Surface Evolution Equations - A level set approach
Exercises

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A. Relaxed limit

Let X be a metric space with metric d. Let D be a subset of X. Let $\{u^\varepsilon\}_{0<\varepsilon<1}$ be a one-parameter family of functions defined on D with values in $\overline{\mathbf{R}}(:= \mathbf{R} \cup \{\pm\infty\}$. We define the upper relaxed limit

$$(\limsup^*_{\varepsilon\to 0} u^\varepsilon)(z) := \lim_{r\downarrow 0} \sup\{u^\varepsilon\ (y)\ :\ y \in D,\ d(y,z) < r,\ 0 < \varepsilon < r\}, z \in \overline{D},$$

where $\overline{D}$ is the closure of D in X. The lower relaxed limit is defined as $\liminf_* u^\varepsilon = -\limsup^*(-u^\varepsilon)$. We often write $\overline{u} = \limsup^* u^\varepsilon$ and $\underline{u} = \liminf_* u^\varepsilon$. Prove the following statements.

1. Let z_0 be a point in $\overline{D}$. Assume that $\overline{u}(z_0) = (\limsup^* u^\varepsilon)(z_0) < \infty$. Then there are sequences $\{\varepsilon_j\}_{j=1}^\infty \subset (0,1)$ and $\{z_j\}_{j=1}^\infty \subset D$ such that $\varepsilon_j \to 0$ as $j \to \infty$, $\lim_{j\to\infty} u^{\varepsilon_j}(z_j) = \overline{u}(z_0)$ and $\lim_{j\to\infty}\ z_j = z_0$.
2. The function $\overline{u} = \limsup^*_{\varepsilon\to 0} u^\varepsilon$ is an upper semicontinuous function defined in $\overline{D}$, i.e., $\overline{u}(z) \geq \limsup_{y\to z} \overline{u}(y)$ for $z \in \overline{D}$. Similarly, the function $\underline{u} = \liminf_{*\varepsilon\to 0}\ u^\varepsilon$ is a lower semicontinuous function in $\overline{D}$.
3. Assume that $\overline{D}$ is locally compact. If $\overline{u}(z) = \underline{u}(z) \in \mathbf{R}$ for all $z \in D$, then u^ε converges to $\overline{u}$ locally uniformly in D as $\varepsilon \to 0$.
4. Assume that D is locally compact. Assume that u^ε is a (real-valued) upper semicontinuous function in D and that $\overline{u}$ attains a strict local maximum ($\neq \infty$) at $z_0 \in \overline{D}$. Then there are sequences $\{\varepsilon_j\}_{j=1}^\infty \subset (0,1)$ and $\{z_j\}_{j=1}^\infty \subset D$ such that z_j is a local maximizer of u^{ε_j} and that $z_j \to z_0$ and $\varepsilon_j \to 0$ as $j \to \infty$. Moreover, $\lim_{j\to\infty} u^{\varepsilon_j}(z_j) = \overline{u}(z_0)$.

5. Assume that X is compact and that u^ε is upper semicontinuous in $D = X$. We set $K = \{z \in X \ : \ \overline{u}(z) \geq 0\}$. Let d_ε be defined by
$$d_\varepsilon = \sup\{d(z, K) \ : \ u^\varepsilon(z) \geq 0, \ z \in X\}.$$
Then $d_\varepsilon \to 0$ as $\varepsilon \to 0$.
6. Assume that $X = \mathbf{R}^n$ and $\overline{u} = \underline{u}$ in X and that u^ε is continuous. Assume that $K = \{a \in X \ : \ \overline{u}(z) \geq 0\}$ is compact and that $K = \overline{H}$ with $H = \{z \in X \ : \ \overline{u}(z) > 0\}$. Then $K_\varepsilon = \{z \in X \ : \ \overline{u}(z) > 0\}$ converges to K as $\varepsilon \to 0$ in the sense of Hausdorff distance topology provided that K_ε is compact. (Here is a definition of the Hausdorff distance for two sets $A, B \subset X$: $d_H(A, B) = \max\{\sup_{x \in A} \ d(x, B), \ \sup_{y \in B} \ d(y, A)\}$.)
7. Give an example that the conclusion of Problem 6 is false if one drops the assumption $K = \overline{H}$ even if one assumes that $\Gamma = \{x \in X \ : \ \overline{u}(z) = 0\}$ has no interior.
8. Assume that u_0 is continuous on a compact set K in $\mathbf{R}^d$. For $x_0 \in K$ and $\lambda > 0$ we set $V_{\lambda, x_0}(x) = \lambda + u_0(x_0) + C|x - x_0|^2$. Then for each $\lambda > 0$ there is a constant C depending only on λ (and u_0) such that
$$u_0(x) \leq V_{\lambda, x_0}(x) \text{ for all } x \ \in K.$$
We shall write such C by $C = C(\lambda)$. Then $u_0(x) = \inf\{V_{\lambda, x_0}(x) : \lambda > 0, \ C = C(\lambda), \ x_0 \in K\}$.
9. Assume that u_0 is continuous in $\overline{\Omega}$, where Ω is a bounded open set in $\mathbf{R}^d$. Let V_{λ, x_0} be as in Problem 8 with $C = C(\lambda)$. Assume that $u^\varepsilon \ : \ \Omega \times (0, T) \to \mathbf{R} \cup \{-\infty\}$ satisfies
$$u^\varepsilon(x, t) \leq V_{\lambda, x_0}(x) + C(\lambda) t.$$
Then $\overline{u}(x, 0) \leq u_0(x)$ for all $x \in \Omega$, where $\overline{u} = \limsup^*_{\varepsilon \to 0} \ u^\varepsilon$.

B. Viscosity Solutions

Prove the following statements.

10. Assume that u^ε is a viscosity subsolution of the level set mean curvature flow equation $u_t - |\nabla u| \ \mathrm{div} \ (\nabla u / |\nabla u|) = 0$ in $\mathbf{R}^d \times (0, T)$. Then $\overline{u} = \limsup^*_{\varepsilon \to 0} u^\varepsilon$ is a viscosity subsolution of the same equation in $\mathbf{R}^d \times (0, T)$ provided that $\overline{u}(z) < \infty$ for all $z \in \mathbf{R}^d \times (0, T)$. One may replace $\mathbf{R}^d$ by an open set Ω in $\mathbf{R}^d$.

11. The stability result in Problem 10 is still vaild for the Neumann boundary value problem in $\Omega \times (0, T)$.

12. Let $u : Q \to \mathbf{R} \cup \{-\infty\}$ be an upper semicontinuous function, where $Q = \Omega \times (0, T)$ and Ω is an open set in $\mathbf{R}^d$. Then u is a viscosity subsolution of a level set mean curvature flow equation in Q if (and only if) $(\phi, \hat{z}) \in C^2(Q) \times Q$ satisfies

    (i) $\phi_t - |\nabla \phi| \mathrm{div}\ (\nabla \phi / |\nabla \phi|) \leq 0$ at $\hat{z} \in Q$ if $\nabla \phi(\hat{z}) \neq 0$ and

    (ii) $\phi_t(\hat{z}) \leq 0$ if $\nabla \phi(\hat{z}) = 0$, $\nabla^2 \phi(\hat{z}) = O$

    whenever $\max_Q(u - \phi) = (u - \phi)(\hat{z})$.

    Hint : Assume that $u - \phi$ takes its strict maximum at $\hat{z} \in Q$ and $\nabla \phi(\hat{z}) = 0$. We consider $u - \phi_\varepsilon$ with $\phi_\varepsilon(x, y, t) = |x - y|^4/4\varepsilon + \phi(y, t)$, $\varepsilon > 0$ and derive several inequalities for ϕ_ε at a maximizer of $u(x, t) - \phi_\varepsilon(x, y, t)$ in Q.

13. The function
$$u(x, t) = \min(0,\ t - |x|)$$
    is a viscosity solution of the Neumann problem
$$\begin{cases} \partial_t u - |\nabla u| = 0 \text{ in } \{|x| < 1\} \times (0, T) \\ \partial u / \partial \nu = 0 \text{ on } \partial\{|x| < 1\} \times (0, T) \end{cases}$$
    (although the slope $\partial u / \partial \nu$ at $|x| = 1$ is not zero.) Here $\partial / \partial \nu$ denotes the exterior normal differential operator.

C. Structure of equations and examples of solutions

14. Write the mean curvature flow equation $V = H$ for $u = u(x_1, t)$, when $\Gamma_t \subset \mathbf{R}^N$ is a hypersurface of rotation of the form
$$\Gamma_t = \{(x_1, \cdots, x_N) \in \mathbf{R}^N | r = u(x_1, t),\ r = (\sum_{j=2}^{N} x_j^2)^{1/2}\}.$$

15. Write the mean curvature flow equation $V = H$ for $u = u(x_1, \cdots, x_{N-1}, t)$, when $\Gamma_t \subset \mathbf{R}^N$ is of the form
$$\Gamma_t = \{(x_1, \cdots, x_N) \in \mathbf{R}^N | x_N = u(x_1, \cdots, x_{N-1}, t)\}.$$

16. Assume that $F = F(p, X)$ (defined in $(\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$) is geometric. Assume that $X \mapsto F(p, X)$ is continuous for each $p \in \mathbf{R}^N\backslash\{0\}$. Assume that F is (degenerate) elliptic. Prove that F satisfies

$$F(p, X + y \otimes p + p \otimes y) = F(p, X) \qquad \text{(SG)}$$

for all $y \in \mathbf{R}^N, X \in \mathbf{S}^N$, $p \in \mathbf{R}^N\backslash\{0\}$. (In other words F is strongly geometric.)

17. Give an example that F is geometric but not fulfills the condition (SG) of Problem 16.

18. Assume that u is a viscosity solution of

$$u_t + F(\nabla u, \nabla^2 u) = 0 \text{ in } Q = \Omega \times (0, T).$$

Assume that F is geometric and continuous in $(\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$. Assume that F can be extended continuously at $(0, O)$. Prove that $\theta \circ u$ is a viscosity subsolution of the above equation in Q provided that θ is continuous and nondecreasing. Here Ω is an open set in $\mathbf{R}^N$.

19. Assume that $\gamma : \mathbf{R}^N \to [0, \infty)$ is positively homogeneous of degree one. Assume that $\gamma \in C^2(\mathbf{R}^N\backslash\{0\})$. Prove that

$$\nabla^2\gamma(p) + p \otimes p > O \text{ for all } p \in \mathbf{R}^N\backslash\{0\}.$$

if and only if $\nabla^2(\gamma^2)(p) > 0$ for all $p \in R^{\cup}\backslash\{0\}$.

20. Under the assumption of Problem 19 prove that

$$\text{Frnak } \gamma = \{p \in \mathbf{R}^N : \gamma(p) \leq 1\}$$

is strictly convex (in the sense that all inward principal curvatures of ∂ (Frank γ) are positive) if and only if $\nabla^2\gamma(p) + p \otimes p > O$ for all $p \in \mathbf{R}^N\backslash\{0\}$.

21. Assume that same hypotheses of Problem 19 concerning γ. Assume that Frank γ is strictly convex. Prove that the Wulff shape

$$W_\gamma = \{p \in \mathbf{R}^N | p \cdot m \leq \gamma(m) \text{ for all } m \in S^{N-1}\}.$$

is strictly convex and C^2.

22. Assume the same hypotheses of Problem 21 concerning γ. Prove that there exists a shrinking self-similar solution of the form $a(t)\partial W_\gamma$ of
$$V = -\gamma(\mathbf{n})\ \mathrm{div}_{\Gamma_t}(\nabla\gamma(\mathbf{n})).$$

23. For $u_0, v_0 \in C(\mathbf{R}^N)$ assume that
$$\{u_0 > 0\} \subset \{v_0 > 0\} (= \{x \in \mathbf{R}^N | v_0(x) > 0\}).$$
Assume that $\overline{\{v_0 > 0\}}$ is compact. Prove that there exists a non-decreasing function $\theta \in C(\mathbf{R})$ such that $\theta(s) = 0$ (for $s \leq 0$) and $\theta(s) > 0$ (for $s > 0$) and
$$u_0 \leq \theta \circ v_0 \text{ in } \mathbf{R}^N.$$

D. Dynamical programing principle

Let K be a compact set in $\mathbf{R}^d$. Assume that $f : \mathbf{R}^N \times K \to \mathbf{R}^N$ is continuous and that there is a constant L satisfying
$$|f(x, a) - f(y, a)| \leq L|x - y|$$
for all $x, y \in \mathbf{R}^N$, $a \in K$. Let T be a positive number (called terminal time). Let A be of the form
$$A = \{\alpha : [0, T] \to K |\ \alpha \text{ is Lebesgue measurable}\}.$$
(An element of this set is called a control.) Let $X^\alpha_{t,x}(s)$ be the solution of the state equation
$$\begin{cases} \frac{dX}{ds} = f(X(s)\ ,\ \alpha(s))\ ,\ T > s > t \\ X(t) = x \in \mathbf{R}^N\ ,\ s = t. \end{cases}$$
Let g be a real-valued continuous function defined in $\mathbf{R}^N$. Let u be the value function (with the terminal data g) of the form
$$u(x, t) = \inf_{\alpha \in A} g(X^\alpha_{x,t}(T)).$$

24. Prove that the dynamical programing principle
$$u(x, t) = \inf_{\alpha \in A} \{u(X^\alpha_{x,t}(t + \delta)\ ,\ t + \delta)\} \text{ for } t + \delta \leq T\ ,\ \delta > 0.$$

25. Prove that $v(x,t) = u(x, T-t)$ is a viscosity solution of

$$v_t - H(x, \nabla v) = 0 \text{ in } \mathbf{R}^N \times (0,T)$$

with $H(x,p) = \min_{a \in K} p \cdot f(x,a)$.

26*. Let Ω be a bounded C^2 convex domain in $\mathbf{R}^2$. Let $S^t(x,v)$, $x \in \Omega$, $v \in S^1$ be the billiard semiflow in Ω. Prove that for any fixed $t \geq 0$, $x \in \overline{\Omega}$ and $v \in S^1$, there exists $d_l \geq 0$, $y_l \in \partial\Omega \cap B_t(x)$ where $l = 1, 2, \cdots$ such that $\sum_{l=0}^{\infty} d_l \nu(y_l)$ converges and

$$\alpha^t(x,v) = \sum_{l=0}^{\infty} d_l \nu(y_l)$$

where $\alpha^t(x,v) = S^t(x,v) - (x + tv)$ is a boundary adjustor. Here ν denotes the unit outward normal of $\partial\Omega$.

27*. Assume that $\{u^\varepsilon\}_{0<\varepsilon<1}$ is uniformly bounded in $\overline{\Omega}\times(0,T)$. Assume that u^ε fullfills

$$u^\varepsilon(x,t) = \inf_{|v|=1} \sup_{b=\pm 1} u^\varepsilon(S^{\sqrt{2}\varepsilon}(x, bv) \ , \ t + \varepsilon^2)$$

for $x \in \overline{\Omega}$, $t \in (0,T)$, $t + \varepsilon^2 \leq T$. Let $\underline{u}$ be $\underline{u} = \liminf_{*\varepsilon\to 0} u^\varepsilon$. Prove that $v(x,t) = \underline{u}(x, T-t)$ is a viscosity supersolution of

$$v_t - |\nabla v| \operatorname{div}(\nabla v / |\nabla v|) = 0 \text{ in } \Omega \times (0,T),$$
$$\partial v / \partial \nu = 0 \text{ on } \partial\Omega \times (0,T).$$

E. Variational problem with obstacles

Let Z be a real-valued C^2 (or $C^{1,1}$) function defined in a bounded interval $\overline{I}$, where $I = (a,b)$. For a given $\Delta > 0$ let $K_\pm$ be the subset of $H^1(I)$ of the form

$$K_\pm = \{\xi \in H^1(I) : Z(x) - \Delta/2 \leq \xi(x) \leq Z(x) + \Delta/2, \ \xi(a) = Z(a) - \Delta/2, \xi(b) = Z(b) \pm \Delta/2\}.$$

Let $J_\pm$ be the functional in $L^2(I)$ defined by

$$J_\pm(\xi) = \begin{cases} \int_a^b |\xi'(x)|^2 dx, & \xi \in K_\pm \\ \infty & , \ \text{otherwise.} \end{cases}$$

28. Prove that $H^1(I) \subset C^{1/2}(\overline{I}) \subset C(\overline{I})$.
29. Prove that $J_\pm$ is lower semicontinuous, convex on $L^2(I)$.
30. Prove that $J_\pm$ admits a unique (absolute) minimizer.
31. Let ξ_+ be the minimizer of J_+. Let $D_\pm$ be the coincidence set defined by

$$D_\pm = \{x \in \overline{I} \ : \ \xi_+ = Z(x) \pm \Delta/2\}.$$

Prove that ξ_+ is concave in a neighborhood of D_- and that ξ_+ is convex in a neighborhood of D_+. Prove that $\xi'_+ = 0$ outside $D_+ \cup D_-$. (We say that ξ satisfies the concave-convex condition if these three properties are fulfilled.)
32. If ξ satisfies the concave-convex condition and $\xi(a) = Z(a) - \Delta/2$, $\xi(b) = Z(b) \pm \Delta/2$, it must be the minimizer of $J_\pm$.
33. Let ξ_+ be the minimizer of J_+. Prove that ξ_+ is $C^{1,1}$ and

$$\sup_{x\in I} |\xi''_+(x)| \le \sup_{x\in I} |Z''(x)|.$$

34. Suppose that the concave hull Z_{cave} of Z in I is smaller than $Z + \Delta/2$ i.e. $Z_{\text{cave}} \le Z + \Delta/2$ in I. Let ξ_- be the minimizer of J_-. Prove that

$$\xi'_-(x) = Z'_{\text{cave}}(x), \ x \in I.$$

35. Suppose that staight line function $\xi(x) = \xi(a) + \frac{Z(b)-Z(a)+\Delta}{b-a}(x-a)$ is in K_+. Prove that ξ is the minimizer of J_+.
36. (Comparison principle)
Let $\xi_\pm$ be the minimizer of $J_\pm$. It is determined by I. Let

$$\Lambda_\pm(x, I) = \xi'_\pm(x).$$

Prove that

$$\Lambda_\pm(x, I_1) \le \Lambda_\pm(x, I_2) \text{ for } x \in I_2$$

if $I_2 \subset I_1$.
37. Let $J^k_\pm$ be the functional defined as J by replacing Z by $Z^k(k = 1, 2, \cdots)$, where Z^k is a real-valued C^2 function defined in $\overline{I}$. Assume that Z^k conveges to Z uniformly with its first derivatie in $\overline{I}$. Prove that for any $\xi_k \to \xi$ in $L^2(I)$

$$J_\pm(\xi) \le \liminf_{k\to\infty} J^k_\pm(\xi_k).$$

38. Assume that the same hypotheses of Problem 37 concerning Z^k. Prove that for each $\xi \in L^2(I)$ there is a sequence $\xi_k \to \xi$ in $L^2(I)$ such that
$$J_\pm(\xi) = \lim_{k\to\infty} J_\pm^k(\xi_k)$$
39. (Convergence of minimizers under (realxed limit) $\Gamma-$ convergence) Assume that same hypotheses of Problem 37 concerning Z^k. Let $\xi_\pm^k$ be the minimizer of $J_\pm^k$ and $\xi_\pm$ be the minimizer of $J_\pm$. Then $\xi_\pm^k \to \xi_\pm$ in $L^2(I)$.
40. Let D be a compact metric space. Assume that u^ε be a real-valued lower semicontinuous function on D. Let z^ε be an (absolute) minimizer of u^ε. Prove that there is a subsequence $z^{\varepsilon_j}(\varepsilon_j \to 0)$ such that it converges to an (absolute) minimizer z of $\underline{u}$ in D.
41. (Stability)
Assume that
$$\sup_{k\ge1}\sup_{x\in I} |(d/dx)^2 Z^k(x)| < \infty \text{ and } Z^k \to Z \text{ in } C^1(\overline{I}).$$
Let
$$\Lambda_\pm^k(x,I) = (d/dx)\xi_\pm^k(x),$$
where $\xi_\pm$ be the minimzier of $J_\pm^k$. Prove that $\Lambda_\pm^k$ converges to $\Lambda_\pm(x,I)$ uniformly in $\overline{I}$ as $k\to\infty$. (Use Problem 33.)
42. Let Z be a C^2 function in $\mathbf{R}$. Prove that $\Lambda_\pm(x,I)$ is continuous with respect to I. (Clarify the meaning of continuity.) Assume furthermore that $|(d/dx)^2 Z|$ is bounded in $\mathbf{R}$. Prove that for each $r>0$
$$\lim_{\mu\to0}\sup_{0<b-a<r}\sup_{a<x<b}|\Lambda_\pm(x,(a,b)) - \Lambda_\pm(x-\mu,\ (a-\mu,\ b-\mu))| = 0.$$

F. Sup-convolution (regularization)

Let ϕ be a function from $\mathbf{R}\times(0,1]$ to $[0,\infty)$. Assume that ϕ fulfills following conditions.

(i) For each $\lambda, 0<\lambda\le1$, $\phi(\cdot,\lambda)$ is Lipschitz continuous on every bounded set in $\mathbf{R}$.

(ii) $\phi(\xi,\lambda)$ is even in ξ, i.e. $\phi(\xi,\lambda)=\phi(-\xi,\lambda)$

(iii) $\phi(\xi, \lambda)$ is nonincreasing in λ for all ξ.

(iv) $\lim_{\xi\to\infty} \phi(\xi, 1) = \infty$ and $\phi(\xi, \lambda)$ is nondecreasing in $\xi \geq 0$, for $0 < \lambda \leq 1$.

(v) $\lim_{\lambda\downarrow 0} \phi(\xi, \lambda) = \infty$ unless $\xi = 0$ and $\phi(0, \lambda) = 0,\ 0 < \lambda \leq 1$.

Let f be a function in $\mathbf{R}$ with values in $\mathbf{R} \cup \{-\infty\}$. We say that

$$f^\lambda(x) = \sup_{\xi\in\mathbf{R}} \{f(\xi) - \phi(\xi - x, \lambda)\}$$

is a *sup-convolution* of f by ϕ. Prove the following statements under assumptions (i)-(v) for ϕ.

43. Let $f(\not\equiv -\infty)$ be a function on $\mathbf{R}$ with values in $\mathbf{R} \cup \{-\infty\}$. Assume that f is locally bounded from above and that

$$\lim_{|\xi|\to\infty} \max(f(\xi), 0)/\phi(\xi - x,\ 1) = 0 \text{ for each } x \in \mathbf{R}.$$

Then f^λ is locally Lipschitz. Moreover,

$$f^\lambda \geq f^\mu \geq f \text{ for } \lambda \geq \mu > 0$$

and $\lim_{\lambda\downarrow 0} f^\lambda(x) = f^*(x)$ for each $x \in \mathbf{R}$. Here f^* denotes the upper semicontinuous envelope of f, i.e.

$$f^*(x) = \lim_{\varepsilon\downarrow 0} \sup\{f(y)\ :\ |x - y| < \varepsilon\}.$$

44. Assume that same hypotheses of Problem 43. Let B and B' be bounded open sets in $\mathbf{R}$ with $\overline{B} \subset B'$. Then for each $K_0 > 0$ there is $\lambda_0(K_0) > 0$ such that

$$\sup_{x\in\overline{B}}\ \sup_{\xi\notin B'} H(\xi, x, \lambda) < -K_0 \text{ for } \lambda < \lambda_0(K_0)$$

with $H(\xi, x, \lambda) = f(\xi) - \phi(\xi - x, \lambda)$. Moreover,

$$f^\lambda(x) = \sup_{\xi\in B'} H(\xi, x, \lambda) \text{ for } x \in \overline{B}$$

provided that $\inf_{\overline{B}} f^* > -\infty$ and $\lambda < \lambda_0 \equiv \lambda_0(\max(0, -\inf_{\overline{B}} f^*)\}$.

45. Assume that same hypotheses of Problem 43. If $\hat{x}$ be a maximizer of f over B', then $f^\lambda(x) \le f(\hat{x})$ for $x \in \overline{B}$ provided that

$$\lambda < \lambda_0'' \equiv \lambda_0(\max(0, -f(\hat{x}))).$$

46. Assume that $f : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$, $f \not\equiv -\infty$ is locally bounded from above. If $\phi(x, \lambda) = |x|^2/\lambda$, then f^λ is semi- convex in $\mathbf{R}$. In fact, $f^\lambda(x) + |x|^2/\lambda$ is convex.
47. Assume that for $0 < \lambda \le 1$

$$\sigma_\lambda := \sup\{|\xi| \ : \ \phi(\xi, \lambda) = 0\} > 0.$$

Assume that same hypotheses of Problem 43. Assume that f has a local maximum at $\hat{x} \in \mathbf{R}$ and that f is not a constant function. Then there is a small $\lambda_1,\ 0 < \lambda_1 \ \le 1$ such that for $\lambda \le \lambda_1$

(i) f^λ is faceted at $\hat{x}$ in $\mathbf{R}$ with slope zero and $f^\lambda(\hat{x}) = f(\hat{x})$.

(ii) $\hat{x}$ is an interior point of the faceted region.

48. Let

$$\vartheta(x, \rho, \lambda) = \begin{cases} (x-\rho)^2/\lambda, & x > \rho \\ 0 \qquad\qquad\ , & |x| \le \rho \\ (x+\rho)^2/\lambda, & x < -\rho \end{cases}$$

Then for each $\rho > 0,\ \phi(x, \lambda) = \vartheta(x, \rho, \lambda)$ satisfies all assumptions (i)-(v) in the begining of Section F and that $\sigma_\lambda > 0$, where σ_λ is defined in Problem 47.
49. Let ϑ be as in Problem 48. Then
    (a) $\vartheta(x, \rho-\alpha, \lambda-\beta) = \sup_{\xi\in\mathbf{R}}\{\vartheta(\xi, \rho, \lambda) - \vartheta(\xi-x, \alpha, \beta)\}$ for $x \in \mathbf{R}$ provided that $0 \le \alpha \le \rho,\ 0 < \beta < \lambda$.
    (b) $\vartheta(x - y, \rho - (\alpha_1 + \alpha_2),\ \lambda - (\beta_1 + \beta_2)) = \sup_\xi \sup_\eta\{\vartheta(\xi - \eta\ ,\ \rho, \lambda) - \vartheta(\xi - x, \alpha_1, \beta_1) - \vartheta(\eta - y, \alpha_2, \beta_2)\}$ for $x, y \in \mathbf{R}$ provided that $0 \le \alpha_i,\ 0 < \beta_i (i = 1, 2)$ and that $\alpha_1 + \alpha_2 \le \rho$ and $\beta_1 + \beta_2 < \lambda$.
50. (Constancy Lemma) Let K be a compact set in $\mathbf{R}^N$ and let h be a real- valued upper semicontinuous function on K. Let φ be a C^2 function in $\mathbf{R}^d$ with $1 \le d < N$. Let G be a bounded domain in $\mathbf{R}^d$. For each $\xi \in G$ assume that there is a maximizer $(r_\xi, \rho_\xi) \in K$ of

$$H_\xi(r, \rho) = h(r, \rho) - \varphi(r - \xi)$$

over K such that $\nabla\varphi(r_\xi - \xi) = 0$. Then

$$h_\varphi(\xi) = \sup\{H_\xi(r,\rho) \ : \ (r,\rho) \in K\}$$

is constant on G.

51. We set $\vartheta(x,\lambda) = \vartheta(x,1,\lambda)$, where ϑ is defined in Problem 48. Let u and $-v$ be upper semicontinuous functions defined in $Q = (0,T) \times \Omega$, where Ω is a bounded open interval with values in $\mathbf{R} \cup \{-\infty\}$. Let S be a real-valued continuous function in $[0,T] \times [0,T]$. Assume that $(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \in Q \times Q$ is a point such that $u(t,x) - v(s,y) - S(t,s) - \vartheta(x - y - (\hat{x} - \hat{y}), \lambda) \leq u(\hat{t}, \hat{x}) - v(\hat{s}, \hat{y}) - S(\hat{t}, \hat{s})$

$$\text{for all } (t,x,s,y) \in \overline{Q} \times \overline{Q} \text{ for all } \lambda \leq \lambda_0,$$

where λ_0 is a positive number. Then $u^\alpha(t,x) - v_\alpha(s,y) \leq u^\alpha(\hat{t}, \hat{x}) - v_\alpha(\hat{s}, \hat{y}) + \vartheta(x - y - (\hat{x} - \hat{y}), \frac{1}{2}\lambda_0) + S(t,s) - S(\hat{t}, \hat{s})$ for all $(t,x), (s,y) \in [0,T] \times Q$ provided that $0 < \alpha \leq \alpha_1 = \min(\alpha_0, \frac{1}{4}\lambda_0)$. Here u^α denotes the sup-convolution of $u(\cdot, t)$ by $\phi(x,\alpha) = \vartheta(x,\alpha)$ and v_α denotes the inf-convolution of $v(\cdot, t)$ by $\phi(x,\alpha)$. Here $\alpha_0 > 0$ is a constant such that $u^\alpha(\hat{t}, \cdot)$, $v_\alpha(\hat{s}, \cdot)$ are faceted at $\hat{x}, \hat{y}$ respectively with slope zero and that $\hat{x}, \hat{y}$ respectively belongs to the interior region of the faceted regions for all $0 < \alpha < \alpha_0$. (Existence of such α_0 is guaranteed by Problem 47.)

G. Doubling variables and comparison principle

Let Ω be a bounded open set in $\mathbf{R}^d$ and $Q = (0,T) \times \Omega$ for $T > 0$. Let u and $-v$ upper semicontiunous in Q with values in $\mathbf{R} \cup \{-\infty\}$. For $z = (t,x)$ and $z' = (s,y) \in Q$ we set

$$w(z,z') = u^*(z) - v_*(z'), z, z' \in \overline{Q}.$$

Let M be the maximum (value) of w over $\overline{Q} \times \overline{Q}$. In other words

$$M = \max\{w(z,z') \ : \ z \in \overline{Q}, z' \in \overline{Q}\}.$$

We consider barrier functions

$$\Phi_\zeta(z,z',\varepsilon,\sigma,\gamma,\gamma') = B_\varepsilon(x - y - \zeta) + S(t,s;\sigma,\gamma,\gamma')$$

$$B_\varepsilon(x) = \frac{|x|^2}{\varepsilon}, S(t,s;\sigma,\gamma,\gamma') = |t-s|^2/\sigma + \gamma/(T-t) + \gamma'/(T-s)$$

for positive parameters $\varepsilon, \sigma, \gamma, \gamma'$ and $\zeta \in \mathbf{R}^d$. We set

$$\Phi_\zeta(z, z') = w(z, z') - \Phi_\zeta(z, z').$$

Let $(z_\zeta, z'_\zeta) = (t_\zeta, x_\zeta, s_\zeta, y_\zeta)$ be a maximizer of Φ_ζ over $\overline{Q} \times \overline{Q}$. Assume that

$$m_0 = \sup\{u(z) - v(z) : z \in Q\} > 0.$$

52. Prove that for each $m'_0 \in (0, m_0)$ there are $\gamma_0, \gamma'_0 > 0$ such that

$$\sup \Phi_\zeta > m'_0 \text{ for all } \varepsilon > 0,\ \sigma > 0,\ \gamma_0 > \gamma > 0,\ \gamma'_0 > \gamma' > 0$$

and $|\zeta| \le \kappa_0(\varepsilon) = \frac{1}{2}(\varepsilon(m_0 - m'_0))^{1/2}$.

53. Prove that

$$|t_\zeta - s_\zeta| \le (M\sigma)^{1/2},\ |x_\zeta - y_\zeta - \zeta| \le (M\varepsilon)^{1/2}$$

for all $\varepsilon > 0,\ \sigma > 0,\ \gamma_0 > \gamma > 0,\ \gamma'_0,\ \gamma' > 0$ and ζ with $|\zeta| \le \kappa_0(\varepsilon)$. Here $\gamma_0, \gamma'_0, \kappa_0$ are defined as in Problem 52.

54. (Boundary condition and maximizers). Assume that $u^* \le v_*$ on $\overline{\partial_p\Omega}$, where $\partial_p\Omega = (0, T) \times \partial\Omega \cup \{0\} \times \overline{\Omega}$. Prove that there are positive numbers ε_0, σ_0 such that (z_ζ, z'_ζ) is an (interior) point of $Q \times Q$ for all $0 < \varepsilon < \varepsilon_0,\ 0 < \sigma < \sigma_0,\ 0 < \gamma < \gamma_0$ and $0 < \gamma < \gamma'_0$ and $|\zeta| \le \kappa_0(\varepsilon)$. Here $\gamma_0, \gamma'_0, \kappa_0$ are defined as in Problem 52 with $m'_0 = m_0/2$.

55. (Comparison principle) Assume that $H = H(x, p)$ is a real-valued continuous function on $\Omega \times \mathbf{R}^d$, where Ω is a bounded domain in $\mathbf{R}^d$. Assume furthermore that there exists a constant C such that

$$|H(x, p) - H(y, p)| \le C(1 + |p|)|x - y|$$

for all $x, y \in \overline{\Omega},\ p \in \mathbf{R}^d$. Let u and v be, respectively, a subsolution and a supersolution of

$$u_t + H(x, \nabla u) = 0 \text{ in } Q.$$

Assume that $u^* \le v_*$ on $\partial_p Q$. Prove that $u^* \le v_*$ in Q.

H. Miscellaneous problems

56. (Level set solution and graph-like solution) Let u be an upper semicontinuous subsolution of

$$u_t - |\nabla u| \text{ div } (\nabla u/|\nabla u|) = 0 \text{ in}$$

in $\mathbf{R}^d \times (0,T)$. For $c \in \mathbf{R}$ let $u^{\#}$ denote the 'height' function of $\{u \geq c\}$i.e.,

$$u^{\#}(t,x') = \sup\{x_d \ : u(x_1,\cdots,x_d,t) \geq c,\ x' = (x_1,\cdots,x_{d-1})\}.$$

Assume that $u^{\#} < \infty$. Prove that $u^{\#}$ is a subsolution of

$$v_t - \sqrt{1+|\nabla' v|^2} \text{ div}' \left(\frac{\nabla' v}{\sqrt{1+|\nabla' v|^2}}\right) = 0 \text{ in } (0,T) \times \mathbf{R}^{d-1}.$$

Here we set $u^{\#}(t,x') = -\infty$ if there is no x_d such that $u(x',x_d,t) \geq c$. Here ∇', div' denote the gradient and the divergence with respect to x'.

57. Let u be a continuous solution of

$$u_t - |\nabla u| \text{ div } \left(\frac{\nabla u}{|\nabla u|}\right) = 0 \text{ in } \mathbf{R}^d \times (0,T).$$

Assume that $c-$level set $\{x \in \mathbf{R}^d | u(x,t) = c\}$ is written as the graph of a continuous function $v(t,x')$ in $U = (t_0,t_1) \times (-L,L)^{d-1}$ with values in $(-L,L)$. Prove that v is a viscosity solution of

$$v_t - \sqrt{1+|\nabla' v|^2} \text{ div}' \left(\frac{\nabla' v}{\sqrt{1+|\nabla' v|^2}}\right) = 0 \text{ in } U.$$

58. Assume that f is a real-valued C^1 function in $\mathbf{R}$. We set

$$u_E(x,t) = \begin{cases} b & x \leq ct, \\ a & x > ct, \end{cases} \quad u_N(x,t) = \begin{cases} a & x < ct, \\ b & x \geq ct, \end{cases}$$

where $a < b,\ a,b \in \mathbf{R}$ and

$$c = \frac{f(b)-f(a)}{b-a}.$$

Prove that u_E is a proper viscosity solution of

$$\text{(C)} \quad u_t + \frac{\partial}{\partial x}(f(u)) = 0$$

in $\mathbf{R} \times (0,\infty)$. Prove that both u_E and u_N is a viscosity solution of (C) in $\mathbf{R} \times (0,\infty)$. Prove that u_N is not a proper viscosity subsolution nor a proper viscosity supersolution.