Consider
\[
\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with \(\Omega\) a bounded regular domain and \(f \in C(\mathbb{R} \times \overline{\Omega})\). Suppose that \(\underline{u} \leq \overline{u}\) are respectively a sub- and a supersolution in appropriate sense. Denote by the set \(\mathcal{F}\) all solutions in \([\underline{u}, \overline{u}]\).

**Lemma 1** If \(\overline{u}\) is a supersolution and \(\underline{u} \leq \overline{u}\) is a subsolution then there exists a solution \(u\) in between:
\[
\underline{u} \leq u \leq \overline{u}.
\]

**Lemma 2** If \(u_1\) and \(u_2\) are supersolutions, then \(u(x) = \min(u_1(x), u_2(x))\) is a supersolution.

**Lemma 3** The set \(\mathcal{F}\) is equicontinuous. The infimum of a totally ordered subset in \(\mathcal{F}\) is a solution, that is every totally ordered subset of \(\mathcal{F}\) has a minimum.

**Proposition 4** There is a minimal solution \(u_{\min}\) in \([\underline{u}, \overline{u}]\): for every solution \(u \in [\underline{u}, \overline{u}]\) it holds that
\[
\underline{u} \leq u_{\min} \leq u \leq \overline{u}.
\]

**Proof.** By Lemma 2 \(\mathcal{F}\) is nonempty. For every \(x \in \Omega\) we define \(U(x) = \inf \{u_\lambda(x) : u_\lambda \in \mathcal{F}\}\). We will show that \(U = u_{\min}\), the minimal solution.

If \(\mathcal{F}\) is finite, Lemma 2 implies that the function \(U^*(x) = \min \{u(x) : u \in \mathcal{F}\}\) is a supersolution and Lemma 1 implies the existence of a solution \(u_1 \in [\underline{u}, \overline{u}]\), that is \(u_1 = U^* \in \mathcal{F}\) is minimal.

Now suppose that \(\mathcal{F}\) is not finite. First we will show that for every \(x \in \Omega\) there is \(u^* \in \mathcal{F}\) with \(u^*(x) = U(x)\). Let us fix \(x \in \Omega\).

1. By the definition of \(U\) there exists \(\{u^n_t\}_{t \in \mathbb{N}} \subset \mathcal{F}\) such that \(\lim_{t \to \infty} u^n_t(x) = U(x)\).

2. Define a new sequence \(\{\overline{u}^n_t\}_{t \in \mathbb{N}} \subset \mathcal{F}\) as follows by iteration:
   \[
   \begin{aligned}
i. & \quad \overline{u}^0_t = u_0 \text{ for some } u_0 \in \mathcal{F}, \\
n. & \quad \text{let } \overline{u}^{n+1}_t \text{ be a solution in } [\underline{u}, \min(\overline{u}^n_t, \overline{u})].
   \end{aligned}
   \]

Note that Lemma 2 implies that \(\min(\overline{u}^n_t, \overline{u}^{n+1}_t)\) is a supersolution and that Lemma 1 gives the existence of a solution \(\overline{u}^{n+1}_t\). The set \(\{\overline{u}^n_t\}_{t \in \mathbb{N}}\) is a totally ordered set in \(\mathcal{F}\) and hence by Lemma 3 its infimum is a solution. Let us denote this function by \(\overline{u}^*\).

3. Next let \(\{x_t\}_{t \in \mathbb{N}}\) be a countable dense subset of \(\Omega\) and define a sequence \(\{u_t\}_{t \in \mathbb{N}}\) of solutions as follows:
   \[
   \begin{aligned}
i. & \quad u_0 = u_0 \text{ for some } u_0 \in \mathcal{F}, \\
n. & \quad \text{let } u_{t+1} \text{ be a solution in } [\underline{u}, \min(u_t, \overline{u}^*_t)].
   \end{aligned}
   \]

Proceeding as for the previous sequence one finds that \(\{u_t\}_{t \in \mathbb{N}}\) is a totally ordered set and that \(u_\infty(x) = \lim_{t \to \infty} u_t(x)\) lies in \(\mathcal{F}\).

Since \(u_\infty(x_t) \leq U(x_t)\) for the dense set \(\{x_t\}_{t \in \mathbb{N}}\) and \(U(x) \leq u_\infty(x)\) for all \(x \in \Omega\) we find that \(U = u_\infty = u_{\min}\).

**References**