NICHOLS ALGEBRAS

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Introduction

Until the late 1980ies, most mathematicians believed that Lie groups and Lie algebras are rigid objects, that is, they don't admit deformations. The situation changed drastically, as Drinfeld has got the Fields medal in 1990 partially for the invention of quantum groups (quantized enveloping algebras). Lusztig developed an approach to quantized enveloping algebras via a pairing of $\mathbb{N}_0$-graded algebras. Motivated by work of Kac, Lusztig's starting point is a Cartan matrix of finite type, and the quantized enveloping algebra is constructed from an algebra satisfying a universal property. In this lecture, I will explain Nichols algebras of diagonal type. These algebras are more general than Lusztig's algebras, and for their definition no Lie theoretical input is used. We will see, how much structure Nichols algebras have, and how to get their combinatorics which generalizes the combinatorics of semisimple Lie algebras.

1. Hopf algebras and braided Hopf algebras

For an introduction to Hopf algebras see [Swe69]. Let us recall the basic definitions. If not stated differently, all algebras are defined over a base field $\Bbbk$, and are unital and associative.

Definition 1.1. A *coalgebra* is a triple (C, Δ, ε), where C is a vector space over $\Bbbk$ and $\varepsilon : C \to \Bbbk$, $\Delta : C \to C \otimes C$ (tensor product over $\Bbbk$) are linear maps such that following diagrams commute (identify C and

$\Bbbk \otimes C$).

$$(1.1) \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle\Delta\otimes \mathrm{id}} \\ C\otimes C & \xrightarrow{\mathrm{id}\otimes\Delta} & C\otimes C\otimes C \end{array}, \quad \begin{array}{ccccc} C & = & C & = & C \\ \Vert & & {\scriptstyle\Delta}\big\downarrow & & \Vert \\ k\otimes C & \xleftarrow{\varepsilon\otimes \mathrm{id}} & C\otimes C & \xrightarrow{\mathrm{id}\otimes\varepsilon} & C\otimes k \end{array}.$$

Sweedler notation: Let C be a coalgebra and $c \in C$. Then there exist $n \in \mathbb{N}_0$ and $c_1, \cdots, c_n, d_1, \cdots, d_n \in C$ such that $\Delta(c) = \sum_{i=1}^n c_i \otimes d_i$. We write $\Delta(c) = c_{(1)} \otimes c_{(2)}$. Coassociativity of Δ allows us to write $(\Delta \otimes \mathrm{id})\Delta(c) = (\mathrm{id} \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$.

Definition 1.2. Let C be a coalgebra. A (left) C-*comodule* is a pair (V, Δ_{L}), where V is a vector space over $\Bbbk$ and $\Delta_{\mathrm{L}} : V \to C \otimes V$ is a map such that the following diagrams are commutative.

$$(1.2) \qquad \begin{array}{ccc} V & \xrightarrow{\Delta_{\mathrm{L}}} & C\otimes V \\ \Vert & & {\scriptstyle\varepsilon\otimes \mathrm{id}}\big\downarrow \\ V & = & \Bbbk\otimes V \end{array}, \quad \begin{array}{ccc} V & \xrightarrow{\Delta_{\mathrm{L}}} & C\otimes V \\ {\scriptstyle\Delta_{\mathrm{L}}}\big\downarrow & & \big\downarrow{\scriptstyle\Delta\otimes \mathrm{id}} \\ C\otimes V & \xrightarrow{\mathrm{id}\otimes\Delta_{\mathrm{L}}} & C\otimes C\otimes V \end{array}.$$

A *coideal* of C is a subspace $D \subset C$ such that $\Delta(D) \subset D \otimes C + C \otimes D$ and $\varepsilon(D) = 0$. A *left coideal of* C is a subspace $D \subset C$ such that $\Delta(D) \subset C \otimes D$, that is, a left C-subcomodule of C with respect to the map $\Delta_{\mathrm{L}} = \Delta$.

Note that a nonzero left coideal is not a coideal: If D is a left coideal, $\varepsilon(D) = 0$, and $d \in D$, then $d = (\mathrm{id} \otimes \varepsilon)\Delta(d) = d_{(1)}\varepsilon(d_{(2)}) = 0$ by Def. 1.2, hence $D = 0$.

Similarly one defines right comodules and right coideals of a coalgebra. The Sweedler notation can be generalized to left comodules V via $\Delta_{\mathrm{L}}(v) = v_{(-1)} \otimes v_{(0)}$ for all $v \in V$.

Let H be an algebra. Then there is a unique algebra structure on $H \otimes H$ such that

$$(1.3) \qquad (a \otimes b)(c \otimes d) = ac \otimes bd \qquad \text{for all } a, b, c, d \in H.$$

Definition 1.3. A *Hopf algebra* is a 6-tuple $H = (H, \mu, 1, \Delta, \varepsilon, S)$, where $(H, \mu, 1)$ is a (unital associative) algebra, (H, Δ, ε) is a coalgebra, and $S : H \to H$ is a linear map such that $\Delta : H \to H \otimes H$ and

$\varepsilon : H \to \Bbbk 1$ are algebra maps and S satisfies

$$\begin{array}{ccccc} H \otimes H & \xleftarrow{\Delta} & H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle (\mathrm{id}\otimes S)}\big\downarrow & & {\scriptstyle 1\varepsilon}\big\downarrow & & {\scriptstyle S\otimes \mathrm{id}}\big\downarrow \\ H \otimes H & \xrightarrow[\mu]{} & H & \xleftarrow[\mu]{} & H \otimes H \end{array} \tag{1.4}$$

Using Sweedler notation, this means that $S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \varepsilon(h)1$ for all $h \in H$.

Let H, H' be Hopf algebras. A linear map $f : H \to H'$ is called a *Hopf algebra map*, if $f(ab) = f(a)f(b)$, $f(a_{(1)}) \otimes f(a_{(2)}) = f(a)_{(1)} \otimes f(a)_{(2)}$ for all $a, b \in H$.

The above axioms without S define a bialgebra structure on H.

Fact: If the antipode of a bialgebra exists, then it is unique, and is an algebra and coalgebra antihomomorphism.

Example 1.4. Let G be a group. The group algebra

$$\Bbbk G = \mathrm{span}_{\Bbbk}\{g \mid g \in G\}$$

is a Hopf algebra with product and unit induced by the product and neutral element of G, and coproduct and counit

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \qquad \text{for all } g \in G.$$

Definition 1.5. Let H be a Hopf algebra. A *Yetter-Drinfel'd module over* H is a triple $(V, \cdot, \Delta_{\mathrm{L}})$, where $\cdot : H \otimes V \to V$, $\Delta_{\mathrm{L}} : V \to H \otimes V$ are linear maps, $(V, \cdot)$ is an H-module, (V, Δ_{L}) is an H-comodule, and

$$\Delta_{\mathrm{L}}(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot v \quad \text{for all } h \in H,\ v \in V. \tag{1.5}$$

The category ${}^H_H\mathcal{YD}$ of Yetter-Drinfel'd modules over a Hopf algebra H (morphisms are linear maps commuting with actions and coactions) is equivalent to the category of H-bimodules H bicomodules with the natural compatibility conditions.

Example 1.6. Let G be a group and (V, Δ_{L}) a left comodule over $\Bbbk G$. For all $g \in G$, $v \in V$ let $v_g \in V$ such that $\Delta_{\mathrm{L}}(v) = \sum_{g\in G} g \otimes v_g$ (finite sum). Then for all $v \in V$,

$$\sum_{g\in G} g \otimes g \otimes v_g = \sum_{g\in G} \Delta(g) \otimes v_g = \sum_{g\in G} g \otimes \Delta_{\mathrm{L}}(v_g)$$

by coassociativity of $\varDelta_{\mathrm{L}}$, hence $\varDelta_{\mathrm{L}}(v_g) = g \otimes v_g$ for all $g \in G$, $v \in V$. On the other hand, $v = \sum_{g\in G} \varepsilon(g) v_g = \sum_{g \in G} v_g$ for all $v \in V$, and hence $V = \oplus_{g\in G} V_g$, where $V_g = \{v \in V \mid \varDelta_{\mathrm{L}}(v) = g \otimes v\}$.

Assume that G is abelian, and that $(V, \cdot, \varDelta_{\mathrm{L}})$ is a Yetter-Drinfel'd module over G. Then $\varDelta_{\mathrm{L}}(h \cdot v) = hgh^{-1} \otimes h \cdot v = g \otimes h \cdot v$ for all $g, h \in G$, $v \in V_g$. Hence V_g is a G-module for all $g \in G$. Assume that G acts on each V_g, $g \in G$, by characters, that is, $V = \oplus_{g\in G, \xi\in\hat{G}} V_g^\xi$, where

$$V_g^\xi = \{v \in V_g \mid h \cdot v = \xi(h) v \text{ for all } h \in G\}.$$

We say that V is of *diagonal type*.

Fact: If $\Bbbk$ is algebraically closed of characteristic 0, and G is a finite abelian group, then all finite-dimensional Yetter-Drinfel'd modules over $\Bbbk G$ are of diagonal type.

Proposition 1.7. *Let H be a Hopf algebra, and let $V, W \in {}^H_H\mathcal{YD}$. Then $V \otimes W \in {}^H_H\mathcal{YD}$, where*

$$\varDelta_{\mathrm{L}}(v \otimes w) = v_{(-1)} w_{(-1)} \otimes (v_{(0)} \otimes w_{(0)}), \quad h \cdot (v \otimes w) = h_{(1)} \cdot v \otimes h_{(2)} \cdot w$$

for all $v \in V$, $w \in W$, $h \in H$. If the antipode of H is bijective and $\dim V < \infty$, *then $V^* \in {}^H_H\mathcal{YD}$, where*

$$(v^*)_{(-1)}\,(v^*)_{(0)}(v) = S^{-1}(v_{(-1)})\, v^*(v_{(0)}), \quad (h \cdot v^*)(v) = v^*(S(h) \cdot v)$$

for all $v \in V$, $w \in W$, $v^* \in V^*$.

For the definition of Nichols algebras we need a more general context [AS02].

Let H be a Hopf algebra and B an algebra in ${}^H_H\mathcal{YD}$. This means that B is a Yetter-Drinfel'd module over H, $\varDelta_{\mathrm{L}}(1) = 1 \otimes 1$, and the multiplication map $\mu : B \otimes B \to B$ is a map of Yetter-Drinfel'd modules. Then $B \otimes B$ becomes an algebra in ${}^H_H\mathcal{YD}$ such that

$$(a \otimes b)(c \otimes d) = a(b_{(-1)} \cdot c) \otimes b_{(0)} d \qquad \text{for all } a, b, c, d \in B,$$

where $\cdot$ denotes the left action of H on B. (In general, $B \otimes B$ with the componentwise multiplication, see Eq. (1.3), is not an algebra in ${}^H_H\mathcal{YD}$.)

Definition 1.8. Let H be a Hopf algebra. A *braided Hopf algebra in* ${}^H_H\mathcal{YD}$ is a 6-tuple $B = (B, \mu, 1, \Delta, \varepsilon, S_B)$, where $(B, \mu, 1)$ is an algebra in ${}^H_H\mathcal{YD}$, (B, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$, and $S_B : B \to B$ is a morphism in ${}^H_H\mathcal{YD}$ such that $\Delta : B \to B \otimes B$ and $\varepsilon : B \to \Bbbk 1$ are algebra maps and S_B satisfies relations similar to Def. 1.3.

Example 1.9. (Important!) Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. The tensor algebra $T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$ is a braided Hopf algebra, where

$$\Delta(v) = 1 \otimes v + v \otimes 1 \in T(V) \otimes T(V), \quad \varepsilon(v) = 0 \qquad \text{for all } v \in V.$$

Exercise: Let $u, v, w \in V$. Then (for brevity, write elements of $T(V)$ without tensor product sign)

$$\begin{aligned}\Delta(u + vw) &= \Delta(u) + \Delta(v)\Delta(w)\\ &= 1 \otimes u + u \otimes 1 + (1 \otimes v + v \otimes 1)(1 \otimes w + w \otimes 1)\\ &= 1 \otimes u + u \otimes 1 + 1 \otimes vw + (v_{(-1)} \cdot w) \otimes v_{(0)} + v \otimes w + vw \otimes 1.\end{aligned}$$

If $H = \Bbbk G$ is a group algebra, V is of diagonal type, $g \in G$, and $v \in V_g$, $w \in V$, then

$$\Delta(vw) = 1 \otimes vw + (g \cdot w) \otimes v + v \otimes w + vw \otimes 1 \tag{1.6}$$

for all $v, w \in V$.

For the coproduct of a braided Hopf algebra, we use modified Sweedler notation: $\Delta(b) = b^{(1)} \otimes b^{(2)}$ for all $b \in B$.

Let H be the group algebra of the trivial group, that is, $H = \Bbbk 1$. Then B is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ if and only if B is a Hopf algebra.

Proposition 1.10. *(Radford's biproduct, Majid's bosonization) Let H be a Hopf algebra and $B \in {}^H_H\mathcal{YD}$ a braided Hopf algebra. Define on $B \otimes H$ a product by*

$$(b \otimes h)(b' \otimes h') = b(h_{(1)} \cdot b') \otimes h_{(2)}h', \quad b, b' \in B, h, h' \in H,$$

and a coproduct by

$$\Delta(b \otimes h) = (b^{(1)} \otimes (b^{(2)})_{(-1)}h_{(1)}) \otimes ((b^{(2)})_{(0)} \otimes h_{(2)}), \quad b \in B, h \in H.$$

Then $B \otimes H$ becomes a Hopf algebra with antipode

$$S(b \otimes h) = (1 \otimes S(h)S(b_{(-1)}))(S_B(b_{(0)}) \otimes 1) \quad \textit{for all } b \in B.$$

Notation: $B\#H$.

Note: $B\#1$ is a subalgebra and $1\#H$ is a Hopf subalgebra of $B\#H$.

Proof. Just check the definition of a Hopf algebra. E.g. antipode: We have to check that $x_{(1)}S(x_{(2)}) = S(x_{(1)})x_{(2)} = \varepsilon(x)1$ for all $x \in B\#H$. For $x = b\#h$, where $b \in B$, $h \in H$, we get

$$\begin{aligned}
&(b^{(1)}\#(b^{(2)})_{(-1)}h_{(1)})S((b^{(2)})_{(0)}\#h_{(2)})\\
&= (b^{(1)}\#(b^{(2)})_{(-2)}h_{(1)})S(h_{(2)})S((b^{(2)})_{(-1)})S_B((b^{(2)})_{(0)})\\
&= \varepsilon(h)(b^{(1)}\#(b^{(2)})_{(-2)}S((b^{(2)})_{(-1)})S_B((b^{(2)})_{(0)})\\
&= \varepsilon(h)\varepsilon((b^{(2)})_{(-1)})(b^{(1)}\#1)S_B((b^{(2)})_{(0)})\\
&= \varepsilon(h)(b^{(1)}\#1)S_B(b^{(2)}) = \varepsilon(h)\varepsilon(b).
\end{aligned}$$

□

Bosonization is the starting point of the Lifting Method (by Andruskiewitsch and Schneider) to classify pointed Hopf algebras (i. e. when all simple left comodules are one-dimensional).

2. Nichols algebras

The main reference is the survey article [AS02].

2.1. **Definitions and examples.**

Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. Recall that

$$T(V) = \oplus_{n=0}^{\infty} V^{\otimes n}$$

is an $\mathbb{N}_0$-graded algebra, where $\deg x = n$ for all $x \in V^{\otimes n}$, $n \in \mathbb{N}_0$.

Lemma 2.1. *There exists a unique maximal coideal $\mathfrak{I}(V)$ among all coideals of $T(V)$ which are contained in $T^{++}(V) = \oplus_{n\ge 2}V^{\otimes n}$. The coideal $\mathfrak{I}(V)$ is homogeneous with respect to the $\mathbb{N}_0$-grading of $T(V)$.*

Proof. Let $(D_i)_{i\in I}$, be the family of all coideals of $T(V)$ contained in $T^{++}(V)$. Then $D = \sum_{i\in I} D_i \subset T^{++}(V)$ is a coideal of $T(V)$. Indeed, let $k \in \mathbb{N}$, $x_i \in D_i$, $i \in \{i_1, \cdots, i_k\} \subset I$. Then

$$\begin{aligned}\Delta\Big(\sum_{j=1}^{k} x_{i_j}\Big) = \sum_{j=1}^{k} \Delta(x_{i_j}) &\in \sum_{j=1}^{k}(D_{i_j} \otimes T(V) + T(V) \otimes D_{i_j}) \\ &\subset D \otimes T(V) + T(V) \otimes D.\end{aligned}$$

Thus $D = D_{i_0}$ for some $i_0 \in I$, and $D_i \subset D$ for all $i \in I$. We get $\mathfrak{I}(V) = D$.

We show that $\mathfrak{I}(V)$ is homogeneous. Let $\pi_n : T(V) \to V^{\otimes n}$ be the natural projection map. Define $D = \oplus_{n \geq 2} \pi_n(\mathfrak{I}(V))$. Then $D \subset T^{++}(V)$. Since $\Delta(1) = 1 \otimes 1$ and $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$, for $n = 0, 1$ we get

$$\Delta(x) = \oplus_{m=0}^{n} \pi_m(x_{(1)}) \otimes \pi_{n-m}(x_{(2)}) \qquad \text{for all } x \in V^{\otimes n}. \tag{2.1}$$

We know: Δ is an algebra map, $V^{\otimes n} \in {}^H_H\mathcal{YD}$, and $v\pi_n(x) = \pi_{n+1}(vx)$ for all $v \in V$, $x \in T(V)$, $n \in \mathbb{N}_0$. Using this, one can show that Eq. (2.1) holds for all $n \in \mathbb{N}_0$, and that $\Delta(D) \subset D \otimes T(V) + T(V) \otimes D$. Since $\mathfrak{I}(V) \subset D$ and $\mathfrak{I}(V)$ is maximal, we get $D = \mathfrak{I}(V)$. □

Definition 2.2. The quotient coalgebra $\mathfrak{B}(V) = T(V)/\mathfrak{I}(V)$ is called the *Nichols algebra of* V.

Similarly to the proof of Lemma 2.1 one can show, that $\mathfrak{I}(V)$ is an ideal (by analyzing the ideal of $T(V)$ generated by $\mathfrak{I}(V)$). This implies that

Theorem 2.3. *The coalgebra $\mathfrak{B}(V)$ is an $\mathbb{N}_0$-graded braided Hopf algebra.*

In particular, $\mathfrak{B}(V)$ is an $\mathbb{N}_0$-graded algebra with homogeneous components $\mathfrak{B}(V)(0) = \mathbb{k}$ and $\mathfrak{B}(V)(1) = V$. The name is chosen in honor of W. Nichols, who studied (bosonozations of) Nichols algebras first [Nic78], following a suggestion of Kaplansky.

Example 2.4. Assume that $H = \Bbbk\{1\}$, the group algebra of the trivial group. Then $V = V_1$ and $1 \cdot v = v$ for all $v \in V$. Hence

$$\begin{aligned}\Delta(vw - wv) =&1 \otimes vw + w \otimes v + v \otimes w + vw \otimes 1\\ &- (1 \otimes wv + v \otimes w + w \otimes v + wv \otimes 1)\\ =&1 \otimes (vw - wv) + (vw - wv) \otimes 1.\end{aligned}$$

by Eq. (1.6). Thus $vw - wv \in \mathfrak{I}(V)$ for all $v, w \in V$. One can show that $\mathfrak{I}(V)$ is the ideal of $T(V)$ generated by $vw - wv$, $v, w \in V$. Thus $\mathfrak{B}(V) = S(V)$, the symmetric algebra of V.

Example 2.5. Let $H = \Bbbk\mathbb{Z}/(2)$ be the group algebra of the group $\{1, -1\}$ of order two (with multiplication as composition). Assume that $V = V_{-1}$ and that $(-1) \cdot v = -v$ for all $v \in V$. Then

$$\begin{aligned}\Delta(v^2) =&1 \otimes v^2 + ((-1) \cdot v) \otimes v + v \otimes v + v^2 \otimes 1\\ =&1 \otimes v^2 + v^2 \otimes 1\end{aligned}$$

for all $v \in V$ by Eq. (1.6). Thus $v^2 \in \mathfrak{I}(V)$ for all $v \in V$. One can show that $\mathfrak{I}(V)$ is the ideal of $T(V)$ generated by the elements v^2, $v \in V$. Thus $\mathfrak{B}(V) = \Lambda(V)$, the exterior algebra of V.

Example 2.6. Assume that $\Bbbk = \mathbb{Q}(q)$, where q is a parameter. Let $n \in \mathbb{N}$, and assume that $H = \Bbbk\mathbb{Z}^n$. Write the elements of $\mathbb{Z}^n$ exponentially: $e^{m_1\alpha_1+\dots+m_n\alpha_n}$, where $m_i \in \mathbb{Z}$ for all $i = 1, \dots, n$, and $\{\alpha_i \,|\, 1 \le i \le n\}$ is the standard basis of $\mathbb{Z}^n$. Let $C = (c_{ij})_{i,j=1,\dots,n} \in \mathbb{Z}^{n\times n}$ be a symmetrizable Cartan matrix and $d_1, \dots, d_n \in \mathbb{N}$ such that $(d_i c_{ij})_{i,j=1,\dots,n}$ is symmetric. Assume that $V = \mathrm{span}_\Bbbk\{x_1, \dots, x_n\} \in {}^H_H\mathcal{YD}$ such that $x_i \in V_{e^{\alpha_i}}$ and $e^{\alpha_i} \cdot x_j = q^{d_i c_{ij}} x_j$ for all $i, j = 1, \dots, n$. Then $\mathfrak{B}(V) = U_q(\mathfrak{n}_+)$, the quantized enveloping algebra of the positive part $\mathfrak{n}_+$ of the Kac-Moody Lie algebra $\mathfrak{g}(C)$. The ideal $\mathfrak{I}(V)$ is generated by the quantum Serre relations

$$\underbrace{[x_i, \dots, [x_i}_{1-c_{ij}\text{ times}}, x_j]_q]_q, \qquad i, j = 1, \dots, n, \quad i \ne j \tag{2.2}$$

where $[x_i, y]_q = x_i y - (e^{\alpha_i} \cdot y)x_i$ for all $i = 1, \dots, n$ and $y \in T(V)$.

In a similar way one can obtain the quantized enveloping algebra of the positive part of a basic classical Lie superalgebra (these are Lie superalgebras which can be defined in terms of a Cartan matrix with zeros and twos on the diagonal).

The above examples show that we have many interesting examples where H is the group algebra of an abelian group and V is of diagonal type. From now on we will always assume this.

Oops: we got many examples related to Lie algebras. Is this an accident? Is there some Lie algebra combinatorics hidden in the structure of Nichols algebras? We are going to look for generalized root systems and generalized Weyl groups for arbitrary Yetter-Drinfeld modules of diagonal type.

First we give characterizations of Nichols algebras of diagonal type. Let G be an abelian group, $H = \Bbbk G$, and $V \in {}^H_H\mathcal{YD}$ finite-dimensional. Recall: there exists $n \in \mathbb{N}$, a basis $\{x_1, \dots, x_n\}$ of V, elements $g_1, \dots, g_n \in G$ and $\xi_1, \dots, \xi_n \in \hat{G}$ such that

$$\Delta_{\mathrm{L}}(x_i) = g_i \otimes x_i, \quad g \cdot x_i = \xi_i(g)x_i \qquad \text{for all } i = 1, \dots, n, g \in G.$$

We fix such a basis and elements g_i, ξ_i, $i = 1, \dots, n$. The following constructions do not depend essentially on this choice. For all $i, j = 1, \dots, n$ let $q_{ij} \in \Bbbk^\times = \Bbbk \setminus \{0\}$ such that

$$g_i \cdot x_j = q_{ij}x_j. \tag{2.3}$$

Let $\{\alpha_1, \dots, \alpha_n\}$ be the standard basis of $\mathbb{Z}^n$.

2.2. **Skew-differential operators.**

Lemma 2.7. *The Nichols algebra $\mathfrak{B}(V)$ is $\mathbb{Z}^n$-graded with* $\deg x_i = \alpha_i$. *There is a unique $\mathbb{Z}^n$-module structure on $\mathfrak{B}(V)$ such that $e^{\alpha_i} \cdot x_j = q_{ji}x_j$ for all $i, j = 1, \dots, n$.*

Proof. The tensor algebra $T(V)$ has a unique $\mathbb{Z}^n$-grading with $\deg x_i = \alpha_i$ for all $i = 1, \dots, n$: $\deg x_{i_1} \cdots x_{i_k} = \alpha_{i_1} + \cdots + \alpha_{i_k}$ for all $k \in \mathbb{N}_0$, $i_1, \dots, i_k = 1, \dots, n$. Now replace in the proof of Lemma 2.1 the projections to the $\mathbb{N}_0$-homogeneous components of $\mathfrak{I}(V)$ by the projections to the $\mathbb{Z}^n$-homogeneous components. It follows that $\mathfrak{I}(V)$ is $\mathbb{Z}^n$-homogeneous, hence $\mathfrak{B}(V)$ is $\mathbb{Z}^n$-graded. For all $i, j = 1, \dots, n$, let $\bar{q}_{i,j} \in \Bbbk^\times$. For all $\beta = \sum_{j=1}^n m_j\alpha_j \in \mathbb{Z}^n$ and $x \in \mathfrak{B}(V)$ with $\deg x = \beta$ define $e^{\alpha_i} \cdot x_\beta = \bar{q}_{i,i_1}^{m_1} \cdots \bar{q}_{i,i_n}^{m_n} x_\beta$. This extends to an action of $\mathbb{Z}^n$ on

$\mathfrak{B}(V)$. Then the claim follows from the special case where $\bar{q}_{i,j} = q_{ji}$ for all i, j. $\square$

Recall the algebra structure of $T(V) \otimes T(V)$. For all $x \in T(V)$ (or $\mathfrak{B}(V)$), $i = 1, \dots, n$ we get (by induction on the degree of x)

$$(1 \otimes x)(x_i \otimes 1) = x_i \otimes (e^{\alpha_i} \cdot x). \tag{2.4}$$

For all $i = 1, \dots, n$ and $k \in \mathbb{N}_0$ let $\partial_i^{\mathrm{L}} : V^{\otimes k} \to V^{\otimes k-1}$, $\partial_i^{\mathrm{R}} : V^{\otimes k} \to V^{\otimes k-1}$ be linear maps such that

$$\partial_i^{\mathrm{L}}(x_{i_1} \cdots x_{i_k}) = \sum_{j=1}^{k} \delta_{i,i_j} q_{i_1 i} q_{i_2 i} \cdots q_{i_{j-1} i} x_{i_1} \cdots x_{i_{j-1}} x_{i_{j+1}} \cdots x_{i_n},$$

$$\partial_i^{\mathrm{R}}(x_{i_1} \cdots x_{i_k}) = \sum_{j=1}^{k} \delta_{i,i_j} q_{i i_{j+1}} q_{i i_{j+2}} \cdots q_{i i_k} x_{i_1} \cdots x_{i_{j-1}} x_{i_{j+1}} \cdots x_{i_n}.$$

Lemma 2.8. *For all $x, y \in T(V)$ and $i, j \in \{1, 2, \dots, n\}$,*

$$\begin{aligned} \partial_i^{\mathrm{L}}(1) =&0, & \partial_i^{\mathrm{L}}(x_j) =&\delta_{ij}, & \partial_i^{\mathrm{L}}(xy) =&\partial_i^{\mathrm{L}}(x)y + (e^{\alpha_i} \cdot x)\partial_i^{\mathrm{L}}(y),\\ \partial_i^{\mathrm{R}}(1) =&0, & \partial_i^{\mathrm{R}}(x_j) =&\delta_{ij}, & \partial_i^{\mathrm{R}}(xy) =&x\partial_i^{\mathrm{R}}(y) + \partial_i^{\mathrm{R}}(x)(g_i \cdot y). \end{aligned}$$

Proof. Direct calculation (for x, y take products of x_i, $1 \le i \le n$). $\square$

Proposition 2.9. *For all $i = 1, \dots, n$, the maps ∂_i^{L}, ∂_i^{R} induce skew-derivations of $\mathfrak{B}(V)$. We have*

$$\partial_i^{\mathrm{L}} \partial_j^{\mathrm{R}} = \partial_j^{\mathrm{R}} \partial_i^{\mathrm{L}} \qquad \text{for all } i, j = 1, \dots, n.$$

Proof. Let $x \in \mathfrak{I}(V)$. Then there exist $a_i \in T(V)$, $1 \le i \le n$, such that $\Delta(x) = 1 \otimes x + \sum_{i=1}^n x_i \otimes a_i +$ terms, where the first tensor factors are $\mathbb{N}_0$-homogeneous of degree ≥ 2. Check by induction on the degree of x that $\partial_i^{\mathrm{L}}(x) = a_i$. Since $\mathfrak{I}(V)$ is a coideal, $a_i \in \mathfrak{I}(V)$ for all i. Hence $\partial_i^{\mathrm{L}}(\mathfrak{I}(V)) \subset \mathfrak{I}(V)$, and ∂_i^{L} induces a map $\partial_i^{\mathrm{L}} \in \mathrm{End}_{\Bbbk}(\mathfrak{B}(V))$. The proof for ∂_i^{R} is similar: Use that $\Delta(x) = x \otimes 1 + \sum_{i=1}^n b_i \otimes x_i + \cdots$ for some $b_i \in \mathfrak{I}(V)$.

If $x \in \Bbbk \oplus V \subset \mathfrak{B}(V)$, then $\partial_i^{\mathrm{L}} \partial_j^{\mathrm{R}}(x) = \partial_j^{\mathrm{R}} \partial_i^{\mathrm{L}}(x) = 0$ for all $i, j = 1, \dots, n$. Check that $\partial_i^{\mathrm{L}} \partial_j^{\mathrm{R}}(xy) = \partial_j^{\mathrm{R}} \partial_i^{\mathrm{L}}(xy)$ for all $x, y \in \mathfrak{B}(V)$ such that $\partial_i^{\mathrm{L}} \partial_j^{\mathrm{R}}(x) = \partial_j^{\mathrm{R}} \partial_i^{\mathrm{L}}(x)$, $\partial_i^{\mathrm{L}} \partial_j^{\mathrm{R}}(y) = \partial_j^{\mathrm{R}} \partial_i^{\mathrm{L}}(y)$. $\square$

Lemma 2.10. *For all $x \in T(V)$, $i = 1, \dots, n$,*

$$\Delta(\partial_i^{\mathrm{L}}(x)) = \partial_i^{\mathrm{L}}(x^{(1)}) \otimes x^{(2)}, \qquad \Delta(\partial_i^{\mathrm{R}}(x)) = x^{(1)} \otimes \partial_i^{\mathrm{R}}(x^{(2)}).$$

Proof. By induction on the $\mathbb{N}_0$-degree of x: decompose x as yx_j resp. x_jy, $j = 1, \dots, n$. Use Eq. (2.4) and Lemma 2.8. $\square$

Theorem 2.11. *The ideal $\mathfrak{I}(V)$ coincides with the largest ideal among all ideals I of $T(V)$ such that $\varepsilon(I) = 0$ and $\partial_i^{\mathrm{L}}(I) \subset I$ for all $i = 1, \dots, n$.*

One can replace ∂_i^{L} by ∂_i^{R} in Thm. 2.11.

Proof. By definition, $\varepsilon(\mathfrak{I}(V)) = 0$. By Prop. 2.9, $\mathfrak{I}(V)$ is stable under ∂_i^{L}, $i = 1, \dots, n$. We have to show that the largest ideal $I_{\max}$ satisfying these properties lies in $\mathfrak{I}(V)$.

The counit ε and the skew-derivations ∂_i^{L}, $i = 1, \dots, n$, are $\mathbb{Z}^n$-graded, hence $I_{\max}$ is $\mathbb{Z}^n$-graded. Let $x \in T(V)$. Assume that x is $\mathbb{Z}^n$-homogeneous, $\varepsilon(x) = 0$ and $\partial_i^{\mathrm{L}}(x) \in I_{\max}$ for all $i = 1, \dots, n$. By Lemma 2.8, $I_{\max} + (x)$ is an ideal contained in $\ker \varepsilon$ and stable under ∂_i^{L}, $i = 1, \dots, n$, hence $x \in I_{\max}$ by maximality of $I_{\max}$.

We show that $I_{\max} \subset T^{++}(V)$ and $I_{\max}$ is a coideal of $T(V)$ (that is,

$$(\mathrm{id} \otimes \pi)\Delta(x) \in I_{\max} \otimes T(V)/I_{\max} \quad \text{for all } x \in I_{\max},$$

where $\pi : T(V) \to T(V)/I_{\max}$ is the canonical map). As noted, $I_{\max}$ is $\mathbb{N}_0$-graded. Assume now that $x \in I_{\max}$ is $\mathbb{N}_0$-homogeneous. Since $\varepsilon(x) = 0$ and $\partial_i^{\mathrm{L}}(x) \in I_{\max}$ for all $i = 1, \dots, n$, we get $\deg x \geq 2$. Assume that $(\mathrm{id} \otimes \pi)\Delta(y) \in I_{\max} \otimes T(V)/I_{\max}$ for all $y \in I_{\max}$ with $\deg y < \deg x$. Then

$$(\partial_i^{\mathrm{L}} \otimes \pi)\Delta(x) = (\mathrm{id} \otimes \pi)\Delta(\partial_i^{\mathrm{L}}(x)) \in I_{\max} \otimes T(V)/I_{\max}$$

for all $i = 1, \dots, n$. Hence $(\mathrm{id} \otimes \pi)\Delta(x) \in I_{\max} \otimes T(V)/I_{\max}$ by the previous paragraph. This proves the theorem. $\square$

This theorem helps to check if a given ideal coincides with $\mathfrak{I}(V)$.

Example 2.12. Assume that $q_{ij}q_{ji} = 1$ for all $1 \leq i < j \leq n$. For all $i = 1, \dots, n$ let $h_i \in \mathbb{N} \cup \{\infty\}$ such that

(1) if $1+q_{ii}+q_{ii}^2+\cdots+q_{ii}^{m-1}=0$ for some $m\in\mathbb{N}$, then h_i is the smallest such number, and
(2) $h_i=\infty$ otherwise.

Then

$$\mathfrak{B}(V)=T(V)/(x_k^{h_k},x_ix_j-q_{ij}x_jx_i\,|\,i<j,h_k<\infty),$$

and the set

$$\{x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}\,|\,0\le m_i<h_i \text{ for all } i=1,\dots,n\} \tag{2.5}$$

is a vector space basis of $\mathfrak{B}(V)$. (We use the convention $m<\infty$ for all $m\in\mathbb{N}_0$.)

Proof: For all $i\ne j$, check that $\partial_l^{\mathrm{L}}(x_ix_j-q_{ij}x_jx_i)=0$ for all $l=1,\dots,n$ (by using that $q_{ij}q_{ji}=1$). Hence $x_ix_j-q_{ij}x_jx_i\in\mathfrak{I}(V)$ for all $i\ne j$. For all $i\in\{1,\dots,n\}$ and $m\in\mathbb{N}$ check that

$$\partial_l^{\mathrm{L}}(x_i^m)=\delta_{i,l}(1+q_{ii}+\cdots+q_{ii}^{m-1})x_i^{m-1}.$$

Thus if $h_i\in\mathbb{N}$ then $x_i^{h_i}\in\mathfrak{I}(V)$. Therefore the set in Eq. (2.5) spans $\mathfrak{B}(V)$.

Let $x=\sum_{0\le m_j<h_j\,\forall j}a_{m_1,\dots,m_n}x_1^{m_1}\cdots x_n^{m_n}\in T(V)$, where $a_{m_1,\dots,m_n}\in\Bbbk$ for all $m_1,\dots,m_n\ge 0$. Assume that $x\ne 0$. Check that there exist $l\ge 0$, $j_1,\dots,j_l=1,\dots,n$ such that $\partial_{j_1}^{\mathrm{L}}\partial_{j_2}^{\mathrm{L}}\cdots\partial_{j_l}^{\mathrm{L}}(x)\in\Bbbk^\times$. Thus $x\notin\mathfrak{I}(V)$ by Thm. 2.11. Hence the set in Eq. (2.5) is a basis of $\mathfrak{B}(V)$.

There are many other characterizations of Nichols algebras of diagonal type (key words: quantum symmetrizer, non-degenerate pairing of braided Hopf algebras, cotensor coalgebra). Let us give one more.

For any braided Hopf algebra $B\in{}^H_H\mathcal{YD}$ let $P(B)$ denote the set of primitive elements:

$$x\in P(B)\quad\Leftrightarrow\quad \varDelta(x)=1\otimes x+x\otimes 1.$$

Note that $\varepsilon(1)x+\varepsilon(x)1=x$ and hence $\varepsilon(x)=0$ for all $x\in P(B)$.

Theorem 2.13. *Let $I\subset T^{++}(V)$, $I\in{}^H_H\mathcal{YD}$ such that $T(V)/I$ is a braided Hopf algebra. Then $T(V)/I=\mathfrak{B}(V)$ if and only if all primitive elements of $T(V)/I$ are contained in V.*

3. Kharchenko's basis

Recall that G is an abelian group, $H = \Bbbk G$ is the group algebra, $V \in {}^H_H\mathcal{YD}$ is of diagonal type, and $n = \dim V$. From now on we consider a more special situation: We assume that the group homomorphism $(\mathbb{Z}^n, +) \to (G, \cdot)$, $\alpha_i \mapsto g_i$ for all $i = 1, \dots, n$, is an isomorphism. This is not an essential restriction, since the structure of the Nichols algebra as an algebra and as a coalgebra depends only on the choice of the algebra structure of $T(V) \otimes T(V)$, that is, only on the numbers q_{ij} given in Eq. (2.3).

A *bicharacter* on an abelian group F is a map $\chi' : F \times F \to \Bbbk^\times$ such that

$$\chi'(a+b, c) = \chi'(a, c)\chi'(b, c), \quad \chi'(c, a+b) = \chi'(c, a)\chi'(c, b) \tag{3.1}$$

for all $a, b, c \in F$. Then $\chi'(0, c) = \chi'(c, 0) = 1$ for all $c \in F$ (put $a = b = 0$ in Eq. (3.1)).

Define a bicharacter χ on $\mathbb{Z}^n$ by

$$\chi(\alpha_i, \alpha_j) = q_{ij} \qquad \text{for all } i, j = 1, \dots, n. \tag{3.2}$$

If V is as in Ex. 2.6, and the Cartan matrix is of finite type, then Lusztig constructed a PBW basis of $\mathfrak{B}(V)$ using the longest element of the Weyl group. Kharchenko [Kha99] (following Rosso's ideas) gave a PBW basis without restricting himself to Ex. 2.6.

Idea: give first a PBW basis for $T(V)$ using Lyndon words, and then show that a subset of this basis is a PBW basis of $\mathfrak{B}(V)$.

3.1. **Lyndon words.** For an introduction see [Lot83].

Let X be a finite set (we say *alphabet*) and let $\mathbb{X}$ be the free monoid generated by X (*dictionary*). That is,

$$\mathbb{X} = \{u_1 u_2 \cdots u_k \,|\, k \geq 0,\ u_i \in X \text{ for all } i = 1, \cdots, k\}.$$

Let $\mathbb{X}^\times = \mathbb{X} \setminus \{1\}$. If $k \geq 0$, $u_1, u_2, \dots, u_k \in X$, and $w = u_1 u_2 \cdots u_k \in \mathbb{X}$, then we write $|w| = k$, and call k the *length of* w.

We fix a total ordering $\leq$ on X. It induces the lexicographic ordering (also denoted by $\leq$) on $\mathbb{X}$: If $u, v \in \mathbb{X}$, then $u \leq v$ if and only if either $v = uu'$ for some $u' \in \mathbb{X}$, or $u = wxu'$, $v = wyv'$, where $x, y \in X$, $x < y$, $w, u', v' \in \mathbb{X}$. In the second case we write $u \ll v$. The lexicographic ordering is a total ordering on $\mathbb{X}$. For all $u, v \in \mathbb{X}$, we write $u < v$ if $u \le v$, $u \ne v$.

Definition 3.1. (Lyndon word) Let $w \in \mathbb{X}^\times$. We say that w is a *Lyndon word*, if $w = uv$, $u, v \in \mathbb{X}^\times$, implies that $w < v$. Let $\mathcal{L}$ denote the set of Lyndon words.

Remember that $\mathcal{L}$ depends on X and the total ordering $\le$ on X.

If $w \in \mathcal{L}$, $w = uv$, $u, v \in \mathbb{X}^\times$, then $w \ll v$, since $|w| < |v|$.

Example 3.2. Let $X = \{1, 2, 3\}$ with the usual ordering. Then 12 is a Lyndon word, since $12 < 2$. Also, 12122 and 132 are Lyndon words. But 11, 1212, 21, 121 are not Lyndon words. The Lyndon words of length ≤ 3 are

$$1, 2, 3, 12, 13, 23, 112, 113, 122, 123, 132, 133, 223, 233.$$

Proposition 3.3. *Let $w \in \mathbb{X}^\times$. Then $w \in \mathcal{L}$ if and only if either $w \in X$ or $w = uv$, $u, v \in \mathcal{L}$, $u < v$. Moreover, if $w = uv$ such that $u \in \mathbb{X}^\times$, $v \in \mathcal{L}$, and $|v|$ is maximal with these properties, then $u \in \mathcal{L}$ and $u < uv < v$.*

Proof. ($\Leftarrow$) If $w \in X$, then $w \in \mathcal{L}$ by definition. Assume now that $w = uv$, where $u, v \in \mathcal{L}$ and $u < v$. Then $w < v$. Indeed, if $v = uv'$, $v' \in \mathbb{X}^\times$, then $v < v'$, since $v \in \mathcal{L}$. Hence $uv < uv'$, that is, $w < v$. Otherwise $u < v$ implies that $u \ll v$. Then $w = uv \ll v$.

Write $u = u'u''$, $u', u'' \in \mathbb{X}^\times$. Then $u \ll u''$, since $u \in \mathcal{L}$. Hence $w = u'u''v = uv \ll u''v$.

Write $v = v'v''$, $v', v'' \in \mathbb{X}^\times$. Then $uv'v'' = w < v < v''$. Hence $w \in \mathcal{L}$.

($\Rightarrow$) Let $w \in \mathcal{L}$ with $|w| \ge 2$. Let $u, v \in \mathbb{X}^\times$ such that $w = uv$, $v \in \mathcal{L}$ and $|v|$ is maximal. Then $u < uv < v$, since $w = uv \in \mathcal{L}$. Note that v is the lexicographically smallest proper end of w. It remains to prove that $u \in \mathcal{L}$. If $u \in X$, then this holds by definition of $\mathcal{L}$. Otherwise let $u = u'u''$, where $u', u'' \in \mathbb{X}^\times$. Since $w \in \mathcal{L}$, we get

$$uv < u''v. \tag{3.3}$$

Suppose first that uv starts with u'', that is, $u = u''u'''$ for some $u''' \in \mathbb{X}^\times$. Then $w = u''u'''v < u''v$, since $w \in \mathcal{L}$. Hence $u'''v < v$, a contradiction to the minimality of v with respect to $<$. Thus $uv \ll u''$ by Eq. (3.3), and hence $u < u''$. Therefore $u \in \mathcal{L}$. □

Definition 3.4. Let $w \in \mathcal{L}$. Assume that $|w| \geq 2$. The pair $(u, v) \in \mathcal{L} \times \mathcal{L}$, where $w = uv$ and v is the longest proper end of w which is in $\mathcal{L}$, is called the *Shirshow decomposition of* w.

For example, let $X = \{1, 2, 3\}$ as above. Then $1231231233 \in \mathcal{L}$, and the Shirshow decomposition of this word is $(123, 1231233)$.

Theorem 3.5. *Any $w \in \mathbb{X}$ can be written uniquely as $w = u_1 u_2 \cdots u_m$, where $m \in \mathbb{N}_0$, $u_1, \dots, u_m \in \mathcal{L}$, and $u_m \leq \cdots \leq u_2 \leq u_1$. Moreover, u_1 is the lexicographically smallest proper end on w.*

One says that w has a unique decomposition as a monotonic product of Lyndon words. For example,

$$1231233123122123 = (1231233)(123)(122123).$$

Fact: Let $u, v \in \mathbb{X}^\times$ and $\bar{u}$, $\bar{v}$ the corresponding monotonic products of Lyndon words. Regard $\mathcal{L}$ as an alphabet with the ordering coming from the ordering on $\mathbb{X}$. Then $u < v$ if and only if $\bar{u} < \bar{v}$.

3.2. **q-brackets.** Recall that $H = \Bbbk G$ and $V = \mathrm{span}_{\Bbbk}\{x_i \,|\, 1 \leq i \leq n\} \in {}^H_H\mathcal{YD}$ is of diagonal type. Let $X = \{1, 2, \dots, n\}$ with the natural ordering, and let $\mathbb{X}$, $\mathcal{L}$ be as in the previous subsection.

For all $w = i_1 \cdots i_k \in \mathbb{X}$, where $k = |w|$, $i_1, \dots, i_k \in X$, let $x_w = x_{i_1} \cdots x_{i_k} \in T(V)$.

Let $[\cdot] : \mathcal{L} \to T(V)$ be the map such that

$$\begin{aligned} [i] &= x_i \quad \text{for all } i, \\ [w] &= [u][v] - \chi(\deg x_u, \deg x_v)[v][u] \quad \text{for all } w \in \mathcal{L} \text{ with } |w| \geq 2, \end{aligned}$$

where $w = (u, v)$ is the Shirshow decomposition of w, and deg means $\mathbb{Z}^n$-degree.

Lemma 3.6. *The elements $[w] \in T(V)$, where $w \in \mathcal{L}$, are $\mathbb{Z}^n$-homogeneous, and if $|w| \geq 2$, then $[w] = [u][v] - ([u]_{(-1)} \cdot [v])[u]_{(0)}$.*

Lemma 3.7. *Let $w \in \mathcal{L}$. Then $[w] = x_w +$ linear combination of terms $x_{w'}$, where $|w'| = |w|$ and $w' > w$.*

Proof. By induction on $|w|$. If $w \in X$, then the claim holds. Otherwise, let (u,v) be the Shirshow decomposition of w. Then $[w] = [u][v] - \chi(\deg x_u, \deg x_v)[v][u]$. Since $w \in \mathcal{L}$, we have $uv = w < v < vu$. By induction hypothesis we obtain that $uv < vu \le w'$ for the terms $x_{w'}$ of $[v][u]$. Further, $w' \le uv$ for the terms $x_{w'}$ of $[u][v]$, and the coefficient of x_{uv} is 1. $\square$

Theorem 3.8. *(PBW theorem) The set*

$$\{[u_1]^{k_1}[u_2]^{k_2}\cdots[u_m]^{k_m} \mid m \in \mathbb{N}_0, u_i \in \mathcal{L} \text{ for all } i, \\ u_m > \cdots > u_2 > u_1,\, k_1,\dots,k_m \ge 0\}$$

is a vector space basis of $T(V)$.

Proof. The set $\{x_w \mid w \in \mathbb{X}\}$ forms a basis of $T(V)$. Then the theorem follows from Thm. 3.5, Lemma 3.7, and since the elements $[w]$, $w \in \mathcal{L}$, are $\mathbb{N}_0$-homogeneous. $\square$

For all $w \in \mathcal{L}$ let $K_w \subset \mathbb{X}$ be the set of products $u_1 \cdots u_k$, where $k \in \mathbb{N}$, $u_i \in \mathcal{L}$, $w \le u_i$ for all i, and $|u_1 \cdots u_k| = |w|$. (Then either $k = 1$, $w = u_1$ or $w \ll u_i$ for all i.) Let

$$K_w^< = \cup_{w' \in \mathbb{X}^\times, |w'|<|w|, w<w'} K_{w'}.$$

The sets K_w, $K_w^<$ are finite for all $w \in \mathcal{L}$.

Proposition 3.9. *Let* $u, v \in \mathcal{L}$. *Assume that* $u < v$. *Then* $[u][v] - \chi(\deg x_u, \deg x_v)[v][u] \in \sum_{w=u_1\cdots u_k \in K_{uv}} [u_1]\cdots[u_k]$.

Proof. (Idea) If (u,v) is the Shirshow decomposition of uv, then

$$[u][v] - \chi(\deg x_u, \deg x_v)[v][u] = [uv].$$

Then the claim holds, since $uv \in K_{uv}$. Otherwise $|u| \ge 2$. Then one can use the Shirshow decomposition of u to prove the claim. One needs more results on q-brackets and Lyndon words. $\square$

By Prop. 3.9, one can assume that $u_1 \ge u_2 \ge \cdots \ge u_k$ in the definition of K_w and in Prop. 3.9. For any $w = w_1 \cdots w_k$, where $k \ge 0$, $w_1, \dots, w_k \in \mathcal{L}$, $w_1 \ge \cdots \ge w_k$, let $[w] = [w_1]\cdots[w_k]$.

Proposition 3.10. *Let* $w \in \mathcal{L}$. *Then* $\Delta([w]) - 1 \otimes [w] - [w] \otimes 1 \in T(V) \otimes \sum_{u \in K_w^<} [u]$.

Proof. (Sketch) By induction on $|w|$. If $w \in X$, then $[w] = x_w$ and $\Delta([w]) = 1 \otimes [w] + [w] \otimes 1$. Assume now that $|w| \geq 2$ and the claim holds for all Lyndon words of smaller length. Let (u, v) be the Shirshow decomposition of w. Then

$$\begin{aligned}
&\Delta([w]) = \Delta([u][v] - \chi(\deg x_u, \deg x_v)[v][u])\\
&= ([u] \otimes 1 + 1 \otimes [u] + \sum_{u' \in K_u^<} a_{u'} \otimes [u'])([v] \otimes 1 + 1 \otimes [v] + \sum_{v' \in K_v^<} b_{v'} \otimes [v'])\\
&\quad - \chi(\deg x_u, \deg x_v)([v] \otimes 1 + 1 \otimes [v] + \sum_{v' \in K_v^<} b_{v'} \otimes [v'])\times\\
&\qquad\qquad\qquad ([u] \otimes 1 + 1 \otimes [u] + \sum_{u' \in K_u^<} a_{u'} \otimes [u'])
\end{aligned}$$

for some $a_{u'}, b_{v'} \in T(V)$. If we remove the parentheses, we get 18 types of summands. Since $(1 \otimes [u])([v] \otimes 1) - \chi(\deg x_u, \deg x_v)[v] \otimes [u] = 0$, only 16 of them remain. Four summands give $[w] \otimes 1 + 1 \otimes [w]$. For the other 12 it can be checked that the second tensor factor lies in K_w. For example,

$$\begin{aligned}
&(1 \otimes [u])(b_{v'} \otimes [v']) - \chi(\deg x_u, \deg x_v)(b_{v'} \otimes [v'])(1 \otimes [u])\\
&\quad = \chi(\deg x_u, \deg b_{v'})b_{v'} \otimes ([u][v'] - \chi(\deg x_u, \deg[v'])[v'][u]),
\end{aligned}$$

and since $u < v'$, (and hence u is smaller than all factors of the monotonic decomposition of v',) the second tensor factor is in $K_{uv}^<$ by Prop. 3.9. □

3.3. **PBW basis for the Nichols algebra.** We use Prop. 3.10 to obtain a PBW basis for $\mathfrak{B}(V)$. The same method works for an arbitrary braided Hopf algebra generated by V. Details can be found in [Kha99], [GH07].

We need the following general fact.

Proposition 3.11. *Let $B \in {}^H_H\mathcal{YD}$ be a braided Hopf algebra. Let $B' \subset B$, $B' \in {}^H_H\mathcal{YD}$ be a subalgebra and $I \subset B \cap \ker \varepsilon$, $I \in {}^H_H\mathcal{YD}$, an ideal of B'. Suppose that*

$$\Delta(B') \subset B' \otimes B' + B \otimes I, \qquad \Delta(I) \subset I \otimes B' + B \otimes I.$$

Then the braided Hopf algebra structure on B induces a braided Hopf algebra structure on $B'/I \in {}^H_H\mathcal{YD}$.

The claim of the proposition means that the algebra and coalgebra maps and the counit and the antipode on B', given by those on B, are well-defined and satisfy the axioms of a braided Hopf algebra.

Let $\pi : T(V) \to \mathfrak{B}(V)$ be the canonical map. For all $w \in \mathcal{L}$, let $\mathcal{V}^{[\geq u]}$ be the subalgebra of $T(V)$ generated by $[v]$, $v \geq u$, and let $\mathcal{I}^{[\geq u]}$ be the ideal of $\mathcal{V}^{[\geq u]}$ generated by $[v]$, $v > u$. By Props. 3.9, 3.10, $\mathcal{V}^{[\geq u]}$ and $\mathcal{I}^{[\geq u]}$ satisfy the conditions in Prop. 3.11. One can show that the powers of $[u]$ form a basis of the quotient $\mathcal{V}^{[\geq u]}/\mathcal{I}^{[\geq u]}$ for all $u \in \mathcal{L}$, and that $[u]$ is a primitive element in $\mathcal{V}^{[\geq u]}/\mathcal{I}^{[\geq u]}$ (but not in $\mathcal{V}^{[\geq u]}$!).

For all $u \in \mathcal{L}$, let $\mathcal{V}_{[\geq u]} = \pi(\mathcal{V}^{[\geq u]}) \subset \mathfrak{B}(V)$ and $\mathcal{I}_{[\geq u]} = \pi(\mathcal{I}^{[\geq u]})$. Using again Prop. 3.11, one can show the following.

Theorem 3.12. *For all $u \in \mathcal{L}$, $\mathcal{V}_{[\geq u]}/\mathcal{I}_{[\geq u]}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ generated by $\pi([u]) + \mathcal{I}_{[\geq u]}$.*

Graded braided Hopf algebras B generated by a nonzero primitive element have a very simple form: either $B = \Bbbk[x]$, the polynomial ring in one variable, or $B = \Bbbk[x]/(x^h)$, $h \geq 2$, where

(3.4)
$$h = h'p^r, \quad h' = \min\{s \in \mathbb{N} \mid q^s = 1\}, \quad p = \operatorname{char}^* \Bbbk\, (\geq 1), \quad r \in \mathbb{N}_0,$$

and $q \in \Bbbk$ such that $x_{(-1)} \cdot x_{(0)} = qx$. If $h' > 1$, then $x^{h'}$ generates a braided Hopf subalgebra of B, and $(x^{h'})_{(-1)} \cdot (x^{h'})_{(0)} = x^{h'}$. If $\operatorname{char} \Bbbk = p > 0$ and $q = 1$, then either $x^p = 0$ or x^p generates a braided Hopf subalgebra of B, and $(x^p)_{(-1)} \cdot (x^p)_{(0)} = x^p$.

A combination of Thms. 3.8, 3.12 and the above observation leads to the following theorem.

Theorem 3.13. *There is a subset $\mathcal{L}' \subset \mathcal{L}$, and for each $u \in \mathcal{L}'$ a number $h_u \in \mathbb{N} \cup \{\infty\}$, $h_u \geq 2$, as in Eq.* (3.4)*, such that the Nichols algebra $\mathfrak{B}(V)$ has a PBW basis of the form*

$$\{[u_1]^{m_1}[u_2]^{m_2} \cdots [u_k]^{m_k} \mid k \geq 0,\, u_1, \dots, u_k \in \mathcal{L}',\, u_1 > u_2 > \cdots > u_k,$$
$$0 \leq m_i < h_{u_i} \quad \textit{for all } i = 1, \dots, k.$$

An important consequence of the discussion above Thm. 3.13 is that (by inserting additional PBW generators $[u]^{h'p^t}$, $h'p^t < h$, if necessary) one can assume, that for each PBW generator x, its exponent is bounded by $h_x = \min\{m \in \mathbb{N} \,|\, 1 + q + \cdots + q^{m-1} = 0\}$, where $q = \chi(\deg x, \deg x)$. Hence the following theorem holds.

Theorem 3.14. *There exist a totally ordered index set $(L, \leq)$ and $\mathbb{Z}^n$-homogeneous elements $X_l \in \mathfrak{B}(V)$, $l \in L$, such that the set*

$$\{X_{l_1}^{m_1} X_{l_2}^{m_2} \cdots X_{l_k}^{m_k} \,|\, k \geq 0,\, l_1, \ldots, l_k \in L,\, l_1 > l_2 > \cdots > l_k, \\ 0 \leq m_i < h_{l_i} \quad \text{for all } i = 1, \ldots, k\}$$

is a vector space basis of $\mathfrak{B}(V)$, where $h_l = \min\{m \in \mathbb{N} \,|\, 1 + q_l + \cdots + q_l^{m-1} = 0\} \cup \{\infty\}$ and $q_l = \chi(\deg X_l, \deg X_l)$ for all $l \in L$.

For any two such ordered sets $(L, \leq)$, $(L', \leq)$ there is a bijection $\phi : L \to L'$ such that $h_{\phi(l)} = h_l$ for all $l \in L$.

The last claim follows from the fact that h_l depends only on the $\mathbb{Z}^n$-degree of X_l, and since $\mathfrak{B}(V)$ is $\mathbb{Z}^n$-graded.

In general, it is not clear how to calculate such a set $(L, \leq)$ and the numbers h_l, $l \in L$, explicitly. But later we will be able to check if L is a finite set. In that case we can determine h_l for all $l \in L$. The key structure for this is the Weyl groupoid of V (more precisely, of χ).

4. Weyl groupoids

We discuss axioms and main properties of Weyl groupoids and root systems. This structure is a generalization of the Weyl group of a Kac-Moody Lie algebra. The main idea is to start with a family of Cartan matrices instead of a single Cartan matrix. The references are [HY08] and [CH08].

Let I be a non-empty finite set and $\{\alpha_i \,|\, i \in I\}$ the standard basis of $\mathbb{Z}^I$. Recall from [Kac90, §1.1] that a generalized Cartan matrix $C = (c_{ij})_{i,j \in I}$ is a matrix in $\mathbb{Z}^{I \times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \leq 0$ for all $i, j, k \in I$ with $j \neq k$,

(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

A groupoid is a category, where all morphisms are isomorphisms. (If there is only one object in the category, then the morphisms of the groupoid form a group.)

Definition 4.1. Let A be a non-empty set, $r_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $r_i^2 = \mathrm{id}$ for all $i \in I$,
(C2) $c^a_{ij} = c^{r_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

We say that $\mathcal{C}$ is *connected*, if the group $\langle r_i \,|\, i \in I\rangle \subset \mathrm{Aut}(A)$ acts transitively on A, that is, if for all $a, b \in A$ with $a \neq b$ there exist $n \in \mathbb{N}$, $a_1, a_2, \dots, a_n \in A$, and $i_1, i_2, \dots, i_{n-1} \in I$ such that

$$a_1 = a, \quad a_n = b, \quad a_{j+1} = r_{i_j}(a_j) \quad \text{for all } j = 1, \dots, n-1.$$

Two Cartan schemes $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ and $\mathcal{C}' = \mathcal{C}'(I', A', (r'_i)_{i\in I'}, (C'^a)_{a\in A'})$ are termed *equivalent*, if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(r_i(a)) = r'_{\varphi_0(i)}(\varphi_1(a)), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij}$$

for all $i, j \in I$ and $a \in A$.

Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{4.1}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are generated by the maps $\sigma^a_i \in \mathrm{Hom}(a, r_i(a))$ with $i \in I$, $a \in A$. Formally, for $a, b \in A$ the set $\mathrm{Hom}(a, b)$ consists of the triples (b, f, a), where

$$f = \sigma^{r_{i_{n-1}}\cdots r_{i_1}(a)}_{i_n} \cdots \sigma^{r_{i_1}(a)}_{i_2}\sigma^a_{i_1}$$

and $b = r_{i_n} \cdots r_{i_2} r_{i_1}(a)$ for some $n \in \mathbb{N}_0$ and $i_1, \dots, i_n \in I$. The composition is induced by the group structure of $\mathrm{Aut}(\mathbb{Z}^I)$:

$$(a_3, f_2, a_2) \circ (a_2, f_1, a_1) = (a_3, f_2 f_1, a_1)$$

for all $(a_3, f_2, a_2), (a_2, f_1, a_1) \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. By abuse of notation we will write $f \in \mathrm{Hom}(a, b)$ instead of $(b, f, a) \in \mathrm{Hom}(a, b)$.

The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$.

The Weyl groupoid $\mathcal{W}(\mathcal{C})$ of a Cartan scheme $\mathcal{C}$ is a groupoid. Indeed, (M1) implies that $\sigma_i^a \in \mathrm{Aut}(\mathbb{Z}^I)$ is a reflection for all $i \in I$ and $a \in A$, and hence the inverse of $\sigma_i^a \in \mathrm{Hom}(a, r_i(a))$ is $\sigma_i^{r_i(a)} \in \mathrm{Hom}(r_i(a), a)$ by (C1) and (C2). Therefore each morphism of $\mathcal{W}(\mathcal{C})$ is an isomorphism.

If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes, then $\mathcal{W}(\mathcal{C})$ and $\mathcal{W}(\mathcal{C}')$ are isomorphic groupoids.

Recall that a groupoid G is called *connected*, if for each $a, b \in \mathrm{Ob}(G)$ the class $\mathrm{Hom}(a, b)$ is non-empty. Hence $\mathcal{W}(\mathcal{C})$ is a connected groupoid if and only if $\mathcal{C}$ is a connected Cartan scheme.

Convention 4.2. Usually, upper indices in symbols related to Cartan schemes refer to elements of A. We will omit these indices, if they are uniquely determined by the context. For example, it is sufficient to write

$$\sigma_{i_n} \cdots \sigma_{i_2} \sigma_{i_1}^a \quad \text{or} \quad \sigma_{i_n} \cdots \sigma_{i_2} \sigma_{i_1} 1_a$$

instead of $\sigma_{i_n}^{r_{i_{n-1}} \cdots r_{i_1}(a)} \cdots \sigma_{i_2}^{r_{i_1}(a)} \sigma_{i_1}^a$.

Definition 4.3. Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i \in I}, (C^a)_{a \in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subset \mathbb{Z}^I$, and define $m_{i,j}^a = |R^a \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a \in A})$$

is a *root system of type* $\mathcal{C}$, if it satisfies the following axioms.

(R1) $R^a = R_+^a \cup -R_+^a$, where $R_+^a = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.
(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.
(R3) $\sigma_i^a(R^a) = R^{r_i(a)}$ for all $i \in I$, $a \in A$.
(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m_{i,j}^a$ is finite, then $(r_i r_j)^{m_{i,j}^a}(a) = a$.

If $\mathcal{R}$ is a root system of type $\mathcal{C}$, then we say that $\mathcal{W}(\mathcal{R}) = \mathcal{W}(\mathcal{C})$ is the *Weyl groupoid of* $\mathcal{R}$. Further, $\mathcal{R}$ is called *connected*, if $\mathcal{C}$ is a connected Cartan scheme. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a \in A})$ is a root system of type $\mathcal{C}$ and

$\mathcal{R}' = \mathcal{R}'(\mathcal{C}', (R'^a_{a\in A'}))$ is a root system of type $\mathcal{C}'$, then we say that $\mathcal{R}$ and $\mathcal{R}'$ are *equivalent*, if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes given by maps $\varphi_0 : I \to I'$, $\varphi_1 : A \to A'$ as in Def. 4.1, and if the map $\varphi_0^* : \mathbb{Z}^I \to \mathbb{Z}^{I'}$ given by $\varphi_0^*(\alpha_i) = \alpha_{\varphi_0(i)}$ satisfies $\varphi_0^*(R^a) = R'^{\varphi_1(a)}$ for all $a \in A$.

Remark 4.4. Reduced root systems with a fixed basis, see [Bou68, Ch. VI,§1.5], are examples of root systems of type $\mathcal{C}$ in the following way. Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme, such that $A = \{a\}$ has only one element, and C^a is a Cartan matrix of finite type. Then $r_i = \mathrm{id}$ for all $i \in I$. Let R^a be the reduced root system associated to C^a, where the basis $\{\alpha_i \,|\, i \in I\}$ of $\mathbb{Z}^I$ is identified with a basis of R^a. Then $\mathcal{R} = \mathcal{R}(\mathcal{C}, R^a)$ is a root system of type $\mathcal{C}$. Note that (R4) holds trivially, since $r_i = \mathrm{id}$ for all i.

Example 4.5. There are many examples of Weyl groupoids, where all Cartan matrices are different. The simplest one is the following.

Let $I = \{1, 2\}$, $A = \{x, y\}$, $r_2 = \mathrm{id}$, $r_1(1) = 2$, $r_1(2) = 1$,

$$C^x = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \qquad C^y = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Let $\mathcal{C}$ be the corresponding Cartan scheme. Then there is a unique root system of type $\mathcal{C}$. The positive roots are

$$R^x_+ = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\},$$
$$R^y_+ = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right\}.$$

This root system is not artificial: there is a Nichols algebra of diagonal type with such a root system.

Let $\mathcal{C}$ be a Cartan scheme, $a, b \in A$, $\omega \in \mathrm{Hom}(a, b) \subset \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. Define the *length* of ω by

$$\ell(\omega) = \min\{m \in \mathbb{N}_0 \,|\, \omega = \sigma_{i_1} \cdots \sigma_{i_m} 1_a, i_1, \dots, i_m \in I\}. \tag{4.2}$$

Let $\mathcal{C}$ be a Cartan scheme and $\mathcal{R}$ a root system of type $\mathcal{C}$.

Proposition 4.6. *Let $a, b \in A$, $\omega \in \mathrm{Hom}(a, b)$, and $i \in I$.*

(1) $\ell(\omega^{-1}) = \ell(\omega)$ and $\ell(\omega\sigma_i) = \ell(\omega) \pm 1$.

(2) We have $\ell(\omega\sigma_i) > \ell(\omega)$ if and only if $\omega(\alpha_i) > 0$, and $\ell(\omega\sigma_i) < \ell(\omega)$ if and only if $\omega(\alpha_i) < 0$.

(3) $\ell(\omega) = |\{\alpha \in R_a^+ \,|\, \omega(\alpha) < 0\}|$.

(4) Let $j \in I$. Then $\omega(\alpha_i) = \alpha_j$ if and only if $\ell(\sigma_j\omega) = \ell(\omega\sigma_i)$, $\ell(\sigma_i\omega\sigma_j) = \ell(\omega)$.

Proposition 4.7. *Let $a \in A$. Assume that R^a is finite. Then there is a unique element $\omega \in \mathrm{Hom}(a, ?) = \cup_{b \in A}\mathrm{Hom}(a, b)$ of maximal length. The length of ω is $|R_+^a|$.*

A very important fact is the following.

Theorem 4.8. *Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Let $\mathcal{W}$ be the abstract groupoid with $\mathrm{Ob}(\mathcal{W}) = A$ such that $\mathrm{Hom}(\mathcal{W})$ is generated by abstract morphisms $s_i^a \in \mathrm{Hom}(a, r_i(a))$, where $i \in I$ and $a \in A$, satisfying the relations*

$$s_i s_i 1_a = 1_a, \quad (s_j s_k)^{m_{j,k}^a} 1_a = 1_a, \qquad a \in A,\, i, j, k \in I,\, j \neq k.$$

Here 1_a is the identity of the object a, and $(s_j s_k)^\infty 1_a$ is understood to be 1_a. The functor $\mathcal{W} \to \mathcal{W}(\mathcal{R})$, which is the identity on the objects, and on the set of morphisms is given by $s_i^a \mapsto \sigma_i^a$ for all $i \in I$, $a \in A$, is an isomorphism of groupoids.

One says that $\mathcal{W}(\mathcal{R})$ is a *Coxeter groupoid*. The most essential difference between Coxeter groupoids and Coxeter groups is the presence of several objects in the former.

WARNING! The generalization of the theory of Coxeter groups is not straightforward. The exchange condition does not hold for all Coxeter groupoids.

Recall the exchange condition for Coxeter groups:

If $k \geq 0$, $i, i_1, \cdots, i_k \in I$, $\omega = \sigma_{i_1} \cdots \sigma_{i_k}$, $\ell(\omega) = k$, $\ell(\sigma_i\omega) = k - 1$, then there exists $m \in \{1, \ldots, k\}$ such that

$$\sigma_i\omega = \sigma_{i_1} \cdots \sigma_{i_{m-1}}\sigma_{i_{m+1}} \cdots \sigma_{i_k}.$$

But a weak version of the exchange condition holds, and it can be used to prove Matsumoto's theorem for Weyl groupoids.

Now we discuss real roots and the finiteness of root systems of type $\mathcal{C}$.

Definition 4.9. Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. For all $a \in A$ let $(R^a)^{\mathrm{re}} = \{\omega(\alpha_i) \,|\, \omega \in \mathrm{Hom}(b, a), b \in A, i \in I\}$, and call $(R^a)^{\mathrm{re}}$ the *set of real roots* of a.

Note that by (R2) and (R3) we have $(R^a)^{\mathrm{re}} \subset R^a$ for all $a \in A$. The sets of real roots are interesting for various reasons, one of them is the following.

Proposition 4.10. *Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme, and let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ be a root system of type $\mathcal{C}$. Then $\mathcal{R}^{\mathrm{re}} = \mathcal{R}^{\mathrm{re}}(\mathcal{C}, ((R^a)^{\mathrm{re}})_{a\in A})$ is a root system of type $\mathcal{C}$, and $\mathcal{W}(\mathcal{R}^{\mathrm{re}}) = \mathcal{W}(\mathcal{R})$.*

Definition 4.11. Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. We say that $\mathcal{R}$ is *finite* if R^a is finite for all $a \in A$.

The finiteness of $\mathcal{R}$ does not mean that $\mathcal{W}(\mathcal{R})$ is finite, since A may be infinite. But the following holds.

Lemma 4.12. *Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Then the following are equivalent.*

(1) *$\mathcal{R}$ is finite.*
(2) *R^a is finite for at least one $a \in A$.*
(3) *$\mathcal{R}^{\mathrm{re}}$ is finite.*
(4) *$\mathcal{W}(\mathcal{R})$ is finite.*

The following proposition tells that if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then all roots are real, that is, $\mathcal{R}$ is uniquely determined by $\mathcal{C}$.

Proposition 4.13. *Let $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Let $a \in A$, $m \in \mathbb{N}_0$,* *and* $i_1,\dots,i_m \in I$ *such that* $\omega = 1_a\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}$ *and* $\ell(\omega)=m$. *Then the elements*

$$\beta_n = 1_a\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{n-1}}(\alpha_{i_n}) \in R^a_+,$$

where $n \in \{1,2,\dots,m\}$ *(and* $\beta_1 = \alpha_{i_1}$*), are pairwise different. In particular, if* $\mathcal{R}$ *is finite and* $\omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{R}))$ *is a longest element, then*

$$\{\beta_n \,|\, 1 \le n \le \ell(\omega) = |R^a_+|\} = R^a_+.$$

5. Root systems of Nichols algebras of diagonal type

We use the PBW theorem 3.14 to define a root system for any Nichols algebra of diagonal type. Under some finiteness assumptions, we also define reflections. Combining these, we get a Cartan scheme $\mathcal{C}$ and a corresponding root system of type $\mathcal{C}$. The main reference is [Hec06b]. The standing assumptions are as in the previous sections.

By Theorem 3.14, the algebra $\mathfrak{B}(V)$ has a (restricted) PBW basis consisting of homogeneous elements with respect to the $\mathbb{Z}^n$-grading of $\mathfrak{B}(V)$. If the exponent of a PBW generator of $\mathbb{Z}^n$-degree α is bounded, then the bound depends on α, but not explicitly on the PBW generator.

Recall that the algebra and coalgebra structures of $\mathfrak{B}(V)$ depend on the bicharacter χ, but not explicitly on the Yetter-Drinfeld module structure of V.

Definition 5.1. Let $(L,\le)$ be a totally ordered set and $\{X_l \,|\, l \in L\} \subset \mathfrak{B}(V)$ given by Thm. 3.14. Let

$$R^\chi_+ = \{\deg X_l \,|\, l \in L\} \subset \mathbb{N}_0^n.$$

We say that R^χ_+ is the *set of positive roots of* $\mathfrak{B}(V)$. For all $\alpha \in R^\chi_+$ let

$$\mathrm{mult}^\chi(\alpha) = |\{l \in L \,|\, \deg X_l = \alpha\}|,$$

and call $\mathrm{mult}^\chi(\alpha)$ the *multiplicity of* α. By Thm. 3.14, R^χ_+ and $\mathrm{mult}^\chi : R^\chi_+ \to \mathbb{N}$ do not depend on the choice of L. Let

$$R^\chi = R^\chi_+ \cup -R^\chi_+ \subset \mathbb{Z}^n. \tag{5.1}$$

Lemma 5.2. *For all* $i=1,\dots,n$, $\alpha_i \in R^\chi_+$ *and* $\mathrm{mult}^\chi(\alpha_i)=1$. *Further, if* $m\alpha_i \in R^\chi_+$, *where* $m \in \mathbb{N}$, *then* $m=1$.

Proof. For all $i = 1, \dots, n$ and $m \in \mathbb{N}$, the $\mathbb{Z}^n$-homogeneous component of degree $m\alpha_i$, $m \geq 1$, is just $\Bbbk x_i^m$, and this is zero if and only if $m \geq h_i$, where

$$h_i = \min\{m' \geq 1 \,|\, 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^{m'-1} = 0\} \cup \{\infty\}. \tag{5.2}$$

This and Thm. 3.14 imply the claim. □

In order to define a root system for $\mathfrak{B}(V)$ in the sense of the previous section, we have to define "reflections".

For all $i \in \{1, 2, \dots, n\}$ let $\mathcal{K}_i$ be the subalgebra of $\mathfrak{B}(V)$ generated by the elements $(\mathrm{ad}_c x_i)^m(x_j)$, $m \geq 0$, $j \neq i$, where

$$(\mathrm{ad}_c x_i)(y) = x_i y - (g_i \cdot y)x_i. \tag{5.3}$$

By choosing the ordering on $\{1, 2, \dots, n\}$ properly, Thm. 3.13 implies the following.

Lemma 5.3. *Let $i \in \{1, 2, \dots, n\}$ and let $h_i \in \mathbb{N} \cup \{\infty\}$ be as in Eq. (5.2). If $h_i = \infty$, then $\mathfrak{B}(V) \simeq \mathcal{K}_i \otimes \Bbbk[x_i]$, and if $h_i \in \mathbb{N}$, then $\mathfrak{B}(V) \simeq \mathcal{K}_i \otimes \Bbbk[x_i]/(x_i^{h_i})$ as $\mathbb{Z}^n$-graded objects in ${}^H_H\mathcal{YD}$.*

Proposition 5.4. *For all $i \in \{1, 2, \dots, n\}$, $\mathcal{K}_i = \ker \partial_i^{\mathrm{R}}$ is a left coideal subalgebra of $\mathfrak{B}(V)$.*

Proof. Let $i \in \{1, 2, \dots, n\}$. For relation $\mathcal{K}_i \subset \ker \partial_i^{\mathrm{R}}$ it suffices to check that $\partial_i^{\mathrm{R}}((\mathrm{ad}_c x_i)^m(x_j)) = 0$ for all $m \in \mathbb{N}_0$. This can be done by induction on m. The inclusion $\ker \partial_i^{\mathrm{R}} \subset \mathcal{K}_i$ follows then from Lemma 5.3. Finally, $\mathcal{K}_i$ is a left coideal of $\mathfrak{B}(V)$ by Lemma 2.10. □

Using the maps ∂_j^{L}, $j = 1, \dots, n$ and the $\mathbb{Z}^n$-grading of $\mathfrak{B}(V)$, one can check the following proposition, which essentially goes back to Rosso.

Proposition 5.5. *Let $i \in \{1, 2, \dots, n\}$. The following are equivalent.*

1. *The algebra $\mathcal{K}_i$ is finitely generated.*
2. *For all $j \in \{1, 2, \dots, n\}$, $j \neq i$, there exists $m_{ij} \geq 0$ such that $(\mathrm{ad}_c x_i)^{m_{ij}+1}(x_j) = 0$.*
3. *For all $j \in \{1, 2, \dots, n\}$, $j \neq i$, there exists $m_{ij} \geq 0$ such that*
$$1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^{m_{ij}} = 0 \quad \text{or} \quad q_{ii}^{m_{ij}} q_{ij} q_{ji} = 1.$$

For all i, j with $j \neq i$, m_{ij} in (2) and m_{ij} in (3) can be chosen to be the same.

Definition 5.6. Let $i \in \{1, 2, \ldots, n\}$. We say that $\mathfrak{B}(V)$ is *i-finite*, if $\mathcal{K}_i$ is a finitely generated algebra. In this case, let

$$c_{ij}^{\chi} = -\min\{m \in \mathbb{N}_0 \mid 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^m = 0 \quad \text{or} \quad q_{ii}^m q_{ij} q_{ji} = 1\}$$

and $c_{ii}^{\chi} = 2$.

Since the algebra structure of $\mathcal{K}_i$ depends only on the bicharacter χ, and not on the Yetter-Drinfeld structure of V, we will also use the terminology "χ is i-finite" instead of "$\mathfrak{B}(V)$ is i-finite".

We are mainly interested in the case when $\mathfrak{B}(V)$ is i-finite for all $i \in \{1, 2, \ldots, n\}$.

Easy exercise: if $\mathfrak{B}(V)$ is i-finite for all $i = 1, 2, \ldots, n$, then $C = (c_{ij}^{\chi})_{i,j=1,\ldots,n}$ is a generalized Cartan matrix.

Note that $\Bbbk\partial_i^{\mathrm{L}} = (\Bbbk x_i)^* \in {}_H^H\mathcal{YD}$ as in Prop. 1.7.

Theorem 5.7. *Let $i \in \{1, 2, \ldots, n\}$. Let $\mathfrak{B}_i$ be the subalgebra of* $\mathrm{End}(\mathfrak{B}(V))$ *generated by the endomorphisms*

$$\partial_i^{\mathrm{L}} \quad \textit{and} \quad L_x : y \mapsto xy, \quad x \in \mathcal{K}_i.$$

Then the elements $(\partial_i^{\mathrm{L}})^m L_x$, where $m \geq 0$ and $x \in \mathcal{K}_i$, span the vector space $\mathfrak{B}_i$. If $\mathfrak{B}(V)$ is i-finite, then $\mathfrak{B}_i$ is generated as an algebra by $V_i \in {}_H^H\mathcal{YD}$, where

$$V_i = \Bbbk\partial_i^{\mathrm{L}} \oplus \bigoplus_{j \neq i} \Bbbk(\mathrm{ad}_c x_i)^{-c_{ij}^{\chi}}(x_j).$$

In this case, there exists a coalgebra structure and an antipode on $\mathfrak{B}_i$ such that the identity on V_i induces an isomorphism $\mathfrak{B}_i \simeq \mathfrak{B}(V_i)$ of graded braided Hopf algebras in ${}_H^H\mathcal{YD}$.

Proof. (Idea) The first part of the theorem follows from the fact that the q-commutator of ∂_i^{L} and L_x, $x \in \mathcal{K}_i$, is $L_{\partial_i^{\mathrm{L}}(x)}$. This can be checked by induction on the degree of x. A similar argument gives that if $\mathfrak{B}(V)$ is i-finite, then $\mathfrak{B}_i$ is generated by V_i.

The second part is proven in two steps, using mainly Thm. 2.11. First one constructs skew-differential operators on $\mathfrak{B}_i$, which form a dual basis of a $\mathbb{Z}^n$-homogeneous basis of V_i. For this one uses skew-differential operators on $\mathfrak{B}(V)$. Let $\mathfrak{B}'_i$ be the quotient of $\mathfrak{B}_i$ by the largest $\mathbb{Z}^n$-homogeneous ideal which has no elements of degree 0 and which is stable under all of these skew-derivations on $\mathfrak{B}_i$. Then $\mathfrak{B}'_i = \mathfrak{B}(V_i)$ by Thm. 2.11. Let χ' be the bicharacter corresponding to V_i. Apply the procedure to $\mathfrak{B}(V_i)$ to get $\mathfrak{B}((V_i)_i)$. It turns out, that $(V_i)_i$ is isomorphic to V in ${}^H_H\mathcal{YD}$, and that $c^\chi_{ij} = c^{\chi'}_{ij}$ for all j. By construction, the dimension of the $\mathbb{Z}^n$-homogeneous component of $\mathfrak{B}((V_i)_i)$ of degree α is at most the dimension of the $\mathbb{Z}^n$-homogeneous component of $\mathfrak{B}(V)$ of degree α, for all $\alpha \in \mathbb{Z}^n$. But since $(V_i)_i \simeq V$ in ${}^H_H\mathcal{YD}$, these dimensions have to be equal, and hence $\mathfrak{B}_i$ is isomorphic to $\mathfrak{B}(V_i)$. □

A first consequence is the following.

Proposition 5.8. *Let $i \in \{1, 2, \dots, n\}$. If $\mathfrak{B}(V)$ is i-finite, then $\dim \mathfrak{B}(V) = \dim \mathfrak{B}(V_i)$ (vector space dimension).*

It comes better.

Theorem 5.9. *Let $i \in \{1, 2, \dots, n\}$. Assume that $\mathfrak{B}(V)$ is i-finite. Let $\sigma^\chi_i \in \mathrm{End}(\mathbb{Z}^n)$ such that $\sigma^\chi_i(\alpha_j) = \alpha_j - c^\chi_{ij}(\alpha_i)$ for all $j = 1, \dots, n$, and let χ' be the bicharacter corresponding to V_i. Then*

$$\sigma^\chi_i(R^\chi_+ \setminus \{\alpha_i\}) = R^{\chi'}_+ \setminus \{\alpha_i\}, \quad \sigma^\chi_i(R^\chi) = R^{\chi'}.$$

Moreover, $\mathrm{mult}^{\chi'}(\sigma^\chi_i(\alpha)) = \mathrm{mult}^\chi(\alpha)$ *for all $\alpha \in R^\chi_+ \setminus \{\alpha_i\}$.*

Proof. This follows by looking at the $\mathbb{Z}^n$-homogeneous PBW-basis of $\mathcal{K}_i$, which is both a subalgebra of $\mathfrak{B}(V)$ and of $\mathfrak{B}_i \simeq \mathfrak{B}(V_i)$. Note also that $\sigma^\chi_i(\alpha_i) = -\alpha_i$. □

Let us calculate the bicharacter corresponding to V_i in Thm. 5.7. For all $w \in \mathrm{Aut}(\mathbb{Z}^n)$, let $w^*\chi$ be the bicharacter on $\mathbb{Z}^n$ defined by $w^*\chi(\alpha, \beta) = \chi(w^{-1}(\alpha), w^{-1}(\beta))$. Recall that $\sigma^\chi_i = (\sigma^\chi_i)^{-1}$ for all $i = 1, \dots, n$.

Proposition 5.10. *Let $i \in \{1, 2, \dots, n\}$. Assume that $\mathfrak{B}(V)$ is i-finite, and let V_i be as in Thm. 5.7. Then the bicharacter corresponding to V_i is $(\sigma^\chi_i)^*\chi$.*

Proof. Let $x_i' = \partial_i^{\mathrm{L}}$ and $x_j' = (\mathrm{ad}_c x_i)^{-c_{ij}^\chi}(x_j)$ for all $j \neq i$. The bicharacter χ' corresponding to V_i is determined by the rule

$$((x_j')_{(-1)} \cdot x_k') \otimes (x_j')_{(0)} = \chi'(\alpha_j, \alpha_k) x_k' \otimes x_j', \quad j,k = 1,\dots,n.$$

Check that

$$g \cdot x_k' = \xi_k(g)\xi_i^{-c_{ik}^\chi}(g) x_k', \qquad \Delta_{\mathrm{L}}(x_j') = g_j g_i^{-c_{ij}^\chi} \otimes x_j'$$

for all $j,k = 1,\dots,n$, $g \in G$. Thus

$$\begin{aligned}\chi'(\alpha_j,\alpha_k)x_k' =& g_j g_i^{-c_{ij}^\chi} \cdot x_k' \\ =& q_{jk} q_{ik}^{-c_{ij}^\chi} q_{ji}^{-c_{ik}^\chi} q_{ii}^{(-c_{ij}^\chi)(-c_{ik}^\chi)} x_k' = \chi(\sigma_i^\chi(\alpha_j), \sigma_i^\chi(\alpha_k)) x_k'\end{aligned}$$

for all $j,k = 1,\dots,n$. $\square$

We are ready to define the Cartan scheme and the root system associated to a large class of Nichols algebras of diagonal type.

Let $\mathcal{X}$ denote the set of all bicharacters on $\mathbb{Z}^n$. For all $i \in \{1,2,\dots,n\}$ let

$$r_i : \mathcal{X} \to \mathcal{X}, \quad r_i(\bar\chi) = \begin{cases} (\sigma_i^{\bar\chi})^* \bar\chi & \text{if } \bar\chi \text{ is } i\text{-finite,} \\ \bar\chi & \text{otherwise.} \end{cases} \tag{5.4}$$

Then $r_i^2 = \mathrm{id}$, and hence the maps r_i are bijections for all $i = 1,\dots,n$. Let $\mathcal{G}$ be the subgroup

$$\mathcal{G} = \langle r_i \,|\, i = 1,\dots,n \rangle$$

of the group of bijections of $\mathcal{X}$. Let $\mathcal{G}(\chi) = \{r(\chi) \,|\, r \in \mathcal{G}\}$ be the $\mathcal{G}$-orbit of χ in $\mathcal{X}$.

Theorem 5.11. *Assume that $\bar\chi$ is i-finite for all $\bar\chi \in \mathcal{G}(\chi)$ and $i = 1,\dots,n$. Let $I = \{1,2,\dots,n\}$, $A = \mathcal{G}(\chi)$, $r_i : A \to A$ for $i \in I$ as in Eq.* (5.4)*, and $C^{\bar\chi} = (c_{ij}^{\bar\chi})_{i,j \in I}$ for $\bar\chi \in A$ such that $c_{ii}^{\bar\chi} = 2$ for all $i \in I$ and*

$$c_{ij}^{\bar\chi} = -\min\{m \in \mathbb{N}_0 \,|\, (1 + \bar q_{ii} + \cdots + \bar q_{ii}^m)(\bar q_{ii}^m \bar q_{ij} \bar q_{ji} - 1) = 0\} \tag{5.5}$$

for all $i,j \in I$, $i \neq j$, where $\bar q_{kl} = \bar\chi(\alpha_k,\alpha_l)$ for all $k,l \in I$. Then $\mathcal{C} = \mathcal{C}(I, A, (r_i)_{i\in I}, (C^{\bar\chi})_{\bar\chi \in A})$ is a Cartan scheme. For all $\bar\chi \in A$ let $R^{\bar\chi} \subset \mathbb{Z}^n$ be as in Eq. (5.1)*. Then $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^{\bar\chi})_{\bar\chi\in A})$ is a root system of type $\mathcal{C}$.*

Proof. The essential point is Thm. 5.9, which establishes Axiom (R3). Axiom (R4) follows from the fact that each $\mathbb{Z}^n$-homogeneous subspace $W \subset \mathfrak{B}(V)_\alpha \subset \mathfrak{B}(V)$ of $\mathbb{Z}^n$-degree α is semisimple, and all simple Yetter-Drinfeld submodules of W are isomorphic. The structure of this simple module is determined uniquely by V and α. □

Definition 5.12. Assume that $\bar\chi$ is i-finite for all $\bar\chi \in \mathcal{G}(\chi)$ and $i = 1, \dots, n$. We write $\mathcal{C}(\chi)$ and $\mathcal{R}(\chi)$ for the Cartan scheme and the root system associated to χ, respectively. The Weyl groupoid $\mathcal{W}(\chi)$ of χ is then just the Weyl groupoid $\mathcal{W}(\mathcal{C})$ of the Cartan scheme $\mathcal{C}$.

Let us calculate two examples. Recall that $(q_{ij})_{i,j=1,\dots,n}$ are the constants associated to V, see Eq. (2.3).

Example 5.13. Assume that V is as in Ex. 2.6, that is, associated to a symmetrizable generalized Cartan matrix $C = (c_{ij})_{i,j=1,\dots,n}$. Let $k \in \{1, 2, \dots, n\}$ and $\chi' = (\sigma_k^\chi)^*\chi$. Then $c_{ij}^\chi = c_{ij}$ for all $i, j = 1, \dots, n$. Thus, similarly to the calculation in the proof of Prop. 5.10, we obtain that

$$\chi'(\alpha_i, \alpha_j) = q_{ij} q_{kj}^{-c_{ki}} q_{ik}^{-c_{kj}} q_{kk}^{c_{ki}c_{kj}} = q_{ij} q^{-d_k c_{kj} c_{ki}} q^{-d_i c_{ik} c_{kj}} q^{2 d_k c_{ki} c_{kj}} = q_{ij}$$

for all $i, j = 1, \dots, n$. Thus $\chi' = \chi$, and hence $\mathcal{G}(\chi) = \chi$. Since the Cartan scheme $\mathcal{C}(\chi)$ has only one object, $A = \{\chi\}$, the Weyl groupoid $\mathcal{W}(\chi)$ is a group generated by the reflections σ_i^χ, $i = 1, 2, \dots, n$, and hence $\mathcal{W}(\chi)$ is isomorphic to the Weyl group of the Kac-Moody Lie algebra associated to C.

Example 5.14. Assume that $n = 2$, $q_{11} = q_{22} = -1$, and $q_{12} = q_{21} = q$, where $q \in \Bbbk^\times$, $q^2 \neq 1$. Let $\chi_1 = r_1(\chi)$, $\chi_2 = r_2(\chi)$. Explicit calculations show that

$$C^{\chi_1} = C^{\chi_2} = C^\chi = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

$$\big(r_1(\chi)(\alpha_i, \alpha_j)\big)_{i,j=1,2} = \begin{pmatrix} -1 & -q^{-1} \\ -q^{-1} & q^2 \end{pmatrix},$$

$$\big(r_2(\chi)(\alpha_i, \alpha_j)\big)_{i,j=1,2} = \begin{pmatrix} q^2 & -q^{-1} \\ -q^{-1} & -1 \end{pmatrix},$$

and $r_2r_1(\chi) = r_1(\chi)$, $r_1r_2(\chi) = r_2(\chi)$. Hence $A = \{\chi, r_1(\chi), r_2(\chi)\}$. This example corresponds to the Lie superalgebra $\mathfrak{sl}(2|1)$ with three different choices of the Cartan subalgebra.

We will see later that there are many examples (even under additional finiteness assumptions) which do not correspond to semisimple Lie algebras or Lie superalgebras.

6. Nichols algebras of diagonal type with finite root system

For the classification of finite-dimensional pointed Hopf algebras it is important to know, which Nichols algebras of diagonal type are finite-dimensional. The root system and the Weyl groupoid of a bicharacter allow us to give a complete answer to this problem. We study the more general case when the root system (in the sense of the previous section) is finite. The references are [Hec08], [Hec05], [Hec06a].

Let G, H, V, χ as before. The classification is based on the following theorems.

Theorem 6.1. *The following are equivalent.*

(1) *R^χ is finite.*
(2) *$\bar{\chi}$ is i-finite for all $i \in \{1, 2, \dots, n\}$ and all $\bar{\chi} \in \mathcal{G}(\chi)$, and $\mathcal{W}(\chi)$ is a finite Weyl groupoid.*

Proof. (2) $\Rightarrow$ (1) follows from Lemma 4.12.

(1) $\Rightarrow$ (2). Assume that R^χ is finite. Let $\bar{\chi} \in \mathcal{G}(\chi)$. Then $R^{\bar{\chi}}$ is finite by Thm. 5.9. Hence for all $i, j = 1, 2, \dots, n$ with $i \neq j$ there exists $m \geq 0$ such that $\alpha_j + m\alpha_i \notin R^{\bar{\chi}}_+$. Thus $\bar{\chi}$ is i-finite for all $i \in \{1, 2, \dots, n\}$. Then $\mathcal{C}(\chi)$ is a connected Cartan scheme and $\mathcal{W}(\chi)$ is a finite Weyl groupoid by Lemma 4.12. □

It is now easy to characterize finite-dimensional Nichols algebras of diagonal type.

Theorem 6.2. *The following are equivalent.*

(1) *The Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional as a vector space.*
(2) *The set R^χ is finite, and for all $\alpha \in R^\chi_+$ there exists $m \in \mathbb{N}$ such that $1 + q_\alpha + q_\alpha^2 + \cdots + q_\alpha^m = 0$, where $q_\alpha = \chi(\alpha, \alpha)$.*

(3) *For all* $\bar{\chi} \in \mathcal{G}(\chi)$ *and all* $i = 1, 2, \ldots, n$ *there exists* $m \in \mathbb{N}$ *such that* $1 + q_\alpha + q_\alpha^2 + \cdots + q_\alpha^m = 0$, *where* $q_\alpha = \chi(\alpha, \alpha)$. $\mathcal{W}(\chi)$ *is a finite Weyl groupoid.*

In order to simplify the classification of bicharacters with finite root system, we use two simplifications. First, the reflections σ_i^χ depend only on the numbers q_{jj} and $q_{jk}q_{kj}$, $j, k = 1, 2, \ldots, n$. Hence it is useful to introduce a new notion.

Definition 6.3. Let χ' be a bicharacter on $\mathbb{Z}^n$. The *generalized Dynkin diagram* $\mathcal{D}^{\chi'}$ of χ' is a non-oriented graph with n vertices $v_1, \ldots, v_n$, where the vertex v_i is labeled by q_{ii}. The graph has no loops. Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. If $q_{ij}q_{ji} = 1$, then there is no edge between v_i and v_j. Otherwise there is precisely one edge between them, and it is labeled by $q_{ij}q_{ji}$.

Example 6.4. The generalized Dynkin diagram of the bicharacter in Ex. 5.14 is $\overset{-1}{\circ}\overset{q^2}{\text{———}}\overset{-1}{\circ}$.

Similarly to the definition in Thm. 5.11, one can define a Cartan scheme $\mathcal{C}_s(\chi)$, where the objects correspond to generalized Dynkin diagrams. The corresponding Weyl groupoid $\mathcal{W}_s(\chi)$ is finite if and only if $\mathcal{W}(\chi)$ is finite. Hence it suffices to classify bicharacters χ, where $\mathcal{W}_s(\chi)$ is finite.

Using Prop. 5.10 we can calculate, how generalized Dynkin diagrams change under reflections. Let $i \in \{1, 2, \ldots, n\}$. Let $\chi' = (\sigma_i^\chi)^*\chi$, and $q'_{jk} = \chi'(\alpha_j, \alpha_k)$ for all $j, k = 1, \ldots, n$. Then

$$q'_{ii} = \chi(\sigma_i^\chi(\alpha_i), \sigma_i^\chi(\alpha)_i) = \chi(-\alpha_i, -\alpha_i) = q_{ii},$$

$$\begin{aligned} q'_{ij}q'_{ji} =& \chi(\sigma_i^\chi(\alpha_i), \sigma_i^\chi(\alpha_j))\chi(\sigma_i^\chi(\alpha_j), \sigma_i^\chi(\alpha_i)) \\ =& \chi(-\alpha_i, \alpha_j - c_{ij}^\chi \alpha_i)\chi(\alpha_j - c_{ij}^\chi\alpha_i, -\alpha_i) = (q_{ij}q_{ji})^{-1} q_{ii}^{2c_{ij}^\chi} \end{aligned}$$

for all $j \neq i$. Similar calculations show the following.

$$\begin{gathered} q'_{ii} = q_{ii}, \quad q'_{ij}q'_{ji} = q_{ii}^{2c_{ij}^\chi}(q_{ij}q_{ji})^{-1}, \quad q'_{jj} = q_{jj}(q_{ij}q_{ji})^{-c_{ij}^\chi} q_{ii}^{(c_{ij}^\chi)^2}, \\ q'_{jk}q'_{kj} = q_{jk}q_{kj}(q_{ij}q_{ji})^{-c_{ik}^\chi}(q_{ik}q_{ki})^{-c_{ij}^\chi} q_{ii}^{2c_{ij}^\chi c_{ik}^\chi} \end{gathered} \tag{6.1}$$

for all $j,k=1,\dots,n$, $j,k\neq i$. One can simplify this a little bit. For all $j=1,\dots,n$ let $p_j\in\Bbbk^\times$,

$$p_j=q_{ii}^{-c^\chi_{ij}}q_{ij}q_{ji}=\begin{cases}1 & \text{if } q_{ii}^{-c^\chi_{ij}}q_{ij}q_{ji}=1,\\ q_{ii}^{-1}q_{ij}q_{ji} & \text{otherwise.}\end{cases}\tag{6.2}$$

Then we get

$$q'_{ii}{=}q_{ii},\ q'_{ij}q'_{ji}{=}q_{ij}q_{ji}p_j^{-2},\ q'_{jj}{=}q_{jj}p_j^{-c^\chi_{ij}},\ q'_{jk}q'_{kj}{=}q_{jk}q_{kj}p_j^{-c^\chi_{ik}}p_k^{-c^\chi_{ij}}\tag{6.3}$$

for all $j,k=1,\dots,n$, $j,k\neq i$.

The second simplification is that we are allowed to look only at connected generalized Dynkin diagrams. The reason is the following proposition.

Proposition 6.5. *Assume that there is a decomposition $\{1,2,\dots,n\}=I'\sqcup I''$ into disjoint non-empty sets, such that $q_{ij}q_{ji}=1$ for all $i\in I'$, $j\in I''$. (Equivalently, the generalized Dynkin dagram of χ is not connected.) Let $V=V'\oplus V''$ be the corresponding decomposition into Yetter-Drinfeld modules. Then $\mathfrak{B}(V)\simeq\mathfrak{B}(V')\otimes\mathfrak{B}(V'')$ as $\mathbb{Z}^n$-graded objects in ${}^H_H\mathcal{YD}$.*

In the situation of Prop. 6.5 we say that χ is *reducible*. Otherwise χ is *irreducible*.

From now on assume that $\operatorname{char}\Bbbk=0$. In this case there is a complete list of bicharacters on $\mathbb{Z}^n$, $n\geq 1$, for which the generalized Dynkin diagram is connected and the root system is finite.

The classification is done by induction on n. The first crucial step is the case $n=2$. So far, there exists only one proof, and this needs some calculations with Lyndon words.

Lemma 6.6. *Assume that $n=2$, χ is irreducible, and R^χ_+ is finite. Then*

$$c^\chi_{12}=-1 \quad \text{or} \quad c^\chi_{21}=-1 \quad \text{or}$$
$$q_{11}q_{12}^2q_{21}^2q_{22}=-1,\quad c^\chi_{12}\in\{-2,-3\}.$$

Proof. (Idea) If R^χ_+ is finite, then $2\alpha_1+2\alpha_2$ is not a root. Hence there is no PBW generator of this degree. Thus

$$[x_1,[[x_1,x_2]_q,x_2]_q]_q$$

is not a PBW generator. By tedious calculations one can show that the latter condition implies the claim. □

Another ingredient of the classification is a small lemma on Weyl groupoids.

Lemma 6.7. *Assuem that χ is irreducible and that R^χ_+ is finite. Then there exists $\bar\chi \in \mathcal{G}(\chi)$ such that $c^{\bar\chi}_{12} = -1$ or $c^{\bar\chi}_{21} = -1$. Moreover, if $c^{\chi'}_{12} = -1$ for all $\chi' \in \mathcal{G}(\chi)$, then there exists $\bar\chi \in \mathcal{G}(\chi)$ such that $c^{\bar\chi}_{21} \in \{-1, -2, -3\}$.*

The general strategy of the classification of irreducible bicharacters of rank two (with $\operatorname{char} \Bbbk = 0$) having finite root system is as follows.

Case 1: There is no object $\bar\chi \in \mathcal{G}(\chi)$ with $c^{\bar\chi}_{12} = -1$ or $c^{\bar\chi}_{21} = -1$. Then R^χ is infinite by Lemma 6.7.

Case 2: If there is an object $\bar\chi \in \mathcal{G}(\chi)$ with $c^{\bar\chi}_{12} = -1$, $c^{\bar\chi}_{21} = -1$, then R^χ is always finite (there are 3 positive roots).

Case 3: If for all $\bar\chi \in \mathcal{G}(\chi)$ one has $c^{\bar\chi}_{12} = -1$, $c^{\bar\chi}_{21} < -1$, then there is $\chi' \in \mathcal{G}(\chi)$ such that $c^{\chi'}_{21} \in \{-2, -3\}$ by Lemma 6.7. Without loss of generality assume that $\chi' = \chi$. If $q_{11} \neq -1$ or $1 + q_{22} + q_{22}^2 \neq 0$, then $R^{\bar\chi}$ s finite. Otherwise apply the reflection σ^χ_2, then σ_1, then σ_2. Since $c^{\bar\chi}_{12} = -1$ for all $\bar\chi \in \mathcal{G}(\chi)$, we get sufficiently many conditions on the numbers q_{11}, $q_{12}q_{21}$, q_{22} to solve the finiteness problem for R^χ.

Case 4: There exists $\bar\chi \in \mathcal{G}(\chi)$ such that $c^{\bar\chi}_{12} = -1$, $c^{\bar\chi}_{21} < -1$, and $c^{\chi'}_{12} < -1$, where $\chi' = (\sigma^{\bar\chi}_2)^*\bar\chi$. Without loss of generality assume that $\bar\chi = \chi$. Then relation $c^\chi_{12} = -1$ gives an equation for the numbers q_{11}, $q_{12}q_{21}$, q_{22}, and Lemma 6.6 gives two other equations. One has to consider several different cases separately, but in each case one can decide easily if $\mathcal{W}(\chi)$ is finite.

The result of the classification is 16 different Weyl groupoids of rank two corresponding to irreducible bicharacters with finite root system. Most of them consist of several objects. The appearing generalized Dynkin diagrams are listed in Table 1. The symbol μ_m, $m \geq 2$, denotes the set of primitive mth roots of 1.

The generalized Dynkin diagrams in Row 13 of Table 1 define a Weyl groupoid as in Ex. 4.5.

WARNING! Not each Weyl groupoid of rank two appears as the Weyl groupoid of a bicharacter of rank two.

7. Drinfeld doubles of Nichols algebras

The main reference for this section is [Hec07].

The Drinfeld double construction is based on a skew-pairing of Hopf algebras, and goes back to Drinfeld. It gives a way to define a Hopf algebra using two other Hopf algebras. We apply this construction to bosonozations of Nichols algebras of diagonal type.

We follow the approach in [Jos95, Sect. 3.2].

7.1. **Drinfeld doubles of Nichols algebras of diagonal type.**

Definition 7.1. Let A, B be two Hopf algebras with bijective antipodes. A *skew-Hopf pairing* of A and B is a bilinear map

$$\eta : A \times B \to \Bbbk, \quad (x, y) \mapsto \eta(x, y),$$

such that

$$\eta(1, y) = \varepsilon(y), \qquad \eta(x, 1) = \varepsilon(x), \tag{7.1}$$

$$\eta(xx', y) = \eta(x', y_{(1)})\eta(x, y_{(2)}), \quad \eta(x, yy') = \eta(x_{(1)}, y)\eta(x_{(2)}, y'), \tag{7.2}$$

$$\eta(S(x), y) = \eta(x, S^{-1}(y)) \tag{7.3}$$

for all $x, x' \in A$ and $y, y' \in B$.

Remark 7.2. There is an analogous notion of a Hopf pairing of two Hopf algebras A, B. The only difference to Def. 7.1 is in the left formula in Eq. (7.2), where x and x' on the right hand side have to be exchanged. One can also say that a skew-Hopf pairing of A and B is a Hopf pairing of A and B^{cop}, where $B^{\mathrm{cop}} = B$ as an algebra and $\mathit{\Delta}_{B^{\mathrm{cop}}}(b) = b_{(2)} \otimes b_{(1)}$ whenever $\mathit{\Delta}_B(b) = b_{(1)} \otimes b_{(2)}$.

Theorem 7.3. *Let A, B be Hopf algebras and let $\eta : A \times B \to \Bbbk$ be a skew-Hopf pairing. Then there is a unique Hopf algebra structure $D(A, B)$ on $A \otimes B$ such that*

1. *$D(A, B) = A \otimes B$ as a vector space,*
2. *the maps $A \to D(A, B)$, where $x \mapsto x \otimes 1$, and $B \to D(A, B)$, where $y \mapsto 1 \otimes y$, are Hopf algebra maps,*

(3) *the product of* $D(A,B)$ *is given by*

$$(x\otimes y)(x'\otimes y') = x\eta(x'_{(1)}, S(y_{(1)}))x'_{(2)} \otimes y_{(2)}\eta(x'_{(3)}, y_{(3)})y' \tag{7.4}$$

for all $x, x' \in A$ *and* $y, y' \in B$.

By letting $x' = y = 1$ in Eq. (7.4), it follows that $D(A,B)$ is generated as an algebra by $A\otimes 1$ and $1\otimes B$.

Definition 7.4. Let A, B be Hopf algebras, and $\eta : A\times B \to \Bbbk$ a skew-Hopf pairing. The *Drinfeld double* of A and B is the vector space $A\otimes B$ with the Hopf algebra structure given in Thm. 7.3.

Let χ be a bicharacter on $\mathbb{Z}^n$ with values in $\Bbbk^\times$. Let $I = \{1,2,\dots,n\}$ and $q_{ij} = \chi(\alpha_i,\alpha_j)$ for all $i,j\in I$. Let $\mathcal{U}^{+0} = \Bbbk[K_i, K_i^{-1}\,|\, i\in I]$ and $\mathcal{U}^{-0} = \Bbbk[L_i, L_i^{-1}\,|\, i\in I]$ be two copies of the group algebra of $\mathbb{Z}^n$. Let

$$V^+(\chi) \in {}^{\mathcal{U}^{+0}}_{\mathcal{U}^{+0}}\mathcal{YD}, \quad V^-(\chi) \in {}^{\mathcal{U}^{-0}}_{\mathcal{U}^{-0}}\mathcal{YD} \tag{7.5}$$

be n-dimensional vector spaces over $\Bbbk$ with basis $\{E_i\,|\, i\in I\}$ and $\{F_i\,|\, i\in I\}$, respectively, such that the left action and the left coaction of $\mathcal{U}^{+0}$ on $V^+(\chi)$ and of $\mathcal{U}^{-0}$ on $V^-(\chi)$, respectively, are determined by the formulas

$$K_i\cdot E_j = q_{ij}E_j, \quad K_i^{-1}\cdot E_j = q_{ij}^{-1}E_j, \quad \Delta_{\mathrm{L}}(E_i) = K_i\otimes E_i, \tag{7.6}$$

$$L_i\cdot F_j = q_{ji}F_j, \quad L_i^{-1}\cdot F_j = q_{ji}^{-1}F_j, \quad \Delta_{\mathrm{L}}(F_i) = L_i\otimes F_i \tag{7.7}$$

for all $i,j\in I$. Let

$$\mathcal{U}^+(\chi) = T\big(V^+(\chi)\big), \qquad \mathcal{U}^-(\chi) = T\big(V^-(\chi)\big) \tag{7.8}$$

denote the tensor algebra of $V^+(\chi)$ and $V^-(\chi)$, respectively. The algebras $\mathcal{U}^+(\chi)$ and $\mathcal{U}^-(\chi)$ are Yetter-Drinfel'd modules over $\mathcal{U}^{+0}$ and $\mathcal{U}^{-0}$, respectively.

Recall the definition of $(\cdot)^{\mathrm{cop}}$ from Rem. 7.2. In this section we study the Drinfel'd double $D(\mathcal{V}^+(\chi), \mathcal{V}^-(\chi))$ of the Hopf algebras

$$\mathcal{V}^+(\chi) = \mathcal{U}^+(\chi)\#\mathcal{U}^{+0}, \qquad \mathcal{V}^-(\chi) = (\mathcal{U}^-(\chi)\#\mathcal{U}^{-0})^{\mathrm{cop}} \tag{7.9}$$

and quotients of it. Here $\#$ denotes Radford's biproduct, see Prop. 1.10. In particular, one has

$$K_iE_j = q_{ij}E_jK_i, \qquad L_iF_j = q_{ji}F_jL_i \tag{7.10}$$

for all $i, j \in I$, and the counits and coproducts are determined by the equations

$$(7.11) \quad \begin{cases} \varepsilon(K_i) = 1, \quad \varepsilon(E_i) = 0, & \varepsilon(L_i) = 1, \quad \varepsilon(F_i) = 0, \\ \Delta(K_i) = K_i \otimes K_i, & \Delta(L_i) = L_i \otimes L_i, \\ \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}, & \Delta(L_i^{-1}) = L_i^{-1} \otimes L_i^{-1}, \\ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, & \Delta(F_i) = 1 \otimes F_i + F_i \otimes L_i \end{cases}$$

for all $i \in I$. The existence of the antipode follows from [Tak71].

The Hopf algebras $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ have a unique $\mathbb{Z}^n$-grading with $\deg E_i = \alpha_i$, $\deg F_i = -\alpha_i$, $\deg K_i = \deg L_i = 0$ for all $i \in I$. This induces unique $\mathbb{N}_0$-gradings with $\deg E_i = \deg F_i = 1$, $\deg K_i = \deg L_i = 0$ for all $i \in I$.

For any given bicharacter χ on $\mathbb{Z}^n$ we fix the skew-Hopf pairing given by the following proposition.

Proposition 7.5. *(i) There exists a unique skew-Hopf pairing η of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ such that for all $i, j \in I$ one has*

$$\eta(E_i, F_j) = -\delta_{i,j}, \quad \eta(E_i, L_j) = 0, \quad \eta(K_i, F_j) = 0, \quad \eta(K_i, L_j) = q_{ij}.$$

(ii) The skew-Hopf pairing η satisfies the equations

$$\eta(EK, FL) = \eta(E, F)\eta(K, L)$$

for all $E \in \mathcal{U}^+(\chi)$, $F \in \mathcal{U}^-(\chi)$, $K \in \mathcal{U}^{+0}$, and $L \in \mathcal{U}^{-0}$.

(iii) Let $\alpha, \beta \in \mathbb{Z}^n$ with $\alpha + \beta \neq 0$. Then $\eta(EK, FL) = 0$ for all $E \in \mathcal{U}^+(\chi)_\alpha$, $F \in \mathcal{U}^-(\chi)_\beta$, $K \in \mathcal{U}^{+0}$, and $L \in \mathcal{U}^{-0}$.

Proof. This is standard and is contained in many textbooks. Let us prove Part (ii).

Let $E \in \mathcal{U}^+(\chi)$, $F \in \mathcal{U}^-(\chi)$, $K \in \mathcal{U}^{+0}$, and $L \in \mathcal{U}^{-0}$. By the definition of η and the coproducts of $\mathcal{V}^+(\chi)$ and $\mathcal{V}^-(\chi)$ one obtains the following equations.

$$\begin{aligned} E_{(1)}K_{(1)}\eta(E_{(2)}K_{(2)},L) &= EK_{(1)}\eta(K_{(2)},L),\\ \eta(EK,F) &= \eta(K,F_{(1)})\eta(E,F_{(2)}) = \varepsilon(K)\eta(E,F),\\ \eta(EK,FL) &= \eta(E_{(1)}K_{(1)},F)\eta(E_{(2)}K_{(2)},L)\\ &= \eta(EK_{(1)},F)\eta(K_{(2)},L)\\ &= \varepsilon(K_{(1)})\eta(E,F)\eta(K_{(2)},L) = \eta(E,F)\eta(K,L). \end{aligned}$$

This proves (ii). □

The *left radical* of the pairing η is the set of all $x \in \mathcal{V}^+(\chi)$ such that $\eta(x,y) = 0$ for all $y \in \mathcal{V}^-(\chi)$. The right radical is the analogous subset of $\mathcal{V}^-(\chi)$. We want to know, what is the radical of the pairing η.

Using Prop. 7.5(ii),(iii) and dual bases, it follows that the left radical of η is a $\mathbb{Z}^n$-homogeneous subspace of $\mathcal{V}^+(\chi)$, and it is spanned by elements EK, where $E \in \mathcal{U}^+(\chi)$, $K \in \mathcal{U}^{+0}$, and E or K are in the radical, Moreover, the restriction of η to $V^+(\chi) \times V^-(\chi)$ is non-degenerate, and hence the $\mathcal{U}^+(\chi)$ part of the left radical has only $\mathbb{N}_0$-homogeneous components of degree ≥ 2.

Proposition 7.6. *Let* $\mathcal{U}^{+0}_{\mathrm{rad}}$ *be the ideal of* $\mathcal{U}^{+0}$ *generated by the elements* $K_1^{m_1}\cdots K_n^{m_n} - 1$, *where* $m_1,\dots,m_n \in \mathbb{Z}$, $q_{1i}^{m_1}q_{2i}^{m_2}\cdots q_{ni}^{m_n} = 1$ *for all* $i \in I$. *The left radical of* η *is*

$$\mathfrak{I}(V^+(\chi))\mathcal{U}^{+0} + \mathcal{U}^+(\chi)\mathcal{U}^{+0}_{\mathrm{rad}}.$$

Proof. The essential point is the observation, following from Props. 1.10, 7.5(ii), that

$$\eta(E,FF') = \eta(E^{(1)},F)\eta(E^{(2)},F'), \qquad E \in \mathcal{U}^+(\chi),\ F,F' \in \mathcal{U}^-(\chi).$$

Thus the $\mathcal{U}^+(\chi)$ part of the left radical is a coideal of $\mathcal{U}^+(\chi)$, and any coideal of $\mathcal{U}^+(\chi)$ contained in $T^{++}\big(V^+(\chi)\big)$ is in the left radical of η. Thus $\mathfrak{I}(V^+(\chi))$ is the $\mathcal{U}^+(\chi)$ part of the left radical of η. The determination of the $\mathcal{U}^{+0}$ part of the left radical is straightforward. □

The right radical has a similar description.

An important corollary of the above claim is the following.

Proposition 7.7. *The pairing η induces a skew-Hopf pairing*

$$\eta : \mathfrak{B}(V^+(\chi))\#\mathcal{U}^{+0} \times \big(\mathfrak{B}(V^-(\chi))\#\mathcal{U}^{-0}\big)^{\mathrm{cop}} \to \Bbbk.$$

The restriction of this pairing to $\mathfrak{B}(V^+(\chi))\times\mathfrak{B}(V^-(\chi))$ is non-degenerate.

Define $\mathcal{U}(\chi) = D(\mathcal{V}^+(\chi), \mathcal{V}^-(\chi))$ with respect to the pairing η. The Drinfeld double $\mathcal{U}(\chi)$ can be described explicitly in terms of generators and relations. A similar presentation for the Nichols algebra $\mathfrak{B}(V^+(\chi))$ is generally an open problem, and hence for the Drinfeld double $U(\chi) = D\big(\mathfrak{B}(V^+(\chi))\#\mathcal{U}^{+0}, (\mathfrak{B}(V^-(\chi))\#\mathcal{U}^{-0})^{\mathrm{cop}}\big)$ we have only the abstract description.

Proposition 7.8. *The algebra $\mathcal{U}(\chi)$ is generated by the elements K_i, K_i^{-1}, L_i, L_i^{-1}, E_i, and F_i, where $i \in I$, and defined by the relations*

$$XY = YX \quad \text{for all } X, Y \in \{K_i, K_i^{-1}, L_i, L_i^{-1} \mid i \in I\}, \tag{7.12}$$

$$K_iK_i^{-1} = 1, \qquad L_iL_i^{-1} = 1, \tag{7.13}$$

$$K_iE_jK_i^{-1} = q_{ij}E_j, \qquad L_iE_jL_i^{-1} = q_{ji}^{-1}E_j, \tag{7.14}$$

$$K_iF_jK_i^{-1} = q_{ij}^{-1}F_j, \qquad L_iF_jL_i^{-1} = q_{ji}F_j, \tag{7.15}$$

$$E_iF_j - F_jE_i = \delta_{i,j}(K_i - L_i). \tag{7.16}$$

The algebra $\mathcal{U}(\chi)$ admits a unique $\mathbb{Z}^n$-grading such that K_i, K_i^{-1}, $L_i, L_i^{-1} \in \mathcal{U}(\chi)_0$, $E_i \in \mathcal{U}(\chi)_{\alpha_i}$, and $F_i \in \mathcal{U}(\chi)_{-\alpha_i}$ for all $i \in I$.

Proposition 7.9. *Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in (\Bbbk^\times)^n$. Then there exists a unique algebra automorphism $\varphi_{\underline{\gamma}}$ of $\mathcal{U}(\chi)$ such that*

$$\varphi_{\underline{\gamma}}(K_i) = K_i,\ \varphi_{\underline{\gamma}}(L_i) = L_i,\ \varphi_{\underline{\gamma}}(E_i) = \gamma_iE_i,\ \varphi_{\underline{\gamma}}(F_i) = \gamma_i^{-1}F_i. \tag{7.17}$$

There exists a unique algebra automorphism ϕ of $\mathcal{U}(\chi)$ such that

$$\begin{aligned} \phi(K_i) &= K_i^{-1}, & \phi(L_i) &= L_i^{-1}, \\ \phi(E_i) &= F_iL_i^{-1}, & \phi(F_i) &= K_i^{-1}E_i. \end{aligned} \tag{7.18}$$

There is a unique algebra antiautomorphism Ω of $\mathcal{U}(\chi)$ such that

$$\Omega(K_i) = K_i, \qquad \Omega(L_i) = L_i, \qquad \Omega(E_i) = F_i, \qquad \Omega(F_i) = E_i. \tag{7.19}$$

All these algebra (anti)automorphisms induce (anti)automorphisms of $U(\chi)$.

Let $\mathcal{U}^0 = \mathcal{U}^{+0}\mathcal{U}^{-0} \subset \mathcal{U}(\chi)$. It is a Hopf subalgebra of $\mathcal{U}(\chi)$, and isomorphic to the group algebra of $\mathbb{Z}^{2n}$.

The commutator $[E, F_i]$, where $E \in \mathcal{U}^+(\chi)$ and $i \in I$, can be expressed in terms of the skew-derivations ∂_i^{L} and ∂_i^{R}. This fact can be used to prove the following proposition.

Proposition 7.10. *Let $\mathcal{I}^+ \subset \mathcal{U}^+(\chi) \cap \ker \varepsilon$ and $\mathcal{I}^- \subset \mathcal{U}^-(\chi) \cap \ker \varepsilon$ be a (not necessarily $\mathbb{Z}$-graded) ideal of $\mathcal{U}^+(\chi)$ and $\mathcal{U}^-(\chi)$, respectively. Then the following statements are equivalent.*

(1) *(Triangular decomposition of $\mathcal{U}(\chi)/(\mathcal{I}^+ + \mathcal{I}^-)$) The multiplication map $\mathrm{m} : \mathcal{U}^+(\chi) \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^-(\chi) \to \mathcal{U}(\chi)$ induces an isomorphism*

$$\mathcal{U}^+(\chi)/\mathcal{I}^+ \otimes \mathcal{U}^0(\chi) \otimes \mathcal{U}^-(\chi)/\mathcal{I}^- \to \mathcal{U}(\chi)/(\mathcal{I}^+ + \mathcal{I}^-)$$

of vector spaces.

(2) *The following equation holds.*

$$\mathcal{U}(\chi)\mathcal{I}^+\mathcal{U}(\chi) + \mathcal{U}(\chi)\mathcal{I}^-\mathcal{U}(\chi) = \mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi) + \mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{I}^-.$$

(3) *The vector spaces $\mathcal{I}^+\mathcal{U}^0(\chi)\mathcal{U}^-(\chi)$ and $\mathcal{U}^+(\chi)\mathcal{U}^0(\chi)\mathcal{I}^-$ are ideals of $\mathcal{U}(\chi)$.*

(4) *For all $X \in \mathcal{U}^0(\chi)$ and $i \in I$ one has*

$$\begin{aligned} X \cdot \mathcal{I}^+ &\subset \mathcal{I}^+, & X \cdot \mathcal{I}^- &\subset \mathcal{I}^-, \\ \partial_i^{\mathrm{R}}(\mathcal{I}^+) &\subset \mathcal{I}^+, & \partial_i^{\mathrm{L}}(\mathcal{I}^+) &\subset \mathcal{I}^+, \\ \partial_i^{\mathrm{R}}(\Omega(\mathcal{I}^-)) &\subset \Omega(\mathcal{I}^-), & \partial_i^{\mathrm{L}}(\Omega(\mathcal{I}^-)) &\subset \Omega(\mathcal{I}^-). \end{aligned}$$

7.2. **Lusztig isomorphisms.** The construction of Lusztig isomorphisms is not possible for all Nichols algebras of diagonal type. The essential condition is again the i-finiteness of the bicharacter χ. Best results can be achieved, if the root system of χ is finite, or at least there is a Cartan scheme associated to χ.

The main idea is, that the root system associated to χ has the symmetry of the Weyl groupoid. The Weyl groupoid has objects labeled by different bicharacters, and there are reflections mapping bicharacters to other bicharacters. Thus, it is natural to ask, if there are algebra isomorphisms between $U(\chi)$ and $U(\chi')$ which on the level of the root system coincide with the generating reflections of the Weyl groupoid. Such isomorphisms indeed exist, and are called Lusztig isomorphisms. The name comes from Lusztig's construction of certain automorphisms of quantized enveloping algebras.

In order to define Lusztig isomorphisms, it is useful to determine some commutation relations in $\mathcal{U}(\chi)$.

Let $p \in I$. For any $i \in I \setminus \{p\}$ let $E^+_{i,0(p)} = E^-_{i,0(p)} = E_i$, and for all $m \in \mathbb{N}$ define recursively

$$E^+_{i,m+1(p)} = E_p E^+_{i,m(p)} - (K_p \cdot E^+_{i,m(p)}) E_p, \tag{7.20}$$

$$E^-_{i,m+1(p)} = E_p E^-_{i,m(p)} - (L_p \cdot E^-_{i,m(p)}) E_p. \tag{7.21}$$

In connection with the letter p we will also write $E^+_{i,m}$ for $E^+_{i,m(p)}$ and $E^-_{i,m}$ for $E^-_{i,m(p)}$, where $m \in \mathbb{N}_0$.

Lemma 7.11. *For all $i \in I \setminus \{p\}$ and all $m \in \mathbb{N}_0$ one has*

$$\Bbbk E^+_{i,m+1} = \Bbbk(E^+_{i,m} E_p - (L_i L_p^m \cdot E_p) E^+_{i,m}),$$
$$\Bbbk E^-_{i,m+1} = \Bbbk(E^-_{i,m} E_p - (K_i K_p^m \cdot E_p) E^-_{i,m}).$$

For $i \in I \setminus \{p\}$ and $m \in \mathbb{N}_0$ let

$$\begin{aligned} F^+_{i,0} &= F^-_{i,0} = F_i, \\ F^+_{i,m+1} &= F_p F^+_{i,m} - (L_p \cdot F^+_{i,m}) F_p, \\ F^-_{i,m+1} &= F_p F^-_{i,m} - (K_p \cdot F^-_{i,m}) F_p \end{aligned} \tag{7.22}$$

for all $i \in I$ and $m \in \mathbb{N}_0$.

For all $m \in \mathbb{N}$, $q \in \Bbbk$ let $(m)_q = 1 + q + q^2 + \cdots + q^{m-1}$, $(0)_q = 0$.

It is not difficult to obtain some formulas about skew-derivatives of the elements E_i^m and $E^\pm_{i,m}$ for $i \in I$, $m \geq 0$. The formulas give rise to commutation rules between these elements and the elements F_i, $i \in I$. Then by induction one can show equations of the following form.

Lemma 7.12. *For all $i \in I \setminus \{p\}$ and all $m, n \in \mathbb{N}_0$ with $m \geq n$ the following equation holds in $\mathcal{U}(\chi)$.*

$$[E^+_{i,m}, F^+_{i,n}] = (-1)^n q_{ip}^{n-m} q_{pp}^{n(n-m)} \prod_{s=0}^{n-1} (m-s)_{q_{pp}} \prod_{s=0}^{m-1} (1 - q_{pp}^s q_{pi} q_{ip}) \times$$
$$(K_p^n K_i - \delta_{m,n} L_p^n L_i) E_p^{m-n}.$$

Lemma 7.13. *Let $m,n\in\mathbb{N}_0$ and $i,j\in I\setminus\{p\}$ such that $i\ne j$. Then*

$$[E^+_{i,m},F^+_{j,n}]=0 \qquad \text{in } \mathcal{U}(\chi).$$

Let $p\in I$. The Lusztig isomorphism should map a generator E_i, $i\ne p$, to $E^+_{i,m}$ for some $m\ge 0$, and F_p to E_p, up to a factor in $\Bbbk[K_i,K_i^{-1},L_i,L_i^{-1}\,|\,i\in I]$. Since $[E_i,F_p]=0$, one can show that the existence of a Lusztig isomorphism requires that $E^+_{i,m+1}=0$ for some $m\ge 0$. We use the following natural ideals for the definition of Lusztig isomorphisms.

Definition 7.14. Let $p\in I$ such that χ is p-finite. Define ideals $\mathcal{I}^+_p(\chi)\subset\mathcal{U}^+(\chi)$ and $\mathcal{I}^-_p(\chi)\subset\mathcal{U}^-(\chi)$ as follows. Assume first that $(m)_{q_{pp}}=0$ and $(m')_{q_{pp}}\ne 0$ for some $m\in\mathbb{N}$ and all m' with $0<m'<m$. Then m is uniquely determined, and we set

$$\begin{aligned}\mathcal{I}^+_p(\chi)=&(E_p^m,E^+_{i,1-c^\chi_{pi}}\,|\,i\in I\setminus\{p\}\text{ such that }1-c^\chi_{pi}<m),\\ \mathcal{I}^-_p(\chi)=&(F_p^m,F^+_{i,1-c^\chi_{pi}}\,|\,i\in I\setminus\{p\}\text{ such that }1-c^\chi_{pi}<m).\end{aligned}$$

Otherwise $(m)_{q_{pp}}\ne 0$ for all $m\in\mathbb{N}$, and we define

$$\mathcal{I}^+_p(\chi)=(E^+_{i,1-c^\chi_{pi}}\,|\,i\ne p),\qquad \mathcal{I}^-_p(\chi)=(F^+_{i,1-c^\chi_{pi}}\,|\,i\ne p).$$

One can show that these ideals satisfy the conditions in Prop. 7.10(4). In particular, the ideals $\mathcal{I}^+_p(\chi)$ and $\Omega(\mathcal{I}^-_p(\chi))$ are contained in the ideal $\mathfrak{I}(V^+(\chi))$.

For all $i\in I$ let

$$\lambda^\chi_i=\prod_{m=1}^{-c^\chi_{pi}}(m)_{q_{pp}}\prod_{s=0}^{-c^\chi_{pi}-1}(q_{pp}^s q_{pi}q_{ip}-1)\in\Bbbk^\times.$$

Lemma 7.15. *Let $p\in I$ such that χ is p-finite, and let $c_{pi}=c^\chi_{pi}$ for all $i\in I$. There are unique algebra maps*

$$T_p,T_p^-:\mathcal{U}(\chi)\to\mathcal{U}(r_p(\chi))/\big(\mathcal{I}^+_p(r_p(\chi)),\mathcal{I}^-_p(r_p(\chi))\big)$$

such that

$$\begin{aligned} T_p(K_p) =& T_p^-(K_p) = K_p^{-1}, & T_p(K_i) =& T_p^-(K_i) = K_i K_p^{-c_{pi}},\\ T_p(L_p) =& T_p^-(L_p) = L_p^{-1}, & T_p(L_i) =& T_p^-(L_i) = L_i L_p^{-c_{pi}},\\ T_p(E_p) =& F_p L_p^{-1}, & T_p(E_i) =& E^+_{i,-c_{pi}},\\ T_p(F_p) =& K_p^{-1} E_p, & T_p(F_i) =& (\lambda_i^{r_p(\chi)})^{-1} F^+_{i,-c_{pi}},\\ T_p^-(E_p) =& K_p^{-1} F_p, & T_p^-(E_i) =& (\lambda_i^{r_p(\chi^{-1})})^{-1} E^-_{i,-c_{pi}},\\ T_p^-(F_p) =& E_p L_p^{-1}, & T_p^-(F_i) =& (-1)^{c_{pi}} F^-_{i,-c_{pi}}. \end{aligned}$$

Notice that there is an asymmetry in the use of constants in the definition. The usual $-$ sign in the definition of $T_p(E_p)$ is also missing. In case of $U_q(\mathfrak{g})$, the interest in the choice of good constants comes from the fact, that certain products of Lusztig automorphisms should map a generator E_i to another generator E_j. It is not clear, if such a property can be achieved in an analogous setting for arbitrary Nichols algebras of diagonal type. With the above definitions it is possible to show that under some natural assumptions, E_i is mapped to a nonzero multiple of E_j. The drawback of this weak property is paired with the advantage of avoiding case by case considerations.

Using some technical results, one shows the following.

Proposition 7.16. *Let p, T_p and T_p^- as in Lemma 7.15.*

(i) The maps T_p, T_p^- induce algebra isomorphisms

$$T_p, T_p^- : \mathcal{U}(\chi)/(\mathcal{I}_p^+(\chi), \mathcal{I}_p^-(\chi)) \to \mathcal{U}(r_p(\chi))/(\mathcal{I}_p^+(r_p(\chi)), \mathcal{I}_p^-(r_p(\chi))).$$

(ii) The maps T_p, T_p^- satisfy the equations

$$T_p T_p^- = T_p^- T_p = \mathrm{id}.$$

Note that Part (ii) makes only sense if one uses appropriate bicharacters. For example, the equation $T_p T_p^- = \mathrm{id}$ means that if T_p^- is defined with respect to χ, then T_p has to be defined with respect to $r_p(\chi)$.

In a next step one calculates commutation rules between the skew-derivations ∂_i^{L}, ∂_i^{R} and the Lusztig isomorphisms. This allows then to prove that the Lusztig isomorphisms in Prop. 7.16 map the ideal $\mathfrak{I}(V^+(\chi))$ to the ideal $\mathfrak{I}(V^+(r_p(\chi)))$.

Theorem 7.17. *Let p, T_p and T_p^- as in Lemma 7.15. The maps T_p, T_p^- induce algebra isomorphisms*

$$T_p, T_p^- : U(\chi) \to U(r_p(\chi)).$$

From now on assume that χ is a bicharacter such that χ' is p-finite for all $\chi' \in \mathcal{G}(\chi)$ and $p \in I$.

The main result on Coxeter relations between Lusztig isomorphisms is based on the following lemma.

Lemma 7.18. *Let $i, j \in I$ with $i \neq j$. Let $i_m = i$ for m odd and $i_m = j$ for m even. Assume that $r, s \in \mathbb{N}_0$ such that*

$$\sigma_{i_s} \cdots \sigma_{i_2} \sigma_{i_1}^{\chi}(\alpha_i + r\alpha_j) = \alpha_{i_{s+1}}. \tag{7.23}$$

Then there exists $t \in \mathbb{N}_0$ such that $\sigma_{i_s} \cdots \sigma_{i_2} \sigma_{i_1}^{\chi}(\alpha_j) = \alpha_{i_s} + t\alpha_{i_{s+1}}$.

Proposition 7.19. *Let $i, j \in I$ with $i \neq j$. Let $i_t = i$ for t odd and $i_t = j$ for t even. Assume that $m, r \in \mathbb{N}_0$ such that*

$$m < |R_+^{\chi} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|, \quad \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1}^{\chi}(\alpha_i + r\alpha_j) \in R_+^{w^*\chi}, \tag{7.24}$$

where $w = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1}^{\chi}$. Then for $E_{i,r(j)}^+, E_{i,r(j)}^- \in U(\chi)$ one gets

$$T_{i_m} \cdots T_{i_2} T_{i_1}(E_{i,r(j)}^+) \in U^+(w^*\chi)_{w(\alpha_i + r\alpha_j)}, \tag{7.25}$$

$$T_{i_m}^- \cdots T_{i_2}^- T_{i_1}^-(E_{i,r(j)}^-) \in U^+(w^*\chi)_{w(\alpha_i + r\alpha_j)}. \tag{7.26}$$

In particular, if $w(\alpha_i + r\alpha_j) = \alpha_{i_{m+1}}$, then one gets

$$T_{i_m} \cdots T_{i_1}(\Bbbk E_{i,r(j)}^+) = \Bbbk E_{i_{m+1}}, \quad T_{i_m}^- \cdots T_{i_1}^-(\Bbbk E_{i,r(j)}^-) = \Bbbk E_{i_{m+1}}. \tag{7.27}$$

This result helps to prove the following theorems.

Theorem 7.20. *Let $m \in \mathbb{N}_0$ and $w = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1}^{\chi} \in \mathcal{W}(\chi)$ a reduced expression. If $p \in I$ such that $w(\alpha_p) \in R_+^{w^*\chi}$, then the algebra isomorphism $T_{i_m} \cdots T_{i_1} : U(\chi) \to U(w^*\chi)$ satisfies the relation*

$$T_{i_m} \cdots T_{i_1}(E_p) \in U^+(w^*\chi). \tag{7.28}$$

Theorem 7.21. *Let $m \in \mathbb{N}_0$ and $w = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1}^{\chi} \in \mathcal{W}(\chi)$, $w = \sigma_{j_m} \cdots \sigma_{j_2} \sigma_{j_1}^{\chi} \in \mathcal{W}(\chi)$ be two reduced expressions of w. Then*

$$T_{i_m} \cdots T_{i_1} = T_{j_m} \cdots T_{j_1} \varphi_{\underline{\gamma}} : U(\chi) \to U(w^*\chi)$$

for some $\underline{\gamma} \in (\Bbbk^\times)^n$.

The proof uses extensively the skew-derivations ∂_i^{L} and ∂_i^{R} and their commutation rules with T_p and T_p^-.

One can also describe the isomorphism corresponding to a longest element of the Weyl groupoid.

One possible application of the Lusztig isomorphisms is the construction of a PBW basis of Nichols algebras of diagonal type with finite root system, using a generalization of Lusztig's approach via a longest element of the Weyl group.

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	generalized Dynkin diagrams	fixed parameters
1	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$	$q \in \Bbbk^* \setminus \{1\}$
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ}$ $\overset{-1}{\circ}\overset{q}{\text{——}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$
3	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{q^2}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{-1}{\circ}$ $\overset{-q^{-1}}{\circ}\overset{q^2}{\text{——}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$, $q \notin \mu_4$
5	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$ $\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{——}}\overset{\zeta q^{-1}}{\circ}$	$\zeta \in \mu_3$, $q \in \Bbbk^* \setminus \{1,\zeta,\zeta^2\}$
6	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_3$
7	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-\zeta^2}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_{12}$
8	$\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-\zeta^2}{\circ}$ $\overset{-\zeta^{2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_{12}$
9	$\overset{-\zeta}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{\zeta^3}{\circ}$ $\overset{\zeta^{3}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_9$
10	$\overset{q}{\circ}\overset{q^{-3}}{\text{——}}\overset{q^3}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$, $q \notin \mu_3$
11	$\overset{\zeta^2}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}$ $\overset{\zeta^{2}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$ $\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_8$
12	$\overset{\zeta^6}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-\zeta^{-4}}{\circ}$ $\overset{\zeta^{6}}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}$ $\overset{-\zeta^{-4}}{\circ}\overset{\zeta^{5}}{\text{——}}\overset{-1}{\circ}$ $\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_{24}$
13	$\overset{\zeta}{\circ}\overset{\zeta^{2}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_5$
14	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_{20}$
15	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{\zeta^5}{\circ}$ $\overset{\zeta^{3}}{\circ}\overset{-\zeta^{4}}{\text{——}}\overset{-\zeta^{-4}}{\circ}$ $\overset{\zeta^{5}}{\circ}\overset{-\zeta^{-2}}{\text{——}}\overset{-1}{\circ}$ $\overset{\zeta^{3}}{\circ}\overset{-\zeta^{2}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_{15}$
16	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mu_7$

TABLE 1. Generalized Dynkin diagrams of irreducible bicharacters of rank two

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