From partitions to modular forms

KWG PhD prize

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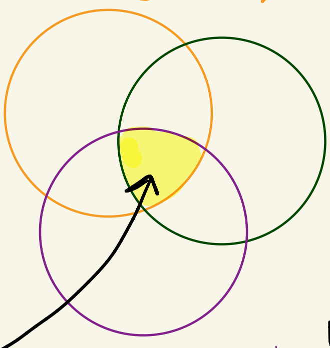
Jan-Willem van Ittersum

Utrecht University

Partitions & modular forms

String theory
↔
Enumerative geometry

Algebraic combinatorics (partitions)



my thesis

Number theory (modular forms)

cake

Ingredients

200g	butter
200g	sugar
4	eggs
200g	self-rising flour
1g	salt

bread

Ingredients

200g	self-rising flour
1 g	salt
150 g	water

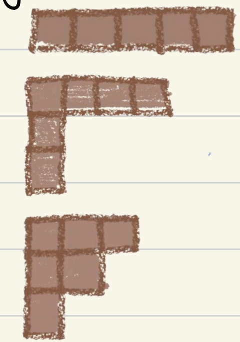
Quasimodular forms

Ingredients

5

6

6



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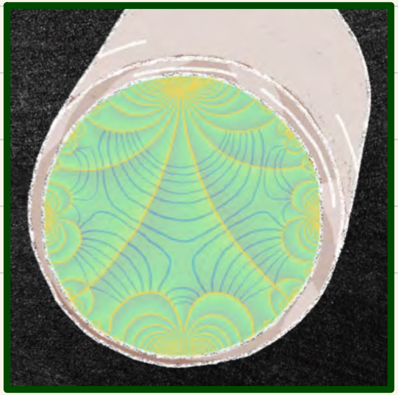
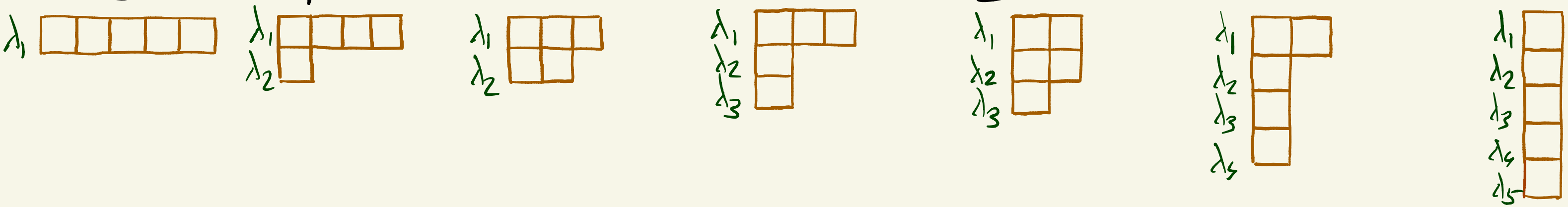


Ingredients: partitions

Ex. Partitions λ of $n=5$ are

$5 \qquad 4+1 \qquad 3+2 \qquad 3+1+1 \qquad 2+2+1 \qquad 2+1+1+1 \qquad 1+1+1+1+1$



Def A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a way to write an integer n as a sum of positive integers $\lambda_1, \lambda_2, \ldots$ (where the order of summation doesn't matter).

Thm (Euler)

integer n that λ is a partition of.

$$\sum_{\lambda \in P} q^{|\lambda|} = 1 + q + 2q^2 + 3q^3 + 5q^4 + \underline{7q^5} + \ldots = \prod_{m=1}^{\infty} \frac{1}{(1-q)^m}.$$

set of all partitions $n = \lambda_1 + \lambda_2 + \ldots$ for all n

Ingredients: how much of each partition?

Recall 200g butter, 200g sugar, ...

set of all partitions
$n = \lambda_1 + \lambda_2 + \cdots$ for all n

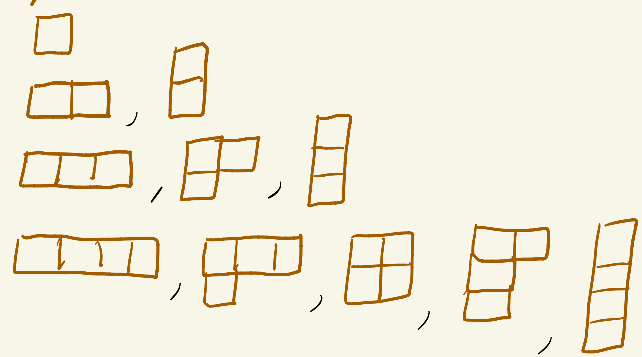
Symmetric functions $S_k : \mathcal{P} \to \mathbb{Q}$ $\qquad S_k(\lambda) = \sum_i \lambda_i^k$

Ex $S_3(2+2+1) = 2^3 + 2^3 + 1$

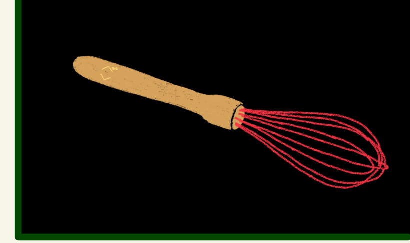
$= 17.$

Ex $S_1(\lambda) = \sum_i \lambda_i = |\lambda|$

0 $\emptyset$
1
2
3
4
⋮



Whisk: the q-bracket



Given $f: \mathcal{P} \to \mathbb{Q}$, the q-bracket of f is given by

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} \in \mathbb{Q}[[q]]$$

formal power series in q.

generating series of partitions (Euler)

Ex

$$\langle S_1 \rangle_q = \frac{\sum_{\lambda \in \mathcal{P}} |\lambda|\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \ldots$$

recall $S_1(\lambda) = |\lambda| = \sum_i \lambda_i$

$$= \sum_n \underbrace{\Big(\sum_{d|n} d \Big)}_{\text{divisor sum}} q^n .$$

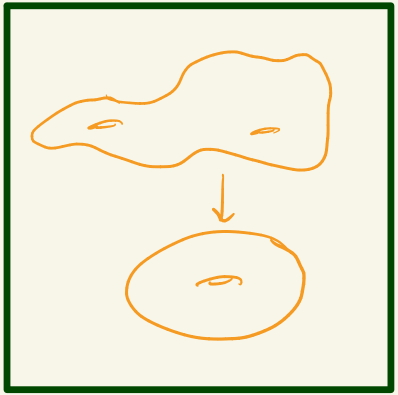
a quasimodular form

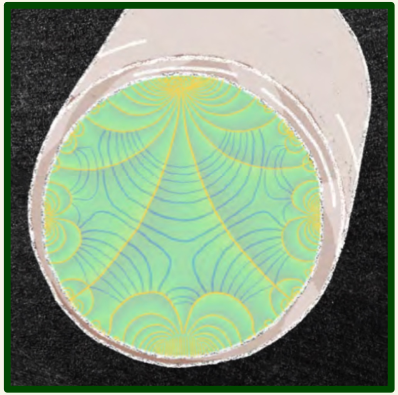
Result: a (quasi) modular form

Quasimodular forms are everywhere:

- Fermat's last theorem
- Sphere packing problem
- Coverings of tori

...





Two key properties of quasimodular forms (QMF):

(i) The sum, difference and product of two QMFs is a QMF
Also, the derivative is a QMF.

(ii) Only one QMF is needed to generate all of them using (i) iteratively.

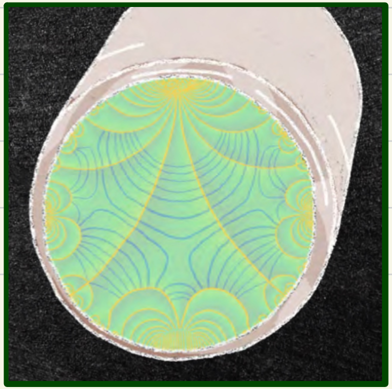
3 recipes : addition

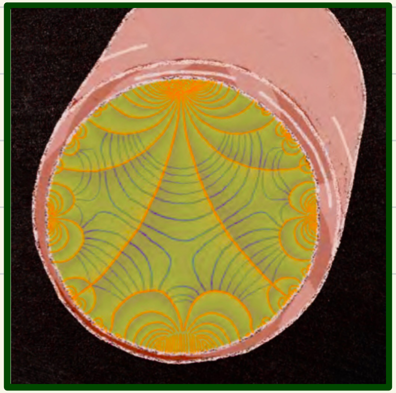
Ingredients cake		Ingredients bread		Ingredients cake bread	
200 g	butter			200 g	butter
200 g	sugar			200 g	sugar
4	eggs			4	eggs
200 g	self-rising flour	200 g	self-rising flour	400 g	self-rising flour
1 g	salt	1 g	salt	2 g	salt
		150 g	water	150 g	water

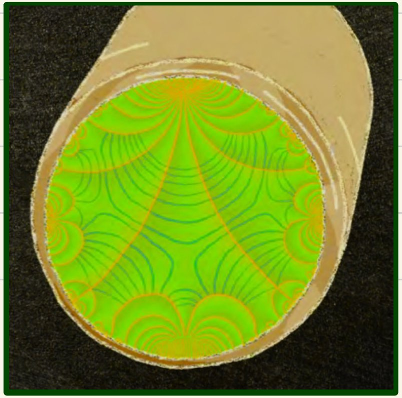
+ =

3 recipes : addition

Ingredients S_1	+	Ingredients S_3	=	Ingredients $S_1 + S_3$
1		1		2
2		8		10
2		2		4
3		27		30
3		9		12
3		3		6
...		...		...







3 recipes: multiplication

Ingredients	cake
200g	butter
200g	sugar
4	eggs
200g	self-rising flour
1 g	salt

×

Ingredients	bread
200g	self-rising flour
1 g	salt
150 g	water

=

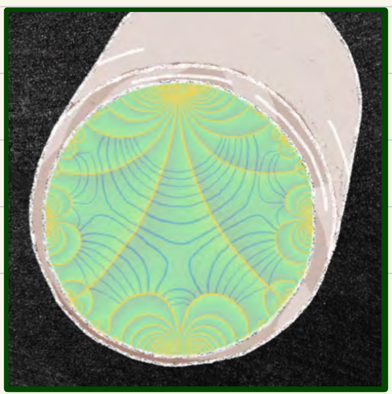
salted flour?!

40000g	self-rising flour?
1g	salt ?

3 recipes: multiplication

Ingredients S_1

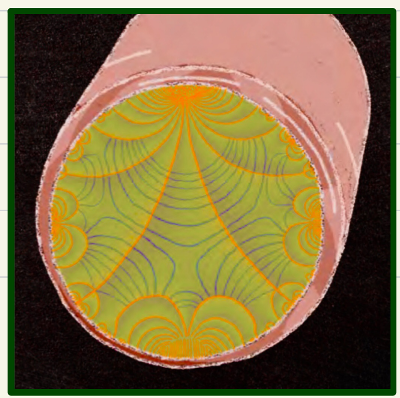
1
2
2
3
3
3
...



×

Ingredients S_3

1
8
2
27
9
3
...



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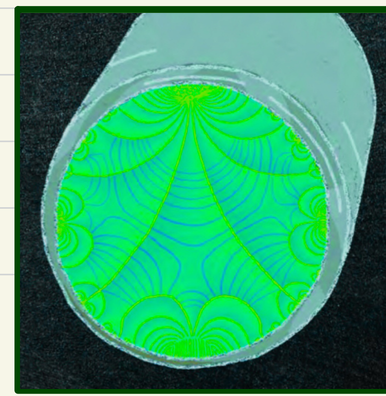
Ingredients $S_1 S_3$

1
16
4
81
27
9
...



Result: a (quasi) modular form

Observation $\langle S_1^2\rangle_q$, $\langle S_3^2\rangle_q$, $\langle S_5^2\rangle_q$, $\langle S_7^2\rangle_q$ are all QMFs. Even combinations as $\langle S_1 S_3\rangle_q$, $\langle S_1 S_5 S_7\rangle_q$ and $\langle S_1^2 S_3 S_5^3 S_7^4\rangle_q$ are QMFs:

Thm (vI)
For all f in the algebra generated by the S_k (k odd)

vector space & ring

$$\langle f\rangle_q \text{ is a QMF.}$$

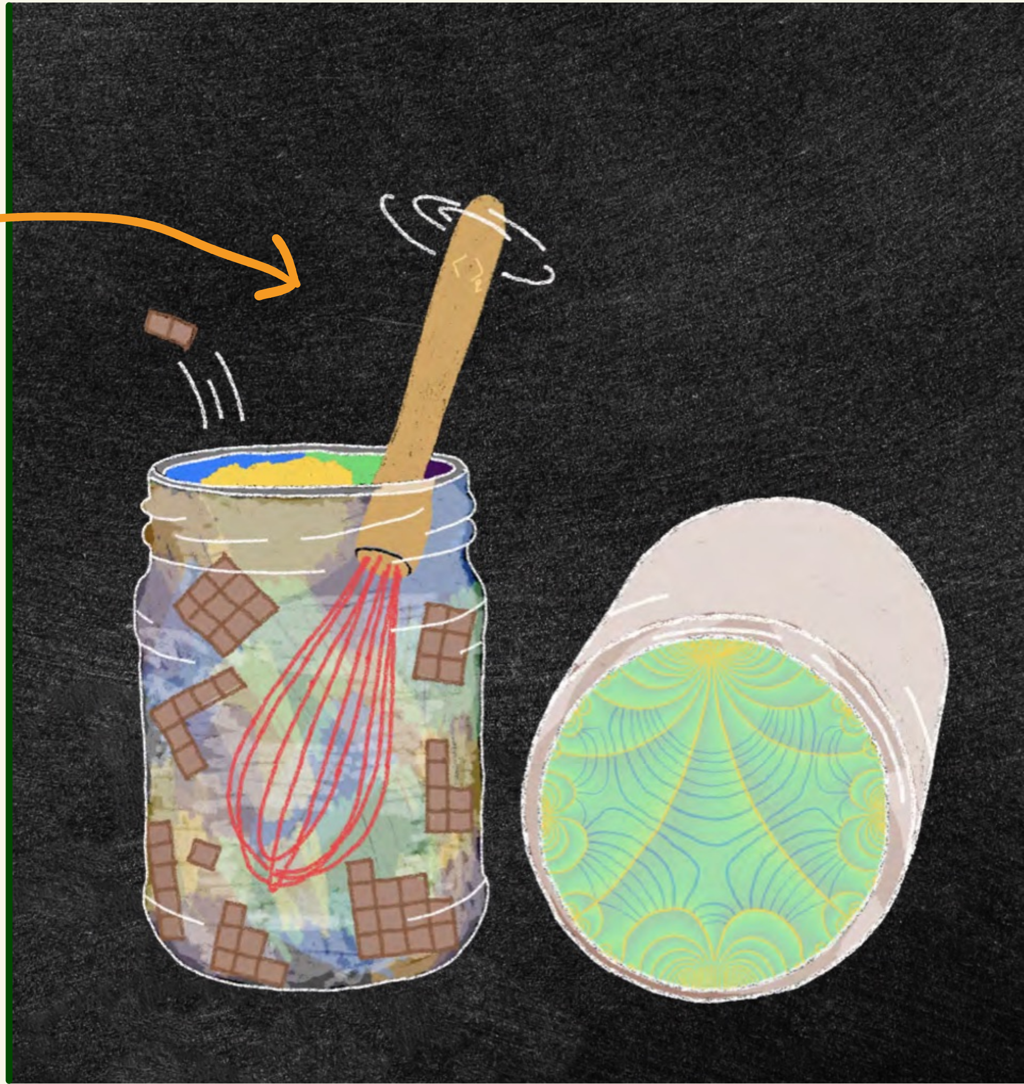
⚠ The q-bracket is not a ring homomorphism

Rk
- Another such algebra, going back to the work of Dijkgraaf in string theory, was found by Bloch–Okounkov.
- How could such a result exist? There is a truly wonderful explanation using convolution products, but it doesn't fit in this margin.

Recipe

From partitions to modular forms

stable under addition and multiplication



Thank you!