Estimates for the expected lifetime of conditioned Brownian motion

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Abstract
Let $\tau_\Omega$ denote the lifetime of Brownian motion in an open connected set $\Omega \subset \mathbb{R}^m$. We obtain the asymptotic behaviour of the expected lifetime $\mathbb{E}_x[\tau_\Omega]$ as $y \to x$, where the Brownian motion is conditioned to start at $x$ and to exit $\Omega \setminus \{y\}$ at $\{y\}$.

1 Introduction

Let $\Omega$ be an open and connected set in Euclidean space $\mathbb{R}^m$ with $m \geq 2$ and let $\Delta$ be the Dirichlet Laplacian for $\Omega$ acting in $L^2(\Omega)$. Let $p_\Omega(x, y; t)$ denote the Dirichlet heat kernel on $\Omega \times \Omega \times (0, \infty)$ associated to the parabolic operator $-\Delta + \frac{\partial}{\partial t}$. It is well known that the resolvent of $-\Delta$ has an integral kernel $G(x, y)$ on $\Omega \times \Omega$ with $x \neq y$ given by

$$G(x, y) = \int_0^\infty p_\Omega(x, y; t)dt,$$  \hspace{1cm} (1.1)

whenever the integral in the right hand side of (1.1) converges. This is always the case if $m \geq 3$ since by monotonicity of the Dirichlet heat kernel

$$0 < p_\Omega(x, y; t) \leq p(x, y; t),$$ \hspace{1cm} (1.2)

where

$$p(x, y; t) = \frac{1}{(4\pi t)^{m/2}} e^{-|x-y|^2/(4t)}.$$ \hspace{1cm} (1.3)

Hence we have that for $m \geq 3$

$$0 < G(x, y) \leq G(x, y) = \frac{\Gamma(\frac{1}{2}m - 1)}{4\pi^{m/2}} |x-y|^{2-m}.$$ \hspace{1cm} (1.4)

For $\Omega$ in $\mathbb{R}^2$ we assume that $\mathbb{R}^2 \setminus \Omega$ contains a compact set with strictly positive logarithmic capacity. This assumption is both necessary and sufficient for the integral in (1.1) to converge off the diagonal (see Section 7.4 in [12]).

The main subject of this paper is the function $H_\Omega: \Omega \times \Omega, x \neq y \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$H_\Omega(x, y) = \int_\Omega \frac{G_\Omega(x, z)G_\Omega(z, y)}{G_\Omega(x, y)}dz.$$ \hspace{1cm} (1.5)

It follows from the positivity and symmetry of $G_\Omega$ that $H_\Omega$ is positive and symmetric. Conditions for $H_\Omega$ to be finite are given in Theorem 2.1. The function $H_\Omega$ shows up in analysis when studying positivity preserving properties of systems of second order elliptic partial differential equations [13, 15, 11]. In probability, $H_\Omega(x, y)$ for $\Omega \subset \mathbb{R}^m$ with $m \geq 2$ and $y \in \Omega$ is the expectation of the lifetime $\tau_\Omega$ of Brownian motion $\{X(t)\}_{t \geq 0}$ killed on exiting $\Omega$, starting in $x$, and conditioned to exit $\Omega \setminus \{y\}$ at $\{y\}$:

$$H_\Omega(x, y) = \mathbb{E}_x^\Omega [\tau_\Omega| \tau_\Omega(B_{\varepsilon}(y)) \in \partial B_{\varepsilon}(y)],$$ \hspace{1cm} (1.6)

where $B_{\varepsilon}(y) = \{x \in \mathbb{R}^m; |x-y| < \varepsilon\}$ [8, 9]. Note that $\{X(t)\}_{t \geq 0}$ and its transition density (1.3) are associated to $-\Delta + \frac{\partial}{\partial t}$. Many authors have investigated the behaviour of $H_\Omega$ in terms of the geometry of $\Omega$. See [5–8,11]. It is well known that $H_\Omega$ is continuous on $\Omega \times \Omega$ whenever $H_\Omega$ is finite. Moreover, if $m \geq 2$ then $\lim_{y \to x} H_\Omega(x, y) = 0$ for any $x \in \Omega$.
The main results of this paper concern the behaviour of $H_\Omega$ near this minimum and are stated in Theorem 2.2. The proof of Theorem 2.2 is deferred to Section 3. The function $H_\Omega(x, y)$ can be extended to $\Omega \times \Omega$ (or $\Omega \times \overline{\Omega}$) for open and connected sets $\Omega$ with $C^2$ boundary. Estimates for $H_\Omega(x, y)$ with $x \in \partial \Omega$ are given in Theorem 2.4. In Section 4 we give a sketch of the proof of Theorem 2.4.

2 Main results

In Theorem 2.1 we will give a sufficient condition for an open and connected set $\Omega$ in $\mathbb{R}^m$, $m \geq 2$ to have $H_\Omega < \infty$. It is possible to weaken the hypothesis if $m = 2, 3, 4$. However, this will not change the asymptotic formulae in Theorems 2.2 and 2.4 respectively.

**Theorem 2.1.** Let $\Omega$ be an open and connected set in $\mathbb{R}^m$. If either $m \geq 5$ or $m = 2, 3, 4$ and

$$\lambda := \inf \text{spec}(-\Delta) > 0$$

then $H_\Omega(x, y) < \infty$.

**Proof.** Since $\Omega$ is connected we have for $x, y \in \Omega$ that $p_\Omega(x, y; t) > 0$ for all $t$. Hence $G_\Omega(x, y) > 0$. By the semigroup property for heat kernels and Cauchy-Schwarz’s inequality

$$p_\Omega(x, y; t) = \int_{\Omega} dz p_\Omega(x, z; t/2)p_\Omega(z, y; t/2)
\leq \left( \int_{\Omega} dz p_\Omega(x, z; t/2)^2 \right)^{1/2} \left( \int_{\Omega} dz p_\Omega(z, y; t/2)^2 \right)^{1/2}
= \left( p_\Omega(x, x; t) p_\Omega(y, y; t) \right)^{1/2}.
$$

By (1.2), (1.3), (2.2) and (2.3)

$$p_\Omega(x, y; t) \leq e^{-t\lambda/2} p_\Omega(x, x; t/2).$$

The assumption $\lambda > 0$ for $m = 2$ implies our hypothesis that if $\Omega \subseteq \mathbb{R}^2$, then $\mathbb{R}^2 \setminus \Omega$ has positive logarithmic capacity. A qualitative upper bound follows by (1.1) and (2.4). By Fubini’s theorem and the semigroup property of heat
kernels
\[ \int_{\Omega} G_\Omega(x, z)G_\Omega(z, y) \, dz \]
\[ = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_{\Omega} p_\Omega(x, z; t_1) p_\Omega(z, y; t_2) \, dz \]
\[ = \int_0^\infty dt_1 \int_0^\infty dt_2 p_\Omega(x, y; t_1 + t_2) = \int_0^\infty t p_\Omega(x, y; t) \, dt. \tag{2.5} \]

Theorem 2.1 follows by (2.4) and (2.5).

The precise asymptotic behaviour of \( H_\Omega(x, y) \) for \( y \) near an interior point \( x \) is given in the following.

Theorem 2.2. Let \( \Omega \) be an open and connected set in \( \mathbb{R}^m \), and let \( x \in \Omega \).

(i) If \( m \geq 5 \) then for \( y \to x \)
\[ H_\Omega(x, y) = \frac{1}{2(m-4)} |y - x|^2 + O(|y - x|^{m-2}). \tag{2.6} \]

(ii) If \( m = 4 \) and \( \lambda > 0 \) then for \( y \to x \)
\[ H_\Omega(x, y) = \frac{1}{2} |y - x|^2 \log \frac{1}{|y - x|} + O(|y - x|^2). \tag{2.7} \]

(iii) If \( m = 3 \) and \( \lambda > 0 \) then
\[ \int_0^\infty t p_\Omega(x, x; t) \, dt < \infty, \tag{2.8} \]
and for \( y \to x \)
\[ H_\Omega(x, y) = 4\pi \left( \int_0^\infty t p_\Omega(x, x; t) \, dt \right) |y - x| + o(|y - x|). \tag{2.9} \]

(iv) If \( m = 2 \) and \( \lambda > 0 \) then (2.8) holds, and for \( y \to x \)
\[ H_\Omega(x, y) = 2\pi \left( \int_0^\infty t p_\Omega(x, x; t) \, dt \right) \frac{-1}{\log |y - x|} + o\left( \frac{1}{\log |y - x|} \right). \tag{2.10} \]

Remark 2.3. It follows from the proof of Theorem 2.2 that the remainder estimates in (2.6) and (2.7) are uniform on compact subsets of \( \Omega \).

The next result concerns the asymptotic behaviour of \( H_\Omega \) when one of the points lies on the boundary. It is well known that if \( \partial \Omega \) is of class \( C^2 \) and if \( x_0 \in \partial \Omega \) then
\[ \tilde{H}_\Omega(x_0, y) := \lim_{x \to x_0} H_\Omega(x, y) \tag{2.11} \]
exists and is non-trivial. Indeed, one has $G_\Omega(x_0, y) = 0$ and $\frac{\partial}{\partial n(x)} G_\Omega(x_0, y) = K_\Omega(x_0, y)$, where $K_\Omega(x_0, y)$ is the Poisson kernel at $x_0 \in \partial \Omega$, and $n(x)$ is the inward pointing unit normal vector at $x$. It follows that

$$\tilde{H}_\Omega(x_0, y) = \int_\Omega \frac{K_\Omega(x_0, z) G_\Omega(z, y)}{K_\Omega(x_0, y)} \, dz. \tag{2.12}$$

For general domains the asymptotic behaviour of $\lim_{x \to x_0} H_\Omega(x, y)$ as $y \to x_0$ will depend on the particular subsequence. We have the following in the case where $y - x_0$ is perpendicular to the tangent plane at $x_0$.

**Theorem 2.4.** Let $\Omega$ be an open and connected set in $\mathbb{R}^m$ with $\partial \Omega$ of class $C^2$. Suppose that $\lambda > 0$ for $m = 2, 3, 4$. Then $\tilde{H}_\Omega(x_0, y)$ in (2.11) exists for any $x_0 \in \partial \Omega$. For $m \geq 3$ and $\eta \to 0$

$$\tilde{H}_\Omega(x_0, x_0 + n(x_0)\eta) = \frac{1}{2m-4}\eta^2(1 + o(1)), \tag{2.13}$$

and for $m = 2$ and $\eta \to 0$

$$\tilde{H}_\Omega(x_0, x_0 + n(x_0)\eta) = \frac{1}{2}\eta^2(\log \eta^{-1})(1 + o(1)). \tag{2.14}$$

In the above we have always assumed that $m > 1$. However, $H(x, y)$ is well defined for an open interval in $\mathbb{R}$. A direct computation yields for $\Omega = (0, 1)$ that

$$G_\Omega(x, y) = (x \wedge y) - xy, \tag{2.15}$$

$$\int_\Omega G_\Omega(x, z) G_\Omega(z, y) \, dz = \frac{1}{2}xy(1-x)(1-y) - \frac{1}{6}(x-y)^2 ((x \wedge y) - xy), \tag{2.16}$$

and

$$H_\Omega(x, y) = \frac{1}{4} (x \lor y) - \frac{1}{6} x^2 - \frac{1}{6} y^2. \tag{2.17}$$

The probabilistic interpretation of $H_\Omega$ is different from the one given for $m > 1$ since one dimensional Brownian motion has a positive probability of hitting any points of $\Omega$. The exit time should be replaced by the quitting or last exit time $\gamma_{\{y\}}$ as defined by Chung in [3, page 209].

### 3 Proof of Theorem 2.2

The proof of Theorem 2.2 is based on some estimates for the Dirichlet heat kernel, which in turn imply precise estimates for the Green function.
Lemma 3.1. Let $\Omega$ an open and connected set in $\mathbb{R}^m$ ($m \geq 1$). Then for $x, y \in \Omega$ and $t > 0$

$$p(x, y; t) \geq p_\Omega(x, y; t) \geq p(x, y; t) - \frac{2m}{(4\pi t)^{m/2}} e^{-c^2(\delta(x)\vee\delta(y))^2/(4t)},$$  \hspace{1cm} (3.1)$$

where $\delta$ is the distance to the boundary

$$\delta(x) = \inf \{|x - y| : y \in \mathbb{R}^m \setminus \Omega\},$$ \hspace{1cm} (3.2)$$

and

$$c = \left(2\sqrt{2} - 2\right) m^{-1/2}.$$ \hspace{1cm} (3.3)$$

Proof. The heat kernel estimates obtained in [1, Theorem 1] for $-\frac{1}{2} \Delta + \frac{\partial}{\partial t}$, yield (3.1), by scaling.

Lemma 3.2. Let $\Omega$ be an open and connected set in $\mathbb{R}^m$, $m \geq 3$. Then for $x, y \in \Omega$

$$G(x, y) \geq G_\Omega(x, y) \geq G(x, y) - m \Gamma \left(\frac{1}{2} m - 1\right) \frac{16}{2\pi^{m/2}} c^{2-m} (\delta(x) \vee \delta(y))^{2-m}.$$ \hspace{1cm} (3.4)$$

Proof. Integrate inequality (3.1) with respect to $t$ over $[0, \infty)$. \hfill \Box

Lemma 3.3. Let $\Omega$ be an open and connected set in $\mathbb{R}^m$, $m \geq 5$. Then for $x, y \in \Omega$

$$\left| \int_0^\infty t p_\Omega(x, y; t) dt - \frac{\Gamma \left(\frac{1}{2} m - 2\right)}{16 \pi^{m/2}} |x - y|^{4-m} \right| \leq \frac{m \Gamma \left(\frac{1}{2} m - 2\right)}{8 \pi^{m/2}} c^{4-m} (\delta(x) \vee \delta(y))^{4-m}.$$ \hspace{1cm} (3.5)$$

Proof. Multiply inequality (3.1) by $t$ and integrate the resulting inequality with respect to $t$ over $[0, \infty)$. \hfill \Box

Proof of Theorem 2.2. (i) The proof of Theorem 2.2 for $m \geq 5$ follows directly from (1.4), (2.5), Lemma 3.2 and Lemma 3.3.

(ii) The proof of Theorem 2.2 for $m = 4$ is more delicate. Let $T = \lambda^{-1}$. By (2.4), we have that for $x, y \in \Omega$

$$\int_T^\infty dt t p_\Omega(x, y; t) \leq \int_T^\infty dt t e^{-t\lambda/2}(2\pi t)^{-2} \leq 1.$$ \hspace{1cm} (3.6)$$

Moreover for all $x, y \in \Omega$ with $|x - y|^2 \leq 4T$ we have that
\[ \int_0^T dt \int_{\Omega} p_\Omega(x,y,t) \leq \int_0^T dt \int_{\Omega} p(x,y,t) \]
\[ = \frac{1}{16\pi^2} \int_{|x-y|^2/(4T)}^\infty ds \frac{1}{s-1} e^{-s} \]
\[ \leq \frac{1}{16\pi^2} \left( \int_1^{\infty} ds \frac{1}{s-1} + \int_1^{\infty} ds e^{-s} \right) \]
\[ \leq \frac{1}{16\pi^2} \log \left( \frac{4T}{|x-y|^2} \right) + 1. \]  
(3.7)

By (3.6–3.7) we have for \( x, y \in \Omega \) with \(|x-y|^2 \leq 4T\)

\[ \int_0^\infty dt \int_{\Omega} p_\Omega(x,y,t) \leq \int_0^\infty dt \int_{\Omega} p(x,y,t) \]
\[ \leq \frac{1}{16\pi^2} \log \left( \frac{4T}{|x-y|^2} \right) + 2. \]  
(3.8)

On the other hand we have by Lemma 3.1 that for \( m = 4 \) and \(|x-y|^2 \leq 4T\)

\[ \int_0^\infty dt \int_{\Omega} p_\Omega(x,y,t) \geq \int_0^T dt \int_{\Omega} p_\Omega(x,y,t) \]
\[ \geq \frac{1}{16\pi^2} \int_{|x-y|^2/(4T)}^1 ds \frac{1}{s-1} e^{-s} - \frac{1}{2\pi^2} \int_{\epsilon^2(\delta(x) \lor \delta(y))^2/(4T)}^\infty ds \frac{1}{s-1} e^{-s} \]
\[ \geq \frac{1}{16\pi^2} \log \left( \frac{4T}{|x-y|^2} \right) - 1 - Te^{-2(\delta(x) \lor \delta(y))^2}. \]  
(3.9)

Combining inequalities (3.8), (3.9), (3.4) with (2.5) and the expression for \( H_\Omega \) we arrive at the conclusion of Theorem 2.2 (ii).

(iii) To prove Theorem 2.2 for \( m = 3 \) we first note that by (2.3) and (1.2)

\[ \int_0^\infty dt \int_{\Omega} p_\Omega(x,x,t) \leq \int_0^\infty dt \int_{\Omega} p(x,x,1/2t) \]
\[ = \frac{1}{(2\pi)^{3/2}} \int_0^\infty dt \frac{1}{t^{1/2}} e^{-t\lambda/2} \leq \lambda^{-1/2}. \]  
(3.10)

This proves (2.8). To prove (2.9) we note that \( y \to p_\Omega(x,y,t) \) is continuous and

\[ \int_0^\infty dt \left| p_\Omega(x,y,t) - p_\Omega(x,x,t) \right| \]
\[ \leq \int_0^\infty dt \left( p_\Omega(x,x,t)p_\Omega(y,y,t) \right)^{1/2} + \int_0^\infty dt \int_{\Omega} p_\Omega(x,y,t) \]
\[ \leq 2\lambda^{-1/2}, \]  
(3.11)
by the estimate in (3.10). Hence by Lebesgue’s dominated convergence theorem we have for \( y \rightarrow x \)

\[
\int_0^\infty dt \, t^p \Omega(x, y; t) = \int_0^\infty dt \, p \Omega(x, x; t) + o(|x - y|).
\] (3.12)

The proof of (2.9) follows directly from (3.12) and (3.4).

(iv) Finally to prove Theorem 2.2 for \( m = 2 \) we first note that by (2.4)

\[
\int_0^\infty dt \, t^p \Omega(x, x; t) \leq \frac{1}{2\pi} \int_0^\infty dt \, e^{-t\lambda/2} \leq \lambda^{-1}.
\] (3.13)

This proves that (2.8) holds for \( m = 2 \). To prove (2.10) we note that

\[
\int_0^\infty dt \, |p \Omega(x, y; t) - p \Omega(x, x; t)| \\
\leq \int_0^\infty dt \left( p \Omega(x, x; t)p \Omega(y, y; t) \right)^{1/2} + \int_0^\infty dt \, p \Omega(x, x; t) \\
\leq 2 \int_0^\infty dt \, e^{-t\lambda/2}(2\pi t)^{-1} \leq 2\lambda^{-1}.
\] (3.14)

Hence by Lebesgue’s dominated convergence theorem we have that (3.12) also holds for \( m = 2 \) and \( y \rightarrow x \). It remains to find the asymptotic behaviour of \( G_\Omega(x, y) \) as \( y \rightarrow x \). By Lemma 3.1 we have for \(|x - y|^2 \leq 4T\)

\[
G_\Omega(x, y) = \int_0^\infty dt \, p \Omega(x, y; t) \\
\geq \int_0^T dt (4\pi t)^{-1} \left( e^{-|x-y|^2/(4t)} - 4e^{-c^2(\delta(x) \lor \delta(y))^2/(4t)} \right) \\
\geq \frac{1}{4\pi} \log \left( \frac{4T}{|x - y|^2} \right) - 1 - 4T/(c^2(\delta(x) \lor \delta(y))^2). \] (3.15)

Secondly

\[
G_\Omega(x, y) \leq \int_0^T dt \, \frac{1}{4\pi} e^{-|x-y|^2/(4t)} + \int_T^\infty dt \left( p \Omega(x, x; t)p \Omega(y, y; t) \right)^{1/2} \\
\leq \frac{1}{4\pi} \log \left( \frac{4T}{|x - y|^2} \right) + 3. \] (3.16)

This concludes the proof of Theorem 2.2 (iv) by (3.12), (3.15) and (3.16).

4 Sketch of the proof of Theorem 2.4

The main idea in the proof of Theorem 2.4 is to approximate the domain by a half space. The cases \( m = 2 \) and \( m = 3 \) will be considered in Lemmas 4.2 and 4.1 respectively.
Lemma 4.1. Let \( \Omega_+ \subseteq \mathbb{R}^m \) be given by
\[
\Omega_+ = \{(x_1, \ldots, x_m) : x_1 > 0\}.
\] (4.1)

Then for \( m \geq 3 \) and \( x_1 > 0 \)
\[
\lim_{x_1 \to 0} H_{\Omega_+}((x_1, 0, \ldots, 0), (y_1, 0, \ldots, 0)) = \frac{1}{2m-4} y_1^2.
\] (4.2)

Proof. By the reflection principle
\[
p_{\Omega_+}((x_1, 0, \ldots, 0), (y_1, 0, \ldots, 0); t) = \frac{e^{-(y_1-x_1)^2/(4t)} - e^{-(y_1+x_1)^2/(4t)}}{(4\pi t)^{m/2}}.
\] (4.3)

Hence
\[
\lim_{x_1 \to 0} \frac{\partial}{\partial x_1} \int_0^\infty dt \; t \; p_{\Omega_+}((x_1, 0, \ldots, 0), (y_1, 0, \ldots, 0); t) = \frac{\Gamma(\frac{1}{2} + m - 1)}{4\pi^{m/2}} y_1^{3-m}.
\] (4.4)

Moreover, by (1.1) and (4.3)
\[
G_{\Omega_+}((x_1, 0, \ldots, 0), (y_1, 0, \ldots, 0)) = \frac{\Gamma(\frac{1}{2} + m - 1)}{4\pi^{m/2}} \left((y_1 - x_1)^{2-m} - |y_1 + x_1|^{2-m}\right).
\] (4.5)

Hence
\[
\lim_{x_1 \to 0} \frac{\partial}{\partial x_1} G_{\Omega_+}((x_1, 0, \ldots, 0), (y_1, 0, \ldots, 0)) = \frac{(m-2)\Gamma(\frac{1}{2} + m - 1)}{2\pi^{m/2}} y_1^{1-m},
\] (4.6)

and Lemma 4.1 follows by L’ Hospital’s rule with (4.4) and (4.6).

\[ \square \]

Lemma 4.2. Let \( m = 2 \) and let \( \Omega_+ \) be given by (4.1). Then for \( y_1 \to 0 \) and \( T > 0 \)
\[
\lim_{x_1 \to 0} \frac{\partial}{\partial x_1} \int_0^T dt \; t \; p_{\Omega_+}((x_1, 0), (y_1, 0); t) = \frac{1}{2\pi} y_1 \left(\log \frac{1}{y_1}\right)(1 + o(1)),
\] (4.7)

and
\[
\lim_{x_1 \to 0} \frac{\partial}{\partial x_1} G_{\Omega_+}((x_1, 0), (y_1, 0)) = \frac{1}{\pi} y_1^{-1}.
\] (4.8)

Proof. By (1.1) and (4.3)
\[
G_{\Omega_+}((x_1, 0), (y_1, 0)) = \int_0^\infty (4\pi t)^{-1}(e^{-(y_1-x_1)^2/(4t)} - e^{-(y_1+x_1)^2/(4t)}) dt
\]
\[
= \frac{1}{2\pi} \log \left(\frac{y_1 + x_1}{y_1 - x_1}\right),
\] (4.9)

and (4.8) follows from (4.9).

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To prove (4.7) we note that we may change the order of differentiation and limit with the integral. Hence the left hand side of (4.7) equals

$$\frac{1}{4\pi} y_1 \int_0^T dt \, t^{-1} e^{-y_1^2/(4t)} = \frac{1}{4\pi} y_1 \left( \log \frac{1}{y_1} \right) (1 + o(1)),$$

(4.10) as \( y_1 \to 0. \)

The main idea in the proof of Theorem 2.4 is to replace \( \partial \Omega \) by the plane tangent to \( \partial \Omega \) at \( x_0 \). This is justified by the fact that the main contributions to the integrals in (1.1) and in (2.5) for \( y \) near \( x \) come from small \( t \) (see [2] for similar approximations). The formulae in Theorem 2.4 can be read-off from (4.2) for \( m \geq 3 \) and from (4.7) and (4.8) for \( m = 2 \) respectively.

References


