Applications of the Bloch-Okounkov theorem

Jan-Willem van Ittersum, Utrecht University / MPIM

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A counting problem in the symmetric group

Problem: Given $b \geq 0$, $d \geq 1$, the number $h_{b,d}$ counts $1/d!$ times the number of

- $\alpha, \beta \in \mathfrak{S}_d$ (symmetric group)
- transpositions $\tau_1, \dots, \tau_b \in \mathfrak{S}_d$

such that
- $\tau_1 \tau_2 \cdots \tau_b = \alpha \beta \alpha^{-1} \beta^{-1}$,
- $\langle \tau_1, \dots, \tau_b, \alpha, \beta \rangle$ acts transitively on $\{1, \dots, d\}$.

Ex ($b=2$)

d	1	2	3	4	5	6	7	8
#	0	2	16	60	160	360	672	1240
$\sigma_1(d)$	1	3	4	7	6	12	8	15
$\sigma_3(d)$	1	9	28	73	126	252	344	585

$\sigma_k(d) = \sum_{d \mid n} d^k$

$$h_{2,d} = \frac{d}{3}\left(\sigma_3(d) - d\sigma(d)\right)$$

Enumerative theories curve → elliptic curve

Idea Let E be an elliptic curve$/\mathbb{C}$. Count mappings

$$X \xrightarrow{f} E.$$

Hurwitz theory $\xleftrightarrow{\text{[Okounkov-Pandharipande]}}$ **Gromov-Witten invariants**

Hurwitz theory:

- X Riemann surface (genus $\frac{b-1}{2}$)
- f ramified covering
- Counting weighted by $1/|\mathrm{Aut}\, f|$
- Computed using monodromy representation

Gromov-Witten invariants:

- X projective connected pointed curve
- f stable map
- Counting by integrating in moduli space of such mappings
- Computed using degeneration formula

Both cases: enumerative problem reduces to counting problem in symmetric group

Quasimodular forms and elliptic curves

Thm (Dijkgraaf, Kaneko-Zagier, Bloch-Okounkov, Eskin-Okounkov, Ochiai)
The generating series in both enum. theories are quasimodular forms

Ex $\sum_d h_{2,d}\, q^d = \frac{1}{3}\left(E_4' - E_2''\right)$ $\quad E_k = -\frac{B_k}{2k} + \sum_d \sigma_{k-1}(d)\, q^d$

Proof sketch ① Express $h_{b,d} = \sum_{\lambda \vdash d} Q_3(\lambda)^b$, where

- sum is over all partitions of size d (cycle types in $\mathfrak{S}_d$)
- $$Q_3(\lambda) = \frac{1}{2}\sum_{i=1}^{\infty}\left(\left(\lambda_i - i - \tfrac{1}{2}\right)^2 - \left(-i+\tfrac{1}{2}\right)^2\right).$$

② For all $b \geq 0$, $\dfrac{\sum_\lambda Q_3(\lambda)^b\, q^{|\lambda|}}{\sum_\lambda q^{|\lambda|}}$ is quasimodular (i.e. in $\mathbb{Q}[E_2, E_4, E_6]$) $\quad |\lambda| = \sum_i \lambda_i$

Upshot

- Computing a Gromov-Witten invariant / Hurwitz number for finitely many degrees d determines it uniquely.

- Growth rate of these invariants can easily be computed

Ex
$$\lim_{q\to 1} (1-q)^5 \tfrac{1}{3}(E_4{}' - E_2{}'') = \lim_{q\to 1} \sum_n n \frac{(1-q)^5}{(1-q^n)^5} \frac{q^n(1+11q^n+11q^{2n}+q^{3n})}{3}$$
$$= 8\zeta(4)$$

Hence,
$$\sum_{d=1}^{N} h_{2,d} = 8\zeta(4)\frac{N^5}{5!} + O(N^4)$$

A relation among multiple zeta values (with Henrik Bachmann)

For $k_1 \geq 2$, $k_2, \ldots, k_n \geq 1$, define the multiple zeta value by

$$\zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n \geq 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \in \mathbb{R}$$

$$\sum_i \beta_i z^{i-1} = \frac{1}{2\sinh(\frac{z}{2})}$$

Generalize $Q_3 : \mathcal{P} \to \mathbb{Q}$ by

$$Q_k(\lambda) = \beta_k + \frac{1}{(k-1)!} \sum_i \left((\lambda_i - i + \tfrac{1}{2})^{k-1} - (-i + \tfrac{1}{2})^{k-1} \right)$$

Computing $\lim\limits_{q \to 1} (1-q)^k \dfrac{\sum_\lambda Q_k(\lambda) q^{|\lambda|}}{\sum_\lambda q^{|\lambda|}}$ in two ways yields:

Cor.

$$\sum_{k \geq 2} \sum_{i=0}^{k-2} (-1)^i \zeta(k-i, \underbrace{1, \ldots, 1}_{i}) z^k = \frac{1}{z^2} \left(1 - \exp\left(\sum_{n \geq 2} \zeta(n) \frac{z^n + (-z)^n}{n} \right) \right).$$

Ohno-Zagier relation

Ex $\zeta(3) - \zeta(2,1) = 0$

The q-bracket

For $f : \mathcal{P} \to \overline{\mathbb{Q}}$, we let

partitions of integers

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} \in \overline{\mathbb{Q}}[[q]]$$

$$\prod_{n \geq 1} (1-q^n) = q^{-1/24}\, \eta$$

Example $f(\lambda) = |\lambda| - \frac{1}{24}$.

$$\langle f \rangle_q = -\frac{1}{24} + q \frac{\partial}{\partial q} \log \Big(\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \Big) = -\frac{1}{24} + q \frac{\partial}{\partial q} \log \eta^{-1} q^{1/24} = E_2$$

Interesting functions on partitions

$$\sum_i \beta_i z^{i-1} = \frac{1}{2\sinh(\frac{z}{2})}$$

(0) <u>shifted symmetric</u> $Q_k(\lambda) = \beta_k + \frac{1}{(k-1)!} \sum_i \left((\lambda_i - i + \tfrac{1}{2})^{k-1} - (-i + \tfrac{1}{2})^{k-1} \right)$

(1) <u>p-adic analogues</u> [Griffin-Jameson-Trebat-Leder, '16]

[Eskin-Okounkov, '06] [Engel, '17] : Orbifold Hurwitz theory

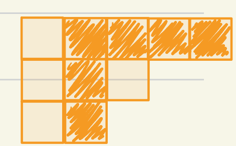
$$Q_k^{(p)}(\lambda) := \beta_k \left(1 - \frac{1}{p}\right) + \frac{1}{(k-1)!} \sum_{(2\lambda_i - 2i + 1, p) = 1} \left((\lambda_i - i + \tfrac{1}{2})^{k-1} - (-i + \tfrac{1}{2})^{k-1} \right)$$

(2) <u>hook-length moments / t-hook functions</u> [Bringmann-Ono-Wagner, '20]

[Chen-Möller-Zagier, '16] : Dynamics of flat surfaces

$$H_k^t(\lambda) := -\frac{B_k}{2k} t^k + \sum_{\substack{\xi \in Y_\lambda \\ t \mid h(\xi)}} h(\xi)^{k-2}$$

$h(\xi)$ hook-length of cell ξ



The generating series of shifted symmetric functions

Let $\lambda \in P$, $\underline{e}(x) = e^{2\pi i x}$, $\tau \in \mathfrak{h}$, $q = \underline{e}(\tau)$ and $z, z_1, \ldots, z_n \in \mathbb{C}$.

$$W_\lambda(z) := \sum_{i=1}^{\infty} \underline{e}\left(\left(\lambda_i - i + \tfrac{1}{2}\right) z\right) \qquad (\operatorname{Im} z < 0)$$

$[z^2] W_\lambda(z) = Q_3(\lambda)$

n-point function

$$\Theta(z) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} (-1)^{\lfloor \nu \rfloor} \underline{e}(\nu z)\, q^{\nu^2/2}$$

Writing $F_n(z_1, \ldots, z_n) := \langle W(z_1) \cdots W(z_n) \rangle_q =: \sum_{\sigma \in \mathfrak{S}_n} V_n(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$, we have

<u>Thm</u> (Bloch–Okounkov)

derivative

$$\sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m)!} \Theta^{(n-m)}(z_1 + \cdots + z_m)\, V_m(z_1 + \cdots + z_m) = 0.$$

<u>Example</u> $V_1(z) = \dfrac{\Theta'(0)}{\Theta(z)}$, $V_2(z_1, z_2) = \dfrac{\Theta'(0)}{\Theta(z_1 + z_2)} \dfrac{\Theta'(z_1)}{\Theta(z_1)}$

Overview

$$\begin{array}{ccc} \mathbb{Q}[W(z_i)] & \xrightarrow{\langle\ \rangle_q} & \widetilde{\mathcal{J}} \\ \downarrow & \circlearrowright & \downarrow \\ \Lambda^* & \xrightarrow{\langle\ \rangle_q} & \widetilde{M} \end{array}$$

"Taylor around $z=0$"

strictly meromorphic quasi-Jacobi forms, e.g. $\frac{\Theta'(0)}{\Theta(z)}$, F_n and their derivatives

Key property: Taylor coeffients at rational points are quasimodular

quasimodular forms $\simeq \overline{\mathbb{Q}}[E_2, E_4, E_6]$, where $E_k = -\frac{B_k}{2k} + \sum_{m,r\geq 1} m^{k-1} q^{mr}$

(B_k = kth Bernoulli number)

shifted symmetric functions

$\Lambda^* := \overline{\mathbb{Q}}[Q_2, Q_3, Q_4, \ldots]$ with

$$Q_k(\lambda) := [z^{k-1}]\, W_\lambda(z)$$
$$= \beta_k + \frac{1}{(k-1)!} \sum_i \left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i+\tfrac{1}{2}\right)^{k-1} \qquad \left(\sum_i \beta_i z^{i-1} = \frac{1}{2\sinh(\frac{z}{2})}\right)$$

Cor For all $f \in \Lambda^*_k$ we have $f \in \widetilde{M}_k$.

p-adic analogues of shifted symmetric functions

Recall $Q_k^{(p)}(\lambda) = \beta_k\left(1-\frac{1}{p}\right) + \frac{1}{(k-1)!}\sum_{(2\lambda_i-2i+1,p)=1}\left(\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i+\tfrac{1}{2}\right)^{k-1}\right)$

Observe $$Q_k^{(p)} = [z^{k-1}]\,W(z) - \frac{1}{p}\sum_{a=0}^{p-1}\left[\left(z-\frac{2a}{p}\right)^{k-1}\right]W(z)$$

$$\begin{tikzcd}
\bar{\mathbb{Q}}[W(z_i)] \arrow[r, "\langle\ \rangle_q"] \arrow[d] & \tilde{\mathcal{J}} \arrow[d] \\
\Lambda^*(p) \arrow[r, "\langle\ \rangle_q"] & \tilde{M}(p)
\end{tikzcd}$$

"Taylor coefficients around $z_i \in \frac{\mathbb{Z}}{p}$".

Write $\Lambda^*(p) = \bar{\mathbb{Q}}\left[Q_k(a) : k \geq 0,\ a \in \frac{\mathbb{Z}}{p}\right]$, $\Lambda^{(p)} = \mathbb{Q}\left[Q_k^{(t)} : t \mid p\right]$ $(p \in \mathbb{N})$

Cor For $f \in \Lambda^{(p)}$, we have $\langle f \rangle_q \in \tilde{M}(\Gamma_0(p^2))$.

t-hook moments

Recall $H_k^{(t)} = -\frac{B_k}{2k} t^k + \sum_{\substack{\xi \in Y_\lambda \\ t \mid h(\xi)}} h(\xi)^{k-2}$

Thm (Chen–Möller–Zagier)

$$\frac{1}{(k-2)!} H_k^{(1)}(\lambda) = -\frac{1}{2} [z^{k-1}] \, W_\lambda(z) W_\lambda(-z)$$

Write $\mathcal{H}^{(N)} = \overline{\mathbb{Q}}[H_k^{(t)} \mid k \geq 0 \text{ even}, t \mid N]$

Cor For $f \in \mathcal{H}^{(N)}$ we have $\langle f \rangle_q \in \widetilde{M}(\Gamma_0(N^2))$.

Note For k odd/negative $\langle H_k^{(t)} \rangle_q$ can be understood as quantum/Harmonic Maass forms. What about say, $\langle H_{k_1}^{(t_1)} H_{k_2}^{(t_2)} \rangle_q$?

A symmetric Bloch-Okounkov theorem

[Zagier, '16] [vI, '20]
[Lee, '20] : Spin Hurwitz theory

<u>Thm (vI)</u> Any polynomial in the following part moment functions has a quasimodular form as its q-bracket:

$SL_2(\mathbb{Z})$

- $S_k = -\frac{B_k}{2k} + \sum_i \lambda_i^{k-1}$

- $S_k(a) = \alpha_k + \frac{1}{2} \sum_i \left(\zeta_a^{\lambda_i} + (-1)^k \zeta_a^{-\lambda_i}\right) \lambda_i^{k-1}$ $\quad (a \in \mathbb{Q}, \zeta_a = e^{2\pi i a})$ $\quad \Gamma_1(N)$

- $S_k^t = -\frac{B_k}{2k} t^k + \sum_{i : t \mid \lambda_i} \lambda_i^{k-1}$ $\quad (t \in \mathbb{Z}_{\geq 1})$ $\quad \Gamma_0(N)$

<u>Proof idea</u>

$$\begin{array}{ccc} \overline{\mathbb{Q}}[\quad] & \xrightarrow{\langle\ \rangle_q} & \widetilde{\mathcal{J}} \\ \downarrow & & \downarrow \\ & \xrightarrow{\langle\ \rangle_q} & \widetilde{M}(N) \end{array}$$

"Taylor coefficients around $z_i \in \frac{\mathbb{Z}}{N}$".

Thank you!

More on Hurwitz numbers & Gromov-Witten invariants

Hurwitz

Given partitions $\eta^1, \ldots, \eta^n$ of d

$$\boxed{H_{g,d}(\eta^1, \ldots, \eta^n) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|}},$$

where

- $f : X \xrightarrow{d} E$ is a degree-d covering map of connected Riemann surfaces, ramified at precisely n points with profiles η^i (genus g)
- $f \sim f'$ if $\exists \phi : X \to X'$ s.t. $f \circ \phi = f'$. The sum is over iso. classes, weighted by $|\mathrm{Aut}(f)|^{-1}$.

Gromov-Witten

Given integers $k_1, \ldots, k_n$

$$\boxed{\langle \tau_{k_1}(\omega) \ldots \tau_{k_n}(\omega) \rangle^X_{g,d} := \int_{[M_{g,n}(X,d)]^{vir}} \psi_1^{k_1} ev_1^*(\omega) \ldots \psi_n^{k_n} ev_n^*(\omega)},$$

where

- $M_{g,n}(X,d)$ moduli space of stable degree-d maps of an n-pointed genus connected curve X
- $\omega \in H^2(X, \mathbb{Q})$ Poincaré dual of a point class
- $ev_i : \overline{M}_{g,n}(X) \longrightarrow X$

  $(C, \pi, x_0, \ldots, x_n) \longmapsto \pi(x_i)$
- $\psi_i \in H^2(\overline{M}_{g,n}(X,d), \mathbb{Q})$

Quasi-Jacobi forms

$z \mapsto \varphi(\tau,z)$ admits a pole at $z = X^t \binom{\tau}{1} \in \mathbb{R}^n\tau + \mathbb{R}^n$

Def A strictly meromorphic quasi-Jacobi form of weight k and index $M \in M_n(\mathbb{Q})$ is a meromorphic function $\varphi : \mathfrak{h} \times \mathbb{C}^n \to \mathbb{C}$ s.t.

(i) For all $X \in M_{2,n}(\mathbb{R})$, either $\varphi|_M X$ admits a pole at $(\tau, 0)$ for all generic $\tau \in \mathfrak{h}$ or $\tau \mapsto (\varphi|_M X)(\tau,0) \in \mathrm{Holo}(\mathfrak{h})$

holomorphic at $\mathfrak{h}$ and all cusps.

(ii) There exist meromorphic $\varphi_{i,\underline{j}} : \mathfrak{h} \times \mathbb{C}^n \to \mathbb{C}$ s.t. $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $X = \begin{pmatrix} \underline{\lambda} \\ \underline{\mu} \end{pmatrix} \in M_{2,n}(\mathbb{Q})$

$$(\varphi|_{k,M}\gamma)(\tau, z_1, \dots, z_n) = \sum_{i,\underline{j}} \varphi_{i,\underline{j}}(\tau, z_1, \dots, z_n) \left(\frac{c}{c\tau+d}\right)^{i+j_1+\dots+j_n} z_1^{j_1} \cdots z_n^{j_n}$$

$$(\varphi|_M X)(\tau, z_1, \dots, z_n) = \sum_{i,\underline{j}} \varphi_{0,\underline{j}}(\tau, z_1, \dots, z_n) (-\lambda_1)^{j_1} \cdots (-\lambda_n)^{j_n}$$

standard slash action as on Jacobi forms

$\boxed{\text{Key property: Taylor coeffient of } \varphi|X \text{ for } X \in M_{2,n}(\mathbb{Q}) \text{ are quasimodular if } \underline{\lambda} = \underline{0}}$