Jahresber Dtsch Math-Ver (2016) 118:275–320
DOI 10.1365/s13291-016-0147-0

SURVEY ARTICLE

On Sign Preservation for Clotheslines, Curtain Rods, Elastic Membranes and Thin Plates

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Published online: 8 September 2016

Abstract All problems mentioned in the title seem to have one thing in common. Whenever a force is applied in one direction, the object moves in that direction. At least this is what one might expect. The corresponding boundary value problems contain differential equations of second order for line and membrane, while rod, or beam, and plate are modeled through fourth order differential equations. How a positive source will give a positive solution crucially depends on this order and the boundary conditions. We will present a survey concerning the so-called positivity preserving property for these various models.

Keywords Positivity · Sign preservation · Second and fourth order elliptic · Kirchhoff plate

Mathematics Subject Classification (2000) 35J40 · 35J08 · 34B27

1 Introduction

What do a curtain rod and a clothesline have in common? An obvious answer could be: if you hang something on it, the rod/line will move downwards. Like with soccer, if you kick a ball in a certain direction, you expect it to go in that direction. In mathematical terms for line and rod: a downwards force should imply a downwards deviation.

▷ **A Short History of the Models** So a differential equation with appropriate boundary conditions that models such a problem should have this sign preserving

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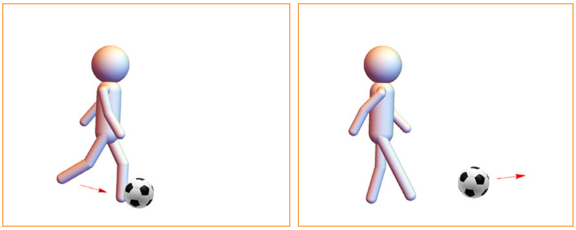


Fig. 1 Kicking the ball forwards gives some hope that the ball moves forwards, but hope may not be enough

property or should be near sign preserving. Finding an appropriate model is a quest in itself. Indeed, the search for the model of a beam has a long history. Leonardo Da Vinci in the 16th and Galileo in the 17th century were interested in the behavior of a beam, see their sketches reproduced in [59], but they lacked the appropriate mathematical tools. Jacob Bernoulli tried to find an appropriate model but was not successful. Only when his nephew, Daniel Bernoulli, wrote a letter to his friend Euler in 1742 suggesting a variational argument, Leonhard Euler [31] came up with the first sound mathematical formulation, now known as the Euler–Bernoulli model. See the surveys of Fraser [32] and Marenholtz [59].

In 2 dimensions spanned membranes and thin plates are problems for which one might expect a similar sign preserving behavior. The model for an elastic thin plate has a similar exciting story. See [19]. A final model appeared somewhere in the vicinity of Sophie Germain [39], Joseph-Louis Lagrange and Denis Poisson [69] at the beginning of the 19th century. A variational formulation of the energy for the rod or plate leads to a differential equation, which nowadays is known as the corresponding Euler–Lagrange equation. Linearizing the equation for a thin plate one arrives at the Kirchhoff–Love model. See [57].

Like beam or string also membrane and plate should be fixed somehow at the boundary in order to have a well-posed problem. The number of physically relevant conditions for fourth order problems is naturally 'much larger' than for second order.

▷ **Maximum Principle or Not?** Clothesline, or string, and membrane are modeled through second order equations, which allow for a maximum principle and a sign preserving result.

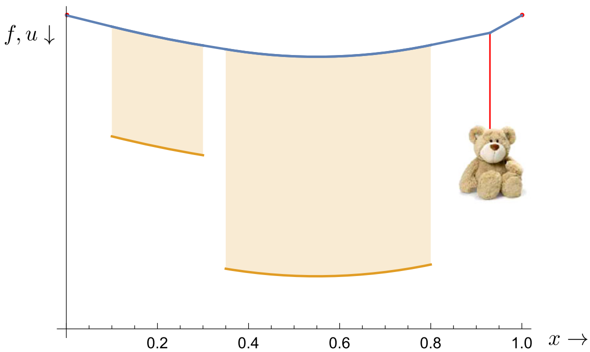


Fig. 2 Not very surprising: a clothesline goes down when you hang something on it. Does the mathematical model confirm this?

Indeed $u''(x_0) > 0$ *implies that* u *will not have a maximum at an interior point* x_0, *and that would imply that the maximum occurs at the boundary. Hence, if at the boundary the function* u *is fixed to be at a zero level, then a fixed sign of* $-u''$ *implies the same sign for* u. *In other words, the corresponding boundary value problem has a sign preserving property.*

So, when thinking of sign preservation, we prefer to write equations with $-u''$ instead of u''. By iteration, a plus sign is again natural in fourth order problems.

A maximum principle not only holds in one dimension and one may replace $-u''$ by $-\Delta u = -\sum_{i=1}^{n} u_{x_i x_i}$ or by Lu, where L is a (purely) second order elliptic operator:

$$Lu = -\sum_{i=1}^{n} a_{ij} u_{x_i x_i} \quad \text{with } a_{ij} = a_{ji} \quad \text{and} \quad \exists c > 0\ \forall \xi \in \mathbb{R}^n\colon \sum_{i=1}^{n} a_{ij}\xi_i \xi_j \geq c|\xi|^2.$$

For positivity question with second order elliptic equations indeed 'Maximum Principle' is the keyword and Protter and Weinberger in 1967 wrote the classic [71] on the subject.

Rods or beams and thin plates are modeled through fourth order elliptic equations, for which such a pointwise argument as above does not directly apply. See Fig. 3. Indeed, here one cannot expect a maximum principle to hold. Nevertheless, from practical experience we still expect that a sign or positivity preserving property (PPP) will hold true for beams on plates. As Fig. 3 makes clear, the corresponding result should follow from the interplay between differential equation and boundary conditions.

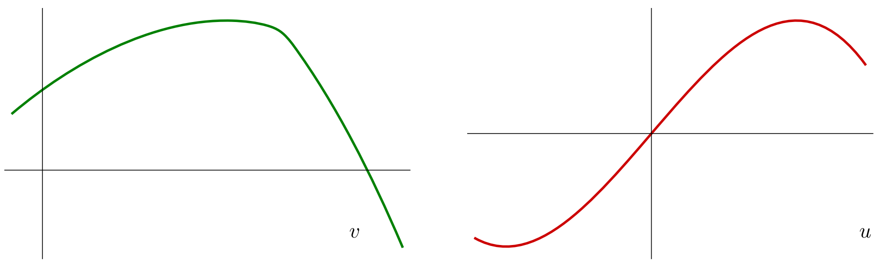


Fig. 3 *On the left*: any function $v : \mathbb{R} \to \mathbb{R}$ satisfying $v'' \leq 0$ has its minimum on the boundary. *On the right*: no such result holds for u with $\pm u'''' \leq 0$. The graph belongs to $u(x) = x - x^3$ and obviously $u''''(x) = 0$

▷ **A Short History on Positivity in Fourth Order Models** For the one-dimensional problem, i.e. $u'''' = f$ with several suitable boundary conditions, the solution can be almost explicitly computed and indeed shows PPP. Around 1900 Boggio and Hadamard got interested in the corresponding mathematical question for the model of a clamped plate in two and its generalization in more dimensions. The differential equation that they considered comes from the Kirchhoff–Love model for pure bending of an isotropic plate: $\Delta^2 u = f$ on $\Omega \subset \mathbb{R}^2$. The equation can also be considered on domains $\Omega \subset \mathbb{R}^n$ but will need to be supplemented by appropriate boundary conditions. For the simplified hinged boundary conditions, i.e. $u = \Delta u = 0$ on $\partial\Omega$ and we will discuss, whether or not these conditions are appropriate, it seems that one finds an iterated Dirichlet–Laplace problem and hence a positivity preserving property. More interesting are the so-called clamped boundary conditions $u = \frac{\partial}{\partial \nu}u = 0$ on $\partial\Omega$. As usual $\frac{\partial}{\partial \nu}$ denotes here the outward normal derivative. Although, with u being only defined in $\overline{\Omega}$, it would have been better if one had agreed upon inward normal. Anyway, these boundary conditions do not allow a repeated second order approach and one has to come up with new techniques. Direct computations for the fourth order model of a clamped plate do not give much in general. Two exceptions are the disk, for which Boggio [10] obtained some results around 1905, and the Limaçon de Pascal studied by Hadamard [50] around 1908.

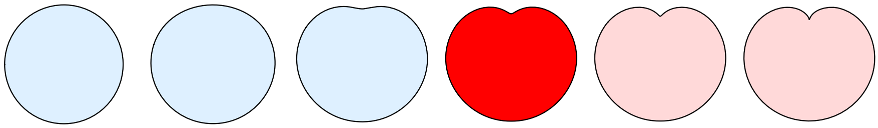


Fig. 4 Between disc and cardioid are the Limaçons de Pascal: defined in polar coordinates by $r = 1 - 2a\sin\varphi$ with $0 \le a \le \frac{1}{2}$. Above are sketches for $a \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{\sqrt{6}}, \frac{4}{5}, \frac{1}{2}\}$. The special role of the fourth (*dark*) limaçon is explained in the item near Fig. 26

For disk and limaçon almost explicit formulas for the solution, so-called Green functions, were found and these explicit functions are or at least seem to be positive. For other domains such functions are not available in a simple form and results concerning positivity are much harder to derive. Nevertheless, Hadamard was aware, see [50], that for a domain with a small hole PPP could not be true. The conjecture named after Boggio–Hadamard, that the clamped plate equation is positivity preserving, became restricted to convex domain. It remained open till Duffin's paper [28] from 1946. The paragraph on the clamped plate problem, i.e. (42), contains more details.

Having PPP on balls and limaçons, but not for general domains, it is natural to ask what happens, when perturbing the domain or the equation starting from a known case. Coffman and Grover in [17] as well as Kozlov and coauthors in [55] followed this strategy for a negative result. A positive result for small perturbations from a disk was found in [44]. Perturbations of the differential operator in arbitrary dimensions are studied in [45].

The simplest perturbations one obtains by adding λu to the differential equation. Not only does this have a physical motivation, but one will also see that the sign preserving question will get more structure. In this paper we will give a survey of positivity preserving properties focused on λ as a parameter, and try to explain what will happen when PPP fails to hold. With such an additional term one obviously will encounter eigenvalue problems.

▷ **Some Remarks** Let us explain the notation convention that we will use for eigenvalues, starting from second order problems.

- *Fix $\Omega \subset \mathbb{R}^n$ to be a bounded domain, i.e. open and connected, and let $f : \Omega \to \mathbb{R}$ be a given force density. The Dirichlet–Laplace problem with the additional feedback term λu, with $\lambda \in \mathbb{R}$ is*

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Generically a solution u exists, except when $\lambda = \lambda_i$ with λ_i some eigenvalue. These eigenvalues will form a countable set $\{\lambda_k\}_{k\in\mathbb{N}^+}$, which is unbounded from below. In fact, they can be ordered:

$$-\infty \longleftarrow \cdots \leq \lambda_{k+1} \leq \lambda_k \leq \cdots \leq \lambda_3 \leq \lambda_2 < \lambda_1 < 0.$$

It is well-known that for (1) a positive force f results in a positive solution u if and only if λ satisfies $\lambda_1 < \lambda$.
- *Whenever we consider a solution operator G_Ω, say $u = G_\Omega f$ solves (1) for $\lambda = 0$, we will use $\{\nu_k\}_{k\in\mathbb{N}}$ for the eigenvalues of G_Ω. So, comparing eigenvalues for (1) and for G_Ω, we get $G_\Omega\varphi_k = \nu_k\varphi_k$ with $\nu_k \downarrow 0$ and $\lambda_k = -(\nu_k)^{-1}$.*

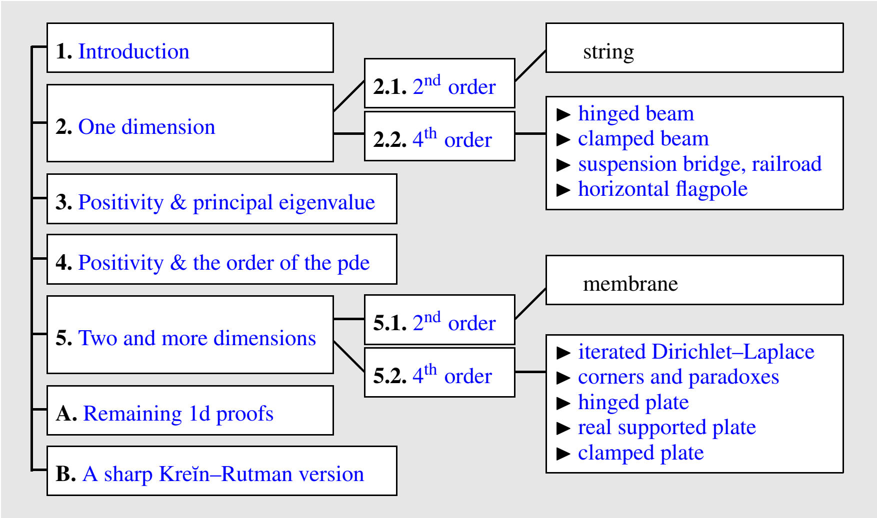
We will keep this notation convention for eigenvalues, i.e. $\lambda_k \downarrow -\infty$ in the differential equation and $\nu_k \downarrow 0$ for the solution operator throughout the paper, also for fourth order models.

Except for curiosity concerning positivity for these models, why study such sign preserving properties? Hadamard formulated the three properties that a well-posed problem should have: existence, uniqueness and continuous dependence. If such a sign preserving property holds, then one finds uniqueness of the solution. Indeed, if there are two solutions, then, at least for linear equations, the pde for the difference has a zero right hand side, which transfers then to the result that this difference should be zero. For nonlinear second order elliptic equations a sign preserving property allows one to obtain a priori estimates by super- and subsolutions. Moreover, in many cases a subsolution below a supersolution implies the existence of a solution in between. Qualitative properties such as regularity for elliptic equations start with estimates and these estimates often start with the maximum principle.

For more information concerning second order elliptic equations see the seminal book by Gilbarg and Trudinger [40]. For the relation between ordering and second order elliptic problems see [2]. Polyharmonic equations and questions concerning positivity can be found in [37].

One may say that the loss of a sign preserving property in general for fourth order elliptic equations, is the major obstacle when moving from second order elliptic problems to fourth order. To come back to the first image: it seems like playing soccer but now without having a clue where the ball might go.

We conclude this introduction with a line-up of the consecutive sections.



2 One Dimension

Simple linearized models for the laundry line and the supported curtain rod, respectively, are

$$\begin{cases} -u'' = f & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \quad \text{and} \quad \begin{cases} u'''' = f & \text{in } (0,1), \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0. \end{cases} \tag{2}$$

Here f is the source term, pulling downwards when $f \geq 0$, and u is the resulting deviation from the horizontal equilibrium, again downwards when $u \geq 0$.

Both problems have an explicit Green function, which is positive also in both cases. In fact the Green function for the problem on the right is just an iteration of the one for the left problem. But even if we would have a clamped rod at both its ends, i.e. $u'' = 0$ is replaced by $u' = 0$ on the boundary, then one would have found a positive Green function. Let us recall that a Green function here is a function $g : [0,1] \times [0,1] \to \mathbb{R}$, such that

$$u(x) = \int_{-1}^{1} g(x,y) f(y) dy$$

solves the corresponding problem in (2). So the question '*when is the problem positivity preserving*' can be rephrased as:

- *When do we have:* $f \geq 0 \Rightarrow u \geq 0$*?*

Or in terms of a Green function, which exists also in the higher dimensional case, that we consider later on:

- *When is the Green function* $g(x,y)$ *nonnegative?*

In order to find a structural answer we will introduce a parameter λ in problems like (2). For $\lambda > 0$ it can be seen as adding elastic springs between the line or rod and a ceiling. For $u = 0$ these springs neither push nor pull. They help the line or rod to stay closer to its equilibrium. Since we move away from the household description, the common terms string and beam are used from now on.

2.1 Second Order: A String with Elastic Feedback

Problem 1 *The linearized second order boundary value problem for a string under tension with elastic feedback*:

$$\begin{cases} -u'' + \lambda u = f & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

For $\lambda > 0$ one models the elastic feedback mentioned above. Mathematically also $\lambda < 0$ is interesting. In Fig. 2 and Fig. 5 solutions of Problem 1 with $\lambda = 0$ and $\lambda = 20$ are sketched for a right hand side f, which is piecewise constant except for a δ-function at the position of the teddy bear. The length of the cloth represents the local value of f.

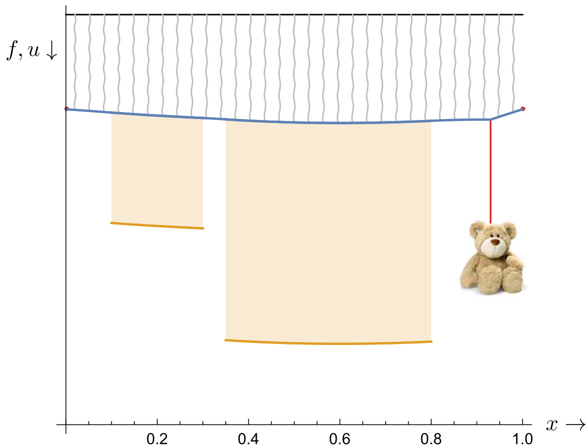


Fig. 5 A clothesline with elastic feedback. Here a sketch of the solution of problem (1) for $\lambda = 20$. For all pictures a scaling of the actual Green function has been used

The following result is standard but we nevertheless write it for comparison with the other problems.

Lemma 2.1 *Problem 1 is positivity preserving if and only if* $\lambda > -\pi^2$.

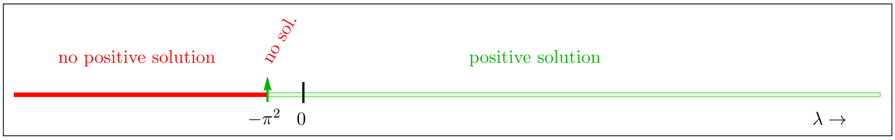


Fig. 6 Suppose that $f \gneq 0$, meaning $0 \neq f \geq 0$. Above is a graphical answer for the dependence from λ of (the sign of) the solution u in Problem 1, the second order model for the clothesline. Note that there is no bound from above for the 'good' λ

The first eigenfunction for Problem 1 is $\varphi_1(x) = \sin(\pi x)$ with $\lambda_1 = -\pi^2$. For a proof see Sect. 3.2 for $\lambda \leq 0$ and Appendix A for $\lambda > 0$.

2.2 Fourth Order: Beams with Elastic Feedback

Three physical relevant boundary conditions, here written for 0 as boundary point, are the following:

- hinged (also called supported): $u(0) = u''(0) = 0$,
- clamped: $u(0) = u'(0) = 0$, and
- free: $u''(0) = u'''(0) = 0$.

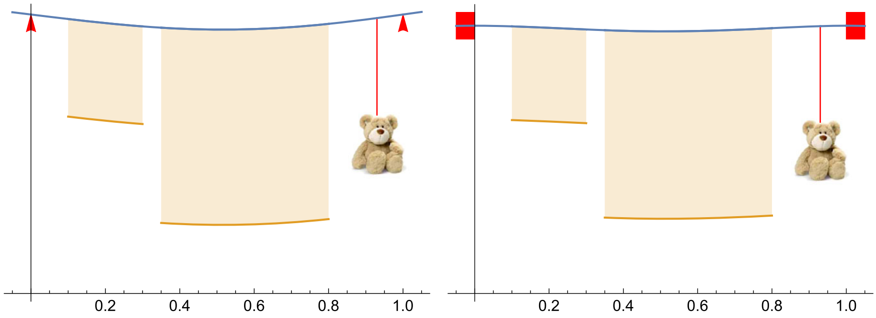


Fig. 7 A hinged (supported) and a clamped beam

In the case that the beam is supported (from below) and the force is downwards, one obtains the hinged boundary condition. For a sketch see Fig. 7. For proofs of the next three lemmas see Sect. 3.2 and Appendix A.

► **The Hinged or Supported Beam** is modeled as follows.

Problem 2 *The linearized fourth order boundary value problem for the supported beam with elastic feedback*:

$$\begin{cases} u'''' + \lambda u = f & in\ (0,1), \\ u(0) = u''(0) = 0, \\ u(1) = u''(1) = 0. \end{cases}$$

Lemma 2.2 *Problem 2 is positivity preserving if and only if* $\lambda \in (-\pi^4, \lambda_c)$, *with* $\lambda_c = 4\mu_c^4$ *and* μ_c *the first positive solution of* $\tan\mu = \tanh\mu$.

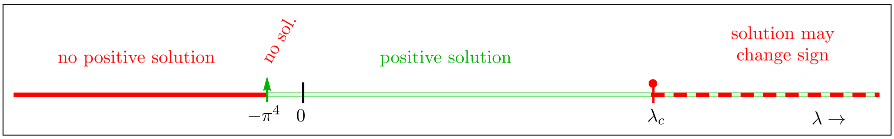


Fig. 8 Suppose that $f \gneq 0$ in Problem 2, the supported beam. What can one say concerning the sign of u? The bound from above is the typical distinction of fourth order problems from second order

The first eigenfunction for Problem 2 is $\varphi_1(x) = \sin(\pi x)$ and the corresponding eigenvalue is $\lambda_1 = -\pi^4$. One finds, see Fig. 8,

$$-\pi^4 \approx -97.409 \quad \text{and} \quad \lambda_c \approx 950.844.$$

▶ **The Clamped Beam** has only two different boundary conditions compared with the hinged beam.

Problem 3 *The linearized fourth order boundary value problem for the clamped beam with elastic feedback*:

$$\begin{cases} u'''' + \lambda u = f & \text{in } (0,1), \\ u(0) = u'(0) = 0, \\ u(1) = u'(1) = 0. \end{cases}$$

Lemma 2.3 *Problem 3 is positivity preserving if and only if* $\lambda \in (-\rho_1, \lambda_c)$, *where*

1. $\rho_1 = (2\mu_1)^4$ *with* μ_1 *the first positive solution of* $\tan\mu + \tanh\mu = 0$, *and*
2. $\lambda_c = 4\mu_c^4$ *with* μ_c *the first positive solution of* $\tan\mu = \tanh\mu$.

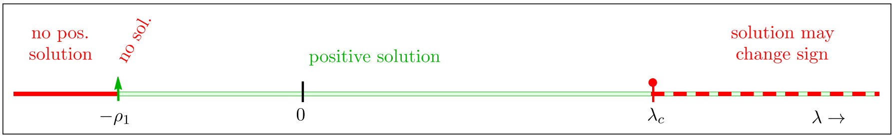


Fig. 9 Suppose $f \gneq 0$ in Problem 3, the clamped beam. What can one say concerning the sign of u? The λ_c-s for the supported and for the clamped boundary conditions are identical

The first eigenfunction for Problem 3 is

$$\varphi_1(x) = \cos\left(2\mu_1\left(x - \frac{1}{2}\right)\right) - \frac{\cos\mu_1}{\cosh\mu_1}\cosh\left(2\mu_1\left(x - \frac{1}{2}\right)\right)$$

with μ_1 given by the lemma and $\lambda_1 = -\rho_1$. One has, see Fig. 9,

$$\rho_1 \approx 500.564 \quad \text{and} \quad \lambda_c \approx 950.844.$$

A sketch for $\lambda > \lambda_c$ is found in Fig. 10.

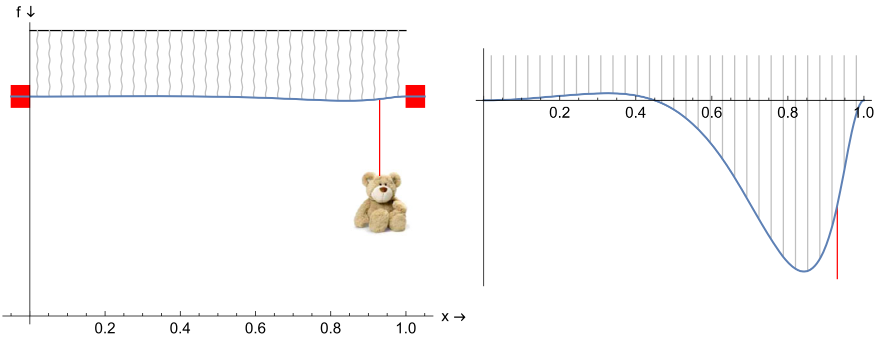


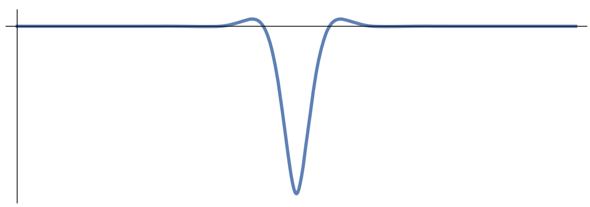
Fig. 10 For $\lambda > \lambda_c$, here $\lambda = 2500$, part of the rod moves upwards under the downwards pulling even with the springs pushing downwards when the rod moves above the neutral horizontal line. *On the right* is a vertically scaled version

► **Suspension Bridges and Railroad Tracks** are two examples of a 'beam', where the elastic feedback is modeled as if many springs push the beam back to the horizontal equilibrium. For a suspension bridge the cables, from which the deck of bridge is hanging, take the place of the springs. For a railroad track the rails are the beam and the soil is the elastic medium. See the paper by Marenholtz [59]. Winkler's model for a railroad with welded rails, meaning of infinite length, would be, with λ some large positive number:

$$\begin{cases} u'''' + \lambda u = f & \text{in } \mathbb{R}, \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{3}$$

Note that for f with compact support $\lim_{|x|\to\infty} u(x) = 0$ implies that all derivatives of u go to 0 at $\pm\infty$.

Fig. 11 A computed solution with a point weight for problem (3). It resembles the measured track displacement of a train as in [5, Fig. 1]



The solution as sketched in Fig. 11 displays a sign change. If this upwards displacement is large enough, it would mean that the rails with its ties/sleepers would come free from the ballast bed. When these come free, smaller stones in the ballast may also come free to resettle, maybe even preventing the sleeper to return to its old position. Such a process will eventually cause track settlement problems. See [58] or [5]. If the ballast particles rearrange itself enough, even a hanging sleeper might develop, which lead to a rapid degradation of the track. By the way, the upwards movement just in front of the train one can notice when standing close to the rails, with the rails on a ballast bed below and not bedded on concrete. Indeed you should have a look at the rails the next time you are waiting at a countryside railway station and a freight train slowly passes. Or one may have a careful look at

http://www.shutterstock.com/de/video/clip-3342815

and notice, that the track moves slightly upwards shortly before the train passes.

▷ **Unilateral Feedback** occurs in a more appropriate model when one replaces the springs by elastic cables, or when the track of the railroad looses its touch with the ballast bed. Indeed, one can pull but not push through a cable and obviously, for a functioning railway the ballast bed should be below the tracks. This leads to the following nonlinear system:

Problem 4 *The fourth order boundary value problem for a clamped beam with a one-sided elastic feedback*:

$$\begin{cases} u'''' + \lambda \max(u, 0) = f \quad \textit{in } (0, 1), \\ u(0) = u'(0) = 0, \\ u(1) = u'(1) = 0. \end{cases}$$

For $\lambda \in (-\rho_1, \lambda_c)$ and f nonnegative the solutions of Problem 3 and 4 coincide. This observation directly implies that the result of Lemma 2.3 for these λ also holds for Problem 4.

As part of a model for a suspension bridge, this fourth order boundary value problem appears in a paper of McKenna and Walter [62]. Not only triggered their paper a lot of consecutive papers, it also triggered some controversy among engineers, whether the investigations after the collapse of the first Tacoma Narrows Bridge led to the correct conclusions. A movie of the actual collapse of this bridge can be seen on:

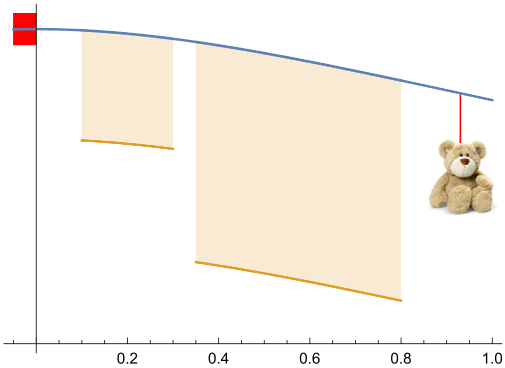
https://www.youtube.com/watch?v=j-zczJXSxnw

When designing the bridge the dynamic behavior was neglected. Soon after finishing the construction an unforeseen interplay between longitudinal and torsional eigenfunctions ruined the bridge. Even today there are new developments concerning the mathematical modeling of suspension bridges. See [36].

► **The Horizontal Flagpole** or the diving board can be seen as a beam clamped on one end and free on the other. The corresponding linear problem with the elastic feedback becomes:

Problem 5 *The fourth order boundary value problem for the beam clamped on one side, free on the other and with elastic feedback*:

$$\begin{cases} u'''' + \lambda u = f & \text{in } (0,1), \\ u(0) = u'(0) = 0, \\ u''(1) = u'''(1) = 0. \end{cases}$$



Lemma 2.4 *Problem 5 is positivity preserving if and only if* $\lambda \in (-d_1, d_c)$, *where*

1. $d_1 = \tau_1^4$ *with* τ_1 *the first positive solution of* $\cos\tau \cosh\tau + 1 = 0$, *and*
2. $d_c = 4\mu_1^4$ *with* μ_1 *the first positive solution of* $\tan\mu + \tanh\mu = 0$.

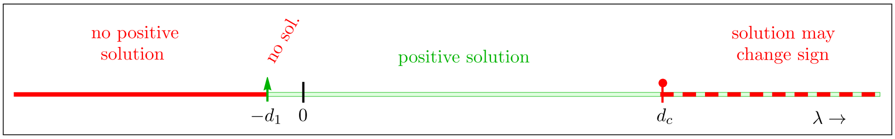


Fig. 12 Suppose that $f \gneq 0$ in the model for the flagpole Problem 5. What can one say concerning the sign of the solution?

The first eigenfunction for Problem 5 is

$$\varphi_1(x) = \sinh(\tau_1 x) - \sin(\tau_1 x) - \frac{\sinh\tau_1 + \sin\tau_1}{\cosh\tau_1 + \cos\tau_1}\big(\cosh(\tau_1 x) - \cos(\tau_1 x)\big)$$

with τ_1 given by the lemma and the corresponding eigenvalue $\lambda_1 = -d_1$. One finds

$$d_1 \approx 12.3624 \quad \text{and} \quad d_c \approx 125.141.$$

3 Positivity and the Principal Eigenvalue

As mentioned before, for both problems in (2) one may compute an explicit Green function $g(x, y)$. Defining the Green operator $\mathcal{G}$ (on an appropriate function space) by

$$(\mathcal{G}f)(x) := \int_\Omega g(x, y) f(y) dy. \tag{4}$$

The solution of (2) is given by $u(x) = (\mathcal{G}f)(x)$. Here $\Omega = (0, 1)$. We recall from [53] for the second order problem on the left in (2) that

$$g_1(x, y) = \begin{cases} x(1-y) & \text{for } x \le y, \\ y(1-x) & \text{for } x > y, \end{cases} \tag{5}$$

while for the one on the right

$$g_2(x, y) = \begin{cases} \frac{1}{6}x(1-y)(2y-x^2-y^2) & \text{for } x \le y, \\ \frac{1}{6}y(1-x)(2x-x^2-y^2) & \text{for } x > y. \end{cases} \tag{6}$$

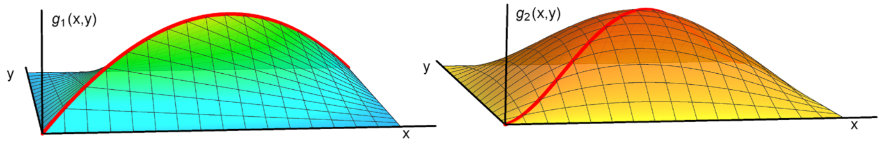


Fig. 13 Graphs of g_1 and g_2

Both Green functions are strictly positive in $\Omega \times \Omega$, which implies that for a nontrivial $f \ge 0$, i.e. $f \gneq 0$, the solution u is strictly positive inside Ω.

In order to show some of the mathematics involved it is convenient to recall some function spaces.

Definition 3.1 The Banach spaces $C^k(\overline{\Omega})$ for bounded $\Omega \subset \mathbb{R}^n$ are defined by

$$C^k(\overline{\Omega}) := \left\{ u : \overline{\Omega} \to \mathbb{R};\ u \text{ is } k \text{ times continuously differentiable} \right\},$$
$$\|u\|_{C^k(\overline{\Omega})} := \sup\left\{ \left|D^\alpha u(x)\right|; |\alpha| \le k \text{ and } x \in \Omega \right\}.$$

Here $D^\alpha u = (\frac{\partial}{\partial x_1})^{\alpha_1} \dots (\frac{\partial}{\partial x_n})^{\alpha_n} u$ with $\alpha \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

- One writes $u \in C_0^k(\overline{\Omega})$ if $u \in C^k(\overline{\Omega})$ and all derivatives of u up to order k are zero at the boundary.

Remark 3.2 Differential equations make most sense, when defined on a domain Ω, meaning an open and connected set, here in $\mathbb{R}^n$. Continuously differentiable on $\overline{\Omega}$ means differentiable in Ω with continuous derivatives and each of these derivatives can be extended to a continuous function on $\overline{\Omega}$.

The vector space $C^k(\Omega)$, being the collection of all k-times continuously differentiable functions on Ω, does not have an obvious norm, since such a function and its derivatives may blow up near $\partial\Omega$.

The operator $\mathcal{G} : C(\overline{\Omega}) \to C(\overline{\Omega})$ in (4), both for g_1 and g_2, is such that for any nontrivial and nonnegative $f \in C(\overline{\Omega})$ there is $c_f > 0$, such that $(\mathcal{G}f)(x) \ge c_f\, d(x, \partial\Omega)$. Here $d(x, \partial\Omega)$ is the distance function to the boundary. For Problems 1 and 2 such $\mathcal{G}$ is called *strongly positive*. For general problems one may define this property as follows.

Definition 3.3 The positive operator $\mathcal{G} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is called *strongly positive*, if for every $0 \lneq f \in C(\overline{\Omega})$ there exists $c_f > 0$ such that $(\mathcal{G}f)(x) \ge c_f(\mathcal{G}1)(x)$.

A second property in the above one-dimensional cases in (2) is that $f \in C(\overline{\Omega})$ implies $u \in C^2(\overline{\Omega})$ and $u \in C^4(\overline{\Omega})$, respectively. Since $C^k(\overline{\Omega})$ with $k \geq 1$ is compactly imbedded in $C(\overline{\Omega})$, the operator $\mathcal{G}$ is also *compact*.

3.1 Kreĭn–Rutman Theorem and Consequences

Positivity and compactness are the main ingredients for the so-called Kreĭn–Rutman Theorem. It generalizes earlier results for positive matrices by Perron–Frobenius [34, 68] and for positive integral kernels by Jentzsch in [51]. Of the Kreĭn–Rutman Theorem one finds many versions in the literature and a common version follows. Strongly positive is easy to state but quite restrictive. A stronger version, which can be applied more easily, is found in Appendix B.

Theorem 3.4 (*Kreĭn–Rutman* [56]) *If $\mathcal{G} \in \mathsf{L}(C(\overline{\Omega}))$ is strongly positive and compact, then the spectral radius satisfies $\nu(\mathcal{G}) > 0$ and is an eigenvalue of $\mathcal{G}$ with a positive eigenfunction. Its algebraic multiplicity is 1 and for all other eigenvalues ν_i of $\mathcal{G}$ it holds that $|\nu_i| < \nu(\mathcal{G})$.*

Remark 3.5 The spectral radius of $\mathcal{G} \in \mathsf{L}(C(\overline{\Omega}))$ is defined by $\nu(\mathcal{G}) = \lim_{n\to\infty} \sqrt[n]{\|\mathcal{G}^n\|}$, where $\|\cdot\|$ is the operator norm for $\mathsf{L}(C(\overline{\Omega}))$, the bounded linear mappings from $C(\overline{\Omega})$ to $C(\overline{\Omega})$.

The condition that $\mathcal{G}$ is strongly positive is easy to explain but needs in general of lot of technical effort to prove it. The version in the appendix replaces 'strongly positive' by irreducibility. This condition needs some explanation but is satisfied in much more general settings and avoids arguments at the boundary like Hopf's boundary point Lemma. Hopf's lemma needs some regularity of the domain. See [7].

We may apply the Kreĭn–Rutman result to

$$\begin{cases} Lu + \lambda u = f & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

with L the differential operator and B denoting some homogeneous boundary conditions as in (2).

Corollary 3.6 *Suppose that for $\lambda = 0$ the solution of (7) is given by $\mathcal{G}f$ with $\mathcal{G} \in \mathsf{L}(C(\overline{\Omega}))$ strongly positive and compact. Set $\lambda_1 := -\nu(\mathcal{G})^{-1}$. Then for $\lambda \in (\lambda_1, 0]$ and $0 \lneq f \in C(\overline{\Omega})$ one finds that the solution of (7) satisfies $u > 0$ in Ω.*

Proof So $\lambda_1 := -\nu(\mathcal{G})^{-1}$ is the first eigenvalue for L with homogeneous boundary conditions $Bu = 0$. Moreover, (7) is equivalent to

$$(I + \lambda\mathcal{G})u = \mathcal{G}f.$$

For λ with $|\lambda\nu(\mathcal{G})| < 1$, that is $|\lambda| < |\lambda_1|$, we find an expression for the solution u of (7) by a Neumann series:

$$u = \sum_{k=0}^{\infty} (-\lambda\mathcal{G})^k \mathcal{G} f. \tag{8}$$

For $\lambda \in (\lambda_1, 0]$ this series not only converges, but the expression even shows that also here the solution operator is strongly positive. □

One may even show that λ_1 is optimal. If $\lambda < \lambda_1$, then one may use the first eigenfunction as the right hand side, $f = \varphi_1 > 0$ and find that

$$u = \frac{1}{\lambda - \lambda_1}\varphi_1 < 0.$$

So no sign preservation for $\lambda < \lambda_1$. In fact one finds not only that for some $f \gneq 0$ the solution will not be positive but even that no positive solution u exists for all $f \gneq 0$ when $\lambda \leq \lambda_1$. If (7) has a selfadjoint setting with respect to the $\mathcal{L}^2(\Omega)$ inner product, then one obtains a contradiction for $\lambda \leq \lambda_1$ assuming $f \gneq 0$ and $u \gneq 0$ through

$$\begin{aligned} 0 < \int_\Omega f\varphi_1 dx &= \int_\Omega (\lambda u + Lu)\varphi_1 dx \\ &= \int_\Omega u(\lambda\varphi_1 + L\varphi_1)dx = (\lambda - \lambda_1)\int_\Omega u\varphi_1 dx \leq 0. \end{aligned} \tag{9}$$

Without selfadjointness one may argue by using the adjoint eigenfunction.

3.2 Proofs for the 1d Case with $\lambda \leq 0$

Proof of Lemma 2.1–2.4 for $\lambda \leq 0$ With the Kreĭn–Rutman result and the argument just explained, it is merely an exercise to prove the claims in these lemmas for $\lambda < 0$. For second order problems, such as in Lemma 2.1, $\mathcal{G}$ is strongly positive by the *strong maximum principle*. Also Lemma 2.2, as an iterated second order problem, can be dealt with this way. For most fourth order problems such a result is in general not available. For the one-dimensional cases in Lemma 2.3–2.4 the most direct way seems to be the explicit Green function. Indeed, the Green functions both for Problems 3 and 5 are positive:

$$g_3(x,y) = \begin{cases} \frac{1}{6}x^2(1-y)^2(3y - x - 2xy) & \text{for } x \leq y, \\ \frac{1}{6}y^2(1-x)^2(3x - y - 2xy) & \text{for } x > y, \end{cases} \tag{10}$$

and

$$g_5(x,y) = \begin{cases} \frac{1}{6}x^2(3y - x) & \text{for } x \leq y, \\ \frac{1}{6}y^2(3x - y) & \text{for } x > y. \end{cases} \tag{11}$$

See Fig. 14. Indeed, $3y - x - 2xy = (y - x) + 2y(1 - x) > 0$ for $0 < x \leq y < 1$. □

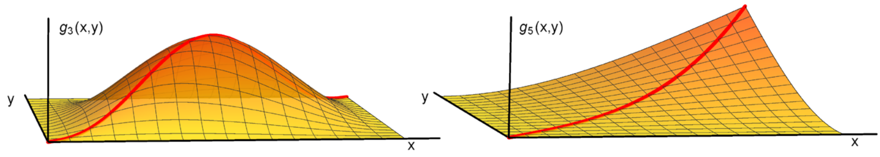


Fig. 14 Graphs of g_3 and g_5

Note that, except for the positivity of $\mathcal{G}$, i.e. $\lambda = 0$, the proof above does not use a maximum principle and purely depends on Kreĭn–Rutman and some operator calculus.

4 Positivity Preserving and the Order of the Differential Equation

As mentioned in the introduction second order elliptic equations have a maximum principle. One may use it to get the positivity preserving property for λ large enough. Suppose that

$$L = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x) \tag{12}$$

is elliptic at $x \in \Omega$, meaning for some $C_x > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C_x|\xi|^2. \tag{13}$$

Generically we assume that $a_{ij}, b_i, c : \overline{\Omega} \to \mathbb{R}$ are smooth enough.

Proposition 4.1 (A rudimentary maximum principle) *Let $\Omega \subset \mathbb{R}^n$ be a domain and let L be as in (12) with $a_{ij}, b_i, c \in C^2(\Omega)$. Suppose that $\lambda > -c(x)$ in Ω and that the C^2-function u satisfies $(L+\lambda)u(x) \geq 0$ in Ω. Then u cannot have a negative minimum inside Ω.*

Proof The proof is standard [71], but also short and simple enough to recall. Suppose that u has a negative minimum in $x \in \Omega$. The ellipticity condition in (13) implies that the symmetric matrix $(a_{ij}(x))$ is positive definite, which allows a diagonalization $T^t(a_{ij}(x))T = D$ in the point x with D a diagonal matrix with positive entries d_{ii} on the diagonal. Using this T for a coordinate transformation $v(y) = u(Ty)$ one finds that, at the point x only,

$$\big((L+\lambda)u\big)(x) = \left(-\sum_{i=1}^{n} d_{ii}\left(\frac{\partial}{\partial y_i}\right)^2 + \sum_{i=1}^{n} \tilde{b}_i\frac{\partial}{\partial y_i} + \lambda + c(x)\right)v(y).$$

In a minimal point one finds $d_{ii}(\frac{\partial}{\partial y_i})^2 v(y) \geq 0$ and $\tilde{b}_i\frac{\partial}{\partial y_i}v(y) = 0$. If $u(x) < 0$ then $v(y) < 0$ and the contradiction follows through $(\lambda + c(x))v(y) < 0$. □

There are many sharper and more general versions of the maximum principle. See [71] or [40].

The result can be used as follows. If the boundary conditions are such that also there no negative minimum can occur, for example when $u = 0$ on $\partial\Omega$, then the solution has to be nonnegative. With the strong maximum principle one even finds $u \equiv 0$ or $u > 0$ in Ω. So if the proposition applies to (7) and if the right hand side, that is the force, has a sign, then the solution will inherit this sign.

What changes for higher order differential equations? Coffman and Grover in [17, Theorem 7.1] proved, for a variational formulation, that if the solution operator is positive for all $\lambda \geq 0$, then the equation is of second order. They mention that for smooth coefficients this result was already proven by Calvert in [11, Proposition 1], who on page 293 again refers to several other authors. So it is not clear, who was the first to prove such a result. In any case, for boundary value problems with higher order equations PPP does not hold for λ large. More precisely:

Proposition 4.2 *If L in (7) is a fourth order elliptic differential operator, then the set of $\lambda \in \mathbb{R}$ for which the boundary value problem in (7) is positivity preserving, is a nontrivial bounded interval or the empty set.*

Proof If (7) is positivity preserving for some λ_0, then λ_0 is not an eigenvalue and one may solve (7) for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$. Moreover, for such λ one finds

$$\begin{cases} \lambda_0 u + Lu = f + (\lambda_0 - \lambda)u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

and a positive solution through a series as in (8). Comparing the solution u_0 for λ_0 and the solution u for λ one finds

$$(\lambda_0 + L)(u - u_0) = (\lambda_0 - \lambda)u$$

and not only $u \geq u_0 \geq 0$ but by the unique continuation principle from [70] that $u > 0$. Hence the conditions of the Krein–Rutman version in the appendix are satisfied. The solution operator for such λ is compact, positive and irreducible and the Krein–Rutman Theorem implies that there is $\lambda_1 < \lambda_0$ such that PPP holds for all $\lambda \in (\lambda_1, \lambda_0]$ and PPP does not hold for $\lambda \leq \lambda_1$.

For the fact that the interval is bounded from above we refer to the result in [17, Theorem 7.1]. □

For the biharmonic operator under homogeneous Dirichlet boundary conditions and with λ large one finds a counterexample for PPP in [46].

Remark 4.3 Some evidence that at most second order differential equations are allowed in order to have PPP for all $\lambda > \lambda_0$ comes by discretization. Approximation of an m-th order derivative needs at least $m + 1$ consecutive points, for example

$$u'''(x) = \lim_{h \downarrow 0} \frac{u(x+2h) - 3u(x+h) + 3u(x) - u(x-h)}{h^3},$$

and the discretization coefficients have at least $m+1$ alternating signs. Replacing the differential equation by a corresponding difference equation leads to a matrix equation $(\lambda I - A)\hat{u} = \hat{f}$ with, in the case that the order satisfies $m > 2$, $(\lambda I - A)$ having positive off-diagonal terms. It is known, that the matrices $(\lambda I - A)$ are inverse positive for all $\lambda \geq \tilde{\lambda}$, if and only if $(\tilde{\lambda} I - A)$ is a so-called non-singular M-matrix. All off-diagonal terms of an M-matrix are nonpositive. See [6]. So the discretization of a boundary value problem with a pde of order $m > 2$ will not be inverse-positive for all large λ.

5 Two and More Dimensions

5.1 Second Order: Membrane with Feedback

The linearized differential equation for a membrane or soap film is $-\Delta u = f$. Fixing u at the boundary gives a Dirichlet condition such as $u = 0$. A soap film might be allowed to slide along a glass wall for some part of the boundary as well. This would imply a Neumann boundary condition $\frac{\partial}{\partial \nu} u = 0$. A sketch of such a problem is found in Fig. 15.

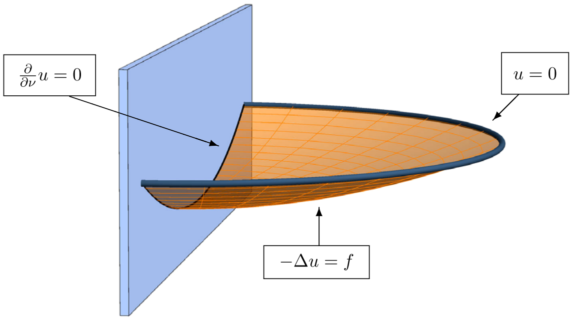


Fig. 15 A soap film spanned on a glass plane and half a ring under a downwards directed force

The linearized problem is as follows:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega \setminus \Gamma, \end{cases} \tag{15}$$

where $\Gamma \subset \partial\Omega$. As long as Γ has a positive $(n-1)$-dimensional measure, the problem in (15) is uniquely solvable in $W^{1,2}(\Omega)$ and the corresponding first eigenvalue λ_1 is strictly negative. That is, there is a nontrivial $\varphi_1 \in W^{1,2}(\Omega)$ satisfying the boundary conditions and $(\lambda_1 - \Delta)\varphi_1 = 0$. Let us assume that $\partial\Omega$ is sufficiently regular, and that the connection between Γ and $\partial\Omega \setminus \Gamma$ is sufficiently nice. Using regularity results and imbedding results between W and C-spaces, one will find that the solution operator for (15) is compact as an operator from $C(\overline{\Omega})$ to $C(\overline{\Omega})$. A general reference for such arguments is [40]. The W-spaces that one meets here are defined for $k \in \mathbb{N}_0$ as follows:

Definition 5.1 Let $p \in (1, \infty)$. The Sobolev spaces $W^{k,p}(\Omega)$ consist of (equivalence classes of) functions:

$$\begin{aligned} W^{k,p}(\Omega) &:= \left\{ u : \Omega \to \mathbb{R}; \|u\|_{W^{k,p}(\Omega)} < \infty \right\}, \\ \|u\|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\alpha| \le k} \int_\Omega \left| D^\alpha u(x) \right|^p dx \right)^{1/p}, \end{aligned} \tag{16}$$

with the multiindex $\alpha \in \mathbb{N}^n$ as in Definition 3.1.

These Banach spaces include the square integrable functions $\mathcal{L}^2(\Omega) = W^{0,2}(\Omega)$.

- One writes $u \in W_0^{k,p}(\Omega)$ if $u \in W^{k,p}(\Omega)$ and all derivatives of u up to order $k-1$ are zero at the boundary (in the sense of traces). One may use the norm of $W^{k,p}(\Omega)$ and even skip all derivatives $|\alpha| < k$.
- One writes $u \in W_{loc}^{k,p}(\Omega)$ if $u_{|A} \in W^{k,p}(A)$ for every open set A with $\overline{A} \subset \Omega$ compact. This space has no decent norm.

Remark 5.2 The derivatives that appear in (16) are defined in a weak sense, that is, $v = \frac{\partial}{\partial x_i} u$ (almost everywhere) if

$$\int_\Omega v \varphi dx = -\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

For more details concerning Sobolev spaces, traces or weak derivatives see [1, 40].

Instead of a homogeneous Dirichlet, which means for a second order elliptic pde that $u = 0$, and Neumann, here $\frac{\partial}{\partial \nu} u = 0$, let us consider more general boundary conditions.

Problem 6 *The second order boundary value problem with homogeneous boundary conditions*:

$$\begin{cases} (-\Delta + \lambda) u = f & \text{in } \Omega, \\ \theta u + (1-\theta) \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

with $\theta : \partial\Omega \to [0, 1]$. *Here* ν *denotes the outward normal* $\partial\Omega$.

As it is, the formulation of Problem 6 is not precise enough to obtain solutions for which a normal derivative is defined at the boundary even in some weak sense. A sufficient condition for a solution in $C^1(\overline{\Omega}) \cap C^2(\Omega)$ would be $\partial\Omega \in C^2$, $\theta \in C^2(\partial\Omega)$ and $f \in C^{0,\gamma}(\overline{\Omega})$ with $\gamma > 0$, but this would rule out the following example depicted in Fig. 15.

Example 5.3 For the problem in Fig. 15 let $D = \{(x_1, x_2);\ x_1 > 0 \text{ and } x_1^2 + x_2^2 < 1\}$ and $\Gamma_1 = \{(0, x_2);\ |x_2| < 1\}$. We consider

$$\begin{cases} -\Delta u = f & \text{in } D, \\ \frac{\partial}{\partial \nu} u = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2 = \partial D \setminus \Gamma_1. \end{cases} \tag{18}$$

Although $\partial\Omega \notin C^1$ and $\theta \notin C(\partial\Omega)$ one may solve this problem in $C^1(\overline{D}) \cap W^{2,p}(D)$ for $f \in C(\overline{D})$. Indeed, one defines $\tilde{f}$ on $\overline{B_1(0)}$ by $\tilde{f}(x_1, x_2) = f(|x_1|, x_2)$, which is continuous if f is, and next one considers the following boundary value problem

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } B_1(0), \\ \tilde{u} = 0 & \text{on } \partial B_1(0). \end{cases} \tag{19}$$

For $\tilde{f} \in C(\overline{B_1(0)})$ problem (19) has a solution in $C^1(\overline{B_1(0)}) \cap W^{2,p}(B_1(0))$ for any $p \in (1, \infty)$, see [40]. Since the solution is unique, it is symmetric with respect to $x_1 = 0$. Hence $\tilde{u}_{x_1}(0, x_2) = 0$ and one finds $u = \tilde{u}_{|D}$ as a solution of (18) in $C^1(\overline{D}) \cap W^{2,p}(D)$.

Instead of finding precise conditions concerning θ and the smoothness of the boundary, we will just assume that the boundary and the function θ are such that there exists a solution operator from $C(\overline{\Omega})$ into $C^1(\overline{\Omega})$. In order that Δu is defined, we will need that $u \in C^2(\Omega)$ or at least $u \in W^{2,p}_{loc}(\Omega)$.

The problem behind this is the following. In one dimension $f \in C(\overline{\Omega})$ implies $u \in C^2(\overline{\Omega})$, but this is not true in higher dimensions. We only find the following interior regularity for second elliptic operators with nice coefficients: If u is a weak solution of (15) and:

- $f \in C^{0,\gamma}(\overline{\Omega})$ for $\gamma \in (0, 1)$, then $u \in C^{2,\gamma}(\Omega)$ (see [40, Theorem 6.2]);
- $f \in \mathcal{L}^p(\Omega)$ for $p \in (1, \infty)$, then $u \in W^{2,p}_{\text{loc}}(\Omega)$ (see [40, Theorem 9.13]).

In general no such result holds for $\gamma \in \{0, 1\}$ or $p \in \{1, \infty\}$.

So the interior regularity allows us to take second derivatives of u in $L^p_{loc}(\Omega)$-sense. Concerning $C^1(\overline{\Omega})$-regularity one finds some results in [40, Sect. 6.7] for smooth boundaries. When the domain has corners as in Example 5.3, see [41]. But, as just mentioned, the $C^1(\overline{\Omega})$-regularity of the solution we assume.

Theorem 5.4 (PPP for the second order Problem 6) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\theta : \partial\Omega \to \mathbb{R}$ be such that for some $\lambda_0 \in \mathbb{R}$ there exists a solution operator $\mathcal{G}_{\lambda_0} \in \mathsf{L}(C(\overline{\Omega}); C^1(\overline{\Omega}))$ of (17) with $\mathcal{G}_{\lambda_0} f \in W^{2,p}_{loc}(\Omega)$ for $f \in C(\overline{\Omega})$.*

Then there is an eigenvalue $\lambda_1 \leq 0$ with a positive eigenfunction φ_1 and moreover the following holds:

- *if $\lambda > \lambda_1$, then for $C(\overline{\Omega}) \ni f \gneq 0$ problem (17) has a solution $u_\lambda \in C^1(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ and $u_\lambda > 0$;*
- *if $\lambda = \lambda_1$, then for $C(\overline{\Omega}) \ni f \gneq 0$ problem (17) has no solution $u_\lambda \in C^1(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$;*

- *if* $\lambda < \lambda_1$, *then for* $C(\overline{\Omega}) \ni f \gneq 0$ *problem* (17) *has no solution* $u_\lambda \in C^1(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ *with* $u_\lambda \geq 0$.

  We recall that $f \gneq 0$ *means* $f \geq 0$ *but not identical* 0.

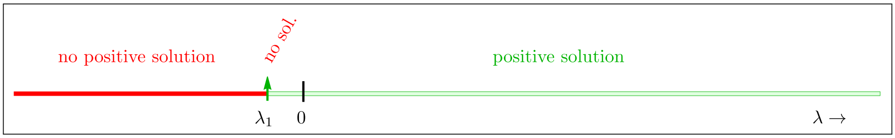


Fig. 16 Suppose that $f \gneq 0$ in Problem 6. What can one say concerning the sign of u?

Proof The result should not be very surprising, but one should notice that the proof has two sides. For $\lambda > 0$ one uses the maximum principle while for $\lambda < 0$ the argument should be called functional analytic.

Step 1. With the natural imbedding $\mathcal{I} : C^1(\overline{\Omega}) \to C(\overline{\Omega})$, which is compact, the operator $\mathcal{G}_{\lambda_0} \circ \mathcal{I}$ is compact. For general λ the solution u_λ of (17) would satisfy

$$\big(I + (\lambda - \lambda_0)\mathcal{G}_{\lambda_0} \circ \mathcal{I}\big)u_\lambda = \mathcal{G}_{\lambda_0} f. \tag{20}$$

Since the operator on the left hand side is Fredholm of index 0, (17) is uniquely solvable for all λ with the exception of countably many eigenvalues.

Step 2. Let $\lambda > 0$ and let u be the solution of (17) for this λ and with $f \geq 0$. Then

$$-\Delta u = f - \lambda u$$

which implies that u cannot have a negative interior minimum. At the boundary part where $\theta = 1$ obviously $u = 0$ holds. At the boundary part where $\theta \in [0, 1)$, one finds if u is negative, that

$$0 > u = -\frac{1-\theta}{\theta}\frac{\partial u}{\partial \nu}$$

and $\frac{\partial u}{\partial \nu} > 0$ implies that also there u cannot have a negative minimum. So $u \geq 0$ in Ω. The strong maximum principle even implies that $u > 0$ in Ω.

Step 3. Take $\tilde{\lambda} > 0$ such that

$$\mathcal{G}_{\tilde{\lambda}} = \big(I + (\tilde{\lambda} - \lambda_0)\mathcal{G}_{\lambda_0} \circ \mathcal{I}\big)^{-1}\mathcal{G}_{\lambda_0}$$

is well-defined through (20). For any $\tilde{\lambda} > 0$ one finds that $\mathcal{I} \circ \mathcal{G}_{\tilde{\lambda}} \in \mathsf{L}(C(\overline{\Omega}))$ is compact, positive and irreducible. Due to Theorem B.5 the spectral radius $\nu(\mathcal{I} \circ \mathcal{G}_{\tilde{\lambda}})$ is strictly positive and supplies us with the principal eigenfunction:

$$(\mathcal{I} \circ \mathcal{G}_{\tilde{\lambda}})\varphi_1 = \nu_1\varphi_1 \quad \text{with } \nu_1 = \nu(\mathcal{I} \circ \mathcal{G}_{\tilde{\lambda}}).$$

One finds that the first eigenvalue for (17) is $\lambda_1 = \lambda_0 - \nu_1^{-1}$ and that

$$\mathcal{G}_\lambda = \mathcal{G}_{\tilde{\lambda}} \sum_{k=0}^{\infty} (\tilde{\lambda} - \lambda)^k (\mathcal{I} \circ \mathcal{G}_{\tilde{\lambda}})^k$$

is convergent for $\lambda \in (\tilde{\lambda} - \nu_1^{-1}, \tilde{\lambda} + \nu_1^{-1})$ and positive for $\lambda \in (\tilde{\lambda} - \nu_1^{-1}, \tilde{\lambda}]$. Since $\tilde{\lambda}$ is an arbitrary positive number one finds that $\mathcal{G}_\lambda$ is positive for all $\lambda > \lambda_1$. Note that $u_\lambda = \mathcal{G}_\lambda f$ solves (17).

Step 4. As in (9) one finds by a duality argument that for $\lambda \leq \lambda_1$ and $f \gneq 0$ no positive solution of (17) exists. □

5.2 Fourth Order: Kirchhoff's thin Plate with Feedback

The most natural way to introduce the boundary value problems for thin plates starts from the energy formulation. Combining the energy due to compression and torsion of the plate together with a perpendicular external force one arrives at the functional

$$J(u) = \iint_\Omega \left(\frac{1}{2} (\Delta u)^2 + (1 - \sigma)\left(u_{xy}^2 - u_{xx} u_{yy}\right) - f u \right) dx dy. \tag{21}$$

Notice that we used (x, y) here instead of (x_1, x_2). The domain $\Omega \subset \mathbb{R}^2$ represents the plate in rest, f is the exterior force and $\sigma \in (-1, 1)$ is the Poisson ratio, which is a physical constant dependent on the material and usually lies around $3/10$.

The hinged case minimizes J over $u \in \mathcal{W}_1 := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and for the clamped case one considers $u \in \mathcal{W}_2 := W_0^{2,2}(\Omega)$. For the hinged problem the boundary condition $u = 0$ on $\partial\Omega$ is not enough for a well-posed boundary value problem and the remaining second boundary condition appears as a natural condition. More precisely, when the solution of the weak formulation of the Euler–Lagrange equations

$$\begin{aligned} \partial J(u; \varphi) &= \iint_\Omega \left(\Delta u \Delta \varphi + (1 - \sigma)(2 u_{xy} \varphi_{xy} - u_{xx} \varphi_{yy} - u_{yy} \varphi_{xx}) - f \varphi\right) dx dy \\ &= 0 \quad \text{for all } \varphi \in \mathcal{W}_i, \end{aligned} \tag{22}$$

has some additional regularity and one may integrate by parts, the boundary term, which appears and does not cancel by the assumptions on φ, gives the second natural boundary condition. Of course one might even refrain from any boundary condition at some part of the boundary and by integrating by parts and would obtain the two natural boundary conditions for the free boundary part. The classical paper by Friedrichs [33] gives the full description on smooth domains. See also [37].

Let us focus on the two classical boundary value problems for the Kirchhoff plate, which are called hinged and clamped, and again introduce a feedback force through an ambient elastic medium with parameter λ. Physically $\lambda \geq 0$ makes sense but the mathematician may take $\lambda \in \mathbb{R}$. The corresponding boundary value problems appear, when we add a term $\frac{1}{2}\lambda u^2$ inside (21), derive the weak Euler–Lagrange equation as for (22), assume that the solution u is four times differentiable, integrate by parts such that all derivatives of φ are moved to u, and apply the main theorem of 'Calculus of Variations' to go from integral equations to pointwise equations.

For the hinged case one has $u, \varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, meaning that $u = 0$ on $\partial\Omega$ is an a priori boundary condition, while

$$-\Delta u = -(1-\sigma)\kappa \tfrac{\partial}{\partial \nu} u \quad \text{on } \partial\Omega$$

comes out of the weak Euler–Lagrange equation as a natural boundary condition. Here κ is the signed curvature of the boundary $\partial\Omega$, positive on convex boundary sections and negative on concave parts and ν the outside normal on $\partial\Omega$. The boundary value problem that we obtain, is as follows:

Problem 7 *The fourth order problem with hinged boundary conditions*:

$$\begin{cases} \Delta^2 u + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\Delta u = -(1-\sigma)\kappa \frac{\partial}{\partial \nu} u & \text{on } \partial\Omega. \end{cases} \tag{23}$$

For the clamped case one has $u, \varphi \in W_0^{2,2}(\Omega)$, meaning that $u = 0$ and $\frac{\partial}{\partial \nu} u = 0$ on $\partial\Omega$ are a priori boundary giving boundary conditions. Since φ satisfies similar boundary conditions in fact all boundary terms disappear for the integration by parts. We find:

Problem 8 *The fourth order problem with clamped boundary conditions*:

$$\begin{cases} \Delta^2 u + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{24}$$

Clamped boundary conditions for the fourth order problem are also called Dirichlet boundary conditions.

► **The Iterated Dirichlet–Laplacian** appears when the boundary is a straight line. Indeed, the curvature satisfies $\kappa = 0$ and the second boundary condition in Problem 7 becomes $\Delta u = 0$. Being naive and forgetting that bounded polygonal domains necessarily have corners one may consider the following problem, which is mathematically interesting even for curvilinear domains. See the next section for a less naive approach.

Problem 9 *The fourth order problem with zero Navier boundary conditions*:

$$\begin{cases} \Delta^2 u + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

The boundary conditions in Problem 9 allow for a kind of iteration, that is, by setting $v = (-\Delta - \sqrt{-\lambda})u$ the boundary conditions split nicely and one obtains:

$$\begin{cases} (-\Delta + \sqrt{-\lambda})v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta - \sqrt{-\lambda})u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

To conclude anything concerning sign preservation at first glance it seems necessary that $\lambda \leq 0$. Then $f \geq 0$ implies $v \geq 0$ since $\sqrt{-\lambda} \geq 0$. Next $v \geq 0$ implies $u \geq 0$ whenever $\sqrt{-\lambda} < \nu_1^{-1}$ with ν_1 the spectral radius of the Green operator $\mathcal{G}_\Omega$ for the Dirichlet Laplacian on Ω.

A somewhat different approach is found in [79]. Using the heat kernel

$$p_\Omega : \mathbb{R}^+ \times \overline{\Omega} \times \overline{\Omega} \to [0, \infty),$$

which is such that $U(t,x) = \int_\Omega p_\Omega(t,x,y)U_0(y)dy$ for U_0 nice enough solves

$$\begin{cases} \partial_t U - \Delta U = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \lim_{t \downarrow 0} U(t,x) = U_0(x) & \text{on } \partial\Omega, \end{cases}$$

one may write the solution u of Problem 9 as follows:

$$u(x) = \int_\Omega \left(\int_0^\infty h(t,\lambda) p_\Omega(t,x,y) dt \right) f(y) dy. \tag{26}$$

The function h is defined by:

$$h(t,\lambda) = \begin{cases} \sinh(\sqrt{-\lambda}t)/\sqrt{-\lambda} & \text{for } \lambda \in (-\nu_1^{-2}, 0), \\ t & \text{for } \lambda = 0, \\ \sin(\sqrt{\lambda}t)/\sqrt{\lambda} & \text{for } \lambda > 0. \end{cases}$$

Here the bound from below for λ appears since for $t \to \infty$

$$p_\Omega(t,x,y) \sim e^{-\nu_1^{-1}t} \varphi_1(x)\varphi_1(y) \quad \text{and} \quad \sinh(\sqrt{-\lambda}t) \sim e^{\sqrt{-\lambda}t}.$$

Its product leads to a convergent integral in (26) if and only if $-\nu_1^{-1} + \sqrt{-\lambda} < 0$, that is, $\lambda > -\nu_1^{-2}$.

So, since h and p_Ω are positive, the kernel in (26) is positive for $\lambda \in (-\nu_1^{-2}, 0]$. One may guess that also for $\lambda > 0$ but small the kernel remains positive by continuity arguments. This indeed holds true. A first hands-on proof for a ball can be found in [76]. A precise argument depends strongly on Ω and uses so called $3G$-estimates. Indeed, by using the results of [18], one can show that $\lambda_c > 0$ for bounded Lipschitz domains. We refer to [46] or [79] for further details.

Theorem 5.5 (PPP for Problem 9) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^1$. Let ν_1 be the spectral radius of $\mathcal{G}_\Omega = (-\Delta)_0^{-1}$. Then $\lambda_1 = -\nu_1^2$ is the principal eigenvalue for Problem 9 and there exists $\lambda_c \in \mathbb{R}^+$, such that for (25):*

1. *if* $\lambda \leq \lambda_1$, *then for* $f \gneq 0$ *there exists no positive solution* u;
2. *if* $\lambda \in (\lambda_1, \lambda_c)$, *the* $f \gneq 0$ *implies* $u > 0$.

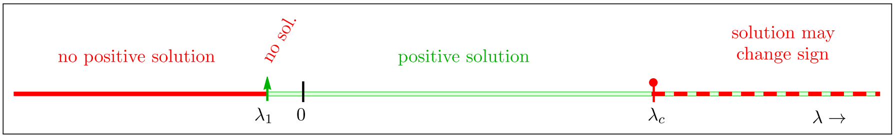


Fig. 17 Suppose that $f \gneq 0$ in Problem 9. What can one say concerning the sign of u?

Remark 5.6 One might expect that it should be sufficient that the boundary is regular in the sense of Perron, see [40], instead of assuming C^1-regularity. There are two obstructions for such a generalization. First of all, in order to apply the 3G-Theorem and hence to satisfy the conditions for that theorem one needs a Lipschitz boundary, see [18]. Secondly, if the domain has nonconvex corners, then one has to specify which type of solutions is considered. We will shortly discuss this in the next section.

► **Corners, Paradoxes and Unexpected Problems** Without the assumption that $\partial\Omega \in C^{4,\gamma}$ one cannot expect that Problem 7, 8 or 9 will have a solution $u \in C^4(\overline{\Omega})$. For well-defined pointwise boundary conditions $\partial\Omega \in C^2$ seems to be necessary. Such a restriction would however leave out natural domains such as polygons, although [52] allows convex corners in some cases.

The two main questions that appear are as follows:

- Can we deal with corners directly?
- Can we approximate by smooth domains?

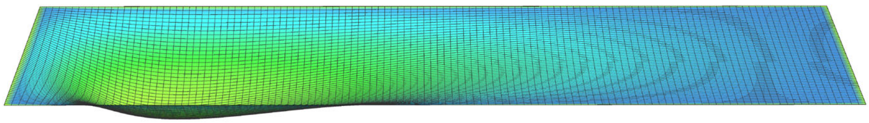


Fig. 18 On straight lines the boundary conditions of hinged and of Navier type are identical. For the polygonal plate under Navier conditions, i.e. Problem 9 one obtains, when all corners are convex as we will see later, an iterated Dirichlet–Laplace problem, which guarantees the sign-preserving property. Indeed pushing downwards will move the plate downwards everywhere in the interior. Here $\Omega = (0, 2) \times (0, 1)$ and f is a point force in $(\frac{1}{4}, \frac{1}{2})$, that is, located on the longer central axis left from the minimum of the solution

▷ **Sapondzhyan** [72] noticed that for the hinged problem as a system of second order problems a singularity appears at reentrant corners in the second order derivatives which is not present in the solution of the original problem. Let us first discuss the change from fourth order to second order system by setting $v = -\Delta u$ for (9). We take $\lambda = 0$.

Problem 10 *Comparison of the biharmonic under Navier conditions and the iterated Laplace under Dirichlet conditions*:

$$I:\begin{cases}\Delta^2 u=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ \Delta u=0 & \text{on } \partial\Omega,\end{cases}\quad \text{and}\quad II:\begin{cases}-\Delta v=f & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega,\\ -\Delta u=v & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}\tag{27}$$

Looking pointwise at both boundary value problems there is no difference. However, for domains with corners there is in general no classical solution and a pointwise setting is in general not appropriate. For a weak setting however much more general domains can be considered, that is, for such domains a weak solution may still exist. The weak setting is usually determined as the first variation of a variational problem. The variational setting for the two boundary value problems in Problem 10 is different:

I. For the left hand side one finds a weak solution by minimizing

$$J(u)=\iint_\Omega \left(\tfrac{1}{2}(\Delta u)^2-fu\right)dxdy \tag{28}$$

for functions $u\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. The condition $\Delta u=0$ appears as a natural boundary condition on smooth boundary parts. For bounded domains, that have a smooth boundary with the exception of finitely many corners, such a minimizer exists. See [64]. We will write u_{I} for this minimizer of the functional J.

II. For the right hand side one finds a weak system solution by first minimizing

$$J_1(u)=\iint_\Omega \left(\tfrac{1}{2}|\nabla v|^2-fv\right)dxdy \tag{29}$$

for functions $v\in W_0^{1,2}(\Omega)$ and use the minimizer v_0 of (29) to find a minimizer u_{II} of

$$J_2(u)=\iint_\Omega \left(\tfrac{1}{2}|\nabla u|^2-v_0\,u\right)dxdy \tag{30}$$

again for functions $u\in W_0^{1,2}(\Omega)$. Both minimizers exist for bounded Ω without further restrictions on the boundary.

Remark 5.7 For a numerical approximation both settings can be directly used for an approximation through finite elements. The weak formulation of the Euler–Lagrange equation for (28), that is

$$\iint_\Omega (\Delta u\Delta\varphi-f\varphi)dxdy=0\quad \text{for all } \varphi\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega),$$

or also (22), lends itself indeed for a direct numerical approximation by finite elements. Since these formulations involve second order derivatives, those finite elements need to be $C^{1,1}(\overline{\Omega})$, which not only increases the complexity of the approximation but also forces one to deal carefully with a curvilinear boundary sections and corners. Or like Ciarlet and Glowinski in [12] call it: *variational crimes are easily made with the biharmonic*. From a practical point of view one should mention that the toolboxes and libraries for the use of $C^{1,1}$-finite elements are less well developed.

In contrast, the system approach on the right hand side can be dealt with by standard piecewise affine finite elements. Moreover, for the Dirichlet–Laplace problem the approximation of curvilinear boundaries through polygons hardly needs any special care. So, from a numerical point of view the system setting in II has its advantages.

So the big question is whether or not these two different settings, that is,

$$\text{I: } u \in W^{2,2}(\Omega) \quad \text{and} \quad \text{II: } u, \Delta u \in W^{1,2}(\Omega),$$

will always give the same solution. In [64] these two settings have been compared.

As one may guess from the lenghty introduction, the answer whether both settings give the same solution, is no, at least generically when the domain has reentrant corners. Indeed, for every reentrant corner there exists a special biharmonic function u_s, which satisfies $u_s = \Delta u_s = 0$ a.e. on the boundary. For a domain with one such corner one finds that

$$u_{\mathrm{I}}(x) = u_{\mathrm{II}}(x) + c_f u_s(x)$$

and $c_f \in \mathbb{R}$ nonzero for most f. By the way, reentrant corner means here that the angle of both tangential directions at the corner point, measured from inside the domain, is larger than π.

Theorem 5.8 (*See* [64]) *Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. Then for each $f \in \mathcal{L}^2(\Omega)$ there exists*:

- *a unique minimizer $u_{\mathrm{I}} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ of* (28), *which is a weak solution of Problem* 10-*I, and*
- *a unique pair $(u_{\mathrm{II}}, v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, with v minimizing* (29) *and u_{II} consecutively minimizing* (30), *which are weak solutions of Problem* 10-*II.*

Moreover,

1. *if Ω has k reentrant corners, then there exist k independent functions $\zeta_i \in \mathcal{L}^2(\Omega)$ such that*

$$u_{\mathrm{I}} = u_{\mathrm{II}} \iff \int_\Omega \zeta_i \, f \, dx = 0 \quad \text{for all } i \in \{1, \ldots, k\}.$$

2. *If $u_{\mathrm{I}} \neq u_{\mathrm{II}}$, then $\Delta u_{\mathrm{I}} \notin W^{1,2}(\Omega)$ and $u_{\mathrm{II}} \notin W^{2,2}(\Omega)$.*
3. *If Ω has precisely one reentrant corner, then $f \gneqq 0$ implies $u_{\mathrm{I}} \neq u_{\mathrm{II}}$.*
4. *If Ω has a reentrant corner with angle $\alpha > \frac{3}{2}\pi$, then there are $f \gneqq 0$ for which u_{I} is sign changing. Recall that $f \gneqq 0$ implies $u_{\mathrm{II}} > 0$.*

Remark 5.9 For $\alpha > \frac{3}{2}\pi$ for some corner, then one finds that u_{I} generically displays a sign change at that corner. Numerical evidence from [64] shows that for $\alpha \in (\pi, \frac{3}{2}\pi)$ the solution u_{I} may have a nodal line, which doesn't start in that corner. In fact, for α just slightly larger than π, indeed for $\alpha \leq 1.2\pi$ in [64, Fig. 6], numerics failed to find a sign changing solution. Up to now there is no analytical explanation. See Fig. 19.

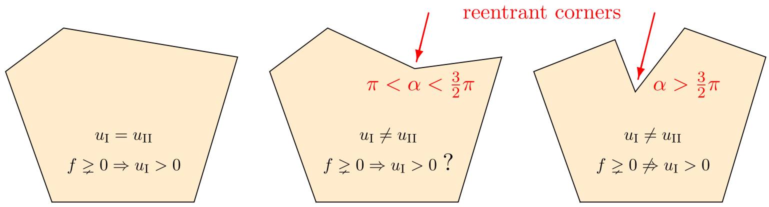


Fig. 19 For Problem 10-I the sign preservation for $u_{\rm I}$ is not clear for reentrant corners between π and $\frac{3}{2}\pi$. The middle one is such a domain

The same results will hold on domains with smooth curvilinear boundary sections connected through corners, but there the technicalities will blur the arguments.

One might wonder that $u_{\rm I} \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ seems to satisfy one boundary condition only, namely $u = 0$ while $u_{\rm II}$ through $u_{\rm II}, \Delta u_{\rm II} \in W_0^{1,2}(\Omega)$ satisfies two. In fact $u_{\rm I}$ satisfies a natural boundary condition inside the straight boundary parts, namely $\Delta u_{\rm I} = 0$. Indeed, away from the corners $u_{\rm I}$ has some extra smoothness and by integrating by parts the weak Euler–Lagrange equation, meaning the one in integral form, and using appropriate testfunctions, this second boundary condition shows up.

One may also ask, which of these two functions $u_{\rm I}$ and $u_{\rm II}$ is the physically appropriate solution. The energy stored in the plate is related with the integral of the second derivatives squared, as this quantity is locally unbounded for $u_{\rm II}$ near corners. That would probably mean that the material would crack. For $u_{\rm I}$ the similar energy integral does not contain such singular terms, but somehow a reentrant corner without 'stress' is not natural either. On the other hand, approximating the domain by a sequence of smooth domains and then using the solutions of (23) does neither approximate $u_{\rm I}$ nor $u_{\rm II}$. See [60, 61, 78] and the next paragraphs.

This phenomenon of two different solutions, unique in the corresponding settings, shows that pointwise defined boundary conditions is not precise enough in the presence of corners. As we just saw, $u_{\rm I}$ does not satisfy the boundary conditions at each point of a nonconvex polygon; $u_{\rm II}$ might not be the right solution. Indeed, a solution satisfying the boundary conditions pointwise on $\partial\Omega$ will not exist in general. Allowing some freedom at individual boundary points may introduce singular solutions. In other words, it will be hard to fulfill the conditions of Hadamard for a well-posed problem in a C-setting. Both settings that consider weak solutions in Sobolev spaces have a unique solution, but non-mathematical arguments are needed to identify the physically relevant solution.

The example that we show next is not a polygonal domain but the same results as in Theorem 5.8 apply and it allows an explicit formula for u_s:

Example 5.10 For the domain $\Omega = \{(r\cos\varphi, r\sin\varphi); 0 < r < 1 \text{ and } 0 < \varphi < \frac{3}{2}\pi\}$ the function u_s is as follows:

$$u_s(x) = \left(\tfrac{3}{5}r^{2/3} + \tfrac{3}{20}r^{8/3} - \tfrac{3}{4}r^{4/3}\right)\sin\left(\tfrac{2}{3}\varphi\right),$$

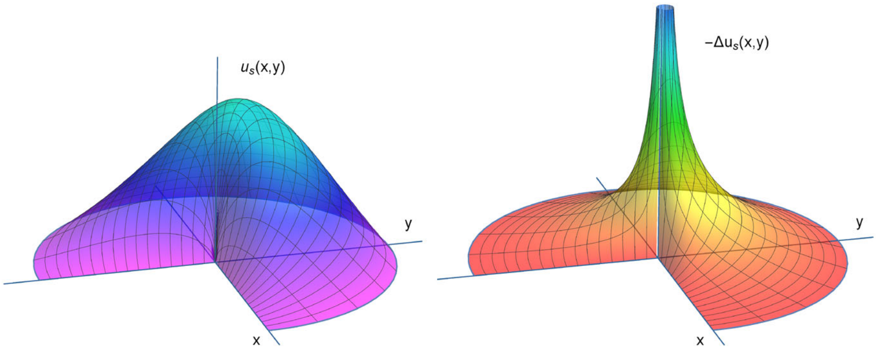


Fig. 20 Sketch of the special biharmonic function u_s and of $-\Delta u_s$ for Ω a three quarter disk

where $x = (r\cos\varphi, r\sin\varphi)$. One finds

$$-\Delta u_s = \left(r^{-2/3} - r^{2/3}\right)\sin\left(\tfrac{2}{3}\varphi\right)$$

and that $u_s \in C^\infty(\overline{\Omega} \setminus \{0\})$ is a nontrivial solution of

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{31}$$

Necessary and sufficient for $r^\alpha \sin(\frac{2}{3}\varphi) \in \mathcal{L}^2(\Omega)$ is $\alpha > -1$. One may check directly that

$$u_s \in W^{1,2}(\Omega) \setminus W^{2,2}(\Omega) \quad \text{and} \quad -\Delta u_s \in \mathcal{L}^2(\Omega) \setminus W^{1,2}(\Omega).$$

A sketch of u_s and of $-\Delta u_s$ can be found in Fig. 20.

$\triangleright$ **Babuška** Ref. [3] remarked that the explicitly known solution of (27) with $f = 1$ on a disk is not equal to the approximation by u_{I}-solutions on regular n-gons for $n \to \infty$. This so-called Babuška-paradox is due to a somewhat other effect as the one by Saponzhyan. The approximation by regular n-gons of the circle is not stable with respect to the second boundary condition in Problem 7 or Problem 10-I. At a convex corner the solution of the Dirichlet–Laplace problem (30) is differentiable and hence $u = 0$ on $\partial\Omega$ implies $\nabla u = 0$ at that corner. As a consequence the limit problem is not hinged but something between hinged and clamped.

For more information we refer to [4, 23, 60, 61, 78] and further references in there.

$\triangleright$ **An Engineering Fault** for thin plates with reentrant corners resulted in two crashes of the first type of passenger jet airliner, the de Havilland DH.106 Comet 1. It had a pressurized cabin and square windows. On January 10, 1954 flight BOAC 781 took off from Rome and exploded above the Mediterranean Sea near Elba. On April 8 of the same year flight SAA 201, again leaving from Rome, crashed soon after take-off into the Mediterranean near Stromboli. Both airplanes were a DH.106 Comet 1.

Wreckage recovered gave evidence that the accidents could be due to a failing of the fuselage due to cracks from metal fatigue. To test this presumption they put the hull of a Comet airplane in a water tank and submitted it to a sequence of pressurizing and depressurizing until it cracked. In the official report of the accidents investigation the result is described as follows:

> *…the cabin structure failed, the starting point of the failure being the corner of one of the cabin windows…*

See also Figs. 7 and 8 on page 32 and 33 and Fig. 12 on page 37 of that report. The report can be found on

http://lessonslearned.faa.gov/Comet1/G-ALYP_Report.pdf

Consecutive airliners with pressure cabins had round windows.

▶ **The Hinged Problem** described in Problem 7 we will here give a closer look:

$$\begin{cases} \Delta^2 u + \lambda u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\Delta u = -(1-\sigma)\kappa \frac{\partial}{\partial \nu} u & \text{on } \partial\Omega. \end{cases} \tag{32}$$

If we assume $\lambda \leq 0$ we may split by setting $v = (-\Delta - \sqrt{-\lambda})u$ similar as in Problem 9, and obtain

$$\begin{cases} (-\Delta + \sqrt{-\lambda})v = f & \text{in } \Omega, \\ v = -(1-\sigma)\kappa \frac{\partial}{\partial \nu} u & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta - \sqrt{-\lambda})u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{33}$$

Obviously this system is not decoupled. However, if the domain is convex, then $\kappa \geq 0$ holds and the system has a cooperative behavior, meaning

$$v \geq 0 \text{ in } \Omega \quad \Rightarrow \quad u \geq 0 \text{ in } \Omega \quad \Rightarrow \quad -(1-\sigma)\kappa \tfrac{\partial}{\partial \nu} u \geq 0 \text{ on } \partial\Omega,$$

and

$$\left.\begin{array}{ll} f \geq 0 & \text{in } \Omega \\ -(1-\sigma)\kappa \frac{\partial}{\partial \nu} u \geq 0 & \text{on } \partial\Omega \end{array}\right\} \quad \Rightarrow \quad v \geq 0 \text{ in } \Omega.$$

If the corresponding principal eigenvalue lies on the appropriate side of 0, then one might prove a similar result as before. In [38] it was proven that there exist $\delta_c \in \mathbb{R}^-$ and $\delta_1 \in \mathbb{R}^+$ such that for $\alpha \in C(\partial\Omega)$ with $\delta_c \lneqq \alpha \lneqq \delta_1$ the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ -\Delta u = -\alpha \frac{\partial}{\partial \nu} u & \text{on } \partial\Omega, \end{cases} \tag{34}$$

is positivity preserving: $f \geq 0 \Rightarrow u \geq 0$. Parini and Stylianou in [67] were able to show for convex $C^{2,1}$-domains and $\sigma \in (1,1)$ that $(1-\sigma)\kappa < \delta_1$ holds. This implies that (32) for $\lambda = 0$ is positivity preserving. By using the Krein–Rutman Theorem it follows that the eigenvalue problem corresponding to Problem 7, i.e.

$$\begin{cases} \Delta^2\varphi + \lambda\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \\ -\Delta\varphi = -(1-\sigma)\kappa \frac{\partial}{\partial\nu}\varphi & \text{on } \partial\Omega, \end{cases} \tag{35}$$

has a first eigenvalue $\lambda_{1,\sigma} < 0$. With the argument for (7) one finds that (32) is positivity preserving for all $\lambda \in (\lambda_{1,\sigma}, 0]$. So by combining the result from Parini and Stylianou in [67] with the argument above one finds:

Theorem 5.11 *Consider Problem 7, i.e. (32), with Ω convex and $\partial\Omega \in C^{2,1}$. Suppose that $\sigma \in (-1,1)$. Then there exist $\lambda_{1,\sigma} < 0$ and $\lambda_{c,\sigma} > 0$ such that the following holds*:

1. *for all $\lambda < \lambda_{1,\sigma}$ and $0 \lneq f \in C(\overline{\Omega})$ there exists no nonnegative solution u*;
2. *for all $\lambda \in (\lambda_{1,\sigma}, \lambda_{c,\sigma})$ and $0 \lneq f \in C(\overline{\Omega})$ the solution satisfies $u > 0$ in Ω.*

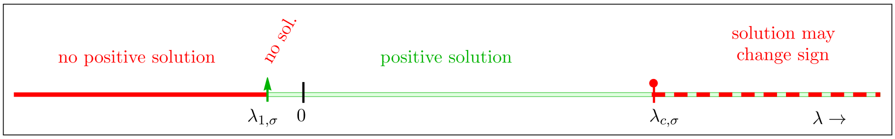


Fig. 21 Suppose that $f \gneq 0$ in Problem 7, which is (32), and that Ω is a convex domains. What can one say for the sign of u?

Proof For $\lambda \leq 0$ the result follows from the arguments above. For $\lambda \geq 0$ we first fix some notations. Let us write $v = \mathcal{G}f + \mathcal{K}_\sigma w$ for the solution of

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = (1-\sigma)\kappa w & \text{on } \partial\Omega, \end{cases} \tag{36}$$

with $\mathcal{G}$ and $\mathcal{K}_\sigma$ the related positive solution operators:

$$\mathcal{G} : C(\overline{\Omega}) \to C_0(\overline{\Omega}) \cap W^{2,p}(\overline{\Omega}),$$
$$\mathcal{K}_\sigma : C(\partial\Omega) \to C(\overline{\Omega}).$$

For p large enough $C_0(\overline{\Omega}) \cap W^{2,p}(\overline{\Omega}) \subset C_0(\overline{\Omega}) \cap C^1(\overline{\Omega})$ and

$$\mathcal{D} : C_0(\overline{\Omega}) \cap C^1(\overline{\Omega}) \to C(\partial\Omega)$$

is well defined by $\mathcal{D}v = -(\nu \cdot \nabla v)_{|\partial\Omega}$, or more shortly $\mathcal{D}v = -\partial_\nu v$. In fact, by Hopf's boundary point Lemma, see [71], $\mathcal{D}\mathcal{G}$ is positive. With these operators and for $\lambda = 0$ the problem in (32) can be written as

$$u = \mathcal{G}^2 f + \mathcal{G}\mathcal{K}_\sigma \mathcal{D}u.$$

Similarly as (8) problem (32) with $\lambda = 0$ is then solved by

$$u = \mathcal{A}_\sigma f, \quad \text{where } \mathcal{A}_\sigma = \mathcal{G}\sum_{k=0}^{\infty}(\mathcal{K}_\sigma \mathcal{D}\mathcal{G})^k \mathcal{G}, \tag{37}$$

whenever the series converges. For convex domains indeed Parini and Stylianou showed convergence of (37). The operator $\mathcal{A}_\sigma$ is positive, since $\mathcal{K}_\sigma\mathcal{D}\mathcal{G}$ and $\mathcal{G}$ are.

For $|\lambda|$ small we use a perturbation argument as before to find

$$u_\lambda = \sum_{k=0}^{\infty}(-\lambda\mathcal{A}_\sigma)^k \mathcal{A}_\sigma f = \sum_{k=0}^{\infty}(\lambda\mathcal{A}_\sigma)^{2k}(I - \lambda\mathcal{A}_\sigma)\mathcal{A}_\sigma f.$$

This series converges for $|\lambda| < \nu_{1,\mathcal{A}_\sigma}^{-1}$, the spectral radius of $\mathcal{A}_\sigma$, which supplies with a first eigenvalue $\lambda_{1,\sigma} < 0$ for (35) With the $3G$-Theorem, see [18] or [79], which shows that $\mathcal{G} - \varepsilon\mathcal{G}^2$ is a positive operator for $\varepsilon > 0$ and small, one may show that also $\mathcal{A}_\sigma - \lambda\mathcal{A}_\sigma^2$ is a positive operator for $\lambda > 0$ and small. For comparing terms with $\mathcal{K}_\sigma\mathcal{D}\mathcal{G}$ see [38]. □

For the unit disk one may almost explicitly compute the relation $\sigma \mapsto \lambda_{1,\sigma}$.

Example 5.12 For the unit disk $B \subset \mathbb{R}^2$ the second boundary condition reads as

$$0 = -\Delta\varphi + (1-\sigma)\kappa\frac{\partial}{\partial\nu}\varphi = -\frac{\partial^2}{\partial\nu^2}\varphi - \sigma\frac{\partial}{\partial\nu}\varphi \quad \text{on } \partial B.$$

Of the four independent radially symmetric solutions of the differential equation in (35) only two do not have a singularity in 0, namely $x \mapsto J_0(\sqrt[4]{-\lambda}|x|)$ and $x \mapsto I_0(\sqrt[4]{-\lambda}|x|)$. Here J_0 and I_0 are the standard and modified, Bessel functions of the first kind, respectively. The principal eigenfunction is

$$\varphi_{1,\sigma}(x) = J_0\big(\rho_\sigma|x|\big)I_0(\rho_\sigma) - J_0(\rho_\sigma)I_0\big(\rho_\sigma|x|\big),$$

with the corresponding first eigenvalue $\lambda_{1,\sigma} = -\rho_\sigma^4$, and where ρ_σ is the first positive zero of

$$r \mapsto r^2\big(J_0''(r)I_0(r) - J_0(r)I_0''(r)\big) + \sigma r\big(J_0'(r)I_0(r) - J_0(r)I_0'(r)\big). \tag{38}$$

In Fig. 22 one finds the relation between σ and $-\lambda_1$. Note that $\sigma = -1$ is indeed critical and one finds $\lambda_{1,\sigma=-1} = 0$ and $\varphi_{1,\sigma=-1}(x) = 1 - |x|^2$.

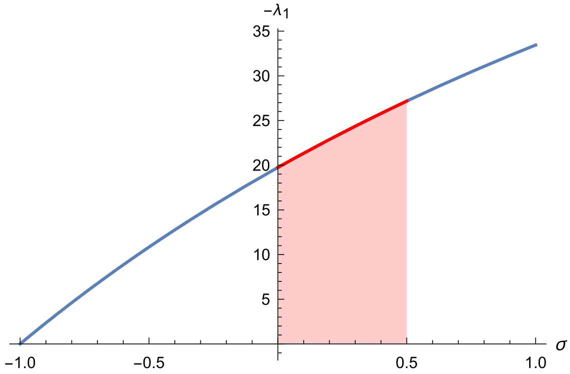


Fig. 22 The relation between σ and λ_1 in Example 5.12. For most materials $\sigma \in [0, .5]$ and the corresponding range of $-\lambda \in [0, -\lambda_1)$ is darkened

► **The Real Supported Plate** hints at a construction such as a flat roof lying on supporting walls at its sides. Nailing that roof to the walls the clamped case seems to be most appropriate. Without nails one expects the weight of the roof keeps the roof on its supporting walls. Indeed, where the roof touches the walls the hinged boundary conditions are appropriate. Although at a larger section the roof will indeed touch its supporting structure, a more careful observation shows that this is not true everywhere. Civil engineers are trained to '*nail down a flat roof at corners*' in order to prevent it from going up. See [8].

An appropriate mathematical formulation for a supported plate is as follows.

Problem 11 *Find the minimum of*

$$J(u) = \iint_\Omega \left(\frac{1}{2} (\Delta u)^2 + (1-\sigma)\left(u_{xy}^2 - u_{xx}u_{yy}\right) - fu \right) dxdy \qquad (39)$$

for $u \in \mathcal{W}^+ = \{u \in W^{2,2}(\Omega); \min(u, 0) \in W_0^{1,2}(\Omega)\}$.

The condition $u \in \mathcal{W}^+$ implies the unilateral condition $u \geq 0$ on the boundary. In [65, Theorem 2.4 and Proposition 2.6] one finds the following results for the case $\lambda = 0$. Before stating the result, let us recall that the affine functions $\xi : \mathbb{R}^2 \to \mathbb{R}$ are $\zeta(x, y) = ax + by + c$ with $a, b, c \in \mathbb{R}$.

Theorem 5.13 (*See* [65]) *Let* $\Omega \subset \mathbb{R}^2$ *have a Lipschitz boundary and suppose that* $|\sigma| < 1$. *Assume that* $f \in \mathcal{L}^2(\Omega)$ *is such that*

$$\iint_\Omega f\zeta dxdy < 0 \quad \text{for all affine functions } \zeta \text{ with } \zeta > 0 \text{ in } \Omega.$$

Then Problem 11 *has a unique solution* $u_\sigma \in \mathcal{W}^+$.

Remark 5.14 In his lectures on mechanics, Blaauwendraad [8] presented a discrete model for a supported plate, and the corners of the discrete model move upwards under a downwards force. Cesaro Davini [24] was the first to give mathematical evidence that solutions for continuous Problem 11 with a downward directed force will move upwards near corners. The analysis of corner behavior, which originates from Kondratiev [54] and Wigley [82], was needed to arrive at the statement above in [65]. A simpler argument but only for rectangular corners was given in [75].

The setting in Problem 11, and the assumption that the solution is sufficiently regular, leads to a boundary value problem with two different boundary conditions on the two corresponding subsets of $\partial\Omega$:

$$\begin{cases} \lambda u + \Delta^2 u = f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \\ \sigma \Delta u + (1-\sigma)\partial_n^2 u = 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \Gamma \subset \partial\Omega, \\ \partial_n(\Delta u) + (1-\sigma)\partial_\tau(\nu \cdot (D^2 u)\tau) = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases} \tag{40}$$

Note that (40) in itself is not well-posed, since it does not tell us where to switch from $u = 0$ to the other boundary condition. In other words, a solution is a pair of a priori unknown (u, Γ). If one assumes that the solution is $C^3(\overline{\Omega})$, then the 'free' boundary points $x \in \partial\Omega$ (should we call them $\partial\Gamma$?) are fixed through

$$u(x) = \left(\sigma\kappa\partial_n + \partial_n^2\right)u(x) = \left(\partial_n(\Delta u) + (1-\sigma)\partial_\tau\left(\nu \cdot \left(D^2 u\right)\tau\right)\right)(x) = 0. \tag{41}$$

This formulation gives no obvious clue that (40) with (41) has a solution or if such a solution is unique.

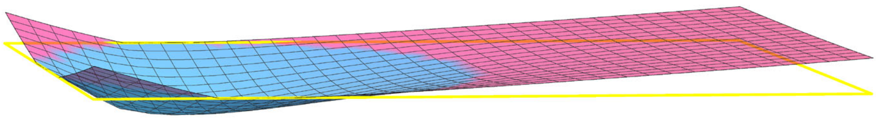


Fig. 23 A supported plate with a downwards force generically moves upwards at corners. One might say *'the real supported plate is really not so supported at all'*. Here the plate, with $\lambda = 0$ and supported by a rectangle, is pushed downwards by a δ-force, located on the longer central axes, as before $\Omega = (0,2) \times (0,1)$ and f is located in $(\frac{1}{8}, \frac{1}{2})$. In *blue* (*light*) the plate below 0; in *red* the part above. The numerical approximation uses an iterative process with finite differences starting from the hinged plate and deciding in each step where the boundary is hinged or free

In the literature one often finds the term 'supported plate' or 'simply supported plate', when one actually means Problem 7, which we call hinged plate. If this mix-up is due to the fact that one expected the plate under a downwards force to touch its supporting boundary frame everywhere, or because of a bad translation between Russian and English, is not clear. But what is in a name; in engineering they are aware that one has to nail down a flat roof at its corners (see [8]) and if you don't, your roof may look like in Fig. 23.

► **The Clamped Plate** that we described in Problem 8, more precisely the question whether or not it is positivity preserving, received a lot of attention around 1900. We recall the problem:

$$\begin{cases} \Delta^2 u + \lambda u = f & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \tag{42}$$

The Boggio–Hadamard conjecture says, that for $\lambda = 0$ PPP holds for (42) on convex domains $\Omega \subset \mathbb{R}^2$.

In 1905 Boggio [10] found explicit Green functions for balls in any dimension and even for every order polyharmonic Dirichlet problems. With Γ_A^B denoting the biharmonic Green function of (42) for $\lambda = 0$, Hadamard writes in conference proceedings from 1908 [50] that:

M. Boggio, qui a, le premier, noté la signification physique de Γ_A^B, en a déduit l'hypothése que Γ_A^B était toujours positif. Malgré l'absence de démonstration rigoureuse, l'exactitude de cette proposition ne paraît pas douteuse pour les aires convexes.

Hadamard also announced in [50] that he had an explicit Green function Γ_B^A for (42) for $\lambda = 0$, when Ω is the interior of a Limaçon de Pascal. After sketching some arguments, he boldly states that:

Ainsi Γ_B^A est toujours positif dans le cas du limaçon de Pascal.

See Fig. 4 for limaçons. He also mentions in these proceedings that the problem on an annulus has been studied by Almansi and Boggio, who found that for some annulus the sign is not preserved. Moreover, it seems that Boggio told him that the conjecture was never meant for such domains. For more details Hadamard refers to a future paper of his, which is most likely [49]. Indeed, the actual Green function for limaçons is computed in the appendix of [49].

It took almost forty years to find out, that the conjecture is not true. Starting in 1946, Duffin ([28]), Garabedian ([35]) and others ([55, 74]) showed that convexity of the domain is not sufficient for the clamped plate equation to be positivity preserving. Duffin in [49] considered an infinitely long rectangle. Garabedian could show sign-change of the Green function for a longer ellipse. Shapiro and Tegmark even constructed an explicit sign-changing function u satisfying Problem 8 on an ellipse such that $f = \Delta^2 u > 0$. Their result can be even be adjusted to higher orders polyharmonic Dirichlet problems [80]. Coming back to a nonconvex domain Engliš and Peetre showed in [30] that the Green function on any annulus is sign-changing. By the way, it took until [21] to notice that the Green function for limaçons by Hadamard was only positive for some limaçons, namely those with $a < 1/\sqrt{6}$. The case $a = 1/\sqrt{6}$ is the dark limaçon in Fig. 4. Note that some nonconvex limaçons are positivity preserving. So, even if Hadamard had worked out details correctly, the result would not have contradicted his conjecture.

So convexity is neither sufficient nor necessary for PPP of Problem 8, although larger nonconvex boundary parts seem to have a bad influence on sign preservation. On the affirmative side is the result in [44] that for small perturbations of a disk PPP holds true.

Not only 'larger nonconvex' boundary sections ruin positivity but also a kind of opposite domain shape is bad for positivity. Coffman and Duffin showed that near rectangular corners the Green function has an oscillatory behavior. In the numerical approximation on the 2 by 1 rectangle in Fig. 24 both sign-changing effects occur. Due to the eccentricity and the isolated force on the left hand side a nodal line occurs somewhat right from the middle. Near the corners the first sign change due to the oscillatory behavior shows itself. In [55] one finds that sign-change also occurs for smooth domains which are close to such domains with corners in even dimensions. Not much is stated for dimensions larger than two, but the tools in [48] can be used to find a counterexample in higher dimensions starting from a counterexample in two dimensions.

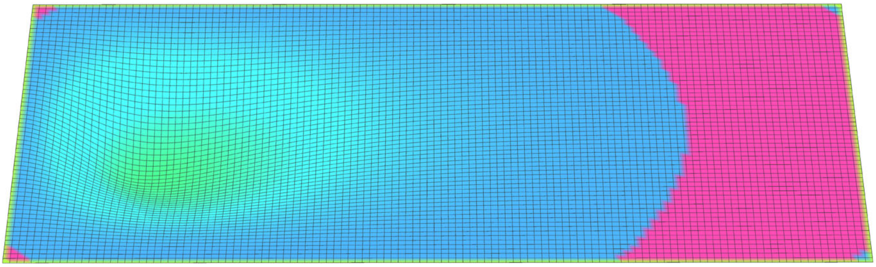


Fig. 24 The rectangular clamped plate pushed in one direction, nevertheless shows an oscillatory behavior. Near corners and on elongated parts away from the downwards directed localized force the solution tends to change sign. Also here the force is a downwards directed force on one grid element located on the longer central axis on $\frac{1}{8}$ from the left. The *red* (*darker*) part is where the plate moves upwards. Notice the sign changes at corners

Concerning positivity preserving for (42) there is not much that would hold for every domain, except the observation, that the negative part is very mildly negative. Indeed, for sufficiently smooth bounded domains in $\mathbb{R}^n$ it has been shown in [43], following [42] and [20], that the Green function for Problem 8 can be estimated from below with some $c_{1,\Omega} \geq 0$ and $c_{2,\Omega} > 0$ by

$$G_\Omega(x, y) + c_{1,\Omega} d(x)^2 d(y)^2 \geq c_{2,\Omega} F_n\big(d(x)d(y), |x-y|\big) \quad \text{for all } x, y \in \Omega. \tag{43}$$

Here is $d : \bar{\Omega} \to \mathbb{R}$ the distance function to the boundary

$$d(x) = \inf\big\{\big|x - x^*\big|; x^* \in \partial\Omega\big\}$$

and F_n is the function derived from Boggio [10, 45] for the biharmonic Dirichlet problem on balls:

$$F_n\big(d(x)d(y), |x-y|\big) = \begin{cases} (d(x)d(y))^{2-n/2} \min(1, \frac{d(x)^2 d(y)^2}{|x-y|^4})^{n/4} & \text{for } n < 4, \\ \log(1 + \frac{d(x)^2 d(y)^2}{|x-y|^4}) & \text{for } n = 4, \\ |x-y|^{4-n} \min(1, \frac{d(x)^2 d(y)^2}{|x-y|^4}) & \text{for } n > 4. \end{cases} \tag{44}$$

The structure of the estimate in (43) is the optimal one possible. Indeed, one may show that there are $C_{2,\Omega}, C_{3,\Omega} \in \mathbb{R}^+$ such that

$$G_\Omega(x,y) \le C_{2,\Omega} F_n\big(d(x)d(y), |x-y|\big) \quad \text{for all } x, y \in \Omega,$$

$$d(x)^2 d(y)^2 \le C_{3,\Omega} F_n\big(d(x)d(y), |x-y|\big) \quad \text{for all } x, y \in \Omega.$$

We end this section by sketching the range of λ where PPP holds for Problem 8 in some special cases. The formulation of the corresponding theorem would be similar to the one for example of Theorem 5.5 and we skip.

- For the disk the formula of Boggio shows that PPP holds for $\lambda = 0$. By Krein–Rutman one may extend this result to $\lambda \in (\lambda_1, 0]$. Perturbation arguments in [45], based on sharp two-sided estimates for the Green functions, show PPP for small positive λ.

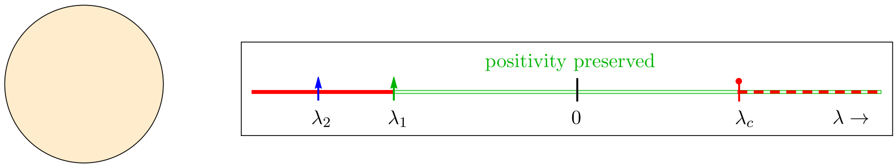


Fig. 25 For the clamped disk PPP holds on an interval that contains 0, similar as in the case of the clamped beam

- Limaçons are defined by $\Omega_a := \{(r\cos\varphi, r\sin\varphi); 0 \le r < 1 + 2a\cos\varphi\}$ with $a \in [0, \frac{1}{2}]$. Hadamard in [49] gave an explicit formula for the Green function, which was shown to be positive if and only if $a \in [0, 1/\sqrt{6}]$ in [21]. For $a \in (1/\sqrt{6}, 1/\sqrt{6} + \varepsilon)$ one uses arguments as in [47]. There it was proven that under a continuous perturbation of the domain a sign change in the Green function occurs before the corresponding first eigenfunction changes sign. For $a = a_0 = 1/\sqrt{6}$ the Green function is still positive but touches 0 for some pair $x \ne y \in \partial\Omega_{a_0}$ in a more than quadratic way. The iterated Green function, which appears in the expansion as in (8), is strictly positive and for $\lambda_0 \in (\lambda_{1,a_0}, 0)$ one finds $G_{\lambda_0,\Omega_{a_0}}(x,y) \simeq F_2(d(x)d(y), |x-y|)$ with F_n as in (44). Perturbing the domain further and by using arguments as in [44] one finds that $G_{\lambda,\Omega_{a_0+\varepsilon}}$ is still positive for $\varepsilon > 0$ but small. So with Krein–Rutman we find a first eigenvalue $\lambda_{1,a_0+\varepsilon} > \lambda_0$ and positivity for $\lambda \in (\lambda_{1,a_0+\varepsilon}, \lambda_c] \subset \mathbb{R}^-$. See Figs. 26 and 4.

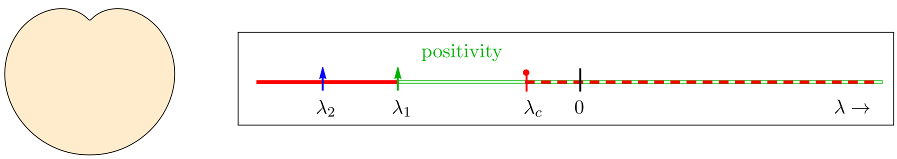


Fig. 26 For a limaçon with $a - a_0 > 0$ but small, PPP holds on an interval in $\mathbb{R}^-$

- For the annulus $\{(r\cos\varphi, r\sin\varphi); \varepsilon < r < 1\}$ with $\varepsilon > 0$ but small Coffman, Duffin and Shaffer [16, 29] could show that the eigenvalue for the positive eigenfunction comes after the eigenvalue for the first sign-changing eigenfunctions, which make a two-dimensional eigenspace. Since λ_2 now corresponds with the positive eigenfunction we expect PPP on (λ_2, λ_c) with $\lambda_c < \lambda_1 < 0$. See Fig. 27.

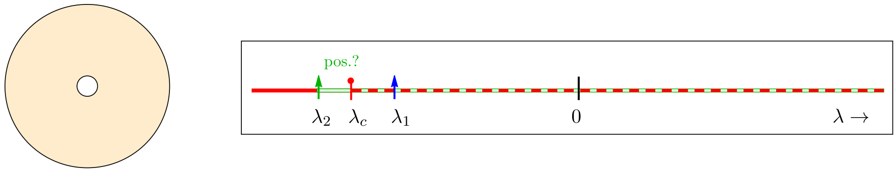


Fig. 27 For an annulus with a small hole the second eigenvalue corresponds with the positive eigenfunction. PPP in a right neighborhood of the second eigenvalue?

- Coffman proved in [14] that for any rectangle there is no eigenfunction with a fixed sign. By the converse of Krein–Rutman it follows that there is no sign preservation for any λ. See also [15].

Remark 5.15 Qualitative results such as PPP may be hard to prove for the clamped boundary value problem. Concerning existence and uniqueness it is however the simplest in the sense that there exists a unique weak solution for any domain. Indeed

$$J(u) = \iint_\Omega \left(\frac{1}{2}(\Delta u)^2 - f\,u\right) dxdy$$

has a unique minimum in $W_0^{2,2}(\Omega)$. This setting can be translated immediately to a numerical approximation by finite elements. However, the piecewise $C^{1,1}$-functions necessary for the direct approach are quite tedious. So one prefers again a system setting, which indeed is possible. Monk in [63], adjusting an idea of Ciarlet in Raviart in [13], considered on $W_0^{1,2}(\Omega) \times W^{1,2}(\Omega)$ the functional

$$J_s(u,v) = \iint_\Omega \left(\nabla u \cdot \nabla v - f\,u - \frac{1}{2}v^2\right) dxdy.$$

One finds as a first variation that for all $(\varphi, \psi) \in W_0^{1,2}(\Omega) \times W^{1,2}(\Omega)$

$$\iint_\Omega (\nabla u \cdot \nabla \psi - v\psi)dxdy = \iint_\Omega (\nabla \varphi \cdot \nabla v - f\,\varphi)dxdy = 0.$$

At least on smooth domains it follows through integrating by parts that $-\Delta u = v$ and $-\Delta v = f$ in Ω and, besides $u = 0$ on $\partial\Omega$ from $u \in W_0^{1,2}(\Omega)$, also through the first equation that

$$\frac{\partial}{\partial \nu} u = 0 \quad \text{on } \partial\Omega.$$

But also here reentrant corners spoil the system approach. See [9, 27]. Another numerical approach is considered by Davini and Pitacco in [25, 26]. See also [22].

Appendix A: 1d Proofs when $\lambda > 0$

The result of Lemma 2.1 is standard. Lemma 2.2 and 2.3 can be found in [81] or [53]. The crucial idea to compute λ_c as 'anti-eigenvalue' or switched eigenvalue goes back to Schröder in [73]. See also [79]. The direct approach is to compute the Green function and see for which λ a sign change occurs. We choose this approach for the first cases, since it will supply us with a formula for the Green function, which has some independent interest. The calculations are tedious and maybe not so interesting, but since the existing results in the literature do not always agree, it might be useful to have them explicitely stated.

Proof of Lemma 2.1 Although we may restrict ourselves to the case $\lambda > 0$ one directly computes the following explicit Green function for (1):

$$g_\lambda(x,y) = \begin{cases} \frac{\sinh(\sqrt{\lambda}\min(x,y))\sinh(\sqrt{\lambda}(1-\max(x,y)))}{\sqrt{\lambda}\sinh(\sqrt{\lambda})} & \text{for } \lambda > 0, \\ \frac{\sin(\sqrt{-\lambda}\min(x,y))\sin(\sqrt{-\lambda}(1-\max(x,y)))}{\sqrt{-\lambda}\sin(\sqrt{-\lambda})} & \text{for } \lambda < 0, \end{cases} \tag{A.1}$$

when $\sqrt{-\lambda}$ is not a multiple of π. Consistent with (5) one finds

$$g_0(x,y) = \lim_{\lambda\to 0} g_\lambda(x,y) = \min(x,y)\big(1-\max(x,y)\big).$$

The denominator in (A.1) turns 0 for $\lambda = -k^2\pi^2$ with $k \in \mathbb{N}^+$. Since the numerators in (A.1) changes sign for some x and y if and only if $\lambda < -\pi^2$, one finds that the Green function is positive if and only if $\lambda > -\pi^2$. □

Proof of Lemma 2.2 Let $\lambda > 0$ and write $\mu = \sqrt[4]{\lambda/4}$. The function

$$\begin{aligned} w(\mu,p) = {}& \sin(\mu p)\cosh\big(\mu(2-p)\big) + \cos(\mu p)\sinh\big(\mu(2-p)\big) \\ & + \sinh(\mu p)\cos\big(\mu(2-p)\big) + \cosh(\mu p)\sin\big(\mu(2-p)\big), \end{aligned} \tag{A.2}$$

satisfies $((\frac{d}{dx})^4 + \lambda)w(\mu,x) = 0$. Then the Green function is

$$g_\lambda(x,y) = \frac{w(\mu,|x-y|) - w(\mu,x+y)}{8\mu^3(\cosh(2\mu)-\cos(2\mu))}. \tag{A.3}$$

See Fig. 28 (right). Indeed, it satisfies the same differential equation as w for $x \neq y$. One may check that

$$w(\mu,p) = w_0(\mu) + w_2(\mu)p^2 + w_3(\mu)p^3 + \mathcal{O}\big(p^4\big) \tag{A.4}$$

with

$$w_0(\mu) = \sinh(2\mu) + \sin(2\mu),$$
$$w_2(\mu) = -\mu^2\big(\sinh(2\mu) - \sin(2\mu)\big),$$
$$w_3(\mu) = \tfrac{2}{3}\mu^3\big(\cosh(2\mu) - \cos(2\mu)\big).$$

The first odd power of p that appears in (A.4) is 3 and with its coefficient this explains

$$\left(\frac{d}{dx}\right)^4 g_\lambda(x,y) + \lambda g_\lambda(x,y) = \delta_y(x).$$

The boundary conditions $g_\lambda(0,y) = 0$ and $(\frac{d}{dx})^2 g_\lambda(0,y) = 0$ follow from (A.3) and (A.2). The ones in $x = 1$ from $w(\mu,p) = w(\mu,2-p)$.

The first eigenvalue is reached for $\lambda = -\pi^4$ and for $\lambda > -\pi^4$ the Green function depends smoothly on λ. Where does positivity break down for $\lambda > 0$? One may show that for $\lambda \mapsto g_\lambda(\cdot,\cdot)$ no interior zero can appear and hence that new 'zeros come in through the boundary' at λ_c if λ_c denotes the value for which positivity of the Green function breaks down. Schröder [73] studied the question and showed that for $\lambda = \lambda_c$ one encounters either a standard eigenvalue or a 'switched' eigenvalue. The standard eigenvalues are negative, so λ_c is, according to Schröder, an eigenvalue with a positive eigenfunction of the 'switched' eigenvalue problem:

$$\begin{cases} \varphi'''' + \lambda\varphi = 0 \quad \text{in } (0,1), \\ \varphi(0) = \varphi'(0) = \varphi''(0) = 0, \\ \varphi(1) = 0. \end{cases} \tag{A.5}$$

Switched, since $\varphi''(1) = 0$ is replaced by $\varphi'(0) = 0$. One could also have replaced $\varphi''(0) = 0$ is replaced by $\varphi'(1) = 0$, but by symmetry one finds the same value λ_c. The first eigenfunction for (A.5) is

$$\varphi(x) = \sin(\mu_c x)\cosh(\mu_c x) - \sinh(\mu_c x)\cos(\mu_c x),$$

with μ_c the first positive solution of $\tan\mu_c = \tanh\mu_c$ and hence $\lambda_c = 4\mu_c^4$. □

Proof of Lemma 2.3 The Green function here is

$$gg(\mu,x,y) = g(\mu,x,y) - rl(\mu,x,y) - rr(\mu,x,y),$$

with $g(\mu,x,y)$ from (A.3),

$$rl(\mu,x,y) = \frac{\sin(\mu(2-y))\sinh(\mu y) - \sin(\mu y)\sinh(\mu(2-y))}{\mu^3(\cosh(2\mu) - \cos(2\mu))(\cosh(2\mu) + \cos(2\mu) - 2)} \times \big(\sin(\mu)\sinh(\mu x)\sin\big(\mu(1-x)\big) - \sinh(\mu)\sin(\mu x)\sinh\big(\mu(1-x)\big)\big)$$

and

$$rr(\mu,x,y) = rl(\mu,1-x,1-y).$$

See Fig. 28 (left). The value $\lambda_1 = -\rho_1 := -(2\mu_1)^4$ corresponds to the first eigenvalue for the clamped problem

$$\varphi_1(x) = \cos\big(2\mu_1\big(x - \tfrac{1}{2}\big)\big) - \frac{\cos\mu_1}{\cosh\mu_1}\cosh\big(2\mu_1\big(x - \tfrac{1}{2}\big)\big)$$

with μ_1 the first positive solution of $\tan\mu_1 + \tanh\mu_1 = 0$. To determine the value λ_c one considers the switched eigenvalue problem where $\varphi'(1) = 0$ is replaced by $\varphi''(0) = 0$, which is again problem (A.5), and hence one obtains the same value λ_c as before. □

Remark A.1 It might appear surprising, that the value λ_c from Lemma 2.2 for the supported beam and the λ_c from Lemma 2.3 for the clamped beam are identical. The reason is that the corresponding switched eigenvalue problems in the sense of [73], which determine λ_c, are identical. Indeed, in both cases one arrives at (A.5).

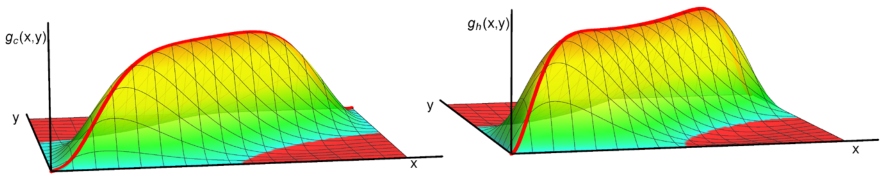


Fig. 28 Sketches of the Green functions for the clamped (*left*) and for the hinged beam with $\lambda = 5184 > \lambda_c$. On the *dark* (*red*) part the Green function is negative. The *line* depicts the diagonal, where $x \mapsto (\frac{d}{dx})^3 g_\lambda(x, y)$ has a jump

Proof of Lemma 2.4 The first eigenfunction that corresponds to Problem 5, is

$$\varphi_1(x) = \sinh(\nu x) - \sin(\nu x) - \frac{\sin\nu + \sinh\nu}{\cos\nu + \cosh\nu}\big(\cosh(\nu x) - \cos(\nu x)\big),$$

with eigenvalue $d_1 = \nu_1^4$, where ν_1 the first positive zero of $\cosh\nu\cos\nu + 1 = 0$. According to [73] the critical value d_c is determined by one of the following two switched eigenvalue problems, namely

$$\begin{cases} \varphi'''' + \lambda\varphi = 0 \quad \text{in } (0,1), \\ \varphi(0) = \varphi'(0) = \varphi''(0) = 0, \\ \varphi''(1) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \varphi'''' + \lambda\varphi = 0 \quad \text{in } (0,1), \\ \varphi(0) = 0, \\ \varphi(1) = \varphi''(1) = \varphi'''(1) = 0. \end{cases} \tag{A.6}$$

The corresponding first eigenfunctions of (A.6) are related through

$$\varphi_{\text{left}}(x) = \varphi''_{\text{right}}(1 - x),$$

and hence give the same eigenvalue. Since

$$\varphi_{\text{left}}(x) = \sinh(\mu_1 x)\cos(\mu_1 x) - \sin(\mu_1 x)\cosh(\mu_1 x),$$

where μ_1 is the first positive number such that $\varphi''_{\text{left}}(1) = 0$, the same μ_1 as for Lemma 2.3 but here implying that $d_c = 4\mu_1^4 \approx 125.141$. □

Appendix B: A Sharper Version of Kreĭn–Rutman

In a more general setting such as for cooperative systems the strong positivity condition, which is used for the version of Kreĭn–Rutman above, i.e. Theorem 3.4, is too restrictive. Instead of strong positivity, some weaker positivity is sufficient, whenever one can show that the spectral radius is strictly positive. Showing that the spectral radius is positive, most of the time will need some tedious constructions in order to obtain a more or less explicit (super)solution. A breakthrough is a theorem of de Pagter in [66]. The slight disadvantage is that one needs the structure of Banach lattices, which may not be a common tool for the analyst working on partial differential equations. The spaces $C(\overline{\Omega})$ and $\mathcal{L}^p(\Omega)$, home turf for analysts, with the natural ordering, however are Banach lattices. Let us briefly introduce the setting.

Definition B.1 The real vectorspace X gives a vector lattice $(X, \leq)$, if:

1. $\leq$ is a partial ordering on X;
2. for all $u, v, w \in X$: $u \leq v$ implies $u + w \leq v + w$;
3. for all $u \in X$ and $t \in \mathbb{R}^+$: $0 \leq u$ implies $0 \leq tu$;
4. for all $u, v \in X$ also $\inf(u, v), \sup(u, v) \in X$, where $\inf(u, v)$ is the largest lower bound and $\sup(u, v)$ the least upper bound for u, v.

Note that $u \in C(\overline{\Omega})$ implies $|u| \in C(\overline{\Omega})$ and a similar result holds for $\mathcal{L}^p(\Omega)$ or $W^{1,2}(\Omega)$. For $C^1(\overline{\Omega})$ and $W^{2,2}(\Omega)$ such a result is not true. One also needs a relation between the norm and the absolute value of a function, $|u| := \sup(u, -u)$.

Definition B.2 If $(X, \leq)$ is a vector lattice and $(X, \|\cdot\|)$ a Banach space, then $(X, \|\cdot\|, \leq)$ is a Banach lattice if for all $u, v \in X$ one has: $|u| \leq |v|$ implies $\|u\| \leq \|v\|$.

The space $(C(\overline{\Omega}), \|\cdot\|_\infty, \leq)$ is a Banach lattice and $(W^{1,2}(\Omega), \|\cdot\|_{W^{1,2}}, \leq)$ is not.

Definition B.3 Let $(X, \|\cdot\|, \leq)$ be a Banach lattice. A subspace $I \subset X$ is called a lattice ideal, if

1. for all $u \in I$ one has $|u| \in I$ and
2. for all $u \in I$ and $v \in X$ with $0 \leq v \leq u$ one has $v \in I$.

Definition B.4 Let $(X, \|\cdot\|, \leq)$ be a Banach lattice and $T \in \mathsf{L}(X)$. T is called irreducible if $\{0\}$ and X are the only closed T-invariant lattice ideals.

The properties of g_1, g_2, g_3 and g_5 in (5), (6), (10), (11), and for more general second order equations the strong maximum principle, imply that the corresponding $\mathcal{G} \in \mathsf{L}(C(\overline{\Omega}))$ is irreducible, without assuming more on the boundary than that $\mathcal{G}$ indeed maps from $C(\overline{\Omega})$ into $C(\overline{\Omega})$.

Using the concept of Banach lattice one may formulate an optimal version of the Kreĭn–Rutman theorem.

Theorem B.5 (Kreĭn–Rutman–de Pagter) *Let* $(X, \|\cdot\|, \leq)$ *be a Banach lattice with* $\dim(X) > 1$ *and let* $T \in \mathsf{L}(X)$ *be compact, positive and irreducible. Then*:

1. *the spectral radius* $\nu(T)$ *is strictly positive*;
2. $\nu(T)$ *is an eigenvalue for* T *with algebraic multiplicity* 1;
3. *all other eigenvalues* ν_i *satisfy* $|\nu_i| < \nu(T)$;
4. *the eigenfunction for* $\nu(T)$ *is of fixed sign and* (*up to multiples*) *it is the only eigenfunction of* T *with a fixed sign*.

The value $\nu(T)$ is called the principal eigenvalue of T just as the corresponding eigenfunction is called the principal eigenfunction of T.

The theorem combines a classical result of Kreĭn and Rutman with a result of de Pagter [66]. See also [37] or [77].

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