Surface Evolution Equations - A level set approach

Exercises

July 2008

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A. Relaxed limit

Let $X$ be a metric space with metric $d$. Let $D$ be a subset of $X$. Let \{${u^\varepsilon}$\}$_{0<\varepsilon<1}$ be a one-parameter family of functions defined on $D$ with values in $\mathbb{R} := \mathbb{R} \cup \{ \pm \infty \}$. We define the upper relaxed limit

$$\left( \limsup_{\varepsilon \to 0} u^\varepsilon \right)(z) := \lim_{r \downarrow 0} \sup \{ u^\varepsilon(y) : y \in D, \; d(y, z) < r, \; 0 < \varepsilon < r \}, \; z \in \overline{D},$$

where $\overline{D}$ is the closure of $D$ in $X$. The lower relaxed limit is defined as $\liminf_{\varepsilon} u^\varepsilon = -\limsup_{\varepsilon} (-u^\varepsilon)$. We often write $u = \limsup_{\varepsilon} u^\varepsilon$ and $\underline{u} = \liminf_{\varepsilon} u^\varepsilon$. Prove the following statements.

1. Let $z_0$ be a point in $\overline{D}$. Assume that $\overline{u}(z_0) = (\limsup_{\varepsilon \to 0} u^\varepsilon)(z_0) < \infty$. Then there are sequences $\{\varepsilon_j\}_{j=1}^\infty \subset (0, 1)$ and $\{z_j\}_{j=1}^\infty \subset D$ such that $\varepsilon_j \to 0$ as $j \to \infty$, $\lim_{j \to \infty} u^{\varepsilon_j}(z_j) = \overline{u}(z_0)$ and $\lim_{j \to \infty} z_j = z_0$.

2. The function $\overline{u} = \limsup_{\varepsilon \to 0} u^\varepsilon$ is an upper semicontinuous function defined in $\overline{D}$, i.e., $\overline{u}(z) \geq \limsup_{y \to z} \overline{u}(y)$ for $z \in \overline{D}$. Similarly, the function $\underline{u} = \liminf_{\varepsilon \to 0} u^\varepsilon$ is a lower semicontinuous function in $\overline{D}$.

3. Assume that $\overline{D}$ is locally compact. If $\overline{u}(z) = \underline{u}(z) \in \mathbb{R}$ for all $z \in D$, then $u^\varepsilon$ converges to $\overline{u}$ locally uniformly in $D$ as $\varepsilon \to 0$.

4. Assume that $D$ is locally compact. Assume that $u^\varepsilon$ is a (real-valued) upper semicontinuous function in $D$ and that $\overline{u}$ attains a strict local maximum ($\neq \infty$) at $z_0 \in \overline{D}$. Then there are sequences $\{\varepsilon_j\}_{j=1}^\infty \subset (0, 1)$ and $\{z_j\}_{j=1}^\infty \subset D$ such that $z_j$ is a local maximizer of $u^{\varepsilon_j}$ and that $z_j \to z_0$ and $\varepsilon_j \to 0$ as $j \to \infty$. Moreover, $\lim_{j \to \infty} u^{\varepsilon_j}(z_j) = \overline{u}(z_0)$. 

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5. Assume that $X$ is compact and that $u^\varepsilon$ is upper semicontinuous in $D = X$. We set $K = \{z \in X : \overline{u}(z) \geq 0\}$. Let $d_\varepsilon$ be defined by

$$d_\varepsilon = \sup\{d(z, K) : u^\varepsilon(z) \geq 0, z \in X\}.$$  

Then $d_\varepsilon \to 0$ as $\varepsilon \to 0$.

6. Assume that $X = \mathbb{R}^n$ and $\overline{u} = u$ in $X$ and that $u^\varepsilon$ is continuous. Assume that $K = \{a \in X : \overline{u}(z) \geq 0\}$ is compact and that $K = \overline{H}$ with $H = \{z \in X : \overline{u}(z) > 0\}$. Then $K_\varepsilon = \{z \in X : \overline{u}(z) > 0\}$ converges to $K$ as $\varepsilon \to 0$ in the sense of Hausdorff distance topology provided that $K_\varepsilon$ is compact. (Here is a definition of the Hausdorff distance for two sets $A, B \subset X$: $d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$.)

7. Give an example that the conclusion of Problem 6 is false if one drops the assumption $K = \overline{H}$ even if one assumes that $\Gamma = \{x \in X : u^\varepsilon = 0\}$ has no interior.

8. Assume that $u_0$ is continuous on a compact set $K$ in $\mathbb{R}^d$. For $x_0 \in K$ and $\lambda > 0$ we set $V_{\lambda, x_0}(x) = \lambda + u_0(x_0) + C|x - x_0|^2$. Then for each $\lambda > 0$ there is a constant $C$ depending only on $\lambda$ (and $u_0$) such that

$$u_0(x) \leq V_{\lambda, x_0}(x) \text{ for all } x \in K.$$  

We shall write such $C$ by $C = C(\lambda)$. Then $u_0(x) = \inf\{V_{\lambda, x_0}(x) : \lambda > 0, \ C = C(\lambda), \ x_0 \in K\}$.

9. Assume that $u_0$ is continuous in $\overline{\Omega}$, where $\Omega$ is a bounded open set in $\mathbb{R}^d$. Let $V_{\lambda, x_0}$ be as in Problem 8 with $C = C(\lambda)$. Assume that $u^\varepsilon : \Omega \times (0, T) \to \mathbb{R} \cup \{-\infty\}$ satisfies

$$u^\varepsilon(x, t) \leq V_{\lambda, x_0}(x) + C(\lambda)t.$$  

Then $\overline{u}(x, 0) \leq u_0(x)$ for all $x \in \Omega$, where $\overline{u} = \limsup_{\varepsilon \to 0} u^\varepsilon$.

B. Viscosity Solutions

Prove the following statements.

10. Assume that $u^\varepsilon$ is a viscosity subsolution of the level set mean curvature flow equation $u_\varepsilon - |\nabla u| \text{ div} (\nabla u / |\nabla u|) = 0$ in $\mathbb{R}^d \times (0, T)$. Then $\overline{u} = \limsup_{\varepsilon \to 0} u^\varepsilon$ is a viscosity subsolution of the same equation in $\mathbb{R}^d \times (0, T)$ provided that $\overline{u}(z) < \infty$ for all $z \in \mathbb{R}^d \times (0, T)$. One may replace $\mathbb{R}^d$ by an open set $\Omega$ in $\mathbb{R}^d$.  

11. The stability result in Problem 10 is still valid for the Neumann boundary value problem in $\Omega \times (0,T)$.

12. Let $u : Q \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function, where $Q = \Omega \times (0,T)$ and $\Omega$ is an open set in $\mathbb{R}^d$. Then $u$ is a viscosity subsolution of a level set mean curvature flow equation in $Q$ if (and only if) $(\phi, \hat{z}) \in C^2(Q) \times Q$ satisfies

(i) $\phi_t - |\nabla \phi| \text{div} (\nabla \phi / |\nabla \phi|) \leq 0$ at $\hat{z} \in Q$ if $\nabla \phi(\hat{z}) \neq 0$ and

(ii) $\phi_t(\hat{z}) \leq 0$ if $\nabla \phi(\hat{z}) = 0$, $\nabla^2 \phi(\hat{z}) = O$

whenever $\max_Q (u - \phi) = (u - \phi)(\hat{z})$.

Hint: Assume that $u - \phi$ takes its strict maximum at $\hat{z} \in Q$ and $\nabla \phi(\hat{z}) = 0$. We consider $u - \phi_\varepsilon$ with $\phi_\varepsilon(x;y,t) = |x-y|^4 / 4 \varepsilon + \phi(y,t)$, $\varepsilon > 0$ and derive several inequalities for $\phi_\varepsilon$ at a maximizer of $u(x,t) - \phi_\varepsilon(x,y,t)$ in $Q$.

13. The function $u(x,t) = \min(0, t - |x|)$ is a viscosity solution of the Neumann problem

$$\begin{align*}
\partial_t u - |\nabla u| &= 0 \quad \text{in } \{|x| < 1\} \times (0,T) \\
\partial u / \partial \nu &= 0 \quad \text{on } \partial\{|x| < 1\} \times (0,T)
\end{align*}$$

(although the slope $\partial u / \partial \nu$ at $|x| = 1$ is not zero.) Here $\partial / \partial \nu$ denotes the exterior normal differential operator.

C. Structure of equations and examples of solutions

14. Write the mean curvature flow equation $V = H$ for $u = u(x_1,t)$, when $\Gamma_t \subset \mathbb{R}^N$ is a hypersurface of rotation of the form

$$\Gamma_t = \{(x_1, \cdots , x_N) \in \mathbb{R}^N | r = u(x_1,t), \ r = (\sum_{j=2}^{N} x_j^2)^{1/2}\}.$$ 

15. Write the mean curvature flow equation $V = H$ for $u = u(x_1, \cdots , x_{N-1}, t)$, when $\Gamma_t \subset \mathbb{R}^N$ is of the form

$$\Gamma_t = \{(x_1, \cdots , x_N) \in \mathbb{R}^N | x_N = u(x_1, \cdots , x_{N-1}, t)\}.$$
16. Assume that $F = F(p, X)$ (defined in $(\mathbb{R}^N \setminus \{0\}) \times \mathbb{S}^N$) is geometric. Assume that $X \mapsto F(p, X)$ is continuous for each $p \in \mathbb{R}^N \setminus \{0\}$. Assume that $F$ is (degenerate) elliptic. Prove that $F$ satisfies

$$F(p, X + y \otimes p + p \otimes y) = F(p, X) \quad \text{(SG)}$$

for all $y \in \mathbb{R}^N, X \in \mathbb{S}^N, p \in \mathbb{R}^N \setminus \{0\}$. (In other words $F$ is strongly geometric.)

17. Give an example that $F$ is geometric but not fulfills the condition (SG) of Problem 16.

18. Assume that $u$ is a viscosity solution of

$$u_t + F(\nabla u, \nabla^2 u) = 0 \text{ in } Q = \Omega \times (0, T).$$

Assume that $F$ is geometric and continuous in $(\mathbb{R}^N \setminus \{0\}) \times \mathbb{S}^N$. Assume that $F$ can be extended continuously at $(0, O)$. Prove that $\theta \circ u$ is a viscosity subsolution of the above equation in $Q$ provided that $\theta$ is continuous and nondecreasing. Here $\Omega$ is an open set in $\mathbb{R}^N$.

19. Assume that $\gamma : \mathbb{R}^N \to [0, \infty)$ is positively homogeneous of degree one. Assume that $\gamma \in C^2(\mathbb{R}^N \setminus \{0\})$. Prove that

$$\nabla^2 \gamma(p) + p \otimes p > O \text{ for all } p \in \mathbb{R}^N \setminus \{0\},$$

if and only if $\nabla^2 (\gamma^2)(p) > 0$ for all $p \in \mathbb{R}^2 \setminus \{0\}$.

20. Under the assumption of Problem 19 prove that Frank $\gamma = \{p \in \mathbb{R}^N : \gamma(p) \leq 1\}$

is strictly convex (in the sense that all inward principal curvatures of $\partial$ (Frank $\gamma$) are positive) if and only if $\nabla^2 \gamma(p) + p \otimes p > O$ for all $p \in \mathbb{R}^N \setminus \{0\}$.

21. Assume that same hypotheses of Problem 19 concerning $\gamma$. Assume that Frank $\gamma$ is strictly convex. Prove that the Wulff shape

$$W_\gamma = \{p \in \mathbb{R}^N | p \cdot m \leq \gamma(m) \text{ for all } m \in \mathbb{S}^{N-1}\}.$$
22. Assume the same hypotheses of Problem 21 concerning $\gamma$. Prove that there exists a shrinking self-similar solution of the form $a(t)\partial W_\gamma$ of

$$V = -\gamma(n) \text{div}_1(\nabla \gamma(n)).$$

23. For $u_0, v_0 \in C(\mathbb{R}^N)$ assume that

$$\{u_0 > 0\} \subset \{v_0 > 0\} (= \{x \in \mathbb{R}^N \mid v_0(x) > 0\}).$$

Assume that $\overline{\{v_0 > 0\}}$ is compact. Prove that there exists a non-decreasing function $\theta \in C(\mathbb{R})$ such that $\theta(s) = 0$ (for $s \leq 0$) and $\theta(s) > 0$ (for $s > 0$) and

$$u_0 \leq \theta \circ v_0 \text{ in } \mathbb{R}^N.$$

D. Dynamical programming principle

Let $K$ be a compact set in $\mathbb{R}^d$. Assume that $f: \mathbb{R}^N \times K \to \mathbb{R}^N$ is continuous and that there is a constant $L$ satisfying

$$|f(x,a) - f(y,a)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}^N$, $a \in K$. Let $T$ be a positive number (called terminal time). Let $A$ be of the form

$$A = \{\alpha : [0, T] \to K \mid \alpha \text{ is Lebesgue measurable}\}.$$

(An element of this set is called a control.) Let $X^\alpha_{x,t}(s)$ be the solution of the state equation

$$\begin{cases}
\frac{dX}{ds} = f(X(s), \alpha(s)), & T > s > t \\
X(t) = x \in \mathbb{R}^N, & s = t.
\end{cases}$$

Let $g$ be a real-valued continuous function defined in $\mathbb{R}^N$. Let $u$ be the value function (with the terminal data $g$) of the form

$$u(x,t) = \inf_{\alpha \in A} g(X^\alpha_{x,t}(T)).$$

24. Prove that the dynamical programming principle

$$u(x,t) = \inf_{\alpha \in A} \{u(X^\alpha_{x,t}(t + \delta), t + \delta) \mid t + \delta \leq T, \delta > 0\}.$$
25. Prove that \( v(x,t) = u(x,T-t) \) is a viscosity solution of

\[
v_t - H(x, \nabla v) = 0 \text{ in } \mathbb{R}^N \times (0,T)
\]

with \( H(x,p) = \min_{a \in K} p \cdot f(x,a) \).

26*. Let \( \Omega \) be a bounded \( C^2 \) convex domain in \( \mathbb{R}^2 \). Let \( S^t(x,v) \), \( x \in \Omega \), \( v \in S^1 \) be the billiard semiflow in \( \Omega \). Prove that for any fixed \( t \geq 0 \), \( x \in \Omega \) and \( v \in S^1 \), there exists \( d_l \geq 0 \), \( y_l \in \partial \Omega \cap B_t(x) \) where \( l = 1, 2, \cdots \) such that \( \sum_{l=0}^{\infty} d_l \nu(y_l) \) converges and

\[
\alpha^t(x,v) = \sum_{l=0}^{\infty} d_l \nu(y_l)
\]

where \( \alpha^t(x,v) = S^t(x,v) - (x + tv) \) is a boundary adjustor. Here \( \nu \) denotes the unit outward normal of \( \partial \Omega \).

27*. Assume that \( \{u^\varepsilon\}_{0 < \varepsilon < 1} \) is uniformly bounded in \( \Omega \times (0,T) \). Assume that \( u^\varepsilon \) fulfills

\[
u_t - |\nabla \nu| \text{div}(\nabla v / |\nabla v|) = 0 \text{ in } \Omega \times (0,T),
\]

\[
\partial \nu / \partial \nu = 0 \text{ on } \partial \Omega \times (0,T).
\]

E. Variational problem with obstacles

Let \( Z \) be a real-valued \( C^2 \) (or \( C^{1,1} \)) function defined in a bounded interval \( I \), where \( I = (a,b) \). For a given \( \Delta > 0 \) let \( K_{\pm} \) be the subset of \( H^1(I) \) of the form

\[
K_{\pm} = \{ \xi \in H^1(I) : Z(x) - \Delta/2 \leq \xi(x) \leq Z(x) + \Delta/2, \xi(a) = Z(a) - \Delta/2, \xi(b) = Z(b) + \Delta/2 \}.
\]

Let \( J_{\pm} \) be the functional in \( L^2(I) \) defined by

\[
J_{\pm}(\xi) = \begin{cases} \int_a^b |\xi(x)|^2 dx, & \xi \in K_{\pm} \\ \infty, & \text{otherwise} \end{cases}
\]
28. Prove that $H^1(I) \subset C^{1/2}(\overline{I}) \subset C(\overline{I})$.

29. Prove that $J_\pm$ is lower semicontinuous, convex on $L^2(I)$.

30. Prove that $J_\pm$ admits a unique (absolute) minimizer.

31. Let $\xi_+$ be the minimizer of $J_+$. Let $D_\pm$ be the coincidence set defined by

$$D_\pm = \{ x \in \overline{I} : \xi_+ = Z(x) \pm \Delta/2 \}.$$

Prove that $\xi_+$ is concave in a neighborhood of $D_-$ and that $\xi_+$ is convex in a neighborhood of $D_+$. Prove that $\xi'_\pm = 0$ outside $D_+ \cup D_-$. (We say that $\xi$ satisfies the concave-convex condition if these three properties are fulfilled.)

32. If $\xi$ satisfies the concave-convex condition and $\xi(a) = Z(a) - \Delta/2$, $\xi(b) = Z(b) + \Delta/2$, it must be the minimizer of $J_\pm$.

33. Let $\xi_+$ be the minimizer of $J_+$. Prove that $\xi_+$ is $C^{1,1}$ and

$$\sup_{x \in I} |\xi''_+(x)| \leq \sup_{x \in I} |Z''(x)|.$$

34. Suppose that the concave hull $Z_{\text{cave}}$ of $Z$ in $I$ is smaller than $Z + \Delta/2$ i.e. $Z_{\text{cave}} \leq Z + \Delta/2$ in $I$. Let $\xi_-$ be the minimizer of $J_-$. Prove that

$$\xi'_-(x) = Z'_{\text{cave}}(x), \ x \in I.$$

35. Suppose that straight line function $\xi(x) = \xi(a) + \frac{Z(b) - Z(a) + \Delta}{b-a}(x-a)$ is in $K_+$. Prove that $\xi$ is the minimizer of $J_+$.

36. (Comparison principle)
Let $\xi_\pm$ be the minimizer of $J_\pm$. It is determined by $I$. Let

$$\Lambda_\pm(x, I) = \xi'_\pm(x).$$

Prove that

$$\Lambda_\pm(x, I_1) \leq \Lambda_\pm(x, I_2) \text{ for } x \in I_2$$

if $I_2 \subset I_1$.

37. Let $J_{k\pm}$ be the functional defined as $J$ by replacing $Z$ by $Z^k(k = 1, 2, \cdots)$, where $Z^k$ is a real-valued $C^2$ function defined in $\overline{I}$. Assume that $Z^k$ converges to $Z$ uniformly with its first derivatie in $\overline{I}$. Prove that for any $\xi_k \rightarrow \xi$ in $L^2(I)$

$$J_\pm(\xi) \leq \liminf_{k \rightarrow \infty} J_{k\pm}(\xi_k).$$
38. Assume that the same hypotheses of Problem 37 concerning $Z^k$.
Prove that for each $\xi \in L^2(I)$ there is a sequence $\xi_k \rightarrow \xi$ in $L^2(I)$
such that

$$J_\pm(\xi) = \lim_{k \rightarrow \infty} J^k_\pm(\xi_k)$$

39. (Convergence of minimizers under (realxed limit) $\Gamma-$ convergence)
Assume that same hypotheses of Problem 37 concerning $Z^k$. Let
$\xi^k_\pm$ be the minimizer of $J^k_\pm$ and $\xi_\pm$ be the minimizer of $J_\pm$. Then
$\xi^k_\pm \rightarrow \xi_\pm$ in $L^2(I)$.

40. Let $D$ be a compact metric space. Assume that $u^\varepsilon$ be a real-valued
lower semicontinuous function on $D$. Let $z^\varepsilon$ be an (absolute) minimizer of $u^\varepsilon$. Prove that there is a subsequence $z^\varepsilon_j(\varepsilon_j \rightarrow 0)$ such
that it converges to an (absolute) minimizer $z$ of $u$ in $D$.

41. (Stability)
Assume that

$$\sup_{k \geq 1} \sup_{x \in I} |(d/dx)^2 Z^k(x)| < \infty \text{ and } Z^k \rightarrow Z \text{ in } C^1(\overline{\mathcal{T}}).$$

Let

$$\Lambda^k_\pm(x, I) = (d/dx)\xi^k_\pm(x),$$

where $\xi_\pm$ be the minimizer of $J^k_\pm$. Prove that $\Lambda^k_\pm$ converges to $\Lambda_\pm(x, I)$ uniformly in $\mathcal{T}$ as $k \rightarrow \infty$. (Use Problem 33.)

42. Let $Z$ be a $C^2$ function in $\mathbb{R}$. Prove that $\Lambda_\pm(x, I)$ is continuous
with respect to $I$. (Clarify the meaning of continuity.) Assume furthermore that $|(d/dx)^2 Z|$ is bounded in $\mathbb{R}$. Prove that for each $r > 0$

$$\lim_{\mu \rightarrow 0} \sup_{0 < b - a < r} \sup_{a < x < b} |\Lambda_\pm(x, (a, b)) - \Lambda_\pm(x - \mu, (a - \mu, b - \mu))| = 0.$$  

F. Sup-convolution (regularization)

Let $\phi$ be a function from $\mathbb{R} \times (0, 1]$ to $[0, \infty)$. Assume that $\phi$ fulfills
following conditions.

(i) For each $\lambda, 0 < \lambda \leq 1$, $\phi(\cdot, \lambda)$ is Lipschitz continuous on every
bounded set in $\mathbb{R}$.

(ii) $\phi(\xi, \lambda)$ is even in $\xi$, i.e. $\phi(\xi, \lambda) = \phi(-\xi, \lambda)$
(iii) $\phi(\xi, \lambda)$ is nonincreasing in $\lambda$ for all $\xi$.

(iv) $\lim_{\xi \to -\infty} \phi(\xi, 1) = \infty$ and $\phi(\xi, \lambda)$ is nondecreasing in $\xi \geq 0$, for $0 < \lambda \leq 1$.

(v) $\lim_{\lambda \downarrow 0} \phi(\xi, \lambda) = \infty$ unless $\xi = 0$ and $\phi(0, \lambda) = 0$, $0 < \lambda \leq 1$.

Let $f$ be a function in $\mathbb{R}$ with values in $\mathbb{R} \cup \{-\infty\}$. We say that

$$f^\lambda(x) = \sup_{\xi \in \mathbb{R}} \{f(\xi) - \phi(\xi - x, \lambda)\}$$

is a sup-convolution of $f$ by $\phi$. Prove the following statements under assumptions (i)-(v) for $\phi$.

43. Let $f(\neq -\infty)$ be a function on $\mathbb{R}$ with values in $\mathbb{R} \cup \{-\infty\}$. Assume that $f$ is locally bounded from above and that

$$\lim_{|\xi| \to \infty} \max(f(\xi), 0)/\phi(\xi - x, 1) = 0$$

for each $x \in \mathbb{R}$.

Then $f^\lambda$ is locally Lipschitz. Moreover,

$$f^\lambda \geq f^\mu \geq f$$

for $\lambda \geq \mu > 0$

and $\lim_{\lambda \downarrow 0} f^\lambda(x) = f^*(x)$ for each $x \in \mathbb{R}$. Here $f^*$ denotes the upper semicontinuous envelope of $f$, i.e.

$$f^*(x) = \lim_{\varepsilon \downarrow 0} \sup_{|x - y| < \varepsilon} \{f(y)\}.$$

44. Assume that same hypotheses of Problem 43. Let $B$ and $B'$ be bounded open sets in $\mathbb{R}$ with $\overline{B} \subset B'$. Then for each $K_0 > 0$ there is $\lambda_0(K_0) > 0$ such that

$$\sup_{x \in \overline{B}} \sup_{\xi \notin B'} H(\xi, x, \lambda) < -K_0$$

for $\lambda < \lambda_0(K_0)$

with $H(\xi, x, \lambda) = f(\xi) - \phi(\xi - x, \lambda)$. Moreover,

$$f^\lambda(x) = \sup_{\xi \in B'} H(\xi, x, \lambda)$$

for $x \in \overline{B}$

provided that $\inf_{\overline{B}} f^* > -\infty$ and $\lambda < \lambda_0 \equiv \lambda_0(\max(0, -\inf_{\overline{B}} f^*))$. 

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45. Assume that same hypotheses of Problem 43. If \( \hat{x} \) be a maximizer of \( f \) over \( B' \), then \( f^\lambda(x) \leq f(\hat{x}) \) for \( x \in \overline{B} \) provided that
\[ \lambda < \lambda'' \equiv \lambda_0(\max(0, -f(\hat{x}))) \].

46. Assume that \( f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}, f \not\equiv -\infty \) is locally bounded from above. If \( \phi(x, \lambda) = |x|^2/\lambda \), then \( f^\lambda \) is semi-convex in \( \mathbb{R} \). In fact, \( f^\lambda(x) + |x|^2/\lambda \) is convex.

47. Assume that for \( 0 < \lambda \leq 1 \)
\[ \sigma_{\lambda} := \sup\{|\xi| : \phi(\xi, \lambda) = 0\} > 0. \]
Assume that same hypotheses of Problem 43. Assume that \( f \) has a local maximum at \( \hat{x} \in \mathbb{R} \) and that \( f \) is not a constant function. Then there is a small \( \lambda_1, 0 < \lambda_1 \leq 1 \) such that for \( \lambda \leq \lambda_1 \)
(i) \( f^\lambda \) is faceted at \( \hat{x} \) in \( \mathbb{R} \) with slope zero and \( f^\lambda(\hat{x}) = f(\hat{x}) \).
(ii) \( \hat{x} \) is an interior point of the faceted region.

48. Let
\[ \vartheta(x, \rho, \lambda) = \begin{cases} 
(x - \rho)^2/\lambda, & x > \rho \\
0, & |x| \leq \rho \\
(x + \rho)^2/\lambda, & x < -\rho
\end{cases} \]
Then for each \( \rho > 0 \), \( \phi(x, \lambda) = \vartheta(x, \rho, \lambda) \) satisfies all assumptions (i)-(v) in the beginning of Section F and that \( \sigma_{\lambda} > 0 \), where \( \sigma_{\lambda} \) is defined in Problem 47.

49. Let \( \vartheta \) be as in Problem 48. Then
(a) \( \vartheta(x, \rho, \lambda, -\beta) = \sup_{\xi \in \mathbb{R}} \{\vartheta(\xi, \rho, \lambda) - \vartheta(\xi - x, \alpha, \beta)\} \) for \( x \in \mathbb{R} \)
providing that \( 0 \leq \alpha \leq \rho, 0 < \beta < \lambda \).
(b) \( \vartheta(x - y, \rho - (\alpha_1 + \alpha_2), \lambda - (\beta_1 + \beta_2)) = \sup_{\xi} \sup_{\eta} \{\vartheta(\xi - \eta, \rho, \lambda) - \vartheta(\xi - x, \alpha_1, \beta_1) - \vartheta(\eta - y, \alpha_2, \beta_2)\} \) for \( x, y \in \mathbb{R} \)
providing that \( 0 \leq \alpha_i, 0 < \beta_i (i = 1, 2) \) and that \( \alpha_1 + \alpha_2 \leq \rho \)
and \( \beta_1 + \beta_2 < \lambda \).

50. (Constancy Lemma) Let \( K \) be a compact set in \( \mathbb{R}^N \) and let \( h \) be a real-valued upper semicontinuous function on \( K \). Let \( \varphi \) be a \( C^2 \) function in \( \mathbb{R}^d \) with \( 1 \leq d < N \). Let \( G \) be a bounded domain in \( \mathbb{R}^d \). For each \( \xi \in G \) assume that there is a maximizer \( (r_{\xi}, \rho_{\xi}) \in K \) of
\[ H_{\xi}(r, \rho) = h(r, \rho) - \varphi(r - \xi) \]
over $K$ such that $\nabla \varphi(r_\xi - \xi) = 0$. Then

$$h_\varphi (\xi) = \sup \{ H_\xi (r, \rho) : (r, \rho) \in K \}$$

is constant on $G$.

51. We set $\vartheta(x, \lambda) = \vartheta(x, 1, \lambda)$, where $\vartheta$ is defined in Problem 48. Let $u$ and $-v$ be upper semicontinuous functions defined in $Q = (0, T) \times \Omega$, where $\Omega$ is a bounded open interval with values in $\mathbb{R} \cup \{-\infty\}$. Let $S$ be a real-valued continuous function in $[0, T] \times [0, T]$. Assume that $(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \in Q \times Q$ is a point such that $u(t, x) - v(s, y) - S(t, s) - \vartheta(x - y - (\hat{x} - \hat{y}), \lambda) \leq u(\hat{t}, \hat{x}) - v(\hat{s}, \hat{y}) - S(\hat{t}, \hat{s})$

for all $(t, x, s, y) \in \overline{Q} \times \overline{Q}$ for all $\lambda \leq \lambda_0$, where $\lambda_0$ is a positive number. Then $u^a(t, x) - v_a(s, y) \leq u^a(\hat{t}, \hat{x}) - v_a(\hat{s}, \hat{y}) + \vartheta(x - y - (\hat{x} - \hat{y}), \frac{\lambda_0}{2}) + S(t, s) - S(\hat{t}, \hat{s})$ for all $(t, x, s, y) \in [0, T] \times Q$ provided that $0 < \alpha \leq \alpha_1 = \min(\alpha_0, \frac{1}{4} \lambda_0)$. Here $u^a$ denotes the sup-convolution of $u(\cdot, t)$ by $\phi(x, \alpha) = \vartheta(x, \alpha)$ and $v_a$ denotes the inf-convolution of $v(\cdot, t)$ by $\phi(x, \alpha)$. Here $\alpha_0 > 0$ is a constant such that $u^a(\cdot, \cdot)$, $v_a(\cdot, \cdot)$ are faceted at $\hat{x}, \hat{y}$ respectively with slope zero and that $\hat{x}, \hat{y}$ respectively belongs to the interior region of the faceted regions for all $0 < \alpha < \alpha_0$. (Existence of such $\alpha_0$ is guaranteed by Problem 47.)

G. Doubling variables and comparison principle

Let $\Omega$ be a bounded open set in $\mathbb{R}^d$ and $Q = (0, T) \times \Omega$ for $T > 0$. Let $u$ and $-v$ upper semicontinuous in $Q$ with values in $\mathbb{R} \cup \{-\infty\}$. For $z = (t, x)$ and $z' = (s, y) \in Q$ we set

$$w(z, z') = u^*(z) - v_*(z'), z, z' \in \overline{Q}.$$

Let $M$ be the maximum (value) of $w$ over $\overline{Q} \times \overline{Q}$. In other words

$$M = \max \{ w(z, z') : z \in \overline{Q}, z' \in \overline{Q} \}.$$

We consider barrier functions

$$\Phi_\xi (z, z', \varepsilon, \sigma, \gamma, \gamma') = B_\varepsilon (x - y - \xi) + S(t, s; \sigma, \gamma, \gamma')$$

$$B_\varepsilon (x) = \frac{|x|^2}{\varepsilon}, S(t, s; \sigma, \gamma, \gamma') = |t - s|^2 / \sigma + \gamma / (T - t) + \gamma' / (T - s)$$
for positive parameters $\varepsilon, \sigma, \gamma, \gamma'$ and $\zeta \in \mathbb{R}^d$. We set

$$\Phi_\zeta(z, z') = w(z, z') - \Phi_\zeta(z, z').$$

Let $(z_\zeta, z'_\zeta) = (t_\zeta, x_\zeta, s_\zeta, y_\zeta)$ be a maximizer of $\Phi_\zeta$ over $Q \times \overline{Q}$. Assume that

$$m_0 = \sup\{u(z) - v(z) : z \in Q\} > 0.$$  

52. Prove that for each $m'_0 \in (0, m_0)$ there are $\gamma_0, \gamma'_0 > 0$ such that

$$\sup \Phi_\zeta > m'_0$$

for all $\varepsilon > 0$, $\sigma > 0$, $\gamma_0 > \gamma > 0$, $\gamma'_0 > \gamma' > 0$ and $|\zeta| \leq \kappa_0(\varepsilon) = \frac{1}{2}(\varepsilon(m_0 - m'_0))^{1/2}$.

53. Prove that

$$|t_\zeta - s_\zeta| \leq (M\sigma)^{1/2}, |x_\zeta - y_\zeta - \zeta| \leq (M\varepsilon)^{1/2}$$

for all $\varepsilon > 0$, $\sigma > 0$, $\gamma_0 > \gamma > 0$, $\gamma'_0 > \gamma' > 0$ and $\zeta$ with $|\zeta| \leq \kappa_0(\varepsilon)$. Here $\gamma_0, \gamma'_0, \kappa_0$ are defined as in Problem 52.

54. (Boundary condition and maximizers). Assume that $u^* \leq v_*$ on $\partial_p \Omega$, where $\partial_p \Omega = (0, T) \times \partial \Omega \cup \{0\} \times \overline{\Omega}$. Prove that there are positive numbers $\varepsilon_0, \sigma_0$ such that $(z_\zeta, z'_\zeta)$ is an (interior) point of $Q \times Q$ for all $0 < \varepsilon < \varepsilon_0$, $0 < \sigma < \sigma_0$, $0 < \gamma < \gamma_0$ and $0 < \gamma < \gamma'_0$ and $|\zeta| \leq \kappa_0(\varepsilon)$. Here $\gamma_0, \gamma'_0, \kappa_0$ are defined as in Problem 52 with $m'_0 = m_0/2$.

55. (Comparison principle) Assume that $H = H(x, p)$ is a real-valued continuous function on $\Omega \times \mathbb{R}^d$, where $\Omega$ is a bounded domain in $\mathbb{R}^d$. Assume furthermore that there exists a constant $C$ such that

$$|H(x, p) - H(y, p)| \leq C(1 + |p|)|x - y|$$

for all $x, y \in \overline{\Omega}$, $p \in \mathbb{R}^d$. Let $u$ and $v$ be, respectively, a subsolution and a supersolution of

$$u_t + H(x, \nabla u) = 0 \text{ in } Q.$$  

Assume that $u^* \leq v_*$ on $\partial_p Q$. Prove that $u^* \leq v_*$ in $Q$.

H. Miscellaneous problems
56. (Level set solution and graph-like solution) Let $u$ be an upper semicontinuous subsolution of
\[ u_t - |\nabla u| \div (\nabla u/|\nabla u|) = 0 \]
in $\mathbb{R}^d \times (0, T)$. For $c \in \mathbb{R}$ let $u^#$ denote the ‘height’ function of $\{u \geq c\}$ i.e.,
\[ u^#(t, x') = \sup \{x_d : u(x_1, \cdots, x_d, t) \geq c, \, x' = (x_1, \cdots, x_{d-1})\}. \]
Assume that $u^# < \infty$. Prove that $u^#$ is a subsolution of
\[ v_t - \sqrt{1 + |\nabla' v|^2} \div' (\frac{\nabla' v}{\sqrt{1 + |\nabla' v|^2}}) = 0 \]
in $(0, T) \times \mathbb{R}^{d-1}$.
Here we set $u^#(t, x') = -\infty$ if there is no $x_d$ such that $u(x', x_d, t) \geq c$. Here $\nabla', \div'$ denote the gradient and the divergence with respect to $x'$.

57. Let $u$ be a continuous solution of
\[ u_t - |\nabla u| \div (\frac{\nabla u}{|\nabla u|}) = 0 \]
in $\mathbb{R}^d \times (0, T)$.
Assume that $c$–level set $\{x \in \mathbb{R}^d | u(x, t) = c\}$ is written as the graph of a continuous function $v(t, x')$ in $U = (t_0, t_1) \times (-L, L)^{d-1}$ with values in $(-L, L)$. Prove that $v$ is a viscosity solution of
\[ v_t - \sqrt{1 + |\nabla' v|^2} \div' (\frac{\nabla' v}{\sqrt{1 + |\nabla' v|^2}}) = 0 \]
in $U$.

58. Assume that $f$ is a real-valued $C^1$ function in $\mathbb{R}$. We set
\[ u_E(x, t) = \begin{cases} b & x \leq ct, \\ a & x > ct, \end{cases} \quad u_N(x, t) = \begin{cases} a & x < ct, \\ b & x \geq ct, \end{cases} \]
where $a < b$, $a, b \in \mathbb{R}$ and
\[ c = \frac{f(b) - f(a)}{b - a}. \]
Prove that $u_E$ is a proper viscosity solution of
\[ (C) \quad u_t + \frac{\partial}{\partial x}(f(u)) = 0 \]
in $\mathbb{R} \times (0, \infty)$. Prove that both $u_E$ and $u_N$ is a viscosity solution of $(C)$ in $\mathbb{R} \times (0, \infty)$. Prove that $u_N$ is not a proper viscosity subsolution nor a proper viscosity supersolution.