Partitions and the Circle Method

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Für Angelina.

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Zusammenfassung

Die vorliegende kumulierte Habilitationsschrift umfasst fünf Originalarbeiten des Autors, die sich alle im Themenkreis der Anwendung der Kreismethode auf Problemstellungen im Bereich der Partitionen bewegen. Die Kreismethode ist eine der bedeutendsten Methoden in der analytischen Zahlentheorie, und extrahiert Informationen über das Wachstum zahlentheoretischer Funktionen aus dem Verhalten ihrer erzeugenden Funktionen in der Einheitskreisscheibe der komplexen Zahlen. Gerade im Umfeld der Partitionen tragen die zugehörigen erzeugenden Funktionen besondere Strukturen, was in vielen Fällen eine tiefgehendere Analyse ermöglicht.

Abstract

The present cumulative habilitation thesis comprises five original works by the author, all centered around the application of the Circle Method to problems in the field of partition theory. The Circle Method is one of the most powerful techniques in analytic number theory, extracting information about the growth of number-theoretic functions from the behavior of their generating functions within the unit disk of the complex plane. Particularly in the context of partitions, the associated generating functions exhibit special structures, which, in many cases, allow for a deeper level of analysis.

Contents

1 Introduction

1.1 The Circle Method

The Circle Method is undoubtedly one of the most powerful tools in analytical number theory. Nevertheless, it can be explained quickly and at least its idea can be fully understood with the knowledge of an undergraduate student in complex analysis. To formulate it, we start with a sequence of, for example, positive integers $c(n)$ that we want to understand better. The characteristic *generating function*

$$f(q) = \sum_{n \geq 0} c(n) q^n \tag{1.1}$$

can be assigned to it. If the coefficients do not grow too strongly, this series will not only be formally defined, but will also converge absolutely in a neighborhood of $q = 0$. In many applications, the radius of convergence of (1.1) will not be greater than 1, and in many cases, for example with moderately increasing integer coefficients, it will even be exactly 1. In the following, we assume without loss of generality that the radius of convergence is 1. Now *Cauchy's integral formula* tells us that we can extract the coefficients from $f(q)$ using a simple orthogonality relation. For every $n \in \mathbb{N}_0$ we have

$$c(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(q)}{q^{n+1}} \mathrm{d}q, \tag{1.2}$$

for any simple closed curve $\mathcal{C}$ contained in the unit disk orientated counterclockwise. Normally, the radius of the integration path is also selected as a function of n, which asymptotically approaches the edge of the unit circle and does not affect the value of the integral. The trick is to find the appropriate radii. This transfers the study of the coefficients to a study of the function f near the edge of the unit circle.

Now it may happen that the function under consideration has good analytical properties. For instance, if the $c(n)$ are positive and monotonically increasing, it is expected that the part close to $q = 1$ provides the dominant contribution to (1.2). This part of the curve is the *major arc* and the complement is the *minor arc*. To obtain an asymptotic expansion for $c(n)$, one then evaluates the major arc to some degree of accuracy and bounds the minor arc in such a way that the error with respect to all main terms is negligible. Depending on the function $f(q)$, both of these tasks present a variety of difficulties. In this context, the particularly diverse approaches to estimating minor arcs from above, depending on the problem, are of special interest. The methodology must be adjusted according to the initial function, and the level of difficulty varies. While in some cases, the approach is rather canonical, in other situation heuristic arguments indicate that there is no way of bounding sufficiently good at all, see below. We will briefly discuss a few approaches below but will then primarily focus on the case where the generating function has a product expansion of the shape

$$f(q) = \sum_{n \geq 0} c(n) q^n = \prod_{n \geq 1} \frac{1}{(1 - q^n)^{a(n)}}. \tag{1.3}$$

This often proves to be highly advantageous, as it enables the usage of logarithms. We encounter this type of product particularly in the context of partition theory. We discuss this in detail below.

Right from the outset, we want to explain the nature of the Circle Method on the basis of a beautiful, quite general related result that continues to find applications in combinatorics and number theory today, for instance in the theory of partitions, see for example [11], [21], [22], [29], and [56]. We recall that $f(x) \sim g(x)$ for $x \to A$ means, that the quotient $\frac{f(x)}{g(x)}$ tends to 1 as $x \to A$.

Theorem 1.1 (Ingham's Tauberian theorem, see [16])**.** *Suppose that $\sum_{n \geq 0} c(n)q^n$ is a power series with non-negative real coefficients and radius of convergence at least 1. If λ, α, β and $\gamma > 0$ are real numbers such that*

$$B\left(e^{-t}\right) \sim \lambda \log\left(\frac{1}{t}\right)^{\alpha} t^{\beta} e^{\frac{\gamma}{t}}, \qquad (t \to 0^+),$$

$$B\left(e^{-z}\right) \ll \log\left(\frac{1}{|z|}\right)^{\alpha} |z|^{\beta} e^{\frac{\gamma}{|z|}}, \qquad (z \to 0),$$

with $z = x + iy$ (where $x, y \in \mathbb{R}$, $x > 0$) in each region of the form $|y| \leq \Delta x$ with $\Delta > 0$, then

$$\sum_{n=0}^{N} c(n) \sim \frac{\lambda \gamma^{\frac{\beta}{2} - \frac{1}{4}} \log(N)^{\alpha}}{2^{\alpha+1}\sqrt{\pi} N^{\frac{\beta}{2} + \frac{1}{4}}} e^{2\sqrt{\gamma N}}, \qquad (N \to \infty).$$

Furthermore, if the $c(n)$ are weakly increasing, then

$$c(n) \sim \frac{\lambda \gamma^{\frac{\beta}{2} + \frac{1}{4}} \log(n)^{\alpha}}{2^{\alpha+1}\sqrt{\pi} n^{\frac{\beta}{2} + \frac{3}{4}}} e^{2\sqrt{\gamma n}}, \qquad (n \to \infty).$$

Remark. Note that the conditions of the theorem actually force the power series to have radius of convergence 1. The theorem originally shown by Ingham [37] is considerably more general, and Theorem 1.1 can be obtained as a special case by translating it into the formalism of Ingham's theorem. This was carried out by Bringmann, Jennings-Shaffer and Mahlburg [16]. For more modern background on Ingham's theorem including a proof the reader may also wish to consult the monograph of Korevaar [39] Chapter IV, Section 21.

We recall that historically the phrase "Tauberian" refers to some kind of converse theorem. To explain this, we remember that a by a classical theorem of Abel convergence of the series $\sum_{n \geq 0} a_n = A$ implies that $\lim_{x \to 1^-} \sum_{n \geq 0} a_n x^n = A$, whereas the second property is called *Abel summability* of the sequence a_n with limit A. In other words, classical summability, i.e., convergence of a series, always implies Abel summability (and the limits are both the same). *Tauberian theorems* now ask for converses of such *Abelian theorems*. This is an interesting question, as examples like

$$\lim_{x \to 1^-} \sum_{n \geq 0} (-1)^n x^n = \lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$$

show that Abel summability does generally not imply classical convergence. Consequently, for such converse theorems, additional conditions must necessarily be assumed, which are usually referred to as *Tauberian conditions*. For instance, Alfred Tauber, after whom the Tauberian theorems are named, showed that a sufficient condition for the convergence of the series $\sum_{n\geq 0} a_n$ is Abel summability of a_n and $na_n = o(1)$ (and then again both limits agree). For an elegant proof of this theorem see [39]. As a consequence, classical Abelian theorems allow conclusions about generating functions based on the properties of their coefficients, while Tauberian theorems, in many practical situations, provide information about sequences under certain assumptions about their generating functions. The Tauberian condition in Theorem 1.1 is that the $c(n)$ are non-negative, and this condition provides an informal explanation for why the Circle Method works. If the coefficients of the power series do not have any sign changes, then the modified power series

$$\frac{1}{1-q}\sum_{n\geq 0} c(n)q^n = \sum_{N\geq 0}\sum_{n=0}^{N} c(n)q^N$$

has weakly increasing coefficients, but a similarly good growth behavior near $q = 1$. With weakly increasing coefficients, it is plausible that the function will have its most dominant growth near $q = 1$ (which can be seen, for example, with partial summation), and this fact is of great importance for the philosophy of the Circle Method. Indeed, the heuristic argument essentially involves placing the major arc around $q = 1$, while the complement defines the minor arc. Thus, although the Tauberian theorem provides at least an asymptotic leading term with minimal effort, its connection to the Circle Method is unmistakable. Indeed, the probably closest relative to Ingham's Tauberian theorem is *Wright's Circle Method*. This method, developed by Wright in [64] and [65], utilizes more detailed information about the generating function at the boundary of the circle of convergence. In return, it yields significantly better results; not only the leading term but also an asymptotic expansion, where the error can generally be made arbitrarily small. However, it should be noted that this always refers to the ratio relative to the leading term; the absolute error, when considered separately, never tends to zero. Therefore, Wright's Circle Method remains a relatively coarse approach to estimating number-theoretic sequences, with its main advantage lying in its great flexibility. A useful tool to find asymptotic expansions of a function near the boundary of the unit circle is Euler–MacLaurin summation. We do not want to go into the technical details concerning Wright's Circle Method at this point, but refer to the literature [13], [64], [65].

1.2 History, Further Examples and Applications of the Circle Method

1.2.1 Distribution of primes

Many mathematicians have been interested in questions regarding the distribution of prime numbers for a long time, some of which remain open problems today. Using elementary methods, one can show that there are infinitely many prime numbers (this is Euclid's celebrated theorem and one of the oldest known results in number theory), and with a few clever additional ideas, one can also see that there must always be at least $c\frac{x}{\log(x)}$ prime numbers below x in the long run, where $c > 0$ is a constant, so $\frac{x}{\log(x)} \ll \pi(x)$, where $\pi(x)$ is the prime-counting function. The young Gauß already conjectured that even

$$\pi(x) \sim \frac{x}{\log(x)}, \qquad (x \to \infty),$$

should hold, but the so-called *prime number theorem* (see [42], p. 2) resisted unproven for about 100 years, until eventually it was shown independently by Hadamard [28] and de la Vallée Poussin [54] in 1896. The starting point of the successful proof was the famous *Riemann zeta function*

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}, \qquad (\mathrm{Re}(s) > 1),$$

where the *Euler product* on the right-hand side extends over all prime numbers p. For this reason, results like the aforementioned prime number theorem are fundamentally attributed to multiplicative number theory. The Riemann zeta function is one of the most studied functions in mathematics and has a number of important properties. It can be shown that it extends to an analytic function to $\mathbb{C} \setminus \{1\}$ with a simple pole in $s = 1$ and satisfies the celebrated functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Here, Γ denotes the *Euler Gamma function*

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} \mathrm{d}x, \qquad (\mathrm{Re}(s) > 0),$$

and integration by parts yields the characteristic functional equation $s\Gamma(s) = \Gamma(s+1)$ and hence a holomorphic continuation to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$ with simple poles in the non-positive integers. Although the proof of the prime number theorem does not use the Circle Method per se, there is at least a philosophical similarity: In the proof of the prime number theorem, the behavior of a "generating function" at the boundary of its domain of convergence is also exploited. In this case, it is a Dirichlet series whose coefficients for n^{-s} correspond to the number of ways the positive integer n can be written as a product of primes. This is, with regard to the question of the exact distribution of prime numbers, a reasonable sequence of numbers. By the Fundamental theorem of arithmetic, this number is always 1, and we obtain precisely the Riemann zeta function along with its Euler product expansion. Now the prime number theorem is essentially equivalent to the statement that the function $s \mapsto \frac{1}{\zeta(s)}$ can be extended holomorphically to the closed strip $\{s \in \mathbb{C} \colon \mathrm{Re}(s) \geq 1\}$, which by the Euler product is to say, $\zeta(s) \neq 0$ for all s with $\mathrm{Re}(s) = 1$. In fact, the prime number theorem can thus be interpreted to mean that the Euler product converges on the closure $\{s \in \mathbb{C} \colon \mathrm{Re}(s) \geq 1\}$ except at $s = 1$, unlike the Dirichlet series. But on top, the behavior of the Riemann zeta function near the point $s = 1$, specifically the fact that it has a pole of order one, is crucial for the distribution of prime numbers. We would like to point out that Erich Hecke emphasized the importance of the behavior at the boundary of the domain of convergence, particularly the singularities that are especially significant also for the Circle Method, as follows:

> „Es ist die Tatsache, daß die genauere Kenntnis des Verhaltens einer analytischen Funktion in der Nähe ihrer singulären Stellen eine Quelle von arithmetischen Sätzen ist.“[1]

– Erich Hecke (see also [20], p. 153)

[1] English translation: The fact of the matter is that a more precise knowledge of the behavior of an analytic function near its singularities is a source of arithmetical results.

Despite these philosophical similarities, the Circle Method is not used in multiplicative number theory; instead, it is replaced by a somewhat different theoretical framework. However, it is traditionally found in additive number theory. The reason for this is simple. The objects of interest for questions of a multiplicative nature are Dirichlet series, as the mapping $n \mapsto n^{-s}$ is strictly multiplicative for all complex numbers s. In contrast, power series are exceptionally well-suited for translating questions of additive number theory into analysis, as $q^{m+n} = q^m q^n$. For more detailed comparisons of both concepts, we refer to [42], Chapter 5. For example, if f_2 represents the number of ways to write a number as the sum of two odd prime numbers, then the identity

$$\left(\sum_{p>2} q^p \right)^2 = \sum_{n \geq 0} f_2(n) q^n = q^6 + 2q^8 + 3q^{10} + 2q^{12} + 3q^{14} + 4q^{16} + 4q^{18} + \cdots \tag{1.4}$$

holds. Of course, this principle can be extended to arbitrary subsets of the non-negative integers, and the number of representations can also vary. It is important to note, however, that the order of representations must always be taken into account. Even this aforementioned example, i.e., the nature of f_2, refers to one of the most prestigious and famous problems in mathematics. It was first formulated in 1742 by Christian Goldbach in a letter to his correspondent Leonhard Euler, making it one of the oldest unsolved problems, despite being so simple to state.

Conjecture 1.2 (Goldbach's conjecture)**.** *Every even integer $n > 2$ is the sum of two primes.*

One initial approach to tackling Goldbach's conjecture might be to gather as much information as possible about the power series $\sum_{p>2} q^p$, and then use the Circle Method to show that every sufficiently large coefficient corresponding to an even exponent in (1.4) must be positive. However, Tao [58] presents heuristic arguments suggesting why the Circle Method might not be suitable for a solution, thus eliminating a very crucial tool of analytic number theory specifically designed for such problems. Similarly, the so-called parity problem also prevents a solution via Sieve theory [60]. From a purely heuristic perspective, it also makes sense that the conjecture holds. However, while no further progress has been made in this area, similar questions have been investigated for a larger number of prime numbers. As early as 2013, Helfgott [34] announced a complete proof of the ternary Goldbach problem, using only the Circle Method.

Theorem 1.3 (Ternary Goldbach conjecture, Helfgott)**.** *Every integer $n > 5$ is the sum of three primes.*

Strictly speaking, Helfgott proves Theorem 1.3 for all values $n > 10^{27}$ a priori. But this yields the result, as the Ternary Goldbach conjecture had been checked numerically up to $8.875 \cdot 10^{30}$ by Helfgott and Platt [35]. It should be emphasized that it has longer been known that the Ternary Goldbach conjecture holds for sufficiently large n. The proof of this weak Ternary Goldbach conjecture is somewhat surpringsingly a more classical application of the Circle Method and the Large Sieve. Central to this is the following asymptotic formula, which goes back to Vinogradov [61] in 1937.

Theorem 1.4. *Let*

$$\varrho_3(n) := \sum_{\substack{k_1,k_2,k_3 \geq 0 \\ k_1+k_2+k_3=n}} \Lambda(k_1) \Lambda(k_2) \Lambda(k_3),$$

where

$$\Lambda(n) := \begin{cases} \log(p), & \text{if } n = p^k \text{for a prime power } p^k, \\ 0, & \text{else,} \end{cases}$$

is the Mangoldt function. Then for any $A \in \mathbb{R}^+$ we have

$$\varrho_3(n) = \frac{1}{2}\left(\prod_{p|n}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p\nmid n}\left(1 + \frac{1}{(p-1)^3}\right)\right)n^2 + O_A\left(\frac{n^2}{\log(n)^A}\right), \quad (n \to \infty).$$

It is an exercise to prove that Theorem 1.4 implies the weak Ternary Goldbach conjecture. A comprehensive and detailed treatment of the topic, along with a proof of Vinogradov's theorem 1.4, can be found in the introductory book by Brüdern [17].

Interestingly, the Circle Method also has applications in the field of prime gaps, at least when it comes to the heuristic justification of certain asymptotic expressions. Perhaps the most famous unresolved problem in this direction is the twin prime conjecture.

Conjecture 1.5 (Twin prime conjecture)**.** *There exist infinitely many prime numbers p such that $p+2$ is also prime.*

To date, no ideas exist on how to prove this. Yet, Hardy and Littlewood were able to significantly refine this conjecture using heuristic application of the Circle Method, by providing an asymptotic estimate for its density.

Conjecture 1.6 (Hardy, Littlewood, see [31] Conjecture B)**.** *Put*

$$C_2 := \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right),$$

where the product runs over all odd primes, and let $\pi_2(x)$ be the number of twin primes up to x. Then we have

$$\pi_2(x) \sim 2C_2\frac{x}{\log(x)^2}, \qquad (x \to \infty).$$

In simple terms, the conjecture suggests that while twin prime numbers occupy a vanishing proportion among prime numbers, there are still quite a few of them. Although a proof of the Twin prime conjecture currently seems unattainably far away, we briefly highlight the progress in the theory regarding "small gaps between primes". The prime number theorem says that the number of primes $\leq x$ is approximately $\frac{x}{\log(x)}$, so the average gap between them is $\log(x)$. As a consequence, one would assume that the gaps between primes should increase over time. However, as the Twin prime conjecture already suggests, prime numbers likely behave much more unpredictably. In a celebrated paper 2014, Zhang [66] had a breakthrough by showing that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 70000000, \tag{1.5}$$

where p_n is the n-th prime in natural order, which means that gaps between prime numbers cannot become arbitrarily large in the long run. Note that the Twin prime conjecture would be settled if we could replace 70000000 by 2 in (1.5). A major ingredient of Zhang's proof is an improved version of the Bombieri–Vinogradov theorem for the case that the involved moduli are not divisible by large powers of primes. Even more astonishing is the fact that Zhang's theorem is just the tip of the iceberg when it comes to strange distribution patterns in the prime numbers. In fact, the following deep theorem, proven independently by Maynard [49] and (in a slightly weaker version) by Tao [59], holds.

Theorem 1.7 (Maynard, Tao)**.** *For each* $m \in \mathbb{N}$*, we have*

$$\liminf_{n\to\infty}(p_{n+m} - p_n) \ll m^3 e^{4m}.$$

It is also worth mentioning that the explicit bounds have been further reduced in some cases (sometimes independently, sometimes depending on the proof of the Elliott–Halberstam conjecture). For example, Maynard [49] showed

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 600$$

independently and

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 12$$

assuming the Elliott–Halberstam conjecture. All of this makes the validity of the Twin prime conjecture at least quite plausible. However, despite all efforts, a proof still seems to be far out of reach.

But even within prime number theory, the Circle Method has recently proved to be extremely flexible. To the great surprise of the mathematical community, Maynard [48] succeeded 2019 in using the Circle Method to show that there are infinitely many prime numbers in which a specific decimal digit is consistently missing. He proved the following.

Theorem 1.8 (Maynard)**.** *Let* $X \geq 4$*,* $a_0 \in \{0, 1, \ldots, 9\}$ *and*

$$\mathcal{A} := \left\{ \sum_{j=0}^{k} n_j 10^j < X : n_j \in \{0, \ldots, 9\} \setminus \{a_0\}, k \geq 0 \right\}$$

be the set of numbers less than X *with no digit in their decimal expansion equal to* a_0*. Then we have*

$$\frac{X^{\frac{\log(9)}{\log(10)}}}{\log(X)} \ll |\mathcal{A} \cap \mathbb{P}| \ll \frac{X^{\frac{\log(9)}{\log(10)}}}{\log(X)}, \qquad (X \to \infty).$$

Here, $\mathbb{P}$ *denotes the set of primes.*

What is particularly remarkable about this theorem is that it identifies an infinite number of prime numbers within a thin set in the natural numbers, that is, a set with asymptotic density zero. Naturally, this approach is not limited to the case of base 10, and can be extended to all bases $\mathcal{B}$.

1.2.2 Diophantine equations

A *Diophantine equation* is an equation of the form

$$F(X_1, \dots, X_n) = 0$$

with some polynomial F with integer coefficients. Normally, one is interested in rational or integer solutions of such equations. In general, it is very difficult to make statements about the solutions of Diophantine equations, even for specific families. For instance, the proof that the *Fermat equation* $X^n + Y^n - Z^n = 0$ has no non-trivial integer solution if $n \geq 3$ is famous for its difficulty. It is also noteworthy that Matiyasevich [45] was able to show in a famous paper that the solvability problem of Diophantine equations (Hilbert's tenth problem) is algorithmically undecidable. However, it is possible to attempt to gain an idea of the number of solutions through analytical methods at least. A particularly famous example concerns Waring's problem. As one can easily see through Fermat's last theorem, but also through significantly more accessible asymptotic methods, not every number is the sum of two k-th powers for general k. However, it is by no means obvious whether there exists a number $g(n)$ such that every natural number is the sum of at least $g(k)$ k-th powers (where $g(k)$ with this property should naturally be minimal). The case $k = 1$ is trivial, with $g(1) = 1$, and through Lagrange's four squares theorem $g(2) \leq 4$ and $g(2) > 3$, see Chapter 1 of [51], we know that $g(2) = 4$ must hold. The general problem was first solved by Hilbert [36] in 1909. While Hilbert's arguments were quite ingenious, they failed to provide a concrete bound for the values of $g(k)$. Instead, they demonstrated their existence.

Theorem 1.9 (Hilbert)**.** *Let $k \geq 1$ be an integer. Then there exists an integer $g(k)$, such that for any integer $n \geq 0$ the equation $x_1^k + x_2^k + \cdots + x_{g(k)}^k = n$ has a solution $(x_1, ..., x_{g(k)}) \in \mathbb{N}_0^{g(k)}$.*

Since the integers of the form $3^k - 1$ can only be represented by k-th powers of the form 1^k and 2^k, one can quickly see that the values of $g(k)$ must grow at least exponentially. However, such phenomena are considered as number theoretical anomalies, which led to the study not only of $g(k)$ but also of the values $G(k)$, defined as the smallest numbers such that every sufficiently large integer can be represented as the sum of at least $G(k)$ k-th powers. These studies bring along their own set of difficulties, and therefore, we do not intend to go further into this issue at this point. We only mention two facts. Firstly, as an infinite number of positive integers is not a sum of three squares (in fact, all numbers of the form $4^a(8b + 7)$ with $a, b \in \mathbb{N}$) we quickly see that $G(2) = g(2) = 4$. Secondly, recently Brüdern and Wooley [18] proved, again using the Circle Method, the upper bound

$$G(k) \leq \lceil k(\log(k) + 4.20032) \rceil,$$

implying that $G(k)$ is much smaller than $g(k)$ for large k.

Shortly after Hilbert's breakthrough, more precise investigations into the values of $g(k)$ followed. Eventually, in 1920, Hardy and Littlewood [30] applied the Circle Method to gain an idea of their order of magnitude. They studied the generating series of the k-th powers,

$$f_k(q) := \sum_{n \geq 0} q^{n^k}$$

near the boundary of the unit circle. However, for $k \geq 3$, no simple relations to well-known special functions and also no transformation laws for the $f_k(q)$ are known, so this approach demanded some technical difficulties and could not provide an exact result. Techniques such as the Euler–Maclaurin formula only provide information on small parts of the unit circle, namely in the neighborhood of roots of unity of low order. If, as in the case of $f_k(q)$ with $k \geq 3$, strong analytical information is not available, then one resorts to dividing the integration contours into major and minor arcs in such a way that the major arcs correspond only to roots of unity with sufficiently low order, and the minor arcs are estimated above. To state the result of Hardy and Littlewood, one needs to consider the so–called *singular series*. It is defined as

$$\mathcal{S}_b(m) := \sum_{h,\ell} \left(\frac{1}{\ell} \sum_{r=0}^{\ell-1} e^{\frac{2\pi i h r^k}{\ell}} \right)^b e^{-\frac{2\pi i m h}{\ell}},$$

where the sum runs over all Farey fractions $\frac{h}{\ell}$ in natural order. For the well-definedness of the singular series, it is exploited that the mean values $\frac{1}{\ell}\sum_{r=0}^{\ell-1} e^{\frac{2\pi i h r^k}{\ell}}$ approach zero fast enough so that for sufficiently large values b there is actually absolute convergence, and even $\mathcal{S}_b(m) \geq \frac{1}{2}$. Also, let $r_{k,\ell}(n)$ be defined by the generating function

$$\sum_{n\geq 0} r_{k,\ell}(n) q^n := (2f_k(q) - 1)^\ell = \left(1 + 2\sum_{m\geq 1} q^{m^k} \right)^\ell.$$

We then have the following theorem on the growth of $r_{k,\ell}(n)$.

Theorem 1.10 (Hardy, Littlewood, 1920)**.** *Let $k \geq 3$ be an integer. There is a number $G_1(k)$, such that*

$$r_{k,\ell}(n) \sim \frac{\left(2\Gamma\left(1+\frac{1}{k}\right)\right)^\ell}{\Gamma\left(\frac{\ell}{k}\right)} n^{\frac{\ell}{k}-1} \mathcal{S}_\ell(n), \qquad (n \to \infty),$$

for all $\ell \geq G_1(k)$.

The estimate $\mathcal{S}_b(m) \geq \frac{1}{2}$ for sufficiently large b then yields an alternative proof of Theorem 1.9. In their proof of Theorem 1.10, Hardy and Littlewood systematically decomposed the integration contour into major and minor arcs via Farey fractions, where roots of unity with small order in some specific relation to n belong to the major arcs. Another centerpiece of the proof is the approximation of the functions $f_k(q)$ by singular series near the edge of the unit circle. A good reference for the details is Chapter 5 in Nathanson's book [51] on additive number theory.

Waring's problem can be generalized to many other Diophantine equations. The analytical approaches using the Circle Method are then usually based on more or less the same ideas and techniques. Estimates of certain exponential sums are particularly central to this. Important examples include the *Weyl sums*, which are generally defined by

$$\sum_{a\leq m\leq b} e^{2\pi i P(m)}$$

with a real polynomial P. A useful estimate is given by Weyl's inequality.

Proposition 1.11 (Weyl's inequality, see [52] p. 132)**.** *Let $P(x) = \alpha x^k + O(x^{k-1})$ be a real polynomial of degree k, i.e. $\alpha \neq 0$. Assume there are rational approximations*

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{c}{b^2}$$

with $b \geq 1$ and $\gcd(a, b) = 1$, where $c > 0$ does not depend on a and b. Then for all $\varepsilon > 0$ we have

$$\left|\sum_{1\leq m\leq N} e^{2\pi i P(m)}\right| \ll_{\varepsilon,c} N^{1+\varepsilon}\left(\frac{1}{N} + \frac{1}{b} + \frac{b}{N^k}\right)^{\frac{1}{2^{k-1}}}.$$

Weyl's inequality is of a fairly general nature, but can be modified and improved, again in the context of Waring's problem. The proof of this is based, among other things, on the Cauchy–Schwarz inequality. The result goes back to Hua.

Proposition 1.12 (Hua's lemma, see [52] p. 135)**.** *The inequality*

$$\int_0^1 \left|\sum_{1\leq m\leq N} e^{2\pi i c m^k}\right|^{2^j} \mathrm{d}x \ll_\varepsilon N^{2^j - j + \varepsilon}$$

holds for all $\varepsilon > 0$ and $1 \leq j \leq k$.

At this point we want to point out that certain exponential sums have a direct relation to the algebraic geometry of curves (and varieties) over finite fields. The proof of the Riemann hypothesis in these cases (by Hasse [33] and Weil [62] in special cases, and Pierre Deligne [25] in the general case) has led to strong estimates of these sums. For example, the *Weil bound*

$$\left|\sum_{x\in\mathbb{F}_p^\times} e^{2\pi i\frac{ax+b\overline{x}}{p}}\right| \leq 2\sqrt{p}$$

holds for all $a, b \in \mathbb{F}_p^\times$, where $\overline{x}$ is defined by the relation $x\overline{x} \equiv 1$ modulo p and $\mathbb{F}_p$ denotes the finite field with p elements. Note that the sum on the left hand side is also called a *Kloosterman sum.* This very sharp estimate has a geometric interpretation: the factor 2 on the right-hand side can be written as $2 = 2g$, where $g = 1$ corresponds to the genus of an elliptic curve. Kloosterman sums have many applications in analytic number theory and automorphic forms; for more background, see [38]. We also want to mention the numerous interesting applications of exponential sums to problems in pure and applied mathematics, for example regarding counting lattice points in convex bodies. For good overviews see the books [40] and [41] of Krätzel.

With tools such as Proposition 1.12, among others, a far-reaching generalization of Waring's problem can be shown using the Circle Method. In this variant even a statement about the number of possibilities of a decomposition for growing arguments n becomes recognizable.

Theorem 1.13. *Let $k \geq 3$ and $\ell \geq 2^k + 1$ be integers, and $c_1, \ldots, c_\ell$ be coprime positive integers. Denote the number of positive integer solutions of the Dipohantine equation*

$$c_1 x_1^k + \cdots + c_\ell x_\ell^k = n$$

by $R_\ell(n)$. Then $R_\ell(n) \to \infty$ as $n \to \infty$.

Of course, this again covers Theorem 1.9. For a detailed proof of this statement, the reader should consult Chapter 4 in [52], for example.

1.2.3 Partitions

Let $n \in \mathbb{N}$. A weakly decreasing sequence of positive integers that sum to n is called a *partition* of n. The number of partitions is denoted by $p(n)$. If $\lambda_1 + \ldots + \lambda_r = n$, then the λ_j are called the *parts* of the partition. For example, we have $p(4) = 5$, as we have the five different decompositions

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

The partition function has no elementary closed formula, nor does it satisfy any finite order recurrence. However, setting $p(0) := 1$, its generating function has the following product expansion

$$\sum_{n \geq 0} p(n) q^n = \prod_{n \geq 1} \frac{1}{1 - q^n}, \tag{1.6}$$

where $|q| < 1$. This can be easily seen by using the geometric series and expanding the product. Further numerical experiments yield the values

$$\begin{aligned} p(10) &= 42, \\ p(100) &= 190\,569\,292, \\ p(200) &= 3\,972\,999\,029\,388, \qquad \text{(MacMahon)} \\ p(1000) &= 24\,061\,467\,864\,032\,622\,473\,692\,149\,727\,991. \end{aligned}$$

We thus observe a significant growth of the partition function, which appears to exceed any form of polynomial growth. For a long time, it was an open question how to describe this growth more precisely, let alone whether there was a way to compute the values $p(n)$ efficiently. However, Leonhard Euler laid important groundwork by proving the following very important theorem.

Theorem 1.14 (Pentagonal number theorem)**.** *We have the identity*

$$\prod_{n \geq 1} (1 - q^n) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(3n-1)}{2}}.$$

Using this power series identity, Euler [27] was able to provide an infinite recursion, as the left-hand side is precisely the inverted generating function of the partition function.

Theorem 1.15. *For the partition function $p(n)$ the following infinite recursion holds:*

$$\begin{aligned} p(n) = &\sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p\left(n - \tfrac{k(3k-1)}{2}\right) \\ &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) - \cdots . \end{aligned}$$

Here we put $p(n) := 0$ if $n < 0$.

The proof this theorem is simple if one knows about the Pentagonal number theorem. One just multiplies the partition function generating function with its inverse in Theorem 1.14, thus

$$\sum_{n \geq 0} p(n) q^n \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{m(3m-1)}{2}} = 1. \tag{1.7}$$

The rest is comparing coefficients in (1.7). MacMahon used the recursion formula in Theorem 1.15 to calculate $p(200)$. However, despite these insights, an asymptotic formula for estimating the growth behavior of the partition function could not yet be achieved. At the beginning of the 20–th century this changed. In [32], Hardy and Ramanujan used (1.6) to show the asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \qquad (n \to \infty), \tag{1.8}$$

which gave birth of the Circle Method. The reader should note how extraordinarily remarkable this asymptotic formula is. It is impressive not only for its brevity but also because it contains several significant mathematical constants at the same time, such as the number π, Euler's constant e, and the square root of 3. The appearance of these values in (1.8) can be readily justified in retrospect by the methods applied. For example, in the case of the number π, it is directly connected to the famous *Basel problem*, stating that:

$$\sum_{n\geq 1} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

Using modular transformations one can describe with high precision the analytic behavior of the product (1.6) if q is near a root of unity. In fact, the product essentially equals the reciprocal of the celebrated *Dedekind eta function*

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n),$$

and we have the following beautiful theorem.

Theorem 1.16. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*, such that* $c > 0$*. Then, for all* $\tau \in \mathbb{H}$*, we have the transformation law*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)(-i(c\tau+d))^{\frac{1}{2}}\eta(\tau),$$

where

$$\varepsilon(a,b,c,d) := \exp\left(\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right),$$

and

$$s(h,k) := \sum_{r=1}^{k-1} \frac{r}{k}\left\{\frac{hr}{k}\right\} \tag{1.9}$$

is the Dedekind sum. Here, $\{x\} := x - \lfloor x \rfloor - \frac{1}{2}$*. As usual, we use the principal branch of the square root.*

A purely complex analytic proof of this theorem can be found in [1]. One further sees directly from the infinite product that dominant singularities occur at such roots of unity with small denominator. These ideas culminate in Rademacher's exact formula for $p(n)$ [55]:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right). \tag{1.10}$$

Here,

$$I_{\frac{3}{2}}(x) := \sqrt{\frac{2}{\pi}} \frac{x\cosh(x) - \sinh(x)}{x^{\frac{3}{2}}}$$

is the so–called *Bessel function of the first kind* and

$$A_k(n) := \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} \exp\left(\pi i s(h,k) - \frac{2\pi i n h}{k}\right) = \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{d \pmod{24k} \\ d^2 \equiv -24n+1 \pmod{24k}}} \left(\frac{12}{d}\right) \exp\left(\frac{2\pi i d}{12k}\right),$$

with the Dedekind sum (1.9) and the Kronecker symbol $(\frac{\cdot}{\cdot})$, see also eq. (9.19) of [13]. It is extremely remarkable that this exact formula is still suitable for numerical calculations even for larger values of n, and this entirely in terms of elementary functions.

Due to its great importance, we will outline the key ideas for the proof of Rademacher's formula in the following. The process begins by determining a suitable contour, see (1.11) below, for applying the Circle Method. Such a contour was already devised by Ford, after whom the so-called *Ford circles* are named. The advantage of this method lies in the fact that the contour increasingly "hugs" the roots of unity, allowing for an effective evaluation of the modular integrand along the integration path. To construct the Ford circles, one first considers the sequence of *Farey fractions*, which, for a fixed positive integer, are defined as all rational numbers in the interval $[0,1]$ whose numerators and denominators are bounded by N. We denote the set of corresponding Farey fractions by $\mathcal{F}_N$. For example, we have

$$\mathcal{F}_5 = \left\{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\right\}.$$

The N-th Rademacher integration path $\mathcal{P}_N$ is a path from i to $i+1$ in the τ-plane, when thinking in terms of the substitution $q = e^{2\pi i\tau}$. It can be decomposed disjointly into segments $\gamma(h,k)$ with $\gcd(h,k) = 1$ and $\frac{h}{k} \in \mathcal{F}_N$, the Farey fractions of order N. Hence, we have

$$\int_{\mathcal{P}_N} = \sum_{\frac{h}{k}\in\mathcal{F}_N} \int_{\gamma(h,k)} = \sum_{k=1}^{N} \sum_{\substack{0\le h<k \\ \gcd(h,k)=1}} \int_{\gamma(h,k)}. \tag{1.11}$$

On the segments $\gamma(h,k)$, the integrand

$$f_n(\tau) := q^{-n-1} \prod_{k\ge1} \frac{1}{1-q^k}$$

is replaced by an elementary function. Due to the modularity of the Dedekind eta function, and more precisely, by Theorem 1.16, the error in this substitution is so small that it tends to 0 as $N \to \infty$ for fixed n. As a result of this analysis, we obtain (1.10).

2 New asymptotic formulas in the theory of partitions

In this habilitation thesis, we explain several new results that the author has developed in collaboration with other authors over the past years. The central link between these results lies in the combinatorial questions surrounding the partition function and, on the other hand, the use of the circle method. Below, we briefly present the key findings.

2.1 Equidistribution results

This is joint work with Walter Bridges and Taylor Garnowski, with all authors contributing each one third to the development of the paper [8]. We study the distribution of partitions whose number of parts is congruent to a residue class a mod b. Denote the number of such partitions by $p(a, b, n)$.

Example. Let $a = 1$ and $b = 5$. Then we have $p(1, 5, 8) = 3$, as the only partitions with one or six parts of 8 are given by

$$8, \qquad 3+1+1+1+1+1, \qquad 2+2+1+1+1+1.$$

It is well-known that these quantities are asymptotically equidistributed.

Theorem 2.1. *For $n \to \infty$, we have the asymptotic formula*

$$p(a, b, n) \sim \frac{p(n)}{b}. \tag{2.1}$$

The key word equidistributed here refers to the fact, that the right-hand side of (2.1) does not depend on a. Theorem 2.1 follows from an analysis of the generating function for $p(a, b, n)$, which can by orthogonality be expressed using twisted eta-products,

$$\sum_{n\geq 0} p(a, b, n)q^n = \frac{1}{b}\left(\frac{q^{\frac{1}{24}}}{\eta(\tau)} + \sum_{j=1}^{b-1} \frac{\zeta_b^{-ja}}{(\zeta_b^j q; q)_\infty}\right), \tag{2.2}$$

where $\zeta_b := e^{2\pi i/b}$ is a primitive b-th root of unity and $(a; q)_\infty := \prod_{n\geq 0}(1 - aq^n)$ is the usual *q-Pochhammer symbol*. Since the first term in the generating function dominates as $n \to \infty$, the asymptotic equidistribution follows as a corollary. Note that questions of equidistribution were considered by Cesana, Craig, and Males [19] in more generality, but without analysis of the secondary term.

We seek to explain the secondary terms in the asymptotic behavior of the $p(a, b, n)$ in more detail. Indeed, the behavior of differences $p(a_1, b, n) - p(a_2, b, n)$ reveals additional structure. A precise asymptotic description of these oscillations is given by the following theorem.

Theorem 2.2. *Let $b \geq 5$ and $a_1 \not\equiv a_2 \pmod{b}$. As $n \to \infty$, the difference $p(a_1, b, n) - p(a_2, b, n)$ satisfies the asymptotic formula*

$$\frac{p(a_1, b, n) - p(a_2, b, n)}{Bn^{-\frac{3}{4}} \exp\left(2\lambda_1\sqrt{n}\right)} = \cos\left(\beta + 2\lambda_2\sqrt{n}\right) + o(1),$$

where $\lambda_1 + i\lambda_2 := \sqrt{\mathrm{Li}_2(\zeta_b)}$, *with* $\mathrm{Li}_2(z) := \sum_{n\geq 1}\frac{z^n}{n^2}$ *denoting the dilogarithm, and the constants* $B>0$ *and* $\beta\in[0,2\pi)$ *are determined by*

$$Be^{i\beta} := \frac{1}{b}\left(\zeta_b^{-a_1} - \zeta_b^{-a_2}\right)\sqrt{\frac{(1-\zeta_b)(\lambda_1+i\lambda_2)}{\pi}}.$$

Remarkable about Theorem 2.2 is that it determines the sign changes of the differences $p(a_1,b,n)-p(a_2,b,n)$ asymptotically. Figure 1 shows the situation for the specific choices $a_1=1$, $a_2=4$ and $b=5$.

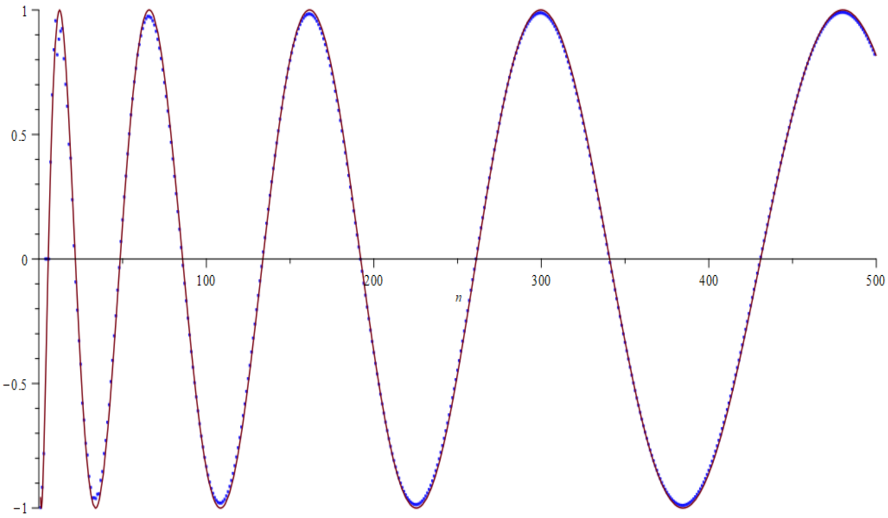

Figure 1: The plot shows the sign changes of $p(1,5,n)-p(4,5,n)$. The blue dots depict the function $\frac{p(1,5,n)-p(4,5,n)}{Bn^{-\frac{3}{4}}e^{2\lambda_1\sqrt{n}}}$ and the red line is the asymptotic prediction $\cos\left(\beta+2\lambda_2\sqrt{n}\right)$. The approximate values of the constants are $B\approx 0.23268$, $\beta\approx 1.4758$, $\lambda_1\approx 0.72984$, and $\lambda_2\approx 0.68327$.

The proof makes use of (2.2) and a detailed study of the coefficients $\sum_{n\geq 0} Q_n(\zeta)q^n := (\zeta q;q)_\infty^{-1}$ if ζ is a root of unity. Prompted by a question of Stanley, a study of the polynomials $Q_n(\zeta)$ and their complex zeroes was undertaken in a series of papers by Boyer, Goh, Keith, and Parry (see [2], [3], [4], [6], [53]), and the functions $(\zeta q;q)_\infty^{-1}$ have also been studied in recent work of Bringmann, Craig, Males, and Ono [12] in the context of distribution of homology of Hilbert schemes and t-hook lengths. Asymptotics for $Q_n(\zeta)$ were studied by Wright [63] if ζ is any positive real number, and then by Boyer and Goh [2], [3] for $|\zeta|\neq 1$. Our results essentially complete this study, proving asymptotics when ζ is any root of unity.

A further key technical result concerns the asymptotics of the coefficients $Q_n(\zeta)$ in the expansion of the twisted eta-products for roots of unity $\zeta=\zeta_b$. For $b\geq 5$, we have the following theorem.

Theorem 2.3. *Let* $b\geq 5$. *As* $n\to\infty$, *the coefficients* $Q_n(\zeta_b)$ *satisfy the asymptotic formula*

$$Q_n(\zeta_b)\sim\frac{\sqrt{1-\zeta_b}\,\mathrm{Li}_2(\zeta_b)^{\frac{1}{4}}}{2\sqrt{\pi}n^{\frac{3}{4}}}\exp\left(2\sqrt{\mathrm{Li}_2(\zeta_b)}\sqrt{n}\right).$$

Finally, we can also extend these results to more general linear combinations of partition functions of the form

$$\sum_{a\in S_1} p(a,b,n) - \sum_{a\in S_2} p(a,b,n),$$

where S_1 and S_2 are non-empty disjoint subsets of $\{0,1,\ldots,b-1\}$. These combinations can be analyzed similarly to simple differences, but with more complex oscillatory behavior.

For example, in the case $b = 12$ and the sets $S_1 = \{1,2,5\}$ and $S_2 = \{0,3,4\}$, we have the following asymptotic description:

Example. Let $b = 12$ and define $\Delta_r(n)$ for any integer shift r by

$$\Delta_r(n) := p(5+r,12,n) - p(4+r,12,n) - p(3+r,12,n) + p(2+r,12,n) + p(1+r,12,n) - p(r,12,n).$$

As $n\to\infty$, the difference $\Delta_r(n)$ satisfies

$$\frac{\Delta_r(n)}{An^{-\frac{3}{4}}\exp\left(2\lambda_1\sqrt{n}\right)} = \cos\left(\alpha - \frac{2\pi r}{6} + 2\lambda_2\sqrt{n}\right) + o(1),$$

where A, α, λ_1, and λ_2 depend only on $\mathrm{Li}_2(\zeta_6)$, but not on the shift r.

These methods can also be applied to the plane partition function. This is joint work with again Walter Bridges and Joshua Males with all authors contributing each one third to the development of the paper [10]. A *plane partition* of n is a two-dimensional array $\pi_{j,k}$ of non-negative integers $j,k\geq 1$, that is non-increasing in both variables, i.e., $\pi_{j,k}\geq\pi_{j+1,k}$, $\pi_{j,k}\geq\pi_{j,k+1}$ for all j and k, and fulfills $|\pi| := \sum_{j,k}\pi_{j,k} = n$. Following MacMahon [44], the number of plane partitions $\mathrm{pp}(n)$ of n is given by the generating function

$$\sum_{n\geq 0}\mathrm{pp}(n)q^n = \prod_{n\geq 1}\frac{1}{(1-q^n)^n}.$$

Associated to the plane partition $\pi = \{\pi_{j,k}\}_{j,k\geq 1}$ is its *trace* $t(\pi)$, which is defined by

$$t(\pi) := \sum_{j\geq 1}\pi_{j,j}.$$

When defining $\mathrm{pp}(m,n)$ the number of plane partitions of n with trace m, we get the two-variable generating function

$$\sum_{m,n\geq 0}\mathrm{pp}(m,n)\zeta^m q^n = \prod_{n\geq 1}\frac{1}{(1-\zeta q^n)^n}.$$

The coefficient of q^n is a polynomial in ζ which we denote by $T_n(\zeta)$.

To extract plane partitions with trace in residue classes, we can substitute roots of unity for ζ and use orthogonality. Let $\mathrm{pp}(a,b;n)$ denote those plane partitions of n with trace congruent to a (mod b). Then

$$\mathrm{pp}(a,b;n) = \frac{\mathrm{pp}(n)}{b} + \frac{1}{b}\sum_{1\leq\nu\leq b-1}\zeta_b^{-a\nu}T_n\left(\zeta_b^{\nu}\right). \tag{2.3}$$

From this identity, again, one can derive equidistribution, as stated in [19].

Theorem 2.4. *For all $a, b \in \mathbb{N}$, we have, as $n \to \infty$,*

$$\mathrm{pp}(a, b; n) \sim \frac{\mathrm{pp}(n)}{b}.$$

Again, secondary terms can be analyzed via a detailed study of the $T_n(\zeta)$ for roots of unity ζ, as we have (2.3). To state the oscillating behavior of the secondary terms, recall the trilogarithm which is defined for $|z| \leq 1$ by the series $\mathrm{Li}_3(z) = \sum_{n\geq 1} \frac{z^n}{n^3}$. Now we can show the following, see [10].

Theorem 2.5. *Let $b \geq 3$, and $a_1 \not\equiv a_2$ be two classes modulo b. Then we have*

$$\frac{\mathrm{pp}(a_1, b; n) - \mathrm{pp}(a_2, b; n)}{B_{a_1,a_2,b} n^{-\frac{2}{3}} \exp\left(\frac{3}{2^{\frac{2}{3}}}\lambda_1 n^{\frac{2}{3}}\right)} = \cos\left(\alpha_{a_1,a_2,b} + \frac{3}{2^{\frac{2}{3}}}\lambda_2 n^{\frac{2}{3}}\right) + o(1),$$

where $\lambda_1 + i\lambda_2 := \mathrm{Li}_3(\zeta_b)^{\frac{1}{3}}$, and $B_{a_1,a_2,b} > 0$ and $\alpha_{a_1,a_2,b} \in [0, 2\pi)$ are defined by

$$B_{a_1,a_2,b} e^{i\alpha_{a_1,a_2,b}} = \frac{2^{\frac{2}{3}}}{b\sqrt{3\pi}} \left(\zeta_b^{-a_1} - \zeta_b^{-a_2}\right) (1 - \zeta_b)^{\frac{1}{12}} (\lambda_1 + i\lambda_2)^{\frac{1}{2}}.$$

2.2 Asymptotics for numbers of representations for the $\mathfrak{su}(3)$

This is joint work with Kathrin Bringmann with both authors contributing one half each to the development of the paper [14].

We explore the connection between the number of representations of the special unitary groups $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ and partition functions. For $\mathfrak{su}(2)$, it is well known that the number of n-dimensional representations corresponds to $p(n)$, the number of partitions of n. We generalize this idea to $\mathfrak{su}(3)$, where the irreducible representations $W_{j,k}$ are indexed by pairs of positive integers, and their dimensions are given by $\dim(W_{j,k}) = \frac{1}{2}jk(j + k)$. The number of n-dimensional representations $r(n)$ for $\mathfrak{su}(3)$ has the generating function

$$G(q) := \sum_{n=0}^{\infty} r(n)q^n = \prod_{j,k\geq 1} \frac{1}{1 - q^{\frac{jk(j+k)}{2}}}.$$

Using probabilistic methods, Romik [57] proved an asymptotic formula for $r(n)$, analogous to the asymptotic for the partition function.

Theorem 2.6. *As $n \to \infty$,*

$$r(n) \sim \frac{C_0}{n^{\frac{3}{5}}} \exp\left(A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}\right),$$

where A_1, A_2, A_3, A_4, C_0 are constants that can be given explicitly.

Romik also posed the question of determining the lower-order terms in the asymptotic expansion of $r(n)$. We provide a detailed answer in the following theorem.

Theorem 2.7. *Let $L \in \mathbb{N}_0$. As $n \to \infty$,*

$$r(n) = \frac{1}{n^{\frac{3}{5}}} \left(\sum_{j=0}^{L} \frac{C_j}{n^{\frac{j}{10}}} + O_L\left(n^{-\frac{L}{10}-\frac{3}{80}}\right) \right) \exp\left(A_1 n^{\frac{2}{5}} + A_2 n^{\frac{3}{10}} + A_3 n^{\frac{1}{5}} + A_4 n^{\frac{1}{10}}\right),$$

where the constants C_j do not depend on L or n and can be computed explicitly.

The proof of Theorem 2.7 requires a detailed study of the related Dirichlet series

$$\omega(s) := 2^{-s}\zeta_{\mathfrak{su}(3)}(s) := \sum_{j,k\geq 1} \frac{1}{(2\dim(W_{j,k}))^s} = \sum_{j,k\geq 1} \frac{1}{j^s k^s (j+k)^s},$$

which converges absolutely for $\mathrm{Re}(s) > \frac{2}{3}$. Matsumoto [46] showed that $\omega(s)$ can be continued meromorphically to the entire complex plane. Romik [57] further studied the properties of $\omega(s)$, showing that it has trivial zeros at $s \in -\mathbb{N}$. A key identity behind this behavior is given by

$$\zeta(6n+2) = \frac{2(4n+1)!}{(6n+1)(2n)!^2} \sum_{k=1}^{n} \frac{\binom{2n}{2k-1}}{\binom{6n}{2n+2k-1}} \zeta(2n+2k)\zeta(4n-2k+2),$$

where $\zeta(s)$ denotes the Riemann zeta function. Romik showed that this identity is equivalent to the fact that $\omega(s)$ has zeros at $s \in -\mathbb{N}$. The trivial zeros and the distribution of poles of $\omega(s)$ play an important role in proving Theorem 2.7.

In proving Theorem 2.7, we use uniform estimates for the function $\omega(s)$ along vertical lines, which is crucial in controlling the error terms in the asymptotic expansion of $r(n)$.

2.3 Asymptotics for coefficients of infinite products and applications

This is joint work with Walter Bridges, Benjamin Brindle, and Kathrin Bringmann with all authors contributing each one quarter to the development of the paper [7]. We focus, as already announced in the introduction (1.3), on generating functions of the form

$$F_f(q) := \sum_{n\geq 0} p_f(n) q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^{f(n)}}, \tag{2.4}$$

which are of great importance in the theory of partitions and representation theory. We regard the p_f as generalized partitions, as the right-hand side product is of course a wide generalization of (1.6). It is natural to concentrate on the cases $f(n) \geq 0$, as the cases $f(n) < 0$ present their own difficulties and in some cases completely elude our methodology. An important example is the celebrated *discriminant function*

$$\frac{\Delta(q)}{q} := \prod_{n\geq 1} (1-q^n)^{24} = 1 - 24q + 252q^2 - 1472q^3 + 4830q^4 - 6048q^5 + \cdots,$$

whose coefficients, despite the seemingly innocuous definition, still pose great challenges to mathematicians today. It is also clear that the $f(n)$ must not grow too quickly, otherwise the generating function will no longer converge for all $|q| < 1$. It is sufficient to commit to polynomial growth. This is equivalent to being able to assign a Dirichlet series associated to the sequence $f(n)$ that converges somewhere. Specifically, when this Dirichlet series, denoted by

$$L_f(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}, \tag{2.5}$$

has specific good properties like a meromorphic continuation, a single simple pole on the positive real axis, and $F_f(q)$ remains bounded in some sense away from $q = 1$, Meinardus [50] provided an asymptotic formula for the coefficients $p_f(n)$. However, Debruyne and Tenenbaum [24] improved these results by relaxing the technical conditions on the growth of $F_f(q)$ and adding some additional assumptions on $f(n)$, leading to a more widely applicable result. In the following we give a brief summary on how we extend these previous works. They provide asymptotic expansions for $p_f(n)$ under milder assumptions on $f(n)$ and lead to new applications in various settings, including the partition function, plane partitions, and representations of Lie algebras.

2.3.1 Summary of results

Our main objective is to establish asymptotic formulas for a general class of partition functions. Let $f : \mathbb{N} \to \mathbb{N}_0$, and define $\Lambda := \mathbb{N} \setminus f^{-1}(\{0\})$, i.e., Λ is the support of f. For $q = e^{-z}$, where $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, we consider $G_f(z) := F_f(q)$ defined in (2.4) and the Dirichlet series $L_f(s)$ defined in (2.5). We impose the following conditions on $L_f(s)$ to obtain the main results:

(P1) All poles of $L_f(s)$ are real. Let $\alpha > 0$ be the largest pole of L_f. There exists $L \in \mathbb{N}$ such that for all primes p, we have $|\Lambda \setminus (p\mathbb{N} \cap \Lambda)| \geq L > \frac{\alpha}{2}$.

(P2) This condition is attached to some $R \in \mathbb{R}_{>0}$. The series $L_f(s)$ converges for some $s \in \mathbb{C}$, has a meromorphic continuation to $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq -R\}$, and is holomorphic on the line $\{s \in \mathbb{C} \colon \mathrm{Re}(s) = -R\}$. The function

$$L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$$

has only real poles $0 < \alpha := \gamma_1 > \gamma_2 > \ldots$, which are simple, except a possible pole at $s = 0$, that may be double.

(P3) For some $a < \frac{\pi}{2}$, in any strip $\sigma_1 \leq \sigma \leq \sigma_2$ in the domain of holomorphicity, $L_f(s)$ satisfies the bound

$$L_f(s) = O_{\sigma_1,\sigma_2}\left(e^{a|t|}\right), \qquad (|t| \to \infty),$$

where as usual $s = \sigma + it$.

Before we can state the main result, we define $\mathcal{P}_R$ to be the set of all poles of L_f^* along with 0, so we assume $0 \in \mathcal{P}_R$. Put

$$\mathcal{L} := \frac{1}{\alpha+1}\mathcal{P}_R + \sum_{\mu \in \mathcal{P}_R} \left(\frac{\mu+1}{\alpha+1} - 1\right)\mathbb{N}_0, \tag{2.6}$$

$$\mathcal{M} := \frac{\alpha}{\alpha+1}\mathbb{N}_0 + \left(-\sum_{\mu\in\mathcal{P}_R}\left(\frac{\mu+1}{\alpha+1}-1\right)\mathbb{N}_0\right) \cap \left[0, \frac{R+\alpha}{\alpha+1}\right), \tag{2.7}$$

$$\mathcal{N} := \left\{\sum_{j=1}^{K} b_j\theta_j : b_j, K \in \mathbb{N}_0, \theta_j \in (-\mathcal{L}) \cap \left(0, \frac{R}{\alpha+1}\right)\right\}. \tag{2.8}$$

We set, with $\omega_\alpha := \mathrm{Res}_{s=\alpha} L_f(s)$,

$$A_1 := \left(1+\frac{1}{\alpha}\right)(\omega_\alpha \Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}, \qquad C := \frac{e^{L_f'(0)}(\omega_\alpha\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{\frac{1}{2}-L_f(0)}{\alpha+1}}}{\sqrt{2\pi(\alpha+1)}}, \tag{2.9}$$

$$b := \frac{1-L_f(0)+\frac{\alpha}{2}}{\alpha+1}. \tag{2.10}$$

Theorem 2.8. *Assume (P1) for $L \in \mathbb{N}$, (P2) for $R > 0$, and (P3). Then, for some $M, N \in \mathbb{N}$,*

$$p_f(n) = \frac{C}{n^b}\exp\left(A_1 n^{\frac{\alpha}{\alpha+1}} + \sum_{j=2}^{M} A_j n^{\alpha_j}\right)\left(1 + \sum_{j=2}^{N}\frac{B_j}{n^{\beta_j}} + O_{L,R}\left(n^{-\min\left\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\right\}}\right)\right),$$

where $0 \le \alpha_M < \alpha_{M-1} < \cdots \alpha_2 < \alpha_1 = \frac{\alpha}{\alpha+1}$ are given by[2] *$\mathcal{L}$ (defined in (2.6)), and $0 < \beta_2 < \beta_3 < \ldots$ are given by $\mathcal{M}+\mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are defined in (2.7) and (2.8), respectively. The coefficients A_j and B_j can be calculated explicitly; the constants A_1, C, and b are provided in (2.9) and (2.10). Moreover, if α is the only positive pole of L_f, then we have $M = 1$.*

Remark. (1) Debruyne and Tenenbaum [24] proved a special case of Theorem 2.8 when f is the indicator function of a subset $\Lambda \subset \mathbb{N}$, and $L_f(s)$ has a single pole at $0 < \alpha \le 1$. Their result is covered by our result.

(2) The complexity of the exponential term strongly depends on the number and positions of the positive poles of L_f, and increases rapidly as this number grows. The case of exactly two positive poles has several applications, and was worked out in [7].

The proof of Theorem 2.8 follows by applying the Circle Method, using the analytic properties of $L_f(s)$ and precise estimates for minor arcs. The result generalizes classical results for partition functions and other generating functions, and they also apply to specific cases such as plane partitions, polygonal numbers, and representation numbers for $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$.

2.3.2 Applications

Straightforward calculations show that Theorem 2.8 covers classical results on asymptotic formulas regarding the partitions function (first term was given in (1.8)), plane partitions, and, more generally, partitions into certain polynomials. For more details, see [7] in the Appendix. In the following we briefly sketch the more subtle applications to numbers of representations of groups.

[2] We can enlarge the discrete exponent sets at will, since we can always add trivial powers with vanishing coefficients to an expansion. Therefore, from now on we always use this expression, even if the set increases tacitly.

We have already seen in Subsection 2.2 that the irreducible representations of $\mathfrak{su}(2)$ correspond to partitions of an integer n, where the number of such representations is $p(n)$. A similar analysis can be performed for $\mathfrak{so}(5)$, where the generating function for the number of representations $r_{\mathfrak{so}(5)}(n)$ is in fact given by

$$\sum_{n\geq 0} r_{\mathfrak{so}(5)}(n)q^n = \prod_{j,k\geq 1} \frac{1}{1-q^{\frac{jk(j+k)(j+2k)}{6}}}.$$

Again, one can consider the corresponding Witten zeta function for $\mathfrak{so}(5)$, which is (for more background to this function, see [46] and [47])

$$\zeta_{\mathfrak{so}(5)}(s) := \sum_{\varphi} \frac{1}{\dim(\varphi)^s} = 6^s \sum_{n,m\geq 1} \frac{1}{m^s n^s (m+n)^s (m+2n)^s}.$$

A careful analysis of the function $\zeta_{\mathfrak{so}(5)}$, especially regarding the proof of the existence of a global meromorphic continuation, growth in vertical strips, zeros and poles is required and carried out in [7]. Very crucial is the following observation.

Theorem 2.9. *The function $\zeta_{\mathfrak{so}(5)}$ has a meromorphic continuation to $\mathbb{C}$ whose positive poles are all simple and occur for $s \in \{\frac{1}{2}, \frac{1}{3}\}$.*

Having this, we can prove the following theorem.

Theorem 2.10. *Let $N \in \mathbb{N}$. We have as $n \to \infty$ the asymptotic expression*

$$r_{\mathfrak{so}(5)}(n) = \frac{C}{n^{\frac{7}{12}}} \exp\left(A_1 n^{\frac{1}{3}} + A_2 n^{\frac{2}{9}} + A_3 n^{\frac{1}{9}} + A_4\right) \left(1 + \sum_{j=2}^{N+1} \frac{B_j}{n^{\frac{j-1}{9}}} + O_N\left(n^{-\frac{N+1}{9}}\right)\right),$$

where the constants C, A_1, A_2, A_3, and A_4 are explicitly by

$$C = \frac{e^{\zeta'_{\mathfrak{so}(5)}(0)} \Gamma\left(\frac{1}{4}\right)^{\frac{1}{6}} \zeta\left(\frac{3}{2}\right)^{\frac{1}{12}}}{2^{\frac{1}{3}} 3^{\frac{11}{24}} \sqrt{\pi}}, \qquad A_1 = \frac{3^{\frac{4}{3}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{3}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{3}}}{2^{\frac{8}{3}}},$$

$$A_2 = \frac{2^{\frac{8}{9}} \left(2^{\frac{1}{3}}+1\right) \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right)}{3^{\frac{7}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{4}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{2}{9}}}, \qquad A_3 = -\frac{2^{\frac{40}{9}} \left(2^{\frac{1}{3}}+1\right)^2 \Gamma\left(\frac{1}{3}\right)^2 \zeta\left(\frac{1}{3}\right)^2 \zeta\left(\frac{4}{3}\right)^2}{3^{\frac{44}{9}} \Gamma\left(\frac{1}{4}\right)^{\frac{20}{9}} \zeta\left(\frac{3}{2}\right)^{\frac{10}{9}}},$$

$$A_4 = \frac{2^8 \left(2^{\frac{1}{3}}+1\right)^3 \Gamma\left(\frac{1}{3}\right)^3 \zeta\left(\frac{1}{3}\right)^3 \zeta\left(\frac{4}{3}\right)^3}{3^8 \Gamma\left(\frac{1}{4}\right)^4 \zeta\left(\frac{3}{2}\right)^2},$$

and the constants B_j can be calculated explicitly.

2.4 Asymptotics of commuting ℓ-tuples in symmetric groups and log-concavity

This is joint work with Kathrin Bringmann and Bernhard Heim with all authors contributing one third each to the development of the paper [9].

We study asymptotics for commuting ℓ-tuples in the symmetric group S_n, where S_n denotes the symmetric group for $n \in \mathbb{N}$. More precisely, let $\ell \in \mathbb{N}$ and

$$C_{\ell,n} := \left\{ (\pi_1, \ldots, \pi_\ell) \in S_n^\ell : \pi_j \pi_k = \pi_k \pi_j \text{ for all } 1 \leq j, k \leq \ell \right\}.$$

The cardinality of this set, $|C_{\ell,n}|$, is divisible by $|S_n|$, and hence the normalized quantity

$$N_\ell(n) := \frac{|C_{\ell,n}|}{|S_n|} = \frac{|C_{\ell,n}|}{n!}$$

is a positive integer. We give asymptotics for the numbers $N_\ell(n)$ for fixed ℓ as $n \to \infty$. A very important first observation in this direction was made by Bryan and Fulman [15], proving the following generating function identity for $|C_{\ell,n}|$.

Theorem 2.11 (Bryan and Fulman)**.** *For $\ell \in \mathbb{N}$, we have*

$$\sum_{n \geq 0} N_\ell(n) q^n = \prod_{n \geq 1} (1 - q^n)^{-g_{\ell-1}(n)} = \exp\left(\sum_{n \geq 1} g_\ell(n) \frac{q^n}{n} \right),$$

where $g_\ell(n)$ is the number of subgroups of $\mathbb{Z}^\ell$ of index n.

For $\ell = 2$, we have $g_2(n) = \sigma_1(n)$, where as usual $\sigma_m(n)$ denotes the divisor sum $\sigma_m(n) = \sum_{d|n} d^m$. Consequently, $N_2(n) = p(n)$, the number of partitions of n, and it counts the number of conjugacy classes in S_n [26]. For $\ell = 3$, $N_3(n)$ counts the number of non-equivalent n-sheeted coverings of a torus, as shown by Liskovets and Medynkh [43].

We investigate the asymptotic behavior and log-concavity of $N_\ell(n)$. Crucial at this point is the Dirichlet generating function

$$\sum_{n \geq 1} \frac{g_{\ell-1}(n)}{n^s} = \zeta(s)\zeta(s-1) \cdots \zeta(s - \ell + 2).$$

This function has a holomorphic continuation to $\mathbb{C} \setminus \{\ell - 1, \ell - 2, \ell - 3\}$, with at most simple poles for $s \in \{\ell - 1, \ell - 2, \ell - 3\}$ (in fact, they are all simple poles in the cases $\ell \geq 4$). The fact that the number of poles is "stationary" is interesting but also very advantageous, as it simplifies the analysis. It comes from the fact that the Riemann zeta function has trivial zeros at the negative even integers. As polynomial growth along vertical strips is well-known by elementary theory of the zeta function, and the $g_{\ell-1}$ are always positive as $\ell > 1$, we can use Theorem 2.8 to find asymptotic formulas for the $N_\ell(n)$.

Theorem 2.12. *For $\ell \geq 6$, as $n \to \infty$, we have the asymptotic expansion*

$$N_\ell(n) \sim \frac{(\ell - 1)!^{\frac{1}{2\ell}} \sqrt{Z_\ell}}{\sqrt{2\pi\ell} n^{\frac{\ell+1}{2\ell}}} \exp\left(\frac{\ell \Gamma(\ell)^{\frac{1}{\ell}} Z_\ell}{\ell - 1} n^{\frac{\ell-1}{\ell}} + \sum_{k=2}^{\ell} A_{\ell,k} n^{\frac{\ell-k}{\ell}} \right) \left(1 + \sum_{j \geq 1} \frac{B_{\ell,j}}{n^{\frac{j}{\ell}}} \right),$$

where $Z_\ell := (\zeta(2)\zeta(3) \cdots \zeta(\ell))^{\frac{1}{\ell}}$, and the constants $A_{\ell,k}$ and $B_{\ell,j}$ can be calculated explicitly.

We can also give similar results for the special cases $\ell \in \{2,3,4,5\}$, see [9]. We further prove that these asymptotics imply log-concavity for $N_\ell(n)$ for large n.

Theorem 2.13. *Assume that $c(n)$ is a sequence with asymptotic expansion*

$$c(n) \sim \frac{C}{n^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda n^\lambda\right) \sum_{\mu \in \mathcal{T}} \frac{\beta_\mu}{n^\mu}, \qquad (n \to \infty),$$

where $\kappa \in \mathbb{R}$, $\mathcal{S} \subset \mathbb{Q}^+ \cap (0,1)$ is finite, and $\mathcal{T} \subset \mathbb{Q}_0^+$, $C, A_\lambda, \beta_\mu, \in \mathbb{R}$ with $\beta_0 = 1$. If $\lambda^* := \max\{\lambda \in \mathcal{S} : A_\lambda \neq 0\}$ and $A_{\lambda^*} > 0$, then $c(n)$ is log-concave for sufficiently large n.

Applying this to $N_\ell(n)$, we obtain the following.

Corollary 2.14. *For $\ell \geq 2$ and sufficiently large n, $N_\ell(n)$ is log-concave.*

We also establish a Bessenrodt-Ono type inequality for the sequences $N_\ell(n)$. We even have the following more general result.

Theorem 2.15. *Let $c(n)$ be a sequence satisfying*

$$c(n) \sim \frac{C}{n^\kappa} \exp\left(\sum_{\lambda \in \mathcal{S}} A_\lambda n^\lambda\right), \qquad (n \to \infty),$$

with $\kappa \in \mathbb{R}$, $\mathcal{S} \in \mathbb{R} \cap (0,1)$ be finite, $C, A_\lambda \in \mathbb{R}$. Let $\lambda^* := \max\{\lambda \in \mathcal{S} : A_\lambda \neq 0\}$ with $A_{\lambda^*} > 0$. If $a, b \gg 1$, then

$$c(a)c(b) > c(a+b).$$

Both of these theorems can be proved using elementary, but tedious calculations. Having Theorem 2.12 and its generalizations to $2 \leq \ell \leq 5$, this applies to the sequences $N_\ell(n)$.

Corollary 2.16. *For $\ell \geq 2$, we have $N_\ell(a)N_\ell(b) > N_\ell(a+b)$ for $a, b \gg 1$.*

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