From partitions to q-analogues of multiple zeta values

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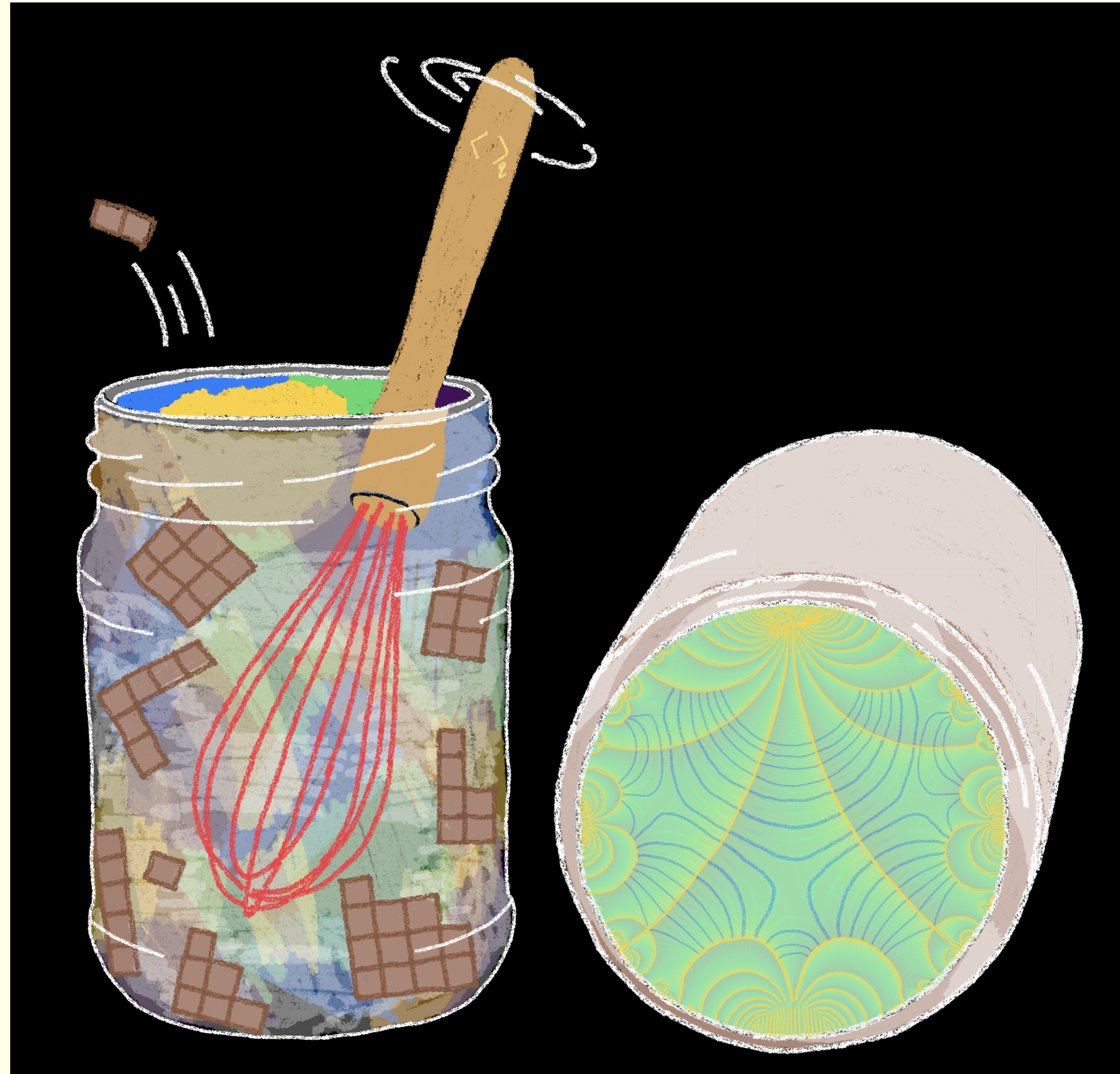


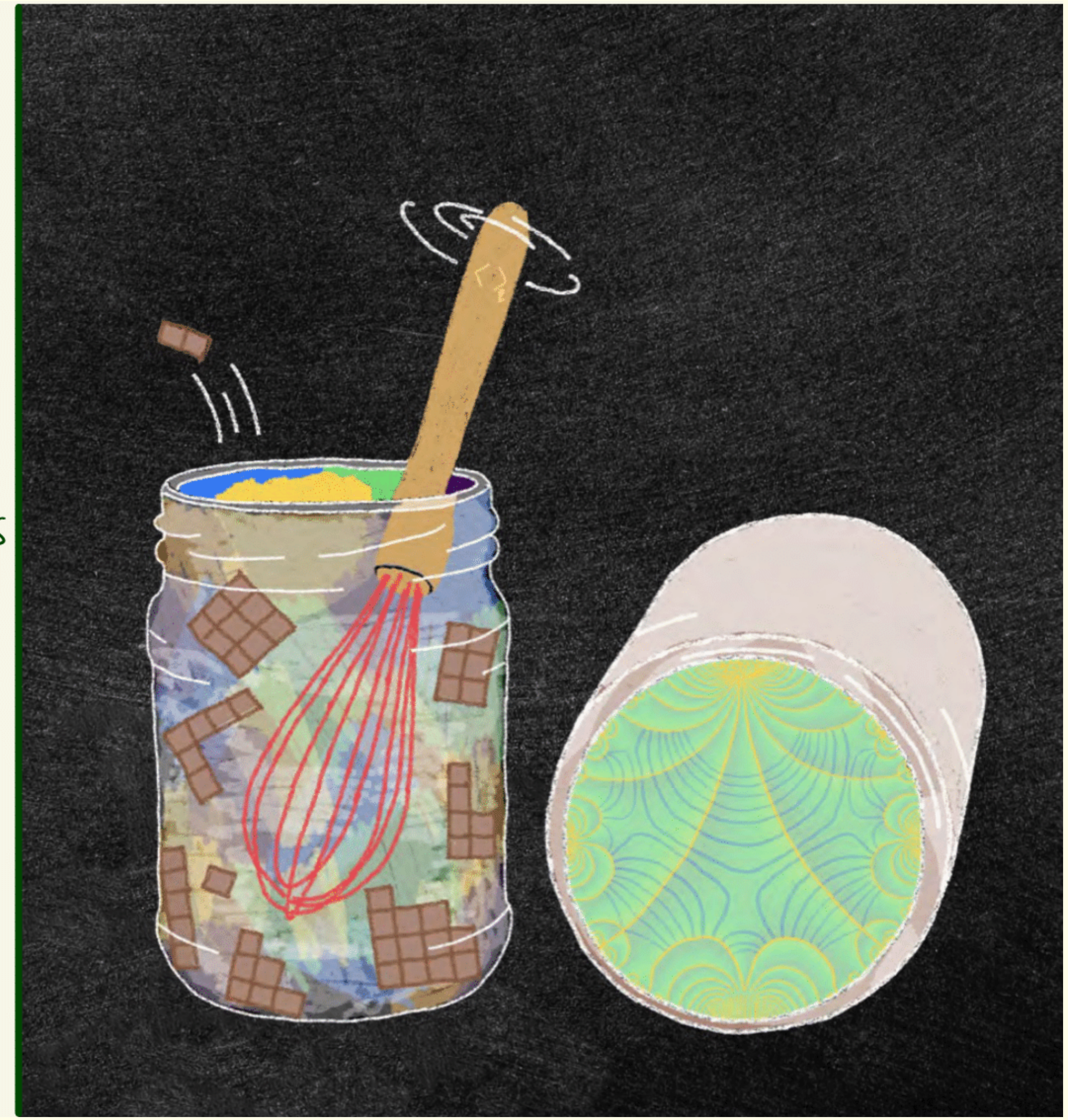
Part I: Partitions & modular forms

String theory
$\longleftrightarrow$
Enumerative geometry

Algebraic combinatorics (partitions)

my thesis

Number theory (modular forms, multiple zeta values)



cake

Ingredients

200g	butter
200g	sugar
4	eggs
200g	self-rising flour
1g	salt

bread

Ingredients

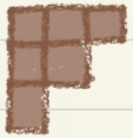
200g	self-rising flour
1 g	salt
150 g	water

Quasimodular forms

Ingredients

5 

6 

6 

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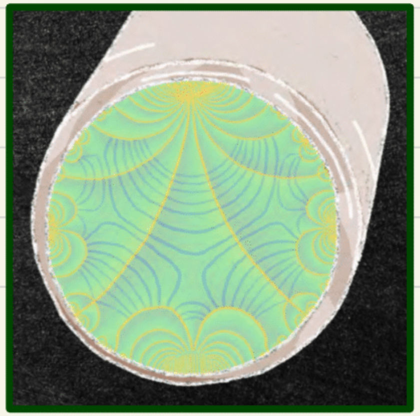
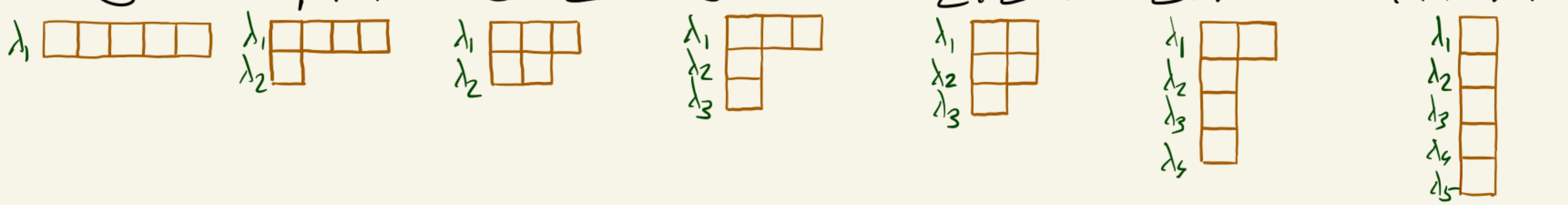


Ingredients: partitions

Ex Partitions λ of $n=5$ are

$5 \quad 4+1 \quad 3+2 \quad 3+1+1 \quad 2+2+1 \quad 2+1+1+1 \quad 1+1+1+1+1$



Def A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a way to write an integer n as a sum of positive integers $\lambda_1, \lambda_2, \ldots$ (where the order of summation doesn't matter).

Thm (Euler)

integer n that λ is a partition of

$$\sum_{\lambda \in P} q^{|\lambda|} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots = \prod_{m=1}^{\infty} \frac{1}{(1-q)^m}.$$

$\sum_n p(n) q^n =$

set of all partitions $n = \lambda_1 + \lambda_2 + \ldots$ for all n

Ingredients: how much of each partition?

Recall 200g butter, 200g sugar, ...

set of all partitions $n = \lambda_1 + \lambda_2 + \cdots$ for all n

Symmetric functions $\quad S_k : \mathcal{P} \to \mathbb{Q} \qquad S_k(\lambda) = \sum_i \lambda_i^k$

Ex $\quad S_3(2+2+1) = 2^3 + 2^3 + 1$

$= 17.$

Ex $\quad S_1(\lambda) = \sum_i \lambda_i = |\lambda|$

0

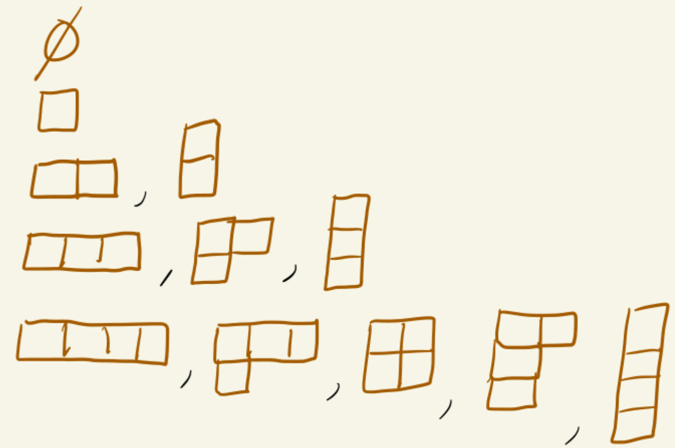
1

2

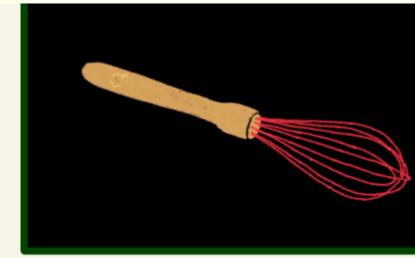
3

4

⋮



Whisk: the q-bracket



Given $f : \mathcal{P} \to \mathbb{Q}$, the q-bracket of f is given by

Ex S_k

formal power series in q.

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} \in \mathbb{Q}[[q]]$$

generating series of partitions (Euler)

Ex

$$\langle S_1 \rangle_q = \frac{\sum_{\lambda \in \mathcal{P}} |\lambda|\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \dots$$

recall $S_1(\lambda) = |\lambda| = \sum_i \lambda_i$

$$= \sum_n \underbrace{\Big(\sum_{d \mid n} d \Big)}_{\text{divisor sum}} q^n .$$

a quasimodular form

Proof $\langle S_1 \rangle_q = q \frac{\partial}{\partial q} \log \left(\sum q^{|\lambda|} \right) = q \frac{\partial}{\partial q} \log \left(\prod_m \frac{1}{1 - q^m} \right)$

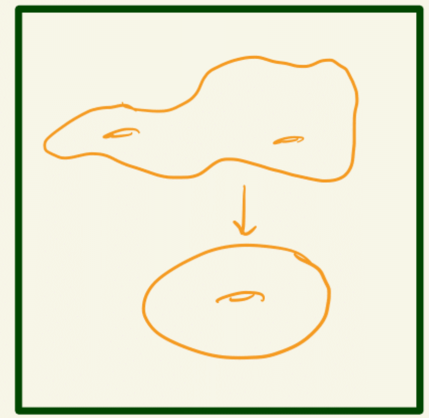
$$= \sum_{m,r} m\, q^{mr} = \sum_n \Big(\sum_{d \mid n} d \Big) q^n$$

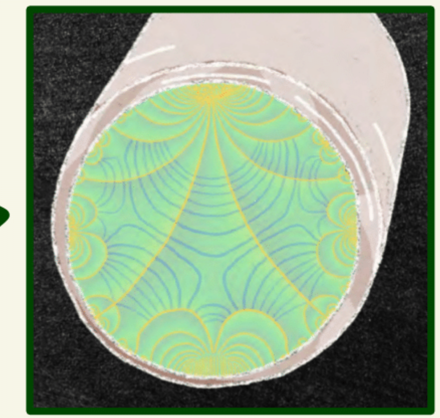
Result: a (quasi) modular form

Quasimodular forms are everywhere:

- Fermat's last theorem
- Sphere packing problem
- Coverings of tori

...





Two key properties of quasimodular forms (QMF):

(i) The sum, difference and product of two QMFs is a QMF
Also, the derivative is a QMF.

(ii) Only one QMF is needed to generate all of them using (i) iteratively.

$$G_k = -\frac{B_k}{2k} + \sum_{m,r} m^{k-1} q^{mr} \in \mathbb{Q}[[q]] \quad \text{for all } k \geq 2 \text{ even}$$

$$\in \widetilde{M}$$

$$\widetilde{M} = \mathbb{Q}[G_2, G_4, G_6]$$

Quasimodula forms and zeta values (intermezzo)

Recall $\zeta(k) = \sum_{m \geq 1} \frac{1}{m^k}$, e.g. $\zeta(2) = \frac{\pi^2}{6}$

Then

$$\zeta(k) = (-1)^{k/2+1} \frac{(2\pi)^k B_k}{2k!} \qquad k \text{ even integer}$$

Thm The map $\mathbb{Q}[G_2, G_4, G_6] \longrightarrow \cancel{\mathbb{R}}\ \mathbb{Q}[\pi^2]$

$$f \longmapsto \lim_{q \to 1} (1-q)^{\mathrm{wt} f} f$$

is a ring homomorphism. Moreover, its image equals $\mathbb{Q}[\pi^2]$

Ex $\lim_{q\to 1} (1-q)^2 G_2 = \lim_{q \to 1} (1-q)^2 \left(-\frac{1}{24} + \sum_{m,r} m q^{mr}\right) = \sum_{m,r} \lim_{q\to 1} (1-q)^2 m q^{mr}$

$$= \sum_r \lim_{q\to 1} \frac{(1-q)^2 q^r}{(1-q^r)^2} = \sum_r \frac{1}{r^2} = \zeta(2) = \frac{\pi^2}{6}$$

3 recipes : addition

Ingredients	cake
200g	butter
200g	sugar
4	eggs
200 g	self-rising flour
1 g	salt

+

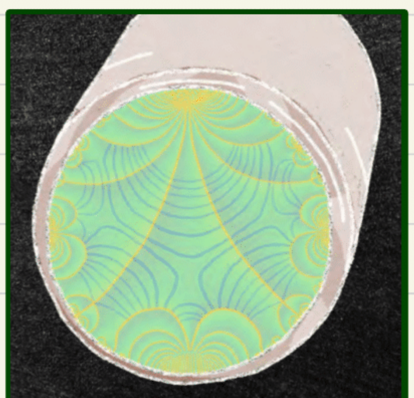
Ingredients	bread
200g	self-rising flour
1 g	salt
150 g	water

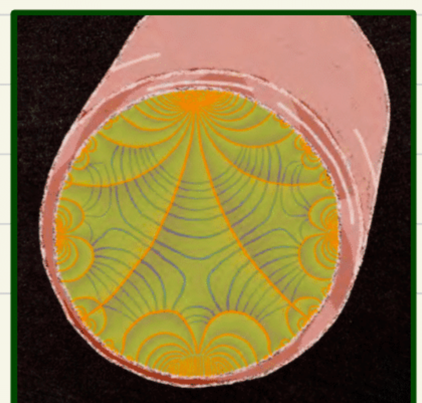
=

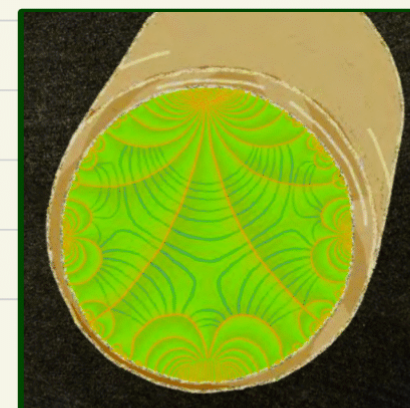
Ingredients	cake bread
200g	butter
200g	sugar
4	eggs
400 g	self-rising flour
2 g	salt
150 g	water

3 recipes : addition

Ingredients S_1	Ingredients S_3	Ingredients $S_1 + S_3$
1	1	2
2	8	10
2	2	4
3	27	30
3	9	12
3	3	6
...	...	...







3 recipes: multiplication

Ingredients cake

200 g	butter
200 g	sugar
4	eggs
200 g	self-rising flour
1 g	salt

×

Ingredients bread

200 g	self-rising flour
1 g	salt
150 g	water

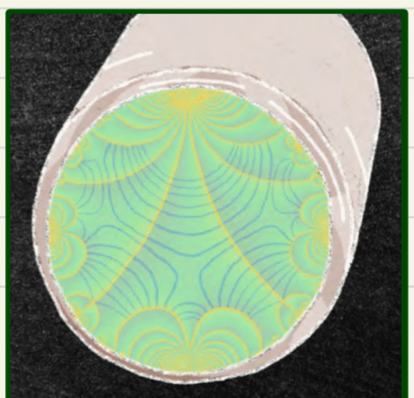
=

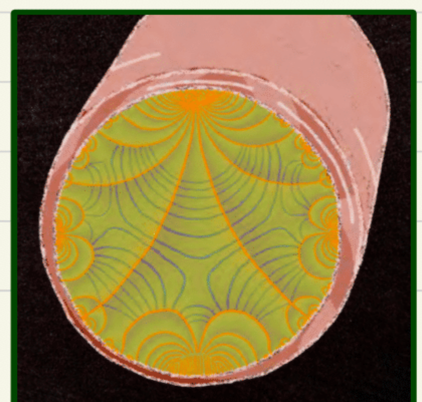
salted flour?!

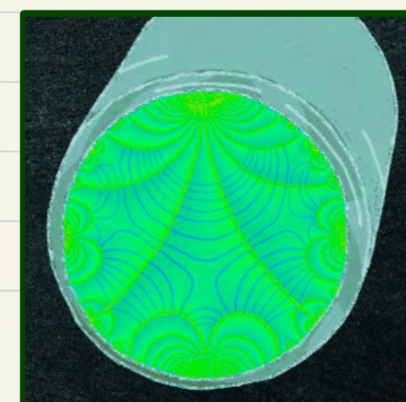
$40000\,g^2$	self-rising flour ?
$1\,g^2$	salt ?

3 recipes: multiplication

Ingredients S_1		Ingredients S_3		Ingredients S_1S_3
1		1		1
2		8		16
2		2		4
3	×	27	=	81
3		9		27
3		3		9
...		...		...







Result: a (quasi) modular form

Observation $\langle S_1^2\rangle_q$, $\langle S_3^2\rangle_q$, $\langle S_5^2\rangle_q$, $\langle S_7^2\rangle_q$ are all QMFs.
Even combinations as $\langle S_1 S_3\rangle_q$, $\langle S_1 S_5 S_7\rangle_q$ and $\langle S_1^2 S_3 S_5^3 S_7^4\rangle_q$ are QMFs:

Thm (vI)
For all f in the algebra generated by the S_k (k odd)

vector space & ring

$$\langle f\rangle_q \text{ is a QMF.}$$

⚠ The q-bracket is not a ring homomorphism

Rk • Another such algebra, going back to the work of Dijkgraaf in string theory, was found by Bloch–Okounkov.

$$Q_k(\lambda) = \beta_k + \sum_{i=1}^{\infty}\left(\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i+\tfrac{1}{2}\right)^{k-1}\right) \quad \left(\frac{1}{z} + \sum_k \beta_k \frac{z^{k-1}}{(k-1)!} = \frac{1}{2\sinh(\frac{z}{2})}\right)$$

$\langle Q_\ell S_\ell\rangle_q$ is not a QMF

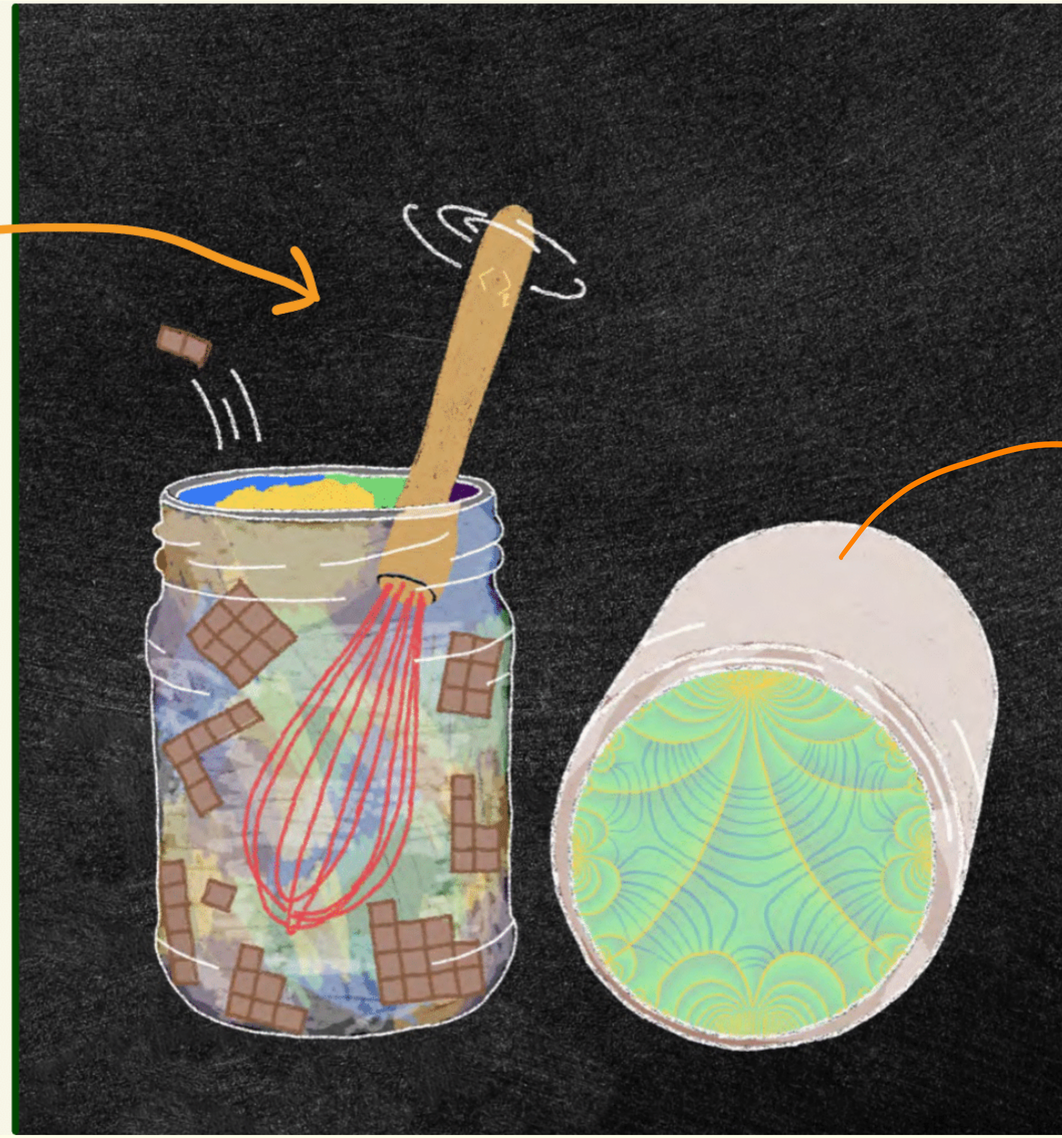
Thm (Bloch–Okounkov) For all f in the algebra generated by the Q_k

$$\langle f\rangle_q \text{ is a QMF.}$$

Recipe

From partitions to modular forms

stable under addition and multiplication



Limit

From modular forms to ζ's.

$$\boxed{\zeta(k) = \sum_{m \geq 1} \frac{1}{m^k}}$$

stable under addition and multiplication

algebra of S_k

algebra of Q_k

$$\mathcal{S}, \Lambda^* \xrightarrow{\langle\ \rangle_q} \widetilde{M} \xrightarrow{\lim_{q \to 1} (1-q)^k f} \mathbb{Q}[\pi^2]$$

quasimodular forms

Part II : partitions and multiple zeta values (jt with Henrik Bachmann)

Def For integers $k_1, \dots, k_r \geq 1$, $k_1 \geq 2$, let

$$\zeta(k_1, \dots, k_r) := \sum_{m_1 > \dots > m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

and let $\mathcal{Z} \subset \mathbb{R}$ be the vector space of MZVs.

Ex There are many relations between MZVs!

$$\zeta(2,3) + \zeta(3,2) + \zeta(5) = \zeta(2)\zeta(3) = \zeta(2,3) + 3\,\zeta(3,2) + 6\,\zeta(4,1) \quad \text{(double shuffle)}$$

$$\zeta(5) - \zeta(4,1) + \zeta(3,1,1) - \zeta(2,1,1,1) = 0 \quad \text{(Ohno–Zagier)}$$

$$\zeta(4,1) + \zeta(3,2) + \zeta(2,3) = \zeta(5) \quad \text{(sum formula)}$$

Overview

$$\begin{array}{ccccc}
\mathbb{Q}^{\mathscr{P}} = \{\mathscr{P} \to \mathbb{Q}\} & \longrightarrow & \mathbb{Q}[[q]] & \longrightarrow & \mathbb{R} \\
\cup & & \cup & & \cup \\
\mathcal{P} & \longrightarrow & \widetilde{\mathcal{Z}}_q & \longrightarrow & \mathcal{Z} \\
\cup & & \cup & & \cup \\
\Lambda^*, \mathcal{S} & \xrightarrow[\langle\,\rangle_q]{} & \widetilde{M} & \xrightarrow[\lim_{q\to 1}(1-q)^k]{} & \mathbb{Q}[\pi^2]
\end{array}$$

no relations $\to \mathcal{P}$

Rk For any $g = 1 + O(q) \in \mathbb{Q}[[q]]$

$$\mathbb{Q}^{\mathbb{N}} \simeq \mathbb{Q}[[q]]$$
$$f \longmapsto \Big(\sum_n f(n)\, q^n\Big) g$$

Def The $\underline{u}$-bracket is given by

$$\langle\,\rangle_{\underline{u}} : \mathbb{Q}^{\mathscr{P}} \xrightarrow{\sim} \mathbb{Q}[[u_1, u_2, u_3, \ldots]]$$
$$f \mapsto \frac{\sum_\lambda f(\lambda)\, u_\lambda}{\sum_\lambda u_\lambda}$$

$$(u_\lambda = u_{\lambda_1} u_{\lambda_2} u_{\lambda_3} \cdots)$$

$$\boxed{\langle\,\rangle_q = \langle\,\rangle_{\underline{u}}\big|_{u_i = q^i}}$$

Def The $\underline{u}$-bracket is given by

$$\langle\ \rangle_{\underline{u}} : \mathbb{Q}^{\mathcal{P}} \rightsquigarrow \mathbb{Q}[u_1, u_2, u_3, \dots]$$

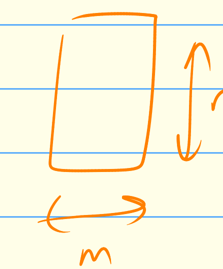
$$f \mapsto \frac{\sum f(\lambda)\, u_\lambda}{\sum u_\lambda}$$

Ex

$$S_k(\lambda) = \sum_i \lambda_i^{k-1} = \sum_m m^{k-1} r_m(\lambda)$$

multiplicity of m in λ

$$\langle S_k \rangle_{\underline{u}} = \sum_{m, r \geq 1} m^{k-1} u_m^r$$



Def Let

$$\psi : \bigoplus_{n \geq 0} \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \to \mathbb{Q}[[u_1, u_2, \dots]]$$

$$g(x_1, \dots, y_n) \mapsto \sum_{\substack{m_1 > \dots > m_n \geq 1 \\ r_1, \dots, r_n \geq 1}} g(m_1, \dots, m_n, r_1, \dots, r_n)\, u_{m_1}^{r_1} \cdots u_{m_n}^{r_n}$$

Then, we let $\langle \mathcal{P} \rangle_{\underline{u}} := \operatorname{Im} \psi$

⚠ ψ is injective

Bi-brackets

Let $g(x_1, \dots, y_n) = \prod_j X_j^{d_j} \frac{Y_j^{k_j-1}}{(k_j-1)!}$ for $d_1, \dots, d_n \geq 0$, $k_1, \dots, k_n \geq 1$

$\langle\ \rangle_{\underline{u}}^{-1}(\psi(g)) : \mathcal{P} \to Q$ is given by

$$P\begin{pmatrix} k_1 & \dots & k_n \\ d_1 & \dots & d_n \end{pmatrix} := \lambda \longmapsto \sum_{m_1 > \dots > m_n \geq 1} \prod_j \left(m_j^{d_j} \sum_{n=1}^{r_{m_j}(\lambda)} \frac{n^{k_j-1}}{(k_j-1)!} \right)$$

Then,

$$\left\langle P\begin{pmatrix} k_1 & \dots & k_n \\ d_1 & \dots & d_n \end{pmatrix} \right\rangle_q = \sum_{\substack{m_1 > \dots > m_n \geq 1 \\ r_1, \dots, r_n \geq 1}} \prod_j \left(m_j^{d_j} \frac{r_j^{k_j-1}}{(k_j-1)!} q^{m_j r_j} \right) \doteq \begin{bmatrix} k_1 & \dots & k_n \\ d_1 & \dots & d_n \end{bmatrix} \in \mathbb{Z}_q$$

Thm For any $f \in \mathcal{P}$, there exists a unique $\deg(f) \in \mathbb{Z}_{\geq 0}$ s.t.

$$\lim_{q \to 1} (1-q)^{\deg(f)} \langle f \rangle_q \quad \in \mathbb{Z}[T]$$

Thm For any $f \in P$, there exists a unique $\deg(f) \in \mathbb{Z}_{\geq 0}$ s.t.

$$Z(f) := \lim_{q \to 1} (1-q)^{\deg f} \langle f \rangle_q \in \mathcal{Z}[T]$$

In fact,

$$\deg P\begin{pmatrix} k_1 & \cdots & k_n \\ d_1 & & d_n \end{pmatrix} = \max_{j \in \{1, \dots, n+1\}} \left(\sum_{i \leq j} (d_i + 1) + \sum_{i > j} k_i \right) \leq \sum (d_i+1) + \sum k_i$$

wt f ;)

$$\boxed{\text{I.e. } \lim_{q \to 1} (1-q)^{\mathrm{wt} f} \langle f \rangle_q \in \mathcal{Z}[T]}$$

Ex

$$\deg P\begin{pmatrix} k_1 & \cdots & k_n \\ d_1 & & d_n \end{pmatrix} = \sum k_i \quad \text{if } k_i > d_i \ \forall i$$

$$\xrightarrow{Z} \zeta(k_1 - d_1, \dots, k_n - d_n)$$

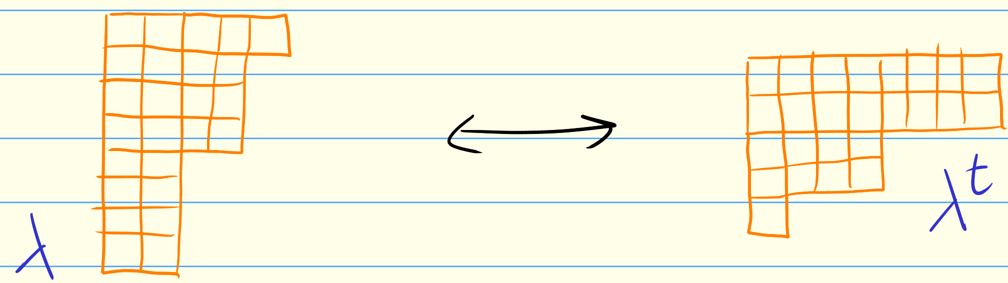
$$\deg P\begin{pmatrix} 1 \cdots 1 & k_t & \cdots & k_n \\ d_1 \cdots d_{t-1} & 0 & \cdots & 0 \end{pmatrix} = \sum_{i=1}^{t-1} (d_i + 1) + \sum_{i > j} k_i = \mathrm{wt}\, P\begin{pmatrix} 1 \cdots 1 & k_t \cdots k_n \\ d_1 \cdots d_{t-1} & 0 \cdots 0 \end{pmatrix}$$

$$\xrightarrow{Z} \xi(d_1, \dots, d_{t-1}) \cdot \zeta(k_1, \dots, k_n)$$

↑ Linear combination of MZVs of wt $\leq \sum_{i=1}^{t-1}(d_i+1)$

Involutions on P, Z_q and Z

Recall The conjugate λ^t of a partition λ



This induces an involution $\iota : \mathbb{Q}[[u_1, u_2, \dots]] \circlearrowleft$

$$u_\lambda \longmapsto u_{\lambda^t} \qquad (u_\lambda = u_{\lambda_1} u_{\lambda_2} \cdots)$$

and hence an involution on $f \in P$ s.t. $\langle f \rangle_q = \langle \iota f \rangle_q$.
In fact, this involution extends to $b \in Z_q$ s.t.

$$b - \iota(b) = 0 \in \mathbb{Q}[[q]]$$

$$\zeta(d_1, \dots, d_n) := Z\, \iota P\begin{pmatrix} 1 & \cdots & 1 \\ d_1 & \cdots & d_n \end{pmatrix}$$

Products on $\mathcal{P}$

On $\mathcal{P}$ there are three natural products

- the pointwise product $\odot$ coming from $\mathcal{P} \to \mathbb{Q} \simeq$
- the harmonic product $\circledast$ coming from $\mathbb{Q}[[u_1, u_2, u_3, \dots]]$
- the shuffle product $\sqcup\!\sqcup$ defined by $f \sqcup\!\sqcup g := \iota(\iota(f) \circledast \iota(g))$

Thm $Z^{reg}(f \circledast g - f \sqcup\!\sqcup g) \stackrel{!}{=} 0$ for all $f, g \in \mathcal{P}$ gives all extended double shuffle relations.

Ex $P\binom{2}{0} \circledast P\binom{3}{0} = P\begin{pmatrix}2 & 3\\ 0 & 0\end{pmatrix} + P\begin{pmatrix}3 & 2\\ 0 & 0\end{pmatrix} + P\binom{5}{0} - \frac{1}{12} P\binom{3}{0}$

"$\zeta(2) \circledast \zeta(3)$" $\xrightarrow{Z_5} \zeta(2,3) + \zeta(3,2) + \zeta(5) + 0$

deg 4

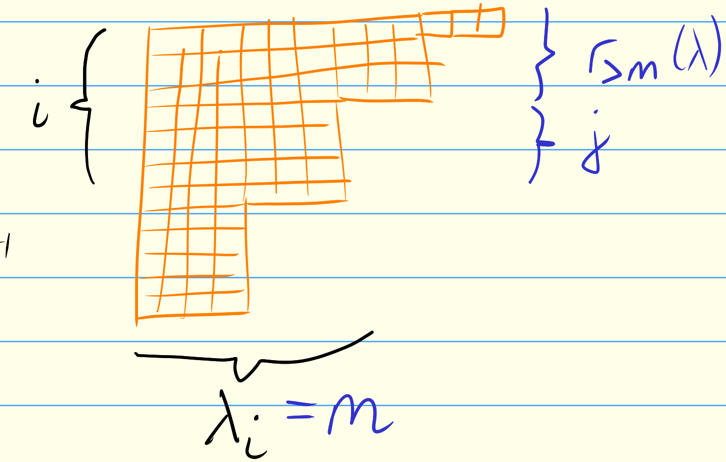
$P\binom{2}{0} \sqcup\!\sqcup P\binom{3}{0} = P\begin{pmatrix}2 & 3\\ 0 & 0\end{pmatrix} + 3P\begin{pmatrix}3 & 2\\ 0 & 0\end{pmatrix} + 6P\begin{pmatrix}4 & 1\\ 0 & 0\end{pmatrix} + 3P\binom{4}{1} - 3P\binom{4}{0}$

$\xrightarrow{Z_5} \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) + 0 + 0$

Relations from shifted symmetric functions

Recall $$Q_k(\lambda) = \beta_k + \sum_i \left(\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i + \tfrac{1}{2}\right)^{k-1}\right)$$

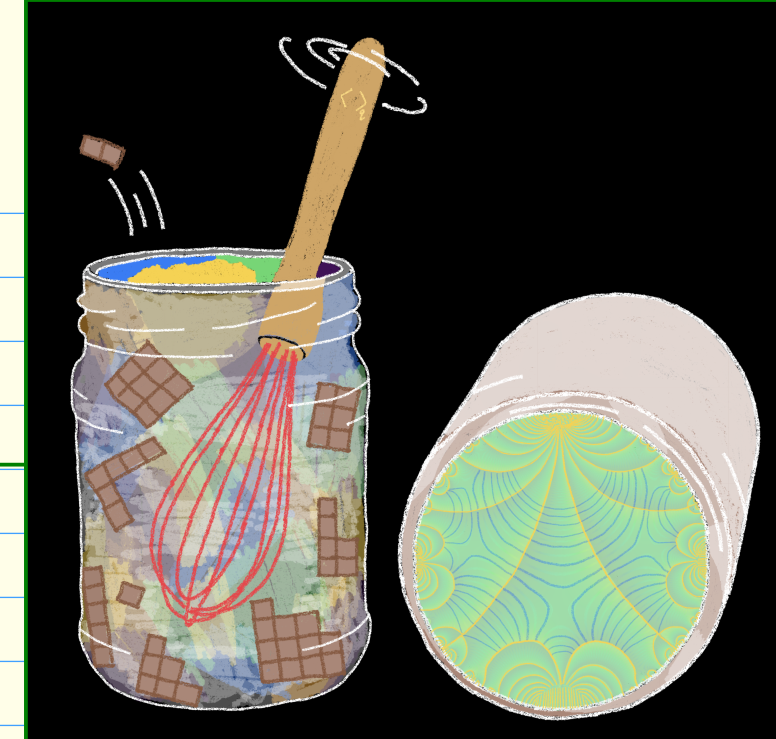
$$= \beta_k + \sum_{m,j} \left(m - (r_{\geq m} + j) + \tfrac{1}{2}\right)^{k-1} - \left(-(r_{\geq m} + j) + \tfrac{1}{2}\right)^{k-1}$$



Thm $$Q_k - \sum_{i=0}^{k-2} \frac{(-1)^i}{(k-1-i)!} P\begin{pmatrix} 1, \dots, 1, & 1 \\ \underbrace{0, \dots, 0}_{i}, & k-1-i \end{pmatrix} \xrightarrow{Z} 0$$

Cor [Ohno-Zagier] $$\sum_{k \geq 2} \sum_{i=0}^{k-2} (-1)^i \zeta(k-i, \underbrace{1, \dots, 1}_{i})\, z^{k+2} = 1 - \exp\left(\sum_{n \geq 2} \zeta(n) \frac{z^n + (-z)^n}{n}\right)$$

Thank you!



no relations

$$\begin{tikzcd}
\mathbb{Q}^{\rho} \arrow[r, phantom, "\overset{\langle\,\rangle_u}{\simeq}"] & \mathbb{Q}[[u_1,u_2,\ldots]] \arrow[r] \arrow[loop above, "\iota"] & \mathbb{Q}[[q]] \arrow[r] & \mathbb{R} \\
\mathcal{P} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\simeq"] \arrow[dr, phantom, "\cup"] & \bigoplus_n \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n] \arrow[u, phantom, "\cup"] \arrow[r] \arrow[loop above, "\iota"] \arrow[d, phantom, "\cup"] & \mathcal{Z}_q \arrow[u, phantom, "\cup"] \arrow[r, two heads] & \mathcal{Z}[T] \quad *, ⧢ \arrow[u, phantom, "\cup"] \\
 & \mathcal{S}, \Lambda^* \arrow[r, "\langle\,\rangle_q"'] & \widetilde{M} \arrow[u, phantom, "\cup"] \arrow[r, "\lim_{q\to 1}(1-q)^k"'] & \mathbb{Q}[\pi^2] \arrow[u, phantom, "\cup"]
\end{tikzcd}$$