Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces

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1 Introduction

In this paper we consider systems of the type

$$\begin{cases} -\Delta u = f(v) & \text{in } \Omega, \\ -\Delta v = g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, with a smooth boundary, and where $f, g : \mathbb{R} \to \mathbb{R}$ are suitable monotone increasing functions satisfying $f(0) = g(0) = 0$. The special case for which f and g are (asymptotically) pure powers has been treated by numerous authors of which we mention [16], [17], [6], [10], [13]. Indeed, if $f(v) = |v|^{\alpha-1} v$ and $g(u) = |u|^{\beta-1} u$ with $\alpha, \beta > 0$, then (1.1) possesses at least one smooth positive solution for dimensions $n \geq 3$ if the following holds:

$$1 > \frac{1}{\alpha+1} + \frac{1}{\beta+1} > 1 - \frac{2}{n}. \tag{1.2}$$

The first inequality corresponds to 'superlinearity' which leads to existence of solutions via a minimax argument. The second inequality corresponds to 'subcriticality' which guarantees the required compactness in the application of a Mountain Pass Lemma as well as regularity of solutions through a bootstrap argument.

The main goal of the present paper is to allow more general nonlinearities. The nonlinearities that we consider still have polynomial growth but are not necessarily asymptotic to a pure power. We will still assume 'superlinearity' and 'subcriticality'. In this present setting a similar condition as in (1.2) is used but the numbers α and β that appear in the right hand side do not need to be the same as the ones in the left hand side.

We obtain a positive solution to (1.1) by inverting the first equation in (1.1) and employing a variant of the Mountain Pass Lemma of Ambrosetti-Rabinowitz ([2]). The right setting for this approach is the use of Sobolev-Orlicz spaces. See e.g [9].

The paper is organized as follows. The exponents mentioned above are introduced in section 2. Our main result is stated in Theorem 2.6. This theorem addresses both existence and regularity. The existence part is based on an abstract result stated in Proposition 3.2 which generalizes [7, Theorem 2]. This result is stated and proved in section 3. The verification of condition ii) in Proposition 3.2 requires an interesting elliptic regularity result in (Sobolev-)Orlicz

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spaces which is based on an interpolation theorem of Boyd ([5]). This is the content of section 4. The regularity part of Theorem 2.6 is established by a bootstrap argument similar as in [6]. Section 5 contains the proof of Theorem 2.6. For the sake of easy reference and completeness we collect some auxiliary results in the Appendix.

2 Preliminaries and main result

Before we state our main result we have to fix the conditions on the nonlinearities f and g.

2.1 Admissible functions

Condition 2.1 *We call φ admissible if:*

i. $\varphi \in C(\mathbb{R};\mathbb{R})$;

ii. *φ is odd: $\varphi(-t) = -\varphi(t)$ for all $t \in \mathbb{R}$;*

iii. *φ is strictly increasing;*

iv. $\varphi(\mathbb{R}) = \mathbb{R}$.

Note that if φ is admissible the inverse exists and is also admissible.

Notation 2.2 *For an admissible function φ we will use the following:*

$$\Phi(s) = \int_0^s \varphi(t)\,dt \qquad \Phi^*(s) = \int_0^s \varphi^{-1}(t)\,dt \tag{2.1}$$

The function Φ^* is called the complementary function to Φ.

Note that $(\Phi^*)^* = \Phi$ and that

$$\Phi(s) + \Phi^*(\varphi(s)) = s\,\varphi(s) \text{ for all } s \in \mathbb{R}. \tag{2.2}$$

We will fix some numbers which will replace the role of the pure powers appearing in Φ respectively Φ^* in the homogeneous case.

Notation 2.3 *For an admissible function φ with Φ as in (2.1) we define:*

$$\begin{array}{lll} m_\varphi = \sup\limits_{t>0} \dfrac{t\varphi(t)}{\Phi(t)}, & m_\varphi^\infty = \limsup\limits_{t\to\infty} \dfrac{t\varphi(t)}{\Phi(t)}, & \tilde{m}_\varphi = \limsup\limits_{t\to\infty} \dfrac{\log(\Phi(t))}{\log t}, \\ \ell_\varphi = \inf\limits_{t>0} \dfrac{t\varphi(t)}{\Phi(t)}, & \ell_\varphi^\infty = \liminf\limits_{t\to\infty} \dfrac{t\varphi(t)}{\Phi(t)}, & \tilde{\ell}_\varphi = \liminf\limits_{t\to\infty} \dfrac{\log(\Phi(t))}{\log t}. \end{array} \tag{2.3}$$

The Boyd exponents for the Orlicz space $L_\Phi(\Omega)$, when Ω is bounded, are given by

$$q_{L_\Phi(\Omega)} = \inf\left\{q;\ \sup_{h,t\geq 1} \frac{\Phi(th)}{\Phi(t)h^q} < \infty\right\} \text{ and } p_{L_\Phi(\Omega)} = \sup\left\{p;\ \inf_{\lambda,t\geq 1} \frac{\Phi(th)}{\Phi(t)h^p} > 0\right\}. \tag{2.4}$$

Lemma 2.4 *For an admissible φ it holds that*

i. $\ell_\varphi \leq \ell_\varphi^\infty \leq q_{L_\Phi(\Omega)} \leq \tilde{\ell}_\varphi \leq \tilde{m}_\varphi \leq p_{L_\Phi(\Omega)} \leq m_\varphi^\infty \leq m_\varphi$;

ii. $\dfrac{1}{m_\varphi} + \dfrac{1}{\ell_{\varphi^{-1}}} = 1, \qquad \dfrac{1}{m_\varphi^\infty} + \dfrac{1}{\ell_{\varphi^{-1}}^\infty} = 1$ *and* $\dfrac{1}{\tilde{m}_\varphi} + \dfrac{1}{\tilde{\ell}_{\varphi^{-1}}} = 1.$

For the proof we refer to Corollary C.5 and Lemma D.1.

Remark 2.4.1 *The numbers defined above all have their specific role:* ℓ_φ, m_φ *will play a role in the Mountain Pass Theorem that we will use. Necessary for the elliptic regularity through interpolation are* $1 < q_{L_\Phi(\Omega)}$ *and* $p_{L_\Phi(\Omega)} < \infty$. *Reflexivity of the spaces involved is related to* $1 < \ell_\varphi^\infty$ *and* $m_\varphi^\infty < \infty$. *Finally, the numbers* $\tilde{\ell}_\varphi$ *and* $\tilde{m}_\varphi$ *will appear in the imbedding results for the Orlicz spaces that we will use.*

Let us finish the introduction with some examples showing some differences in these numbers.

Example I *In case of a pure power, that is* $\varphi(t) = |t|^{\alpha-1} t$ *with* $\alpha > 0$ *one finds*

$$\ell_\varphi = \ell_\varphi^\infty = \tilde{\ell}_\varphi = \alpha + 1 = \tilde{m}_\varphi = m_\varphi^\infty = m_\varphi.$$

Ex. II *For* $\varphi(t) = t^\alpha \log\left(1 + t^\beta\right)$ *for* $t \geq 0$ *with* $\alpha, \beta > 0$ *one finds*

$$\ell_\varphi = \tilde{\ell}_\varphi = \alpha + 1 = \tilde{m}_\varphi = m_\varphi^\infty < m_\varphi = \alpha + \beta + 1.$$

Ex. III *For* $\varphi(t) = \frac{|t|^\alpha}{\log\left(1+|t|^\beta\right)}$ *for* $t \geq 0$ *with* $0 < \beta < \alpha$ *one finds*

$$\ell_\varphi = \alpha - \beta + 1 < \tilde{\ell}_\varphi = \alpha + 1 = \tilde{m}_\varphi = m_\varphi.$$

Ex. IV *The next* φ *is not admissible since it is not continuous and, although increasing, not strictly increasing. A slightly perturbed* φ_ε *however will be admissible. Set*

$$\varphi(t) = e^{[\log(t)]} \text{ for } t > 0$$

with $[x] = \sup\{n \in \mathbb{Z}; n \leq x\}$. *Then*

$$\Phi(t) = te^{[\log(t)]} - \frac{e}{e+1} e^{2[\log(t)]} \text{ for } t > 0.$$

Straightforward computations show that

$$1 + e^{-1} = \ell_\varphi^\infty < \tilde{\ell}_\varphi = 2 = \tilde{m}_\varphi < m_\varphi^\infty = 1 + e.$$

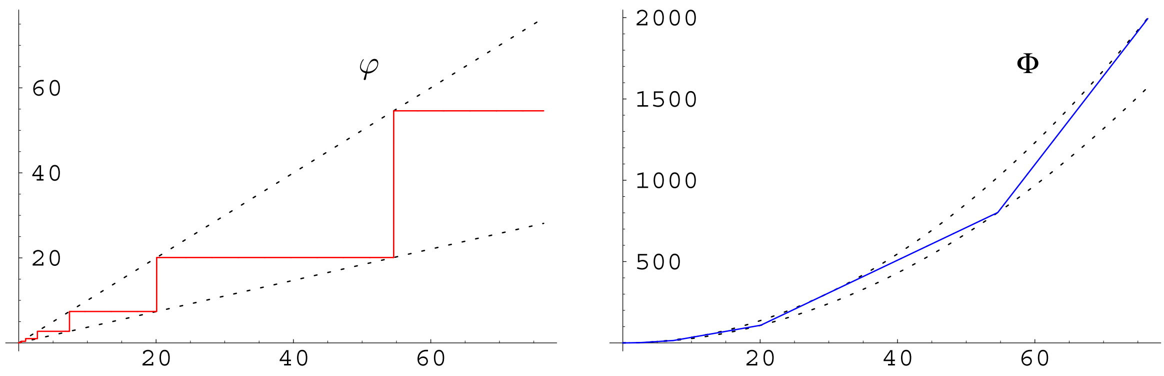
We remark that

$$\begin{array}{rcccll} e^{-1}\, t & \leq & \varphi(t) & \leq & t & \text{for all } t \geq 0, \\ \frac{1}{e+1}\, t^2 & \leq & \Phi(t) & \leq & \frac{e+1}{4e}\, t^2 & \text{for all } t \geq 0, \end{array} \tag{2.5}$$

and that these constants are optimal, even when restricting to large values of t. *One finds*

$$\tfrac{4e}{(e+1)^2} h^{2-p} \leq \frac{\Phi(th)}{\Phi(t) h^p} \leq \tfrac{(e+1)^2}{4e} h^{2-p},$$

which shows that $q_{L_\Phi(\Omega)} = p_{L_\Phi(\Omega)} = 2$.



2.2 The main result

Throughout the paper we assume the following.

Condition 2.5 *The admissible functions f and g satisfy:*

$$i)\quad m_f^\infty < \infty, \qquad ii)\quad \ell_f > 1,$$
$$iii)\quad m_g^\infty < \infty, \qquad iv)\quad \ell_g > 1.$$

Notice that we hence find that both for $\varphi = f$ and $\varphi = g$:

$$1 < \ell_\varphi \le \ell_\varphi^\infty \le \tilde{\ell}_\varphi \le \tilde{m}_\varphi \le m_\varphi^\infty < \infty.$$

There is no restriction from above for m_f and m_g.

Theorem 2.6 *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Suppose that f and g are admissible functions satisfying Condition 2.5.*

- *If*
$$1 > \frac{1}{\ell_f} + \frac{1}{\ell_g} \text{ and } \frac{1}{\tilde{m}_f} + \frac{1}{\tilde{m}_g} > 1 - \frac{2}{n} \tag{2.6}$$
then system (1.1) has a positive solution $(u, v) \in \left(W^{2,p}(\Omega) \cap C_0(\bar{\Omega})\right)^2$ for all $p \in (1, \infty)$.
- *If (2.6) holds and if $f, g \in C^\gamma(\mathbb{R})$ and $\partial\Omega \in C^{2,\gamma}$ for some $\gamma \in (0, 1)$, then*
$$(u, v) \in \left(C^{2,\gamma}(\bar{\Omega}) \cap C_0(\bar{\Omega})\right)^2. \tag{2.7}$$

The existence part of this theorem is a consequence of an abstract result that is proved in the next section. The proof of the theorem above will be postponed accordingly.

Remark 2.6.1 *Note that since $\ell_\varphi \le \tilde{m}_\varphi$ for an admissible function φ we may reformulate (2.6) as*

$$1 > \frac{1}{\ell_f} + \frac{1}{\ell_g} \ge \frac{1}{\tilde{m}_f} + \frac{1}{\tilde{m}_g} > 1 - \frac{2}{n}. \tag{2.8}$$

Remark 2.6.2 *In case that f and g are pure powers: $f(t) = |t|^{\alpha-1}t$ and $g(t) = |t|^{\beta-1}t$, the condition in (2.6) reduces to the well-known (see [6]) inequalities (1.2).*

3 An abstract existence result

In this section we give an existence result for an abstract variational problem which is used in the proof of our main theorem. Let f, g be two admissible functions with F and G as in (2.1) and such that Condition 2.5 is satisfied.

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $L_F(\Omega)$, $L_{F^*}(\Omega)$, $L_G(\Omega)$ and $L_{G^*}(\Omega)$ be the corresponding Orlicz spaces as defined in (A.3). Supposing that X is a real Banach space and $A \in \mathrm{Isom}\,(X, L_{F^*}(\Omega))$ we define the following functionals:

$$\begin{aligned} I_1(u) &:= \int_\Omega F^*(Au)\, d\mu \quad && \text{for } u \in X, \\ I_2(w) &:= \int_\Omega G(w)\, d\mu && \text{for } w \in L_G(\Omega), \\ I_{2,+}(w) &:= \int_\Omega G\left(w^+\right) d\mu && \text{for } w \in L_G(\Omega) \text{ where } w^+ = w \vee 0. \end{aligned} \tag{3.1}$$

In view of Lemma A.5 we have $I_1 \in C^1(X; \mathbb{R})$, I_2, $I_{2,+} \in C^1(L_G(\Omega); \mathbb{R})$. Supposing that X is continuously imbedded in $L_G(\Omega)$ we may define

$$\begin{aligned} I(u) &:= I_1(u) - I_2(u) \quad && \text{for } u \in X, \\ I_+(u) &:= I_1(u) - I_{2,+}(u) && \text{for } u \in X. \end{aligned} \tag{3.2}$$

We also have I, $I_+ \in C^1(X; \mathbb{R})$. Notice that $I(0) = I_+(0) = 0$.

Lemma 3.1 *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, and let f, g be two admissible functions which are such that Condition 2.5 is satisfied. Assuming $A \in \mathrm{Isom}(X, L_{F^*}(\Omega))$ and $X \hookrightarrow L_G(\Omega)$ let I and I_+ be defined as in (3.2). Suppose moreover that*

i. $\dfrac{1}{\ell_f} + \dfrac{1}{\ell_g} < 1$;

ii. X is compactly imbedded in $L_G(\Omega)$.

Then there exist $\bar{u}$, $\bar{u}_+ \in X$, such that

$$\begin{aligned} I(\bar{u}) > 0 \quad &\textit{and} \quad I'(\bar{u}) = 0, \\ I_+(\bar{u}_+) > 0 \quad &\textit{and} \quad I_+'(\bar{u}_+) = 0. \end{aligned}$$

Since we have

$$\frac{1}{\ell_g} < 1 - \frac{1}{\ell_f} = \frac{1}{m_{f^{-1}}}$$

a scheme for the situation is as follows:

$$\begin{array}{ccc} X & \Subset & L_G \\ & {\scriptstyle A}\searrow & \cap \\ & & L_{F^*} \end{array}$$

Proof. The proof consists of two major steps. First we will show that the assumptions of the Mountain Pass Theorem as in Proposition B.1 are fulfilled. Next we shall establish the existence of the critical point $\bar{u}$ (respectively $\bar{u}_+$).

Step 1.a: Verification that $r, \alpha > 0$ exist such that $\|u\|_X = r$ implies $I(u) \geq \alpha$.

Since $A \in \operatorname{Isom}(X, L_{F^*}(\Omega))$ we may choose $\|u\|_X := \|Au\|_{L_{F^*}}$. In view of the second assumption of Lemma 3.1 there exists $c > 0$ such that

$$\|u\|_{L_G} \le c \|Au\|_{L_{F^*}}.$$

For $u \in X$ with $0 < \|u\|_X \le \frac{1}{1+c}$ we have both $\|u\|_{L_G} < 1$ as well as $\|Au\|_{L_{F^*}} < 1$. From Lemma C.7 item (C.11) we have for such u that

$$\int_\Omega F^*(Au)\, d\mu \ge \|Au\|_{L_{F^*}}^{\ell_f^*}, \tag{3.3}$$

with $\frac{1}{\ell_f^*} = 1 - \frac{1}{\ell_f} = \frac{1}{m_{f-1}}$. Hence

$$I_1(u) = \int_\Omega F^*(Au)\, d\mu \ge \left(\frac{1}{c}\right)^{\ell_f^*} \|u\|_{L_G}^{\ell_f^*}. \tag{3.4}$$

On the other hand in view of Lemma C.7 item (C.9) we have

$$\int_\Omega G(u)\, d\mu \le \|u\|_{L_G}^{\ell_g}. \tag{3.5}$$

We may even assume that $\|Au\|_{L_{F^*}} = r < \min\left(1, c^{-1}\right)$ and find for such u by combining (3.4) and (3.5) that

$$\begin{aligned} I(u) &= \int_\Omega F^*(Au(x))\, dx - \int_\Omega G(u(x))\, dx = \\ &\ge \|Au\|_{L_{F^*}}^{\ell_f^*} - \|u\|_{L_G}^{\ell_g} = \|Au\|_{L_{F^*}}^{\ell_f^*} \left(1 - c^{-\ell_g} \|Au\|_{L_{F^*}}^{\ell_g - \ell_f^*}\right). \end{aligned}$$

Since $\ell_f^* = \left(1 - \frac{1}{\ell_f}\right)^{-1} < \ell_g$ one finds that appropriate $r \in \left(0, \min\left(1, c^{-1}\right)\right) > 0$ and $\alpha > 0$ exist.

Since for all $u \in X$

$$\int_\Omega G\left(u^+\right)\, d\mu \le \int_\Omega G(u)\, d\mu$$

the same r and α can be taken for I_+.

Step 1.b: The verification of the second condition of Proposition B.1.

Take $u_0 \in X$ with $\int_\Omega G\left(u_0^+\right)\, d\mu > 0$ and $\lambda > 1$. Then, in view of Lemma C.3.iv,

$$\int_\Omega F^*(\lambda A u_0)\, d\mu \le \lambda^{\ell_f^*} \int_\Omega F^*(A u_0)\, d\mu$$

and by Lemma C.3.ii

$$\int_\Omega G\left(\lambda u_0^+\right)\, d\mu \ge \lambda^{\ell_g} \int_\Omega G\left(u_0^+\right)\, d\mu.$$

Then

$$I(\lambda u_0) \le I_+(\lambda u_0) \le \lambda^{\ell_f^*} \int_\Omega F^*(A u_0)\, d\mu - \lambda^{\ell_g} \int_\Omega G\left(u_0^+\right)\, d\mu < -1$$

for λ sufficiently large since $\ell_g > \ell_f^*$ and $\int_\Omega G\left(u_0^+\right)\, d\mu > 0$.

Step 2: With the result of the first step we may apply Proposition B.1. Let $\{u_n\}_{n=1}^{\infty} \subset X$ be a sequence as in the conclusion which is such that $I(u_n) \to c > 0$ and $I'(u_n) \to 0$ in X' for $n \to \infty$.

Step 2.a: First we will show that $\{u_n\}_{n=1}^{\infty}$ is bounded in X.

We will proceed by contradiction and suppose that $\|Au_n\|_{L_{F^*}} \to \infty$ for $n \to \infty$. Then one finds by Lemma C.7 item (C.10) for $\varphi = f^{-1}$ that for n large

$$I_1(u_n) = \int_\Omega F^*(Au_n)\,dx \geq \|Au_n\|_{L_{F^*}}^{\ell_f^*} \tag{3.6}$$

and hence $I_1(u_n) \to \infty$. Since $I(u_n)$ is bounded one finds that also $\int_\Omega G(u_n)\,dx = I_2(u_n) \to \infty$. One can even conclude that

$$\lim_{n\to\infty} \frac{\int_\Omega G(u_n)\,dx}{\int_\Omega F^*(Au_n)\,dx} = 1 - \lim_{n\to\infty} \frac{I(u_n)}{I_1(u_n)} = 1. \tag{3.7}$$

The assumption $I'(u_n) \to 0$ in X' for $n \to \infty$ means that there are $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$\left| \int_\Omega f^{-1}(Au_n)\,Avdx - \int_\Omega g(u_n)\,vdx \right| \leq \varepsilon_n \|Av\|_{L_{F^*}} \text{ for all } v \in X. \tag{3.8}$$

It follows from the definition of ℓ_f^* and ℓ_g, (3.7), (3.8) and (3.6) that

$$\begin{aligned}
\ell_g &\leq \limsup_{n\to\infty} \frac{\int_\Omega g(u_n)\,u_n dx}{\int_\Omega G(u_n)\,dx} = \limsup_{n\to\infty} \frac{\int_\Omega g(u_n)\,u_n dx}{\int_\Omega F^*(Au_n)\,dx} \leq \\
&\leq \limsup_{n\to\infty} \left(\frac{\int_\Omega f^{-1}(Au_n)\,Au_n dx}{\int_\Omega F^*(Au_n)\,dx} + \varepsilon_n \frac{\|Au_n\|_{L_{F^*}}}{\int_\Omega F^*(Au_n)\,dx} \right) \leq \\
&\leq \ell_f^* + \limsup_{n\to\infty} \varepsilon_n \|Au_n\|_{L_{F^*}}^{1-\ell_f^*} = \ell_f^*,
\end{aligned}$$

contradicting the restriction for ℓ_g and ℓ_f in this lemma. Indeed $\ell_f^* = \left(1 - \frac{1}{\ell_f}\right)^{-1} < \ell_g$. So we may assume that $\|Au_n\|_{L_{F^*}} \leq C$ for some $C \in \mathbb{R}^+$.

Step 2.b: Existence of a limit $\bar{u} \in X$ with the desired properties.

Since X is a reflexive Banach space we may assume that there exists a subsequence, again denoted by $\{u_n\}_{n=1}^{\infty}$ and a $\bar{u} \in X$ such that $u_n \rightharpoonup \bar{u}$ in X. Since X is compactly imbedded in $L_G(\Omega)$ by condition ii, we have $u_n \to \bar{u}$ in $L_G(\Omega)$.

Since $I_2 \in C^1(L_G(\Omega);\mathbb{R})$ it follows that $I_2(u_n) \to I_2(\bar{u})$ and $I_2'(u_n) \to I_2'(\bar{u})$. Since F^* is convex it follows that

$$I_1(v) - I_1(u_n) \geq \left\langle I_1'(u_n), v - u_n \right\rangle \text{ for all } v \in X.$$

Hence

$$I_1(v) - I_1(u_n) - \left\langle I_2'(u_n), v - u_n \right\rangle \geq \left\langle I'(u_n), v - u_n \right\rangle \geq$$

$$\geq - \left\| I'(u_n) \right\|_{X'} \|v - u_n\|_X \to 0 \tag{3.9}$$

since $I'(u_n) \to 0$ and $\|v - u_n\|_X$ is bounded. Since $I_2'(u_n) \to I_2'(\bar{u})$ in $(L_G(\Omega))'$ and $v - u_n \to v - \bar{u}$ in $L_G(\Omega)$ it follows that

$$\left\langle I_2'(u_n), v - u_n \right\rangle \to \left\langle I_2'(\bar{u}), v - \bar{u} \right\rangle.$$

By (3.9) we obtain

$$I_1(v) - \limsup_{n\to\infty} I_1(u_n) \geq \left\langle I_2'(\bar{u}), v - \bar{u} \right\rangle$$

and taking $v = \bar{u}$

$$\limsup_{n\to\infty} I_1(u_n) \leq I_1(\bar{u}).$$

Since I_1 is lower semi-continuous $\liminf_{n\to\infty} I_1(u_n) \geq I_1(\bar{u})$ holds and hence we find that $\lim_{n\to\infty} I_1(u_n) = I_1(\bar{u})$.

Since $\lim_{n\to\infty} I_2(u_n) = I_2(\bar{u})$ and $\lim_{n\to\infty} I_1(u_n) = I_1(\bar{u})$ we have

$$I(\bar{u}) = \lim_{n\to\infty} I(u_n) = c.$$

By $I'(u_n) \to 0$ and $I_2'(u_n) \to I_2'(\bar{u})$ one finds $I_1'(u_n) \to I_2'(\bar{u})$ and hence by (3.9) it follows that

$$I_1(v) - I_1(u_n) \geq \left\langle I_1'(\bar{u}), v - u_n \right\rangle =$$

$$= \left\langle I_2'(\bar{u}), v - u_n \right\rangle + \left\langle I_1'(u_n) - I_2'(\bar{u}), v - u_n \right\rangle \text{ for all } v \in X.$$

We find as in [9] that

$$I_1(v) - I_1(\bar{u}) \geq \left\langle I_2'(\bar{u}), v - \bar{u} \right\rangle \text{ for all } v \in X$$

and hence $I_1'(\bar{u}) = I_2'(\bar{u})$ and $I'(\bar{u}) = 0$. Moreover $I(\bar{u}) = c > 0$.

Notice that the last part of this proof is identical when I is replaced by I_+. ■

We conclude this section by giving an existence result for an abstract system of the form

$$\begin{cases} Au = f(v), \\ Bv = g(u). \end{cases} \tag{3.10}$$

Proposition 3.2 *Let f and g in system (3.10) be admissible functions satisfying Condition 2.5. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let X, Y be two real Banach spaces.*
Suppose that

i. *X and Y are continuously imbedded in respectively $L_G(\Omega)$ and $L_F(\Omega)$;*

ii. *$A \in Isom(X; L_{F^*}(\Omega))$, $B \in Isom(Y; L_{G^*}(\Omega))$;*

iii. $\displaystyle\int_\Omega Au\, v\, d\mu = \int_\Omega u\, Bv\, d\mu$ *for all $u \in X$, $v \in Y$.*

If moreover

iv. $\dfrac{1}{\ell_f} + \dfrac{1}{\ell_g} < 1$ *and*

v. *X is compactly imbedded in $L_G(\Omega)$,*

then system (3.10) possesses at least one nontrivial solution $(u, v) \in X \times Y$.

If in addition

vi. *A is inverse positive: $0 \leq z \in L_{F^*}(\Omega)$ implies $0 \leq A^{-1}z$;*

then system (3.10) possesses at least one positive solution $(u,v) \in X \times Y$.

Proof. In view of Lemma 3.1 there exists $u \in X \setminus \{0\}$ such that

$$\int_\Omega f^{-1}(Au)\, A\hat{u}\, d\mu = \int_\Omega g(u)\, \hat{u}\, d\mu \text{ for all } \hat{u} \in X. \tag{3.11}$$

Set $v := f^{-1}(Au)$. Since $Au \in L_{F^*}(\Omega)$ it follows from Lemma A.4 that $v \in L_F(\Omega)$. Since $u \in X \subset L_G(\Omega)$ we have $g(u) \in L_{G^*}(\Omega)$ and hence $\tilde{v} := B^{-1}g(u) \in Y$ is well-defined. Therefore $v, \tilde{v} \in L_F(\Omega)$ and from the third condition in the proposition and (3.11) we have

$$\int_\Omega v\, A\hat{u}\, d\mu = \int_\Omega g(u)\, \hat{u}\, d\mu = \int_\Omega B\tilde{v}\, \hat{u}\, d\mu = \int_\Omega \tilde{v}\, A\hat{u}\, d\mu \text{ for all } \hat{u} \in X.$$

Since A is surjective we obtain $v = \tilde{v}$. Hence, $(u,v) \in X \times Y$ is a solution of (3.10).

In case that the operator A is inverse-positive we replace I_2 by $I_{2,+}$ and obtain $(u,v) \in X \times Y$ such that $Au = f(v)$ and $Bv = g(u^+)$. We are done provided that also B is also inverse-positive. For the sake of convenience we give a proof.

Let $v \in Y$ be such that $Bv \geq 0$. Then for every $z \in L_G(\Omega)$ with $z \geq 0$, we have $A^{-1}z \geq 0$ and

$$\int_\Omega v\, z\, d\mu = \int_\Omega v\, A\left(A^{-1}z\right)\, d\mu = \int_\Omega Bv\, A^{-1}z\, d\mu \geq 0.$$

Hence $v \geq 0$. ■

4 Elliptic regularity in Orlicz spaces

Let us consider

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with Ω a bounded domain in $\mathbb{R}^n$. Let $L_\Phi(\Omega)$ be the Orlicz space associated with the Lebesgue measure on Ω. If the Boyd indices $p_{L_\Phi(\Omega)}$ and $q_{L_\Phi(\Omega)}$ are such that $1 < p_{L_\Phi(\Omega)}$ and $q_{L_\Phi(\Omega)} < \infty$ one may use interpolation theory in order to show that the solution operator for (4.1) is an isomorphism from $L_\Phi(\Omega)$ into $W^{2,\Phi}(\Omega) \cap W_0^{1,\Phi}(\Omega)$.

Lemma 4.1 *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Let φ be admissible with $1 < \ell_\varphi^\infty$ and $m_\varphi^\infty < \infty$. Then for every $f \in L_\Phi$ there exists exactly one solution $w \in W^{2,\Phi}(\Omega) \cap W_0^{1,\Phi}(\Omega)$ of (4.1). Moreover, there exists a constant c, independent of f, such that*

$$\|w\|_{W^{2,\Phi}} \leq c\, \|f\|_{L_\Phi}.$$

Proof. Uniqueness. Let $u \in W^{2,\Phi}(\Omega) \cap W_0^{1,\Phi}(\Omega)$ be such that $\Delta u = 0$. Then by (A.5) and Definition A.2 $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for $q \in \left(1, \ell_\varphi^\infty\right)$. By standard results for elliptic p.d.e. ([12]) it follows that $u = 0$.

Existence. Since the Boyd indices are strictly between 1 and ∞ there are p and q with $1 < q < p < \infty$ such that $L^p(\Omega) \hookrightarrow L_\Phi(\Omega) \hookrightarrow L^q(\Omega)$. Moreover, Boyd's interpolation Theorem (see [15, part II, Theorem 2.b.11, page 145]) applied to $Z \in \mathcal{L}(L^q(\Omega))$ with $Z_{|L^p(\Omega)} \in \mathcal{L}(L^p(\Omega))$ yields that $Z_{|L_\Phi(\Omega)} \in \mathcal{L}(L_\Phi(\Omega))$. For $f \in L^q(\Omega)$ let the function $Kf := u$ denote the unique solution of $-\Delta u = f$ in $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. By elliptic regularity ([12]) one finds that $Z_\alpha := \mathcal{D}^\alpha K \in \mathcal{L}(L^q(\Omega))$ for $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$ and $\mathcal{D}^\alpha = \prod_{i=1}^n (\frac{\partial}{\partial x_i})^{\alpha_i}$ and also that $Z_{\alpha|L^p(\Omega)} \in \mathcal{L}(L^p(\Omega))$. So $Z_{\alpha|L_\Phi(\Omega)} \in \mathcal{L}(L_\Phi(\Omega))$ holds. It remains to show that $Z_{\alpha|L_\Phi(\Omega)} f = \mathcal{D}^\alpha (Kf)$ for $f \in L_\Phi(\Omega)$. Since $L_\Phi(\Omega) \hookrightarrow L^q(\Omega)$ one has $f \in L^q(\Omega)$ and $Z_{\alpha|L_\Phi(\Omega)} f = Z_\alpha f = \mathcal{D}^\alpha (Kf)$ as a weak derivative in $L^q(\Omega)$ and hence in $L_\Phi(\Omega)$. Therefore $Kf \in W^{2,\Phi}(\Omega) \cap W_0^{1,\Phi}(\Omega)$. The inequality follows from the boundedness of $\mathcal{D}^\alpha K$ in $L_\Phi(\Omega)$. ∎

Whenever we are proceeding with inequalities for the coefficients of the spaces involved it will be sufficient to proceed through imbeddings and we may not need maximal regularity. Then we could use $\tilde{\ell}_\varphi \geq p_{L_\Phi(\Omega)}$ and $\tilde{m}_\varphi \leq q_{L_\Phi(\Omega)}$.

Lemma 4.2 *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$ and let φ be an admissible function with*

$$1 < \tilde{\ell}_\varphi \leq \tilde{m}_\varphi < \infty. \tag{4.2}$$

Then for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that for every $f \in L_\Phi(\Omega)$ problem (4.1) has a solution $w \in W^{\tilde{\ell}_\varphi - \varepsilon,2}(\Omega) \cap W_0^{\tilde{\ell}_\varphi,1}(\Omega)$, and moreover

$$\|w\|_{W^{\tilde{\ell}_\varphi - \varepsilon,2}(\Omega)} \leq c_\varepsilon \|f\|_{L_\Phi}.$$

Proof. Since $L_\Phi(\Omega)$ is continuously imbedded in $L^{\tilde{\ell}_\varphi - \varepsilon}(\Omega)$ for any $\varepsilon > 0$ one finds $\|f\|_{L^{\tilde{\ell}_\varphi - \varepsilon}} \leq \tilde{c}_\varepsilon \|f\|_{L_\Phi}$. By regularity theory (see [12]) there is a unique solution $w \in W^{\tilde{\ell}_\varphi - \varepsilon,2}(\Omega) \cap W_0^{\tilde{\ell}_\varphi,1}(\Omega)$ with $\|w\|_{W^{\tilde{\ell}_\varphi - \varepsilon,2}} \leq \tilde{c} \|f\|_{L^{\tilde{\ell}_\varphi - \varepsilon}}$. ∎

5 Proof of the Theorem 2.6

Let $X = W^{2,F^*}(\Omega) \cap W_0^{1,F^*}(\Omega)$ and $Y = W^{2,G^*}(\Omega) \cap W_0^{1,G^*}(\Omega)$ supplied with the Lebesgue measure. See Definition A.1.

5.1 Existence through Proposition 3.2

We will verify the conditions of Proposition 3.2.

I. X is compactly imbedded in $L_G(\Omega)$. Indeed, this result follows from the assumption in the right hand side of (2.6), Lemma 2.4 ii. and Corollary D.4. By symmetry Y is compactly imbedded in $L_F(\Omega)$.

II. $(-\Delta)_0^{-1} : L_{F^*}(\Omega) \to X$ and $(-\Delta)_0^{-1} : L_{G^*}(\Omega) \to Y$ are well-defined and continuous. This result immediately follows from Lemma 4.1.

III. By the assumption in (2.6) and by Lemma 2.4 we find

$$\frac{1}{\tilde{m}_f} + \frac{1}{\tilde{m}_g} > 1 - \frac{2}{n} \text{ and } \frac{1}{\tilde{m}_f} + \frac{1}{\tilde{\ell}_{f^{-1}}} = 1.$$

Hence the conditions of Corollary D.4 are satisfied and there are $p \in (\tilde{m}_g, \infty)$ and $q \in (1, \tilde{\ell}_{f^{-1}})$ such that

$$W^{2,F^*}(\Omega) \subset W^{2,q}(\Omega) \Subset L^p(\Omega) \subset L_G(\Omega).$$

The following relation holds:

$$\frac{1}{\tilde{m}_g} > \frac{1}{p} > \frac{1}{q} - \frac{2}{n} > \frac{1}{\tilde{\ell}_{f^{-1}}} - \frac{2}{n}$$

which is, due to Lemma 2.4, equivalent to

$$\frac{1}{\tilde{\ell}_{g^{-1}}} - \frac{2}{n} < 1 - \frac{1}{p} - \frac{2}{n} < 1 - \frac{1}{q} < \frac{1}{\tilde{m}_f}.$$

Defining p^* and q^* by $\frac{1}{p} + \frac{1}{p^*} = 1$, respectively $\frac{1}{q} + \frac{1}{q^*} = 1$ we find

$$\frac{1}{\tilde{\ell}_{g^{-1}}} - \frac{2}{n} < \frac{1}{p^*} - \frac{2}{n} < \frac{1}{q^*} < \frac{1}{\tilde{m}_f}$$

and hence by Lemma D.2, respectively Rellich-Kondrachov (see [1, Theorem 6.2]), it follows that

$$W^{2,G^*}(\Omega) \subset W^{2,p^*}(\Omega) \Subset L^{q^*}(\Omega) \subset L_F(\Omega).$$

So for $u \in X$ we have $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \subset L^p(\Omega)$ and, by symmetry, for $v \in Y$ that $v \in W^{2,p^*}(\Omega) \cap W_0^{1,p^*}(\Omega) \subset L^{q^*}(\Omega)$. Hence the following integrals are well defined and the identity holds:

$$\int_\Omega (\Delta u)\, v \; dx = \int_\Omega u \,(\Delta v) \; dx.$$

Since $(-\Delta)_0^{-1} : L_{F^*}(\Omega) \to X$ and $(-\Delta)_0^{-1} : L_{G^*}(\Omega) \to Y$ are positive operators one even has $u \geq 0$ and $v \geq 0$.

This completes the verification that the conditions of Proposition 3.2 hold. We find that (1.1) has a positive solution $(u, v) \in X \times Y$.

5.2 Bootstrapping to regularity

Here is the result for one step in the bootstrapping argument in $L^p(\Omega)$ spaces.

Lemma 5.1 *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$ and let $(u, v) \in X \times Y$ be a solution of (1.1) with $u \in L^p(\Omega)$ and $p > \tilde{m}_g$.*

- *If* $\frac{p}{\tilde{m}_g} < \left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right)\frac{n}{2}$, *then $u \in L^{\hat{p}}(\Omega)$ for every $\hat{p}$ satisfying*

$$\hat{p} < \left(1 + \frac{\frac{1}{\tilde{m}_g} + \frac{1}{\tilde{m}_f} + \frac{2}{n}\frac{p}{\tilde{m}_g} - 1}{\left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right) - \frac{2}{n}\frac{p}{\tilde{m}_g}}\right) p.$$

- *If* $\frac{p}{\tilde{m}_g} = \left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right)\frac{n}{2}$, *then $u \in L^{\hat{p}}(\Omega)$ for every $\hat{p} \in (1, \infty)$.*
- *If* $\frac{p}{\tilde{m}_g} > \left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right)\frac{n}{2}$, *then $u \in C(\bar{\Omega})$.*

Remark 5.1.1 *Notice that whenever $p > \tilde{m}_g$ the assumption in (2.6) guarantees that*

$$\frac{1}{\tilde{m}_g} + \frac{1}{\tilde{m}_f} + \frac{2}{n}\frac{p}{\tilde{m}_g} - 1 > 0.$$

Proof. By Remark C.1.1 one finds that for any $\varepsilon > 0$

$$g(s) \leq C_G \; s^{-1} G(s) \leq C_{G,\varepsilon} \; s^{\tilde{m}_g - 1 + \varepsilon} \text{ for } s \text{ large enough.}$$

Hence, if $u \in L^p(\Omega)$ then one finds $g(u) \in L^{\tilde{p}}(\Omega)$ for any number $\tilde{p} \in (1, p/(\tilde{m}_g - 1))$. Standard regularity, see [12], implies that $v \in W^{2,\tilde{p}}(\Omega)$ and by Sobolev imbedding

$$\begin{array}{ll} v \in L^{\frac{n\tilde{p}}{n-2\tilde{p}}}(\Omega) & \text{if } \tilde{p} < \frac{1}{2}n, \\ v \in C(\bar{\Omega}) & \text{if } \tilde{p} > \frac{1}{2}n. \end{array}$$

Repeating similar steps for v if $p/(\tilde{m}_g - 1) \leq \frac{1}{2}n$ we find for any

$$\tilde{q} < \frac{np/(\tilde{m}_g - 1)}{n - 2p/(\tilde{m}_g - 1)} / (\tilde{m}_f - 1) = \frac{p}{\left(\tilde{m}_g - 1 - 2\frac{p}{n}\right)(\tilde{m}_f - 1)}$$

that $f(v) \in L^{\tilde{q}}(\Omega)$ and hence $u \in W^{2,\tilde{q}}(\Omega)$. By Sobolev imbedding

$$\begin{array}{ll} u \in L^{\frac{n\tilde{q}}{n-2\tilde{q}}}(\Omega) & \text{if } \tilde{q} < \frac{1}{2}n, \\ u \in C(\bar{\Omega}) & \text{if } \tilde{q} > \frac{1}{2}n. \end{array}$$

If $\frac{p}{\left(\tilde{m}_g - 1 - 2\frac{p}{n}\right)(\tilde{m}_f - 1)} < \frac{1}{2}n$, then $u \in L^{\hat{p}}(\Omega)$ for all $\hat{p}$ satisfying

$$\hat{p} < \frac{n \frac{p}{\left(\tilde{m}_g - 1 - 2\frac{p}{n}\right)(\tilde{m}_f - 1)}}{n - 2\frac{p}{\left(\tilde{m}_g - 1 - 2\frac{p}{n}\right)(\tilde{m}_f - 1)}} = \frac{1}{(\tilde{m}_g - 1)(\tilde{m}_f - 1) - 2\frac{p}{n}\tilde{m}_f} p$$

Notice that $\hat{p} > p$ whenever $(\tilde{m}_g - 1)(\tilde{m}_f - 1) - 2\frac{p}{n}\tilde{m}_f < 1$. This inequality is equivalent to

$$1 < \frac{1}{\tilde{m}_f} + \frac{1}{\tilde{m}_g} + \frac{2}{n}\frac{p}{\tilde{m}_g}.$$

■

Corollary 5.2 *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$ and let $(u, v) \in X \times Y$ be a solution of (1.1) with $u \in L^p(\Omega)$ and $p > \tilde{m}_g$. If*

$$\frac{1}{\tilde{m}_g} + \frac{1}{\tilde{m}_f} > 1 - \frac{2}{n}\frac{p}{\tilde{m}_g}, \tag{5.1}$$

then $u \in C_0(\bar{\Omega})$.

Proof. Since (5.1) holds and $p > \tilde{m}_g$ one obtains for $\frac{p}{\tilde{m}_g} < \left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right)\frac{n}{2}$ that

$$1 + \frac{\frac{1}{\tilde{m}_g} + \frac{1}{\tilde{m}_f} + \frac{2}{n}\frac{p}{\tilde{m}_g} - 1}{\left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right) - \frac{2}{n}\frac{p}{\tilde{m}_g}} > 1 + \frac{\frac{1}{\tilde{m}_g} + \frac{1}{\tilde{m}_f} + \frac{2}{n} - 1}{\left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right) - \frac{2}{n}} > 1$$

independently of p, and hence, after finitely many iterations using the first item in Lemma 5.1, one comes to the second or third item of this lemma. If $\frac{p}{\tilde{m}_g} = \left(1 - \frac{1}{\tilde{m}_g}\right)\left(1 - \frac{1}{\tilde{m}_f}\right)\frac{n}{2}$ holds then after a single step one arrives to the third item. This third item in Lemma 5.1 yields $u \in C(\bar{\Omega})$. Since $W_0^{1,p}(\Omega) \cap C(\bar{\Omega}) \subset C_0(\bar{\Omega})$ the conclusion follows. ■

Whenever one reaches an L^∞-bound one continues by standard arguments to find higher regularity. We have the following result.

Lemma 5.3 *Fix $\gamma \in (0,1)$. Let $f, g \in C^{\gamma}(\mathbb{R})$ and suppose that Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{2,\gamma}$. Let $(u,v) \in X \times Y$ be a solution of (1.1) with $u \in C_0(\bar{\Omega})$. Then $u \in C^{2,\gamma}(\bar{\Omega})$.*

Proof. If $u \in C(\bar{\Omega})$, then $g(u) \in C(\bar{\Omega}) \subset L^p(\Omega)$ for any $p \in (1,\infty)$, and one finds (see [12, Th. 9.15]) that $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $p \in (1,\infty)$. Taking $p > n$ the Sobolev imbedding gives $v \in C^1(\bar{\Omega})$ which implies that $f(v) \in C^{\gamma}(\bar{\Omega})$. Since also $\partial\Omega \in C^{2,\gamma}$ holds, regularity results (see [12, Th. 6.14]) yield $u \in C^{2,\gamma}(\bar{\Omega})$. ■

Now we may complete the proof of Theorem 2.6. In the previous section we found that there exist a positive nontrivial solution $(u,v) \in X \times Y$. Moreover by Corollary D.4 one finds that there are appropriate p and q such that $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \subset L^p(\Omega)$ with $p > \tilde{m}_g$. Next Corollary 5.2 implies that $u \in C_0(\bar{\Omega})$. Similarly $v \in C_0(\bar{\Omega})$ holds. With the additional assumption that $f, g \in C^{\gamma}(\mathbb{R})$ and $\partial\Omega \in C^{2,\gamma}$ for some $\gamma \in (0,1)$ one finds by Lemma 5.3 that (2.7) holds.

Appendices

A Orlicz space setting

Let us shortly recall the setup for Orlicz spaces. Every convex function $\Phi : \mathbb{R} \to \mathbb{R}_0^+$ with $\Phi(0) = 0$ can be represented by $\Phi(s) = \int_0^s \varphi(t)\,dt$ where $\varphi : \mathbb{R} \to \mathbb{R}$ is right continuous and nondecreasing (see [14, Theorem 1.1]). If $\varphi(0) = 0$, $\varphi(t) > 0$ or $t > 0$, φ nondecreasing and such that $\lim_{t\to\infty} \varphi(t) = \infty$ the function Φ is called an N-function (see also [1, Chapter VIII]). If we assume the somewhat stronger condition above that φ is admissible then φ^{-1} is admissible and Φ^* is also an N-function.

The Orlicz class $K_\Phi(\Omega)$ is defined by

$$K_\Phi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable; } \int_\Omega \Phi(u(x))\,dx < \infty \right\} \tag{A.1}$$

and the Orlicz spaces $E_\Phi(\Omega)$ and $L_\Phi(\Omega)$ by

$$E_\Phi(\Omega) = \text{the maximal linear subspace of } K_\Phi(\Omega); \tag{A.2}$$

$$L_\Phi(\Omega) = \text{the linear hull of } K_\Phi(\Omega). \tag{A.3}$$

The Luxemburg norm for $L_\Phi(\Omega)$ is defined by

$$\|u\|_{L_\Phi} = \inf \left\{ k > 0; \int_\Omega \Phi\left(\frac{u(x)}{k}\right) dx < 1 \right\}. \tag{A.4}$$

Assuming that Ω has a finite volume and that the $\triangle_2$-condition holds for large numbers one finds (see[1, page 240]) that

$$E_\Phi(\Omega) = K_\Phi(\Omega) = L_\Phi(\Omega).$$

Moreover, if Ω has finite volume, then $L_\Phi(\Omega)$ is reflexive if and only if the $\triangle_2$-condition for large numbers holds both for Φ and Φ^*. In that case the mapping $R : L_{\Phi^*}(\Omega) \to (L_\Phi(\Omega))'$, defined for $u \in L_{\Phi^*}(\Omega)$ by

$$(Ru)(v) = \int_\Omega u\,v\,d\mu, \text{ for every } v \in L_\Phi(\Omega),$$

is an isomorphism.

Higher order Sobolev-Orlicz spaces are defined as follows.

Definition A.1 *Let $k \in \mathbb{N}$ and $L_\Phi(\Omega)$ as above. Set*

$$\begin{aligned} W^{k,\Phi}(\Omega) &= \{u : \Omega \to \mathbb{R}\,; \mathcal{D}^\kappa u \in L_\Phi(\Omega) \text{ for all } \kappa \in \mathbb{N}^n \text{ with } |\kappa| \leq k\}\,, \\ \|u\|_{W^{k,\Phi}} &= \sum_{0 \leq |\kappa| \leq k} \|D^\kappa u\|_{L_\Phi}\,, \end{aligned}$$

where $\kappa \in \mathbb{N}^n$ is a multi-index and $|\kappa| = \sum_{i=1}^n \kappa_i$.

In order to define the Sobolev-Orlicz spaces of functions that vanish on the boundary we remark that the trace operator $T_p : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ with $1 \leq p < \infty$ is uniquely defined (as the only continuous operator with $T_p u = u_{|\partial\Omega}$ for $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$; see [11, Section 5.5]) and is such that $T_p u = T_1 u$ for all $u \in W^{1,p}(\Omega)$ with $p \in (1, \infty)$ whenever Ω is bounded and $\partial\Omega \in C^1$. Indeed, in that case $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$ and $T_p = T_1$ on $C^1(\bar{\Omega})$.

So we have

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega); T_1 u = 0\right\}, \tag{A.5}$$

and may define $W_0^{k,\Phi}(\Omega)$ in a similar way.

Definition A.2 *Assume that Ω is bounded and $\partial\Omega \in C^1$ (see [1, page 67]). We will define the Sobolev-Orlicz space of functions vanishing on the boundary by*

$$W_0^{1,\Phi}(\Omega) = \left\{u \in W^{1,\Phi}(\Omega); T_1 u = 0\right\}. \tag{A.6}$$

Remark A.2.1 *For Ω is bounded and $\partial\Omega \in C^1$ one may also define*

$$W_0^{k,\Phi}(\Omega) = \left\{u \in W^{k,\Phi}(\Omega); T_1 \mathcal{D}^\kappa u = 0 \text{ for all } \kappa \in \mathbb{N}^n \text{ with } |\kappa| \leq k-1\right\}.$$

Next we state a lemma relating convergence in the mean to convergence in norm.

Lemma A.3 *Let φ be an admissible function. Suppose that $\mu(\Omega) < \infty$ and that Φ satisfies the $\triangle_2$-condition for large numbers. Let $\{u_n\}_{n=1}^\infty$ in $L_\Phi(\Omega)$. Then the following are equivalent:*

- $\lim_{n\to\infty} \int_\Omega \Phi(u_n)\, d\mu = 0$;
- $\lim_{n\to\infty} \|u_n\|_{L_\Phi(\Omega)} = 0$.

Proof. ($\Rightarrow$) It is sufficient to show that a subsequence $\{u_{n_k}\}_{k=0}^\infty$ tends to 0 in norm. Since $\Phi(u_n)$ converges in $L^1(\Omega)$ there is a subsequence $\{u_{n_k}\}_{k=0}^\infty$ and a $g \in L^1(\Omega)$ such that

$$\begin{aligned} u_{n_k} &\to 0 \text{ in } \Omega \ \mu\text{-a.e.}, \\ |u_{n_k}| &\leq g \text{ in } \Omega \ \mu\text{-a.e.} \end{aligned}$$

Let $\varepsilon \in (0,1)$. Since Φ satisfies the $\triangle_2$-condition for large numbers there are $m, R > 0$ such that

$$\Phi(h\,t) \leq h^m \Phi(t) \text{ for all } t > R \text{ and } h \geq 1.$$

Let $\Omega_k := \{x \in \Omega; |u_{n_k}(x)| \leq R\}$. Then

$$\int_{\Omega\setminus\Omega_k} \Phi\left(\frac{u_{n_k}(x)}{\varepsilon}\right) d\mu \leq \varepsilon^{-m} \int_{\Omega\setminus\Omega_k} \Phi(u_{n_k}(x))\, d\mu \leq \varepsilon^{-m} \int_\Omega \Phi(u_{n_k}(x))\, d\mu \to 0 \text{ as } k \to \infty.$$

On the other hand $\mathbf{1}_{\Omega_k}(x)\,\Phi\left(\frac{u_{n_k}(x)}{\varepsilon}\right)\to 0$ in Ω μ-a.e. for $k\to\infty$ and

$$\left|\mathbf{1}_{\Omega_k}(x)\,\Phi\left(\frac{u_{n_k}(x)}{\varepsilon}\right)\right|\le\begin{cases}0 & \text{if } x\notin\Omega_k,\\ \Phi\left(\frac{R}{\varepsilon}\right) & \text{if } x\in\Omega_k,\end{cases}$$

which is integrable, imply that

$$\int_{\Omega_k}\Phi\left(\frac{u_{n_k}(x)}{\varepsilon}\right)d\mu\to 0 \text{ as } k\to\infty.$$

Therefore there exists $K\in\mathbb{N}$ such that for $k>K$

$$\int_{\Omega}\Phi\left(\frac{u_{n_k}(x)}{\varepsilon}\right)d\mu\le 1,$$

and hence $\|u_{n_k}\|_{L_\Phi(\Omega)}\le\varepsilon$ for $k>K$.

($\Leftarrow$) Since $\Phi(0)=0$ and Φ is convex $\Phi(s)\le\varepsilon\,\Phi\left(\frac{1}{\varepsilon}s\right)$ for $\varepsilon\in(0,1)$. Hence

$$\beta_n:=\inf\left\{\beta>0;\int_\Omega\Phi\left(\frac{u_n(x)}{\beta}\right)d\mu\le 1\right\}\to 0,$$

implies that for n large ($\beta_n\le\frac{1}{2}$)

$$\int_\Omega\Phi(u_n(x))\,d\mu\le 2\beta_n\int_\Omega\Phi\left(\frac{u_n(c)}{2\beta_n}\right)d\mu\le 2\beta_n\to 0.$$

■

Lemma A.4 *Let* (Ω) *be a finite measure space and let* φ *be an admissible function with* $m_\varphi^\infty<\infty$*. Then*

i. $(u\mapsto\Phi(u))\in C\left(L_\Phi(\Omega);L^1(\Omega)\right)$;

ii. $\varphi(u)\in L_{\Phi^*}(\Omega)$ *for every* $u\in L_\Phi(\Omega)$;

If moreover $\ell_\varphi^\infty>1$, *then*

iii. $(u\mapsto\varphi(u))\in C\left(L_\Phi(\Omega);L_{\Phi^*}(\Omega)\right)$.

Proof. First let us recall that $m_\varphi^\infty<\infty$ implies the $\triangle_2$-condition of Φ for large numbers. See Remark C.6.

i. Let $u_n,u\in L_\Phi(\Omega)$ be such that $u_n\to u$ in $L_\Phi(\Omega)$, in other words,

$$\beta_n:=\inf\left\{\beta>0;\int_\Omega\Phi\left(\frac{u_n-u}{\beta}\right)d\mu\le 1\right\}\to 0.$$

Since $m_\varphi^\infty<\infty$ it follows by Lemma C.3, interchanging the role of Φ and Φ^* and using $\beta_n=h^{-1}$, that (n large implies $2\beta_n<1$)

$$\int_\Omega\Phi(u_n-u)\,d\mu\le(2\beta_n)^\ell\int_\Omega\Phi\left(\frac{u_n-u}{2\beta_n}\right)d\mu\le(2\beta_n)^\ell\to 0.$$

Hence $\Phi(u_n - u) \to 0$ in $L^1(\Omega)$ and hence there is a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ and $g \in L^1(\Omega)$ such that

$$\Phi(u_{n_k}(x) - u(x)) \to 0 \qquad \text{for } x \in \Omega \ \mu\text{-a.e.,} \tag{A.7}$$
$$\Phi(u_{n_k}(x) - u(x)) \le g(x) \qquad \text{for } x \in \Omega \ \mu\text{-a.e.} \tag{A.8}$$

Since Φ^{-1} is continuous we find $u_{n_k}(x) \to u(x)$ for $x \in \Omega$ μ-a.e. and by the continuity of Φ hence $\Phi(u_{n_k}(x)) \to \Phi(u(x))$ for $x \in \Omega$ μ-a.e. By convexity and the $\triangle_2$-condition of Φ for large numbers for $x \in \Omega$ μ-a.e.

$$\begin{aligned}\Phi(u_{n_k}(x)) &= \Phi(u(x) + u_{n_k}(x) - u(x)) \le \tfrac{1}{2}\Phi(2u(x)) + \tfrac{1}{2}\Phi(2(u_{n_k}(x) - u(x))) \\ &\le C + c_1\Phi(u(x)) + c_1\Phi(u_{n_k}(x) - u(x)) \le C + c_1\Phi(u(x)) + c_1 g(x).\end{aligned}$$

The dominated convergence theorem implies:

$$\lim_{k\to\infty} \int \Phi(u_{n_k})\, d\mu = \int \Phi(u)\, d\mu.$$

Since this holds for any subsequence $\{u_{n_k}\}$ of $\{u_n\}$ one finds continuity of Φ.

ii. For the second claim notice that

$$\Phi^*(\varphi(s)) \le \Phi(s) + \Phi^*(\varphi(s)) = s\,\varphi(s) \le \int_s^{2s} \varphi(t)\, dt \le \Phi(2s),$$

and hence by the $\triangle_2$-condition for large numbers

$$\int_\Omega \Phi^*(\varphi(u))\, d\mu \le \int_\Omega \Phi(2u)\, d\mu \le C + c_1 \int_\Omega \Phi(u)\, d\mu,$$

implying $\int_\Omega \Phi^*(\varphi(u))\, d\mu < \infty$ and hence $\varphi(u) \in L_{\Phi^*}(\Omega)$.

iii. We proceed as in the proof of i. Let $\{u_{n_k}\}_{k=1}^{\infty}$ be a subsequence in $L_\Phi(\Omega)$ satisfying (A.7-A.8) we have $\varphi(u_{n_k}(x)) \to \varphi(u(x))$ for $x \in \Omega$ μ-a.e. and hence

$$\Phi^*(\varphi(u_{n_k}(x)) - \varphi(u(x))) \text{ for } x \in \Omega \ \mu\text{-a.e.}$$

Observe that $\varphi(u_{n_k}), \varphi(u) \in L_{\Phi^*}(\Omega)$ by ii. and hence since $\ell_\varphi^\infty > 1$ that $\varphi(u_{n_k}) - \varphi(u) \in L_{\Phi^*}(\Omega)$. Moreover we have by convexity and the $\triangle_2$-condition of Φ^* for large numbers (since $\ell_\varphi^\infty > 1$) that for $x \in \Omega$ μ-a.e.

$$\Phi^*(\varphi(u_{n_k}(x)) - \varphi(u(x))) \le C + c_1^*\Phi^*(\varphi(u_{n_k}(x))) + c_1^*\Phi^*(\varphi(u(x))).$$

The right hand side is integrable since $\mu(\Omega) < \infty$ and since $\Phi^*(\varphi(v)) \le \Phi(2v) \le C + c_1\Phi(v)$ for $v \in L_\Phi(\Omega)$. Notice that

$$\Phi^*(\varphi(u_{n_k}(x))) \le C + c_1\Phi(u_{n_k}(x)) \le C' + c_1' g(x) + c_1'\Phi(u(x)).$$

Again the dominated convergence theorem implies

$$\lim_{k\to\infty} \int \Phi^*(\varphi(u_{n_k}(x)) - \varphi(u(x)))\, d\mu = 0.$$

By Lemma A.3 we find

$$\lim_{k\to\infty} \|\varphi(u_{n_k}) - \varphi(u)\|_{L_{\Phi^*}(\Omega)} = 0.$$

■

We end this section by a differentiability result.

Lemma A.5 *Let φ be admissible and let $\mu(\Omega) < \infty$. Set*

$$\begin{aligned} I_\Phi(u) &:= \int_\Omega \Phi(u)\, d\mu \text{ for } u \in L_\Phi(\Omega), \\ I_{\Phi,+}(u) &:= \int_\Omega \Phi\left(u^+\right) d\mu \text{ for } u \in L_\Phi(\Omega). \end{aligned}$$

Then the following holds:

i. $I_\Phi, I_{\Phi,+} \in C\left(L_\Phi(\Omega); \mathbb{R}\right);$

ii. $I_\Phi, I_{\Phi,+}$ *are everywhere Gateaux-differentiable and*

$$\begin{aligned} I'_\Phi(u)(v) &= \int_\Omega \varphi(u)\, v\, d\mu \text{ for } u, v \in L_\Phi(\Omega), \\ I'_{\Phi,+}(u)(v) &= \int_\Omega \varphi\left(u^+\right) v\, d\mu \text{ for } u, v \in L_\Phi(\Omega). \end{aligned}$$

If moreover $\ell_\varphi^\infty > 1$, then

iii. $I_\Phi, I_{\Phi,+} \in C^1\left(L_\Phi(\Omega); \mathbb{R}\right).$

Proof of i. This follows from Lemma A.4.i for I_Φ. For $I_{\Phi,+}$ one uses that

$$\left\|u^+ - v^+\right\|_{L_\Phi(\Omega)} \le \|u - v\|_{L_\Phi(\Omega)}.$$

ii. Let $u, v \in L_\Phi(\Omega)$ and take $t \neq 0$ with $|t| \le 1$. Then

$$\frac{1}{t}\Big(I_\Phi(u + tv) - I_\Phi(u)\Big) - \int_\Omega \varphi(u)\, v\, d\mu = \int_\Omega \frac{1}{t} \int_0^{t\, v(x)} \Big(\varphi(u(x) + s) - \varphi(u(x))\Big)\, ds\, d\mu$$

and for all $x \in \Omega:$

$$\left|\frac{1}{t}\int_0^{t\, v(x)} \left(\varphi(u(x) + s) - \varphi(u(x))\right) ds\right| \le \Big(\varphi(|u(x)| + |v(x)|) + \varphi(|u(x)|)\Big)\, |v(x)|.$$

The right hand side belongs to $L^1(\Omega)$ and the left hand side tends to 0 for every $x \in \Omega$ which proves claim ii for I_Φ. For $I_{\Phi,+}$ one observes that

$$I_{\Phi,+}(u) = \int_\Omega \int_0^{u^+(x)} \varphi(s)\, ds\, d\mu = \int_\Omega \int_0^{u(x)} \varphi^+(s)\, ds\, d\mu,$$

where

$$\varphi^+(s) = \begin{cases} \varphi(s) & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases}$$

The proof is similar as for I_Φ.

iii. Since $(u \mapsto \varphi(u)), (u \mapsto \varphi^+(u)) \in C\left(L_\Phi(\Omega); L_{\Phi^*}(\Omega)\right)$ both $I_\Phi, I_{\Phi,+}$ are continuously Fréchet differentiable. ■

B A Mountain Pass Theorem

Proposition B.1 *(Ambrosetti, Rabinowitz, Ekeland) Let X be a real Banach space and $I \in C^1(X;\mathbb{R})$ with $I(0)=0$. Suppose that for some $r>0$:*

- *There is $\alpha > 0$ such that $\|u\|_X = r$ implies $I(u) \geq \alpha$.*
- *There is $e \in X$ such that $\|e\|_X > r$ and $I(e) \leq 0$.*

Let $\Gamma = \{\gamma \in C([0,1];X); \gamma(0)=0,\ \gamma(1)=e\}$ and set

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)).$$

Note that $c \geq \alpha$.
Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ in X'.

This particular version can be found in [9].

C A zoo of growth conditions

Let us recall the following condition for Φ.

Definition C.1 ($\triangle_2$-condition) *Suppose that $\Phi : \mathbb{R} \to \mathbb{R}_0^+$ is convex, even and such that $\Phi(0) = 0$. Then Φ is said to satisfy the $\triangle_2$-condition on $[R,\infty)$ if for some $C_{\Phi,R} > 0$ it holds that*

$$\Phi(2s) \leq C_{\Phi,R}\,\Phi(s) \text{ for all } s > R. \tag{C.1}$$

Remark C.1.1 *If φ is admissible and Φ is as in this definition then*

$$\Phi(s) \leq s\varphi(s) \leq C_{\Phi,R}\Phi(s) \text{ for } s > R.$$

Indeed $\Phi(s) \leq \Phi(s) + \Phi^*(\varphi(s)) = s\varphi(s)$ for all s, and $s\varphi(s) \leq \int_s^{2s}\varphi(t)dt \leq \Phi(2s)$ for all $s > R$.

The $\triangle_2$-condition for Φ is related with superhomogeneity of Φ^*. In fact both conditions give growth restrictions respectively from above and from below for Φ.

Definition C.2 (superhomogeneous) *Suppose that $\Phi : \mathbb{R} \to \mathbb{R}_0^+$ is convex, even and such that $\Phi(0) = 0$. Then Φ is said to be superhomogeneous of degree $\ell > 1$ on $[R,\infty)$ if it holds that*

$$\Phi(h\,s) \geq h^\ell\,\Phi(s) \text{ for all } h \in [1,\infty) \text{ and } s \in [R,\infty). \tag{C.2}$$

In the first section we defined $\ell_\varphi, \ell_\varphi^\infty, \tilde{\ell}_\varphi, \tilde{m}_\varphi, m_\varphi^\infty$ and m_φ which all represented some growth rate of the nonlinearity involved. A technical lemma that relates the different growth rates is the following.

Lemma C.3 *Suppose that Φ is an N-function with $\Phi(s) = \int_0^s \varphi(\sigma)\,d\sigma$ and with φ admissible and let Φ^* be the complementary function as in (2.1). Let $\ell \in (1,\infty)$ with ℓ^* defined by $\frac{1}{\ell} + \frac{1}{\ell^*} = 1$. and suppose that $R \in [0,\infty)$.*

- *Then the following four statements are equivalent:*

i. $\Phi(s) \leq \dfrac{s\,\phi(s)}{\ell}$ *for all* $s \in [R,\infty)$;

ii. $\Phi(h\,s) \geq h^{\ell}\,\Phi(s)$ *for all* $h \in [1,\infty)$ *and* $s \in [R,\infty)$;

iii. $\Phi^*(t) \geq \dfrac{t\,\phi^{-1}(t)}{\ell^*}$ *for all* $t \in [\varphi(R),\infty)$;

iv. $\Phi^*(h\,t) \leq h^{\ell^*}\,\Phi^*(t)$ *for all* $h \in [1,\infty)$ *and* $t \in [\varphi(R),\infty)$.

- *Moreover each of the above conditions implies that:*

v. Φ^* *satisfies the* $\triangle_2$*-condition on* $[\varphi(R),\infty)$ *with constant* $C_{\Phi^*} = 2^{\ell^*}$.

vi. for some $c_1 > 0$ *it holds that* $\varphi^{-1}(t) \leq c_1\,t^{\ell^*-1}$ *for* $t \geq \varphi(R)$.

vii. for some $c_2 > 0$ *it holds that* $\varphi(s) \geq c_2\,s^{\ell-1}$ *for* $s \geq R$.

- *Each of the above conditions is implied by:*

viii. Φ^* *satisfies the* $\triangle_2$*-condition on* $[\varphi(R),\infty)$ *with constant* $C_{\Phi^*} = \ell^* + 1$.

Remark C.3.1 *Results as above can be found in [14, Chapter I. §4].*

Proof. For the sake of easy reference we will give a complete proof.

i ⇒ ii. If $\Phi(s) \leq \dfrac{s\,\phi(s)}{\ell}$ for all $s \in [R,\infty)$, then for $h \in [1,\infty)$

$$\log(\Phi(hs)) - \log(\Phi(s)) = \int_s^{hs} \frac{\varphi(\sigma)}{\Phi(\sigma)}d\sigma \geq \int_s^{hs} \frac{\ell}{\sigma}d\sigma = \log h^{\ell}. \tag{C.3}$$

ii ⇒ *i.* If $\Phi(h\,s) - h^{\ell}\,\Phi(s) \geq 0$ for all $h \in [1,\infty)$ and $s \in [R,\infty)$ then

$$s\,\varphi(s) - \ell\,\Phi(s) = \frac{\partial}{\partial h}\Big(\Phi(h\,s) - h^{\ell}\,\Phi(s)\Big)_{h=1} \geq 0. \tag{C.4}$$

i ⇔ *iii.* With (2.2) one finds for $s \geq R$ that

$$\Phi^*(\varphi(s)) = s\,\varphi(s) - \Phi(s) \geq s\,\varphi(s) - \frac{s\,\phi(s)}{\ell} = \frac{s\,\phi(s)}{\ell^*}$$

and hence *iii* for $t \geq \varphi(R)$ and vice versa.

iii ⇔ *iv.* As before in (C.3) and (C.4) but now with reversed inequality signs:

$$\log(\Phi^*(ht)) - \log(\Phi^*(s)) = \int_t^{ht} \frac{\varphi^{-1}(\tau)}{\Phi^*(\tau)}d\tau \leq \int_t^{ht} \frac{\ell^*}{\tau}d\tau = \log h^{\ell^*},$$

and

$$t\,\varphi^{-1}(t) - \ell^*\,\Phi^*(t) = \frac{\partial}{\partial h}\Big(\Phi^*(h\,t) - h^{\ell^*}\,\Phi(t)\Big)_{h=1} \geq 0.$$

iv ⇒ *v.* This comes straightforwardly by using $h = 2$.

iii & iv ⇒ *vii.* One finds for $t \geq \varphi(R)$ that

$$\varphi^{-1}(t) \leq \frac{\ell^*\;\Phi^*(t)}{t} \leq \ell^*\left(\frac{t}{\varphi(R)}\right)^{\ell^*}\frac{\Phi^*(\varphi(R))}{t} = \frac{\ell^*\;\Phi^*(\varphi(R))}{\varphi(R)^{\ell^*}}t^{\ell^*-1}. \tag{C.5}$$

vii ⇒ *vi.* Taking $t = \varphi(s)$ the result follows from (C.5).

viii ⇒ *iii.* By assumption and since φ^{-1} is increasing one finds

$$\ell^*\,\Phi^*(t) \geq \Phi^*(2t) - \Phi^*(t) = \int_t^{2t} \varphi^{-1}(\tau)\,d\tau \geq t\,\varphi^{-1}(t).$$

■

Lemma C.4 *Suppose that φ is admissible and let $\tilde{\ell}_\varphi, \tilde{m}_\varphi, \tilde{\ell}_{\varphi^{-1}}, \tilde{m}_{\varphi^{-1}}$ be defined as in Definition 2.3. Then*

$$\frac{1}{\tilde{\ell}_\varphi} + \frac{1}{\tilde{m}_{\varphi^{-1}}} = 1 = \frac{1}{\tilde{\ell}_{\varphi^{-1}}} + \frac{1}{\tilde{m}_\varphi}. \tag{C.6}$$

Proof. Since φ is admissible Φ and Φ^* are strictly increasing on $[0,\infty)$ and are hence invertible. Since $t\,s \le \Phi(t) + \Phi^*(s)$ for all $s,t \ge 0$ we have

$$\Phi^{-1}(t)\ \Phi^{*,-1}(t) \le \Phi\left(\Phi^{-1}(t)\right) + \Phi^*\left(\Phi^{*,-1}(t)\right) = 2t \text{ for all } t \ge 0. \tag{C.7}$$

Since $\Phi(t) \le t\,\varphi(t)$ and $\Phi^*(s) \le s\,\varphi^{-1}(s)$ it follows that

$$\Phi^*\left(\frac{\Phi(t)}{t}\right) \le \frac{\Phi(t)}{t}\,\varphi^{-1}\left(\frac{\Phi(t)}{t}\right) \le \frac{\Phi(t)}{t}\,\varphi^{-1}(\varphi(t)) = \Phi(t) \text{ for all } t \ge 0$$

and hence, setting $s = \Phi(t)$:

$$\Phi^*\left(\frac{s}{\Phi^{-1}(s)}\right) \le \Phi\left(\Phi^{-1}(s)\right) = s \text{ for all } s \ge 0,$$

which implies

$$\frac{s}{\Phi^{-1}(s)} \le \Phi^{*,-1}(s) \text{ for all } s \ge 0. \tag{C.8}$$

Combining (C.8) and (C.7) one finds (see also [1, p. 230]) that

$$t \le \Phi^{-1}(t)\ \Phi^{*,-1}(t) \le 2t \text{ for all } t \ge 0.$$

Consequently $\log(t) \le \log\left(\Phi^{-1}(t)\ \Phi^{*,-1}(t)\right) \le \log(2t)$ and

$$1 \le \frac{\log\left(\Phi^{-1}(t)\right)}{\log(t)} + \frac{\log\left(\Phi^{*,-1}(t)\right)}{\log(t)} \le \frac{\log 2}{\log(t)} + 1 \text{ for all } t > 1.$$

Hence we obtain

$$\begin{aligned} \limsup_{t\to\infty} \frac{\log\left(\Phi^{-1}(t)\right)}{\log(t)} &= 1 - \liminf_{t\to\infty} \frac{\log\left(\Phi^{*,-1}(t)\right)}{\log(t)}, \\ \liminf_{t\to\infty} \frac{\log\left(\Phi^{-1}(t)\right)}{\log(t)} &= 1 - \limsup_{t\to\infty} \frac{\log\left(\Phi^{*,-1}(t)\right)}{\log(t)}, \end{aligned}$$

and since

$$\limsup_{t\to\infty} \frac{\log\left(\Phi^{-1}(t)\right)}{\log(t)} = \left(\liminf_{s\to\infty} \frac{\log(\Phi(s))}{\log(s)}\right)^{-1},$$

the claim follows. ■

Corollary C.5 *For an admissible φ it holds that*

i. $\dfrac{1}{m_\varphi} + \dfrac{1}{\ell_{\varphi^{-1}}} = 1, \qquad \dfrac{1}{m_\varphi^\infty} + \dfrac{1}{\ell_{\varphi^{-1}}^\infty} = 1$ *and* $\dfrac{1}{\tilde{m}_\varphi} + \dfrac{1}{\tilde{\ell}_{\varphi^{-1}}} = 1;$

ii. $\ell_\varphi \le \ell_\varphi^\infty \le \tilde{\ell}_\varphi \le \tilde{m}_\varphi \le m_\varphi^\infty \le m_\varphi.$

Proof. The first three identities follow by Lemma C.3 respectively Lemma C.4.

For the inequalities the only ones which are not immediate from the definition are $\tilde{m}_\varphi \leq m_\varphi^\infty$ and the dual $\ell_\varphi^\infty \leq \tilde{\ell}_\varphi$. Since for all $\varepsilon > 0$ and $t > t_1$ large enough

$$\log(\Phi(t)) - \log(\Phi(t_1)) = \int_{t_1}^{t} \frac{\varphi(s)}{\Phi(s)} ds \leq \int_{t_1}^{t} \frac{m_\varphi^\infty + \varepsilon}{s} ds \leq \left(m_\varphi^\infty + \varepsilon\right) \log(t)$$

one finds $\tilde{m}_\varphi = \limsup_{t\to\infty} \frac{\log(\Phi(t))}{\log t} \leq m_\varphi^\infty + \varepsilon$. Similarly $\ell_\varphi^\infty \leq \tilde{\ell}_\varphi$. ■

Lemma C.6 *Suppose φ is admissible.*

i. $m_\varphi^\infty < \infty$ if and only if Φ satisfies the $\triangle_2$-condition for large numbers.

ii. $\ell_\varphi^\infty > 1$ if and only if Φ^* satisfies the $\triangle_2$-condition for large numbers.

Proof. Lemma C.3 states that Φ satisfies the $\triangle_2$-condition for large numbers if and only if there is $m < \infty$ such that

$$t\,\varphi(t) < m\,\Phi(t) \text{ for large } t.$$

Moreover, if $m_\varphi^\infty < \infty$ then there is $R < \infty$ such that $t\,\varphi(t) < \left(m_\varphi^\infty + 1\right)\Phi(t)$ for $t > R$. Conversely, if $t\,\varphi(t) < m\,\Phi(t)$ for large t, then $m_\varphi^\infty \leq m < \infty$.
Similarly $\ell_\varphi^\infty > 1$ holds, which is equivalent to $m_{\varphi^{-1}}^\infty < \infty$, if and only if Φ^* satisfies the $\triangle_2$-condition for large numbers. ■

Lemma C.7 *Suppose that Φ is an N-function with $\Phi(s) = \int_{\sigma=0}^{s} \varphi(\sigma)\,d\sigma$ and with φ admissible and that Φ^* is the complementary function. If for some $\ell > 1$*

$$\Phi(s) \leq \frac{s\,\varphi(s)}{\ell} \text{ for all } s \in \mathbb{R},$$

then, with $\frac{1}{\ell} + \frac{1}{\ell^*} = 1$:

$$\int_\Omega \Phi(u(x))\,dx \leq \|u\|_{L_\Phi}^\ell \quad \text{for all } u \in L_\Phi(\Omega) \text{ with } \|u\|_{L_\Phi} < 1; \tag{C.9}$$

$$\int_\Omega \Phi(u(x))\,dx \geq \|u\|_{L_\Phi}^\ell \quad \text{for all } u \in L_\Phi(\Omega) \text{ with } \|u\|_{L_\Phi} > 1; \tag{C.10}$$

$$\int_\Omega \Phi^*(u(x))\,dx \geq \|u\|_{L_{\Phi^*}}^{\ell^*} \quad \text{for all } u \in L_{\Phi^*}(\Omega) \text{ with } \|u\|_{L_{\Phi^*}} < 1; \tag{C.11}$$

$$\int_\Omega \Phi^*(u(x))\,dx \leq \|u\|_{L_{\Phi^*}}^{\ell^*} \quad \text{for all } u \in L_{\Phi^*}(\Omega) \text{ with } \|u\|_{L_{\Phi^*}} > 1. \tag{C.12}$$

Proof. Assuming that $\|u\|_{L_\Phi} < 1$ we may take $\beta \in \left(\|u\|_{L_\Phi}, 1\right)$ and find that for any such β by Lemma C.3-*ii* respectively the definition of the Luxemburg-norm that

$$\int_\Omega \Phi(u(x))\,dx \leq \beta^\ell \int_\Omega \Phi\left(\frac{u(x)}{\beta}\right) dx \leq \beta^\ell.$$

The estimate in (C.9) follows letting $\beta \downarrow \|u\|_{L_\Phi}$.

Assuming that $\|u\|_{L_\Phi} > 1$ we may take $\beta \in \left(1, \|u\|_{L_\Phi}\right)$ and find that for any such β by Lemma C.3-*iv* respectively the definition of the Luxemburg-norm that

$$\int_\Omega \Phi\left(u\left(x\right)\right) dx \geq \beta^\ell \int_\Omega \Phi\left(\frac{u\left(x\right)}{\beta}\right) dx \geq \beta^\ell.$$

The estimate in (C.10) follows letting $\beta \uparrow \|u\|_{L_\Phi}$.

For (C.11), assuming that $\|u\|_{L_{\Phi^*}} < 1$, we take β such that $\beta \uparrow \|u\|_{L_{\Phi^*}}$. By Lemma C.3-*iv* for any $\beta \leq 1$

$$\int_\Omega \Phi^*\left(u\left(x\right)\right) dx \geq \beta^{\ell^*} \int_\Omega \Phi^*\left(\frac{u\left(x\right)}{\beta}\right) dx.$$

Since $\int_\Omega \Phi^*\left(\frac{u(x)}{\beta}\right) dx \geq 1$ for such β the estimate in (C.11) follows.

For (C.12), assuming that $\|u\|_{L_{\Phi^*}} > 1$, we again let $\beta \downarrow \|u\|_{L_{\Phi^*}}$ in order to find by Lemma C.3-*ii*

$$\int_\Omega \Phi^*\left(u\left(x\right)\right) dx \leq \beta^{\ell^*} \int_\Omega \Phi^*\left(\frac{u\left(x\right)}{\beta}\right) dx \leq \beta^{\ell^*}. \qquad \blacksquare$$

D Orlicz and L^p-spaces

Let Ω be a bounded domain in $\mathbb{R}^n$. The Boyd exponents for $L_\Phi(\Omega)$, defined in (2.4), have the following property. If for any $p \in \left[1, p_{L_\Phi(\Omega)}\right)$ and $q \in \left(q_{L_\Phi(\Omega)}, \infty\right)$ a linear operator $T : L^p(\Omega) + L^q(\Omega) \to L^p(\Omega) + L^q(\Omega)$ is such that both $T : L^p(\Omega) \to L^p(\Omega)$ and $T : L^q(\Omega) \to L^q(\Omega)$ are bounded, then so is $T : L_\Phi(\Omega) \to L_\Phi(\Omega)$. See [4] or [15, Th. 2.b.11].

Lemma D.1 *Let φ be admissible, let ℓ_φ, $\tilde{\ell}_\varphi$, m_φ and $\tilde{m}_\varphi$ be as in Definition 2.3 and suppose that Ω is bounded. Then*

$$\ell_\varphi^\infty \leq p_{L_\Phi(\Omega)} \leq \tilde{\ell}_\varphi \leq \tilde{m}_\varphi \leq q_{L_\Phi(\Omega)} \leq m_\varphi^\infty.$$

Proof. Let $m > m_\varphi^\infty$. Then there is $K_m \geq 1$ such that

$$\frac{t\varphi\left(t\right)}{\Phi\left(t\right)} \leq m \text{ for all } t \geq K_m.$$

By Lemma C.3 one finds

$$\Phi\left(h\,t\right) \leq h^m \Phi\left(t\right) \text{ for all } t \geq K_m,\ h \geq 1,$$

and hence

$$\sup_{h,t\geq 1} \frac{\Phi\left(t\,h\right)}{\Phi\left(t\right)h^m} = \max\left\{\sup_{\substack{h\geq 1\\ 1\leq t\leq K_m}} \frac{\Phi\left(t\,h\right)}{\Phi\left(t\right)h^m}, \sup_{\substack{h\geq 1\\ t\geq K_m}} \frac{\Phi\left(t\,h\right)}{\Phi\left(t\right)h^m}\right\} \leq$$

$$\leq \max\left\{\sup_{h\geq 1} \frac{\Phi\left(K_m\,h\right)}{\Phi\left(1\right)h^m}, 1\right\} = \max\left\{\frac{\Phi\left(K_m\right)}{\Phi\left(1\right)}, 1\right\} = \frac{\Phi\left(K_m\right)}{\Phi\left(1\right)}.$$

By definition $q_{L_\Phi(\Omega)} \leq m$ and since $m > m_\varphi^\infty$ is arbitrary, it follows that

$$q_{L_\Phi(\Omega)} \leq m_\varphi^\infty.$$

Next we take $q > q_{L_\Phi(\Omega)}$. Then there exists K_q such that

$$\sup_{h,t\geq 1} \frac{\Phi\,(t\,h)}{\Phi\,(t)\,h^q} \leq K_q$$

and hence $\Phi\,(t) \leq K_q \Phi\,(1)\,t^q$ for $t \geq 1$, which implies that

$$\frac{\log\,(\Phi\,(t))}{\log t} \leq \frac{\log\,(K_q \Phi\,(1))}{\log t} + q \text{ for } t \geq 2.$$

It follows that

$$\tilde{m}_\varphi = \limsup_{t\to\infty} \frac{\log\,(\Phi\,(t))}{\log t} \leq q,$$

and, again since $q > q_{L_\Phi(\Omega)}$ is arbitrary, $\tilde{m}_\varphi \leq q_{L_\Phi(\Omega)}$. For the other estimates one may proceed similarly. ■

Lemma D.2 *Let φ be admissible and let $\tilde{\ell}_\varphi$ and $\tilde{m}_\varphi$ be as in Definition 2.3. Suppose that $\mu(\Omega) < \infty$. Then for any $p \in \left(1, \tilde{\ell}_\varphi\right)$ and $q \in (\tilde{m}_\varphi, \infty)$ the following continuous imbeddings exist:*

$$L^q(\Omega) \subset L_\Phi(\Omega) \subset L^p(\Omega).$$

Remark D.2.1 *This result implies that $p_{L_\Phi(\Omega)} \leq \tilde{\ell}_\varphi$ and $\tilde{m}_\varphi \leq q_{L_\Phi(\Omega)}$ where $p_{L_\Phi(\Omega)}$ and $q_{L_\Phi(\Omega)}$ are the Boyd exponents for $L_\Phi(\Omega)$. Both inequalities can be strict (see [15, Prop. 2.b.3. and Remark 3 page 134]).*

Proof. By [1, Theorem 8.12] one finds that for $|\Omega| < \infty$ the imbedding $L_{\Phi_1}(\Omega) \subset L_{\Phi_2}(\Omega)$ holds if Φ_1 dominates Φ_2 near infinity, that is, for some $c, t_0 > 0$

$$\Phi_2\,(t) \leq \Phi_1\,(c\;t) \text{ for all } t \geq t_0.$$

Since $\liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log t} = \tilde{\ell}_\varphi$ implies that for any $\varepsilon > 0$ (take $\varepsilon = \tilde{\ell}_\varphi - p$) there is a number t_ε with

$$\Phi\,(t) \geq t^{\tilde{\ell}_\varphi - \varepsilon} \text{ for } t > t_\varepsilon,$$

one finds $L^p(\Omega) \subset L_\Phi(\Omega)$. A similar argument with reversed inequality signs yields $L^q(\Omega) \subset L_\Phi(\Omega)$. ■

Corollary D.3 *Let φ and ψ be admissible and let $1 < \tilde{\ell}_{\psi^{-1}}$ and $\tilde{m}_\varphi < \infty$ be as in Definition 2.3. Suppose that Ω has the cone property and $\mu(\Omega) < \infty$. If*

$$\frac{1}{\tilde{\ell}_{\psi^{-1}}} - \frac{1}{n} < \frac{1}{\tilde{m}_\varphi} \tag{D.1}$$

then there exists $p \in (\tilde{m}_\varphi, \infty)$ and $q \in \left(1, \tilde{\ell}_{\psi^{-1}}\right)$ such that the following continuous imbedding exists:

$$W^{1,\Psi^*}(\Omega) \subset W^{1,q}(\Omega) \Subset L^p(\Omega) \subset L_\Phi(\Omega). \tag{D.2}$$

The one denoted by $\Subset$ is compact.

Proof. Since the inequality is strict we may take $p \in (\tilde{m}_\varphi, \infty)$ and $q \in \left(1, \tilde{\ell}_{\psi^{-1}}\right)$ such that

$$\frac{1}{\tilde{m}_\varphi} > \frac{1}{p} > \frac{1}{q} - \frac{1}{n} > \frac{1}{\tilde{\ell}_{\psi^{-1}}} - \frac{1}{n}. \tag{D.3}$$

Since Ω is bounded Lemma D.2 implies the continuity of $W^{1,\Psi^*}(\Omega) \subset W^{1,q}(\Omega)$ and of $L^p(\Omega) \subset L_\Phi(\Omega)$. By (D.3) it follows that

$$\frac{nq}{n-q} = \frac{1}{\frac{1}{q} - \frac{1}{n}} > p$$

and we find by Rellich-Kondrachov (see [1, Theorem 6.2]) that $W^{1,q}(\Omega) \Subset L^p(\Omega)$. ■

Corollary D.4 *Let φ and ψ be admissible and let $1 < \tilde{\ell}_{\psi^{-1}}$ and $\tilde{m}_\varphi < \infty$ be as in Definition 2.3. Suppose that Ω has the cone property and that $\mu(\Omega) < \infty$. If*

$$\frac{1}{\tilde{\ell}_{\psi^{-1}}} - \frac{2}{n} < \frac{1}{\tilde{m}_\varphi} \tag{D.4}$$

then there exists $p \in (\tilde{m}_\varphi, \infty)$ and $q \in \left(1, \tilde{\ell}_{\psi^{-1}}\right)$ such that the following continuous imbedding exists:

$$W^{2,\Psi^*}(\Omega) \subset W^{2,q}(\Omega) \Subset L^p(\Omega) \subset L_\Phi(\Omega). \tag{D.5}$$

The one denoted by $\Subset$ is compact.

Proof. Similar as for the previous Corollary with $p \in (\tilde{m}_\varphi, \infty)$ and $q \in \left(1, \tilde{\ell}_{\psi^{-1}}\right)$ such that

$$\frac{1}{\tilde{m}_\varphi} > \frac{1}{p} > \frac{1}{q} - \frac{2}{n} > \frac{1}{\tilde{\ell}_{\psi^{-1}}} - \frac{2}{n}. \tag{D.6}$$

■

The last two corollaries are sharp when considering general $\tilde{\ell}_\psi$ and $\tilde{m}_\varphi$. However, for some specific N-functions Φ and Ψ the imbeddings before could be compact even with an equality sign in (D.1) or (D.4).

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