with $c > 0$.*
If $\Lambda_{-1} > \ell + 1 - \beta - \frac{1}{p}n$ then solutions are not unique since $V_\beta^{\ell+1,p}(\Omega)$ is not restrictive enough and allows harmonic functions that behave like $|x|^{\Lambda_{-1}}\,\phi_1\left(\frac{x}{|x|}\right)$.

So, in general no more regularity for a solution u, even for $f \in C_c^\infty(\Omega)$, as what the cone allows. The result of Kondrat'ev tells us, that when we want more regularity, one has to single out individual singular components. From [29] one finds an almost explicit formula:

$$u(x) = \sum_{h_1 < \operatorname{Im}\lambda_j < h} \sum_{s=0}^{k_j} c_{js}\,|x|^{\lambda_j}\,(\ln|x|)^s\,\phi_{js}\left(\frac{x}{|x|}\right) + w(x).$$

Kondrat'ev seminal paper [29] treated the general $2m$-th elliptic boundary value problems although in a Hilbert space setting. He already refers to earlier results by Èskin for a general setting and others for special cases. Maz'ya and and collaborators [30, 34], Nazarov, Plamenevskij [38], the french school, Grisvard [25, 26], Dauge [14], Nicaise [42] have considered special cases and extended settings. Most of their articles balance between the awfull technicalities of the general case and explicit results for special cases motivated by applications. The role of the special functions was also noted by Williams in [57]. In 2d-settings one may proceed through conformal mappings [56, 18], which is useful for numerics.

For the Dirichlet laplacian or Navier bilaplacian self-adjointness implies that no logarithmic terms appear:

$$u(x) = \sum_{h_1 < \operatorname{Im} \lambda_j < h} c_j \, |x|^{\lambda_j} \, \phi_j \left(\frac{x}{|x|} \right) + w_{h,h_1}(x) , \tag{41}$$

with h and h_1 some bounds as before implied by values $\Lambda_{\pm k}$. Noite that the formula of Kondrat'ev is not asymptotic expansion. Even ny making the interval $(h_, h_1)$ larger the corresponding w_{h,h_1} is not necessarily going to zero. The formula identifies the singular terms that might appear in the stated range for any finite range.

Returning to the semilinear problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases} \tag{42}$$

of Section 3. With $f(u) = f(0) + cu + h.o.t$ we obtain approximately in the 2d case near the corner now located in 0 for $\alpha \neq \frac{1}{2}\pi$ that

$$u(x_1, x_2) = -\tfrac{1}{2} f(0) \frac{x_2^2 - \left(\tan(\frac{1}{2}\alpha)\right)^2 x_1^2}{1 - \left(\tan(\frac{1}{2}\alpha)\right)^2} + c\phi_1 + ...$$

The first term is regular and corresponds to w in (41) while the second term resembles the singular part. Notice that the sign near the corner is determined by:

1. The sign of the first term depends on $f(0)$ and whether α larger or less than $\frac{1}{2}\pi$ is. That first term is always quadratic.

2. The second term $\tilde{c}\phi_1$, that looks like $r^{\alpha/\pi}\cos(\frac{\alpha}{\pi}\theta)$, is less than quadratic if and only if $\alpha > \frac{1}{2}\pi$.

8 Two examples and their power type functions

Let us look in more detail to the singular functions that appear for the Dirichlet laplacian and Navier bilaplacian. In this section we recall two graphs relating in two dimensions the opening angle α with the corresponding power type functions $r^{\lambda_k}\phi_k(\theta)$ in the case of

$$\begin{cases} -\Delta u = f & \text{in } \mathsf{C}_\alpha, \\ u = 0 & \text{on } \partial\mathsf{C}_\alpha, \end{cases} \quad \text{and} \quad \begin{cases} \Delta^2 u = f & \text{in } \mathsf{C}_\alpha, \\ u = \Delta u = 0 & \text{on } \partial\mathsf{C}_\alpha. \end{cases}$$

Here as before $\mathsf{C}_\alpha = \left\{(r,\theta)\,; r > 0 \text{ and } |\theta| < \frac{1}{2}\alpha\right\}$.

In the horizontal direction one finds α and in the vertical direction the corresponding λ_k. Recall that for nice ϕ the function u_λ, defined by

$$u_\lambda(x) = r^\lambda \phi(\theta)$$

satisfies $u_\lambda \in W^{k,p}(\mathsf{C}_{\alpha,1}) \Leftrightarrow \lambda > k - \frac{2}{p}$. So the vertical direction not just displays the possible λ related to the angle α but also corresponds to order of regularity.

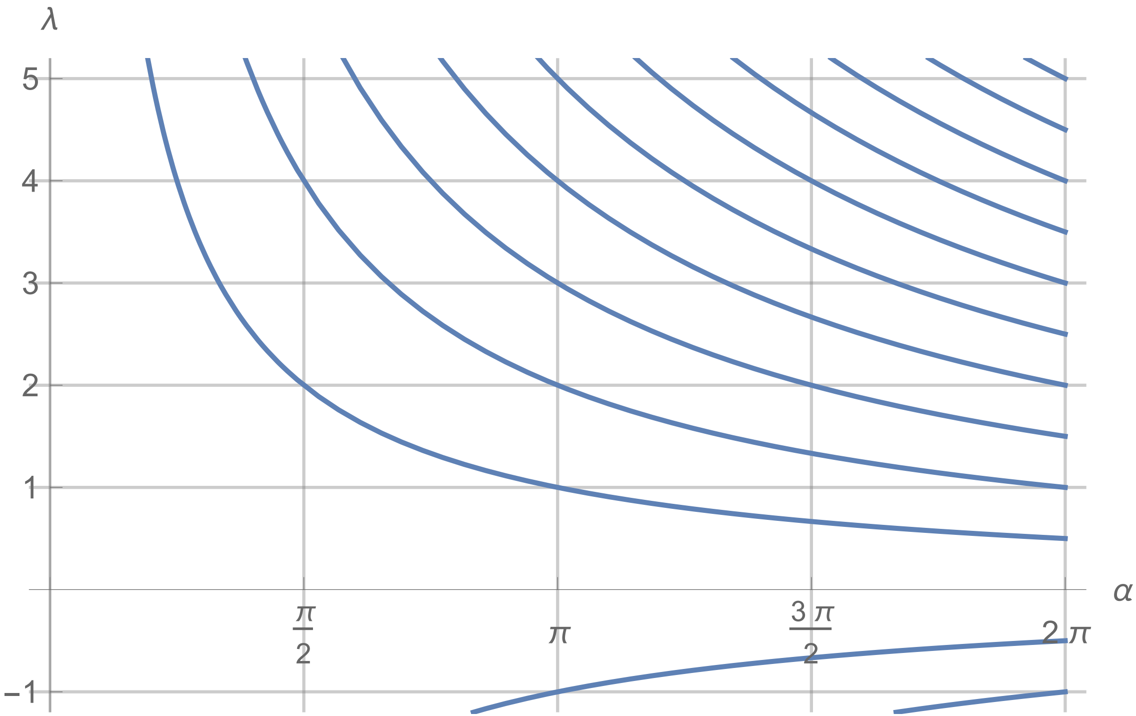


Figure 22: The Dirirchlet Laplace case

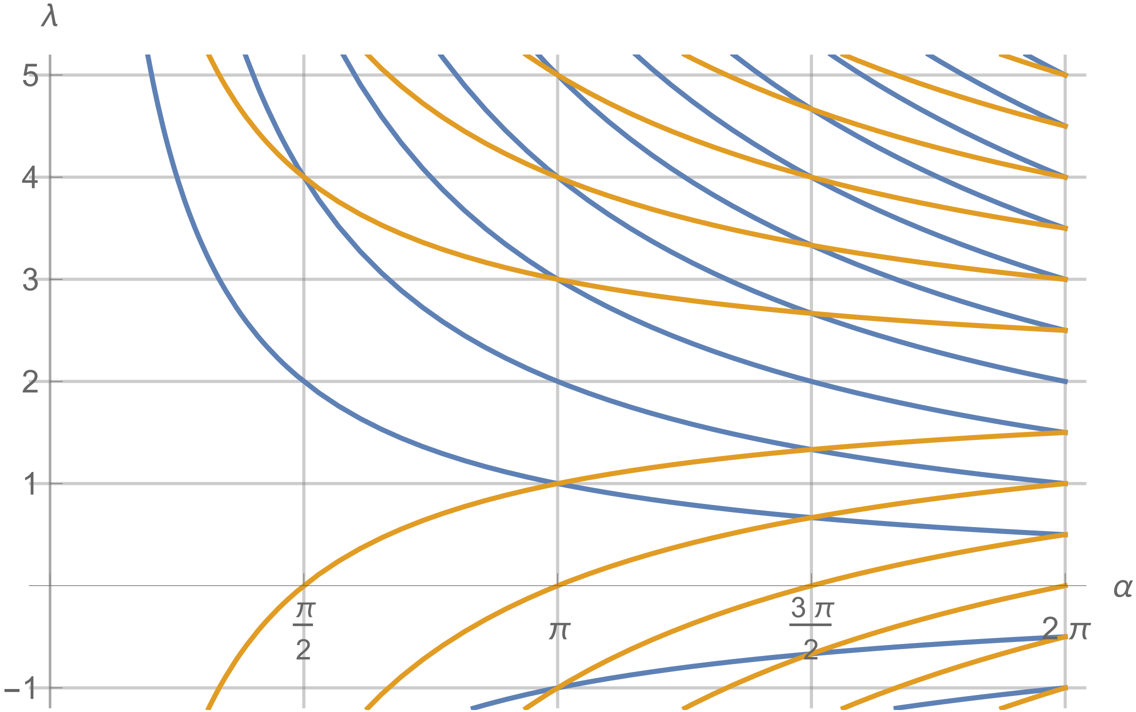


Figure 23: The Navier bilaplace case

A unilateral constraint and corners

9 The Kirchhoff-Love model for a supported plate

The following model was made plausible in Section 5 for a hinged plate

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - (1-\sigma)\,\kappa u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{43}$$

or in the case of a polygonal domains where $\kappa = 0$ except at corners and we seem to arrive at (27), (30), which is (43) with $\sigma = 0$. Whenever one finds in the literature 'Kirchhoff-Love model for a supported plate' one usually finds either one of these boundary value problems. Is the name appropriate? A discrete model in [6] should be a warning. One might guess that by pushing the plate downwards, the plate should go downwards. On the other hand, this guess could also apply to the clamped plate model:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{44}$$

for which it is known since Duffin [19] that no such result holds. See also [54]. On the other hand, for problem (30), i.e.

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{45}$$

on smooth domains, and even on domains with only convex corners one may split it in a system and still find a solution u with $u, \Delta u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and hence $f \geq 0$ implies $u \geq 0$. This however is a circular reasoning. One should go back to the energy functional and allow functions $u \in W^{2,2}(\Omega)$ with

$$u^- = -\min(0, u) \in W_0^{1,2}(\Omega).$$

and minimize over all those admissable functions.

For an overview of sign preservation for simple membrane and plate models see [54]. The real supported plate problem, namely to find the minimize of the functional in (25), that is

$$J_\sigma(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 + (1-\sigma) \left(u_{xx} u_{yy} - u_{xy}^2 \right) - f\ u \right) dx \tag{46}$$

on the space

$$\mathcal{W}_+(\Omega) := \left\{ u \in W^{2,2}(\Omega);\ u^- \in W_0^{1,2}(\Omega) \right\},$$

has been considered in [49, 41]. We recall some results from those papers:

Theorem 36 ([41]) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega$ Lipschitz and let $\sigma \in (-1, 1)$. For $f \in L^2(\Omega)$ satisfying*

$$\int_\Omega f\ \zeta\ dx < 0 \text{ for all nontrivial affine functions } \zeta > 0 \text{ in } \Omega$$

there is a unique minimizer $u_s \in \mathcal{W}_+(\Omega)$.

Let u_h be the solution of the hinged problem, that is, a minimizer of J from (46) in

$$\mathcal{W}_0(\Omega) := \left\{ u \in W^{2,2}(\Omega) ;\ u \in W_0^{1,2}(\Omega) \right\}.$$

For the hinged problem on polygonal domains see [39]. The hinged problem on curvilinear domains has been considered in [13]. Analytic results for a correct numerical approach are obtained by Davini [16], [17].

Only if u_s and u_h are equal one can mix the names hinged and supported. But unfortunately for the literature one finds:

Theorem 37 ([41]) *If $\Omega \subset \mathbb{R}^2$ is a bounded polygon and $0 \gneq f \in L^2(\Omega)$, then $u_s \neq u_h$.*

We will first recall the proof from [49] in the case of a rectangle and $f \in C_0^\infty(\Omega)$.

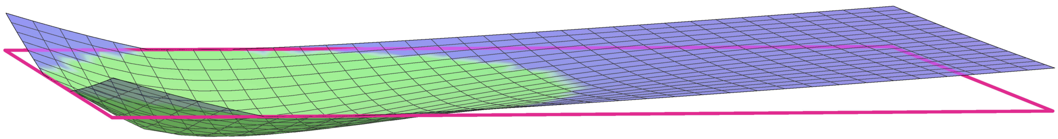


Figure 24: A rectangular supported plate pushed down on the left side. Picture taken from [54].

Proof. For $u \in \mathcal{W}_h$ in the case of a rectangle $\Omega = R$, say $R = (0,a) \times (0,b)$ we can use some reflection arguments. Define the extension operator for functions on $R = (0,a) \times (0,b)$ to functions on

$$R_2 = (-a,a) \times (-b,b),$$

neglecting the axes, by

$$E(u)(x,y) = \operatorname{sign}(xy)\, u(|x|, |y|).$$

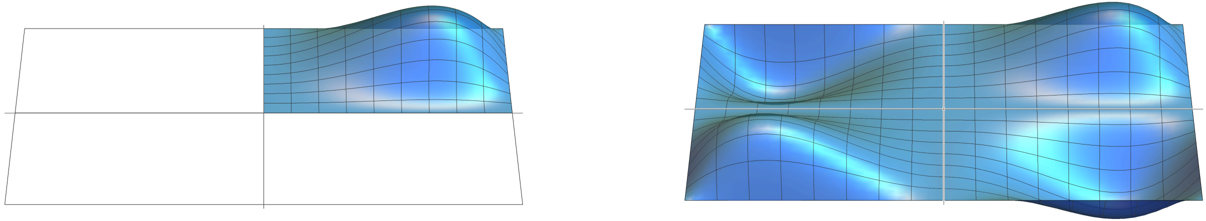


Figure 25: Extending by reflecting

If $u \in \mathcal{W}_0(R)$, then the Dirichlet condition implies that $E(u) \in \mathcal{W}_0(R_2)$. Moreover, if u is a (weak) solution of (45) for $\Omega = R$, then $E(u)$ is a (weak) solution of (45) for $\Omega = R_2$ with $E(f)$ on the right hand side. And last but not least, $E(u) = u$ on R which implies that the regularity of the interior is transferred to the boundary. So $f \in L^2(R)$ results in $u \in W^{4,2}(R)$ and we may do some partial integration such as

$$K(u) := \int_R \left(u_{xx}u_{yy} - u_{xy}^2\right) dxdy = \int_{\partial R} \left(u_{xx}u_y n_2 - u_{xy}u_y n_1\right) d\tau = 0.$$

So u_h is a minimizer of J_σ independent of σ. But then

$$\partial K(u_h; v) = 0 \text{ for all } v \in \mathcal{W}_0(R).$$

When is u_s a minimizer of J_σ for functions in $\mathcal{W}_+(R)$? From [28] one finds that, assuming J_σ is convex as a functional and $\mathcal{W}_+(R)$ is convex and closed as a set, the equivalent condition is

$$\partial J\left(u_s; v-u_s\right) \geq 0 \text{ for all } v \in \mathcal{W}_+(R). \tag{47}$$

If $u_s = u_h$, then for $v \in \mathcal{W}_+(R)$

$$\begin{aligned} \partial J_\sigma\left(u_h; v-u_h\right) &= \partial J_\sigma\left(u_h; v\right) - \partial J_\sigma\left(u_h; u_h\right) = \partial J_\sigma\left(u_h; v\right) \\ &= \int_R \left(\Delta u_h \Delta v - fv\right) dxdy - \left(1-\sigma\right) \partial K\left(u_h; v\right). \end{aligned} \tag{48}$$

Integrating by parts gives, since $\Delta u_h = 0$ on ∂R:

$$\begin{aligned} \int_R \left(\Delta u_h \Delta v - fv\right) dxdy &= \int_{\partial R} \left(\Delta u_h\ \partial_n v - \partial_n \Delta u_h\ v\right) d\tau + \int_R \left(\Delta^2 u_h - f\right) v dxdy \\ &= -\int_{\partial R} \partial_n \Delta u_h\ v\ d\tau \end{aligned} \tag{49}$$

and also

$$\begin{aligned} \partial K\left(u_h; v\right) &= \int_R \left(u_{h,xx} v_{yy} + u_{h,yy} v_{xx} - 2u_{h,xy} v_{xy}\right) dxdy \\ &= -\int_{\partial R} u_{h,\tau n} v_\tau d\tau = 2\left[u_{h,xy} v\right]_{(0,0) \text{ and } (a,b)}^{(a,0) \text{ and } (0,b)} + \int_{\partial R} u_{h,\tau\tau n} v d\tau. \end{aligned} \tag{50}$$

Combining (47), (48), (49) and (50) we obtain that

$$\int_{\partial R} \left(\partial_n \Delta u_h + \left(1-\sigma\right) u_{h,\tau\tau n}\right)\ v\ d\tau + 2\left(1-\sigma\right)\left[u_{h,xy} v\right]_{(0,0) \text{ and } (a,b)}^{(a,0) \text{ and } (0,b)} \leq 0. \tag{51}$$

For the rectangle we can skip the Kondrat'ev functions and use Serrin's corner point lemma, which tells us that for $g \gneq 0$ and u the solution of

$$\begin{cases} -\Delta u = g & \text{in } R, \\ u = 0 & \text{on } \partial R, \end{cases}$$

a type of Hopf' boundary point lemma holds, namely

$$u_{xy} > 0 \text{ in } \{(0,0), (a,b)\} \text{ and } u_{xy} < 0 \text{ in } \{(a,0), (0,b)\}.$$

Applying this to $-u_h$, indeed u_h solves the system in (15) with $f \lneq 0$ and hence $-\Delta u_h < 0$ and $u_h < 0$. So we find for $v > 0$ near the corners that

$$2(1-\sigma)\left[u_{h,xy} v\right]_{(0,0) \text{ and } (a,b)}^{(a,0) \text{ and } (0,b)} > 0. \tag{52}$$

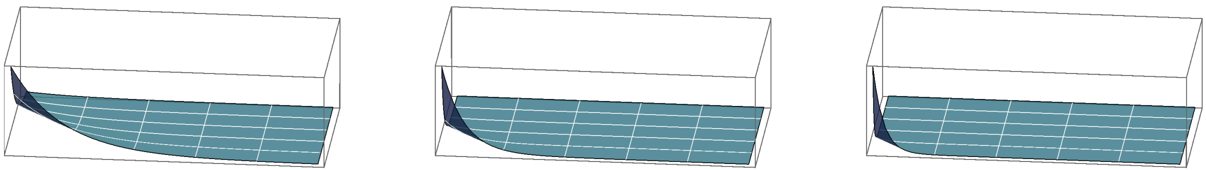


Figure 26: A series of v

Since the left hand side of (51) is an integral, we may play with the order of a pointwise contribution and an L^1-contribution to find $0 \leq v \in \mathcal{W}_+(R)$ such that the sign of (51) is determined by the corner contribution in (52), and that gives the contradiction: $u_s \neq u_h$. ∎

If the domain has a corner with an angle different from $\frac{1}{2}\pi$, then the solution in general is not in $W^{4,2}(\Omega)$ and one has to proceed with more care. In [41] it is shown that at each corner the solution of the supported plate under a downwards force moves upwards at each corner. For convex corners it moves upwards at the corner and for reentrant corners the deviation at the corner itself maybe 0 but nearby the plate will be 'loose'. To prove this result one compares with the lowest order singular function(s) that is (are) still in $W^{2,2}(\Omega)$. And as before $r^{\lambda}\phi(\theta)$ belongs to this space whenever $\lambda > 1$. See Figure 27.

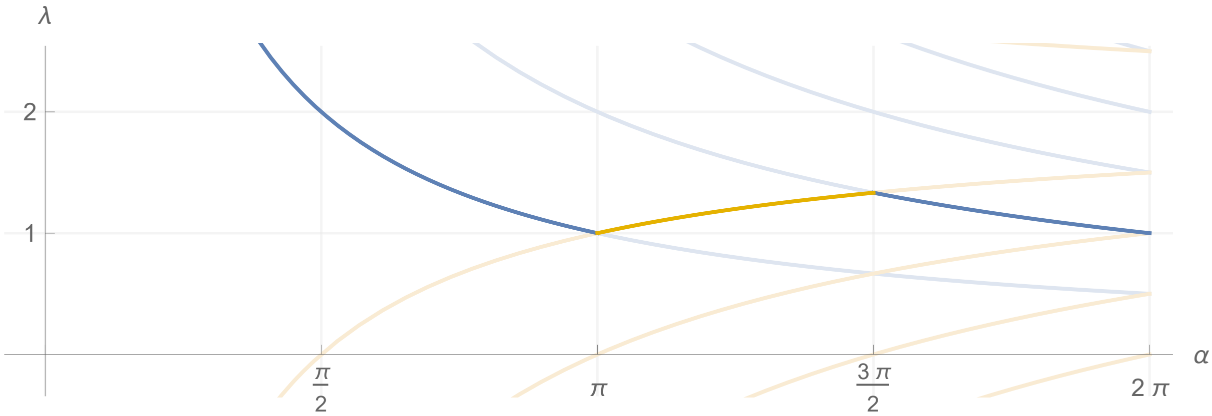


Figure 27: The lowest relevant singular functions

10 Two first eigenfunctions

In this section we will recall a result from [55]. Let us first come back to a famous result named after Krein and Rutman [31]. For integral operators the result was already stated by Jentzsch [27] in 1912. A more perfect statement needs an additional result of de Pagter [43].

Theorem 38 (Krein-Rutman-De Pagter [22]) *A compact, positive and irreducible operator $A \in L(E;E)$ for a Banach lattice E with dimension larger 1, has a positive spectral radius, which is the unique eigenvalue with a positive eigenfunction and it is algebraically simple.*

For second order elliptic boundary value problems the maximum principle gives strict positivity, which includes irreducibility, and when the domain is bounded, the solution operator for regular problems is compact. So one may apply the theorem.

The simple model for the hinged plate we stated before in (27) and (30), namely

$$\left\{ \begin{array}{cl} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{53}$$

is of fourth order, so a priori no maximum principle applies. Nevertheless, when writing this equation as a system as in (15), we find strict positivity and the above result should apply.

Let us consider the corresponding eigenvalue problem

$$\begin{cases} \Delta^2\phi = \lambda\phi & \text{in } \mathsf{C}_\alpha, \\ \phi = \Delta\phi = 0 & \text{on } \partial\mathsf{C}_\alpha. \end{cases} \tag{54}$$

for the non-convex sectors

$$\mathsf{C}_\alpha = \left\{(r\cos\varphi, r\sin\varphi)\,; 0 < r < 1 \text{ and } |\varphi| < \tfrac{1}{2}\alpha\right\} \tag{55}$$

with $\alpha \in (\pi, 2\pi)$. The specialty of this case is that one may find the solutions of (54) by explicit formulas using Bessel functions. Doing so, one finds the two eigenfunctions sketched in Figure 28.

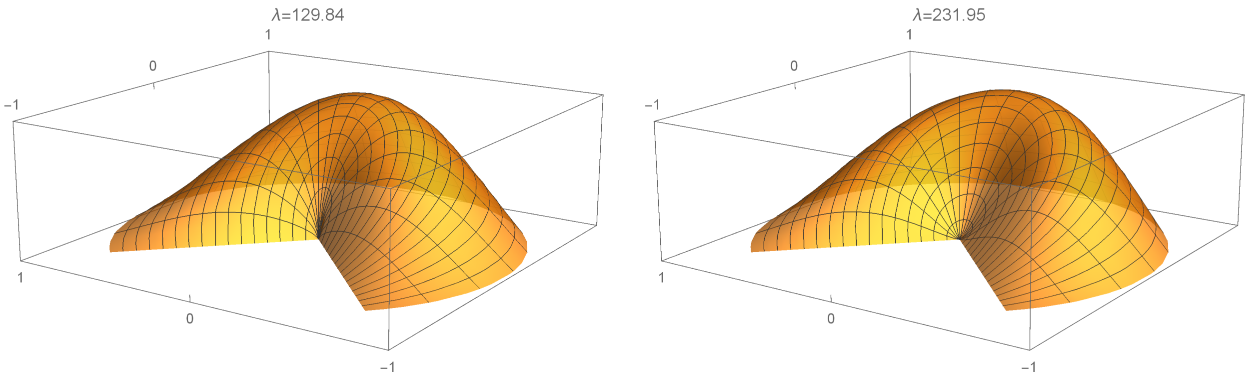


Figure 28: Two positive eigenfunctions for (54) for $\alpha = \frac{3}{2}\pi$. Picture taken from [55].

If we let ϕ_1 be the first eigenfunction of

$$\begin{cases} -\Delta\phi = \mu\phi & \text{in } \mathsf{C}_{3\pi/2}, \\ \phi = 0 & \text{on } \partial\mathsf{C}_{3\pi/2}, \end{cases}$$

one may compute

$$\phi_1(r,\varphi) = J_{2/3}\left(\nu_{2/3,1}\ r\right)\cos\left(\tfrac{2}{3}\varphi\right), \tag{56}$$

with $\nu_{2/3,1}$ the first positive root of $J_{2/3}$. This function is sketched on the left of Figure 28. Here $J_{2/3}$ is the Bessel function of the first kind. Next we will also need $I_{2/3}$, the modified Bessel function of the first kind.

With ν the first positive number satisfying

$$I_{-2/3}(\nu)\,J_{2/3}(\nu) = I_{2/3}(\nu)\,J_{-2/3}(\nu),$$

one finds the second positive eigenfunction $\tilde{\phi}_1$ defined by

$$\tilde{\phi}_1(r,\varphi) = \Big(\left(I_{-2/3}(\nu) - J_{-2/3}(\nu)\right)\left(I_{2/3}(\nu r) - J_{2/3}(\nu r)\right) \tag{57}$$

$$- \left(I_{2/3}(\nu) - J_{2/3}(\nu)\right)\left(I_{-2/3}(\nu r) - J_{-2/3}(\nu r)\right)\Big)\cos\left(\tfrac{2}{3}\varphi\right), \tag{58}$$

which is sketched on the left of Figure 28.

Both functions are in $C_0\left(\overline{\mathsf{C}_{3\pi/2}}\right) \cap C^\infty\left(\mathsf{C}_{3\pi/2}\right)$ and not in $C^2\left(\overline{\mathsf{C}_{3\pi/2}}\right)$, so the condition $\Delta\phi(0) = 0$ would need some explanation. Both functions satisfy $\Delta\phi(x) = 0$ for $x \in \partial\mathsf{C}_{3\pi/2} \setminus \{0\}$. By the way, there is no positive eigenfunction in $C^2\left(\overline{\mathsf{C}_{3\pi/2}}\right)$.

Another paradox?

No, the answer lies in space.

Through direct computations one finds that

$$\begin{array}{llcccl} (\phi_1,\Delta\phi_1) & \in & \Big(W_0^{1,2}(\mathsf{C}_{3\pi/2}) & \times & W_0^{1,2}(\mathsf{C}_{3\pi/2})\Big) & =: \mathcal{A}_{1,1}, \\ & & \bigcup & & \bigcap & \\ (\tilde\phi_1,\Delta\tilde\phi_1) & \in & \Big(W^{2,2}(\mathsf{C}_{3\pi/2})\cap W_0^{1,2}(\mathsf{C}_{3\pi/2}) & \times & L^2(\mathsf{C}_{3\pi/2})\Big) & =: \mathcal{A}_{2,0}. \end{array}$$

Moreover, neither space $\mathcal{A}_{i,j}$ is contained in the other and even $(\phi_1,\Delta\phi_1)\notin\mathcal{A}_{2,0}$ and $(\tilde\phi_1,\Delta\tilde\phi_1)\notin\mathcal{A}_{1,1}$.

Remark 39 *The eigenfunction ϕ_1 is the eigenfunction from the system setting in (15), for which the solution operator is positive due to the maximum principle applied twice. Hence one may apply Theorem 38. The eigenfunction $\tilde\phi_1$ is the minimizer for the functional in (17). It is positive, although that cannot be concluded from Theorem 38, since the solution operator is not known to be positive.*

The same result, namely the existence of 'two' first eigenfunctions one can show for all cones C_α with $\alpha>\pi$. See [55]. In [39] one finds that the solution operator for the second setting is not positive for angles larger than $\frac{3}{2}\pi$. For angles in $(\pi,\frac{3}{2}\pi)$ (non)positivity is unknown.

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