Critical points of modular forms

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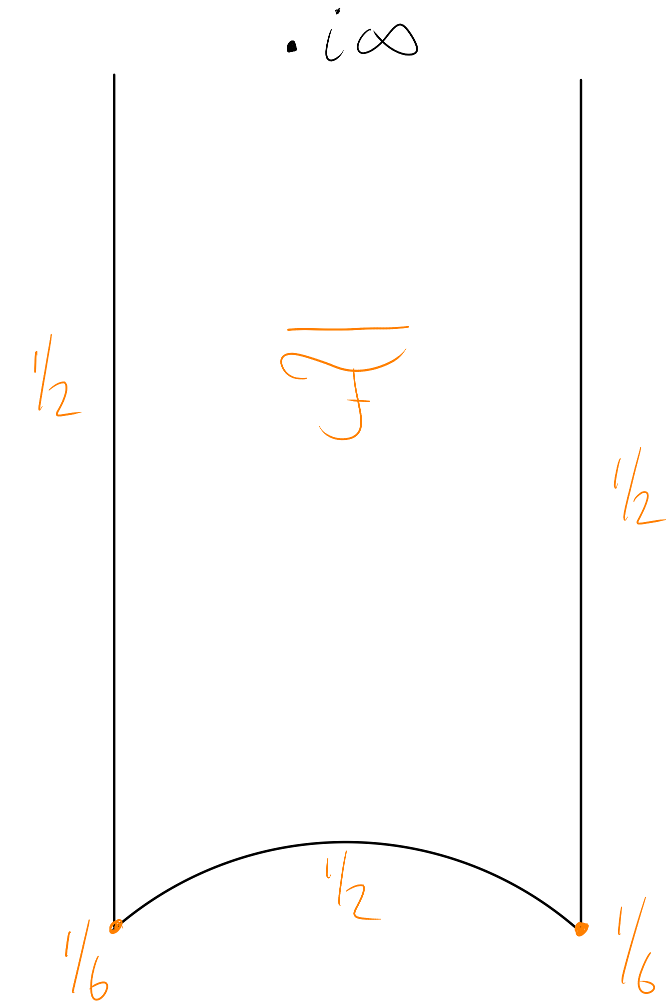
joint with Berend Ringeling

Zeros of modular forms



Valence formula $f \in M_k \leftarrow$ modular forms of weight k

$$N_\gamma(f) := \sum_{z \in \gamma \overline{\mathcal{F}}} \omega_z \, v_z(f) = k/12$$

weight of z

multiplicity of zero of f at z

Thm (F. Rankin, P. Swinnerton-Dyer '70) – Wohlfahrt conjecture '64
All zeros of the Eisenstein series E_k inside $\overline{\mathcal{F}}$ are on the lower arc
($k \geq 4$ even)

$$E_k(\tau) := \sum_{(m,n)=1} \frac{1}{(m\tau+n)^k}$$

Thm (Holowinsky-Soundararajan '10) Zeros of Hecke eigenforms are equidistributed w.r.t. hyperbolic measure as $k \to \infty$

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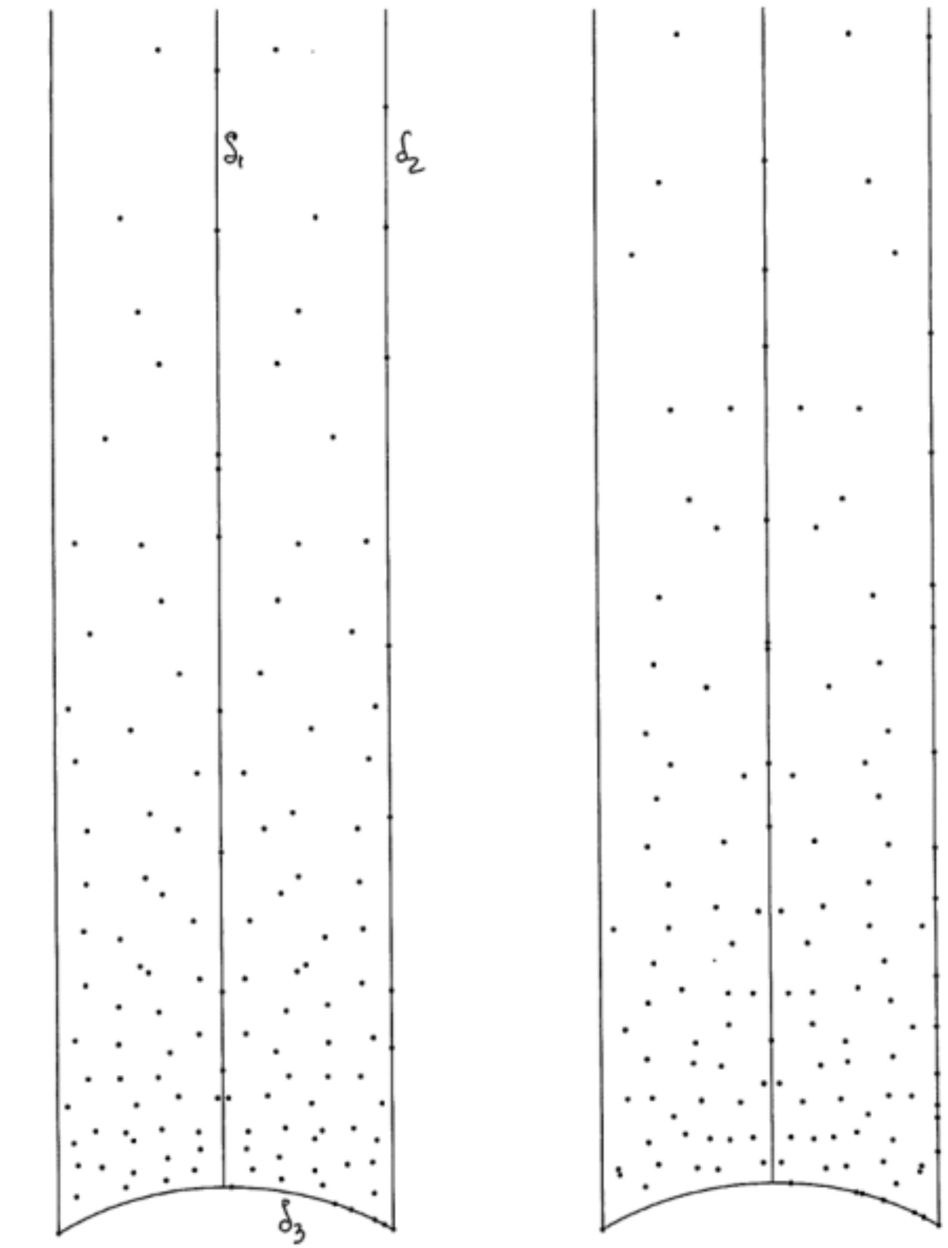


Zeros for y < 4.5 of two cuspforms of weight 2000 as computed by F. Stromberg. All the zeros found for y > 10 were real.

(copied from P. Sarnak)

Thm (Holowinsky–Soundararajan '10) Zeros of Hecke eigenforms are equidistributed w.r.t. hyperbolic measure as $k \to \infty$

Two observations

(i) Write $f|_k\gamma := (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Modular forms $f \in M_k$ satisfy $\quad f|_k\gamma = f \quad \forall \gamma \in SL_2(\mathbb{Z})$

$' = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}$

Note $$f'|_{k+2}\gamma = f' + \frac{k}{2\pi i}\frac{c}{c\tau+d} f$$

Ex $$\Delta'|_{14}\gamma = \Delta' + \frac{12}{2\pi i}\frac{c}{c\tau+d}\Delta$$

$$E_2|_4\gamma = E_2 + \frac{12}{2\pi i}\frac{c}{c\tau+d} \qquad \left(E_2 = \frac{\Delta'}{\Delta}\right)$$

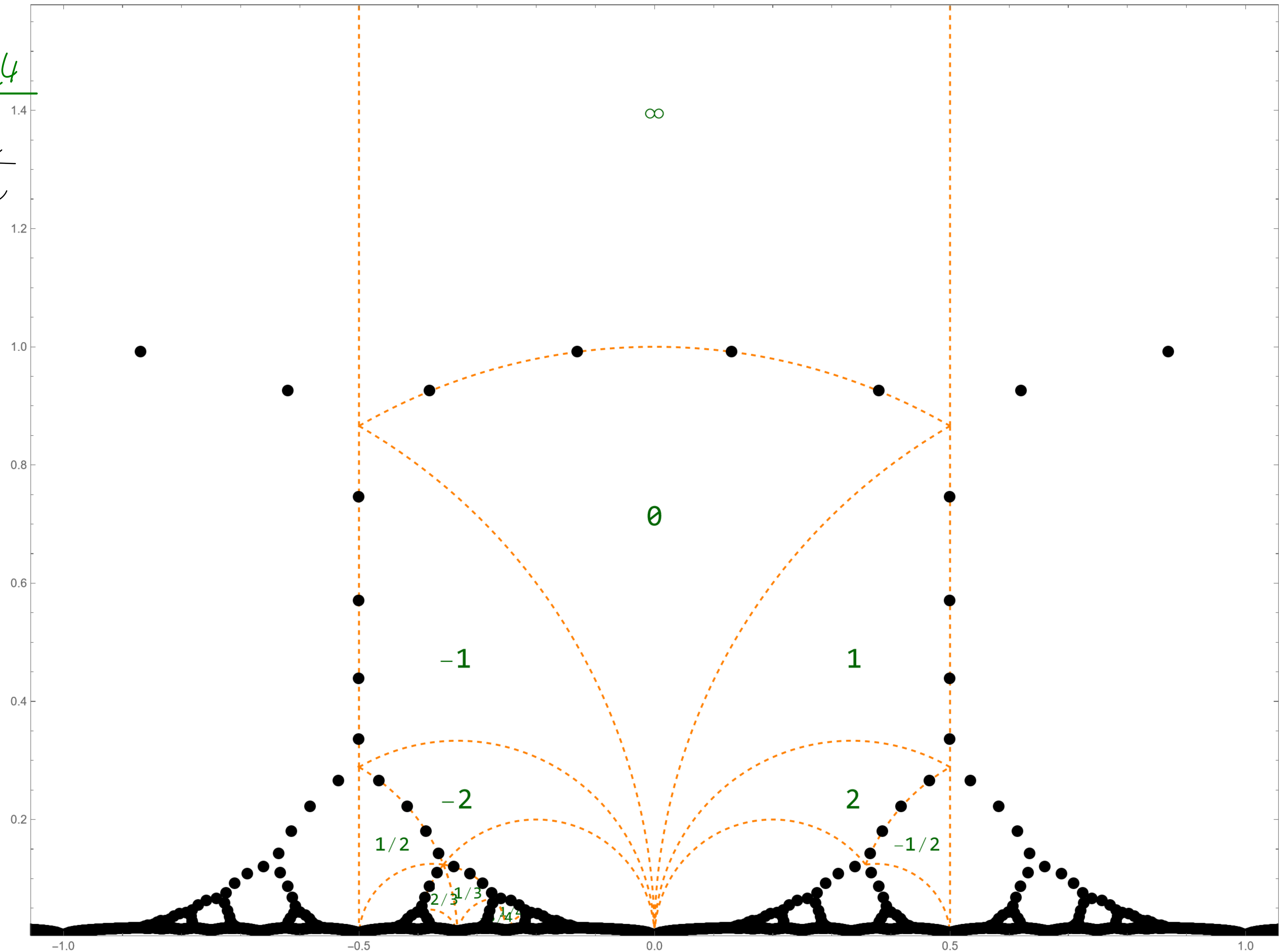
(ii) $N_\gamma(g) := \sum_{z \in \gamma\mathcal{F}} \omega_z v_z(g)$ only depends on g and $\lambda := -\frac{d}{c} = \gamma^{-1}(\infty)$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

for g 1-periodic + conditions at cusps.

Write $N_\lambda(g) = N_\gamma(g)$.

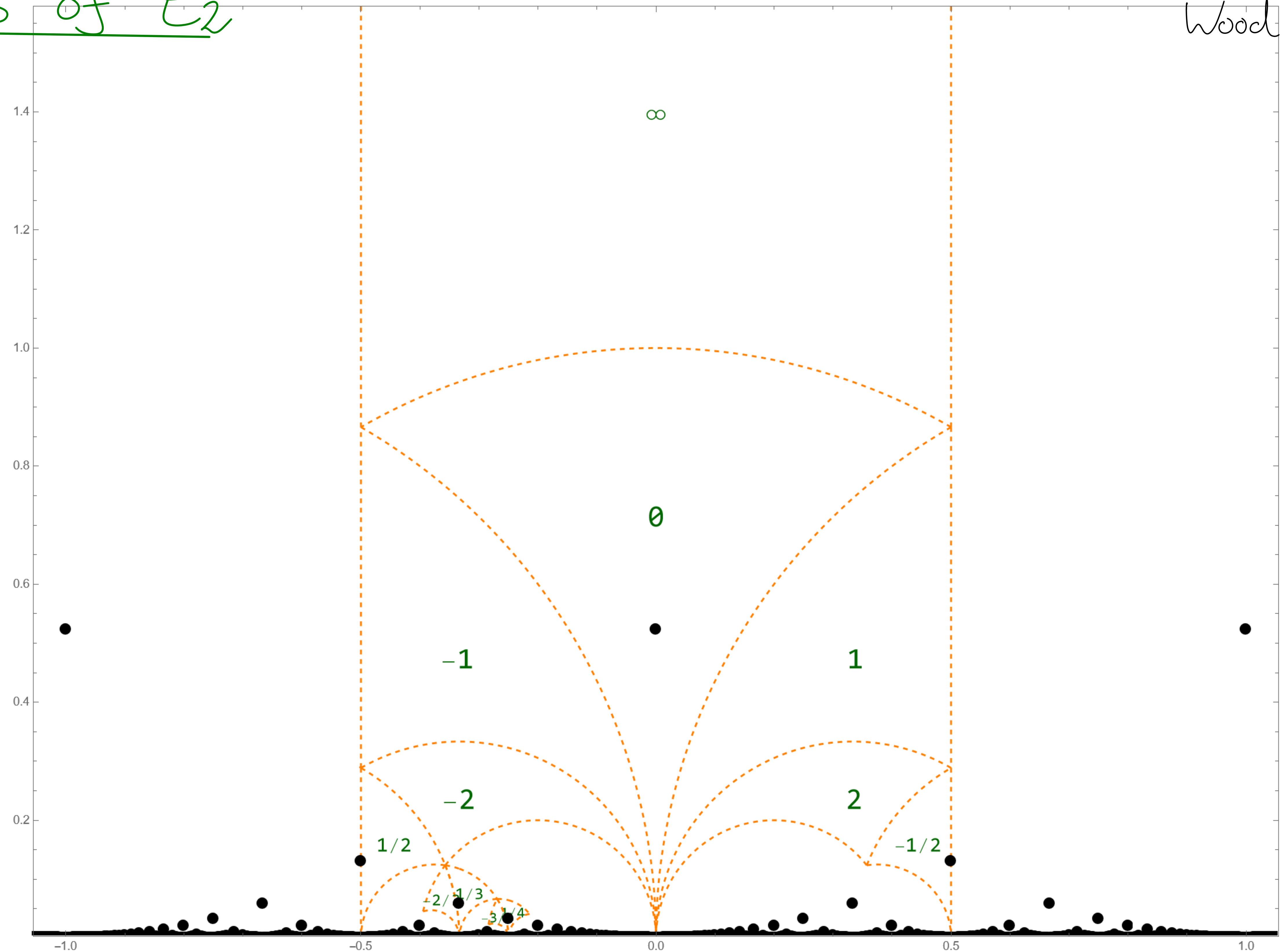
Zeros of E_{24}

$$N_\lambda(E_k) = \frac{k}{12}$$



Zeros of E_2

Imamoḡlu–Arpad–Toth '14
Wood–Young '14

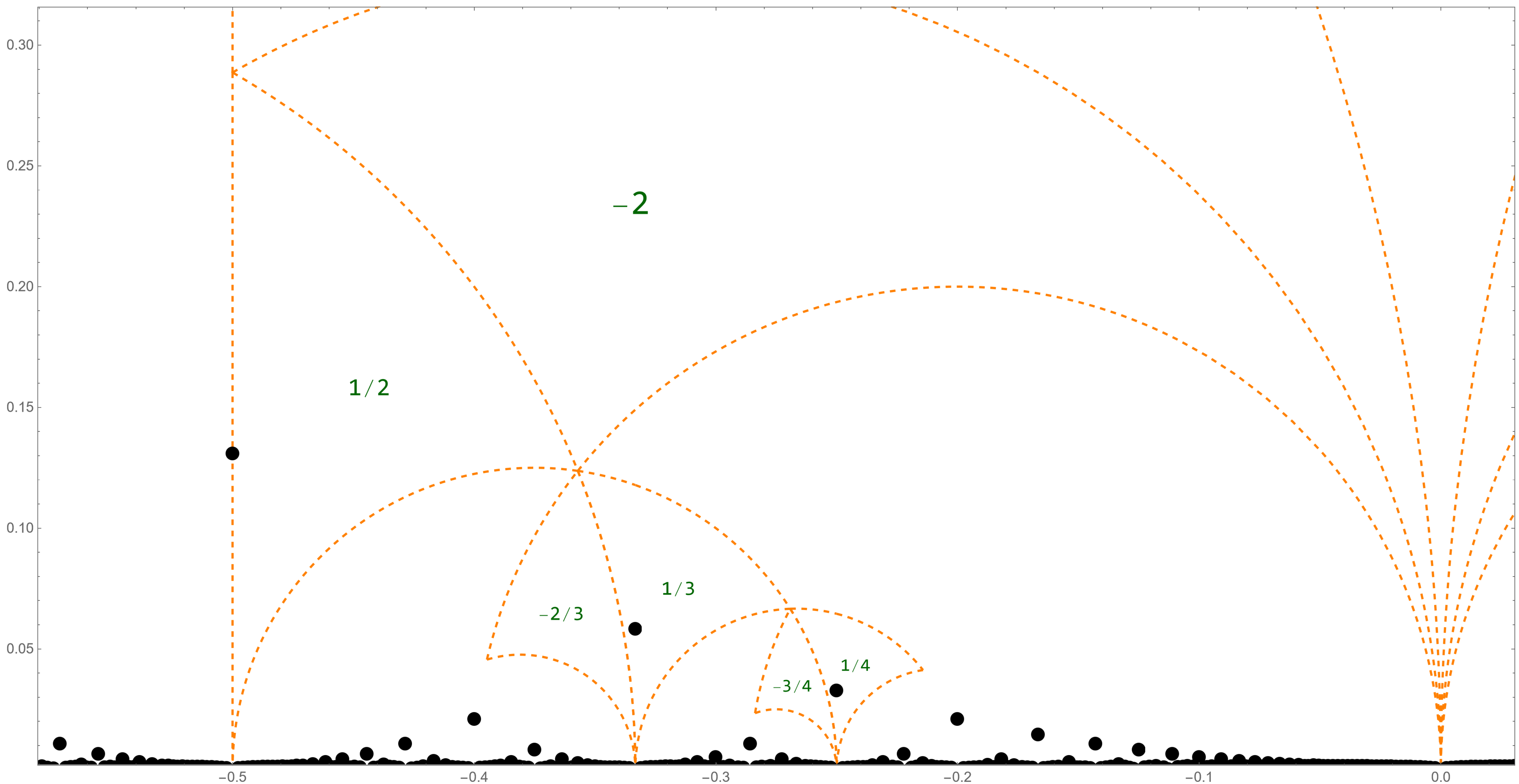


Zeros of E_2

Imamoḡlu-Arpad-Toth '14
Wood-Young '14

$$N_\lambda(E_2) = \begin{cases} 0 & |\lambda| > \frac{1}{2} \\ 1 & |\lambda| < \frac{1}{2} \end{cases}$$

$N_{\frac{1}{2}}(E_2) = \frac{1}{2}$



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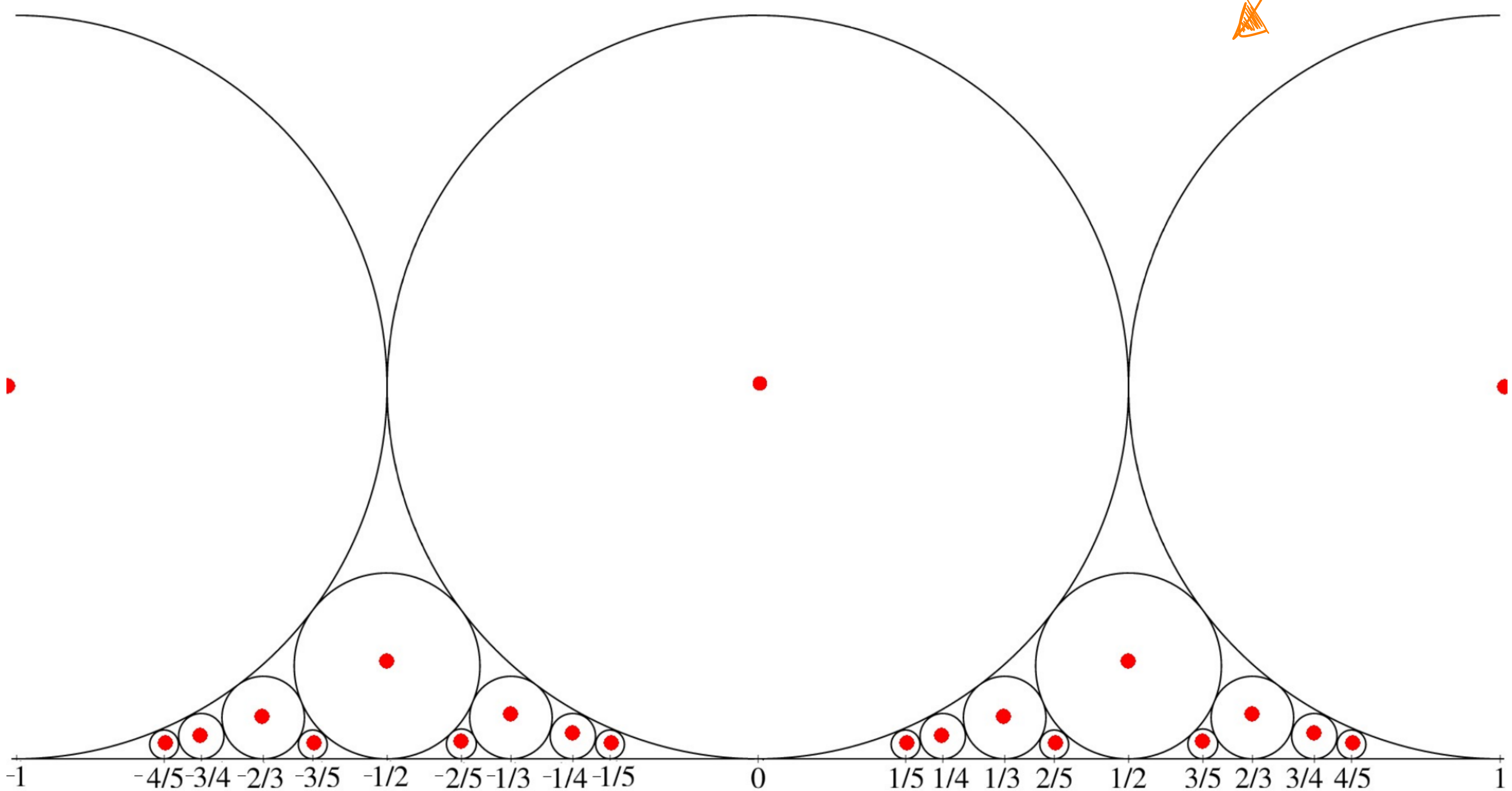
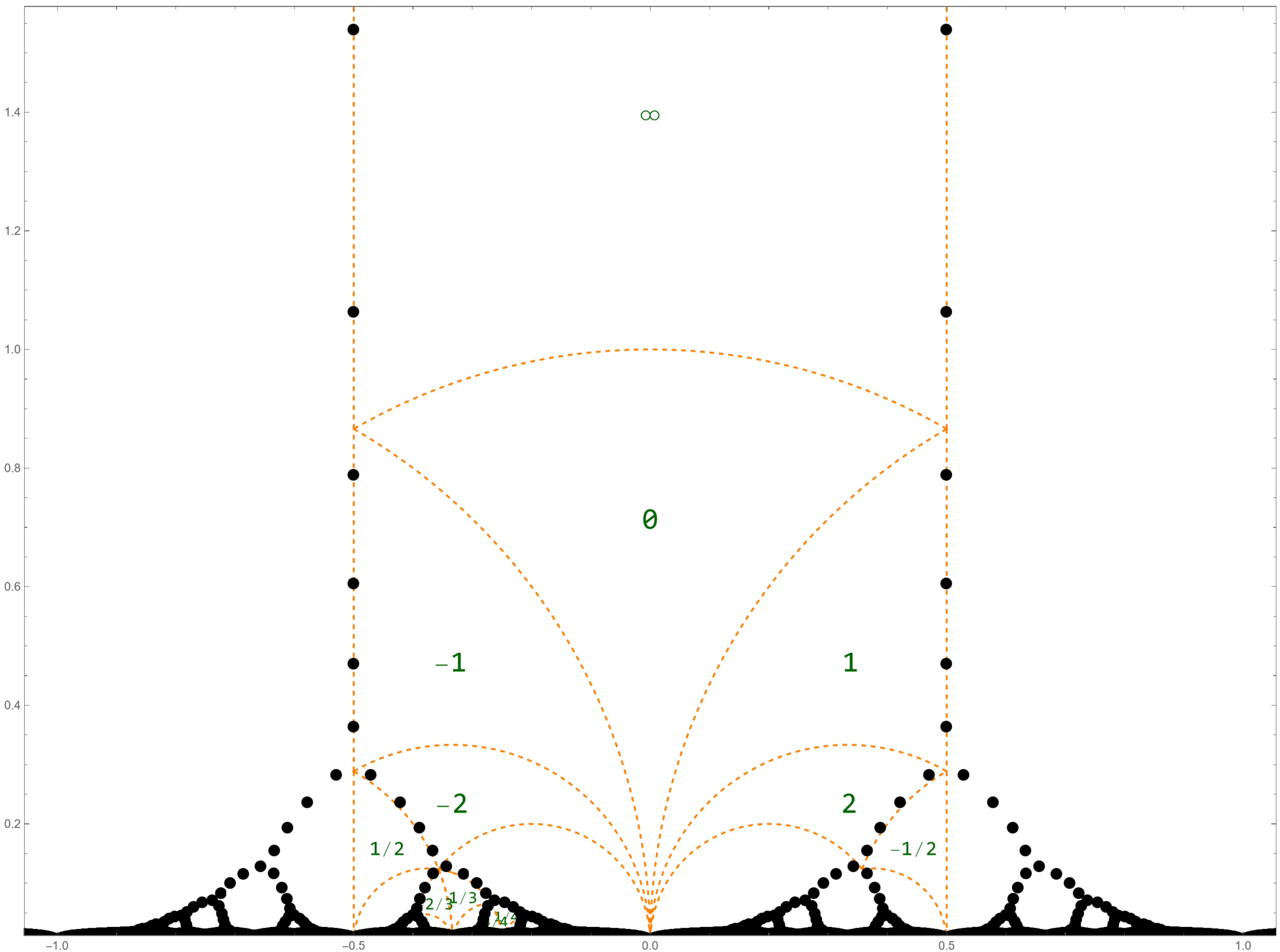


Zeros of E_{24}'

Gun-Oesterlé '22



Zeros of E_{24}'

Gun–Oesterlé '22

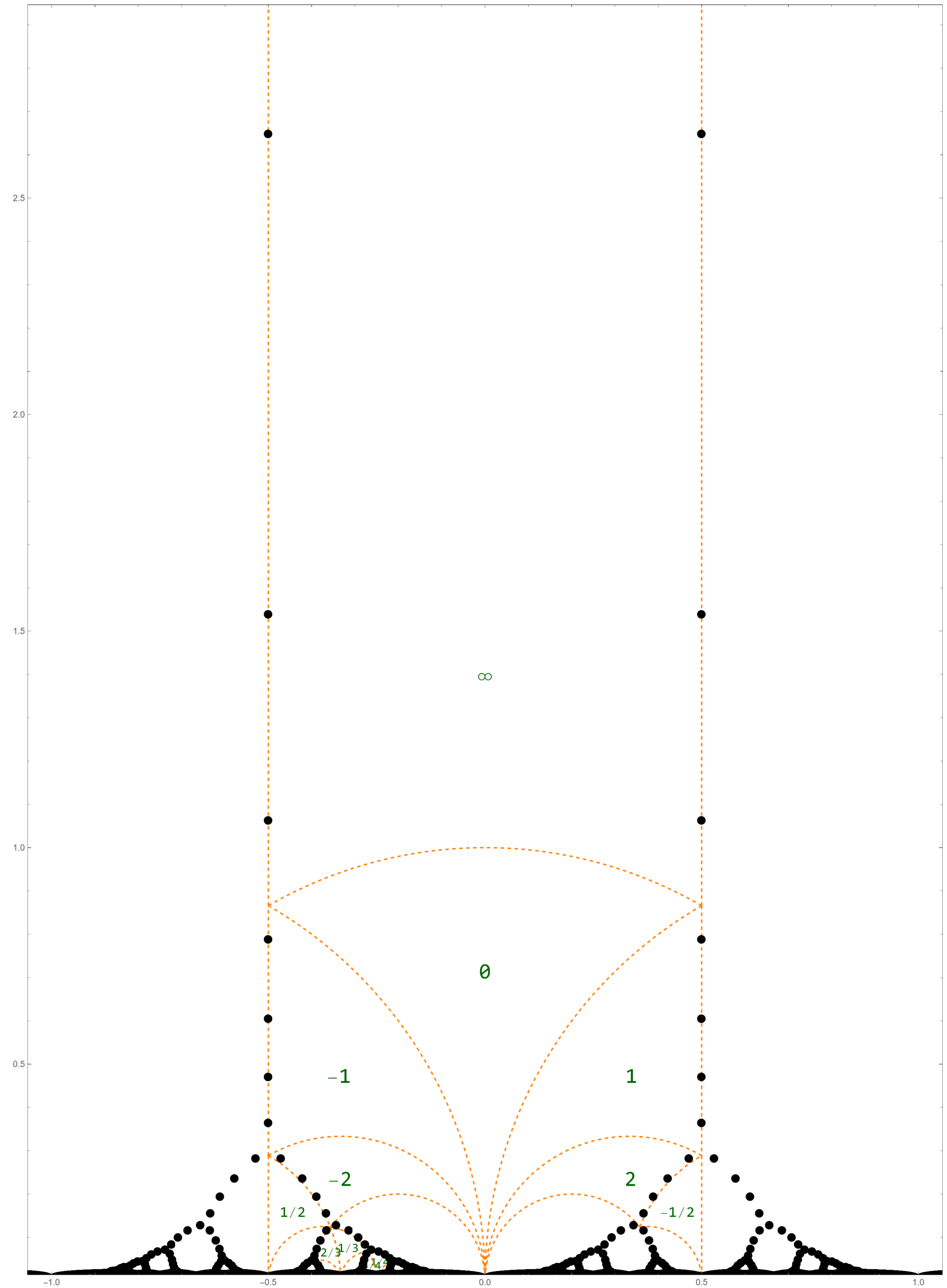
$$N_\lambda(E_{24}') = \begin{cases} 4 & |\lambda|>1 \\ 0 & |\lambda|<1 \end{cases}$$

$N_1(E_{24}')=2$

$$N_\lambda(E_k') = \begin{cases} \left\lfloor \frac{k+2}{6} \right\rfloor & |\lambda|>1 \\ 0 & |\lambda|<1 \end{cases}$$

$(k \not\equiv 2\ (6))$

$N_1(E_k') = \frac{1}{2}\left\lfloor \frac{k+2}{6} \right\rfloor$



Main results (I)

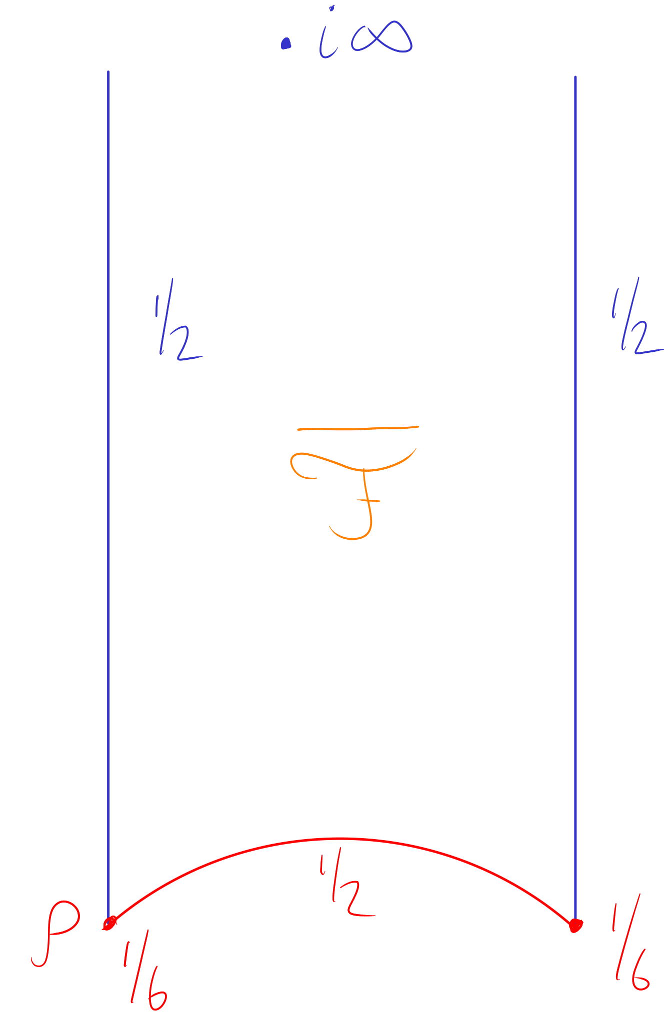
Thm (vI–Ringeling) For $f \in M_k$ with real Fourier coefficients

$$N_\lambda(f') = \frac{k}{12} + \begin{cases} C(f) + \frac{1}{3}\delta_{g(\rho)=0} & 1<|\lambda| \\ -C(f) & \frac{1}{2}<|\lambda|<1 \\ -C(f) + L(f) & |\lambda|<\frac{1}{2} \end{cases},$$

where

$C(f)$ is wt # of distinct zeros on lower arc,

$L(f)$ is wt # of distinct zeros on vertical boundary, including cusp at infinity, excluding ρ.



Ex $C(\Delta) = 0$, $L(\Delta) = 1 \rightsquigarrow N_\lambda(E_2) = \begin{cases} 0 & |\lambda|>1/2 \\ 1 & |\lambda|<1/2 \end{cases}$

$C(E_k) = \frac{k}{12}$, $L(E_k) = 0 \rightsquigarrow N_\lambda(E_k') = \begin{cases} \lfloor \frac{k+2}{6} \rfloor & |\lambda|>1 \\ 0 & |\lambda|<1 \end{cases}$ $\quad (k \not\equiv 2\ (6))$

Cor
- $N_\lambda(f') = N_\lambda(f)$ if all zeros of $f \in M_k$ are on the interior of $\mathcal{F}$.
- $\dfrac{N_\lambda(f_k')}{N_\lambda(f_k)} \longrightarrow 1$ for any sequence of Hecke eigenforms f_k.

Main results (II)

Write $\widetilde{M}_k^{\le p}$ for the space of quasimodular forms, i.e. polynomials in E_2 of degree $\le p$ with modular coefficients

Thm (vI-Ringeling) Let $f = f_0 + f_1 E_2 \in M_k^{\le 1}$, s.t. f_0 and f_1 have real Fourier coefficients and no common zeros. Then,

$$N_\lambda(f) = \begin{cases} N_\infty(f) & 1 < |\lambda| \\ N_{3/4}(f) & \frac{1}{2} < |\lambda| < 1 \\ N_0(f) & |\lambda| < \frac{1}{2} \end{cases}$$

+ closed formulas for $N_\infty(f)$, $N_{3/4}(f)$ and $N_0(f)$...

In particular,

$$N_\lambda(f) \le \dim \widetilde{M}_k^{\le 1} + \begin{cases} -1 & |\lambda| > \frac{1}{2} \\ 0 & |\lambda| < \frac{1}{2} \end{cases}$$

if $E_4 \nmid f$.

Best possible:

E_k'

E_2

Q If $f \in M_k^{\le p}$ s.t. $E_4 \nmid f$ and f has real Fourier coefficients, then

$$v_{i\infty}(f) \le N_\lambda(f) \overset{?}{\le} \dim \widetilde{M}_k^{\le p} + \begin{cases} -1 & |\lambda| > \frac{1}{2} \\ 0 & |\lambda| < \frac{1}{2} \end{cases} \quad ?$$

Rk Such a result would lead to Sturm bounds for quasimodular forms

Main results (III)

Thm Let $f \in \widetilde{M}_k$ with real Fourier coefficients. Then $\exists$ fin. many disjoint intervals I s.t. $\mathbb{R} = \bigcup I$, and $\exists\, N_I(f) \in \frac{1}{6}\mathbb{Z}$ s.t.

$$N_\lambda(f) = N_I(f) \qquad \lambda \in I$$

Ex (Chen, Lin '19)

$$N_\lambda(E_2') = \begin{cases} 1 & |\lambda| \in (\frac{1}{\varpi}, \frac{1}{2}) \cup (\varpi, \infty) \\ 0 & |\lambda| \in [0, \frac{1}{\varpi}) \cup (\frac{1}{2}, \varpi) \end{cases}$$

with $\varpi = 5.555295\ldots$

Main tool in the proofs

Saber–Sebbar: $h_f(\tau) := \tau + k\dfrac{f(\tau)}{f'(\tau)}$ for $f \in M_k$ with real coefficients

Then

$$h_f\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{a h_f(\tau)+b}{c h_f(\tau)+d}, \qquad h_f(-\bar{\tau}) = -\overline{h_f(\tau)}$$

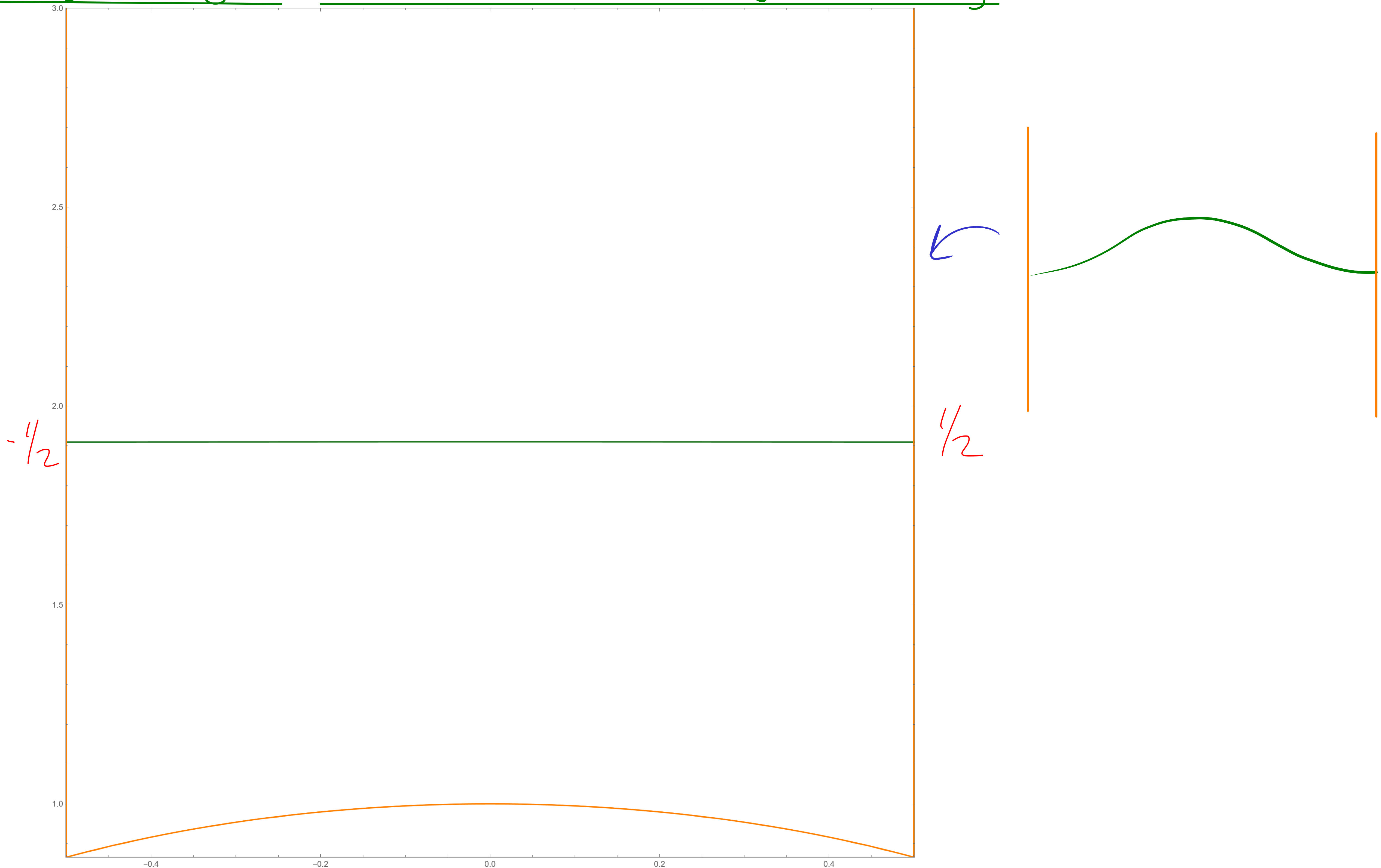
is an <u>equivariant form</u>.

Critical points of f in $\gamma^{-1}\overline{\mathcal{F}}$ $\longleftrightarrow$ poles of h_f in $\gamma^{-1}\overline{\mathcal{F}}$

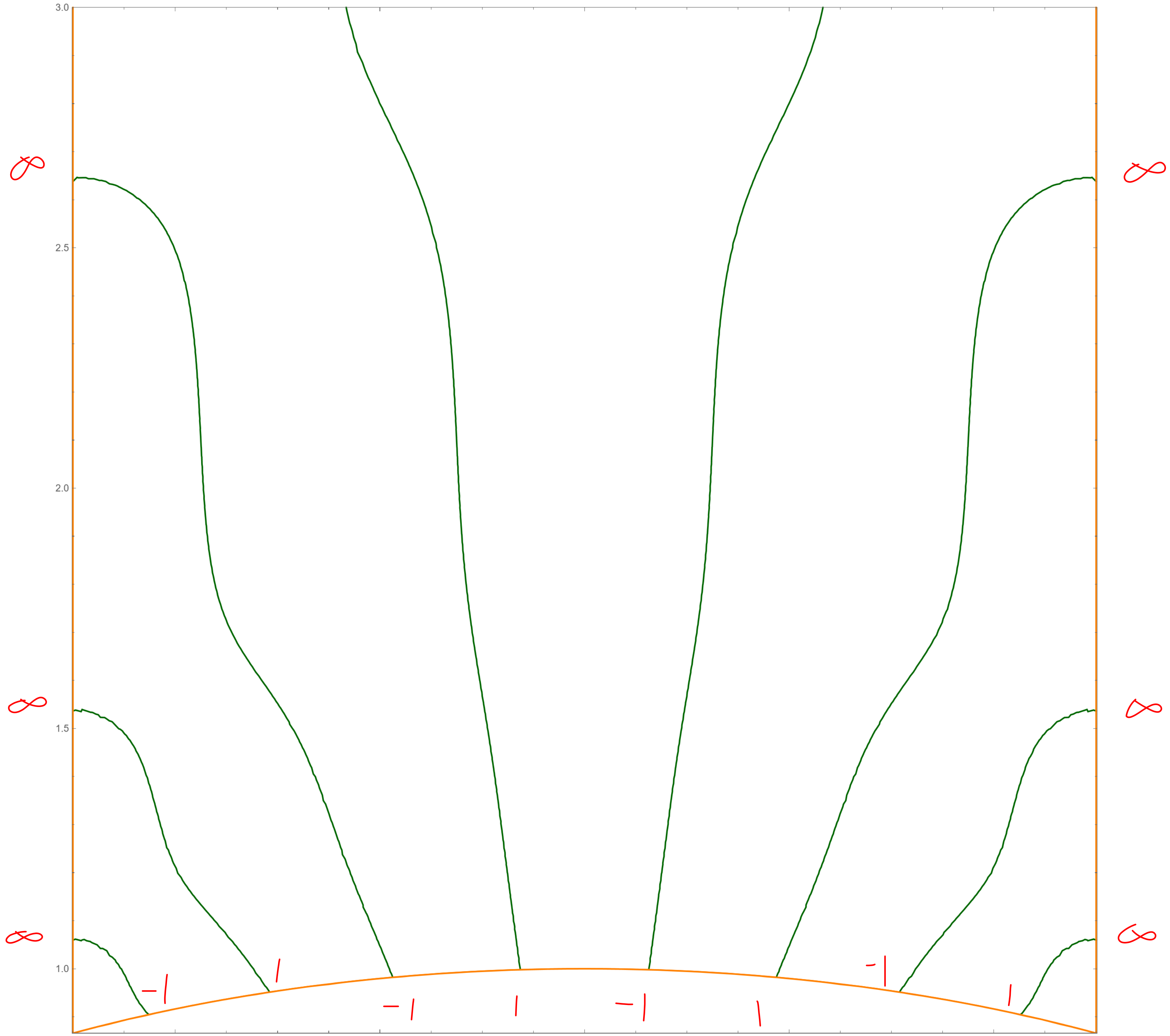
$\longleftrightarrow$ $h_f(\tau) = -d/c$ in $\overline{\mathcal{F}}$ $\qquad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$

<u>Upshot</u> Study the real locus of f in $\overline{\mathcal{F}}$

Real locus of h_Δ: $\overline{\mathcal{F}} \cap \overline{SL_2(\mathbb{Z}) \cdot \{z \in \mathfrak{h} \mid E_2(z) = 0\}}$



Real locus of $h_{E_{24}}$: $\mathcal{F} \cap \overline{SL_2(\mathbb{Z}) \cdot \{z \in \mathfrak{h} \mid E_{24}'(z) = 0\}}$



$h_f(z) = \lambda + r_\lambda(z)\, e^{2\pi i \alpha_\lambda(z)}$

$r_\lambda, \alpha_\lambda : \partial\mathcal{F} \longrightarrow \hat{\mathbb{C}}$

real analytic

Proof sketch

Assume $h_f(z) \neq \lambda$ for $z \in \partial\mathcal{F}$

$$N_\lambda(f) - N_\infty(f) = \frac{1}{2\pi i}\int_{\partial\mathcal{F}} \frac{h_f'(z)}{h_f(z)-\lambda}\, dz$$

$$= \frac{1}{2\pi}\,\mathrm{Im}\int_{\partial\mathcal{F}} \frac{h_f'(z)}{h_f(z)-\lambda}\, dz$$

$$= \frac{1}{2\pi}\,\mathrm{Im}\int_{\partial\mathcal{F}} \frac{r_\lambda'(z)\, e^{2\pi i\alpha_\lambda(z)} + 2\pi i\,\alpha_\lambda'(z)\, r_\lambda(z)\, e^{2\pi i \alpha_\lambda(z)}}{r_\lambda(z)\, e^{2\pi i\alpha_\lambda(z)}}$$

$$= \alpha_\lambda\left(\tfrac{1}{2}+i\infty\right) - \alpha_\lambda\left(-\tfrac{1}{2}+i\infty\right)$$

Observe α_λ is constant as a function of λ, as long as assumption holds.

Note

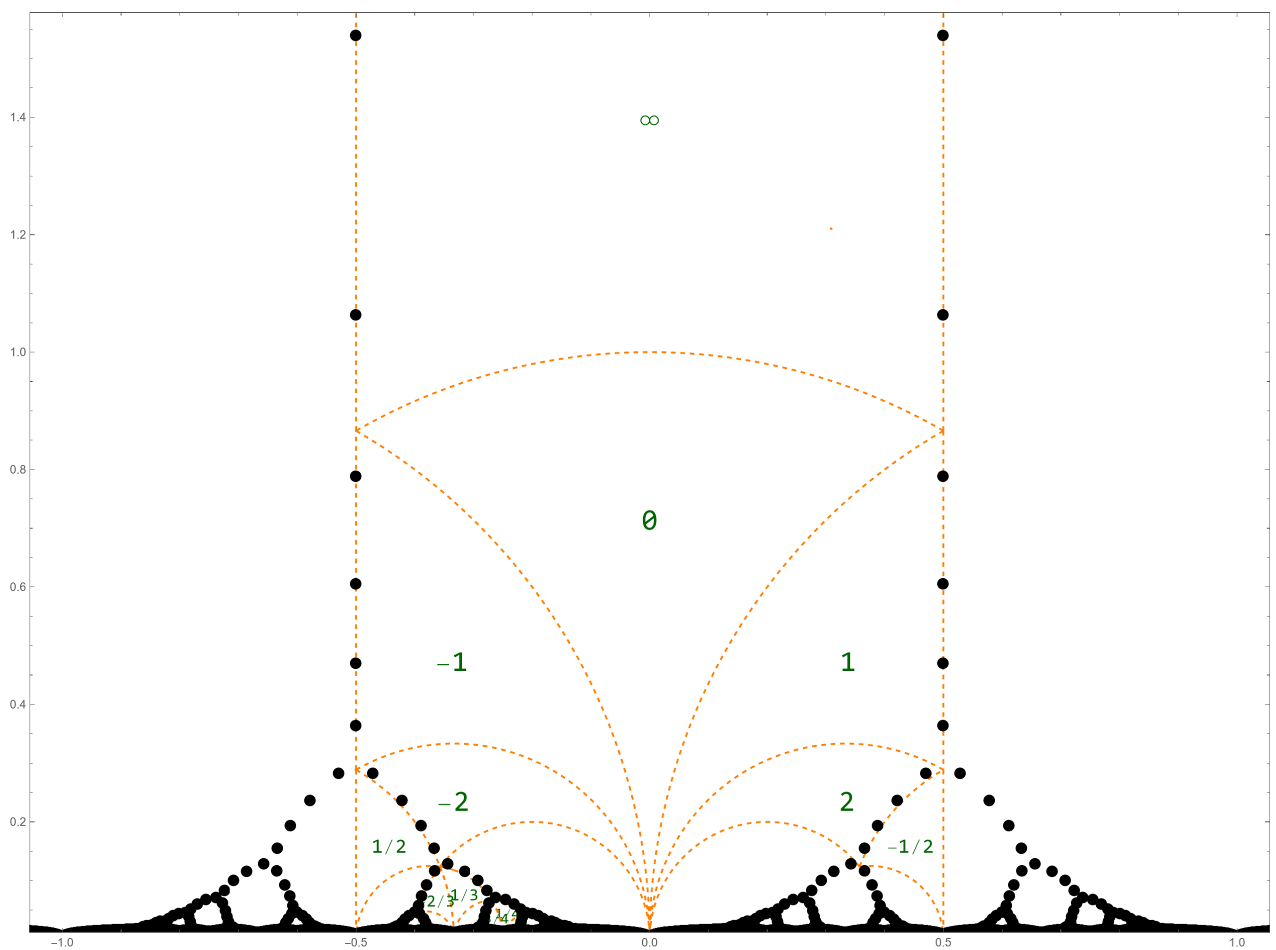
$(t\in\mathbb{R})$ $\quad h_f\left(\tfrac{1}{2}+it\right) \overset{T^{-1}}{=} h_f\left(-\tfrac{1}{2}+it\right) - 1 \overset{C}{=} -\overline{h_f\left(\tfrac{1}{2}+it\right)} - 1 \rightsquigarrow$

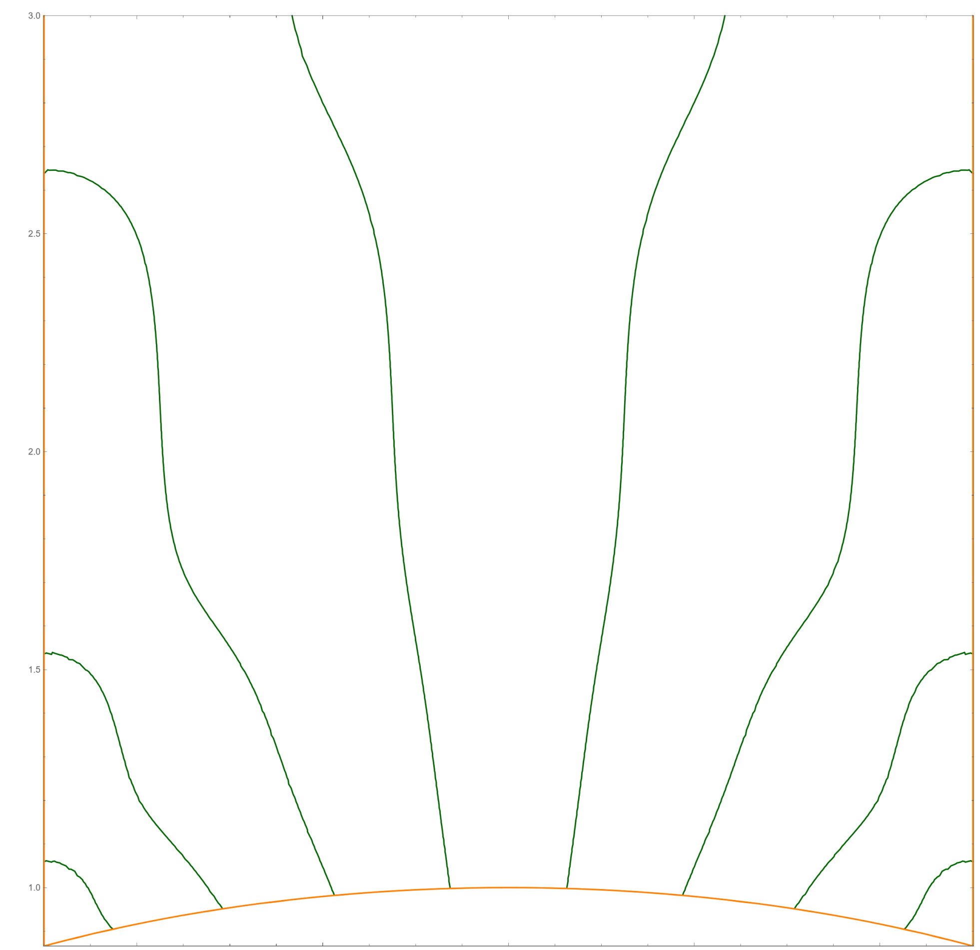
$(|z|=1)$ $\quad h_f(z) \overset{S}{=} -h_f(-z^{-1})^{-1} \overset{C}{=} \overline{h_f(\overline{z}^{-1})}^{-1} = \overline{h_f(z)}^{-1} \rightsquigarrow$

$$\boxed{\begin{array}{l} h_f\left(\tfrac{1}{2}+it\right) \in \tfrac{1}{2} + i\mathbb{R} \\ |h_f(z)| = 1 \qquad (|z|=1) \end{array}}$$

explains intervals $(0,\tfrac{1}{2})$, $(\tfrac{1}{2},1)$ & $(1,\infty)$

Quasimodular forms, for which zeros are not $SL_2(\mathbb{Z})$-invariant, admit surprisingly pretty distributions of zeros in $\gamma\mathcal{F}$ for $\gamma \in SL_2(\mathbb{Z})$





Thank you!